Complete Synchronization of Hyperchaotic Systems via Novel Sliding Mode Control

Sundarapandian Vaidyanathan and Sivaperumal Sampath

Abstract Chaos in nonlinear dynamics occurs widely in physics, chemistry, biology, ecology, secure communications, cryptosystems and many scientific branches. Synchronization of chaotic systems is an important research problem in chaos theory. Sliding mode control is an important method used to solve various problems in control systems engineering. In robust control systems, the sliding mode control is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as insensitivity to parameter uncertainties and disturbance. In this work, we derive a novel sliding mode control method for the complete synchronization of identical chaotic or hyperchaotic systems. The general result is derived using novel sliding mode control method. The general result is established using Lyapunov stability theory. As an application of the general result, the problem of complete synchronization of identical hyperchaotic Vaidyanathan systems (2014) is studied and a new sliding mode controller is derived. The Lyapunov exponents of the hyperchaotic Vaidyanathan system are obtained as $L_1 = 1.4252$, $L_2 = 0.2445$, $L_3 = 0$ and $L_4 = -17.6549$. Since the Vaidyanathan hyperjerk system has two positive Lyapunov exponents, it is hyperchaotic. Also, the Kaplan-Yorke dimension of the Vaidyanathan hyperjerk system is obtained as $D_{KY} = 3.0946$. Numerical simulations using MATLAB have been shown to depict the phase portraits of the hyperchaotic Vaidyanathan system and the sliding mode controller design for the anti-synchronization of identical hyperchaotic Vaidyanathan systems.

Keywords Chaos · Chaotic systems · Hyperchaos · Hyperchaotic systems · Sliding mode control · Synchronization

S. Vaidyanathan (✉)
Research and Development Centre, Vel Tech University, Avadi, Chennai 600062, Tamil Nadu, India
e-mail: sundarvtu@gmail.com

S. Sampath
School of Electrical and Computing, Vel Tech University, Avadi, Chennai 600062, Tamil Nadu, India
e-mail: sivaperumals@gmail.com

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1 Introduction

Chaos theory describes the quantitative study of unstable aperiodic dynamic behaviour in deterministic nonlinear dynamical systems. For the motion of a dynamical system to be chaotic, the system variables should contain some nonlinear terms and the system must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions.

Chaos theory and control systems have many important applications in science and engineering [1, 8–11, 102]. Some commonly known applications are oscillators [23, 50], lasers [28, 96], chemical reactions [16, 35, 68, 69, 71, 73, 74, 78], biology [13, 25, 64–67, 70, 72, 76, 77], ecology [17, 51], encryption [26, 100], cryptosystems [43, 57], mechanical systems [3–7], secure communications [14, 33, 98], robotics [32, 34, 90], cardiology [39, 94], intelligent control [2, 30], neural networks [19, 22, 31], memristors [38, 91], etc.

A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents [8]. Thus, the dynamics of a hyperchaotic system can expand in several different directions simultaneously. Thus, the hyperchaotic systems have more complex dynamical behaviour and they have miscellaneous applications in engineering such as secure communications [15, 27, 95], cryptosystems [18, 42, 101], fuzzy logic [49, 99], electrical circuits [93, 97], etc.

The minimum dimension of an autonomous, continuous-time, hyperchaotic system is four. The first 4-D hyperchaotic system was found by Rössler [44]. Many hyperchaotic systems have been reported in the chaos literature such as hyperchaotic Lorenz system [20], hyperchaotic Lü system [12], hyperchaotic Chen system [29], hyperchaotic Wang system [92], hyperchaotic Vaidyanathan systems [61, 63, 79, 86, 88, 89], hyperchaotic Pham system [36], etc.

The synchronization of chaotic systems aims to synchronize the states of master and slave systems asymptotically with time. There are many methods available for chaos synchronization such as active control [21, 46, 47, 81, 83], adaptive control [37, 45, 48, 52–54, 60, 75, 80, 82], sliding mode control [55, 62, 85, 87], backstepping control [40, 41, 56, 84], etc.

The design goal of complete synchronization of chaotic systems is to use the output of the master system to control the slave system so that the states of the slave system coincide with the states of the master system asymptotically, i.e.

$$\lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0, \quad \forall \mathbf{x}(0), \mathbf{y}(0) \in \mathbf{R}^n \tag{1}$$

In this research work, we derive a general result for the complete synchronization of chaotic systems using sliding mode control (SMC) theory [58, 59]. The sliding mode control approach is recognized as an efficient tool for designing robust controllers for linear or nonlinear control systems operating under uncertainty conditions. A major advantage of sliding mode control is low sensitivity to parameter variations in the plant and disturbances affecting the plant, which eliminates the necessity of exact modeling of the plant.

In the sliding mode control theory, the control dynamics will have two sequential modes, viz. the reaching mode and the sliding mode. Basically, a sliding mode controller (SMC) design consists of two parts: hyperplane design and controller design. A hyperplane is first designed via the pole-placement approach in the modern control theory and a controller is then designed based on the sliding condition. The stability of the overall system is guaranteed by the sliding condition and by a stable hyperplane.

This work is organized as follows. In Sect. 2, we discuss the problem statement for the complete synchronization of identical chaotic or hyperchaotic systems. In Sect. 3, we derive a general result for the complete synchronization of identical chaotic or hyperchaotic systems using novel sliding mode control. In Sect. 4, we describe the hyperchaotic Vaidyanathan system and its phase portraits. In Sect. 5, we describe the qualitative properties of the hyperchaotic Vaidyanathan system. The Lyapunov exponents of the hyper system are obtained as $L_1 = 1.4252$, $L_2 = 0.2445$, $L_3 = 0$ and $L_4 = -17.6549$, which shows that the hyperchaotic Vaidyanathan system is hyperchaotic.

In Sect. 6, we describe the sliding mode controller design for the complete synchronization of identical hyperchaotic Vaidyanathan systems using novel sliding mode control and its numerical simulations using MATLAB. Section 7 contains the conclusions of this work.

2 Problem Statement

As the *master* system, we consider the chaotic or hyperchaotic system given by

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}) \tag{2}$$

where $\mathbf{x} \in \mathbf{R}^n$ denotes the state of the system, $A \in \mathbf{R}^{n \times n}$ denotes the matrix of system parameters and $f(\mathbf{x}) \in \mathbf{R}^n$ contains the nonlinear parts of the system.

As the *slave* system, we consider the controlled identical system given by

$$\dot{\mathbf{y}} = A\mathbf{y} + f(\mathbf{y}) + \mathbf{u} \tag{3}$$

where $\mathbf{y} \in \mathbf{R}^n$ denotes the state of the system and $\mathbf{u}$ is the control.

The complete synchronization error is defined as

$$\mathbf{e} = \mathbf{y} - \mathbf{x} \tag{4}$$

The error dynamics is easily obtained as

$$\dot{\mathbf{e}} = A\mathbf{e} + \psi(\mathbf{x}, \mathbf{y}) + \mathbf{u}, \tag{5}$$

where

$$\psi(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - f(\mathbf{y}) \tag{6}$$

Thus, the complete synchronization problem between the systems (2) and (3) can be stated as follows: Find a controller $\mathbf{u}(\mathbf{x}, \mathbf{y})$ so as to render the anti-synchronization error $\mathbf{e}(t)$ to be globally asymptotically stable for all values of $\mathbf{e}(0) \in \mathbf{R}^n$, i.e.

$$\lim_{t \to \infty} \|\mathbf{e}(t)\| = 0 \;\text{ for all }\; \mathbf{e}(0) \in \mathbf{R}^n \tag{7}$$

3 A Novel Sliding Mode Control Method for Solving Complete Synchronization Problem

This section details the main results of this work, viz. novel sliding mode controller design for achieving complete synchronization of chaotic or hyperchaotic systems.

First, we start the design by setting the control as

$$\mathbf{u}(t) = -\psi(\mathbf{x}, \mathbf{y}) + Bv(t) \tag{8}$$

In Eq. (8), $B \in \mathbf{R}^n$ is chosen such that (A, B) is completely controllable.

By substituting (8) into (5), we get the closed-loop error dynamics

$$\dot{\mathbf{e}} = A\mathbf{e} + Bv \tag{9}$$

The system (9) is a linear time-invariant control system with single input v.

Next, we start the sliding controller design by defining the sliding variable as

$$s(\mathbf{e}) = C\mathbf{e} = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n, \tag{10}$$

where $C \in \mathbf{R}^{1 \times n}$ is a constant vector to be determined.

The sliding manifold S is defined as the hyperplane

$$S = \{\mathbf{e} \in \mathbf{R}^n \;:\; s(\mathbf{e}) = C\mathbf{e} = 0\} \tag{11}$$

We shall assume that a sliding motion occurs on the hyperplane S.

In sliding mode, the following equations must be satisfied:

$$s = 0 \tag{12a}$$

$$\dot{s} = CA\mathbf{e} + CBv = 0 \tag{12b}$$

We assume that

$$CB \neq 0 \tag{13}$$

The sliding motion is influenced by equivalent control derived from (12b) as

$$v_{\text{eq}}(t) = -(CB)^{-1}\, CA\mathbf{e}(t) \tag{14}$$

By substituting (14) into (9), we obtain the equivalent error dynamics in the sliding phase as follows:

$$\dot{\mathbf{e}} = A\mathbf{e} - (CB)^{-1}CA\mathbf{e} = E\mathbf{e}, \tag{15}$$

where

$$E = \left[I - B(CB)^{-1}C\right]A \tag{16}$$

We note that E is independent of the control and has at most $(n-1)$ non-zero eigenvalues, depending on the chosen switching surface, while the associated eigenvectors belong to $\ker(C)$.

Since (A, B) is controllable, we can use sliding control theory [58, 59] to choose B and C so that E has any desired $(n-1)$ stable eigenvalues.

This shows that the dynamics (15) is globally asympotically stable.

Finally, for the sliding controller design, we apply a novel sliding control law, viz.

$$\dot{s} = -ks - qs^2\,\text{sgn}(s) \tag{17}$$

In (17), $\text{sgn}(\cdot)$ denotes the *sign* function and the SMC constants $k > 0$, $q > 0$ are found in such a way that the sliding condition is satisfied and that the sliding motion will occur.

By combining Eqs. (12b), (14) and (17), we finally obtain the sliding mode controller $v(t)$ as

$$v(t) = -(CB)^{-1}\left[C(kI + A)\mathbf{e} + qs^2\,\text{sgn}(s)\right] \tag{18}$$

Next, we establish the main result of this section.

Theorem 1 *The sliding mode controller defined by (8) achieves complete synchronization between the identical chaotic systems (2) and (3) for all initial conditions* $\mathbf{x}(0), \mathbf{y}(0)$ *in* $\mathbf{R}^n$*, where* v *is defined by the novel sliding mode control law (18),* $B \in \mathbf{R}^{n\times 1}$ *is such that* (A, B) *is controllable,* $C \in \mathbf{R}^{1\times n}$ *is such that* $CB \neq 0$ *and the matrix* E *defined by (16) has* $(n-1)$ *stable eigenvalues.*

Proof Upon substitution of the control laws (8) and (18) into the error dynamics (5), we obtain the closed-loop error dynamics as

$$\dot{\mathbf{e}} = A\mathbf{e} - B(CB)^{-1}\left[C(kI + A)\mathbf{e} + qs^2\,\text{sgn}(s)\right] \tag{19}$$

We shall show that the error dynamics (19) is globally asymptotically stable by considering the quadratic Lyapunov function

$$V(\mathbf{e}) = \frac{1}{2}\, s^2(\mathbf{e}) \tag{20}$$

The sliding mode motion is characterized by the equations

$$s(\mathbf{e}) = 0 \ \text{ and } \ \dot{s}(\mathbf{e}) = 0 \tag{21}$$

By the choice of E, the dynamics in the sliding mode given by Eq. (15) is globally asymptotically stable.

When $s(\mathbf{e}) \neq 0$, $V(\mathbf{e}) > 0$.

Also, when $s(\mathbf{e}) \neq 0$, differentiating V along the error dynamics (19) or the equivalent dynamics (17), we get

$$\dot{V}(\mathbf{e}) = s\dot{s} = -ks^2 - qs^3 \,\text{sgn}(s) < 0 \tag{22}$$

Hence, by Lyapunov stability theory [24], the error dynamics (19) is globally asymptotically stable for all $\mathbf{e}(0) \in \mathbf{R}^n$.

This completes the proof. ■

4 Hyperchaotic Vaidyanathan System

The hyperchaotic Vaidyanathan system [63] is described by the 4-D dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_3 + x_4 \\ \dot{x}_2 &= cx_1 - x_1x_3 + x_4 \\ \dot{x}_3 &= -bx_3 + x_1x_2 \\ \dot{x}_4 &= -d(x_1 + x_2) \end{aligned} \tag{23}$$

where x_1, x_2, x_3, x_4 are the states and a, b, c, d are constant, positive, parameters.

In [63], it was shown that the system (23) is hyperchaotic when the parameters take the values

$$a = 12, \quad b = 4, \quad c = 100, \quad d = 5 \tag{24}$$

For numerical simulations, we take the initial values of the hyperchaotic Vaidyanathan system (23) as

$$x_1(0) = 1.5, \quad x_2(0) = 0.6, \quad x_3(0) = 1.8, \quad x_4(0) = 2.5 \tag{25}$$

Figures 1, 2, 3 and 4 show the 3-D projections of the hyperchaotic Vaidyanathan system (23) on (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4) and (x_2, x_3, x_4) spaces, respectively.

The 3-D projection of the hyperchaotic Vaidyanathan system (23) on (x_1, x_2, x_3) space has the shape of a *two-scroll attractor* or *butterfly attractor*. Thus, we may also call the hyperchaotic Vaidyanathan system (23) as *hyperchaotic butterfly attractor*.

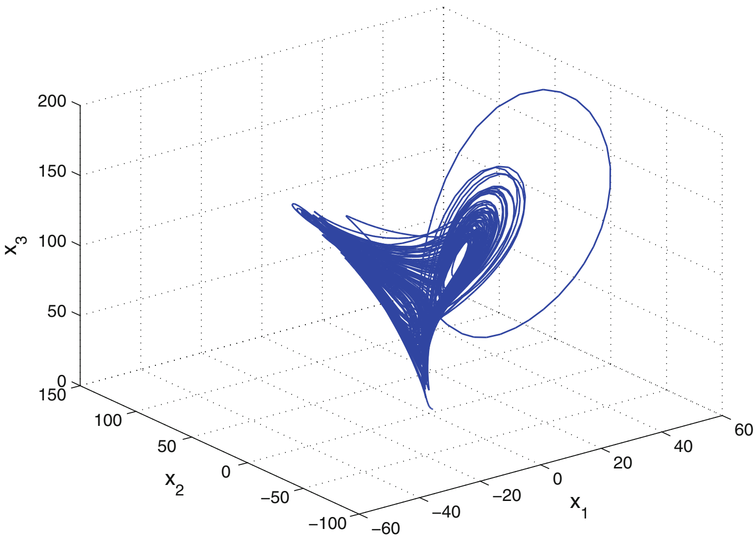

Fig. 1 3-D projection of the hyperchaotic Vaidyanathan system on the (x_1, x_2, x_3) space

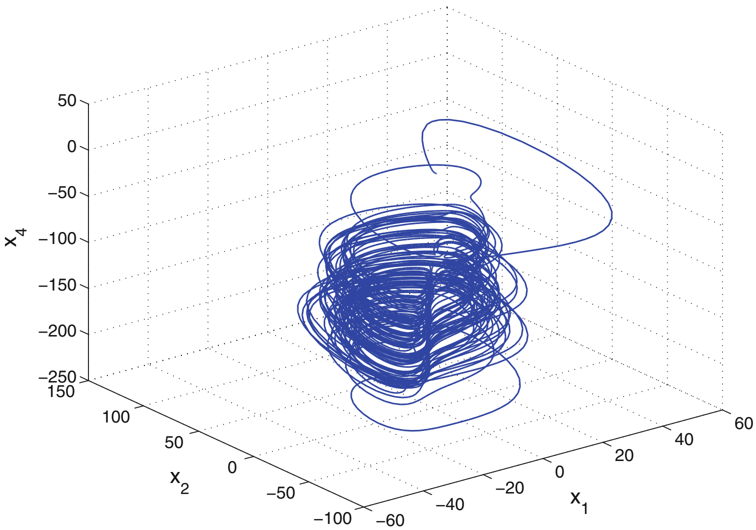

Fig. 2 3-D projection of the hyperchaotic Vaidyanathan system on the (x_1, x_2, x_4) space

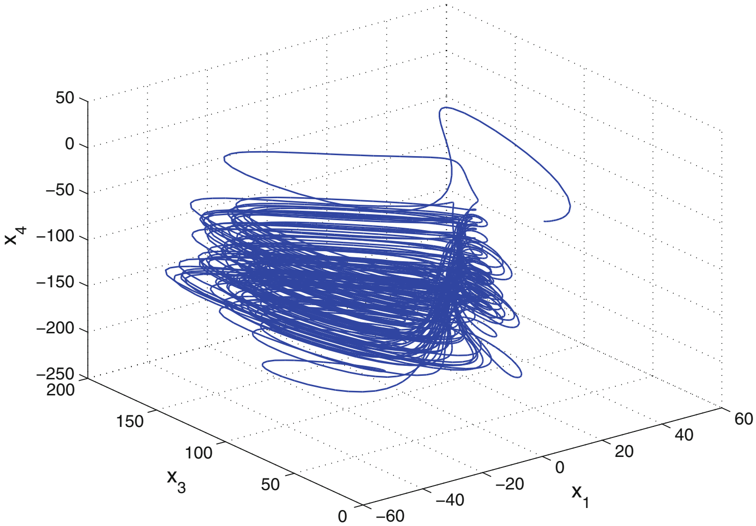

Fig. 3 3-D projection of the hyperchaotic Vaidyanathan system on the (x_1, x_3, x_4) space

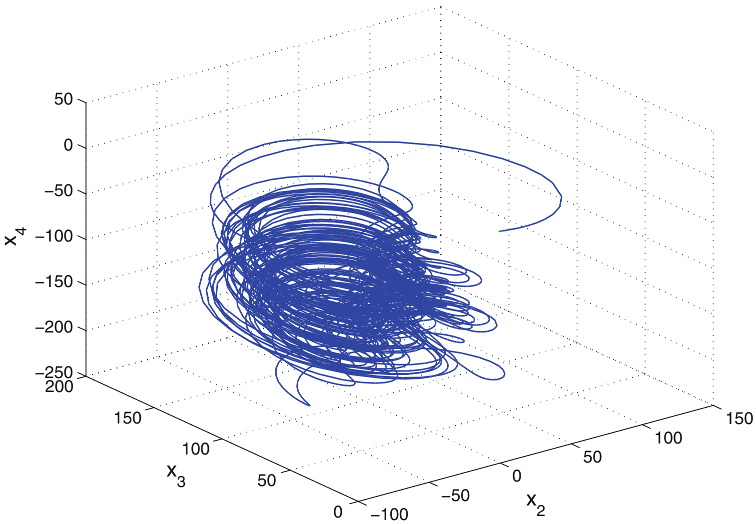

Fig. 4 3-D projection of the hyperchaotic Vaidyanathan system on the (x_2, x_3, x_4) space

5 Qualitative Properties of the Hyperchaotic Vaidyanathan System

5.1 Dissipativity

In vector notation, the hyperchaotic Vaidyanathan system (23) can be expressed as

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2, x_3, x_4) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3, x_4) \\ f_4(x_1, x_2, x_3, x_4) \end{bmatrix}, \tag{26}$$

where

$$\begin{cases} f_1(x_1, x_2, x_3, x_4) = a(x_2 - x_1) + x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) = cx_1 - x_1 x_3 + x_4 \\ f_3(x_1, x_2, x_3, x_4) = -bx_3 + x_1 x_2 \\ f_4(x_1, x_2, x_3, x_4) = -d(x_1 + x_2) \end{cases} \tag{27}$$

We take the parameter values as in the hyperchaotic case (24).

Let Ω be any region in $\mathbf{R}^4$ with a smooth boundary and also, $\Omega(t) = \Phi_t(\Omega)$, where Φ_t is the flow of f. Furthermore, let $V(t)$ denote the hypervolume of $\Omega(t)$.

By Liouville's theorem, we know that

$$\dot{V}(t) = \int_{\Omega(t)} (\nabla \cdot f)\, dx_1\, dx_2\, dx_3\, dx_4 \tag{28}$$

The divergence of the hyperchaotic Vaidyanathan system (26) is found as:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} = -(a + b) = -\mu < 0 \tag{29}$$

where $\mu = a + b = 12 + 4 = 16 > 0$.

Inserting the value of $\nabla \cdot f$ from (29) into (28), we get

$$\dot{V}(t) = \int_{\Omega(t)} (-\mu)\, dx_1\, dx_2\, dx_3\, dx_4 = -\mu V(t) \tag{30}$$

Integrating the first order linear differential equation (30), we get

$$V(t) = \exp(-\mu t) V(0) \tag{31}$$

Since $\mu > 0$, it follows from Eq. (31) that $V(t) \to 0$ exponentially as $t \to \infty$. This shows that the hyperchaotic Vaidyanathan system (23) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the hyperchaotic Vaidyanathan system (23) settles onto a strange attractor of the system.

5.2 Equilibrium Points

The equilibrium points of the hyperchaotic Vaidyanathan system (23) are obtained by solving the equations

$$\left.\begin{array}{lcl} f_1(x_1,x_2,x_3,x_4) = a(x_2-x_1)+x_3+x_4 & = 0 \\ f_2(x_1,x_2,x_3,x_4) = cx_1 - x_1x_3 + x_4 & = 0 \\ f_3(x_1,x_2,x_3,x_4) = -bx_3 + x_1x_2 & = 0 \\ f_1(x_1,x_2,x_3,x_4) = -d(x_1+x_2) & = 0 \end{array}\right\} \tag{32}$$

We take the parameter values as in the hyperchaotic case (24), viz. $a = 12, b = 4$, $c = 100$ and $d = 5$.

Solving the system (32), we see that the system (23) has a unique equilibrium point at the origin, i.e.

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

To test the stability type of the equilibrium point E_0, we calculate the Jacobian of the system (23) at E_0 as

$$J_0 = J(E_0) = \begin{bmatrix} -a & a & 1 & 1 \\ c & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ -d & -d & 0 & 0 \end{bmatrix} = \begin{bmatrix} -12 & 12 & 1 & 1 \\ 100 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ -5 & -5 & 0 & 0 \end{bmatrix} \tag{34}$$

The matrix J_0 has the eigenvalues

$$\lambda_1 = -41.2284, \quad \lambda_2 = -4, \quad \lambda_3 = 0.5239 \text{ and } \lambda_4 = 28.7045 \tag{35}$$

This shows that the equilibrium point E_0 is a saddle point, which is unstable.

5.3 Lyapunov Exponents and Kaplan-Yorke Dimension

We take the parameter values of the 4-D system (23) as

$$a = 12, \quad b = 4, \quad c = 100, \quad d = 5 \tag{36}$$

We take the initial values of the 4-D system (23) as

$$x_1(0) = 1.5, \quad x_2(0) = 0.6, \quad x_3(0) = 1.8, \quad x_4(0) = 2.5 \tag{37}$$

Then the Lyapunov exponents of the 4-D system (23) are numerically obtained using MATLAB as

$$L_1 = 1.4252, \quad L_2 = 0.2445, \quad L_3 = 0, \quad L_4 = -17.6549 \tag{38}$$

Equation (38) shows that the 4-D system (23) is hyperchaotic, since it has two positive Lyapunov exponents.

The dynamics of the Lyapunov exponents is depicted in Fig. 5. From Fig. 5, we see that the maximal Lyapunov exponent of the hyperchaotic Vaidyanathan system is given by $L_1 = 1.4252$. Since the sum of the Lyapunov exponents is negative, the system (23) is a dissipative hyperchaotic system.

Also, the Kaplan-Yorke dimension of the hyperchaotic Vaidyanathan system (23) is obtained as

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.0946, \tag{39}$$

which is fractional.

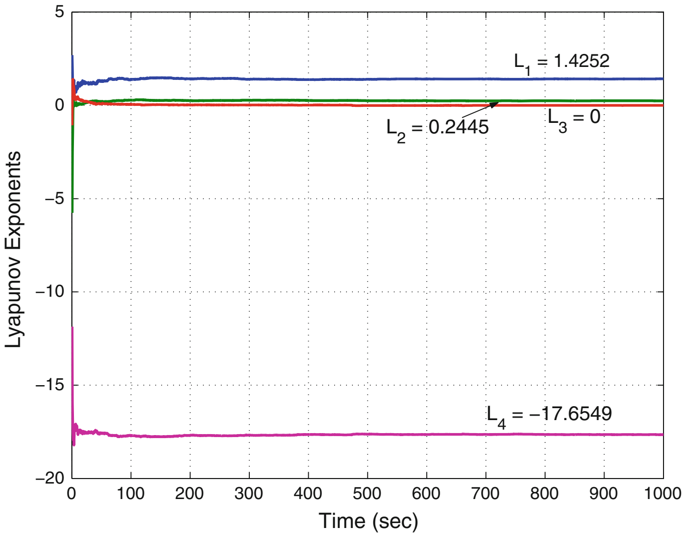

Fig. 5 Dynamics of the Lyapunov exponents of the hyperchaotic Vaidyanathan system

6 Sliding Mode Controller Design for the Complete Synchronization of Hyperchaotic Vaidyanathan Systems

In this section, we describe the sliding mode controller design for the complete synchronization of identical hyperchaotic Vaidyanathan systems [63] by applying the novel method described by Theorem 1 in Sect. 3.

As the master system, we consider the hyperchaotic Vaidyanathan system given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_3 + x_4 \\ \dot{x}_2 &= cx_1 - x_1x_3 + x_4 \\ \dot{x}_3 &= -bx_3 + x_1x_2 \\ \dot{x}_4 &= -d(x_1 + x_2) \end{aligned} \tag{40}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, d are positive parameters.

As the slave system, we consider the controlled hyperchaotic Vaidyanathan system given by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_3 + y_4 + u_1 \\ \dot{y}_2 &= cy_1 - y_1y_3 + y_4 + u_2 \\ \dot{y}_3 &= -by_3 + y_1y_2 + u_3 \\ \dot{y}_4 &= -d(y_1 + y_2) + u_4 \end{aligned} \tag{41}$$

where y_1, y_2, y_3, y_4 are the state variables and u_1, u_2, u_3, u_4 are the controls.

The complete synchronization error between (40) and (41) is defined as

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 - x_4 \end{aligned} \tag{42}$$

Then the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + e_3 + e_4 + u_1 \\ \dot{e}_2 &= ce_1 + e_4 - y_1y_3 + x_1x_3 + u_2 \\ \dot{e}_3 &= -be_3 + y_1y_2 - x_1x_2 + u_3 \\ \dot{e}_4 &= -d(e_1 + e_2) + u_4 \end{aligned} \tag{43}$$

In matrix form, we can write the error dynamics (43) as

$$\dot{\mathbf{e}} = A\mathbf{e} + \psi(\mathbf{x}, \mathbf{y}) + \mathbf{u} \tag{44}$$

The matrices in (44) are given by

$$A = \begin{bmatrix} -a & a & 1 & 1 \\ c & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ -d & -d & 0 & 0 \end{bmatrix} \text{ and } \psi(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} 0 \\ -y_1 y_3 + x_1 x_3 \\ y_1 y_2 - x_1 x_2 \\ 0 \end{bmatrix} \tag{45}$$

We follow the procedure given in Sect. 3 for the construction of the novel sliding controller to achieve complete synchronization of the identical hyperchaotic Vaidyanathan systems (40) and (41).

First, we set $\mathbf{u}$ as

$$\mathbf{u}(t) = -\psi(\mathbf{x}, \mathbf{y}) + Bv(t) \tag{46}$$

where B is selected such that (A, B) is completely controllable.

A simple choice of B is

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{47}$$

It can be easily checked that (A, B) is completely controllable.

The hyperchaotic Vaidyanathan system (40) displays a strange attractor when the parameter values are selected as

$$a = 12, \quad b = 4, \quad c = 100, \quad d = 5 \tag{48}$$

Next, we take the sliding variable as

$$s(\mathbf{e}) = C\mathbf{e} = \begin{bmatrix} 10 & 8 & -1 & -2 \end{bmatrix} \mathbf{e} = 10e_1 + 8e_2 - e_3 - 2e_4 \tag{49}$$

Next, we take the sliding mode gains as

$$k = 6, \quad q = 0.2 \tag{50}$$

From Eq. (18) in Sect. 3, we obtain the novel sliding control v as

$$v(t) = -50e_1 - 11.8667e_2 - 0.5333e_3 - 0.4e_4 - 0.0133s^2 \operatorname{sgn}(s) \tag{51}$$

As an application of Theorem 1 to the identical hyperchaotic Vaidyanathan systems, we obtain the following main result of this section.

Theorem 2 *The identical hyperchaotic Vaidyanathan systems (40) and (41) are globally and asymptotically synchronized for all initial conditions* $\mathbf{x}(0), \mathbf{y}(0) \in \mathbf{R}^4$ *with the sliding controller* $\mathbf{u}$ *defined by (46), where* $\psi(\mathbf{x}, \mathbf{y})$ *is defined by (45),* B *is defined by (47) and* v *is defined by (51).* ■

For numerical simulations, we use MATLAB for solving the systems of differential equations using the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$.

The parameter values of the hyperchaotic Vaidyanathan systems are taken as in the hyperchaotic case, viz. $a = 12$, $b = 4$, $c = 100$ and $d = 5$.

The sliding mode gains are taken as $k = 6$ and $q = 0.2$.

As an initial condition for the master system (40), we take

$$x_1(0) = 4.9, \quad x_2(0) = -2.5, \quad x_3(0) = 6.9, \quad x_4(0) = 12.4 \tag{52}$$

As an initial condition for the slave system (41), we take

$$y_1(0) = 8.1, \quad y_2(0) = 5.3, \quad y_3(0) = 1.3, \quad y_4(0) = -5.1 \tag{53}$$

Figures 6, 7, 8 and 9 show the complete synchronization of the states of the identical hyperchaotic Vaidyanathan systems (40) and (41).

From Fig. 6, it is clear that the states x_1 and y_1 are synchronized in 1 s.

From Fig. 7, it is clear that the states x_2 and y_2 are synchronized in 2 s.

From Fig. 8, it is clear that the states x_3 and y_3 are synchronized in 1 s.

From Fig. 9, it is clear that the states x_4 and y_4 are synchronized in 2 s.

Figure 10 shows the time-history of the complete synchronization errors e_1, e_2, e_3, e_4.

From Fig. 10, it is clear that all the synchronization errors converge to zero in 2 s.

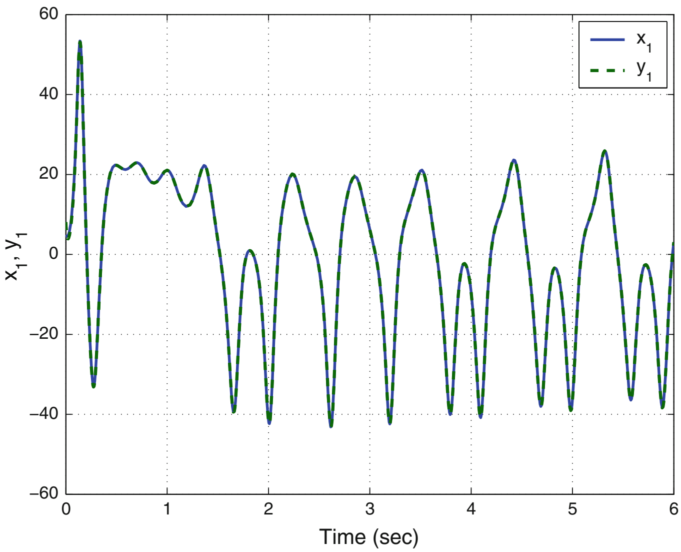

Fig. 6 Complete synchronization of the states x_1 and y_1

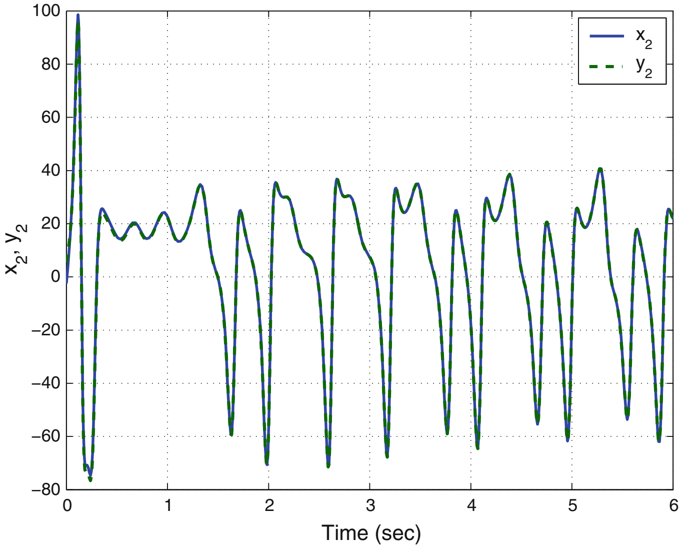

Fig. 7 Complete synchronization of the states x_2 and y_2

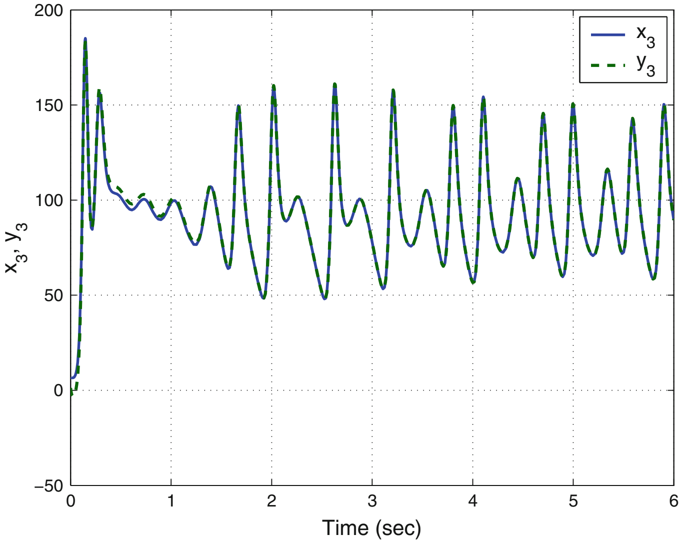

Fig. 8 Complete synchronization of the states x_3 and y_3

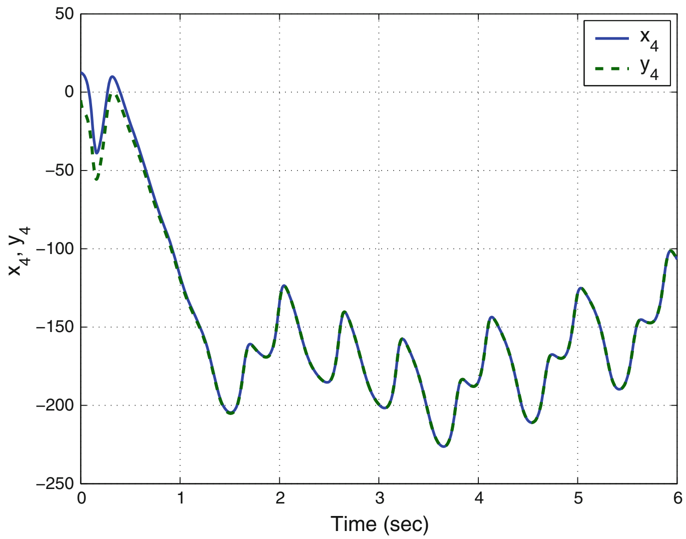

Fig. 9 Complete synchronization of the states x_4 and y_4

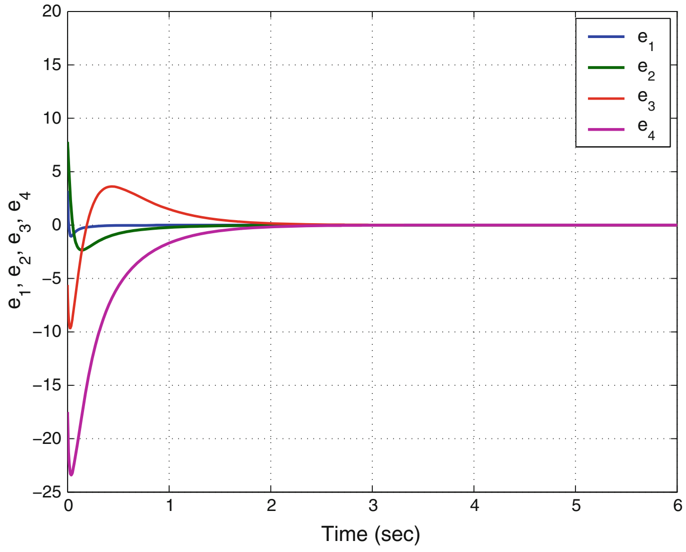

Fig. 10 Time-history of the complete synchronization errors e_1, e_2, e_3, e_4

7 Conclusions

Chaos and hyperchaos have important applications in science and engineering. Hyperchaotic systems have more complex behaviour than chaotic systems and they have miscellaneous applications in areas like secure communications, cryptosystems, etc. In robust control systems, the sliding mode control is commonly used due to its inherent advantages of easy realization, fast response and good transient performance as well as insensitivity to parameter uncertainties and disturbance. In this work, we derived a novel sliding mode control method for the complete synchronization of identical chaotic or hyperchaotic systems. We proved the main result using Lyapunov stability theory. As an application of the general result, the problem of complete synchronization of identical hyperchaotic Vaidyanathan systems (2014) was studied and a new sliding mode controller has been derived. Numerical simulations using MATLAB were shown to depict the phase portraits of the hyperchaotic Vaidyanathan system and the sliding mode controller design for the complete synchronization of identical hyperchaotic Vaidyanathan systems.

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