Chapter 7
What Is Normal About Normal Modes?

Thomas J.S. Abrahamsson and Randall J. Allemang

Abstract A normal mode of a vibrating system is a mode that is orthogonal to all other normal modes of that system. The orthogonality is in a weighted sense. For an undamped discretized linear mechanical system, the orthogonality is defined with respect to stiffness and mass such that normal modes are mutually stiffness and mass orthogonal. Another commonly used definition of an oscillating normal mode is that it is a pattern of motion in which all parts of the system vibrate harmonically with the same frequency and therefore with fixed relative phase relations between parts. The normality of a mode is thus not in a statistical sense. For lightly damped linear systems, a normal observation, i.e. one very common observation in the statistical sense, is that the phase relation between the motion of different parts of the system deviates very little from zero or pi. However, this normally occurring behavior should not lead us to think that that always has to be the case. Here it is shown by example that the normal modes of an undamped system may have arbitrary phase relations. One such mode of vibration may then possess the property of moving nodal lines, which is often attributed to non-proportionally damped or non-selfadjoint systems. The proper normalization of such modes is discussed and their relation to the well-known modal mass and MAC concepts and also to state-space based normalizations that are usually being used for complex-valued eigenmodes.

Keywords Normal mode • Vibrational mode • Undamped eigenmode • Repeated eigenvalues • Eigenmode orthogonality

7.1 Introduction

Already in the nineteenth century inspired by the work of Jacques Charles François Sturm and Joseph Liouville, the use of eigenfunctions has played a central role for the solution of partial and ordinary differential equations. The remarkable property of the eigenfunctions to decouple the equations to more easily solvable sets of equations was the key that they explored. The eigenfunctions, also known as eigenmodes, were shown by Sturm and Liouville to form a complete set for the solution. The decoupled solutions for each individual eigenmode can be superimposed to form the complete solution of the (Sturm-Liouville) problem. The orthogonality property of the eigenmodes is what makes them agents for decoupling. The non-uniqueness of the eigenmodes is well known and it is common knowledge that an eigenmode scaled by any number (except for zero) is again an eigenmode. This has led to the fixing of eigenmodes with the concept of eigenmode normalization. In structural dynamics it is common to normalize the eigenfunctions such that the mass distribution weighted by the square of the eigenfunction produces the unit mass. Such normalized eigenmodes are called (mass) orthonormalized modes. That fixes the size of the mode which is then unique to within a multiplication factor of -1. The use of the name normal mode stems from that normalization and is somewhat misleading. The orthogonality property of the modes is more important than the possibility to normalize these modes. This is even more misleading since the term “normal mode” is also used for modes that are not normalized. Again in structural dynamics, and similar in other fields with system dynamics, the meaning of a normal mode has been the idea of a free vibration pattern in which all structural particles move in an harmonic motion around their neutral positions with a single frequency that is common for all particles.

In vibration testing of structural systems it is often found that the system damping is very low. In such systems, it is often found that the system displacements or acceleration responses are either in-phase or completely anti-phase with a harmonic excitation force. Since the system response is the superposition of responses from each normal mode this has led

T.J. Abrahamsson (✉)
Department of Applied Mechanics, Chalmers University of Technology, Gothenburg, Sweden
e-mail: thomas.abrahamsson@chalmers.se

R.J. Allemang
Structural Dynamics Lab, University of Cincinnati, Cincinnati, OH, USA

M. Mains (ed.), *Topics in Modal Analysis & Testing, Volume 10*, Conference Proceedings of the Society for Experimental Mechanics Series, DOI 10.1007/978-3-319-30249-2_7

to the impression that the eigenmodes are associated with (close to) in-phase and anti-phase motion only. Such modes can be approximated by real-valued functions. Since low damped systems are very common it is easy to believe that the (almost pure) realness of the modes is the most normal physical behavior in a statistical sense. The authors of this paper have heard this meaning of normality in conjunction to eigenmodes. With this paper we hope to share some light of the properties of eigenmodes of lightly damped systems and by that discourage the understanding of "normal" as the attribute of an eigenmode to be associated to an often observed (= normal) behavior.

7.1.1 Limitations

In this paper, only eigenmodes or modal vectors of the undamped system are considered. This is done to limit the discussion and to eliminate the confusion that might exist when damped systems are considered. Once damped systems are considered, the possibility of non-proportional damping will yield the possibility of complex-valued eigenmodes that can occur whether repeated roots are present or not.

However, this paper is not limiting the discussion to common constraints that may be imposed in numerical solutions or common use. For example, it is possible to define an eigenmode that is the product of a complex-valued scalar and a real-valued vector thus yielding a complex-valued eigenmode that is rotated compared to the real-valued vector. It is also possible, for the repeated root case, to define an eigenmode that is a linear combination of two real vectors, each multiplied by different complex scalars. In both of these cases, the resulting eigenmode will mathematically satisfy the necessary orthogonality constraints but will yield complex-valued modal vectors that could be defined alternatively as real valued modal vectors. These complex-valued eigenmodes will yield generalized (modal) mass that will be negative or complex valued if evaluated by current practices. While this is acceptable from the viewpoint that modal mass is just a scaling factor between somewhat arbitrary scaled eigenmodes and the physical properties of the system being studied, most users would probably not find this result acceptable. We are not advocating that this representation be used, simply pointing out the possibility of alternate mathematical forms that fully satisfy all relationships of physical system and of the mathematical solution.

7.2 Theory

A linear mechanical system with viscous damping is often represented by partial differential equations subsequently discretized into a system of ordinary differential equations as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{V}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{Q}(t) = \boldsymbol{P}_u\boldsymbol{u}(t) \tag{7.1}$$

Here $\boldsymbol{M}$, $\boldsymbol{V}$ and $\boldsymbol{K}$ are the system's positive semi-definite mass, viscous-damping and stiffness matrices, $\boldsymbol{q}$ and $\boldsymbol{Q}$ are the generalized displacements and forces and a dot represents time differentiation. This paper focusses on systems with negligible damping, *i.e.* $\boldsymbol{V} = \boldsymbol{0}$. In many situations, a vast amount of elements of $\boldsymbol{Q}$ are zero and the remaining non-zero elements are collected in the stimulus vector $\boldsymbol{u}$ that relate to $\boldsymbol{Q}$ by the distribution matrix $\boldsymbol{P}_u$. Many systems are self-adjoint and follows Maxwell-Betti's reciprocity rule which leads to symmetric $\boldsymbol{K}$ and $\boldsymbol{M}$. For such systems it is well known that they can be transformed into a decoupled system of equations using the real-valued eigenmodes, collected as columns of the modal matrix $\boldsymbol{\Phi}$ of the eigenvalue equation

$$\boldsymbol{K}\boldsymbol{\Phi} = \boldsymbol{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2 \tag{7.2}$$

in which $\boldsymbol{\Omega}^2$ is a diagonal matrix in which the positive roots $\omega_1^2, \omega_2^2, \ldots$ of the characteristic equation $\left|\boldsymbol{K} - \omega^2\boldsymbol{M}\right| = 0$ appear along its diagonal. It is assumed that $\omega_1, \omega_2, \ldots$ is an ordered sequence such that $\omega_1 \leq \omega_2 \leq \ldots$. Using the transformation $\boldsymbol{q} = \boldsymbol{\Phi}\boldsymbol{z}$ and after a pre-multiplication of the transpose of the modal matrix on Eq. (7.1) one obtains the decoupled differential equation of the undamped system as

$$\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}\ddot{\mathbf{z}} + \boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi}\mathbf{z} = \boldsymbol{\Phi}^T\boldsymbol{Q} = \boldsymbol{\Phi}^T\boldsymbol{P}_u\boldsymbol{u} \tag{7.3}$$

with the mass and stiffness matrices being diagonalized into $\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi} = \text{diag}\,(m_n)$ and $\boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi} = \text{diag}\,(k_n)$. Using mass-normalized modes to fill $\boldsymbol{\Phi}$, all modal masses m_n become unity and thus $\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi} = \boldsymbol{I}$ and the diagonal matrix of modal stiffnesses $k_n = m_n\omega_n^2$ then becomes $\boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi} = \Omega^2 = \text{diag}\left(\omega_n^2\right)$. Note that this is simply one form of normalization that can be chosen. Another common form when performing finite element analysis is to normalize the eigenmode so that the largest element in the eigenmode is unity. Another normalization that uses coefficient matrices of an equivalent state-space system is given later.

For the following it is noted that the symmetric and positive semi-definite mass matrix can be decomposed into

$$\boldsymbol{M} = \boldsymbol{M}^{1/2}\boldsymbol{M}^{1/2} \tag{7.4}$$

where the symmetric square root of the mass matrix is

$$\boldsymbol{M}^{\frac{1}{2}} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^T \tag{7.5}$$

in which the eigenmode matrix $\boldsymbol{V}$ and the diagonal matrix of positive eigenvalues $\boldsymbol{D}$ are given by the solution of the eigenproblem for mass matrix formulated as

$$\boldsymbol{M}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D}^2 \tag{7.6}$$

It can be observed that $\boldsymbol{M} = \boldsymbol{V}\boldsymbol{D}^2\boldsymbol{V}^T$ and the relation for the mass matrix (7.4) and its square root (7.5) holds since; $\boldsymbol{M} = \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{M}^{\frac{1}{2}} = \left(\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^T\right)\left(\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^T\right) = \boldsymbol{V}\boldsymbol{D}\left(\boldsymbol{V}^T\boldsymbol{V}\right)\boldsymbol{D}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{D}\boldsymbol{I}\boldsymbol{D}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{D}^2\boldsymbol{V}^T$ with self-orthonormal $\boldsymbol{V}$, *i.e* with $\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$.

For the computation of the eigensolution $\{\boldsymbol{\Phi}, \boldsymbol{\Omega}^2\}$, computational methods like the block-Lanczos or the subspace iteration methods can be used to the advantage. In the presence of repeated roots some eigenproblem solution techniques just provide a linearly independent set of normal modes that belongs to these roots. That linearly independent set needs to be made mass and stiffness orthogonal before the eigenvector set can be used for decoupling. An illustration of this is given in Sect. 7.8.1.

Let now $\boldsymbol{v}$ be a matrix holding a linearly independent set of eigenvectors that belong to a set of repeated roots as columns. Then the Gram-Schmidt orthogonalization procedure produces a mass orthonormal set of modes in $\boldsymbol{V}$ as

$$\boldsymbol{V}^H = \boldsymbol{C}^{-1}\boldsymbol{v}^H \tag{7.7}$$

with $\boldsymbol{C}$ being the lower triangular Schur factor of a matrix $\boldsymbol{S}$. These both matrices are defined by the relations

$$\boldsymbol{S} = \boldsymbol{C}\boldsymbol{C}^H = \boldsymbol{v}^H\boldsymbol{M}\boldsymbol{v} = \left(\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{v}\right)^H\left(\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{v}\right) \tag{7.8}$$

By its construction it can be noted that $\boldsymbol{S}$ is Hermitian, a condition for the Schur decomposition to exist. To see that $\boldsymbol{V}$ forms a mass-orthonormal basis one notes that

$$\boldsymbol{V}^H\boldsymbol{M}\boldsymbol{V} = \boldsymbol{C}^{-1}\boldsymbol{v}^H\boldsymbol{M}\boldsymbol{v}\boldsymbol{C}^{-H} = \boldsymbol{C}^{-1}\boldsymbol{C}\boldsymbol{C}^H\boldsymbol{C}^{-H} = \boldsymbol{I} \tag{7.9}$$

In fact, the normal modes does not have to be real-valued. Let $\boldsymbol{v} = \overline{\boldsymbol{v}}$ be the set of linearly independent complex-valued normal modes associated to the repeated roots of the undamped eigenvalue problem. The Gram-Schmidt orthogonalization procedure as formulated above is not affected by the complexity of the modes.

It is considered a well-known fact that for repeated roots, any linear combination of the associated eigenvectors is also an eigenvector. The commonly used procedure for establishing such linear combinations is by introducing real-valued combination factors. It is considered to be less well-known that also complex-valued combinations of modes of repeated roots are fully valid. Let again $\boldsymbol{v}$ be a matrix holding a linearly independent set of real-valued eigenvectors that belong to a set of repeated roots. Then let

$$\overline{\boldsymbol{v}} = \boldsymbol{v}\boldsymbol{\alpha} \tag{7.10}$$

with $\boldsymbol{\alpha}$ being a square matrix of proper size with combination factors as elements and note that $\overline{\boldsymbol{v}}$ is a matrix of eigenvectors associated to the same roots. If any of the factors are complex-valued, a complex-valued eigenvector will result. After such an operation, however, the modes of $\overline{\boldsymbol{v}}$ are not necessarily orthogonal but may be orthogonalized by use of the Gram-Schmidt procedure explained above. The physical interpretation of such complex-valued eigenvectors is illustrated in Sect. 7.8.

7.3 System Decoupling

The Gram-Schmidt orthogonalization procedure that involves a Schur decomposition as used above suggest that the proper force transformation to decouple the system equations is by the Hermitian (conjugate transpose) of the modal matrix, i.e. $\boldsymbol{\Phi}^{\boldsymbol{H}} = \text{conj}\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\right)$. The decoupled second order equation associated with undamped version of Eq. (7.1) is then (c.f. Eq. (7.3))

$$\boldsymbol{\Phi}^{\boldsymbol{H}}\boldsymbol{M}\boldsymbol{\Phi}\ddot{\mathbf{z}} + \boldsymbol{\Phi}^{\boldsymbol{H}}\boldsymbol{K}\boldsymbol{\Phi}\mathbf{z} = \boldsymbol{\Phi}^{\boldsymbol{H}}\mathbf{Q} \tag{7.11}$$

Note that the use of the Hermitian in this case is justified by the orthogonalization procedure and the fact that any complex-valued modal vectors are related to a set of real-valued modal vectors. This would not be the case for a set of complex-valued modal vectors arising from a non-proportionally damped system for which no real-valued modal vectors that decouple the system equations exist.

With mass-normalized real-valued eigenmodes related to unique eigenvalues and possibly complex-valued eigenmodes related to repeated eigenvalues in $\boldsymbol{\Phi}$, the mass matrix is diagonalized into the real-valued identity matrix, i.e. $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{\Phi} = \boldsymbol{I}$. In order to see that complex-valued mass-orthonormal modes associated to repeated roots also transform the stiffness matrix into a real-valued diagonalized matrix it can be noted that the eigenvalue problem of the associated repeated roots of value $\overline{\Omega}^2$ is

$$\boldsymbol{K}\boldsymbol{v} = \boldsymbol{M}\boldsymbol{v}\boldsymbol{\Omega}^2 = \overline{\Omega}^2\boldsymbol{M}\boldsymbol{v}\boldsymbol{I} \tag{7.12}$$

By pre-multiplying this by $\overline{\boldsymbol{v}}^{\boldsymbol{H}} = \boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}$ and post-multiplying by $\boldsymbol{\alpha}$ one obtains

$$\boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{v}\boldsymbol{\alpha} = \overline{\Omega}^2\boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{v}\boldsymbol{I}\boldsymbol{\alpha} = \overline{\Omega}^2\boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{v}\boldsymbol{\alpha} \tag{7.13}$$

The mass-orthonormal $\overline{\boldsymbol{v}} = \boldsymbol{v}\boldsymbol{\alpha}$ leads to that $\boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{v}\boldsymbol{\alpha} = \boldsymbol{I}$ and thus

$$\boldsymbol{\alpha}^{\boldsymbol{H}}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{v}\boldsymbol{\alpha} = \overline{\Omega}^2\boldsymbol{I} \tag{7.14}$$

The complex transformation of the stiffness matrix thus provides a real-valued diagonal matrix with the repeated roots along the diagonal just as the transformation with real-valued eigenmodes.

7.4 Perturbation of Repeated Eigenvalues

It is well known, see e.g. [1, 2] that systems with close or coalescent eigenvalues are very sensitive to parametric perturbations. To investigate the effect of such perturbations an introduction of a suitable system descriptor with damping possibility is useful. One such model that can accommodate the constant damping behavior of structures, as can be observed in testing, is the frequency domain hysteretic model

$$\left(\boldsymbol{K} + i\text{sign}\,(\omega)\,\boldsymbol{H} - \omega^2\boldsymbol{M}\right)\widehat{\boldsymbol{q}} = \widehat{\boldsymbol{Q}} \tag{7.15}$$

Here $\boldsymbol{H}$ is the hysteresis matrix and $\left(\widehat{\boldsymbol{q}}, \widehat{\boldsymbol{Q}}\right)$ are time-invariant complex-valued amplitudes of the harmonic displacements and loads defined by

$$\boldsymbol{q}(t) = \widehat{\boldsymbol{q}}\exp\,(i\omega t) \qquad \boldsymbol{Q}(t) = \widehat{\boldsymbol{Q}}\exp\,(i\omega t) \tag{7.16a, b}$$

For this perturbation study one eigenvector complexity parameter ξ and one eigenvalue separation parameter ϱ are introduced. A subset of the full eigenvector solution $\boldsymbol{\Phi}$ is introduced as the perturbed set of eigenvectors $\acute{\boldsymbol{v}}$ related to set of real-valued multiple eigenvalues $\boldsymbol{v}$ of the undamped eigenvalue problem as

$$\acute{\boldsymbol{v}} = \acute{\boldsymbol{v}}\,(\boldsymbol{v}, \xi) \tag{7.17}$$

with $\acute{\boldsymbol{v}}(\boldsymbol{v},0) = \boldsymbol{v}$. The perturbed full set of eigenmodes is denoted $\acute{\boldsymbol{\Phi}}$. For illustration purpose the following perturbation study is limited to systems with maximum multiplicity of their repeated roots being two. For the eigenvalue separation perturbation of such systems the perturbed eigenvalues are defined as

$$\acute{\boldsymbol{\omega}}_i = (1-\varrho)\,\omega_i \qquad \acute{\boldsymbol{\omega}}_{i+1} = (1+\varrho)\,\omega_{i+1} \tag{7.18a, b}$$

for all the undamped system's pairs of multiple eigenvalues (ω_i, ω_{i+1}).

It can be noted from above that complex-valued eigenmodes of undamped reciprocal systems can only exist for repeated roots. The eigenvalue separation perturbation with associated complex-valued modes will thus lead to non-reciprocity. Here mass-orthonormalized modes are used throughout, i.e. $\acute{\boldsymbol{\Phi}}^{\boldsymbol{H}}\boldsymbol{M}\acute{\boldsymbol{\Phi}} = \boldsymbol{I}$, and the perturbed model is established from the decoupled equations as

$$\boldsymbol{I}\ddot{\boldsymbol{z}} + \text{diag}\left(\omega_n^2\right)\boldsymbol{z} = \acute{\boldsymbol{\Phi}}^H\boldsymbol{Q} \tag{7.19}$$

The full set of eigenmodes is used with $\acute{\boldsymbol{\Phi}}$ being square and invertible and thus $\boldsymbol{z} = \acute{\boldsymbol{\Phi}}^{-1}\boldsymbol{q}$. After pre-multiplication of (7.19) by $\acute{\boldsymbol{\Phi}}^{-H}$ one has

$$\acute{\boldsymbol{\Phi}}^{-H}\acute{\boldsymbol{\Phi}}\ddot{\boldsymbol{q}} + \acute{\boldsymbol{\Phi}}^{-H}\text{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\boldsymbol{q} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \acute{\boldsymbol{\Phi}}^{-H}\text{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\boldsymbol{q} = \boldsymbol{Q} \tag{7.20}$$

Let the perturbation stiffness matrix $\acute{\boldsymbol{K}}$ and hysteretic damping matrix $\acute{\boldsymbol{H}}$ be defined as

$$\acute{\boldsymbol{K}} = \left\{\Re e\acute{\boldsymbol{\Phi}}^{-H}\text{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\right\} \qquad \acute{\boldsymbol{H}} = \left\{\Im m\acute{\boldsymbol{\Phi}}^{-H}\text{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\right\} \tag{7.21a, b}$$

One observes that these two matrices are symmetric ($\acute{\boldsymbol{K}}$) and anti-symmetric ($\acute{\boldsymbol{H}}$) by construction. The perturbed system is thus

$$\left(\acute{\boldsymbol{K}} + i\text{sign}\,(\omega)\,\acute{\boldsymbol{H}} - \omega^2\boldsymbol{M}\right)\widehat{\boldsymbol{q}} = \widehat{\boldsymbol{Q}} \tag{7.22}$$

and the eigenvalue problem for positive frequencies ω associated with the perturbed system is

$$\left(\acute{\boldsymbol{K}} + i\acute{\boldsymbol{H}}\right)\boldsymbol{\Phi} = \boldsymbol{M}\acute{\boldsymbol{\Phi}}\text{diag}\left(\acute{\lambda}_{\boldsymbol{n}}\right) \tag{7.23}$$

By construction, the solution vectors to this eigenvalue problem consists of the perturbed eigenvectors.

7.5 Mode Normalization and Modal Correlation

It was noted above, shown by use of the Schur factorization, that the proper transformation of loading is by the Hermitian transpose of the eigenvectors. It was shown that this transformation leads to the undamped system's decoupled equations in (7.11). Let $\boldsymbol{\Psi}$ be the modal matrix of mass and stiffness orthogonal but not necessarily mass-orthonormal modes. The associated modal normalization factors, the modal masses m_n and the modal stiffnesses k_n, then become the real-valued diagonal elements of

$$\boldsymbol{\Psi}^H\boldsymbol{M}\boldsymbol{\Psi} = \text{diag}\,(m_n) \qquad \boldsymbol{\Psi}^H\boldsymbol{K}\boldsymbol{\Psi} = \text{diag}\,(k_n) \tag{7.24a, b}$$

Modes from testing needs another procedure to be normalized. In the test situation, the mass and stiffness distributions of the structure is most often not known or not reliable and thus no $\boldsymbol{K}$ and $\boldsymbol{M}$ matrices are available. In vibration testing, following from signal processing of the real-world time signals from our sensors, one gets the system's frequency response functions $\boldsymbol{G}$ at discrete frequencies Ω_k. For a receptance frequency response function relating the response at i to the loading at j of the undamped systems this has the form [3]

$$G_{ij}\left(\Omega_k\right) = \Sigma_n \frac{\overline{\psi}_i^{(n)} \psi_j^{(n)}}{m_n\left(\Omega_k^2 - \omega_n^2\right)} \tag{7.25}$$

With mass-orthonormal modes $\boldsymbol{\phi}^{(n)}$ related to the orthogonal modes by $\boldsymbol{\psi}^{(n)} = \alpha_n \boldsymbol{\phi}^{(n)}$ the modal masses m_n are unity and thus for the direct receptance in which $i = j$ one has

$$G_{jj}\left(\Omega_k\right) = \Sigma_n \frac{\left(\mathcal{R}e^2\left\{\alpha_n\right\} + \Im m^2\left\{\alpha_n\right\}\right)\left(\mathcal{R}e^2\left\{\psi_j^{(n)}\right\} + \Im m^2\left\{\psi_j^{(n)}\right\}\right)}{\Omega_k^2 - \omega_n^2} \tag{7.26}$$

Let Ω_k be the discrete frequencies in a close surrounding of ω_n. Let further m be the multiplicity of the root ω_n and let $n = 1, \ldots, m$ be the given numbers of these roots. For the undamped system, the receptance in a frequency range around these roots are totally dominated by the contribution of the associated eigenmodes.

This leads to the approximation

$$G_{jj}\left(\Omega_k\right) \approx \sum\nolimits_{n=1}^{m} \frac{\left(\mathcal{R}e^2\left\{\alpha_n\right\} + \Im m^2\left\{\alpha_n\right\}\right)\left(\mathcal{R}e^2\left\{\psi_j^{(n)}\right\} + \Im m^2\left\{\psi_j^{(n)}\right\}\right)}{\Omega_k^2 - \omega_n^2} \tag{7.27}$$

and the complex-valued factors α_n can be determined, e.g. by non-linear least squares fitting, provided the repeated modes are identifiable from the measured direct receptances. One necessary requirement for such identifiability is that the number of receptance functions ($G_{jj}, j = 1,2,\ldots$) used is at least equal to the root multiplicity. Note that this normalization procedure also works for modes with unique roots.

After mass-orthonormalization of the modes and provided that the set of modes is as large as the set of sensors in the test, the modal matrix is square and invertible. A dynamically equivalent second-order $\{\boldsymbol{K},\boldsymbol{M},\boldsymbol{H}\}$ model can thus be constructed using

$$\boldsymbol{M} = \left\{\acute{\boldsymbol{\Phi}}\acute{\boldsymbol{\Phi}}^H\right\}^{-1} \quad \boldsymbol{K} = \Re e\left\{\acute{\boldsymbol{\Phi}}^{-H}\mathrm{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\right\} \quad \boldsymbol{H} = \Im m\left\{\acute{\boldsymbol{\Phi}}^{-H}\mathrm{diag}\left(\omega_n^2\right)\acute{\boldsymbol{\Phi}}^{-1}\right\} \tag{7.28a–c}$$

These relations will be used in the numerical example below to show the effect of the skew-symmetric $\boldsymbol{H}$ on the reconstruction of test data, mimicked from a perturbed modal model with complex-valued eigenvectors and non-repeated roots.

For the purpose of correlation analysis of eigenvectors from different sources, the Weighted Modal Assurance Criterion (WMAC) is often employed as a correlation metric. The sources may be from repeated testing, from finite element analysis, from repeated use of various modal parameter extraction methods on same test data, etc. The correlation analysis is most often realized by the use of a condensed mass matrix as a positive definite weighting matrix. This metric requires that eigenvectors from different sources have the same length (same number of elements) and that the system's inertia is condensed to the degrees-of-freedom that correspond to the eigenvector elements. One such mass condensation techniques is the Guyan reduction technique, see e.g. [4]. With the mass factorization given above, the mass weighted eigenvectors associated with the ith eigenvector ψ_i from one source and the jth eigenvector ψ_j from another source are

$$\boldsymbol{\varphi_i} = \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{\psi_i} \quad \boldsymbol{\varphi_j} = \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{\psi_j} \tag{7.29a, b}$$

The associated WMAC correlation index $WMAC_{ij}$ is then [5]

$$WMAC_{ij} = \boldsymbol{\varphi_i^H}\boldsymbol{\varphi_j}\,\boldsymbol{\varphi_j^H}\boldsymbol{\varphi_i} \,/\left(\|\boldsymbol{\varphi_i}\|_2^2\|\boldsymbol{\varphi_j}\|_2^2\right) \tag{7.30}$$

The auto and cross WMAC are given in the numerical example below for modal models with real-valued and complex-valued modes.

7.6 First Order Modeling

The second order differential Eq. (7.1) can be brought to first order form into

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \tag{7.31}$$

Together with the response (output) equation

$$\boldsymbol{y} = \boldsymbol{Cx} \tag{7.32}$$

it forms what is often called a state-space model with the state vector $\boldsymbol{x} = \left\{\boldsymbol{q}^T\ \dot{\boldsymbol{q}}^T\right\}^T$. It can be verified that it relates to the coefficient matrices of the second order model by

$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{V} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{M}^{-1}\boldsymbol{P}_u \end{bmatrix} \tag{7.33a, b}$$

Note that this form of the $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices is one form that can be used to convert from an $N \times N$ second order model into a $2N \times 2N$ first order model. This approach, however, is inconsistent with the definition of Modal A and Modal B that is commonly used for vector scaling of the generally viscously damped problem. The different formulations are discussed in Sect. 7.7.

The output equation gives the output $\boldsymbol{y}$ which can be any quantities that linearly relate to the state $\boldsymbol{x}$. One specific example is when $\boldsymbol{C} = \boldsymbol{I}$ and $\boldsymbol{D} = \boldsymbol{0}$ which gives the output vector $\boldsymbol{y}$ being the collection of (generalized) displacement and velocity vectors of the system. The state-space counterpart of the undamped decoupled (modal) differential equation with displacement and velocity output is given by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{\Omega}^2 & \boldsymbol{0} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \text{diag}\,(m_n)\,\boldsymbol{\Phi}^T\boldsymbol{P}_u \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Phi} \end{bmatrix} \tag{7.34a–c}$$

It can easily be verified that the system transfer function $\boldsymbol{H}$ (from stimulus to response) is

$$\boldsymbol{H} = \boldsymbol{C}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} \tag{7.35}$$

Consider the system with complex eigenvectors stored in $\boldsymbol{\psi}$. Its transfer function from force stimulus to displacement-velocity output is

$$\boldsymbol{H}_{\psi} = \boldsymbol{C}_{\psi}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\psi} \tag{7.36}$$

with

$$\boldsymbol{C}_{\psi} = \begin{bmatrix} \boldsymbol{\psi} & 0 \\ 0 & \boldsymbol{\psi} \end{bmatrix} \tag{7.37}$$

The corresponding system with real modes and same system poles is given by its transfer function

$$\boldsymbol{H}_{\Phi} = \boldsymbol{C}_{\Phi}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\Phi} \tag{7.38}$$

with

$$\boldsymbol{C}_{\Phi} = \begin{bmatrix} \boldsymbol{\Phi} & 0 \\ 0 & \boldsymbol{\Phi} \end{bmatrix} \quad \boldsymbol{B}_{\Phi} = \begin{bmatrix} \boldsymbol{0} \\ \text{diag}\,(m_n)\,\boldsymbol{\Phi}^T\boldsymbol{P}_u \end{bmatrix} \tag{7.39a, b}$$

Now assume that there is a sensor system with state-dependent properties that gives the signal output

$$\boldsymbol{y}_s = \boldsymbol{C}_s\tilde{\boldsymbol{y}} = \boldsymbol{C}_s\boldsymbol{S}\boldsymbol{y} \tag{7.40}$$

with $\boldsymbol{S}$ being the square calibration matrix that for an ideal sensor system should be $\boldsymbol{S}_{\text{ideal}} = \boldsymbol{I}$ and $\boldsymbol{C}_{\text{s}}$ is the channel selection matrix. The internal states of the sensor system are the elements of $\tilde{\boldsymbol{y}} = \boldsymbol{S}\boldsymbol{y}$. As an example consider the displacement sensor system with the system output $\boldsymbol{y} = \left\{ \boldsymbol{q}^T \ \dot{\boldsymbol{q}}^T \right\}^T$ is identical to the state vector and $\boldsymbol{C}_{\text{s}}$ connects that to the output signals $\boldsymbol{y}_s$ of the sensor system as

$$\boldsymbol{y}_s = \boldsymbol{q}_s = \begin{bmatrix} \boldsymbol{C}_{sq} & \boldsymbol{C}_{s\dot{q}} \end{bmatrix} \boldsymbol{S}\boldsymbol{y} = \begin{bmatrix} \boldsymbol{C}_{sq} & \boldsymbol{0} \end{bmatrix} \boldsymbol{S}\boldsymbol{C}\boldsymbol{x} \tag{7.41}$$

One notes that the partition that relates to the sensor system's internal velocity state is zero (i.e. $\boldsymbol{C}_{s\dot{q}} = \boldsymbol{0}$).

7.7 State-Space Modal Normalization

A state-space model that is mathematical equivalent to the state-space model of Eqs. (7.31) and (7.33a, b) is

$$\widehat{\boldsymbol{A}}\dot{\boldsymbol{x}} + \widehat{\boldsymbol{B}}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{P}_{\text{u}} \\ \boldsymbol{0} \end{bmatrix} u \tag{7.42}$$

with

$$\widehat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \quad \widehat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} \tag{7.43a, b}$$

That the models are equivalent can easily be verified using that

$$\widehat{\boldsymbol{A}}^{-1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M}^{-1} \\ \boldsymbol{M}^{-1} & -\boldsymbol{M}^{-1}\boldsymbol{V}\boldsymbol{M}^{-1} \end{bmatrix} \tag{7.44}$$

The related eigenvalue problem is

$$\widehat{\boldsymbol{A}}\Psi = -\widehat{\boldsymbol{B}}\Psi\boldsymbol{\Lambda} \tag{7.45}$$

One notes that $\hat{\boldsymbol{A}}$ and $\widehat{\boldsymbol{B}}$ are both symmetric and $\widehat{\boldsymbol{B}}$ is indefinite since $\boldsymbol{K}$ and $\boldsymbol{M}$ are both positive definite or $\boldsymbol{K}$ is just semi-definite. Because of symmetry the eigenvalue problem is self-adjoint, meaning that the left-hand and right-hand eigenvectors are identical. Because of the indefiniteness, the eigenvalue problem (7.45) does not normally render real-valued eigenvalues in $\boldsymbol{\Lambda}$. In fact, the eigenvalue problem may be deficient leading to that no orthogonal eigenvector set can be found at all.[1] For the non-deficient case, the eigenvalue matrix Λ is diagonal. In that case one has

$$\boldsymbol{\Psi}^{\text{T}}\widehat{\boldsymbol{A}}\boldsymbol{\Psi} \equiv \text{diag}\,(a_n) = -\boldsymbol{\Psi}^{\text{T}}\widehat{\boldsymbol{B}}\boldsymbol{\Psi}\boldsymbol{\Lambda} \tag{7.46}$$

In which the modal normalization constants a_n are usually referred to as the Modal A constants. The $\widehat{\boldsymbol{B}}$-orthogonality property of $\boldsymbol{\Psi}$ leads to

$$\boldsymbol{\Psi}^{\text{T}}\widehat{\boldsymbol{B}}\boldsymbol{\Psi} = \text{diag}\,(b_n) \tag{7.47}$$

[1]This is a rare case in practice, however mathematically interesting. The simplest such case is probably the critically damped single-degree-of-freedom mass-stiffness-damper system.

The modal scale factors b_n are usually called Modal B constants[2]. The $\widehat{\boldsymbol{B}}$-orthogonality can be used to find the relation between the Modal A constant, the Modal B constant and the eigenvalues as

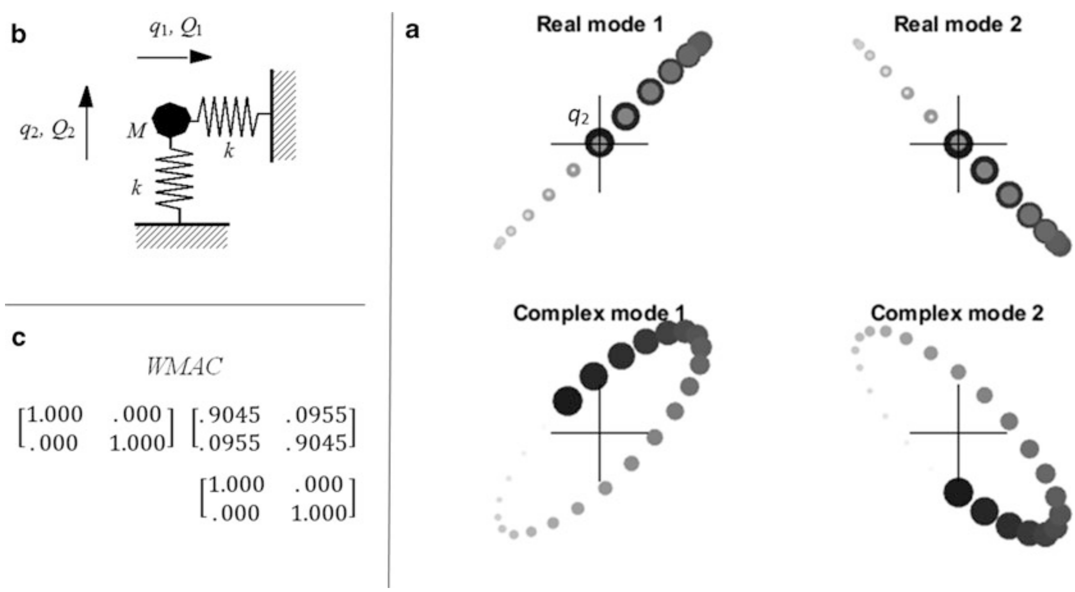
$$\boldsymbol{\Psi}^{\mathrm{T}}\widehat{\boldsymbol{A}}\boldsymbol{\Psi} = -\boldsymbol{\Psi}^{\mathrm{T}}\widehat{\boldsymbol{B}}\boldsymbol{\Psi}\boldsymbol{\Lambda} = -\mathrm{diag}\,(b_n)\ \mathrm{diag}\,(\lambda_n) = \mathrm{diag}\,(-b_n\lambda_n) = \mathrm{diag}\,(a_n) \tag{7.48}$$

7.8 Numerical Examples

The following numerical examples demonstrate the character of normal modes that correspond to repeated or almost coalescent roots of undamped systems. Some real-valued and some complex-valued such modes of the infinite set of possible modes of a system with truly repeated roots are illustrated in Sect. 7.8.1. In Sect. 7.8.2 this system is perturbed such that its modes are complex-valued and its roots do not longer coalesce. Further, in Sect. 7.8.3, the impact response of a non-reciprocal system and the response of a corresponding reciprocal system with test inaccuracies are compared and show similar behavior. This illustrates that a system with complex-valued modes and a system with real-valued modes can look identical from the viewpoint of test data.

7.8.1 Free Vibration Motion of a System with Repeated Roots

The simplest possible system with repeated roots is shown embedded in Fig. 7.1. It consists of an undamped two-dof sprung particle mass system (mass m) with two perpendicular springs of same stiffness $\boldsymbol{k}$. The repeated roots are $\omega_1^2 = \omega_2^2 = k/m$. Two proper sets of linearly independent eigenvectors are the columns of $\boldsymbol{v}$ or $\overline{\boldsymbol{v}}$ with



$$\begin{bmatrix}1.000 & .000\\ .000 & 1.000\end{bmatrix} \quad \begin{bmatrix}.9045 & .0955\\ .0955 & .9045\end{bmatrix}$$
$$\begin{bmatrix}1.000 & .000\\ .000 & 1.000\end{bmatrix}$$

Fig. 7.1 (**a**) Four examples of free single-eigenmode vibration during an almost full eigenperiod. *Upper two graphs* illustrate motion in each of two mass-orthonormal real-valued modes. *Lower graphs* illustrate motion in each of two mass-orthonormal complex-valued modes. For increasing time the mass particle is made larger and darker. **NB!** Since the eigenmodes of repeated roots are not unique, these are just samples from an infinite set of possible eigenmodes of the system. (**b**) The two-degree-of-freedom system with particle mass m in planar motion (q_1, q_2) supported by two similar massless linear springs of stiffness k. (**c**) Auto-*WMAC* 2×2 matrix of $\boldsymbol{\Phi}$ (*upper left*), cross-*WMAC* matrix between $\boldsymbol{\Phi}$ and $\overline{\boldsymbol{\Phi}}$ (*upper right*), and auto-*WMAC* matrix of $\overline{\boldsymbol{\Phi}}$ (*lower right*)

[2]Ref. [6] advocates the use of *modal Foss damping* and *modal Foss stiffness* for the a_n and b_n respectively and gives an abstraction of systems with decoupled complex-valued states. One reason for this naming is that they carry the units of damping and stiffness and the other reason for it is a tribute to K.A. Foss who was one of the first, if not the first, that used the form given by Eq. (7), see [7].

$$\boldsymbol{v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \overline{\boldsymbol{v}} = \begin{bmatrix} 1 + .325\mathrm{i} & 1 \\ 1 - .325\mathrm{i} & 0 \end{bmatrix} \tag{7.49a, b}$$

Following the Gram-Schmidt orthogonalization procedure (see Eqs. (7.7) and (7.8) one obtains the associated mass-orthonormal (i.e. $m_1 = m_2 = 1$) bases

$$\boldsymbol{\Phi} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \quad \acute{\boldsymbol{\Phi}} = \begin{bmatrix} .549 + .178\mathrm{i} & .408 \\ .549 - .178\mathrm{i} & .661 + .480\mathrm{i} \end{bmatrix} \tag{7.50a, b}$$

The associated auto-*WMAC* and cross-*WMAC* indices are embedded in Fig. 7.1.

In free vibration from arbitrary initial conditions, the system is set into motion that is fully characterized by its eigenmodes. By initial conditions that are orthogonal to one of the two eigenmodes, the system motion can be fully characterized by the other eigenmode. As an example, let us consider initial conditions that are orthogonal to the second eigenmode (second column of $\boldsymbol{\Phi}$, i.e. $\boldsymbol{\Phi}_{.2}$). Let the first column of $\boldsymbol{\Phi}$, i.e. $\boldsymbol{\Phi}_{.1}$, be the first displacement eigenmode. The free vibration displacements are then given by, see e.g. [6],

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \beta\, \boldsymbol{\Phi}_{.1} e^{i\omega t} + \beta \mathrm{conj}\left(\boldsymbol{\Phi}_{.1} e^{i\omega t}\right) = 2\beta\, \Re e\left(\boldsymbol{\Phi}_{.1} e^{i\omega t}\right) \tag{7.51}$$

with $\omega = \omega_1 = \omega_2$ and the real-valued β given by the magnitude of the initial displacement and velocity state. A couple of the infinitely many possible eigenmode patterns are illustrated in Fig. 7.1. The free vibration motion of the system subjected to initial conditions that are orthogonal to either the real-valued modes $\boldsymbol{\Phi}_{.1}$ or $\boldsymbol{\Phi}_{.2}$ are illustrated in Fig. 7.1 (upper part). The lower part of the figure shows the free vibration of the system vibrating purely in its complex-valued modes $\overline{\boldsymbol{\Phi}}_{.1}$ or $\overline{\boldsymbol{\Phi}}_{.2}$ given above.

7.8.2 *Frequency Responses of Systems with Closely Spaced Roots*

This example demonstrate the close correspondence of the responses of a non-selfadjoint system with complex-valued eigenmodes and a reciprocal system with real-valued eigenmodes. Both systems have closely spaced roots with identical eigenfrequency separation. The systems are created as perturbations of the two-dof system illustrated in Sect. 7.8.1. For the reciprocal system an eigenfrequency perturbation $\varrho = 0.01$ is made and the real-valued eigenvectors are kept, i.e. $\xi = 0$. This leads to mass-orthonormalized eigenvectors as in Eq. (7.50a, b). The non-selfadjoint system is based on the same eigenfrequency perturbation, i.e. $\varrho = 0.01$, with an added eigenvector perturbation $\xi = 0.10$ such that

$$\acute{\boldsymbol{v}} = \begin{bmatrix} 1 + i\xi & 1 \\ 1 - i\xi & -1 \end{bmatrix} \tag{7.52}$$

The corresponding mass-orthonormalized eigenmodes are

$$\acute{\boldsymbol{\Phi}} = \begin{bmatrix} .7036 + .0704i & .7036 + .0704i \\ .7036 - .0704i & -.7036 + .0704i \end{bmatrix} \tag{7.53}$$

Equations (7.21a, b) give coefficient matrices of the perturbed system. For the selfadjoint system these evaluates to

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 1.0001 & -.0200 \\ -.0200 & 1.0001 \end{bmatrix} \quad \boldsymbol{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.54a–c}$$

and for the non-selfadjoint system they are

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 1.0001 & -.0196 \\ -.0196 & 1.0001 \end{bmatrix} \quad \boldsymbol{H} = \begin{bmatrix} 0 & -.0040 \\ .0040 & 0 \end{bmatrix} \tag{7.55a–c}$$

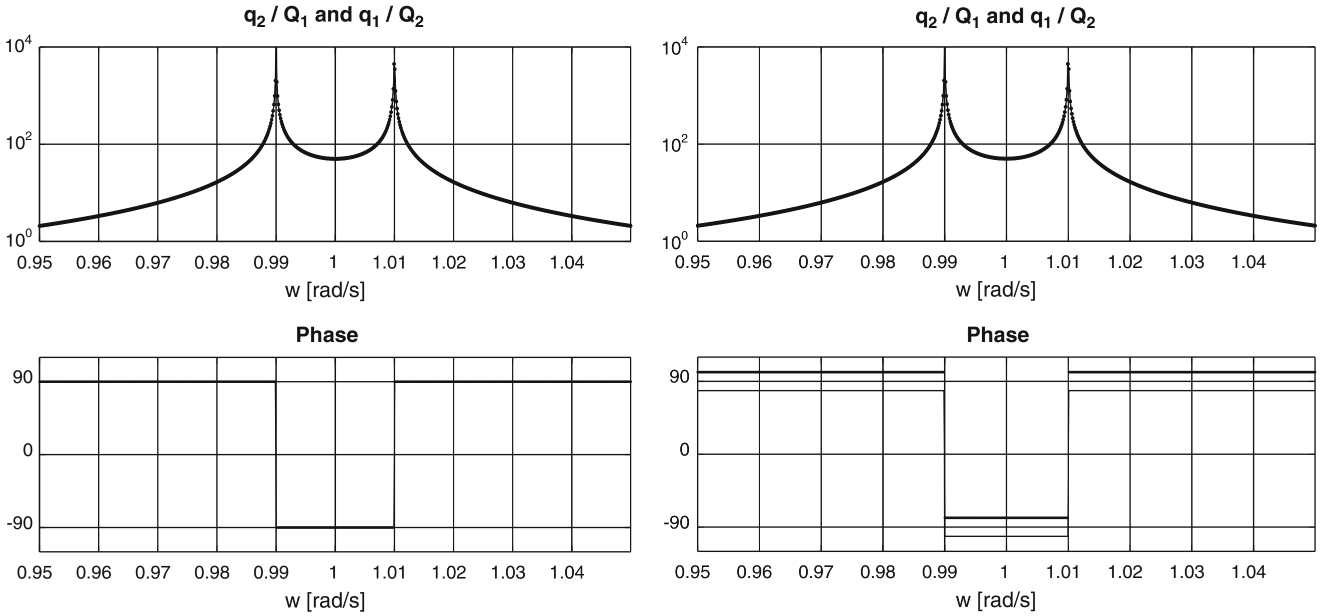


Fig. 7.2 Bode-like plots of the two cross-transfer functions of the selfadjoint (*left*) and non-selfadjoint (*right*) systems. *Dotted curves* are for $\dot{q}_1/Q_2$ and *solid curves* are for $\dot{q}_2/Q_1$

The cross mobility frequency response functions $Q_1 \rightarrow \dot{q}_2$ and $Q_2 \rightarrow \dot{q}_1$ of the non-selfadjoint system can be seen in Fig. 7.2. It can be observed that the magnitude of the two functions coalesce but that their phase functions differ. This different phase relation is not present for the selfadjoint system based on real-valued modes.

7.8.3 Imperfectly Measured Impact Response of Selfadjoint System with Closely Spaced Roots

A condition in which the real signals from a selfadjoint system with imperfect instrumentation may misleadingly result in the conclusion that the system is non-selfadjoint, and thus possesses complex-valued normal modes, is investigated next. Note that the transfer function to the sensor system's internal states $\tilde{\boldsymbol{y}}$ from a stimulus of a selfadjoint system with real-valued modes $\boldsymbol{\Phi}$ are given by Eqs. (7.38) and (7.40) as

$$\boldsymbol{H}_s = \boldsymbol{S}\boldsymbol{C}_{\boldsymbol{\Phi}}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\boldsymbol{\Phi}} \tag{7.56}$$

On the other hand, a spurious non-selfadjoint system with complex-valued modes $\boldsymbol{\psi}$ and same $\boldsymbol{A}$ would have the transfer function

$$\boldsymbol{H}_{\psi} = \boldsymbol{C}_{\psi}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\psi} \tag{7.57}$$

By setting the transfer function to the observed internal state of the sensor system equal to the transfer functions of a false non-selfadjoint system with results in

$$\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{\Phi}}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\boldsymbol{\Phi}} = \boldsymbol{C}_{\psi}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\psi} \tag{7.58}$$

This leads to

$$\boldsymbol{B}_{\psi} = (i\omega\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{C}_{\psi}^{-1}\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{\Phi}}(i\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_{\Phi} \tag{7.59}$$

One notes that the only condition that do not lead to frequency dependence of $\boldsymbol{B}_{\psi}$ is that

$$\boldsymbol{C}_{\psi}^{-1}\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{\Phi}} = \boldsymbol{I} \tag{7.60}$$

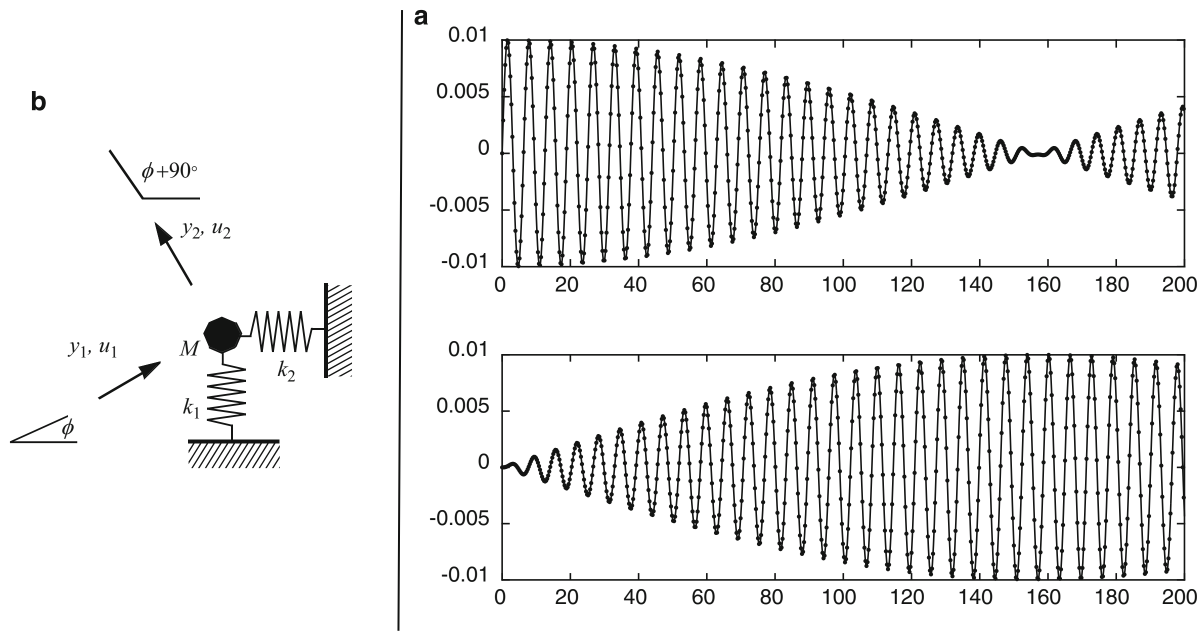


Fig. 7.3 (**a**) Impulse responses of selfadjoint (*solid*) with imperfect sensing and non-selfadjoint (*dots*) two-dof systems with closely spaced eigenfrequencies. Figures show beating velocity responses y_1 (*top*) and y_2 (*bottom*). (**b**) A two-dof system with small eigenvalue separation ($k_1 \approx k_2$). Orientation angle φ of inputs $\boldsymbol{u}$ and responses $\boldsymbol{y}$ are shown

and thus $\boldsymbol{B}_\Psi = \boldsymbol{B}_\Phi$ and

$$\boldsymbol{S} = \boldsymbol{C}_\Psi \boldsymbol{C}_\Phi^{-1} \tag{7.61}$$

For the selfadjoint system shown in Fig. 7.3, the system matrices related to the local coordinate system ($\varphi = 45°$, $k_1 = 0.9801$ N/m, $k_2 = 1.0201$ N/m, m = 1 kg) are

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 1.0001 & -.0200 \\ -.0200 & 1.0001 \end{bmatrix} \tag{7.62a, b}$$

Let the system be excited by u_1 only and thus $\boldsymbol{P}_u = [1\ 0]^T$. This leads to state-space system matrices with displacement and velocity output as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.0001 & .0200 & 0 & 0 \\ .0200 & -1.0001 & 0 & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{C} = \boldsymbol{I}_{4\times 4} \tag{7.63a–c}$$

which can be brought to block-diagonal real form by a similarity transformation as

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 0 & 1.01 & 0 & 0 \\ -1.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & .99 \\ 0 & 0 & -.99 & 0 \end{bmatrix} \boldsymbol{B}_\Phi = \begin{bmatrix} .9951 \\ 0 \\ 0 \\ -1.0051 \end{bmatrix} \boldsymbol{C}_\Phi = \begin{bmatrix} 0 & +.4975 & -.5025 & 0 \\ 0 & -.4975 & -.5025 & 0 \\ -.5025 & 0 & 0 & -.4975 \\ +.5025 & 0 & 0 & -.4975 \end{bmatrix} \tag{7.64a–c}$$

The corresponding eigenvectors for displacements and velocities as observed by the output are the columns of $\boldsymbol{\Phi}$ with

$$\boldsymbol{\Phi} = \begin{bmatrix} +.4975i & -.4975i & -.5025 & -.5025 \\ -.4975i & +.4975i & -.5025 & -.5025 \\ -.5025 & -.5025 & -.4975i & +.4975i \\ +.5025 & +.5025 & -.4975i & +.4975i \end{bmatrix} \quad (7.65)$$

The real-valued modes that are related to displacements are thus $[+.4975\ -.4975]^T$ and $[.5025\ .5025]^T$.

One arbitrarily chosen complex-valued perturbation of $\boldsymbol{\Phi}$ leads to the complex valued modes of

$$\boldsymbol{\psi} = \begin{bmatrix} -.0495 + .4950i & -.0495 - .4950i & -.5025 & -.5025 \\ -.0495 - .4975i & -.0495 + .4950i & -.5025 & -.5025 \\ -.5000 - .0500i & -.5000 + .0500i & -.4975i & +.4975i \\ +.5000 - .0500i & +.5000 + .0050i & -.4975i & +.4975i \end{bmatrix} \quad (7.66)$$

which in turn leads to a non-selfadjoint state-space model that can be brought to block-diagonal real form with the same dynamic matrix $\boldsymbol{A}$ as given in Eq. (7.64a) and output matrix $\boldsymbol{C}_{\psi}$ given by

$$\boldsymbol{C}_{\psi} = \begin{bmatrix} -.0495 & +.4950 & -.5025 & 0 \\ -.0495 & -.4950 & -.5025 & 0 \\ -.5000 & -.0500 & 0 & -.4975 \\ +.5000 & -.0500 & 0 & -.4975 \end{bmatrix} \quad (7.67)$$

A non-perfect sensor system that would give the same output from the selfadjoint system as the perfectly measured output from the non-selfadjoint perturbed system would have the sensitivity matrix

$$\boldsymbol{S} = \begin{bmatrix} 1.0025 & -.0025 & -.0495 & +.0495 \\ -.0025 & 1.0025 & -.0495 & +.0495 \\ +.0505 & -.0505 & 1.0025 & -.025 \\ +.0505 & -.0505 & -.0025 & 1.0025 \end{bmatrix} \quad (7.68)$$

with elements that are off the ideal sensor's by less than 5.1 %. It is seen that the sensitivity matrix has a full structure with off-diagonal elements that give coupling between displacement and velocity states.

The response of the non-selfadjoint model, see Eq. (7.64a, b) and (7.67), is simulated at the action of a 10 ms 1 N impact impulse $u_1(t)$. The velocity responses along the loading direction y_1 and in a perpendicular orientation y_2 are shown in Fig. 7.3. It is compared with the output of an imperfect sensor system with sensitivity matrix given by Eq. (7.68) sitting on a selfadjoint system given by Eq. (7.64a–c). It comes without surprise that the curves are indistinguishable.

7.9 Concluding Remarks

It has been shown that, under a certain condition, an undamped self-adjoint system may vibrate in a normal mode pattern that can best be described by a complex-valued eigenmode. Since the condition is that the system possesses repeated eigenvalues, and these occur with a probability that is almost zero in practice, this is a vibration pattern that is not normally observed. If such non-normally observed modes should occur they are still normal modes. Systems with high degree of symmetry, such as church bells or turbine rotors, are good candidates for studies of such modes. This condition does sometimes occur in experimental data when the modal parameter estimation problem is poorly conditioned (when insufficient references are included in the FRF data set). An example of this is when two modal vectors are very close in frequency (pseudo-repeated roots) but only one mode can be well estimated from data in the frequency region of the pseudo-repeated roots.

Although the conditions for repeated roots (exactly repeated eigenvalues) are seldom met in practice, the spurious output of a non-perfect sensor system may lead to that a system without repeated eigenvalues is mistakenly seen as a system with complex eigenmodes. An example showed a situation with a sensor system that had spurious mixing of displacement and velocity states. In vibration testing the sensors are most often accelerometers. While these are known to have spurious mixing of acceleration states, known as cross-sensitivity, it is not known by the authors if they can also possess cross-sensitivity to other states. However since an angular velocity of a system results in absolute acceleration away from its rotational center, it might not be impossible to find such velocity-acceleration couplings in accelerometers that are based on the acceleration state of a small seismic mass should those sensors sit on a test-piece with local angular motion.

References

1. Wilkinson, J.H.: The Algebraic Eigenvalue Problem. Oxford University Press, Oxford (1965)
2. Golub, G.H., Van Loan, C.F.: Matrix Computation, 4th edn. John Hopkins University Press, Baltimore (2013)
3. Ewins, D.J.: Modal Testing: Theory, Practice and Application. Research Studies Press, Baldock (2000)
4. Craig Jr., R.R., Kurdila, A.J.: Fundamentals of Structural Dynamics. Wiley, Hoboken, NJ (2006)
5. Allemang, R.J.: The modal assurance criterion—twenty years of use and abuse. J. Sound Vib. **37**(8), 14–23 (2003)
6. Abrahamsson, T.J.S.: Modal-parameter extraction for nonproportionally damped linear structures. Int. J. Anal. Exp. Modal Anal. **3**(2) (1988)
7. Foss, K.A.: Co-ordinates which uncouple the equations of motion of damped linear systems. Technical Report 25–20, Aeroelastic and Structures Research Laboratory, Massachusetts Institute of Technology (1956)