Variational Analysis of a Quasistatic Contact Problem

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Abstract We start by proving an existence and uniqueness result for a new class of variational inequalities which arise in the study of quasistatic models of contact. The novelty lies in the special structure of these inequalities which involve history-dependent operators. The proof is based on arguments of monotonicity, convexity and fixed point. Then, we consider a mathematical model which describes the frictional contact between an elastic-viscoplastic body and a moving foundation. The mechanical process is assumed to be quasistatic, and the contact is modeled with a multivalued normal compliance condition with unilateral constraint and memory term, associated to a sliding version of Coulomb's law of dry friction. We prove that the model casts in the abstract setting of variational inequalities, with a convenient choice of spaces and operators. Further, we apply our abstract result to prove the unique weak solvability of the contact model.

1 Introduction

Contact phenomena involving deformable bodies abound in industry and everyday life. They lead to nonsmooth and nonlinear mathematical problems. Their analysis, including existence and uniqueness results, was carried out in a large number of works, see for instance [3, 4, 6, 9, 16, 17] and the references therein. The numerical analysis of the problems, including error estimation for discrete schemes and numerical simulations, can be found in [10, 11, 13, 14, 22]. The state of the art in the field, including applications in engineering, could be found in the recent special issue [15].

The study of both the qualitative and numerical analysis of various mathematical models of contact is made by using various mathematical tools, including the theory of variational inequalities. At the heart of this theory is the intrinsic inclusion of free boundaries in an elegant mathematical formulation. Existence and uniqueness

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results in the study of variational inequalities can be found in [1, 2, 12, 16, 20], for instance. References concerning their numerical analysis of include [5, 11, 13].

The large variety of frictional or frictionless models of quasistatic contact led to different classes of time-dependent or evolutionary variational inequalities which, on occasion, have been studied in an abstract framework. Examples could be found in [7, 8, 19, 20]. Nevertheless, it was recently recognized that some models of contact lead to weak formulations expressed in terms of variational inequalities which are more general than those studied in the above-mentioned papers. Therefore, in order to prove the unique solvability of these models, there is a need to extend these results to a more general classes of inequalities.

The first aim of the present paper is to provide such extension. Thus, we provide here an abstract existence and uniqueness result in the study of a new class of history-dependent variational inequalities. Our second aim is to illustrate how this result is useful in the analysis of a new model of contact with viscoplastic materials.

The rest of the paper is structured as follows. In Sect. 2, we introduce some notation and preliminary material. Then, we state and prove our main abstract result, Theorem 2. In Sect. 3 and we describe the frictional contact problem, list the assumption on the data, derive its variational formulation and state its unique weak solvability, Theorem 3. The proof of Theorem 3, based on the abstract result provided by Theorem 2, is presented in Sect. 4.

2 An Abstract Existence and Uniqueness Result

Everywhere in this paper, we use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a normed space $(X, \|\cdot\|_X)$ we use the notation $C(\mathbb{R}_+; X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values on K. The following result, obtained in [18], will be used twice in this paper.

Theorem 1 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator. Assume that for all $n \in \mathbb{N}$ there exist two constants $c_n \geq 0$ and $d_n \in [0, 1)$ such that*

$$\|\Lambda u(t) - \Lambda v(t)\|_X \leq c_n \int_0^t \|u(s) - v(s)\|_X \, ds + d_n \, \|u(t) - v(t)\|_X$$

for all $u, \ v \in C(\mathbb{R}_+; X)$ and all $t \in [0, n]$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

The proof of Theorem 1 was carried out in several steps, based on the fact that the space $C(\mathbb{R}_+; X)$ can be organized as a Fréchet space with a convenient distance function.

We assume in what follows that X is real Hilbert space and Y is a real normed space. Let K be a subset of X, $A : K \subset X \to X$ and $\mathscr{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$. Moreover, let $j : Y \times X \times K \to \mathbb{R}$ and $f : \mathbb{R}_+ \to X$. We consider the following assumptions.

$$K \text{ is a closed, convex, nonempty subset of } X. \tag{1}$$

$$\begin{cases} \text{(a) There exists } L > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \le L\|u_1 - u_2\|_Y \quad \forall\, u_1,\, u_2 \in K. \\ \text{(b) There exists } m > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \ge m\|u_1 - u_2\|_X^2 \quad \forall\, u_1\,, u_2 \in K. \end{cases} \tag{2}$$

$$\begin{cases} \text{(a) For all } y \in Y \text{ and } u \in X,\ j(y, u, \cdot) \text{ is convex and l.s.c on } K. \\ \text{(b) There exists } \alpha > 0 \text{ and } \beta > 0 \text{ such that} \\ \quad j(y_1, u_1, v_2) - j(y_1, u_1, v_1) + j(y_2, u_2, v_1) - j(y_2, u_2, v_2) \\ \qquad \le \alpha\|y_1 - y_2\|_Y\, \|v_1 - v_2\|_X + \beta\, \|u_1 - u_2\|_X\, \|v_1 - v_2\|_X \\ \qquad \forall\, y_1,\ y_2 \in Y,\ \forall\, u_1,\ u_2 \in X,\ \forall\, v_1,\ v_2 \in K. \end{cases} \tag{3}$$

$$\begin{cases} \text{For all } n \in \mathbb{N} \text{ there exists } s_n > 0 \text{ such that} \\ \|\mathscr{S}u_1(t) - \mathscr{S}u_2(t)\|_Y \le s_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X\, ds \\ \forall\, u_1,\ u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{cases} \tag{4}$$

$$f \in C(\mathbb{R}_+; X). \tag{5}$$

Concerning these assumptions we have the following comments. First, assumption (2) show that A is a Lipschitz continuous strongly monotone operator on K. Next, in (3) we use the abbreviation l.s.c. for a lower semicontinuous function. Finally, following the terminology introduced in [19] and used in various papers, condition (4) show that the operator $\mathscr{S}$ is a *history-dependent operator*. Example of operators which satisfies this condition could be find in [19, 20]. Variational inequalities involving history-dependent operators are also called *history-dependent variational inequalities*. In their study we have the following existence and uniqueness result.

Theorem 2 *Assume that (1)–(5) hold. Moreover, assume that*

$$m > \beta, \tag{6}$$

where m and β are the constants in (2) and (3), respectively. Then, there exists a unique function $u \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_X + j(\mathscr{S}u(t), u(t), v) \\ & -j(\mathscr{S}u(t), u(t), u(t)) \ge (f(t), v - u(t))_X \qquad \forall\, v \in K. \end{aligned} \tag{7}$$

Proof The proof of Theorem 2 is based on argument similar to those presented in [19] and, for this reason, we skip the details. The main step in the proof are the followings.

(i) Let $\eta \in C(\mathbb{R}_+; X)$ be fixed and denote by $y_\eta \in C(\mathbb{R}_+; Y)$ the function given by

$$y_\eta(t) = \mathscr{S}\eta(t) \qquad \forall\, t \in \mathbb{R}_+. \tag{8}$$

In the first step we use standard arguments on time-dependent elliptic variational inequalities to prove that there exists a unique function $u_\eta \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$\begin{aligned} u_\eta(t) \in K, \quad & (Au_\eta(t), v - u_\eta(t))_X + j(y_\eta(t), \eta(t), v) \\ & -j(z_\eta(t), \eta(t), u_\eta(t)) \geq (f(t), v - u_\eta(t))_X \qquad \forall\, v \in K. \end{aligned} \tag{9}$$

(ii) Next, in the second step, we consider the operator $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; K) \subset C(\mathbb{R}_+; X)$ defined by equality

$$\Lambda\eta = u_\eta \qquad \forall\, \eta \in C(\mathbb{R}_+; X) \tag{10}$$

and we prove that it has a unique fixed point $\eta^* \in C(\mathbb{R}_+; K)$. Indeed, let $\eta_1,\ \eta_2 \in C(\mathbb{R}_+; X)$, and let y_i, be the functions defined by (8) for $\eta = \eta_i$, i.e. $y_i = y_{\eta_i}$, for $i = 1, 2$. Also, denote by u_i the solution of the variational inequality (9) for $\eta = \eta_i$, i.e. $u_i = u_{\eta_i}$, $i = 1, 2$. Let $n \in \mathbb{N}$ and $t \in [0, n]$. Then, using (9), (2) and (3) is easy to see that

$$m\, \|u_1(t) - u_2(t)\|_X \leq \alpha\, \|y_1(t) - y_2(t)\|_Y + \beta\, \|\eta_1(t) - \eta_2(t)\|_X. \tag{11}$$

Moreover, by the assumptions (4) on the operator $\mathscr{S}$ one has

$$\|y_1(t) - y_2(t)\|_Y = \|\mathscr{S}\eta_1(t) - \mathscr{S}\eta_2(t)\|_Y \leq s_n \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds. \tag{12}$$

Thus, using (10)–(12) yields

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X &= \|u_1(t) - u_2(t)\|_X \\ &\leq \frac{\alpha s_n}{m} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds + \frac{\beta}{m}\, \|\eta_1(t) - \eta_2(t)\|_X \end{aligned}$$

which, together with the smallness assumption (6) and Theorem 1, implies that the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$. Moreover, since Λ has values on $C(\mathbb{R}_+; K)$, we deduce that $\eta^* \in C(\mathbb{R}_+; K)$.

(iii) Let $\eta^* \in C(\mathbb{R}_+; K)$ be the fixed point of the operator Λ. It follows from (8) and (10) that

$$y_{\eta^*}(t) = \mathscr{S}\eta^*(t), \qquad u_{\eta^*}(t) = \eta^*(t). \tag{13}$$

for all $t \in \mathbb{R}_+$. Now, letting $\eta = \eta^*$ in the inequality (9) and using (13) we conclude that $\eta^* \in C(\mathbb{R}_+; K)$ is a solution to the variational inequality (7). This proves the existence part in Theorem 2.

(iv) The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ and can be proved as follows. Denote by $\eta^* \in C(\mathbb{R}_+; K)$ the solution of the variational inequality (7) obtained above, and let $\eta \in C(\mathbb{R}_+; K)$ be a different solution of this inequality, which implies that

$$\begin{aligned}&(A\eta(t), v - \eta(t))_X + j(\mathscr{S}\eta(t), \eta(t), v)\\&\quad - j(\mathscr{S}\eta(t), \eta(t), \eta(t)) \geq (f(t), v - \eta(t))_X \qquad \forall\, v \in K, \quad t \in \mathbb{R}_+.\end{aligned} \tag{14}$$

Letting $y_\eta = \mathscr{S}\eta \in C(\mathbb{R}_+; Y)$, inequality (14) implies that η is solution to the variational inequality (9). On the other hand, by step (i) this inequality has a unique solution u_η and, therefore,

$$\eta = u_\eta. \tag{15}$$

This shows that $\Lambda\eta = \eta$ where Λ is the operator defined by (10). Therefore, by Step (i) it follows that $\eta = \eta^*$, which concludes proof.

3 The Contact Model and Main Result

We turn now to an application of Theorem 2 in Contact Mechanics and, to this end, we start by presenting some notations and preliminaries. Let Ω a regular domain of $\mathbb{R}^d$ $(d = 2, 3)$ with surface Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}\,(\Gamma_1) > 0$ and, in addition, Γ_3 is plane. We use the notation $\mathbf{x} = (x_i)$ for a typical point in Ω and $\boldsymbol{\nu} = (\nu_i)$ for the outward unit normal at Γ. In order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x}$. Let $\mathbb{R}^d$ be d-dimensional real linear space and the let $\mathbb{S}^d$ denote the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned}&\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \qquad \forall\, \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^d,\\&\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,\end{aligned}$$

respectively. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

We use standard notation for the Lebesgue and the Sobolev spaces associated to Ω and Γ. Also, we introduce the spaces

$$V = \{ \mathbf{v} = (v_i) \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1 \},$$
$$Q = \{ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \},$$
$$Q_1 = \{ \boldsymbol{\tau} = (\tau_{ij}) \in Q : \mathrm{Div}\boldsymbol{\tau} \in L^2(\Omega)^d \}.$$

Here and below $\mathrm{Div}\boldsymbol{\tau} = (\tau_{ij,j})$ denotes the divergence of the field $\boldsymbol{\tau}$, where the index that follows a coma indicates a partial derivative with the corresponding component of the spatial variable $\mathbf{x}$, i.e. $\tau_{ij,j} = \partial\tau_{ij}/\partial x_j$. The spaces Q and Q_1 are real Hilbert spaces with the canonical inner products given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx \qquad \forall\,\boldsymbol{\sigma},\ \boldsymbol{\tau} \in Q,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx + \int_\Omega \mathrm{Div}\boldsymbol{\sigma}\cdot\,\mathrm{Div}\boldsymbol{\tau}\,dx \qquad \forall\,\boldsymbol{\sigma},\ \boldsymbol{\tau} \in Q_1.$$

In addition, since meas$\,(\Gamma_1) > 0$, it is well known that V is a real Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,dx \qquad \forall\,\mathbf{u}, \mathbf{v} \in V$$

where $\boldsymbol{\varepsilon}$ is the deformation operator, i.e. $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ij}(\mathbf{u})$, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i})$, $u_{i,j} = \partial u_i/\partial x_j$. The associated norms on the spaces V, Q and Q_1 will be denoted by $\|\cdot\|_V$, $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$, respectively.

For all $\mathbf{v} \in V$ we still write $\mathbf{v}$ for the trace of $\mathbf{v}$ to Γ. We recall that, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \le c_0\,\|\mathbf{v}\|_V \qquad \forall\,\mathbf{v} \in V. \tag{16}$$

For $\mathbf{v} \in V$ we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ, in the sense of traces, given by $v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}$, $\ \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}$. Moreover, for $\boldsymbol{\sigma} \in Q_1$ we denote by $\sigma_\nu \in H^{-\frac{1}{2}}(\Gamma)$ its normal component, in the sense of traces. Let $R : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ be a linear continuous operator. Then, there exists a positive constant $c_R > 0$ which depends on R, Ω and Γ_3 such that

$$\|R\sigma_\nu\|_{L^2(\Gamma_3)} \le c_R\,\|\boldsymbol{\sigma}\|_{Q_1} \qquad \forall\,\boldsymbol{\sigma} \in Q_1. \tag{17}$$

Next, we recall that if $\boldsymbol{\sigma}$ is a regular function, then its normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ and the following Green's formula holds:

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \int_{\Omega} \text{Div}\boldsymbol{\sigma} \cdot \mathbf{v}\, dx = \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v}\, da \qquad \forall\, \mathbf{v} \in V. \tag{18}$$

With these notation, we formulate the following problem.

Problem $\mathscr{P}$. *Find a displacement field* $\mathbf{u} = (u_i) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathscr{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{in } \Omega, \tag{19}$$

$$\text{Div}\boldsymbol{\sigma}(t) + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \tag{20}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \tag{21}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \tag{22}$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu |R\sigma_\nu(t)|\, \mathbf{n}^* \quad \text{on } \Gamma_3, \tag{23}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\begin{cases} u_\nu(t) \le g, \qquad \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ 0 \le \xi(t) \le F\Big(\displaystyle\int_0^t u_\nu^+(s)\, ds\Big), \\ \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ \xi(t) = F\Big(\displaystyle\int_0^t u_\nu^+(s)\, ds\Big) \ \text{ if } \ u_\nu(t) > 0 \end{cases} \quad \text{on } \Gamma_3, \tag{24}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \tag{25}$$

Problem $\mathscr{P}$ represents a mathematical model which describes the quasistatic process of contact between a viscoplastic body and a moving foundation. Here Ω represents the reference configuration of a the body and the dot above denotes the derivative with respect the time variable, i.e. $\dot{f} = \frac{\partial f}{\partial t}$. Equation (19) represents the viscoplastic constitutive law. Details and various mechanical interpretation concerning such kind of laws can be found in [9, 20], for instance. Equation (20) represents the equation of equilibrium in which $\mathbf{f}_0$ denotes the density of body forces, assumed to be time-independent. We use this equation since the process is quasistatic and, therefore, the inertial term in the equation of motion is neglected. Conditions (21) and (22) are the displacement and the traction boundary condition, respectively. They describe the fact that the body is fixed on Γ_1 and prescribed traction of density $\mathbf{f}_2$ act on Γ_2, during the contact process.

Conditions (23) and (24) represent a sliding version of Coulomb's law of dry friction and a normal compliance contact condition with unilateral constraint and memory term, respectively. Their are obtained from arguments presented in our recent paper [21] and, for this reason, we do not describe them with details. We just mention that μ denotes the coefficient of friction, $\mathbf{n}^*$ denotes a given unitary vector in the plane on Γ_3 and $v^* < 0$ is given. In addition, p and F are given function which describe the deformability and the memory effects of the foundation, $g > 0$ is a given depth and r^+ represent the positive part of r, i.e. $r^+ = \max\{r, 0\}$. Finally, conditions (25) represent the initial conditions for the displacement and the stress field, respectively.

In the study of Problem $\mathscr{P}$ we assume that the elasticity operator $\mathscr{E}$ and the nonlinear constitutive function $\mathscr{G}$ satisfy the following conditions.

$$\left\{\begin{array}{l}\text{(a) } \mathscr{E} = (\mathscr{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } \mathscr{E}_{ijkl} = \mathscr{E}_{klij} = \mathscr{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i, j, k, l \le d.\\ \text{(c) There exists } m_{\mathscr{E}} > 0 \text{ such that}\\ \quad \mathscr{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathscr{E}} \|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega.\end{array}\right. \tag{26}$$

$$\left\{\begin{array}{l}\text{(a) } \mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) There exists } L_{\mathscr{G}} > 0 \text{ such that}\\ \quad \|\mathscr{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathscr{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \le L_{\mathscr{G}} (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|)\\ \qquad \text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega.\\ \text{(c) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \text{ is measurable on } \Omega,\\ \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d.\\ \text{(d) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q.\end{array}\right. \tag{27}$$

The densities of body forces and surface traction are such that

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{28}$$

The normal compliance function p and the surface yield function F satisfy

$$\left\{\begin{array}{l}\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\ \text{(b) There exists } L_p > 0 \text{ such that}\\ \quad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \le L_p |r_1 - r_2| \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ \text{(c) } (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \ge 0 \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ \text{(d) The mapping } \mathbf{x} \mapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}.\\ \text{(e) } p(\mathbf{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \mathbf{x} \in \Gamma_3.\end{array}\right. \tag{29}$$

$$\left\{\begin{array}{l}\text{(a) } F : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\ \text{(b) There exists } L_F > 0 \text{ such that}\\ \quad |F(\mathbf{x}, r_1) - F(\mathbf{x}, r_2)| \le L_F |r_1 - r_2| \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ \text{(c) The mapping } \mathbf{x} \mapsto F(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}.\\ \text{(d) } F(\mathbf{x}, 0) = 0 \text{ a.e. } \mathbf{x} \in \Gamma_3.\end{array}\right. \tag{30}$$

Also, the the coefficient of friction verifies

$$\mu \in L^\infty(\Gamma_3), \ \ \mu(t, \mathbf{x}) \geq 0 \ \ \text{a.e.}\ \mathbf{x} \in \Gamma_3, \tag{31}$$

and the initial data are such that

$$\mathbf{u}_0 \in V, \qquad \boldsymbol{\sigma}_0 \in Q. \tag{32}$$

In what follows we consider the set of admissible displacements fields and the set of admissible stress fields defined by

$$U = \{\, \mathbf{v} \in V \ : \ v_\nu \leq g \ \text{ on } \Gamma_3 \,\}, \tag{33}$$

$$\Sigma = \{\, \boldsymbol{\tau} \in Q \ : \ \ \mathrm{Div}\boldsymbol{\tau} + \mathbf{f}_0 = \mathbf{0} \ \text{ in } \Omega \,\}. \tag{34}$$

respectively. Note that assumptions $g > 0$ and $\mathbf{f}_0 \in L^2(\Omega)^d$ imply that U and Σ are closed, convex nonempty subsets of the spaces V and Q, respectively.

Assume in what follows that $(\mathbf{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (19)–(24) and let $\mathbf{v} \in U$ and $t > 0$ be given. First, we use the equilibrium equation (20) and the contact condition (23) to see that

$$\mathbf{u}(t) \in U, \qquad \boldsymbol{\sigma}(t) \in \Sigma. \tag{35}$$

Then, we use Green's formula (18), the equilibrium equation (20) and the friction law (23) to obtain that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\, dx \\
&\quad = \int_\Omega \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}(t))\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}(t))\, da \\
&\qquad + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\, da - \int_{\Gamma_3} \mu |R\sigma_\nu(t)|\, \mathbf{n}^* \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau(t))\, da.
\end{aligned} \tag{36}$$

We now use the contact conditions (24) and the definition (33) of the set U to see that

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \geq -(p(u_\nu(t)) + \xi(t))(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{37}$$

Next, we use (24), again, and the hypothesis (30)(a) on function F to deduce that

$$F\left(\int_0^t u_\nu^+(s) ds\right)\, (v_\nu^+ - u_\nu^+(t)) \geq \xi(t)(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{38}$$

We now add the inequalities (37) and (38) and integrate the result on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\, da \geq -\int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\, da \tag{39}$$

$$-\int_{\Gamma_3} F\left(\int_0^t u_\nu^+(s)ds\right)(v_\nu^+ - u_\nu^+(t))\, da.$$

Finally, we combine (36) and (39) to deduce that

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\, dx + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\, da \tag{40}$$

$$+\int_{\Gamma_3} F\left(\int_0^t u_\nu^+(s)ds\right)(v_\nu^+ - u_\nu^+(t))\, da + \int_{\Gamma_3} \mu|R\boldsymbol{\sigma}_\nu(t)|\mathbf{n}^*\cdot(\mathbf{v}_\tau - \mathbf{u}_\tau(t))\, da$$

$$\geq \int_\Omega \mathbf{f}_0\cdot(\mathbf{v} - \mathbf{u}(t))\, dx + \int_{\Gamma_2} \mathbf{f}_2(t).(\mathbf{v} - \mathbf{u}(t))\, da.$$

We now integrate the the constitutive law (19) with the initial conditions (25), then we gather the resulting equation with the regularity (35) and inequality (40) to obtain the following variational formulation of Problem $\mathscr{P}$.

Problem $\mathscr{P}_V$. *Find a displacement field* $\mathbf{u} : \mathbb{R}_+ \to U$ *and a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to \Sigma$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))\, ds + \boldsymbol{\sigma}_0 - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_0), \tag{41}$$

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\, dx + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\, da \tag{42}$$

$$+\int_{\Gamma_3} F\left(\int_0^t u_\nu^+(s)ds\right)(v_\nu^+ - u_\nu^+(t))\, da + \int_{\Gamma_3} \mu|R\boldsymbol{\sigma}_\nu(t)|\mathbf{n}^*\cdot(\mathbf{v}_\tau - \mathbf{u}_\tau(t))\, da$$

$$\geq \int_\Omega \mathbf{f}_0\cdot(\mathbf{v} - \mathbf{u}(t))\, dx + \int_{\Gamma_2} \mathbf{f}_2(t).(\mathbf{v} - \mathbf{u}(t))\, da$$

for all $t \in \mathbb{R}_+$.

Our main existence and uniqueness result in the study of the Problem $\mathscr{P}$, that we state here and prove in the next section is the following.

Theorem 3 *Assume that (26)–(32) hold. Then there exists a positive constant* μ_0 *which depends only on* Ω, Γ_1, Γ_3, R *and* $\mathscr{E}$ *such that Problem* $\mathscr{P}_V$ *has a unique solution, if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0. \tag{43}$$

Moreover, the solution satisfies $\mathbf{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; \Sigma)$.

Note that Theorem 3 provides the unique weak solvability of Problem $\mathscr{P}$, under the smallness assumption (43) on the coefficient of friction.

4 Proof of Theorem 3

The proof of the theorem will be carried out in several steps. To present it we assume in what follows that (26)–(32) hold. We start with the following existence and uniqueness result.

Lemma 4 *For each function* $\mathbf{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\Theta\mathbf{u} \in C(\mathbb{R}_+; Q)$ *such that*

$$\Theta\mathbf{u}(t) = \int_0^t \mathscr{G}(\Theta\mathbf{u}(s) + \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{44}$$

Moreover, the operator $\Theta : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ *is history-dependent, i.e. for all* $n \in \mathbb{N}$ *there exists* $\theta_n > 0$ *such that*

$$\|\Theta\mathbf{u}_1(t) - \Theta\mathbf{u}_2(t)\|_Q \le \theta_n \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V\, ds \tag{45}$$
$$\forall\, \mathbf{u}_1,\ \mathbf{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Proof Let $\mathbf{u} \in C(\mathbb{R}_+; V)$ and consider the operator $\Lambda : C(\mathbb{R}_+; Q) \to C(\mathbb{R}_+; Q)$ defined by

$$\Lambda\boldsymbol{\tau}(t) = \int_0^t \mathscr{G}(\boldsymbol{\tau}(s) + \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)))ds + \boldsymbol{\sigma}_0 - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_0) \tag{46}$$
$$\forall\, \boldsymbol{\tau} \in C(\mathbb{R}_+; Q),\ t \in \mathbb{R}_+.$$

The operator Λ depends on $\mathbf{u}$ but, for the sake of simplicity, we do not indicate it explicitly. Let $\boldsymbol{\tau}_1,\ \boldsymbol{\tau}_2 \in C(\mathbb{R}_+; Q)$ and let $t \in \mathbb{R}_+$. Then, using (46) and (27) we have

$$\|\Lambda\boldsymbol{\tau}_1(t)-\Lambda\boldsymbol{\tau}_2(t)\|_Q \le L_{\mathscr{G}}\int_0^t \|\boldsymbol{\tau}_1(s)-\boldsymbol{\tau}_2(s)\|_Q\,ds.$$

This inequality combined with Theorem 1 shows that the operator Λ has a unique fixed point in $C(\mathbb{R}_+;Q)$. We denote by $\Theta\mathbf{u}$ the fixed point of Λ and we combine (46) with the equality $\Lambda(\Theta\mathbf{u})=\Theta\mathbf{u}$ to see that (44) holds.

To proceed, let $n\in\mathbb{N}$, $t\in[0,n]$ and let $\mathbf{u}_1,\mathbf{u}_2\in C(\mathbb{R}_+;V)$. Then, using (44) and taking into account (27), (26) we write

$$\begin{aligned}&\|\Theta\mathbf{u}_1(t)-\Theta\mathbf{u}_2(t)\|_Q\\ &\quad=L_0\Big(\int_0^t\|\Theta\mathbf{u}_1(s)-\Theta\mathbf{u}_2(s)\|_Q\,ds+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\,ds\Big),\end{aligned}$$

where L_0 is a positive constant which depends on $\mathscr{G}$ and $\mathscr{E}$. Using now a Gronwall argument we deduce that

$$\|\Theta\mathbf{u}_1(t)-\Theta\mathbf{u}_2(t)\|_Q\le L_0\,e^{L_0 n}\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\,ds.$$

This inequality shows that (45) holds with $\theta_n=L_0\,e^{L_0 n}$.

Next, we consider the operators $A:V\to V$ and $\mathscr{R}:V\times Q\to L^2(\Gamma_3)$ defined by

$$(A\mathbf{u},\mathbf{v})_V=(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q+\int_{\Gamma_3}p(u_\nu)v_\nu\,da\qquad\forall\,\mathbf{u},\mathbf{v}\in V, \tag{47}$$

$$\mathscr{R}(\mathbf{u},\mathbf{z})=|R\big(P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u})+\mathbf{z})\big)_\nu|\qquad\forall\,\mathbf{u}\in V,\ \ \mathbf{z}\in Q, \tag{48}$$

where $P_\Sigma:Q\to\Sigma$ represents the projection operator. Note that, since $\Sigma\subset Q_1$, the operator $\mathscr{R}$ is well defined. Denote $Y=Q\times L^2(\Gamma_3)\times Q$ where, here and below, $X_1\times\ldots\times X_m$ represents the product of the Hilbert spaces $X_1,\ldots,X_m$ $(m=2,3)$, endowed with its canonical inner product. Besides the operator $\Theta: C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;Q)$ defined in Lemma 4, let $\Phi:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;L^2(\Gamma_3))$ and $\mathscr{S}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;Y)$ be the operators given by

$$(\Phi\mathbf{v})(t)=F\left(\int_0^t v_\nu^+(s)ds\right), \tag{49}$$

$$\mathscr{S}\mathbf{v}(t)=(\Theta\mathbf{v}(t),\Phi\mathbf{v}(t),\Theta\mathbf{v}(t)) \tag{50}$$

for all $\mathbf{v} \in C(\mathbb{R}_+; V)$, $t \in \mathbb{R}_+$. Finally, let $j : Y \times V \times V \to \mathbb{R}$ and $\mathbf{f} : \mathbb{R}_+ \to V$ denote the functions defined by

$$\begin{aligned} j(\mathbf{w}, \mathbf{u}, \mathbf{v}) &= (\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + (y, v_\nu^+)_{L^2(\Gamma_3)} + (\mu \mathscr{R}(\mathbf{u}, \mathbf{z})\mathbf{n}^*, \mathbf{v}_\tau)_{L^2(\Gamma_3)^d} \\ &\qquad \forall\, \mathbf{w} = (\mathbf{x}, y, \mathbf{z}) \in Y, \ \ \mathbf{u}, \mathbf{v} \in V, \end{aligned} \tag{51}$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0 \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da \qquad \forall\, \mathbf{v} \in V,\ t \in \mathbb{R}_+. \tag{52}$$

We have the following equivalence result.

Lemma 5 *Assume that* $\mathbf{u} \in C(\mathbb{R}_+; U)$ *and* $\boldsymbol{\sigma} \in C(\mathbb{R}_+; \Sigma)$. *Then, the couple* $(\mathbf{u}, \boldsymbol{\sigma})$ *is a solution of Problem* $\mathscr{P}_V$ *if and only if*

$$\boldsymbol{\sigma}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \Theta\mathbf{u}(t), \tag{53}$$

$$\begin{aligned} &(A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V + j(\mathscr{S}\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) \\ &\quad - j(\mathscr{S}\mathbf{u}(t), \mathbf{u}(t), \mathbf{u}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \qquad \forall\, \mathbf{v} \in U \end{aligned} \tag{54}$$

for all $t \in \mathbb{R}_+$.

Proof Let $(\mathbf{u}, \boldsymbol{\sigma}) \in C(\mathbb{R}_+; U \times \Sigma)$, be a solution of Problem $\mathscr{P}_V$ and let $t \in \mathbb{R}_+$. By (41) we have

$$\boldsymbol{\sigma}(t) - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) = \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)) + \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_0),$$

and, using the definition (44) of the operator Θ, we obtain (53). Moreover, we substitute (41) in (42), then we use (49) and equality $P_\Sigma \boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(t)$. As a result, we deduce that

$$\begin{aligned} &\int_\Omega \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\, dx + \int_\Omega \Theta\mathbf{u}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\, dx \\ &\quad + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\, da + \int_{\Gamma_3} \Phi\mathbf{u}(t)\, (v_\nu^+ - u_\nu^+(t))\, da \\ &\quad + \int_{\Gamma_3} \mu \big| R\big(P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \Theta\mathbf{u}(t))\big)_\nu \big| \, \mathbf{n}^* \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau(t))\, da \\ &\quad \geq \int_\Omega \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}(t))\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}(t))\, da \qquad \forall\, \mathbf{v} \in U. \end{aligned} \tag{55}$$

Using now the definitions (47), (48) and (52) yields

$$\begin{aligned}&(A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V + (\Theta\mathbf{u}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \\ &\quad + (\Phi\mathbf{u}(t))\,(v_\nu^+ - u_\nu^+(t)))_{L^2(\Gamma_3)} + (\mu\mathscr{R}(\mathbf{u}(t), \Theta(\mathbf{u}(t))\mathbf{n}^*, \mathbf{v}_\tau - \mathbf{u}_\tau(t))_{L^2(\Gamma_3)^d} \\ &\quad\quad \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \qquad \forall\, \mathbf{v} \in U.\end{aligned}$$

This inequality combined with the definitions (50) and (51) shows that the variational inequality (54) holds.

Conversely, assume that $(\mathbf{u}, \boldsymbol{\sigma}) \in C(\mathbb{R}_+; U \times \Sigma)$ is a couple of functions which satisfies (53) and (54) and let $t \in \mathbb{R}_+$. Then, using the definitions (47), (48), (50)–(52) it follows that (55) holds. Moreover, recall that the regularity $\boldsymbol{\sigma} \in C(\mathbb{R}_+; \Sigma)$ implies that $P_\Sigma\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(t)$ and, in addition, (53) yields $\boldsymbol{\sigma}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \Theta\mathbf{u}(t)$. Substituting these equalities in (55) and using (49) we see that (42) holds. Finally, to conclude, we note that (41) is a direct consequence of (53) and the definition of the operator Θ in Lemma 5.

The interest in Lemma 5 arrises in the fact that it decouples the unknowns $\mathbf{u}$ and $\boldsymbol{\sigma}$ in the system (41)–(42). The next step is to provide the unique solvability of the variational inequality (54) in which the unknown is the displacement field. To this end we need the following intermediate result on the operator $\mathscr{R}$.

Lemma 6 *There exists $L_\mathscr{R} > 0$ which depends only on Ω, Γ_3 and R, such that*

$$\begin{aligned}&\|\mathscr{R}(\mathbf{u}_1, \mathbf{z}_1) - \mathscr{R}(\mathbf{u}_2, \mathbf{z}_2)\|_{L^2(\Gamma_3)} \leq L_\mathscr{R}\left(\|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|\mathbf{z}_1 - \mathbf{z}_2\|_Q\right) \\ &\qquad\qquad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in V,\ \mathbf{z}_1, \mathbf{z}_2 \in Q.\end{aligned} \tag{56}$$

Proof Let $\mathbf{u}_1, \mathbf{u}_2 \in V$, $\mathbf{z}_1, \mathbf{z}_2 \in Q$. Then, by the definition (48) of the operator $\mathscr{R}$ combined with inequality (17) we have

$$\begin{aligned}&\|\mathscr{R}(\mathbf{u}_1, \mathbf{z}_1) - \mathscr{R}(\mathbf{u}_2, \mathbf{z}_2)\|_{L^2(\Gamma_3)} \\ &\quad \leq c_R\|P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) + \mathbf{z}_1) - P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_2) + \mathbf{z}_2)\|_{Q_1}.\end{aligned} \tag{57}$$

On the other hand, the definition of the set Σ and the nonexpansivity of the operator P_Σ yields

$$\begin{aligned}&\|P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) + \mathbf{z}_1) - P_\Sigma(\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_2) + \mathbf{z}_2)\|_{Q_1} \\ &\quad \leq \|\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_2) + \mathbf{z}_1 - \mathbf{z}_2\|_Q\end{aligned} \tag{58}$$

We now combine inequalities (57) and (58) to see that

$$\|\mathscr{R}(\mathbf{u}_1, \mathbf{z}_1) - \mathscr{R}(\mathbf{u}_2, \mathbf{z}_2)\|_{L^2(\Gamma_3)} \leq c_R\left(\|\mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathscr{E}\boldsymbol{\varepsilon}(\mathbf{u}_2) + \mathbf{z}_1 - \mathbf{z}_2\|_Q\right) \tag{59}$$

Lemma 6 is now a consequence of inequality (59) and assumption (26).

We proceed with the following existence and uniqueness result.

Lemma 7 *The variational inequality (54) has a unique solution with regularity* $\mathbf{u} \in C(\mathbb{R}_+, U)$.

Proof It is straightforward to see that inequality (54) represents a variational inequality of the form (7) in which $X = V$, $K = U$ and $Y = Q \times L^2(\Gamma_3) \times Q$. Therefore, in order to prove its unique solvability, we check in what follows the assumptions of Theorem 2.

First, we note that assumption (1) is obviously satisfied. Next, we use the definition (47), assumptions (26), (29)(b) and inequality (16) to obtain that

$$\|A\mathbf{u} - A\mathbf{v}\|_V \le (L_{\mathscr{E}} + c_0^2 L_p)\|\mathbf{u} - \mathbf{v}\|_V \qquad \forall\, \mathbf{u}, \mathbf{v} \in V, \tag{60}$$

where $L_{\mathscr{E}}$ is a positive constant which depends on the elasticity operator $\mathscr{E}$. On the other hand, from (26)(c) and (29)(c) the we deduce that

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_V \ge m_{\mathscr{E}}\|\mathbf{u} - \mathbf{v}\|_V^2. \tag{61}$$

We conclude from above that the operator A satisfies condition (2) with $L = L_{\mathscr{E}} + c_0^2 L_p$ and $m = m_{\mathscr{E}}$.

Let $\mathbf{w} = (\mathbf{x}, y, \mathbf{z}) \in Y$ and $\mathbf{u} \in V$ be fixed. Then, using the properties of the traces it is easy to see that the function $\mathbf{v} \mapsto j(\mathbf{w}, \mathbf{u}, \mathbf{v})$ is convex and continuous and, therefore, it satisfies condition (3)(a). We now consider the elements $\mathbf{w}_1 = (\mathbf{x}_1, y_1, \mathbf{z}_1)$, $\mathbf{w}_2 = (\mathbf{x}_2, y_2, \mathbf{z}_2) \in Y$, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. Then, using inequality (56), assumption (31) and inequality (16) we find that

$$\begin{aligned} & j(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_2) \\ &\quad \le \|\mathbf{x}_1 - \mathbf{x}_2\|_Q \|\mathbf{v}_1 - \mathbf{v}_2\|_V + c_0\|y_1 - y_2\|_{L^2(\Gamma_3)}\|\mathbf{v}_1 - \mathbf{v}_2\|_V \\ &\qquad + c_0 L_{\mathscr{R}}\|\mu\|_{L^\infty(\Gamma_3)} \left(\|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|\mathbf{z}_1 - \mathbf{z}_2\|_Q\right) \|\mathbf{v}_1 - \mathbf{v}_2\|_V \\ &\qquad\quad \le \alpha\, \|\mathbf{w}_1 - \mathbf{w}_2\|_Z \|\mathbf{v}_1 - \mathbf{v}_2\|_V + \beta\, \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V \end{aligned}$$

where $\alpha = 2\max\{1,\ c_0,\ c_0 L_{\mathscr{R}}\|\mu\|_{L^\infty(\Gamma_3)}\}$ and $\beta = c_0 L_{\mathscr{R}}\|\mu\|_{L^\infty(\Gamma_3)}$. It follows from here that j satisfies condition (3)(b). Let

$$\mu_0 = \frac{m_{\mathscr{E}}}{c_0 L_{\mathscr{R}}}, \tag{62}$$

which, clearly, depends only on Ω, Γ_1, Γ_3, R and $\mathscr{E}$. Then, it is easy to see that if the smallness assumption $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ is satisfied we have $\beta < m$ and, therefore, condition (6) holds.

Next, let $\mathbf{u}, \mathbf{v} \in C(\mathbb{R}_+; V)$, $n \in \mathbb{N}$ and let $t \in [0, n]$. Then, using (49) and taking into account (30)(b) and (16) we obtain that

$$\|\Phi\mathbf{u}(t) - \Phi\mathbf{v}(t)\|_{L^2(\Gamma_3)} = \left\| F\left(\int_0^t u_\nu^+(s)ds\right) - F\left(\int_0^t v_\nu^+(s)ds\right)\right\|_{L^2(\Gamma_3)}$$

$$\leq L_F \left\|\int_0^t (u_\nu^+(s) - v_\nu^+(s))ds\right\|_{L^2(\Gamma_3)} \leq c_0 L_F \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V\, ds.$$

Therefore, using this the definition (50) of the operator $\mathscr{S}$ and (45) we have

$$\|\mathscr{S}\mathbf{u}(t) - \mathscr{S}\mathbf{v}(t)\|_{Q\times L^2(\Gamma_3)\times Q} \leq (2\,\theta_n + c_0 L_F\,) \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V\, ds.$$

It follows from here that the operator $\mathscr{S}$ satisfies condition (4). Finally, we note that assumption (28) on the body forces and traction and definition (52) imply that $\mathbf{f} \in C(\mathbb{R}_+; V)$.

We conclude from above that all the assumptions of Theorem 2 are satisfied. Therefore, we deduce that inequality (54) has a unique solution $\mathbf{u} \in C(\mathbb{R}_+; U)$ which concludes the proof.

We now have all the ingredients to provide the proof of Theorem 3.

Proof (Proof of Theorem 3) Let $\mathbf{u} \in C(\mathbb{R}_+; U)$ be the unique solution of inequality (54) obtained in Lemma 7 and let $\boldsymbol{\sigma}$ the function defined by (53). Then, using assumption (26) it follows that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Let $t \in \mathbb{R}_+$ be given. Arguments similar to those used in the proof of Lemma 7 show that

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))\,dx + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\,da$$

$$+ \int_{\Gamma_3} F\left(\int_0^t u_\nu^+(s)ds\right)(v_\nu^+ - u_\nu^+(t))\,da$$

$$+ \int_{\Gamma_3} \mu|R(P_\Sigma(\boldsymbol{\sigma}_\nu(t))|\,\mathbf{n}^*\cdot(\mathbf{v}_\tau - \mathbf{u}_\tau(t))\,da$$

$$\geq \int_\Omega \mathbf{f}_0\cdot(\mathbf{v} - \mathbf{u}(t))\,dx + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot(\mathbf{v} - \mathbf{u}(t))\,da \qquad \forall\,\mathbf{v} \in U.$$

Let $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$. We test in this inequality with $\mathbf{v} = \mathbf{u}(t) \pm \boldsymbol{\varphi}$ to deduce that

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \mathbf{f}_0 \cdot \boldsymbol{\varphi}\, dx$$

which implies that $\text{Div}\boldsymbol{\sigma}(t) + \mathbf{f}_0 = \mathbf{0}$ in Ω. It follows from here that $\boldsymbol{\sigma}(t) \in \Sigma$ and, moreover, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; \Sigma)$.

We conclude from above that $(\mathbf{u}, \boldsymbol{\sigma})$ represents a couple of functions which satisfies (53)–(54) and, in addition, it has the regularity $(\mathbf{u}, \boldsymbol{\sigma}) \in C(\mathbb{R}_+; U \times \Sigma)$. The existence part in Theorem 3 is now a direct consequence of Lemma 5. The uniqueness part follows from the uniqueness of the solution of the variational inequality (54), guaranteed by Lemma 7.

References

1. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems. John Wiley, Chichester (1984)
2. Brezis, H.: Equations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18**, 115–175 (1968)
3. Capatina, A.: Variational Inequalities and Frictional Contact Problems, Advances in Mechanics and Mathematics, vol. 31. Springer, New York (2014)
4. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer-Verlag, Berlin (1976)
5. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer, New York (1984)
6. Eck, C., Jarušek, J., Krbeč, M.: Unilateral Contact Problems: Variational Methods and Existence Theorems, Pure and Applied Mathematics, vol. 270. Chapman/CRC Press, New York (2005)
7. Han, W., Sofonea, M.: Time-dependent variational inequalities for viscoelastic contact problems. J. Comput. Appl. Math. **136**, 369–387 (2001)
8. Han, W., Sofonea, M.: Evolutionary variational inequalities arising in viscoelastic contact problems. SIAM J. Numer. Anal. **38**, 556–579 (2000)
9. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, Studies in Advanced Mathematics, vol. 30. American Mathematical Society, Providence, RI-International Press, Somerville, MA (2002)
10. Haslinger, J., Hlavácek, I., Nečas, J.: Numerical methods for unilateral problems in solid mechanics. In: Lions, J.-L., Ciarlet, P. (eds.) Handbook of Numerical Analysis, vol. IV, pp. 313–485. North-Holland, Amsterdam (1996)
11. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Springer, New York (1988)
12. Kikuchi, N., Oden, J.T.: Theory of variational inequalities with applications to problems of flow through porous media. Int. J. Eng. Sci. **18**, 1173–1284 (1980)
13. Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods. SIAM, Philadelphia (1988)
14. Laursen, T.: Computational Contact and Impact Mechanics. Springer, Berlin (2002)
15. Migórski, S., Shillor, M., Sofonea, M. (Guest Editors): Special section on Contact Mechanics, Nonlinear Analysis: Real World Applications **22**, 435–680 (2015)
16. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Birkhäuser, Boston (1985)
17. Shillor, M., Sofonea, M., Telega, J.: Models and Variational Analysis of Quasistatic Contact. Lecture Notes in Physics, vol. 655. Springer, Heidelberg (2004)

18. Sofonea, M., Avramescu, C., Matei, A.: A fixed point result with applications in the study of viscoplastic frictionless contact problems. Commun. Pure Appl. Anal. **7**, 645–658 (2008)
19. Sofonea, M., Matei, A.: History-dependent quasivariational inequalities arising in contact mechanics. Eur. J. Appl. Math. **22**, 471–491 (2011)
20. Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Note Series, vol. 398. Cambridge University Press, Cambridge (2012)
21. Sofonea, M., Souleiman, Y.: A viscoelastic sliding contact problem with normal compliance, unilateral constraint and memory term. Mediterr. J. Math. (2015). doi:10.1007/s00009-015-0661-9 (to appear)
22. Wriggers, P.: Computational Contact Mechanics. Wiley, Chichester (2002)