Chapter 7
Convergence to a Continuous State Model

The aim of this chapter is to take the limit in the renormalized version of the model of the previous chapter, i.e., let $N \to \infty$ in the models of sections 6.5 and 6.6, respectively. In section 7.1, we shall take the limit in the model for the evolution of the population size, as a function of the two parameters x (the ancestral population size) and t (time). Since we want to stick to our rather minimal assumptions on the function f, checking tightness requires some care. In section 7.2, we will take the limit in the renormalized contour process of section 6.6. Here there are two difficulties. One is the fact that since we do not want to restrict ourself to the (sub)critical case, it is not clear whether the contour process will accumulate an arbitrary amount of local time at level 0. In order to circumvent this difficulty, we use as in chapter 5 a trick due to Delmas [16] which consists in considering the population process killed at an arbitrary time a, which amounts to reflect the contour process below a. The behavior of the contour process below a (and of its local time accumulated below level a) is described by the solution of the corresponding equation reflected below any $a' > a$. The second difficulty comes from the fact that f' is not assumed to be bounded from below. We will prove our convergence result by a combination of Theorem 7 and Girsanov's theorem. For that sake, we shall first consider the case where $|f'|$ is bounded, and then the general case.

7.1 Convergence of $Z^{N,x}$

The aim of this section is to prove the convergence in law as $N \to \infty$ of the two-parameter process $\{Z_t^{N,x},\ t \geq 0, x \geq 0\}$ defined in section 6.5 towards the process $\{Z_t^x,\ t \geq 0, x \geq 0\}$ solution of the SDE (4.10). We need to make precise the topology for which this convergence will hold. We note that the process $Z_t^{N,x}$ (resp., Z_t^x) is a Markov process indexed by x, with values in the space of càdlàg (resp., continuous) functions of t $D([0,\infty);\mathbb{R}_+)$ (resp., $C([0,\infty);\mathbb{R}_+)$). So it will be natural to consider a topology of functions of x, with values in a space of functions of t.

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For each fixed x, the process $t \to Z_t^{N,x}$ is càdlàg, constant between its jumps, with jumps of size $\pm N^{-1}$, while the limit process $t \to Z_t^x$ is continuous. On the other hand, both $Z_t^{N,x}$ and Z_t^x are discontinuous as functions of x. The mapping $x \to Z_\cdot^x$ has countably many jumps on any compact interval, but the mapping $x \to \{Z_t^x,\ t \geq \varepsilon\}$, where $\varepsilon > 0$ is arbitrary, has finitely many jumps on any compact interval, and it is constant between its jumps. This fact is well-known in the case where f is linear, see section 4.4, and has been proved in the general case in Corollary 2. Recall that $D([0,\infty);\mathbb{R}_+)$, equipped with the distance d_∞^0 defined by (16.4) in [10], is separable and complete, see Theorem 16.3 in [10]. We have the following statement

Theorem 13. *Suppose that Assumption (H1) is satisfied. Then as $N \to \infty$,*

$$\{Z_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty);D([0,\infty);\mathbb{R}_+))$, equipped with the Skorokhod topology of the space of càdlàg functions of x, with values in the Polish space $D([0,\infty);\mathbb{R}_+)$ equipped with the metric d_∞^0, where $\{Z_t^x,\ t \geq 0, x \geq 0\}$ is the unique solution of the SDE (4.10).

7.1.1 Tightness of $Z^{N,x}$

Recall (6.3) and (6.5). We first establish a few Lemmas.

Lemma 20. *For all $T > 0$, $x \geq 0$, there exists a constant $C_0 > 0$ such that for all $N \geq 1$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(Z_t^{N,x}\right) \leq C_0.$$

Moreover, for all $t \geq 0$, $N \geq 1$,

$$\mathbb{E}\left(-\int_0^t f(Z_r^{N,x})dr\right) \leq x.$$

PROOF: Let $(\tau_n, n \geq 0)$ be a sequence of stopping times such that τ_n tends to infinity as n goes to infinity and for any n, $\left(M_{t\wedge\tau_n}^{N,x}, t \geq 0\right)$ is a martingale and $Z_{t\wedge\tau_n}^{N,x} \leq n$. Taking the expectation on both sides of equation (6.3) at time $t \wedge \tau_n$, we obtain

$$\mathbb{E}\left(Z_{t\wedge\tau_n}^{N,x}\right) = \frac{\lfloor Nx \rfloor}{N} + \mathbb{E}\left(\int_0^{t\wedge\tau_n} f(Z_r^{N,x})dr\right). \tag{7.1}$$

It follows from the Assumption (H1) on f that

$$\mathbb{E}\left(Z_{t\wedge\tau_n}^{N,x}\right) \leq \frac{\lfloor Nx \rfloor}{N} + \beta \int_0^t \mathbb{E}(Z_{r\wedge\tau_n}^{N,x})dr$$

From Gronwall's and Fatou's Lemmas, we deduce that there exists a constant $C_0 > 0$ which depends only upon x and T such that

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\left(Z_t^{N,x}\right)\leq C_0.$$

From (7.1), we deduce that

$$-\mathbb{E}\left(\int_0^{t\wedge\tau_n} f(Z_r^{N,x})dr\right)\leq\frac{\lfloor Nx\rfloor}{N}.$$

Since $-f(Z_r^{N,x})\geq -\beta Z_r^{N,x}$, the second statement follows using Fatou's Lemma and the first statement. □

We now have the following Lemma.

Lemma 21. *For all $T>0$, $x\geq 0$, there exists a constant $C_1>0$ such that*

$$\sup_{N\geq 1}\mathbb{E}\left(\langle M^{N,x}\rangle_T\right)\leq C_1.$$

PROOF: For any $N\geq 1$ and $k,k'\in\mathbb{Z}_+$ such that $k\leq k'$, we set $z=\frac{k}{N}$ and $z'=\frac{k'}{N}$. We deduce from (6.6) that

$$||f||_{N,z,z'}=\sum_{i=k+1}^{k'}\left\{2\left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)^+-\left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)\right\}.$$

Hence it follows from Assumption (H1) that

$$||f||_{N,z,z'}\leq 2\beta(z'-z)+f(z)-f(z'). \tag{7.2}$$

We deduce from (7.2), (6.5), and Lemma 20 that

$$\begin{aligned}\mathbb{E}\left(\langle M^{N,x}\rangle_T\right)&\leq\int_0^T\left\{\left(\sigma^2+\frac{2\beta}{N}\right)\mathbb{E}(Z_r^{N,x})-\frac{1}{N}\mathbb{E}\left(f(Z_r^{N,x})\right)\right\}dr\\&\leq\left(\sigma^2+\frac{2\beta}{N}\right)C_0T+\frac{x}{N}.\end{aligned}$$

Hence the Lemma. □

It follows from this that $M^{N,x}$ is in fact a square integrable martingale. We also have

Lemma 22. *For all $T>0$, $x\geq 0$, there exist two constants $C_2,C_3>0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\left[\left(Z_t^{N,x}\right)^2\right]\leq C_2,$$

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\left(-\int_0^t Z_r^{N,x}f(Z_r^{N,x})dr\right)\leq C_3.$$

PROOF: We deduce from (6.3), (A.4), and the fact that $\langle M^{N,x}\rangle_t - [M^{N,x}]_t$ is a local martingale

$$\left(Z_t^{N,x}\right)^2 = \left(\frac{\lfloor Nx \rfloor}{N}\right)^2 + 2\int_0^t Z_r^{N,x} f(Z_r^{N,x})dr + \langle M^{N,x}\rangle_t + M_t^{N,x,(2)}, \tag{7.3}$$

where $M^{N,x,(2)}$ is a local martingale. Let $(\tau_n, n \geq 1)$ be a sequence of stopping times such that $\lim_{n\to\infty} \tau_n = +\infty$ a.s. and for each $n \geq 1$, $\left(M_{t\wedge\tau_n}^{N,x,(2)}, t \geq 0\right)$ is a martingale. Taking the expectation on the both sides of (7.3) at time $t \wedge \tau_n$ and using Assumption (H1), Lemma 21, and the Gronwall and Fatou Lemmas, we obtain that for all $T > 0$, there exists a constant $C_2 > 0$ such that

$$\sup_{N\geq 1} \sup_{0\leq t\leq T} \mathbb{E}\left(Z_t^{N,x}\right)^2 dr \leq C_2.$$

We also have that

$$2\mathbb{E}\left(-\int_0^{t\wedge\tau_n} Z_r^{N,x} f(Z_r^{N,x})dr\right) \leq \left(\frac{\lfloor Nx \rfloor}{N}\right)^2 + C_1$$

From Assumption (H1), we have $-Z_r^{N,x} f(Z_r^{N,x}) \geq -\beta(Z_r^{N,x})^2$. The second result now follows from Fatou's Lemma. □

We want to check tightness of the sequence $\{Z^{N,x},\ N \geq 0\}$. Because of the very weak assumptions upon f, we cannot use Proposition 37 below. Instead, we now show directly how we can use Aldous' criterion (A), see section A.7. Let $\{\tau_N,\ N \geq 1\}$ be a sequence of stopping times in $[0,T]$. We deduce from Lemma 22

Proposition 21. *For any $T > 0$ and η, $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{N\geq 1} \sup_{0\leq\theta\leq\delta} \mathbb{P}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \geq \eta\right) \leq \varepsilon.$$

PROOF: Let c be a nonnegative constant. Provided $0 \leq \theta \leq \delta$, we have

$$\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \leq \sup_{0\leq r\leq c} |f(r)|\delta + \int_{\tau_N}^{(\tau_N+\theta)\wedge T} \mathbf{1}_{\{Z_r^{N,x}>c\}} |f(Z_r^{N,x})|dr$$

But

$$\begin{aligned}\int_{\tau_N}^{(\tau_N+\theta)\wedge T} \mathbf{1}_{\{Z_r^{N,x}>c\}} |f(Z_r^{N,x})|dr &\leq c^{-1}\int_0^T Z_r^{N,x}\left(f^+(Z_r^{N,x}) + f^-(Z_r^{N,x})\right)dr \\ &\leq c^{-1}\int_0^T \left(2Z_r^{N,x} f^+(Z_r^{N,x}) - Z_r^{N,x} f(Z_r^{N,x})\right)dr \\ &\leq c^{-1}\int_0^T \left(2\beta(Z_r^{N,x})^2 - Z_r^{N,x} f(Z_r^{N,x})\right)dr.\end{aligned}$$

From this and Lemma 22, we deduce that $\forall\, N \geq 1$, again with $\theta \leq \delta$,

$$\sup_{0\leq\theta\leq\delta} \mathbb{P}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \geq \eta\right) \leq \eta^{-1}\mathbb{E}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right|\right)$$
$$\leq \sup_{0\leq r\leq c} \frac{|f(r)|\delta}{\eta} + \frac{A}{c\eta},$$

with $A = 2\beta C_2 T + C_3$. The result follows by choosing $c = 2A/\varepsilon\eta$, and then $\delta = \varepsilon\eta/2\sup_{0\leq r\leq c}|f(z)|$. □

From Proposition 21, the Lebesgue integral term in the right-hand side of (6.3) satisfies Aldous's condition (A). The same Proposition, Lemma 20, (6.5), and (7.2) imply that $< M^{N,x} >$ satisfies the same condition, hence so does $M^{N,x}$, according to Rebolledo's theorem, see [21]. We have proved

Proposition 22. *For any fixed $x \geq 0$, the sequence of processes $\left\{Z^{N,x},\ N \geq 1\right\}$ is tight in $D([0,\infty);\mathbb{R}_+)$.*

We deduce from Proposition 22 the following Corollary.

Corollary 6. *For any $0 \leq x < y$ the sequence of processes $\left\{V^{N,x,y},\ N \geq 1\right\}$ is tight in $D([0,\infty);\mathbb{R}_+)$*

PROOF: For any x fixed the process $Z_t^{N,x}$ has jumps equal to $\pm\frac{1}{N}$ which tend to zero as $N \to \infty$. It follows from this, Proposition 22, and Proposition 37 that any weak limit of a converging subsequence of $Z^{N,x}$ is continuous. We deduce that for any $x,y \geq 0$, the sequence $\left\{Z^{N,y} - Z^{N,x}, N \geq 1\right\}$ is tight since $\left\{Z^{N,x}, N \geq 1\right\}$ and $\left\{Z^{N,y}, N \geq 1\right\}$ are tight and both have a continuous limit as $N \to \infty$. □

7.1.2 Proof of Theorem 13

The next two Propositions will be the main steps in the proof of Theorem 13.

Proposition 23. *For any $n \in \mathbb{N}$, $0 \leq x_1 < x_2 < \cdots < x_n$,*

$$\left(Z^{N,x_1}, Z^{N,x_2}, \cdots, Z^{N,x_n}\right) \Rightarrow \left(Z^{x_1}, Z^{x_2}, \cdots, Z^{x_n}\right)$$

as $N \to \infty$, for the topology of locally uniform convergence in t.

PROOF: We prove the statement in the case $n = 2$ only. The general statement can be proved in a very similar way. For $0 \leq x_1 < x_2$, we consider the process $\left(Z^{N,x_1}, V^{N,x_1,x_2}\right)$, using the notations from section 6.5. The argument preceding the statement of Proposition 22 implies that the sequences of martingales M^{N,x_1} and M^{N,x_1,x_2} are tight. Hence
$\left(Z^{N,x_1}, V^{N,x_1,x_2}, M^{N,x_1}, M^{N,x_1,x_2}\right)$ is tight. Thanks to (6.3), (6.5), (6.8), (6.9), and (6.10), any converging subsequence of

$\{Z^{N,x_1}, V^{N,x_1,x_2}, M^{N,x_1}, M^{N,x_1,x_2}, N \geq 1\}$ has a weak limit $(Z^{x_1}, V^{x_1,x_2}, M^{x_1}, M^{x_1,x_2})$ which satisfies

$$Z_t^{x_1} = x_1 + \int_0^t f(Z_s^{x_1})ds + M_t^{x_1}$$
$$V_t^{x_1,x_2} = x_2 - x_1 + \int_0^t [f(Z_s^{x_1} + V_s^{x_1,x_2}) - f(Z_s^{x_1})]\, ds + M_t^{x_1,x_2},$$

where the continuous martingales M^{x_1} and M^{x_1,x_2} satisfy

$$\langle M^x \rangle_t = \sigma^2 \int_0^t Z_s^{x_1} ds,\ \langle M^{x_1,x_2} \rangle_t = \sigma^2 \int_0^t V_s^{x_1,x_2} ds,\ \langle M^{x_1}, M^{x_1,x_2} \rangle_t = 0.$$

This implies that the pair (Z^{x_1}, V^{x_1,x_2}) is a weak solution of the system of SDEs (4.10) and (4.18), driven by the same space-time white noise. The result follows from the uniqueness of the system, see Theorem 5. □

Proposition 24. *There exists a constant C, which depends only upon θ and T, such that for any $0 \leq x < y < z$, which are such that $y - x \leq 1$, $z - y \leq 1$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Z_t^{N,y} - Z_t^{N,x}|^2 \times \sup_{0\leq t\leq T} |Z_t^{N,z} - Z_t^{N,y}|^2\right] \leq C|z-x|^2.$$

We first prove the

Lemma 23. *For any $0 \leq x < y$, we have*

$$\sup_{0\leq t\leq T} \mathbb{E}\left(Z_t^{N,y} - Z_t^{N,x}\right) = \sup_{0\leq t\leq T} \mathbb{E}(V_t^{N,x,y}) \leq \left(\frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}\right) e^{\beta T},$$

PROOF: Let $(\tau_n, n \geq 0)$ be a sequence of stopping times such that $\lim_{n\to\infty} \tau_n = +\infty$ and for each $n \geq 1$, $\left(M_{t\wedge\tau_n}^{N,x,y},\ t \geq 0\right)$ is a martingale. Taking the expectation on both sides of (6.8) at time $t \wedge \tau_n$, we obtain that

$$\mathbb{E}(V_{t\wedge\tau_n}^{N,x,y}) \leq \left(\frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}\right) + \beta \int_0^t \mathbb{E}(V_{r\wedge\tau_n}^{N,x,y})dr \tag{7.4}$$

Using Gronwall's and Fatou's Lemmas, we obtain that

$$\sup_{0\leq t\leq T} \mathbb{E}(V_t^{N,x,y}) \leq \left(\frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}\right) e^{\beta T}.$$

□

PROOF OF PROPOSITION 24 From equation (6.8), using a stopping time argument as above, Lemma 23, and Fatou's Lemma, where we take advantage of the inequality $f(Z_r^{N,x}) - f(Z_r^{N,x} + V_r^{N,x,y}) \geq -\beta V_r^{N,x,y}$, we deduce that

$$\mathbb{E}\left(\int_0^t \left[f(Z_r^{N,x}) - f(Z_r^{N,x} + V_r^{N,x,y})\right] dr\right) \leq \frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}. \tag{7.5}$$

We now deduce from (6.9), Lemma 23, and inequalities (7.5) and (7.2) that for each $t > 0$, there exists a constant $C(t) > 0$ such that

$$\mathbb{E}\left(\langle M^{N,x,y} \rangle_t\right) \leq C(t) \left(\frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}\right). \tag{7.6}$$

This implies that $M^{N,x,y}$ is in fact a square integrable martingale. For any $0 \leq x < y < z$, we have $Z_t^{N,z} - Z_t^{N,y} = V_t^{N,y,z}$ and $Z_t^{N,y} - Z_t^{N,x} = V_t^{N,x,y}$ for any $t \geq 0$. On the other hand we deduce from (6.8) and Assumption (H1) that

$$\sup_{0\leq t\leq T} (V_t^{N,x,y})^2 \leq 3\left(\frac{\lfloor Ny \rfloor}{N} - \frac{\lfloor Nx \rfloor}{N}\right)^2 + 3\beta^2 T \int_0^T \sup_{0\leq s\leq r} (V_s^{N,x,y})^2 dr + 3 \sup_{0\leq t\leq T} \left(M_t^{N,x,y}\right)^2$$

and

$$\sup_{0\leq t\leq T} (V_t^{N,y,z})^2 \leq 3\left(\frac{\lfloor Nz \rfloor}{N} - \frac{\lfloor Ny \rfloor}{N}\right)^2 + 3\beta^2 T \int_0^t \sup_{0\leq s\leq r} (V_s^{N,y,z})^2 dr + 3 \sup_{0\leq t\leq T} \left(M_t^{N,y,z}\right)^2.$$

Now let $\mathscr{G}^{x,y} := \sigma\left(Z_t^{N,x}, Z_t^{N,y}, t \geq 0\right)$ be the filtration generated by $Z^{N,x}$ and $Z^{N,y}$. It is clear that for any t, $V_t^{N,x,y}$ is measurable with respect to $\mathscr{G}^{x,y}$. We then have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |V_t^{N,x,y}|^2 \times \sup_{0\leq t\leq T} |V_t^{N,y,z}|^2\right] = \mathbb{E}\left[\sup_{0\leq t\leq T} |V_t^{N,x,y}|^2 \mathbb{E}\left(\sup_{0\leq t\leq T} |V_t^{N,y,z}|^2 | \mathscr{G}^{x,y}\right)\right].$$

Conditionally upon $Z^{N,x}$ and $Z^{N,y} = u(.)$, $V^{N,y,z}$ solves the following SDE

$$V_t^{N,y,z} = \frac{\lfloor Nz \rfloor - \lfloor Ny \rfloor}{N} + \int_0^t \left[f(V_r^{N,y,z} + u(r)) - f(u(r))\right] dr + M_t^{N,y,z},$$

where $M^{N,y,z}$ is a martingale conditionally upon $\mathscr{G}^{x,y}$, hence the arguments used in Lemma 23 lead to

$$\sup_{0\leq t\leq T} \mathbb{E}\left(V_t^{N,y,z} | \mathscr{G}^{x,y}\right) \leq \left(\frac{\lfloor Nz \rfloor}{N} - \frac{\lfloor Ny \rfloor}{N}\right) e^{\beta T},$$

and those used to prove (7.5) yield

$$\mathbb{E}\left(\int_0^t f(Z_r^{N,y}) - f(Z_r^{N,y} + V_r^{N,y,z}) dr | \mathscr{G}^{x,y}\right) \leq \frac{\lfloor Nz \rfloor}{N} - \frac{\lfloor Ny \rfloor}{N}.$$

From this we deduce (see the proof of (7.6)) that

$$\mathbb{E}\left(\langle M^{N,y,z}\rangle_t|\mathscr{G}^{x,y}\right) \le C(t)\left(\frac{\lfloor Nz\rfloor}{N}-\frac{\lfloor Ny\rfloor}{N}\right).$$

From Doob's inequality we have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|M_t^{N,y,z}|^2|\mathscr{G}^{x,y}\right) &\le 4\mathbb{E}\left(\langle M^{N,y,z}\rangle_T|\mathscr{G}^{x,y}\right)\\ &\le C(T)\left(\frac{\lfloor Nz\rfloor}{N}-\frac{\lfloor Ny\rfloor}{N}\right).\end{aligned}$$

Since $0<z-y<1$, we deduce that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|V_t^{N,y,z}|^2|\mathscr{G}^{x,y}\right) &\le 3(1+C(T))\left(\frac{\lfloor Nz\rfloor}{N}-\frac{\lfloor Ny\rfloor}{N}\right)\\ &\quad+3\beta^2T\int_0^T\mathbb{E}\left(\sup_{0\le s\le r}(V_s^{N,y,z})^2|\mathscr{G}^{x,y}\right)dr,\end{aligned}$$

From this and Gronwall's Lemma we deduce that there exists a constant $K_1>0$ such that

$$\mathbb{E}\left(\sup_{0\le t\le T}|V_t^{N,y,z}|^2|\mathscr{G}^{x,y}\right) \le K_1\left(\frac{\lfloor Nz\rfloor}{N}-\frac{\lfloor Ny\rfloor}{N}\right). \tag{7.7}$$

Similarly we have

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(V_s^{N,x,y}\right)^2\right] \le K_1\left(\frac{\lfloor Ny\rfloor}{N}-\frac{\lfloor Nx\rfloor}{N}\right).$$

Since $0\le y-x<z-x$ and $0\le z-y<z-x$, we deduce that

$$\mathbb{E}\left[\sup_{0\le t\le T}|V_t^{N,x,y}|^2\times\sup_{0\le t\le T}|V_t^{N,y,z}|^2\right]\le K_1^2\left(\frac{\lfloor Nz\rfloor}{N}-\frac{\lfloor Nx\rfloor}{N}\right)^2,$$

hence the result.

Proof of Theorem 13 We now show that for any $T>0$,

$$\{Z_t^{N,x},\ 0\le t\le T,\ x\ge 0\}\Rightarrow\{Z_t^x,\ 0\le t\le T,\ x\ge 0\}$$

in $D([0,\infty);D([0,T],\mathbb{R}_+))$. From Theorems 13.1 and 16.8 in [10], since from Proposition 23, for all $n\ge 1$, $0<x_1<\cdots<x_n$,

$$(Z_\cdot^{N,x_1},\dots,Z_\cdot^{N,x_n})\Rightarrow(Z_\cdot^{x_1},\dots,Z_\cdot^{x_n})$$

in $D([0,T];\mathbb{R}^n)$, it suffices to show that for all $\bar{x}>0$, ε, $\eta>0$, there exists $N_0\ge 1$ and $\delta>0$ such that for all $N\ge N_0$,

$$\mathbb{P}(w_{\bar{x},\delta}(Z^N) \geq \varepsilon) \leq \eta, \tag{7.8}$$

where for a function $(x,t) \to z(x,t)$

$$w_{\bar{x},\delta}(z) = \sup_{0 \leq x_1 \leq x \leq x_2 \leq \bar{x}, x_2 - x_1 \leq \delta} \inf\{\|z(x,\cdot) - z(x_1,\cdot)\|, \|z(x_2,\cdot) - z(x,\cdot)\|\},$$

with the notation $\|z(x,\cdot)\| = \sup_{0 \leq t \leq T} |z(x,t)|$. But from the proof of Theorem 13.5 in [10], (7.8) for Z^N follows from Proposition 24. □

7.2 Convergence of the Contour Process H^N

In this section, we assume that $f \in C^1$. In this case, Assumption (H1) is equivalent to

Assumption (H1') There exists a constant $\beta > 0$ such that for all $x \geq 0$, $f'(x) \leq \beta$,

which we assume to be in force in this section.

7.2.1 The Case Where f' Is Bounded

We assume in this subsection that $|f'(x)| \leq C$ for all $x \geq 0$ and some $C > 0$. This constitutes the first step of the proof of convergence of H^N.

As explained at the end of section 6.6, in this case we can use Girsanov's theorem to bring us back to the situation studied in section 5.3.

Recalling equations (6.13) and (6.15), we note that

$$H_s^{a,N} = \mathscr{M}_s^{1,a,N} - \mathscr{M}_s^{2,a,N} + 2^{-1}[L_s^{a,N}(0) - L_s^{a,N}(a^-)] + \varepsilon_N, \text{ where}$$
$$\varepsilon_N = (4N)^{-1}(1 - V_s^{a,N}) - 2^{-1}L_{0^+}^{a,N}(0).$$

Moreover, from (6.12), (6.14), and (6.15),

$$[\mathscr{M}^{1,a,N}]_s = \frac{4}{N^2\sigma^4} Q_s^{1,a,N}, \quad [\mathscr{M}^{2,a,N}]_s = \frac{4}{N^2\sigma^4} Q_s^{2,a,N}$$
$$\langle \mathscr{M}^{1,a,N} \rangle_s = \frac{4}{\sigma^2} \int_0^s \mathbf{1}_{V_r^N = -1} dr, \ \langle \mathscr{M}^{2,a,N} \rangle_s = \frac{4}{\sigma^2} \int_0^s \mathbf{1}_{V_r^N = 1} dr,$$
$$[Y^{a,N}]_s = \frac{4}{N^2\sigma^4} \int_0^s |Y_{r^-}^{a,N}|^2 \Bigg[\left| (f_N')^+ \left(\frac{\sigma^2}{4} L_{r^-}^{a,N}(H_r^{a,N}) \right) \right|^2 dQ_r^{1,a,N}$$
$$+ \left| (f_N')^- \left(\frac{\sigma^2}{4} L_{r^-}^{a,N}(H_r^{a,N}) \right) \right|^2 dQ_r^{2,a,N} \Bigg]$$

$$\langle Y^{a,N}\rangle_s = \frac{4}{\sigma^2}\int_0^s |Y_r^{a,N}|^2 \Big[\Big|(f_N')^+\left(\frac{\sigma^2}{4}L_r^{a,N}(H_r^{a,N})\right)\Big|^2 \mathbf{1}_{V_r^{a,N}=-1} + \Big|(f_N')^-\left(\frac{\sigma^2}{4}L_r^{a,N}(H_r^{a,N})\right)\Big|^2 \mathbf{1}_{V_r^{a,N}=1}\Big] dr$$

$$[Y^{a,N},\mathscr{M}^{1,a,N}]_s = \frac{4}{N^2\sigma^4}\int_0^s Y_{r^-}^{a,N}(f_N')^+\left(\frac{\sigma^2}{4}L_{r^-}^{a,N}(H_r^{a,N})\right)dQ_r^{1,a,N}$$

$$[Y^{a,N},\mathscr{M}^{2,a,N}]_s = \frac{4}{N^2\sigma^4}\int_0^s Y_{r^-}^{a,N}(f_N')^-\left(\frac{\sigma^2}{4}L_{r^-}^{a,N}(H_r^{a,N})\right)dQ_r^{2,a,N}$$

$$\langle Y^{a,N},\mathscr{M}^{1,a,N}\rangle_s = \frac{4}{\sigma^2}\int_0^s Y_r^{a,N}(f_N')^+\left(\frac{\sigma^2}{4}L_r^{a,N}(H_r^{a,N})\right)\mathbf{1}_{V_r^{a,N}=-1}dr$$

$$\langle Y^{a,N},\mathscr{M}^{2,a,N}\rangle_s = \frac{4}{\sigma^2}\int_0^s Y_r^{a,N}(f_N')^-\left(\frac{\sigma^2}{4}L_r^{a,N}(H_r^{a,N})\right)\mathbf{1}_{V_r^{a,N}=1}dr,$$

while

$$[\mathscr{M}^{1,a,N},\mathscr{M}^{2,a,N}]_s = \langle \mathscr{M}^{1,a,N},\mathscr{M}^{1,a,N}\rangle_s = 0.$$

Recall Corollary 3 and Lemma 15. Since f' is bounded, the same is true for $(f')_N(x) = N[f(x+1/N) - f(x)]$, uniformly with respect to N. It is not difficult to deduce from the above formulae and Proposition 37 that $\{(H^{a,N},\mathscr{M}^{1,a,N},\mathscr{M}^{2,a,N},Y^{a,N}),\ N \ge 1\}$ is a tight sequence in $C([0,\infty)) \times D([0,+\infty))^3$. Hence at least along a subsequence (but we do not distinguish between the notation for the subsequence and for the sequence),

$$(H^{a,N},\mathscr{M}^{1,a,N},\mathscr{M}^{2,a,N},Y^{a,N}) \Rightarrow (H^a,\mathscr{M}^1,\mathscr{M}^2,Y^a)$$

as $N\to\infty$ in $C([0,\infty))\times D([0,+\infty))^3$, the limit being continuous (since the jumps of $\mathscr{M}^{1,a,N}$, $\mathscr{M}^{2,a,N}$, and $Y^{a,N}$ tend to zero). Moreover

$$\langle Y^{a,N}\rangle_s \Rightarrow \frac{2}{\sigma^2}\int_0^s |Y_r^a|^2 \times \left| f'\left(\frac{\sigma^2}{4}L_r(H_r)\right)\right|^2 dr$$

$$\langle \mathscr{M}^{1,a,N}\rangle_s \Rightarrow \frac{2}{\sigma^2}s,$$

$$\langle \mathscr{M}^{2,a,N}\rangle_s \Rightarrow \frac{2}{\sigma^2}s,$$

$$\langle Y^{a,N},\mathscr{M}^{1,a,N}\rangle_s \Rightarrow \frac{2}{\sigma^2}\int_0^s Y_r^a f'^+\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr,$$

$$\langle Y^{a,N},\mathscr{M}^{2,a,N}\rangle_s \Rightarrow \frac{2}{\sigma^2}\int_0^s Y_r^a f'^-\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr.$$

It follows from the above that Corollary 3 can be enriched as follows

Proposition 25. *For each $a > 0$, as $N \to \infty$,*

$$\left(H^{a,N}, M^{1,a,N}, M^{2,a,N}, L_{\cdot}^{a,N}(0), L_{\cdot}^{a,N}(a^-), Y_{\cdot}^{a,N}\right) \Longrightarrow \left(H^a, \frac{\sqrt{2}}{\sigma}B^1, \frac{\sqrt{2}}{\sigma}B^2, L_{\cdot}^a(0), L_{\cdot}^a(a^-), Y_{\cdot}^a\right),$$

where B^1 and B^2 are two mutually independent standard Brownian motions, $L_{\cdot}^a(0)$ (resp., $L_{\cdot}^a(a^-)$) denotes the local time of the continuous semimartingale H^a at level 0 (resp., at level a^-). Moreover

$$H_s^a = \frac{\sqrt{2}}{\sigma}(B_s^1 - B_s^2) + \frac{1}{2}[L_s^a(0) - L_s^a(a^-)], \quad \textit{and}$$
$$Y_s^a = 1 + \frac{\sqrt{2}}{\sigma}\int_0^s Y_r^a\left[f'^+\left(\frac{\sigma^2}{4}L_r(H_r)\right)dB_r^1 + f'^-\left(\frac{\sigma^2}{4}L_r(H_r)\right)dB_r^2\right].$$

We clearly have

$$Y_s^a = \exp\left(\frac{\sqrt{2}}{\sigma}\int_0^s\left[f'^+\left(\frac{\sigma^2}{4}L_r(H_r)\right)dB_r^1 + f'^-\left(\frac{\sigma^2}{4}L_r(H_r)\right)dB_r^2\right]\right.$$
$$\left. - \frac{1}{\sigma^2}\int_0^s\left|f'\left(\frac{\sigma^2}{4}L_r(H_r)\right)\right|^2 dr\right). \tag{7.9}$$

Since f' is bounded, it is plain that $\mathbb{E}(Y_s^a) = 1$ for all $s > 0$. Let now $\tilde{\mathbb{P}}^a$ denote the probability measure such that

$$\left.\frac{d\tilde{\mathbb{P}}^a}{d\mathbb{P}}\right|_{\mathscr{G}_s^a} = Y_s^a, \tag{7.10}$$

where $\mathscr{G}_s^a := \sigma\{H_r^a,\ 0 \le r \le s\}$. It follows from Girsanov's theorem (see Proposition 35 below) that there exist two mutually independent standard $\widetilde{\mathbb{P}}^a$-Brownian motions $\widetilde{B}^1$ and $\widetilde{B}^2$ such that

$$B_s^1 = \frac{\sqrt{2}}{\sigma}\int_0^s f'^+\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr + \widetilde{B}_s^1,$$
$$B_s^2 = \frac{\sqrt{2}}{\sigma}\int_0^s f'^-\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr + \widetilde{B}_s^2.$$

Consequently

$$\frac{\sqrt{2}}{\sigma}(B_s^1 - B_s^2) = \frac{2}{\sigma}B_s + \frac{2}{\sigma^2}\int_0^s f'\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr,$$

where $B_s = (\sqrt{2})^{-1}(\widetilde{B}_s^1 - \widetilde{B}_s^2)$ is a standard Brownian motion under $\widetilde{\mathbb{P}}^a$. Consequently H^a is a weak solution of the SDE

$$H_s^a = \frac{2}{\sigma^2}\int_0^s f'\left(\frac{\sigma^2}{4}L_r(H_r)\right)dr + \frac{2}{\sigma}B_s + \frac{1}{2}[L_s^a(0) - L_s^a(a^-)], \tag{7.11}$$

where $L_s^a(t)$ denotes the local time accumulated at level t up to time s by the process H^a.

We note that under $\widetilde{\mathbb{P}}^{a,N}$, $Y^{a,N}$ solves (6.13), and under $\widetilde{\mathbb{P}}^{a}$, Y^a solves the SDE (7.11).

Let us establish a general Lemma

Lemma 24. *Let (ξ_N,η_N), (ξ,η) be random pairs defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, with η_N, η nonnegative scalar random variables, and ξ_N, ξ taking values in some complete separable metric space $\mathscr{X}$. Assume that $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$. Write $(\tilde{\xi}_N,\tilde{\eta}_N)$ for the random pair (ξ_N,η_N) defined under the probability measure $\tilde{\mathbb{P}}^N$ which has density η_N with respect to $\mathbb{P}$, and $(\tilde{\xi},\tilde{\eta})$ for the random pair (ξ,η) defined under the probability measure $\tilde{\mathbb{P}}$ which has the density η with respect to $\mathbb{P}$. If (ξ_N,η_N) converges in distribution to (ξ,η), then $(\tilde{\xi}_N,\tilde{\eta}_N)$ converges in distribution to $(\tilde{\xi},\tilde{\eta})$.*

PROOF: Due to the equality $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$ and Scheffé's theorem (see Theorem 16.12 in [9]), the sequence η_N is uniformly integrable. Hence for all bounded continuous $F : \mathscr{X} \times \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}[F(\tilde{\xi}_N,\tilde{\eta}_N)] = \mathbb{E}[F(\xi_N,\eta_N)\eta_N] \to \mathbb{E}[F(\xi,\eta)\eta] = \mathbb{E}[F(\tilde{\xi},\tilde{\eta})].$$

□

It follows readily from Proposition 25 and Lemma 24

Proposition 26. *For any $a > 0$, as $N \to \infty$, $H^{a,N}$, solution of* (6.13) *where the intensity of P^N is σ^2N^2, converges in law towards the solution H^a of the SDE* (7.11).

We now define for each $a,x > 0$ the stopping time

$$\tau_x^a = \inf\left\{s > 0,\ L_s^a(0) > \frac{4}{\sigma^2}x\right\}.$$

Combining the above arguments with those of Proposition 18, we deduce that

Lemma 25. *For any $k \geq 1$, $0 < x_1 < x_2 < \cdots < x_k$, $a > 0$, as $N \to \infty$,*

$$(H^{a,N},\tau_{x_1}^{a,N},\tau_{x_2}^{a,N},\ldots,\tau_{x_k}^{a,N},Y^{a,N}) \Rightarrow (H^a,\tau_{x_1}^a,\tau_{x_2}^a,\ldots,\tau_{x_k}^a,Y^a)$$

weakly in $C(\mathbb{R}_+;\mathbb{R}_+) \times \mathbb{R}_+^k \times C(\mathbb{R}_+;\mathbb{R}_+)$.

We can now prove an extension of the Ray–Knight theorem

Proposition 27. *Assume that f' is bounded. Then for any $a > 0$, the process*

$$\left\{\frac{\sigma^2}{4}L_{\tau_x^a}^a(t),\ 0 \leq t < a,\ x > 0\right\}$$

is a weak solution of equation (4.10) *on the time interval* $[0,a)$.

PROOF: Fix an arbitrary integer $k \geq 1$ and let $0 < x_1 < x_2 < \cdots < x_k$, $g_1, g_2, \ldots, g_k \in C([0,a];\mathbb{R})$. It follows from Corollary 5 that we have the identity in law

$$\left(\int_0^a g_1(t)Z_t^{N,x_1}dt, \ldots, \int_0^a g_k(t)Z_t^{N,x_k}dt\right)$$
$$\stackrel{(d)}{=} \left(\frac{\sigma^2}{4}\int_0^a g_1(t)L^{a,N}_{\tau^{a,N}_{x_1}}(t)dt, \ldots, \frac{\sigma^2}{4}\int_0^a g_k(t)L^{a,N}_{\tau^{a,N}_{x_k}}(t)dt\right)$$
$$= \left(\int_0^{\tau^{a,N}_{x_1}} g_1(H_r^{a,N})dr, \ldots, \int_0^{\tau^{a,N}_{x_k}} g_k(H_r^{a,N})dr\right),$$

where the second equality follows from the "occupation times formula" for zigzag curves, see (5.15). It follows from Proposition 23 that the term on the left converges in law as $N \to \infty$ towards

$$\left(\int_0^a g_1(t)Z_t^{x_1}dt, \ldots, \int_0^a g_k(t)Z_t^{x_k}dt\right),$$

while Lemma 25 implies that the last term on the right converges to

$$\left(\int_0^{\tau^a_{x_1}} g_1(H_r^a)dr, \ldots, \int_0^{\tau^a_{x_k}} g_k(H_r^a)dr\right)$$
$$= \left(\frac{\sigma^2}{4}\int_0^a g_1(t)L^a_{\tau^a_{x_1}}(t)dt, \ldots, \frac{\sigma^2}{4}\int_0^a g_k(t)L^a_{\tau^a_{x_k}}(t)dt\right),$$

where the last identity follows from the occupation times formula (see Proposition 34 below). Consequently for any $k \geq 1$, $0 < x_1 < \cdots < x_k$, $g_1, \ldots, g_k \in C([0,a];\mathbb{R})$,

$$\left(\int_0^a g_1(t)Z_t^{x_1}dt, \ldots, \int_0^a g_k(t)Z_t^{x_k}dt\right) \stackrel{(d)}{=} \left(\frac{\sigma^2}{4}\int_0^a g_1(t)L^a_{\tau^a_{x_1}}(t)dt, \ldots, \frac{\sigma^2}{4}\int_0^a g_k(t)L^a_{\tau^a_{x_k}}(t)dt\right),$$

which implies the result, since Z is the unique solution of (4.10). □

7.2.2 *The General Case ($f \in C^1$ and $f' \leq \beta$)*

Y^a is still defined by (7.9). However, it is not clear a priori that $\mathbb{E}[Y_s^a] = 1$ for all $s > 0$ and we need to justify the fact that we can apply Girsanov's theorem.

For each $x > 0$, $a > 0$, and $n \geq 1$, let

$$T_n^a = \inf\{s > 0, \sup_{0 \leq t < a} L_s^a(t) > n\}.$$

It is plain that the process $f'(L_r^a(H_r^a))$ is bounded on the random interval $[0, T_n^a]$, hence $\mathbb{E}[Y^a_{s\wedge T_n^a}] = 1$ for all $s > 0$, and from Proposition 35 below that we can define

$\tilde{\mathbb{P}}^a$ on $\cup_n \mathscr{F}_{T_n^a}$, which is a probability on each $\mathscr{F}_{T_n^a}$, by

$$\left.\frac{d\tilde{\mathbb{P}}^a}{d\mathbb{P}}\right|_{\mathscr{F}_{T_n^a}} = Y^a_{T_n^a}. \tag{7.12}$$

We now establish

Lemma 26. *For any $x > 0$, $a > 0$, $\mathbb{P}(T_n^a < \tau_x^a) \to 0$, and $\tilde{\mathbb{P}}^a(T_n^a < \tau_x^a) \to 0$, as $n \to \infty$.*

PROOF: It follows from Theorem 8 that

$$\mathbb{P}(T_n^a < \tau_x^a) = \mathbb{P}(\sup_{0 \le t < a} X_t^x > n),$$

where X_t^x is critical Feller diffusion, solution of the SDE

$$X_t^x = x + 2\int_0^t \sqrt{X_r^x} dB_r.$$

But from Doob's inequality and Gronwall's Lemma,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \le r \le t} (X_r^x)^2\right] &\le 2x^2 + 4\mathbb{E}\left(\sup_{0 \le r \le t}\left|\int_0^r \sqrt{X_s^x} dB_s\right|^2\right) \\
&\le 2x^2 + 16\mathbb{E}(\int_0^t X_r^x dr) \\
&\le 2x^2 + 16tx.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(T_n^a < \tau_x^a) &\le \frac{\mathbb{E}\left[\left(\sup_{0 \le t \le a} X_t^x\right)^2\right]}{n^2} \\
&\le \frac{2x^2 + 16ax}{n^2},
\end{aligned}$$

which tends to 0 as $n \to \infty$.

Now let $f_n \in C_b^1(\mathbb{R})$ be such that $f_n(z) = f(z)$, for any $0 \le z \le n$. Applying Proposition 27 with f_n, and noting that on the random interval $[0, T_n^a]$, $f_n'(L_s^a(H_s^a)) = f'(L_s^a(H_s^a))$, we have that

$$\tilde{\mathbb{P}}^a(T_n^a < \tau_x^a) = \mathbb{P}(\sup_{0 \le t < a} Z_t^x > n),$$

where Z_t^x solves the SDE

$$Z_t^x = x + \int_0^t f(Z_r^x)dr + 2\int_0^t \sqrt{Z_r^x}dB_r.$$

Now since $f(z) \le \beta z$, $Z_t^x \le Y_t^x$, solution of the SDE

$$Y_t^x = x + \int_0^t \beta Y_r^x dr + 2\int_0^t \sqrt{Y_r^x}dB_r.$$

A slight extension of the above argument shows that for some constant $C(x,\beta,a)$,

$$\tilde{\mathbb{P}}^a(T_n^a < \tau_x^a) \le \frac{C(x,\beta,a)}{n^2}.$$

□

We can now prove

Proposition 28. $\tilde{\mathbb{P}}^a$ *being defined by* (7.12)*, we have that* $\tilde{\mathbb{P}}^a << \mathbb{P}$ *on* $\mathscr{F}^a_{\tau_x^a}$ *for any* $x > 0$*, and moreover*

$$\frac{d\tilde{\mathbb{P}}^a}{d\mathbb{P}}|_{\mathscr{F}^a_{\tau_x^a}} = Y^a_{\tau_x^a}.$$

PROOF: For any $A \in \mathscr{F}^a_{\tau_x^a}$, $A \cap \{\tau_x^a \le T_n^a\} \in \mathscr{F}^a_{T_n^a \wedge \tau_x^a} \subset \mathscr{F}^a_{T_n^a}$,

$$\begin{aligned}\tilde{\mathbb{P}}^a(A \cap \{\tau_x^a \le T_n^a\}) &= \int_{A\cap\{\tau_x^a \le T_n^a\}} Y^a_{T_n^a \wedge \tau_x^a} d\mathbb{P} \\ &= \int_{A\cap\{\tau_x^a \le T_n^a\}} Y^a_{\tau_x^a} d\mathbb{P}.\end{aligned}$$

Taking the limit as $n \to \infty$ in this identity with the help of Lemma 26 and the monotone convergence theorem, we deduce that

$$\tilde{\mathbb{P}}^a(A) = \int_A Y^a_{\tau_x^a} d\mathbb{P}.$$

□

We can now extend Proposition 27 to our standard assumptions.

Proposition 29. *Assume that f satisfies Assumption (H1'). Then for any* $a > 0$*, the process*

$$\left\{ \frac{\sigma^2}{4} L_{\tau_x^a}^a(t),\ 0 \le t < a,\ x > 0 \right\}$$

is a weak solution of equation (4.10) *on the time interval* $[0,a)$.

PROOF: Consider a sequence $\{f_n,\ n \geq 1\} \subset C_b^1(\mathbb{R}_+)$, as introduced in the proof of Lemma 26. Let $Z^{n,x}$, $H^{n,a}$, and $L^{n,a}$ denote the corresponding population process, contour process, and its local time. From Proposition 27 follows the identity in law

$$\left\{ \frac{\sigma^2}{4} L^{n,a}_{\tau^a_x}(t),\ 0 \leq t < a,\ x > 0 \right\} \stackrel{(d)}{=} \{Z^{n,x}_t,\ 0 \leq t < a,\ x > 0\}.$$

For each $x > 0$, both $\{L^{n,a}_{\tau^a_{x'}}(t),\ 0 \leq t < a,\ 0 < x' \leq x\}$ and $\{Z^{n,x'}_t,\ 0 \leq t < a,\ 0 < x' \leq x\}$ converge a.s. towards $\{L^a_{\tau^a_{x'}}(t),\ 0 \leq t < a,\ 0 < x' \leq x\}$ and $\{Z^{x'}_t,\ 0 \leq t < a,\ 0 < x' \leq x\}$ (which are associated with the original function f), in the sense that the set where the sequence equals its limit increases a.s. to Ω as $n \to \infty$, as a consequence of Lemma 26. The result follows, since $x > 0$ is arbitrary. $\square$