Chapter 5
Genealogies

It is not obvious how to trace genealogies for the individuals whose progeny survives for a given duration of time, also we have seen there is only a finite number of those, see section 4.4. Another point of view, which we now develop, is to use the so-called contour or height process. In the case of Feller's diffusion, the bijection between the contour process and the branching process is given by a Ray–Knight theorem, see section 5.4 below. Here we shall give an independent derivation of this theorem, starting with the contour process of a binary continuous time Galton–Watson process.

There are various forms of bijection between a contour (or height) process and a random binary tree. This section starts with a description of such a bijection, and a rather simple proof that a certain law on the contour paths is in bijection with the law of a continuous time binary Galton–Watson random tree, see Ba, Pardoux, Sow [6]. The result in the critical case was first established by Le Gall [26], and in the subcritical case by Pitman and Winkel [39], see also Geiger and Kersting [19], Lambert [24], where the contour processes are jump processes, while ours are continuous. For similar results in the case where the approximating process is in discrete time and the tree is not necessarily binary, see Duquesne and Le Gall [17]. We consider also the supercritical case. Inspired by the work of Delmas [16], we note that in the supercritical case, the random tree killed at time $a > 0$ is in bijection with the contour process reflected below a. Moreover, one can define a unique local time process, which describes the local times of all the reflected contour processes, and has the same law as the supercritical Galton–Watson process. We next renormalize our Galton–Watson tree and height process, and take the weak limit, thus providing a new proof of Delmas' extension [16] of the second Ray–Knight theorem. The classical version of this theorem establishes the identity in law between the local time of reflected Brownian motion considered at the time when the local time at 0 reaches x, and at all levels, and a Feller critical branching diffusion. The same result holds in the subcritical (resp. supercritical) case, Brownian motion being replaced by Brownian motion with drift (in the supercritical case, reflection

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below an arbitrary level, as above, is needed). The contour process in fact describes the genealogical tree (in the sense of Aldous [2]) of the population, whose total mass follows a Feller SDE. Our proof by approximation makes this interpretation completely transparent.

5.1 Preliminaries

The *artificial* time for the evolution of the contour process of our trees will be labelled s, while the *real* time of the evolution of the population is t. t will also label the values taken by the contour process. See the various figures below, where s is always the coordinate of the horizontal axis, and t the coordinate of the vertical axis.

We fix an arbitrary $p > 0$. Consider a continuous piecewise linear function H from a subinterval of $\mathbb{R}_+$ into $\mathbb{R}_+$, which possesses the following properties : its slope is either p or $-p$; it starts at time $s = 0$ from 0 with the slope p; whenever $H(s) = 0$, $H'_-(s) = -p$, and $H'_+(s) = p$; H is stopped at the time T_m of its m-th return to 0, which is supposed to be finite. We shall denote by $\mathscr{H}_{p,m}$ the collection of all such functions. We shall write $\mathscr{H}_p$ for $\mathscr{H}_{p,1}$. We add the restriction that between two consecutive visits to zero, any function from $\mathscr{H}_{p,m}$ has all its local minima at distinct heights.

We denote by $\mathscr{T}$ the set of continuous time finite rooted binary trees which are defined as follows. An ancestor is born at time 0. This is the root of the tree. Until she eventually dies, she gives birth to an arbitrary number of offsprings, but only one at a time. The same happens to each of her offsprings, and to the offsprings of her offsprings, etc... until eventually the population dies out (assuming for simplicity that we are in the (sub)critical case). For instance, the picture on the right of Figure 5.1 shows a binary tree where the ancestor gives birth to two children before dying. The first child dies childless, while the second one has one child, who dies at the same time as herself. Note that we do not distinguish between the two trees shown in Figure 5.1. We denote by $\mathscr{T}_m$ the set of forests which are the union of m elements of $\mathscr{T}$.

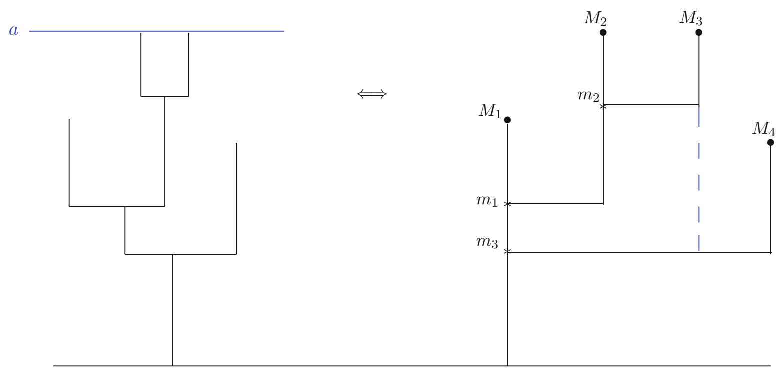

Fig. 5.1 Two equivalent ways of drawing a binary tree

There is a well-known bijection between binary trees and contour processes. Under the curve representing an element $H \in \mathscr{H}_p$, we can draw a tree as follows. The height h_{lfmax} of the leftmost local maximum of H is the lifetime of the ancestor and the height h_{lowmin} of the lowest non zero local minimum is the time of the birth of the first offspring of the ancestor. If there is no such local minimum, the ancestor dies before giving birth to any offspring. We draw a horizontal line at level h_{lowmin}. H has two excursions above h_{lowmin}. The right one is used to represent the fate of the first offspring and of her progeny. The left one is used to represent the fate of the ancestor and of the rest of her progeny, excluding the first offspring and her progeny. If there is no other local minimum of H neither at the left nor at the right of the first explored one, it means that there is no further birth: we draw a vertical line up to the unique local maximum, whose height is a death time. Continuing until there is no further local minimum-maximum to explore, we define by this procedure a bijection $\boldsymbol{\Phi}_p$ from $\mathscr{H}_p$ into $\mathscr{T}$ (see Figure 5.2). Repeating the same construction m times, we extend $\boldsymbol{\Phi}_p$ as a bijection between $\mathscr{H}_{p,m}$ and $\mathscr{T}_m$. Note that drawing the contour process of a tree is obvious (since the horizontal distances between the vertical branches are arbitrary, the choice of p is arbitrary). See the top of Figure 5.2.

We now define probability measures on $\mathscr{H}_p$ (resp. $\mathscr{H}_{p,m}$) and $\mathscr{T}$ (resp. $\mathscr{T}_m$). We describe first the (sub)critical case. Let $0 < b \leq d$ be two parameters. We define a stochastic process whose trajectories belong to $\mathscr{H}_p$ as follows. Let $\{U_k,\ k \geq 1\}$ and $\{V_k,\ k \geq 1\}$ be two mutually independent sequences of i.i.d exponential random variables with means $1/d$ and $1/b$, respectively. Let $Z_k = U_k - V_k$, $k \geq 1$. $\mathbb{P}_{b,d}$ is the law of the random element of $\mathscr{H}_p$, which is such that the height of the first local maximum is U_1, that of the first local minimum is $(Z_1)^+$. If $(Z_1)^+ = 0$, the process is stopped. Otherwise, the height of the second local maximum is $Z_1 + U_2$, the height of the second local minimum is $(Z_1 + Z_2)^+$, etc. Because $b \leq d$, $\mathbb{E}Z_1 \leq 0$, hence the process returns to zero a.s. in finite time. The random trajectory which we have constructed is an excursion above zero (see the bottom of Figure 5.2). We define similarly a law on $\mathscr{H}_{p,m}$ as the concatenation of m i.i.d. such excursions, and denote it by $\mathbb{P}_{b,d}$. This thus defined random element of $\mathscr{H}_{p,m}$ is called a contour or height process. We associate the continuous time Galton–Watson tree (which is a random element of $\mathscr{T}$) with the same pair of parameters (b,d) as follows. The lifetime of each individual is exponential with expectation $1/d$, and during her lifetime, independently of it, each individual gives birth to offsprings according to a Poisson process with rate b. The behaviors of the various individuals are i.i.d. We denote by $\mathbb{Q}_{b,d}$ the law on $\mathscr{T}_m$ of a forest of m i.i.d. random trees whose law is as just described.

In the supercritical case, the case where $b > d$, the contour process defined above does not come back to zero *a.s.*. To overcome this difficulty, we use a trick which is due to Delmas [16], and reflect the process H below an arbitrary level $a > 0$ (which amounts to kill the whole population at time a). The height process $H^a = \{H^a_s,\ s \geq 0\}$ reflected below a is defined as above, with the addition of the rule that whenever the process reaches the level a, it stops and starts immediately going down with slope $-p$ for a duration of time exponential with expectation $1/b$. Again the

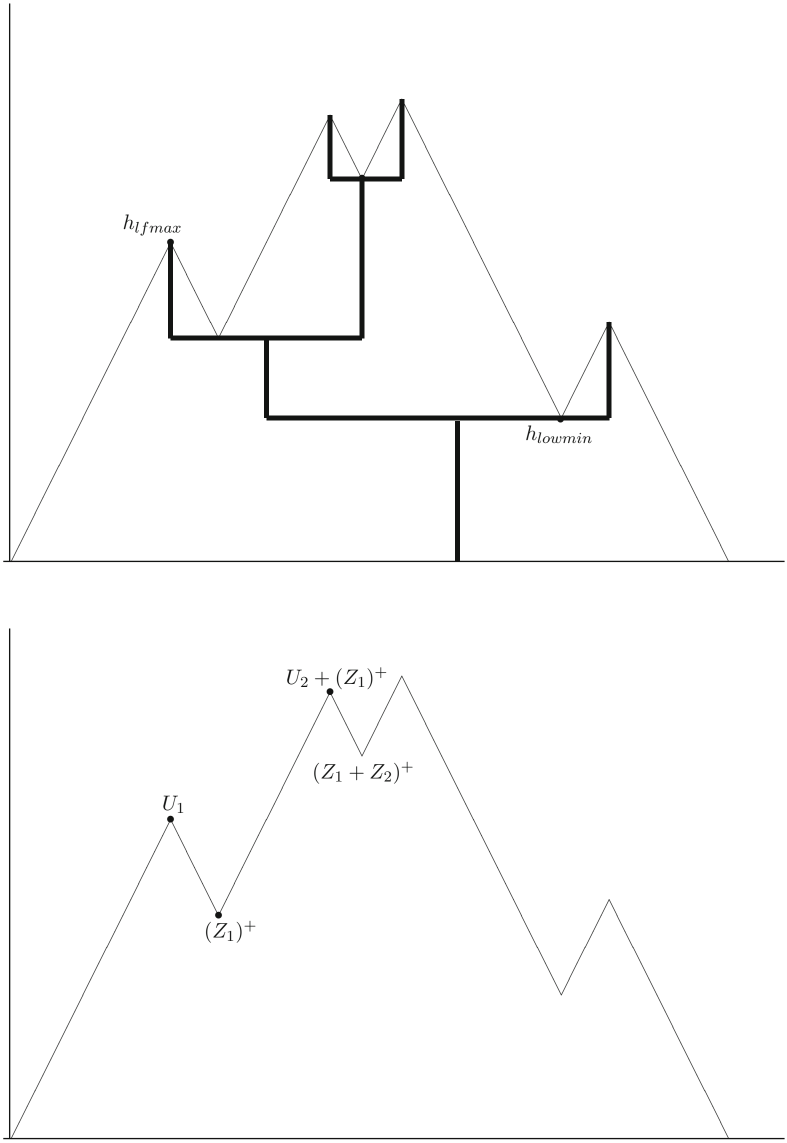

Fig. 5.2 Bijection between $\mathscr{H}_2$ and $\mathscr{T}$ (see above), and a trajectory of a contour process (see below)

process stops when first going back to zero. The reflected process H^a comes back to zero almost surely. Indeed, let A^a_n denote the event "H^a does not reach zero during its n first descents." Since the levels of the local maxima are bounded by a, clearly $\mathbb{P}(A_n) \leq (1-\exp(-ba))^n$, which goes to zero as $n \longrightarrow \infty$. Hence the result. For each $a \in (0,+\infty)$, and any pair (b,d) of positive numbers, denote by $\mathbb{P}_{b,d,a}$ the law of the process H^a. Define $\mathbb{Q}_{b,d,a}$ to be the law of a binary Galton–Watson tree with birth rate b and death rate d, killed at time $t = a$ (i. e. all individuals alive at time a^- are killed at time a). $\mathbb{P}_{b,d,+\infty}$ makes perfect sense in case $b \leq d$, $\mathbb{Q}_{b,d,+\infty}$ is always well defined.

5.2 Correspondence of Laws

The aim of this section is to prove that, for any $b,d > 0$ and $a \in (0,+\infty)$ [including possibly $a = +\infty$ in the case $b \leq d$], $\mathbb{P}_{b,d,a}\Phi_p^{-1} = \mathbb{Q}_{b,d,a}$. Let us state some basic results on homogeneous Poisson process, which will be useful in the sequel.

5.2.1 Preliminary Results

Let $(T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity (or rate) b. This means that $T_0 = 0$, and $(T_{k+1} - T_k,\ k \geq 0)$ are i.i.d exponential r.v.'s with mean $1/b$. Let $(N_t,\ t \geq 0)$ be the counting process associated with $(T_k)_{k\geq 0}$, that is,

$$\forall t \geq 0,\ N_t = \sup\{k \geq 0,\ T_k \leq t\}.$$

We shall also call $(N_t,\ t \geq 0)$ a Poisson process. This process has independent increments, and for any $0 \leq s < t$, $N_t - N_s$, which is the number of points of the above PPP in the interval $(s,t]$, follows the Poisson law with parameter $b(t-s)$. In particular, the intensity b is the mean number of points of the PPP in an interval of length 1.

The first result is well-known and elementary.

Lemma 8. *Let M be a nonnegative random variable independent of $(T_k)_{k\geq 0}$, and define*

$$R_M = \sup_{k\geq 0}\{T_k; T_k \leq M\}. \tag{5.1}$$

Then $M - R_M \overset{(d)}{=} V \wedge M$ where V and M are independent, V has an exponential distribution with mean $1/b$.
Moreover, on the event $\{R_M > s\}$, the conditional law of $N_{R_M^-} - N_s$ given R_M is Poisson with parameter $b(R_M - s)$.

The second one is (in the next statement, we use again the definition (5.1)) :

Lemma 9. *Let $(T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity b. M a positive random variable which is independent of $(T_k)_{k\geq 0}$. Consider the integer-valued* *random variable K such that $T_K = R_M$ and a second Poisson point process $\left(T'_k\right)_{k\geq 0}$ with intensity b, which is jointly independent of the first and of M. Then $\left(\overline{T}_k\right)_{k\geq 0}$ defined by:*

$$\overline{T}_k = \begin{cases} T_k & \text{if } k < K \\ T_K + T'_{k-K+1} & \text{if } k \geq K \end{cases}$$

is a Poisson point process on $\mathbb{R}_+$ with intensity b, which is independent of R_M.

PROOF: Let $(N_t, t \geq 0)$, $\left(\overline{N}_t, t \geq 0\right)$ and $(N'_t, t \geq 0))$ be the counting processes associated to $(T_k)_{k\geq 0}$, $\left(\overline{T}_k\right)_{k\geq 0}$ and $\left(T'_k\right)_{k\geq 0}$, respectively. It suffices to prove that for any $n \geq 1$, $0 < t_1 < \cdots < t_n$ and $k_1 < k_2 < \cdots < k_n \in \mathbb{N}^*$,

$$\xi_M = \mathbb{P}\left(\overline{N}_{t_1} = k_1, \ldots, \overline{N}_{t_n} = k_n | R_M\right) = e^{-bt_n} \prod_{i=1}^{n} \frac{(b(t_i - t_{i-1}))^{k_i - k_{i-1}}}{(k_i - k_{i-1})!}.$$

Since there is no harm in adding $t'_i s$, we only need to do that computation on the event that there exists $2 \leq i \leq n$ such that $t_{i-1} < R_M < t_i$, in which case a standard argument yields easily the claimed result, thanks to Lemma 8. Indeed we have that

$$\xi_M = \mathbb{P}\left(N_{t_1} = k_1, \cdots, N_{t_{i-1}} = k_{i-1}, N_{R_M^-} + N'_{t_i - R_M} = k_i, \cdots, N_{R_M^-} + N'_{t_n - R_M} = k_n\right)$$

$$= \mathbb{P}\Bigg(N_{t_1} = k_1, \cdots, N_{t_{i-1}} - N_{t_{i-2}} = k_{i-1} - k_{i-2}, N_{R_M^-} - N_{t_{i-1}} + N'_{t_i - R_M} = k_i - k_{i-1},$$

$$N'_{t_{i+1} - R_M} - N'_{t_i - R_M} = k_{i+1} - k_i, \cdots, N'_{t_n - R_M} - N'_{t_{n-1} - R_M} = k_n - k_{n-1}\Bigg)$$

$$= e^{-bt_n} \prod_{i=1}^{n} \frac{(b(t_i - t_{i-1}))^{k_i - k_{i-1}}}{(k_i - k_{i-1})!}.$$

□

5.2.2 *Basic Theorem*

We are now in a position to prove the next theorem, which says that the tree associated to the contour process H^a defined in section 5.1 is a continuous time binary Galton–Watson tree with death rate d and birth rate b, killed at time a, and vice versa.

Theorem 6. *For any* $b,d > 0$ *and* $a \in (0,+\infty)$ $[$*including possibly* $a = +\infty$ *in the case* $b \leq d$ $]$,

$$\mathbb{Q}_{b,d,a} = \mathbb{P}_{b,d,a} \boldsymbol{\Phi}_p^{-1}.$$

PROOF: The individuals making up the population represented by the tree whose law is $\mathbb{Q}_{b,d,a}$, will be numbered: $\ell = 1,2,\ldots$. 1 is the ancestor of the whole family. The subsequent individuals will be identified below. We will show that this tree is explored by a process whose law is precisely $\mathbb{P}_{b,d,a}$. We introduce a family $(T_k^\ell, k \geq 0, \ell \geq 1)$ of mutually independent Poisson point processes with intensity b. For any $\ell \geq 1$, the process T_k^ℓ describes the times of birth of the offsprings of the individual ℓ. We define U_ℓ to be the lifetime of individual ℓ.

- **Step 1**: We start from $H_0^a = 0$ at the initial time $s = 0$ and climb up with slope p to the level $M_1 = U_1 \wedge a$, where U_1 follows an exponential law with mean $1/d$. H_s^a goes down from M_1 with slope $-p$ until we find the most recent point of the Poisson process (T_k^1) which gives the times of birth of the offsprings of individual 1. So from Lemma 8, H_s^a has descended by $V_1 \wedge M_1$, where V_1 follows an exponential law with mean $1/b$, and is independent of M_1. We hence reach the level $m_1 = M_1 - V_1 \wedge M_1$.
  If $m_1 = 0$, we stop, else we turn to
- **Step 2**: We give the label 2 to this last offspring of the individual 1, born at the time m_1. Let us define $(\bar{T}_k^2)$ by:

$$\bar{T}_k^2 = \begin{cases} T_k^1 & \text{if } k < K_1 \\ T_{K_1}^1 + T_{k-K_1+1}^2 & \text{otherwise} \end{cases}$$

  where K_1 is such that $T_{K_1}^1 = m_1$.
  Thanks to Lemma 9, $(\bar{T}_k^2)$ is a Poisson process with intensity b on $\mathbb{R}_+$, which is independent of m_1 and in fact also of (U_1, V_1).
  Starting from m_1, the contour process climbs up to level $M_2 = (m_1 + U_2) \wedge a$, where U_2 is an exponential r.v. with mean $1/d$, independent of (U_1, V_1). Starting from level M_2, we go down a height $M_2 \wedge V_2$ where V_2 follows an exponential law with mean $1/b$ and is independent of (U_2, U_1, V_1), to find the most recent point of the Poisson process $(\bar{T}_k^2)$. At this moment we are at the level $m_2 = M_2 - V_2 \wedge M_2$.
  If $m_2 = 0$ we stop. Otherwise we give the label 3 to the individual born at time m_2, and repeat step 2 until we reach 0. See Figure 5.1.

Since either we have a reflection at level a or $b \leq d$, zero is reached *a.s.* after a finite number of iterations. It is clear that the random variables M_i and m_i determine fully the law $\mathbb{Q}_{b,d,a}$ of the binary tree killed at time $t = a$ and they both have the same joint distribution as the levels of the successive local maxima and minima of the process H^a under $\mathbb{P}_{b,d,a}$. □

5.2.3 A Discrete Ray–Knight Theorem

For any $a,b,d > 0$, we consider the contour process $\{H^a_s,\ s \geq 0\}$ defined in section 5.1 which is reflected in the interval $[0,a]$ and stopped at the first moment it reaches zero for the m-th time. To this process we can associate a forest of m binary trees with birth rate b and death rate d, killed at time $t = a$, which all start with a single individual at the initial time $t = 0$. Consider the continuous time branching process $\left(X^{a,m}_t,\ t \geq 0\right)$ describing the number of offsprings alive at time t of the m ancestors born at time 0, whose progeny is killed at time $t = a$. Every individual in this population, independently of the others, lives for an exponential time with parameter d and gives birth to offsprings according to a Poisson process of intensity b. We now choose the slopes of the piecewise linear process H^a to be ± 2 (i.e., $p = 2$). We define the local time accumulated by H^a at level t up to time s:

$$L^a_s(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \leq H^a_r < t+\varepsilon\}} dr. \tag{5.2}$$

$L^a_s(t)$ equals the number of pairs of branches of H^a which cross level t between times 0 and s. Note that a local minimum at level t counts for two crossings, while a local maximum at level t counts for none. We have the following "occupation times formula," whose proof is an easy exercise. For any integrable function g,

$$\int_0^s g(H^a_r) dr = \int_0^a g(r) L^a_s(r) dr. \tag{5.3}$$

Let

$$\tau^a_m = \inf\{s > 0 : L^a_s(0) \geq m\}. \tag{5.4}$$

$L^a_{\tau_m}(t)$ counts the number of descendants of m ancestors at time 0, which are alive at time t. Then we have

Lemma 10. *For all $b,d > 0$ and $a \in (0,+\infty)$ [including possibly $a = +\infty$ in the case $b \leq d$].*

$$\left\{L^a_{\tau^a_m}(t),\ t \geq 0, m \geq 1\right\} \equiv \left\{X^{a,m}_t,\ t \geq 0, m \geq 1\right\} \ \ a.s.$$

We now want to establish a similar statement without the arbitrary parameter a. There remains a difficulty only in the supercritical case, in which case we cannot choose $a = +\infty$ in the above construction. For any $0 < a < b$, we define the function $\Pi^{a,b}$ which maps continuous trajectories with values in $[0,b]$ into trajectories with values in $[0,a]$ as follows. If $u \in C(\mathbb{R}_+,[0,b])$,

$$\rho_u(s) = \int_0^s \mathbf{1}_{\{u(r) \leq a\}} dr; \quad \Pi^{a,b}(u)(s) = u(\rho_u^{-1}(s)). \tag{5.5}$$

Lemma 11.

$$\Pi^{a,b}(H^b) \stackrel{(d)}{=} H^a$$

PROOF: It is in fact sufficient to show that the conditional law of the level of the first local minimum of H^b after crossing the level a downwards, given the past of H^b, is the same as the conditional law of the level of the first local minimum of H^a after a reflection at level a, given the past of H^a. This identity follows readily from the "lack of memory" of the exponential law. □

This last Lemma says that reflecting under a, or chopping out the pieces of trajectory above level a, yields the same result (at least in law).

We now consider the case $p = 2$. To each $b,d > 0$, $m \geq 1$, we associate the process $\{X_t^m,\ t \geq 0\}$ which describes the evolution of the number of descendants of m ancestors, with birth rate b and death rate d. For each $a > 0$ [including possibly $a = +\infty$ in the case $b \leq d$], we let $(H_s^a,\ s \geq 0)$ denote the contour process of the genealogical tree of this population killed at time a, L^a denotes its local time and τ_m^a is defined by (5.4). It follows readily from Lemma 11 that for any $0 < a < b$,

$$\left(L^b_{\tau^b_m}(t), 0 \leq t < a, m \geq 1\right) \stackrel{(d)}{=} \left(L^a_{\tau^a_m}(t), 0 \leq t < a, m \geq 1\right). \tag{5.6}$$

The compatibility relation (5.6) implies the existence of a projective limit $\{\mathscr{L}_m(t), t \geq 0, m \geq 1\}$ with values in $\mathbb{R}_+$, which is such that for each $a > 0$,

$$\{\mathscr{L}_m(t), 0 \leq t < a, m \geq 1\} \stackrel{(d)}{=} \{L^a_{\tau^a_m}(t), 0 \leq t < a, m \geq 1\}. \tag{5.7}$$

We have the following "discrete Ray–Knight Theorem."

Proposition 17.

$$\{\mathscr{L}_m(t), t \geq 0, m \geq 1\} \stackrel{(d)}{=} \{X_t^m,\ t \geq 0, m \geq 1\}.$$

PROOF: It suffices to show that for any $a \geq 0$,

$$\{\mathscr{L}_m(t), 0 \leq t < a, m \geq 1\} \stackrel{(d)}{=} \{X_t^m, 0 \leq t < a, m \geq 1\}.$$

This result follows from (5.7) and Lemma 10. □

5.2.4 Renormalization

Let $x > 0$ be arbitrary, and $N \geq 1$ be an integer which will eventually go to infinity. Let $(X_t^{N,x})_{t \geq 0}$ denote the branching process which describes the number of descendants at time t of $[Nx]$ ancestors, in the population with birth rate $b_N = \sigma^2 N/2 + \alpha$ and death rate $d_N = \sigma^2 N/2 + \beta$, where $\alpha, \beta \geq 0$. We set for $t \geq 0$

$$Z_t^{N,x} = N^{-1} X_t^{N,x}.$$

In particular we have that $Z_0^{N,x} = \frac{[Nx]}{N} \to x$ when $N \to +\infty$. Let $H^{a,N}$ be the contour process associated to $\{X_t^{N,x},\ 0 \le t < a\}$ defined in the same way as previously, but with slopes $\pm 2N$, and b and d are replaced by b_N and d_N. We define also $L_s^{a,N}(t)$, the local time accumulated by $H^{a,N}$ at level t up to time s, as

$$L_s^{a,N}(t) = \frac{4}{\sigma^2} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \le H_r^{a,N} < t+\varepsilon\}} dr \tag{5.8}$$

$\frac{\sigma^2}{4} L_s^{a,N}(t)$ equals $1/N$ times the number of pairs of t-crossings of $H^{a,N}$ between times 0 and s. Let

$$\tau_x^{a,N} = \inf\left\{ s > 0 : L_s^{a,N}(0) \ge \frac{4}{\sigma^2} \frac{[Nx]}{N} \right\}. \tag{5.9}$$

We define for all $N \ge 1$ the projective limit $\{\mathscr{L}_x^N(t), t \ge 0, x > 0\}$, which is such that for each $a > 0$,

$$\{\mathscr{L}_x^N(t), 0 \le t < a, x > 0\} \overset{(d)}{=} \{L^{a,N}_{\tau_x^{a,N}}(t), 0 \le t < a, x > 0\}.$$

Proposition 17 translates as (note that the factor N^{-1} in the definition of $Z_t^{N,x}$ matches the slopes $\pm 2N$ of $H^{a,N}$, which introduces a factor N^{-1} in the local times defined by (5.8))

Lemma 12. *We have the identity in law*

$$\{\mathscr{L}_x^N(t), t \ge 0, x > 0\} \overset{(d)}{=} \left\{ \frac{4}{\sigma^2} Z_t^{N,x}, t \ge 0, x > 0 \right\}.$$

We will need to write precisely the evolution of $\{H^{a,N},\ s \ge 0\}$, the contour process of the forest of trees representing the population $\{X_t^{N,x},\ 0 \le t < a\}$. Let $\{V_s^{a,N},\ s \ge 0\}$ be the $\{-1,1\}$-valued process which is such that s-a.e. $\frac{dH_s^{a,N}}{ds} = 2NV_s^{a,N}$. The $\mathbb{R}_+ \times \{-1,1\}$-valued process $\{(H_s^{a,N}, V_s^{a,N}),\ s \ge 0\}$ is a Markov process, which solves the following SDE :

$$\begin{aligned} \frac{dH_s^{a,N}}{ds} &= 2NV_s^{a,N}, \quad H_0^{a,N} = 0, V_0^{a,N} = 1, \\ dV_s^{a,N} &= 2\mathbf{1}_{\{V_{s^-}^{a,N} = -1\}} dP_s^+ - 2\mathbf{1}_{\{V_{s^-}^{a,N} = 1\}} dP_s^- + \frac{\sigma^2}{2} N dL_s^{a,N}(0) - \frac{\sigma^2}{2} N dL_s^{a,N}(a^-), \end{aligned} \tag{5.10}$$

where $\{P_s^+,\ s \ge 0\}$ and $\{P_s^-,\ s \ge 0\}$ are two mutually independent Poisson processes, with intensities (given by $2N \times$ the rate of birth (resp. death))

$$\sigma^2 N^2 + 2\alpha N \quad \text{and} \quad \sigma^2 N^2 + 2\beta N,$$

$L_s^{a,N}(0)$ and $L_s^{a,N}(a^-)$ denote, respectively, the number of visits to 0 and a by the process $H^{a,N}$ up to time s, multiplied by $4/N\sigma^2$ (see (5.8)). These two terms in the expression of $V^{a,N}$ stand for the reflection of $H^{a,N}$ above 0 and below a. Note that our definition of $L^{a,N}$ makes the mapping $t \longrightarrow L_s^{a,N}(t)$ right-continuous for each $s > 0$. Hence $L_s^{a,N}(t) = 0$ for $t \geq a$, while $L_s^{a,N}(a^-) = \lim_{n\to\infty} L_s^{a,N}(a - \frac{1}{n}) > 0$ if $H^{a,N}$ has reached the level a by time s.

5.3 Weak Convergence

5.3.1 Tightness of $H^{a,N}$

We deduce from (5.10) that

$$\begin{aligned}
H_s^{a,N} &= 2N\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}}dr - 2N\int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}}dr,\\
\frac{V_s^{a,N}}{\sigma^2 N} &= \frac{1}{\sigma^2 N} - \left(2N + \frac{4\beta}{\sigma^2}\right)\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}}dr + \left(2N + \frac{4\alpha}{\sigma^2}\right)\int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}}dr\\
&\quad - \frac{2}{\sigma^2 N}\int_0^s \mathbf{1}_{\{V_{r^-}^{a,N}=1\}}dM_r^- + \frac{2}{\sigma^2 N}\int_0^s \mathbf{1}_{\{V_{r^-}^{a,N}=-1\}}dM_r^+\\
&\quad + \frac{1}{2}[L_s^{a,N}(0) - L_{0^+}^{a,N}(0)] - \frac{1}{2}L_s^{a,N}(a^-),
\end{aligned}$$

where

$$M_s^+ = P_s^+ - (\sigma^2 N^2 + 2\alpha N)s, \quad M_s^- = P_s^- - (\sigma^2 N^2 + 2\beta N)s$$

are two martingales. Consequently

$$\begin{aligned}
H_s^{a,N} + \frac{V_s^{a,N}}{\sigma^2 N} &= \frac{1}{\sigma^2 N} + \frac{4}{\sigma^2}\int_0^s \left(\alpha \mathbf{1}_{\{V_r^{a,N}=-1\}} - \beta \mathbf{1}_{\{V_r^{a,N}=1\}}\right)dr + M_s^{a,N}\\
&\quad + \frac{1}{2}[L_s^{a,N}(0) - L_{0^+}^{a,N}(0)] - \frac{1}{2}L_s^{a,N}(a^-),
\end{aligned} \tag{5.11}$$

where $M_s^{a,N}$ is a martingale such that

$$\begin{aligned}
[M^{a,N}]_s &= \frac{4}{\sigma^4 N^2}\left(\int_0^s \mathbf{1}_{\{V_{r^-}^{a,N}=1\}}dP_r^- + \int_0^s \mathbf{1}_{\{V_{r^-}^{a,N}=-1\}}dP_r^+\right),\\
\langle M^{a,N}\rangle_s &= \frac{4}{\sigma^2}s + \frac{8}{\sigma^4 N}\int_0^s \left(\alpha \mathbf{1}_{\{V_r^{a,N}=-1\}} + \beta \mathbf{1}_{\{V_r^{a,N}=1\}}\right)dr.
\end{aligned}$$

Lemma 13. *For any $a > 0$, the sequence $\{H_s^{a,N},\ s \geq 0\}_{N\geq 1}$ is tight in $C([0,\infty))$.*

PROOF: Let us rewrite (5.11) in the form

$$H_s^{a,N} = K_s^{a,N} + \frac{1}{2}[L_s^{a,N}(0) - L_{0^+}^{a,N}(0)] - \frac{1}{2}L_s^{a,N}(a^-).$$

It follows readily from Proposition 37 below that $K^{a,N}$ is tight in $D([0,\infty))$, and all its jumps converge to 0 as $N \to \infty$. It then follows from Theorem 13.2 and (12.9) in [10] that for any $T > 0$, all $\varepsilon, \eta > 0$, there exist N_0 and $\delta > 0$ such that for all $N \geq N_0$,

$$\mathbb{P}\left(\sup_{0\leq r,s\leq T,\ |s-r|\leq\delta} |K_s^{a,N} - K_r^{a,N}] > \varepsilon \right) \leq \eta.$$

Since $L^N(0)$ (resp. $L^N(a^-)$) increases only when $H_s^N = 0$ (resp. when $H_s^N = a$), it is not hard to conclude that, provided $\varepsilon < a$, the above implies that for all $N \geq N_0$,

$$\mathbb{P}\left(\sup_{0\leq r,s\leq T,\ |s-r|\leq\delta} |H_s^{a,N} - H_r^{a,N}] > \varepsilon \right) \leq \eta.$$

In view of (12.7) in [10], this implies that $H^{a,N}$ is tight in $D([0,\infty))$. □

5.3.2 Weak Convergence of $H^{a,N}$

Let us state our convergence result.

Theorem 7. *For any $a > 0$ [including possibly $a = +\infty$ in the case $\alpha \leq \beta$], $H^{a,N} \Rightarrow H^a$ in $C([0,\infty))$ as $N \to \infty$, where $\{H_s^a,\ s \geq 0\}$ is the process*

$$\frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s$$

reflected in $[0,a]$. In other words, H^a is the unique weak solution of the reflected SDE[1]

$$H_s^a = \frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s + \frac{1}{2}L_s^a(0) - \frac{1}{2}L_s^a(a^-), \tag{5.12}$$

where $L_s^a(t)$ denotes the local time accumulated by $(H_r^a,\ r \geq 0)$ up to time s at level t.

We first prove

Lemma 14. *For any sequence $\left(U^N, N \geq 1\right) \subset C([0,+\infty))$ which is such that $U^N \Rightarrow U$ as $N \to \infty$, for all $s > 0$,*

$$\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}} U_r^N dr \Rightarrow \frac{1}{2}\int_0^s U_r dr, \quad \int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}} U_r^N dr \Rightarrow \frac{1}{2}\int_0^s U_r dr.$$

[1] The fact that Brownian motion with drift reflected in the interval $[0,a]$ takes this form is explained at the end of section A.4 below.

PROOF: It is an easy exercise to check that the mapping

$$\Phi : C([0,+\infty)) \times C_{\uparrow}([0,+\infty)) \to C([0,+\infty))$$

defined by

$$\Phi(x,y)(t) = \int_0^t x(s)dy(s),$$

where $C_{\uparrow}([0,+\infty))$ denotes the set of increasing continuous functions from $[0,\infty)$ into $\mathbb{R}$, and the three spaces are equipped with the topology of locally uniform convergence, is continuous. Consequently it suffices to prove that locally uniformly in $s > 0$,

$$\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}} dr \to \frac{s}{2}$$

in probability, as $N \to \infty$. In fact since both the sequence of processes and the limit are continuous and monotone, it follows from an argument "à la Dini" that it suffices to prove

Lemma 15. *For any $s > 0$,*

$$\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}} dr \to \frac{s}{2}, \int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}} dr \to \frac{s}{2}$$

in probability, as $N \to \infty$.

PROOF: We have (the second line follows from (5.10))

$$\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}} dr + \int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}} dr = s,$$
$$\int_0^s \mathbf{1}_{\{V_r^{a,N}=1\}} dr - \int_0^s \mathbf{1}_{\{V_r^{a,N}=-1\}} dr = (2N)^{-1} H_s^{a,N}.$$

It follows readily from Lemma 13 that $(2N)^{-1} H_s^{a,N} \to 0$ in probability, as $n \to \infty$. We conclude by adding and subtracting the two above identities. □

PROOF OF THEOREM 7 Let us prove that

$$\left(H^{a,N}, M^{N,a}, L_{\cdot}^{a,N}(0), L_{\cdot}^{a,N}(a^-)\right) \Longrightarrow \left(H^a, \frac{2}{\sigma}B, L_{\cdot}^a(0), L_{\cdot}^a(a^-)\right).$$

Concerning tightness, we only need to take care of the third and fourth terms in the quadruple. We notice that Lemma 13 implies that $(L^{a,N}(0) - L^N(a^-))_{N\geq 1}$ is tight in $D([0,\infty))$. Since $L^N(0)$ (resp. $L^N(a^-)$) increases only when $H_s^N = 0$ (resp. when $H_s^N = a$), it is not hard to deduce that both $(L^{a,N}(0))_{N\geq 1}$ and $(L^N(a^-))_{N\geq 1}$ are tight in $D([0,\infty))$. Alternatively tightness of $L^N(0)$ can be deduced from the identity (5.13) below, and that of $L^N(a^-)$ from a similar formula.

Then $\left(H^{a,N}, M^{N,a}, L_{\cdot}^{a,N}(0),\ L_{\cdot}^{a,N}(a^-)\right)_{N\geq 1}$ is tight in $C([0,\infty)) \times [D([0,\infty))]^3$. Moreover any weak limit of $M^{N,a}$ along a converging subsequence equals $2B/\sigma$, since $< M^{N,a} >_s \to 4s/\sigma^2$ and the jumps of $M^{N,a}$ are equal in amplitude to $\frac{2}{\sigma^2 N}$.

Let $f \in C^2(\mathbb{R})$ such that $f'(0) = 1$ and $f'(a) = 0$, and define $f^N(h,v) = f(h) + \frac{v}{\sigma^2 N} f'(h)$. We deduce from (5.10)

$$L_s^{a,N}(0) = 2f(H_s^{a,N}) + \frac{2V_s^{a,N}}{\sigma^2 N} f'(H_s^{a,N}) - 2f(0) - \frac{2}{\sigma^2 N} f'(0) - \frac{4}{\sigma^2} \int_0^s f''(H_r^{a,N}) dr$$
$$- \frac{8}{\sigma^2} \int_0^s f'(H_r^{a,N})(\alpha \mathbf{1}_{\{V_r^N = -1\}} - \beta \mathbf{1}_{\{V_r^N = 1\}}) dr - 2M_s^{f,N} - 2\tilde{M}_s^{f,N}, \tag{5.13}$$

where $M^{f,N}$ and $\tilde{M}^{f,N}$ are martingales such that

$$\langle M^{f,N} \rangle_s = \frac{4}{\sigma^2} \int_0^s [f'(H_r^{a,N})]^2 dr, \quad \langle \tilde{M}^{f,N} \rangle_s \leq \frac{c(f)}{N} s.$$

It follows by taking the limit in (5.13) (and in a similar formula for $L_s^{a,N}(a^-)$) that we have a limit of the form $(H^a, 2B/\sigma, L^a(0), L^a(a^-))$ along a converging subsequence of the sequence $\left(H^{a,N}, M^{N,a}, L^{a,N}(0), L^{a,N}(a^-)\right)$.

For any $0 < \varepsilon < a$, let $f_\varepsilon \in C^2(\mathbb{R})$ be such that $f'_\varepsilon(0) = 1$, $f'_\varepsilon(x) = 0$ for all $\varepsilon \leq x \leq a$, $f'(x) \geq 0$ and $f''(x) \leq 0$ for all $x \geq 0$. Taking the limit along the converging subsequence in (5.13) with $f_\varepsilon^N(h,v) = f_\varepsilon(h) + \frac{v}{\sigma^2 N} f'_\varepsilon(h)$, we deduce that

$$L_s^a(0) = 2f_\varepsilon(H_s^a) - 2f_\varepsilon(0) - \frac{4}{\sigma^2} \int_0^s f''_\varepsilon(H_r^a) dr - \frac{4}{\sigma^2}(\alpha - \beta) \int_0^s f'_\varepsilon(H_r^a) dr - 2M_s^{f_\varepsilon},$$
$$\langle M^{f_\varepsilon} \rangle_s = \frac{4}{\sigma^2} \int_0^s [f'_\varepsilon(H_r^a)]^2 dr.$$

It then follows that $\int_0^s \mathbf{1}_{\{H_r^a \geq \varepsilon\}} dL_r^a(0) = 0$. This being true for all $0 < \varepsilon < a$, we have that $L_s^a(0) = \int_0^s \mathbf{1}_{\{H_r^a = 0\}} dL_r^a(0)$. We prove similarly that $L_s^a(a^-) = \int_0^s \mathbf{1}_{\{H_r^a = a\}} dL_r^a(a^-)$. Moreover it is plain that both $L_s^a(0)$ and $L_s^a(a^-)$ are continuous and increasing. Now (5.12) follows by taking the limit in (5.11). It is plain that H^a, being a limit (along a subsequence) of $H^{a,N}$, takes values in $[0,a]$. The fact that $L^a(0)$ (resp. $L^a(a^-)$) is continuous and increasing, and increases only on the set of time when $H_s^a = 0$ (resp. $H_s^a = a$) proves that $\frac{\sigma}{2} H^a$ is a Brownian motion with drift $(\alpha - \beta)s/\sigma$, reflected in $[0,a]$, which characterizes its law. We can refer, e.g., to the formulation of reflected SDEs in [30]. Hence the whole sequence converges, and the Theorem is proved. □

We have proved in particular

Corollary 3. *For each $a > 0$ (including $a = +\infty$ in the case $\alpha \leq \beta$),*

$$\left(H^{a,N}, M^{N,a}, L^{a,N}(0), L^{a,N}(a^-)\right) \Rightarrow \left(H^a, \frac{2}{\sigma} B, L^a(0), L^a(a^-)\right)$$

as $N \to \infty$, where B is a standard Brownian motion, $L^a(0)$ (resp. $L^a(a^-)$) is the local time of H^a at level 0 (resp. at level a), and

$$H_s^a = \frac{2}{\sigma^2}(\alpha-\beta)s + \frac{2}{\sigma}B_s + \frac{1}{2}\left[L_s^a(0) - L_s^a(a^-)\right],$$

i.e., H^a equals $2/\sigma$ multiplied by Brownian motion with drift $(\alpha-\beta)s/\sigma$, reflected in the interval $[0,a]$.

5.4 A Ray–Knight Theorem

In this section we give a new proof of Delmas' generalization of the second Ray–Knight Theorem, see [16]. In case $\alpha \le \beta$, we can choose $a = +\infty$, let $L_\cdot(0)$ denote the local time of H at level 0, and define

$$\tau_x = \inf\left\{s > 0,\ L_s(0) > \frac{4}{\sigma^2}x\right\}.$$

In the supercritical case, of course the construction is more complex. It follows from Lemma 11 and Corollary 3 (see also Lemma 2.1 in [16]) that for any $0 < a < b$,

$$\Pi^{a,b}(H^b) \overset{(d)}{=} H^a, \tag{5.14}$$

where H^a [resp. H^b] is $2/\sigma$ multiplied by Brownian motion, with drift $(\alpha-\beta)s/\sigma$, reflected in the interval $[0,a]$ [resp. $[0,b]$], see Theorem 7. Now define for each $a, x > 0$,

$$\tau_x^a = \inf\left\{s > 0,\ L_s^a(0) > \frac{4}{\sigma^2}x\right\}.$$

It follows from (5.14) that, as in the discrete case, $\forall\ 0 < a < b$,

$$\{L^b_{\tau_x^b}(t),\ 0 \le t < a, x > 0\} \overset{(d)}{=} \{L^a_{\tau_x^a}(t),\ 0 \le t < a, x > 0\}.$$

Consequently we can define the projective limit, which is a process $\{\mathscr{L}_x(t),\ t \ge 0, x > 0\}$ such that for each $a > 0$,

$$\{\mathscr{L}_x(t),\ 0 \le t < a, x > 0\} \overset{(d)}{=} \{L^a_{\tau_x^a}(t),\ 0 \le t < a, x > 0\}.$$

We have the (see also Theorem 3.1 in Delmas [16])

Theorem 8 (Generalized Ray–Knight theorem).

$$\{\mathscr{L}_x(t), t \ge 0, x > 0\} \overset{(d)}{=} \left\{\frac{4}{\sigma^2}Z_t^x, t \ge 0, x > 0\right\},$$

where Z^x is the Feller branching diffusion process, solution of the SDE

$$Z_t^x = x + (\alpha-\beta)\int_0^t Z_r^x dr + \sigma\int_0^t \sqrt{Z_r^x}\, dB_r,\ t \ge 0.$$

PROOF: Since both sides have stationary independent increments in x, it suffices to show that for any $x > 0$,

$$\{\mathscr{L}_x(t),\ t \geq 0\} \stackrel{(d)}{=} \left\{\frac{4}{\sigma^2} Z_t^x,\ t \geq 0\right\}.$$

Fix an arbitrary $a > 0$. By applying the elementary "occupation times formula" to $H^{a,N}$ (which differs from (5.3) since (5.8) differs from (5.2)), and Lemma 12, we have for any $g \in C(\mathbb{R}_+)$ with support in $[0,a]$,

$$\begin{aligned}\frac{4}{\sigma^2}\int_0^{\tau_x^{a,N}} g(H_r^{a,N})dr &= \int_0^\infty g(t)L^{a,N}_{\tau_x^{a,N}}(t)dt \\ &\stackrel{(d)}{=} \frac{4}{\sigma^2}\int_0^\infty g(t)Z_t^{N,x}dt \end{aligned} \tag{5.15}$$

We deduce clearly from Proposition 7 that

$$\int_0^\infty g(t)Z_t^{N,x}dt \Longrightarrow \int_0^\infty g(t)Z_t^x dt. \tag{5.16}$$

Let us admit for a moment that as $N \to \infty$

$$\int_0^{\tau_x^{a,N}} g(H_r^{a,N})dr \Longrightarrow \int_0^{\tau_x^a} g(H_r^a)dr, \tag{5.17}$$

where $\tau_x^a = \inf\{s > 0,\ L_s^a(0) > x\}$.

From the occupation times formula for the continuous semimartingale $(H_s^a, s \geq 0)$ (see Proposition 34 below), we have that

$$\frac{4}{\sigma^2}\int_0^{\tau_x^a} g(H_r^a)dr = \int_0^\infty g(t)L^a_{\tau_x^a}(t)dt. \tag{5.18}$$

We deduce from (5.15), (5.16), (5.17), and (5.18) that for any $g \in C(\mathbb{R}_+)$ with compact support in $[0,a]$,

$$\frac{4}{\sigma^2}\int_0^\infty g(t)Z_t^x dt \stackrel{(d)}{=} \int_0^\infty g(t)\mathscr{L}_x(t)dt.$$

Since both processes $(Z_t^x, t \geq 0)$ and $(\mathscr{L}_x(t), t \geq 0)$ are *a.s.* continuous, the theorem is proved. □

It remains to prove (5.17), which clearly is a consequence of (recall the definition (5.9) of $\tau_x^{a,N}$)

Proposition 18.

$$(H^{a,N}, \tau_x^{a,N}) \Longrightarrow (H^a, \tau_x^a).$$

PROOF: For the sake of simplifying the notations, we suppress the superscript a. Let us define the function ϕ from $\mathbb{R}_+ \times C_\uparrow([0,+\infty))$ into $\mathbb{R}_+$ by

$$\phi(x,y) = \inf\{s > 0 : y(s) > x\}.$$

For any fixed x, the function $\phi(x,.)$ is continuous in the neighborhood of a function y which is strictly increasing at the time when it first reaches the value x. Define

$$\tau_x'^N := \phi\left(x, \frac{\sigma^2}{4} L^N_\cdot(0)\right).$$

We note that for any $x > 0$, $s \longmapsto L_s(0)$ is a.s. strictly increasing at time τ_x, which is a stopping time. This follows from the strong Markov property, the fact that $H_{\tau_x} = 0$, and $L_\varepsilon(0) > 0$, for all $\varepsilon > 0$. Consequently τ_x is *a.s.* a continuous function of the trajectory $L_\cdot(0)$, and from Corollary 3

$$(H^N, \tau_x'^N) \Longrightarrow (H, \tau_x).$$

It remains to prove that $\tau_x'^N - \tau_x^N \longrightarrow 0$ in probability. For any $y < x$, for N large enough

$$0 \le \tau_x'^N - \tau_x^N \le \tau_x'^N - \tau_y'^N.$$

Clearly $\tau_x'^N - \tau_y'^N \Longrightarrow \tau_x - \tau_y$, hence for any $\varepsilon > 0$,

$$0 \le \limsup_N \mathbb{P}\big(\tau_x'^N - \tau_x^N \ge \varepsilon\big) \le \limsup_N \mathbb{P}\big(\tau_x'^N - \tau_y'^N \ge \varepsilon\big) \le \mathbb{P}\big(\tau_x - \tau_y \ge \varepsilon\big).$$

The result follows, since $\tau_y \uparrow \tau_x$ a.s. as $y \uparrow x$, $y < x$. $\square$