Chapter 3
Convergence to a Continuous State Branching Process

If one wants to understand the evolution of a large population (e.g., in order to study its extinction time), it may be preferable to consider the limit, as the population size tends to infinity, of the rescaled $\mathbb{Z}_+$-valued branching process. The limit, which is $\mathbb{R}_+$-valued, inherits a branching property, that of the so-called continuous state branching process (in short CSBP). The formal statement of the CSBP property, which is very similar to the formulation of the branching property as stated before Proposition 1 in Chapter 2, will be given at the start of Chapter 4. In the present chapter, we will show convergence results of rescaled branching processes towards the solution of a Feller SDE. Note that we consider only convergence towards CSBPs with continuous trajectories, hence towards a Feller diffusion. More general CSBPs will be alluded to below in Remark 2 of Chapter 4. For the convergence of branching processes towards those general CSBPs, we refer to Duquesne, Le Gall [17] and Grimvall [20].

3.1 Convergence of Discrete Time Branching Processes

Let $x > 0$ be a given real number. To each integer N, we associate a Bienaymé–Galton–Watson process $\{X_n^{N,x},\ n \geq 0\}$ starting from $X_0^{N,x} = [Nx]$. We now define the rescaled continuous time process

$$Z_t^{N,x} := N^{-1} X_{[Nt]}^{N,x}.$$

We shall let the p. g. f. of the Bienaymé–Galton–Watson process depend upon N in such a way that

$$\mathbb{E}[\xi^N] = f_N'(1) = 1 + \frac{\gamma_N}{N},$$
$$\mathrm{Var}[\xi^N] = \sigma_N^2,$$

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where as $N \to \infty$,

$$\gamma_N \to \gamma \in \mathbb{R}, \quad \sigma_N \to \sigma.^1 \tag{3.1}$$

We assume in addition the following Lindeberg type condition

$$\mathbb{E}\left[|\xi^N|^2; \xi^N \geq a\sqrt{N}\right] \to 0 \text{ as } N \to \infty, \text{ for all } a > 0, \tag{3.2}$$

where we have used the notation $\mathbb{E}[X;A] = \mathbb{E}[X\mathbf{1}_A]$.

We denote by $\xi_j^{N,i}$ the number of offsprings of the j-th individual from generation i. $(\xi_j^{N,i})_{i\geq 0,\ j\geq 1}$ are i.i.d. with the above law. We have

$$\begin{aligned} Z_t^{N,x} &= Z^{N,x}_{\frac{[Nt]}{N}} \\ &= \frac{[Nx]}{N} + \frac{1}{N}\sum_{i=0}^{[Nt]-1}\sum_{j=1}^{NZ^{N,x}_{i\Delta t}} (\xi_j^{N,i} - 1). \end{aligned}$$

Hence adding and subtracting γ_N/N in each term of the last double sum, we deduce that

$$Z_t^{N,x} = \frac{[Nx]}{N} + \gamma_N \int_0^{\frac{[Nt]}{N}} Z_s^N ds + M_t^N, \tag{3.3}$$

where $M_t^N = M^N_{\frac{[Nt]}{N}}$, with $M^N_{k\Delta t} = \tilde{M}^N_k$, and $\{\tilde{M}^N_k, k \geq 0\}$ is a discrete time martingale given by

$$\tilde{M}_k^N = \frac{1}{N}\sum_{i=0}^{k-1}\sum_{j=1}^{NZ^N_{i\Delta t}} \left[\xi_j^{N,i} - \left(1 + \frac{\gamma_N}{N}\right)\right].$$

It is shown in [20] that under conditions (3.1) and (3.2)[2] (for the definition of the space $D([0,+\infty);\mathbb{R}_+)$, see section A.7 below),

Proposition 4. *$Z^{N,x} \Rightarrow Z^x$ in $D([0,+\infty);\mathbb{R}_+)$ equipped with the Skohorod topology, where $\{Z_t^x,\ t \geq 0\}$ solves the SDE*

$$dZ_t^x = \gamma Z_t^x dt + \sigma\sqrt{Z_t^x} dB_t,\ t \geq 0, \quad Z_0^x = x. \tag{3.4}$$

The proof in [20] is based on Laplace transform calculations. We will now give a proof based on martingale arguments. We deduce easily from (3.3)

Lemma 2. *For any $N \geq 1$, $t > 0$,*

$$\mathbb{E}[Z_t^{N,x}] \leq x\exp(\gamma_N t).$$

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[1] The particular choice $\sigma = 2$ would introduce simplifications in many formulas of Chapters 5, 6, 7, and 8 below.

[2] In fact the result is proved in [20] under the slightly weaker assumption $\mathbb{E}\left[|\xi^N|^2; \xi^N \geq aN\right] \to 0$ as $N \to \infty$, for all $a > 0$.

PROOF: Taking the expectation in (3.3), we obtain the inequality

$$\mathbb{E}[Z_t^{N,x}] \leq x + \gamma_N \int_0^t \mathbb{E}[Z_s^{N,x}]ds,$$

from which the result follows, thanks to Gronwall's Lemma. □

Note that M_t^N is not a continuous time martingale, but $M_{k\Delta t}^N = \tilde{M}_k^N$ is a discrete time martingale. Let

$$[\tilde{M}^N]_k = \sum_{j=0}^{k-1} (\tilde{M}_{j+1}^N - \tilde{M}_j^N)^2.$$

It is easily shown that $\{(\tilde{M}_k^N)^2 - [\tilde{M}^N]_k,\ k \geq 1\}$ is a martingale. Moreover, from the fact that with $\mathscr{G}_i^N = \sigma\{Z_{j\Delta t}^{N,x},\ j \leq i\}$,

$$\mathbb{E}\left[(\tilde{M}_{j+1}^N - \tilde{M}_j^N)^2 | \mathscr{G}_i^N\right] = \sigma_N^2 Z_{j\Delta t}^{N,x} \Delta t,$$

we deduce that

$$(\tilde{M}_k^N)^2 - \langle \tilde{M}^N \rangle_k \ \text{ is a martingale,} \tag{3.5}$$

where

$$\langle \tilde{M}^N \rangle_k = \sigma_N^2 \int_0^{k\Delta t} Z_s^N ds,$$

We can now prove the

Lemma 3. *For any $T > 0$,*

$$\sup_{N \geq 1} \mathbb{E}\left(\sup_{0 \leq t \leq T} Z_t^{N,x} \right) < \infty.$$

PROOF: In view of (3.3) and Lemma 2, it suffices to estimate

$$\begin{aligned}
\left[\mathbb{E}\left(\sup_{0\leq k\leq NT} |\tilde{M}_k^N|\right)\right]^2 &\leq \mathbb{E}\left(\sup_{0\leq k\leq NT} |\tilde{M}_k^N|^2\right) \\
&\leq 4\mathbb{E}\left(|\tilde{M}_{[NT]}^N|^2\right) \\
&= 4\mathbb{E}\langle \tilde{M}^N \rangle_{[NT]} \\
&\leq 4\sigma_N^2 \int_0^T \mathbb{E}(Z_s^{N,x})ds \\
&\leq 4x\sigma_N^2 \frac{\exp[\gamma_N T] - 1}{\gamma_N},
\end{aligned}$$

where we have used Doob's inequality on the second line, (3.5) on the third, and with the understanding that the last ratio equals T if $\gamma_N = 0$. □

Remark 1. It is not very hard to show that

$$\mathbb{E}\left[(Z_t^{N,x})^2\right] \le \left(x^2 + x\sigma_N^2 \frac{e^{\gamma_N t}-1}{\gamma_N}\right) e^{\gamma_N(2+\frac{\gamma_N}{N})t}$$

$$\mathbb{E}\left[\sup_{0\le s\le t}(Z_s^{N,x})^2\right] \le C(t)(x+x^2).$$

However, we do not need those estimates, and we leave their proof to the reader.

We can now proceed to the

PROOF OF PROPOSITION 4 Since M_t^N is not really a martingale, the arguments of Propositions 37 and 38 need to be slightly adapted. We omit the details of those adaptations. It follows from (3.3), (3.5), Lemma 3, Proposition 37 (see also Remark 14) and (3.1) that $\{Z_t^{N,x},\ t\ge 0\}_{N\ge 1}$ is tight in $D([0,\infty);\mathbb{R}_+)$. In order to show that

$$Z_t^x = x + \gamma\int_0^t Z_s^x ds + M_t,$$

where M is a continuous martingale such that

$$\langle M\rangle_t = \sigma^2\int_0^t Z_s^x ds,$$

see Proposition 38, it remains to prove that the last condition of Proposition 37 holds, namely that

Lemma 4. *For any $T>0$, as $N\to\infty$,*

$$\sup_{0\le t\le T}|Z_t^{N,x}-Z_{t^-}^{N,x}| \to 0 \ \textit{in probability}.$$

If we admit for a moment this Lemma, it follows from the martingale representation Theorem 19 that there exists a standard Brownian motion $\{B_t,\, t\ge 0\}$ such that

$$Z_t^x = x + \gamma\int_0^t Z_s^x ds + \sigma\int_0^t \sqrt{Z_s^x}dB_s,\ t\ge 0.$$

It follows from Corollary 1 below that this SDE has a unique solution, hence the limiting law of $\{Z_t^x, t\ge 0\}$ is uniquely characterized, and the whole sequence $\{Z^{N,x}\}$ converges to Z^x as $N\to\infty$. □

PROOF OF LEMMA 4 For any $\varepsilon' < \varepsilon$, provided N is large enough,

$$\mathbb{P}\left(\sup_{0\le t\le T}|Z_t^{N,x}-Z_{t^-}^{N,x}|>\varepsilon\right) \le \mathbb{P}\left(\sup_{i\le NT}|\tilde{M}_i^N-\tilde{M}_{i-1}^N|>\varepsilon'\right)$$

Now define

$$\tilde{M}_k^{N,K} = \frac{1}{N}\sum_{i=0}^{k-1}\sum_{j=1}^{NZ_{i\Delta t}^{N,K}}\left[\xi_j^{N,i} - \left(1+\frac{\gamma_N}{N}\right)\right],$$

where $Z^{N,K}_{i\Delta t} = Z^N_{i\Delta t} \wedge K$. It follows from Lemma 3 that for all $T > 0$,

$$\lim_{K\to\infty} \mathbb{P}(\tilde{M}^{N,K}_k = \tilde{M}^N_k \text{ for all } k \le NT, \text{ all } N \ge 1) = 1.$$

It thus suffices to show that for each fixed $K > 0$, $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{i<NT} |\tilde{M}^{N,K}_{i+1} - \tilde{M}^{N,K}_i| > \varepsilon\right) \to 0,$$

as $N \to \infty$. But

$$|\tilde{M}^{N,K}_{i+1} - \tilde{M}^{N,K}_i| \le \sqrt{\frac{K}{N}}|U^N_i|, \text{ where } U^N_i = \frac{1}{\sqrt{NZ^{N,K}_{i\Delta t}}} \sum_{j=1}^{NZ^{N,K}_{i\Delta t}} \bar{\xi}^{N,i}_j,$$

with $\bar{\xi}^{N,i}_j = \xi^{N,i}_j - (1 + \gamma_N/N)$. We have with $\varepsilon_K = \varepsilon/\sqrt{K}$,

$$\begin{aligned}\mathbb{P}\left(\sup_{i\le NT} |\tilde{M}^{N,K}_i - \tilde{M}^{N,K}_{i-1}| > \varepsilon\right) &\le \mathbb{P}\left(\bigcup_{i<NT} \left\{|U^N_i| > \varepsilon_K\sqrt{N}\right\}\right)\\ &= 1 - \mathbb{P}\left(\bigcap_{i<NT} \left\{|U^N_i| \le \varepsilon_K\sqrt{N}\right\}\right)\end{aligned}$$

It is plain that

$$\begin{aligned}\mathbb{P}\left(|U^N_i| \le \varepsilon_K\sqrt{N} \,\middle|\, \mathscr{G}^N_i\right) &= 1 - \mathbb{P}\left(|U^N_i| > \varepsilon_K\sqrt{N} \,\middle|\, \mathscr{G}^N_i\right)\\ &\ge 1 - \frac{C_{N,K}}{\varepsilon_K^2 N},\end{aligned}$$

where

$$C_{N,K} = \sup_{0<z\le K} \mathbb{E}\left(\frac{1}{[Nz]}\left|\sum_{j=1}^{[Nz]} \bar{\xi}^{N,i}_j\right|^2 ; \frac{1}{\sqrt{[Nz]}}\left|\sum_{j=1}^{[Nz]} \bar{\xi}^{N,i}_j\right| > \varepsilon_K\sqrt{N}\right).$$

Conditioning first upon $\mathscr{G}^N_{[NT]}$, then upon $\mathscr{G}^N_{[NT]-1}$, etc., and using repeatedly the last computations, we deduce that, provided $2\varepsilon_K^2 C_{N,K} \le N$,

$$\begin{aligned}\mathbb{P}\left(\sup_{i<NT} |\tilde{M}^{N,K}_{i+1} - \tilde{M}^{N,K}_i| > \varepsilon\right) &\le 1 - \left(1 - \frac{C_{N,K}}{\varepsilon_K^2 N}\right)^{[NT]}\\ &\le 1 - \exp(-(2\log 2)\varepsilon_K^{-2} C_{N,K} T).\end{aligned}$$

It remains to show that for each $K > 0$, $C_{N,K} \to 0$, as $N \to \infty$. This follows readily from the fact that our assumption (3.2) allows us to deduce from Lindeberg's theorem that if $X_N := \frac{1}{\sqrt{N}}\sum_{j=1}^N \bar{\xi}^{N,i}_j$, $X_N \Rightarrow X$ as $N \to \infty$, where X is a centered normal r.v. with variance 2. But we also have that $\mathbb{E}[X_N^2] \to \mathbb{E}[X^2]$, hence the collection of r.v.'s $\{X_N^2,\ N \geq 1\}$ is uniformly integrable, from which we deduce that $C_{N,K} \to 0$, as $N \to \infty$. □

3.2 The Individuals with an Infinite Line of Descent

Consider again the collection indexed by N of BGW processes $\{X_n^{N,x}\}$ introduced at the beginning of the previous section, but this time with $\gamma_N = \gamma > 0$, $\sigma_N = \sigma$, for all $N \geq 1$. For each $t \geq 0$, let Y_t^N denote the individuals in the population $X_{[Nt]}^{N,x}$ with an infinite line of descent. Let us describe the law of Y_0^N. Each of the $[Nx]$ individuals living at time $t = 0$ has the probability $1 - q_N$ of having an infinite line of descent, if q_N is the probability of extinction for a population with a unique ancestor at the generation 0. It then follows from the branching property that the law of Y_0^N is the binomial law $B([Nx], 1 - q_N)$. It remains to evaluate q_N, the unique solution in the interval $(0,1)$ of the equation $f_N(x) = x$. Note that

$$f_N''(1) = \mathbb{E}[\xi_N(\xi_N - 1)] = \sigma^2 + \frac{\gamma}{N} + \left(\frac{\gamma}{N}\right)^2.$$

We deduce from a Taylor expansion of f near $x = 1$ that

$$1 - q_N = \frac{2\gamma}{\sigma^2 N} + \circ\left(\frac{1}{N}\right).$$

Consequently

Proposition 5. *In the above model, the number Y_0^N of individuals at time 0 with an infinite line of descent converges in law, as $N \to \infty$, towards $Poi(2x\gamma/\sigma^2)$.*

3.3 Convergence of Continuous Time Branching Processes

Consider a continuous time $\mathbb{Z}_+$-valued branching process $X_t^{N,x}$, with initial condition $X_0^{N,x} = [Nx]$ and reproduction measure μ_N such that $\mu_N(\mathbb{N}) = N$,

$$N^{-1}\sum_{k\geq 0} k\mu_N(k) = 1 + \frac{\gamma_N}{N}, \quad \mathrm{Var}(N^{-1}\mu_N) = \sigma_N^2,$$

with $\gamma_N \to \gamma$ and $\sigma_N \to \sigma$, as $N \to \infty$. We define $Z_t^{N,x} = N^{-1}X_t^{N,x}$.

$\{P(t),\ t \geq 0\}$ being a standard Poisson process, let

$$Q_t^N = P\left(N^2 \int_0^t Z_s^{N,x} ds\right).$$

It is fair to decide that Q_t^N is the number of birth events which have happened between time 0 and time t, since $N^2 Z_t^{N,x} = N X_t^{N,x}$ is the rate at which birth events occur. Now

$$Z_t^{N,x} = \frac{[Nx]}{N} + N^{-1} \sum_{n=1}^{Q_t^N} (\xi_n^N - 1),$$

where ξ_n^N denotes the number of offsprings at the n-th birth event. Those constitute an i.i.d. sequence with the common law $N^{-1}\mu_N$, which is globally independent of the Poisson process $P(t)$. We have

$$\begin{aligned}
Z_t^{N,x} &= \frac{[Nx]}{N} + \frac{\gamma_N}{N^2} Q_t^N + N^{-1} \sum_{n=1}^{Q_t^N} (\xi_n^N - \mathbb{E}\xi_n^N) \\
&= \frac{[Nx]}{N} + \gamma_N \int_0^t Z_s^{N,x} ds + \gamma_N \left[N^{-2} Q_t^N - \int_0^t Z_s^{N,x} ds \right] \\
&\quad + N^{-1} \sum_{n=1}^{Q_t^N} (\xi_n^N - \mathbb{E}\xi_n^N) \\
&= \frac{[Nx]}{N} + \gamma_N \int_0^t Z_s^{N,x} ds + \varepsilon_N(t) + M_t^N, \qquad (3.6)
\end{aligned}$$

where $\varepsilon_N(t) \to 0$ and M_t^N are martingales, and their quadratic variations satisfy

$$\begin{aligned}
[M^N]_t &= N^{-2} \sum_{n=1}^{Q_t^N} (\xi_n^N - \mathbb{E}\xi_n^N)^2, \\
\langle M^N \rangle_t &= \sigma_N^2 \int_0^t Z_s^{N,x} ds, \qquad (3.7)
\end{aligned}$$

hence

$$\mathbb{E}\langle M^N \rangle_t = \sigma_N^2 \int_0^t \mathbb{E}[Z_s^{N,x}] ds, \qquad (3.8)$$

while

$$\mathbb{E}\langle \varepsilon_N \rangle_t = N^{-2} \gamma_N^2 \int_0^t \mathbb{E}[Z_s^{N,x}] ds. \qquad (3.9)$$

From (3.6),

$$\mathbb{E}[Z_t^{N,x}] \le x e^{\gamma_N t}.$$

And from this, (3.6), (3.8), and (3.9), we deduce that for all $T > 0$,

$$\sup_{N \ge 1} \mathbb{E} \left(\sup_{0 \le t \le T} Z_t^{N,x} \right) < \infty.$$

It is plain that $\varepsilon_N(t) \to 0$ in probability locally uniformly in t. It follows from these last statements, (3.6) and (3.7), Propositions 37 and 38 that the sequence $\{Z_t^{N,x},\ t \ge 0\}$ is tight in $D([0,+\infty))$, and moreover that any limit of a converging subsequence is a solution of the SDE (3.4). In other words we have the

Proposition 6. *$Z^{N,x} \Rightarrow Z^x$ as $N \to \infty$ for the topology of locally uniform convergence, where Z^x is the unique solution of the following Feller SDE*

$$Z_t^x = x + \gamma \int_0^t Z_r^x dr + \sigma \int_0^t \sqrt{Z_r^x} dB_r,\ t \geq 0.$$

3.4 Convergence of Continuous Time Binary Branching Processes

We now restrict ourselves to continuous time binary branching processes. We refer to subsection 2.2.2, and consider for each $N \geq 1$ a continuous time $\mathbb{Z}_+$-valued Markov birth and death process $X_t^{N,x}$ with birth rate $b_N = \sigma^2 N/2 + \alpha$ and death rate $d_N = \sigma^2 N/2 + \beta$, where $\alpha, \beta \geq 0$, and initial condition $X_0^{N,x} = [Nx]$.

We define $Z_t^{N,x} = N^{-1} X_t^{N,x}$. It is not hard to see that there exist two mutually independent standard (i.e., rate 1) Poisson processes $P_b(t)$ and $P_d(t)$, such that

$$\begin{aligned} Z_t^{N,x} = \frac{[Nx]}{N} &+ N^{-1} P_b\left(\left(\frac{\sigma^2}{2}N + \alpha\right)\int_0^t NZ_s^{N,x} ds\right) \\ &- N^{-1} P_d\left(\left(\frac{\sigma^2}{2}N + \beta\right)\int_0^t NZ_s^{N,x} ds\right). \end{aligned}$$

Define the two martingales $M_b(t) = P_b(t) - t$ and $M_d(t) = P_d(t) - t$. We have

$$\begin{aligned} Z_t^{N,x} &= \frac{[Nx]}{N} + (\alpha - \beta)\int_0^t Z_s^{N,x} ds + M^N(t), \text{ where} \\ M^N(t) &= N^{-1}\left[M_b\left(\left(\frac{\sigma^2}{2}N + \alpha\right)\int_0^t NZ_s^{N,x} ds\right) - M_d\left(\left(\frac{\sigma^2}{2}N + \beta\right)\int_0^t NZ_s^{N,x} ds\right)\right]. \end{aligned}$$

Consequently its quadratic variation is given as

$$\begin{aligned} [M^N]_t &= N^{-2}\left[P_b\left(\left(\frac{\sigma^2}{2}N + \alpha\right)\int_0^t NZ_s^{N,x} ds\right) + P_d\left(\left(\frac{\sigma^2}{2}N + \beta\right)\int_0^t NZ_s^{N,x} ds\right)\right], \\ \langle M^N \rangle_t &= \left(\sigma^2 + \frac{\alpha + \beta}{N}\right)\int_0^t Z_s^{N,x} ds. \end{aligned}$$

It is plain that $\mathbb{E}(Z_t^{N,x}) \leq x\exp((\alpha - \beta)t)$, and moreover that for any $T > 0$, $\sup_{N\geq 1} \mathbb{E}\left(\sup_{0\leq t\leq T} Z_t^{N,x}\right) < \infty$. We deduce from the above arguments

Proposition 7. *$Z^{N,x} \Rightarrow Z^x$ as $N \to \infty$ for the topology of locally uniform convergence, where Z^x is the unique solution of the following Feller SDE*

$$Z_t^x = x + \gamma \int_0^t Z_r^x dr + \sigma \int_0^t \sqrt{Z_r^x} dB_r,\ t \geq 0,$$

where $\gamma = \alpha - \beta$.

3.5 Convergence to an ODE

It can be noted that the conditions for convergence towards a Feller diffusion are rather rigid. In the last case which we considered, we need order (N) intensities for both Poisson processes, with a difference in the intensities which is allowed to be of order 1 only. Consider again the case of a continuous time binary branching process as in the previous section, but this time we assume that the birth rate is constant equal to α, and the death rate is constant equal to β. Assume again that $X_0^{N,x} = [Nx]$, and define as above $Z_t^{N,x} = N^{-1}X_t^{N,x}$. Then

$$Z_t^{N,x} = \frac{[Nx]}{N} + N^{-1}P_b\left(\alpha N\int_0^t Z_s^{N,x}ds\right) - N^{-1}P_d\left(\beta N\int_0^t Z_s^{N,x}ds\right).$$

With again $M_b(t) = P_b(t) - t$ and $M_d(t) = P_d(t) - t$,

$$Z_t^{N,x} = \frac{[Nx]}{N} + (\alpha-\beta)\int_0^t Z_s^{N,x}ds + M^N(t), \quad \text{where}$$
$$M^N(t) = N^{-1}\left[M_b\left(\alpha N\int_0^t Z_s^{N,x}ds\right) - M_d\left(\beta N\int_0^t Z_s^{N,x}ds\right)\right].$$

Now

$$[M^N]_t = N^{-2}\left[P_b\left(\alpha N\int_0^t Z_s^{N,x}ds\right) + P_d\left(\beta N\int_0^t Z_s^{N,x}ds\right)\right],$$
$$\langle M^N\rangle_t = \frac{\alpha+\beta}{N}\int_0^t Z_s^{N,x}ds.$$

We again have that $\mathbb{E}(Z_t^{N,x}) \le x\exp((\alpha-\beta)t)$, and moreover that for any $T > 0$, $\sup_{N\ge 1}\mathbb{E}\left(\sup_{0\le t\le T} Z_t^{N,x}\right) < \infty$. For any $t > 0$, $\mathbb{E}[(M_t^N)^2] \to 0$ as $N \to \infty$. Hence we have the following law of large numbers:

Proposition 8. *As $N \to \infty$, $Z_t^{N,x}$ converges in probability, locally uniformly in t, towards the solution of the ODE*

$$\frac{dZ_t^x}{dt} = (\alpha-\beta)Z_t^x, \quad Z_0^x = x.$$