Analysis of Boundary States of Multi-state System by Direct Partial Logic Derivatives

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Abstract. Multi-State System (MSS) is mathematical model that is used in reliability engineering for the representation of initial investigated object (system). In a MSS, both the system and its components may experience more than two states (performance levels). One of possible description of MSS is a structure function that is defined correlation between a system components states and system performance level. The investigation of a structure function allows obtaining different properties, measures and indices for MSS reliability. For example, boundary system's states, probabilities of a system performance levels and other measures are calculated based a structure function. In this paper mathematical approach of Direct Partial Logical Derivatives is proposed for calculation of boundary states of MSS.

Keywords: Multi-state system · Multiple-valued logic · Direct partial logic derivatives · Boundary state

1 Introduction

Reliability is considered as an important characteristic of any modern system. How-ever, modern systems are very complex and, therefore, their reliability analysis is a challenging task. The complexity of these systems results from the fact that they consist of a huge amount of elements with various behaviour. Examples of such systems include electrical transmission systems, gas grids [1, 2] or healthcare systems [3]. Distribution grids are typical examples of network systems. They represent large systems consisting of many (not necessarily) hardware elements with different properties, e.g. generating units, different types of transmission lines, and demand centres in the case of electrical grids. Similarly, a healthcare system is a typical instance of socio-technical systems composing of a lot of highly variable elements (components) that can be classified as hardware, software, organizational, and human [1]. This variety of components of complex systems causes that such systems can operate at many different levels of performance and, therefore, their analysis is more difficult than the analysis of other systems.

One of the principal tasks of reliability engineering is investigation of influence of individual system components on system activity. Results of such analysis are useful in

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optimization of system availability or in planning system maintenance. This investigation requires creation of a model of the considered system. Several types of mathematical models are used in reliability analysis. The first ones are known as *Binary-State Systems* (BSSs) because they permit defining only two states in system/components performance – functioning and failure. BSSs have been widely used in classical approaches of reliability engineering [4]. However, they are not very suitable for the analysis of complex systems because their use requires drawing the line between situations in which the system is functioning and when it is considered to be failure. Very often, this is impossible and, therefore, other types of mathematical models are used. According to [1], *Multi-State Systems* (MSSs) are one of the most prospective approaches.

MSSs have been introduced in reliability engineering since 1975 [5–7]. Their main advantage lies in the fact that they allow introducing more than two states in system/components performance [1, 8], i.e. from perfectly functioning, through functioning with restrictions, etc. to completely failed. This indicates that they are very suitable for modelling of complex systems. On the other hand, growing number of states of system components results in dramatic increase of model size, what causes increase of time complexity of the analysis. Therefore, use of MSSs requires development of new and effective methods that could be used to analyse them.

There are several approaches that are used in proposing methods and algorithms for the investigation of MSSs. As a rule, these approaches are based on one of the following four mathematical backgrounds [8]: extensions of Boolean methods, stochastic processes, universal generating function, and Monte Carlo simulation. Each of these approaches is used for some specific tasks relating to evaluation of MSSs. For example, stochastic processes (such as Markov processes) are used to analyse system state transition process [9], universal generating function is useful in optimization problems [10], while Monte Carlo simulation represents a universal tool for reliability assessment of systems consisting of a huge amount of components [11]. If we want to investigate influence of individual system components on the system activity, then the approach based on extensions of Boolean methods can be viewed as one of the most suitable.

The idea of extensions of Boolean methods for the analysis of MSSs lies in the fact that the tools of Boolean algebra have been widely used in reliability analysis of BSSs. This has resulted from the fact that the system structure function, which defines dependency between system state and states of system components, can be viewed as a Boolean function in the case of BSSs [4, 8]. However, this is not true for MSSs. Therefore, some works, e.g. [12, 13], have tried to transform the structure function of a MSS into a Boolean function using Boolean algebra with restrictions [12]. Such transformation allows us to use methods of Boolean algebra in reliability analysis of MSSs. However, this transformation can result in increase of model complexity. To avoid these complications, other authors have proposed using *Multiple-Valued Logic* (MVL) in reliability analysis of MSSs since there are some correlations between the structure function of a MSS and a MVL function [14, 15]. One of the tools of MVL that can be used in the analysis of MSSs is logical differential calculus [16, 17].

Logical differential calculus has been developed for the analysis of dynamic behaviour of Boolean and MVL functions. The central concept of this tool is a logic derivative. There are several types of logic derivatives from which the most interesting are *Direct Partial Logic Derivatives* (DPLDs). Use of DPLDs in reliability analysis of MSSs has been considered in several works, e.g. [15–18]. In those works, it has been shown that DPLDs are useful in calculation of special reliability indices that are named as importance measures. In this paper we propose the application of DPLD for the calculation of exact boundary states that indicate the system state for which the change of one or some of fixed components cause the system performance level decrease or increase. Therefore, they can be used to identify situations in which a degradation of a given component results in a decrease of system performance. Identification of such situations is very important for estimation of component influence on system activity. New indices as probabilities of exact boundary states are introduced in the paper. These indices allow estimating the stability of system in point of view of its availability.

The paper has next structure. Section 2 recalls some basics about structure function and introduce a concept of boundary states of a MSS. Section 3 provides short description of DPLDs computed with respect to one variable and with respect to a vector of several variables (variable vector). Finally, new indices for estimation of MSS boundary states are proposed in Sect. 4, the calculation of these indices are provided based on DPLD.

2 MSS Structure Function

2.1 Structure Function of MSS

A MSS is mathematical model that is used for the description of the system of n components. The i-th system component state is denoted as x_i ($i = 1, \ldots, n$). Consider simple variant of a MSS where the system components and system have equal number of states and performance levels. Each component in such mathematical representation has m states that are indicated as 0 for the complete failure and as m-1 for perfect functioning. Suppose, that the system has m performance level too: from the complete failure (it is 0) to the perfect functioning (it is m-1). The system performance levels depend on components states and this dependency is defined by the structure function ϕ ($\boldsymbol{x}$) identically:

$$\phi(x_1, x_2, \ldots, x_n) = \phi(x) : \{0, 1, \ldots, m-1\}^n \rightarrow \{0, 1, \ldots, m-1\}. \quad (1)$$

In the mathematical point of view the structure function (1) corresponds with the definition of a MVL function [19]. Therefore the mathematical approaches of MVL can be used in qualification and quantification analysis of MSS. But the structure function (1) allows representing the very small class of real system for which the number of system performance levels and number of every component states are equal. As a rule, the real-world system has different numbers of states for different components and these numbers are not equal to number of performance levels and the structure function of such system is defined as:

$$\phi(x) : \{0, 1, \ldots, m_1 - 1\} \times \ldots \times \{0, 1, \ldots, m_n - 1\} \rightarrow \{0, 1, \ldots, M - 1\}, \quad (2)$$

where m_i is number of states for i-th system component ($i = 1, \ldots, n$) and M is number of a system performance levels.

The Eq. (2) is not a MVL function. The interpretation of the Eq. (2) as a MVL function needs additional transformation. Some formal changes in the interpretation of the structure function (2) allow to consider this function as an *incompletely specified MVL function*. The first of them it is definition of value of incompletely specified MVL function. In this case the maximal value of numbers of components states and number of system performance level is interpreted as an value of incompletely specified MVL function $m_{\max} = \mathrm{MAX}\{m_1, \ldots, m_n, M\}$. The second changes in formal interpretation of the structure function is addition of real values of m_i ($i = 1,.., n$) and M to value of incompletely specified MVL function $m_{\max}$. And the structure function (2) as an incompletely specified MVL function can be defined as:

$$\phi'(x) : \{0, 1, \ldots, m_{\max} - 1\}^n \rightarrow \{0, 1, \ldots, m_{\max} - 1\}. \quad (3)$$

The interpretation of the structure function (2) as an incompletely specified MVL function (3) permits to use mathematical approaches of MVL without principal restriction for analysis of properties of the structure function (2). Therefore components states (x_i, $i = 1, \ldots, n$) are interpreted as values of variables of MVL function and system performance levels are considered as values of the MVL function (Fig. 1). And changes of the i-th system component state agrees with changes of the i-th variable value at that the changes of a MSS performance level can be considered as changes of a MVL function. The introduction of these correlations is important for consideration of a MSS boundary states that will be investigated below.

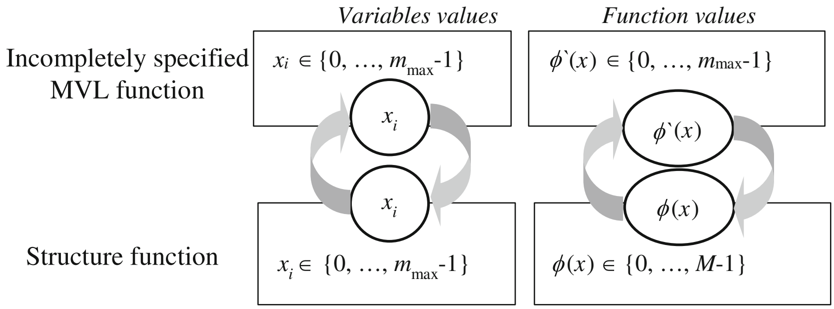
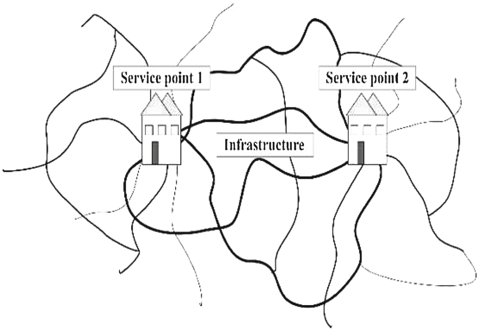


Fig. 1. The correlation of MVL function and structure function.

For example, consider the simple service system (Fig. 2) in a region from paper [18]. This system consists of three components ($n = 3$) – service point 1 (x_1), service point 2 (x_2) and infrastructure (x_3). This system has four performance levels: 0 – non-operational (no customer is satisfied), 1 – partially non-operational (some customers are not satisfied), 2 – partially operational (some customers are satisfied), 3 – fully operational (all customers are satisfied). Next, we assume that the service points are only functional (state 1) or dysfunctional (state 0). The infrastructure can be modelled by 4 quality levels, i.e. from 0 (the quality of the infrastructure is poor) to 3 (the quality is perfect). The structure function of this system according to (2) is defined in Table in Fig. 2 (where m_1 = m_2 = 2, m_3 = 4 and M = 4). Interpret this structure function as incompletely specified MVL function. The value of this function is defined as maximal value of m_1, m_2, m_3, and M and it is $m_{\max}$ = 4. Therefore two variables x_1 and x_2 must be added by two values: 2 and 3. But values of the structure function are not known and are indicated by "–". The structure function of this system as incompletely specified MVL function is in Table 1.

The system



The system structure function

Components states					
x_1	x_2	x_3			
		0	1	2	3
0	0	**0**	**0**	**0**	**0**
0	1	**0**	**1**	**1**	**2**
1	0	**0**	**1**	**1**	**2**
1	1	**0**	**2**	**3**	**3**

Fig. 2. A simple service system and its structure function

Application of the mathematical approach of MVL in reliability engineering has one assumption. This mathematical approach can be used for the analysis of MSS reliability in stationary state or availability only because MVL function is time-independent function [14, 19]. But this assumption is not restriction for the definition of boundary states of a MSS and other reliability indices. Consider the calculation of most important indices in reliability analysis as availability and unavailability. The availability and unavailability of MSS depend on components states. MSS components states are defined in mathematical model by the probabilities of states:

$$p_{i,s} = \Pr\{x_i = s\},\ s = 0, \ldots, m_i - 1. \tag{4}$$

A MSS availability and unavailability based on the conception of the structure function (2) and taking into account the components states probabilities (4) are defined as follows [8]:

$$A(j) = \Pr\{\phi(x) \geq j\},\ j = 1, \ldots, M - 1, \tag{5}$$

$$U = \Pr\{\phi(x) = 0\}. \tag{6}$$

Table 1. The structure function of the simple service system represented as an incompletely specified MVL function

Components states					
x_1	x_2	x_3			
		0	1	2	3
0	0	**0**	**0**	**0**	**0**
0	1	**0**	**1**	**1**	**2**
0	2	–	–	–	–
0	3	–	–	–	–
1	0	**0**	**1**	**1**	**2**
1	1	**0**	**2**	**3**	**3**
1	2	–	–	–	–
1	3	–	–	–	–
2	0	–	–	–	–
2	1	–	–	–	–
2	2	–	–	–	–
2	3	–	–	–	–
3	0	–	–	–	–
3	1	–	–	–	–
3	2	–	–	–	–
3	3	–	–	–	–

But there is the special case of the definition of the system availability for MSS as [8, 15]:

$$A_j = \Pr\{\phi(x) = j\}, \ j = 1, \ldots, M - 1 \tag{7}$$

In this paper Definitions (7) and (6) for a MSS availability and unavailability will be used. Consider a coherent MSS. There are next assumptions for structure function of a coherent MSS [8]: (*a*) the structure function (2) is monotone and $\phi(\boldsymbol{s}) = s$ ($s \in \{0, \ldots, m\text{-}1\}$), and (*b*) all components are *s*-independent and are relevant to the system.

In papers [6, 7] authors shown that any system state j ($j = 1, \ldots, M$ -1) of a coherent MSS according to the assumption (*b*) can be calculated as the product of probabilities of components states (4) and the system availability for performance level j is sum of probabilities of all possible states for the performance level j. In terms of structure function it means that the system availability (7) can be calculate as the sum of probabilities of all values j of the structure function $\phi(\boldsymbol{x})$ that are computed as product of probabilities of components states.

Illustrate the calculation of the availability (7) by example. Consider the simple service system (Fig. 2) and calculate the availability of this system for the performance level 3. The structure function of this system has two values 3 for the state vectors (x_1 x_2 x_3) = (1, 1, 2) and (x_1 x_2 x_3) = (1, 1, 3). Therefore the availability of the simple service system for the performance level 3 is:

Table 2. The components states probabilities of the simple service system

Components states								
	x_1		x_2		x_3			
	0	1	0	1	0	1	2	3
$p_{i,s}$	0.3	0.7	0.2	0.8	0.2	0.6	0.1	0.1

$$A_3 = \Pr\{\phi(x) = 3\} = \Pr\{\phi(1,1,2) + \Pr\{\phi(1,1,3) = p_{1,1} \cdot p_{2,1} \cdot \left(p_{3,2} + p_{3,3}\right)$$

Taking into account the components states probabilities for this system shown in Table 2 the availability of this system for the performance level 3 is $A_3 = 0.112$.

The system availability for the performance levels 2 and 1, and unavailability are calculated similar:

$$A_2 = \Pr\{\phi(x) = 2\} = p_{1,1} \cdot p_{2,1} \cdot p_{3,1} + (p_{1,0} \cdot p_{2,1} + p_{1,1} \cdot p_{2,0}) \cdot p_{3,3} = 0.374,$$

$$A_1 = \Pr\{\phi(x) = 1\} = (p_{1,0} \cdot p_{2,1} + p_{1,1} \cdot p_{2,0}) \cdot \left(p_{3,1} + p_{3,2}\right) = 0.266$$

$$U = \Pr\{\phi(x) = 0\} = p_{3,0} + p_{1,0} \cdot p_{2,0} \cdot \left(p_{3,1} + p_{3,2} + p_{3,3}\right) = 0.248$$

Note, the structure function in Table 1 has one more performance level that is indicated as incompletely specified and marked by "–". These states are depended on the 1-th and 2-nd components states 2 and 3 for which the component state probabilities are equal zero, because these states are not possible: $p_{1,2} = p_{1,3} = p_{2,2} = p_{2,3} = 0$. Therefore the probability of this performance level is zero too.

The system availability for the performance level 2 has maximal value of probability, therefore the service system functioning is more possible as partially operational (some customers are satisfied). The system fault or non-operational (where no customer is satisfied) is characterized by unavailability $U = 0.248$. In comparison with other available performance level this probability is not large.

The system availability and unavailability indicate the probability of the system performance level been, but don't represent the critical states/situation for which a modification of one of components states causes the change of the system performance level, first of all the system degradations. The definition such states are possible by the boundary states of MSS.

2.2 Boundary States of MSS

The conception of boundary states has been proposed for Binary-State System firstly. The boundary state of BSS is defined as system state for which the failure of one system components causes of a system failure [23]. There are different types of boundary states as minimal cut/path sets [14, 23]; exact boundary states [21]. In paper [8] boundary states have been generalized for MSS. The boundary state of MSS must be defined for every system performance level. In papers [24] the boundary states of MSS are interpreted as minimal cut/path sets. Authors of [22] introduced conception of Lower (Upper) Boundary Points of MSS for system performance level j (j = 0, …, M-1). The boundary states for system performance level j and component i (i = 1, …, n) (named as exact boundary states) has been proposed and considered in papers [20, 21]. In paper [18] and [25] the correlations of these boundary states with minimal cut/path sets and Lower (Upper) Boundary Points are shown accordingly.

The exact boundary states have been considered in paper [15]. These states are system states for which the change of the i-th component state from s to $\tilde{s}$ causes the system performance level change from j to $\tilde{j}$ (s, $\tilde{s} \in$ {0,…, m_i -1}, $s \neq \tilde{s}$ and j, $\tilde{j} \in$ {0, …, M -1}, $j \neq \tilde{j}$). The exact boundary state is defined by the exact boundary vector unambiguously. Therefore the exact boundary vectors must be calculated for the definition of exact boundary states. Illustrate the correlation of a system exact boundary state and an exact boundary vector by the example for the service system in Fig. 2.

Determine the exact boundary states of this service system for which the failure of the first component causes the system failure as the change of the system performance level from state 1 to 0. According to Table 1, there are two situations that correspond to this condition. They are possible for the failure of the second component and the third component state 1 or 2. These exact boundary states can be presented as vector states: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,2)$. Note that the boundary state $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,3)$ does not satisfy the condition because the system performance level in this case changes from 1 to 2 depending on the failure of the first component.

One of possible mathematical approaches for the definition of the exact boundary states in MVL is Logical Differential Calculus, in particular the DPLDs [19]. Consider the application of this mathematical approach for analysis of structure function of MSS.

3 Direct Partial Logical Derivatives

3.1 Direct Partial Logical Derivative with Respect to One Variable

The mathematical tool of DPLDs has been proposed in [15] for calculation of an exact boundary states of a MSS. In this paper the definition of DPLDs for MVL function has been adapted for a structure function (1). This definition has been generalized for the structure function (2) in paper [16]. According to [16] DPLD with respect to variable x_i for the structure function (2) permits analyse the system performance level change from j to $\tilde{j}$ when the i-th component state changes from s to $\tilde{s}$:

$$\partial\phi(j \rightarrow \tilde{j})/\partial x_i(s \rightarrow \tilde{s}) = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) = j \text{ and } \phi(\tilde{s}_i, \boldsymbol{x}) = \tilde{j} \\ 0, & \text{other} \end{cases} \tag{8}$$

where $\phi(s_i, \boldsymbol{x}) = \phi(x_1,\ldots, x_{i\text{-}1}, s, x_{i+1},\ldots, x_n)$; $\phi(\tilde{s}, \boldsymbol{x}) = \phi(x_1,\ldots, x_{i\text{-}1}, \tilde{s}, x_{i+1},\ldots, x_n)$; s, $\tilde{s} \in$ {0,…, m_i -1}, $s \neq \tilde{s}$ and j, $\tilde{j} \in$ {0,…, M -1}, $j \neq \tilde{j}$.

There is correlation between exact boundary states and DPLD. Therefore these derivatives can be used for the calculation of exact boundary states and the investigation of influence of the i-th system component changes from s to $\tilde{s}$ to performance level j.

For example, investigate the influence of the first component failure to the fault of the simple service system in Fig. 2. DPLD $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)$ allows to calculate the system state for which this failure causes the system break down. The calculation of this derivative is shown in Fig. 3 in form of flow graph. The derivative $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)$ has two non-zero values that agrees with state vectors: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,2)$. According to the definition of DPLD (8) for these system states the failure of the first system component causes the system failure too. Therefore the service system fails after the failure of the first service point if the second service point isn't functioning and the functioning of the infrastructure conforms state 1 or state 2. The system states $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,2)$ are exact boundary states for the first system component failure and the system performance level 1.

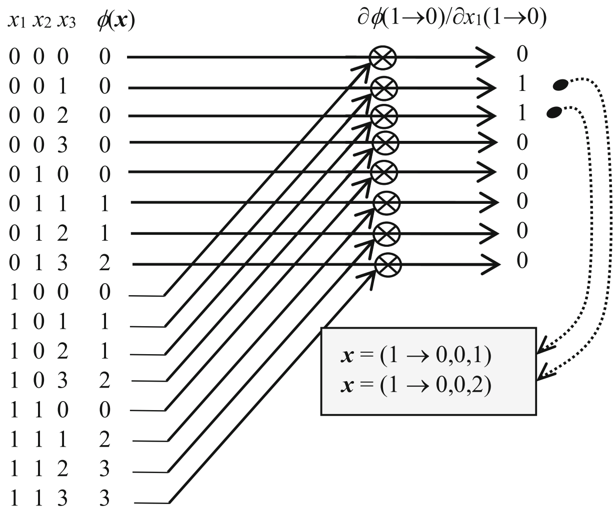


Fig. 3. Calculation of the direct partial logic derivative $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)$.

DPLD (8) allows investigating boundary states of a MSS for which component state x_i change from s to $\tilde{s}$ causes the system performance level change from j to $\tilde{j}$. Therefore, this derivative allows calculating exact boundary states of the i-th system component for MSS performance level j that agree to state vectors $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$. All possible changes of the i-th system component and their influence to MSS performance level can be investigated based on DPLD (8). But this derivative permits to investigate the influence of one component only. DPLD with respect of variable vector investigates the system state changes depending on changes of states of some system components.

3.2 Direct Partial Logical Derivative with Respect to Variable Vector

DPLD of a structure function $\phi(\boldsymbol{x})$ of n variables with respect to variables vector $x^{(p)} = \left(x_{i_1}, x_{i_2}, \ldots, x_{i_p}\right)$ reflects the fact of changing of function from j to $\tilde{j}$ when the value of every variable of vector $\boldsymbol{x}^{(\boldsymbol{p})}$ is changing from s to $\tilde{s}$ [16]:

$$\partial\phi(j \to \tilde{j}) \Big/ \partial x_i(\boldsymbol{s}^{(\boldsymbol{p})} \to \tilde{\boldsymbol{s}}^{(\boldsymbol{p})}) = \begin{cases} 1, & \text{if } \phi(s_{i_1}, \ldots, s_{i_p}, \boldsymbol{x}) = j \text{ and } \phi(\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_p}, \boldsymbol{x}) = \tilde{j} \\ 0, & \text{other} \end{cases} \quad (9)$$

In (9) a change of value of i_q-th variable x_{i_q} form s_{i_q} to $\tilde{s}_{i_q}$ agrees with a change of i_q-th MSS component state form s_{i_q} to $\tilde{s}_{i_q}$ ($q = 1, \ldots, p$ and $p < n$). Changes of some components states correspond with change of a variables vector $x^{(p)} = \left(x_{i_1}, \ldots, x_{i_p}\right)$. Every variable values of this vector changes form s_{i_q} to $\tilde{s}_{i_q}$. So, vector $\boldsymbol{x}^{(\boldsymbol{p})}$ can be interpreted as components states vector or components efficiencies vector.

For example, consider the simple service system (Fig. 2) failure depending on fault of the first service point and reduction of functioning of infrastructure from state 2 to state 1. This system behavior can be presented by the Direct Partial Logic Derivative $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)\ \partial x_3(2 \to 1)$. The calculation of this derivative is shown in Fig. 4.

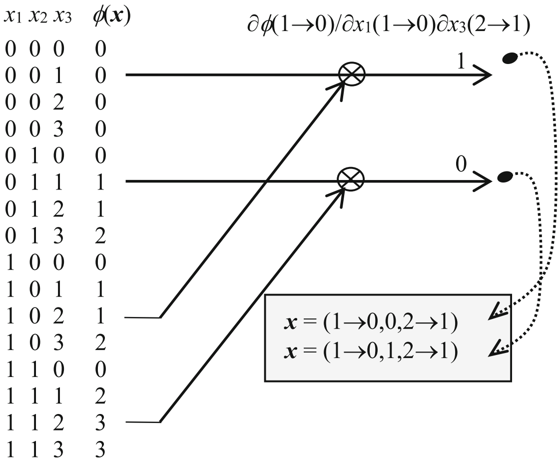


Fig. 4. Calculation of the direct partial logic derivative $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)\ \partial x_3(2 \to 1)$.

The derivative $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)\ \partial x_3(2 \to 1)$ has two values and one of them is non-zero value that agrees with state vector: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0, 0, 2 \to 1)$. This state vector define of the service system failure depending on the failure of the first service point and deterioration of the infrastructure functioning from state 2 to state 1. Therefore the system state $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0, 0, 2 \to 1)$ can be interpreted as exact boundary state for the first and the third system components of the system performance level 1.

The Direct Partial Logic Derivative with respect to variable vector (15) allows investigating boundary states of a MSS for which simultaneous changes of p components states from s_{i_q} to $\tilde{s}_{i_q}$ ($q = 1, \ldots, p$ and $p < n$) causes the system performance level change from j to $\tilde{j}$. Therefore, the Direct Partial Logic Derivative with respect to variable vector allows calculating exact boundary states for MSS performance level j of the i-th system component.

4 The Calculation and Estimation of Exact Boundary States of MSS Based on Direct Partial Logic Derivatives

The exact boundary state of MSS are defined based on the condition that fixed system performance level change depending on the appointed change of one system component state or specified changes of some components states. DPLD with respect to one variable (8) and DPLD with respect to variable vector (9) can be used to investigate change of the system performance level from j to $\tilde{j}$ that are caused by specified changes of one or some system components states. These derivatives have non-zero values of the structure function for system states that satisfy for specified condition: the system performance level change from j to $\tilde{j}$ depending on specified changes of one or some system components states. *Therefor the exact boundary states can be defined as system states that conform to non-zero values of derivatives* (8) *and* (9). In paper [26] new probabilistic indices for exact boundary state that allow estimating the probability of the system boundary/critical states. In this paper this investigation is continued and some new indices are introduced.

Use the symbol $\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix}$ for the exact boundary state of MSS performance level j depending on the i-th system component has been introduced in paper [26]. This state is indicated by vector state $\boldsymbol{x} = (x_1,\ldots, x_i,\ldots, x_n) = (a_1,\ldots, s_i,\ldots, a_n)$ for which $\phi(a_1, \ldots, s_i,\ldots, a_n) = j$ and $\phi(a_1,\ldots, \tilde{s}_i,\ldots, a_n) = \tilde{j}$. Therefore this state can be calculated as non-zero value of DPLD (8). The exact boundary state for MSS performance level j depends on p components $x_{i1}, x_{i2}, \ldots, x_{ip}$ $\begin{pmatrix} j\to\tilde{j} \\ x_{i_1}\ldots x_{i_p} \\ s_{i_1}\to\tilde{s}_{i_1}\, s_{i_p}\to\tilde{s}_{i_p} \end{pmatrix}$ is indicated by vector state $\boldsymbol{x} = (x_1,\ldots, x_{i1},\ldots, x_{ip},\ldots, x_n) = (a_1,\ldots, s_{i1},\ldots, s_{ip},\ldots, a_n)$. This state is calculated as non-zero value of DPLD (9).

In paper [26] some probabilistic indices of the exact boundary states for the coherent MSS have been introduced (Table 3).

The probability of every boundary state $(a_1, \ldots, s_i, \ldots, a_n)$ for MSS performance level j depending on the i-th system component change from s to $\tilde{s}$ is calculated based on the probabilities of components states [26]:

$$p_{(a_1\ldots a_n)}\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix} = p_{1,a_1}\cdot\ldots\cdot p_{i-1,a_{i-1}}\cdot p_{i,s_i}\cdot p_{i+1,a_{i+1}}\cdot\ldots\cdot p_{n,a_n} \tag{10}$$

Table 3. Probabilities for the estimation of exact boundary states of MSS defined in paper [26]

The index description	Equation for calculation
The probability of exact boundary state of the i-th component change from s to $\tilde{s}$ and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix} = \sum_{\phi(a_1,\ldots,s_i,\ldots,a_n)=j} p_{(a_1\ldots a_n)}\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix}$
The probability of exact boundary states of the i-th component for all changes and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \end{pmatrix} = \sum_{s,\tilde{s}} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix}$
The probability of exact boundary states of the i-th component change from s to $\tilde{s}$ and for all performance level j changes	$p\begin{pmatrix} x_i \\ s\to\tilde{s} \end{pmatrix} = \sum_{j,\tilde{j}} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix}$
The probability of exact boundary states of the i-th component all changes	$p\begin{pmatrix} x_i \end{pmatrix} = \sum_{s,\tilde{s}} p\begin{pmatrix} x_i \\ s\to\tilde{s} \end{pmatrix}$

The practical application of estimation of a MSS boundary states supposes the calculation of other probabilities, for example, as probability of MSS performance level change depending on all possible changes of the i-th system component. These probabilities are calculated based on the probability of the system boundary state (10) and are presented in Table 3.

In the Table 3 new indices for the analysis of MSS availability are proposed. These indices are probabilities of exact boundary states that allow estimate the critical state/situation of investigated system/object. In particular, these indices allow investigating the influence of different changes the i-th component state s to the fixed performance level.

The similar indices for probabilistic estimation of exact boundary states (Table 3 and 4) can be defined for estimation of exact boundary state for MSS performance level j of p components $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ $\begin{pmatrix} j\to\tilde{j} \\ x_{i_1}\ldots x_{i_p} \\ s_{i_1}\to\tilde{s}_{i_1}\, s_{i_p}\to\tilde{s}_{i_p} \end{pmatrix}$.

Table 4. New probabilities for the estimation of exact boundary states of MSS

The index description	Equation for calculation
Probability of exact boundary state of i-th component for all decreases from state s and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\downarrow \end{pmatrix} = \sum_{r=s-1}^{0} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to r \end{pmatrix}$
Probability of exact boundary state of i-th component for all increases from state s and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\uparrow \end{pmatrix} = \sum_{r=s+1}^{m-1} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to r \end{pmatrix}$
Probability of exact board state of i-th component for all changes of state s and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s \end{pmatrix} = p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\downarrow \end{pmatrix} + p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\uparrow \end{pmatrix}$
Probability of exact board state of i-th component for all decreases from state s and for performance level change from j to $\tilde{j}$	$p\begin{pmatrix} j\to\tilde{j} \\ x_i \end{pmatrix} = \sum_{r=0}^{m-1} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ r \end{pmatrix}$

Consider the examples of the estimation of exact boundary states for the simple service system (Fig. 2). The components states probabilities for this system are defined in Table 2. Investigate this system failure depending on the first components. DPLD $\partial\phi(1 \rightarrow 0)/\partial x_1(1 \rightarrow 0)$ represents this system behavior (Fig. 3). This derivative has two non-zero values that conform to two boundary states $\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix}$: $\boldsymbol{x} = (1, 0, 1)$ and $\boldsymbol{x} = (1, 0, 2)$.

The probabilities of boundary states for the system failure depending on the first component break down $\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix}$ are calculate according to (16) and are:

$$p\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix} = p_{(101)}\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix} + p_{(102)}\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix} = p_{1,1} \cdot p_{2,0} \cdot p_{3,1} + p_{1,1} \cdot p_{2,0} \cdot p_{3,2} = 0.098 \quad (11)$$

By the similar way the probability of this system failure depending on the breakdown of the second service point is calculated and this probability is $p\begin{pmatrix} 1\to0 \\ x_2 \\ 1\to0 \end{pmatrix} = 0.168$ too. The influence of the infrastructure failure to the fault of the system by the critical state is estimated as $p\begin{pmatrix} 1\to0 \\ x_3 \\ 1\to0 \end{pmatrix} = 0.228$. Therefore the infrastructure failure has maximal influence to the stop of the service system.

Investigate other critical states of this system failure. The critical states can be indicated as exact boundary states of the system. These states correlate to non-zero values of DPLDs $\partial\phi(j \rightarrow 0)/\partial x_i(s \rightarrow s-1)$ for $j, s \in \{1, 2, 3\}$ (Table 5).

Table 5. The exact boundary states of simple service system calculated based on DPLD

DPLDs	Boundary states for the components		
	x_1	x_2	x_2
$\partial\phi(1 \rightarrow 0)/\partial x_i(3 \rightarrow 2)$			–
$\partial\phi(1 \rightarrow 0)/\partial x_i(3 \rightarrow 1)$			–
$\partial\phi(1 \rightarrow 0)/\partial x_i(3 \rightarrow 0)$			–
$\partial\phi(1 \rightarrow 0)/\partial x_i(2 \rightarrow 1)$			–
$\partial\phi(1 \rightarrow 0)/\partial x_i(2 \rightarrow 0)$			(0, 1, 2), (1, 0, 2)
$\partial\phi(1 \rightarrow 0)/\partial x_i(1 \rightarrow 0)$	(1, 0, 1), (1, 0, 2)	(0, 1, 1), (0, 1, 2)	(0, 1, 1), (1, 0, 1)
$\partial\phi(2 \rightarrow 0)/\partial x_i(3 \rightarrow 2)$			–
$\partial\phi(2 \rightarrow 0)/\partial x_i(3 \rightarrow 1)$			–
$\partial\phi(2 \rightarrow 0)/\partial x_i(3 \rightarrow 0)$			(0, 1, 3), (1, 0, 3)
$\partial\phi(2 \rightarrow 0)/\partial x_i(2 \rightarrow 1)$			–
$\partial\phi(2 \rightarrow 0)/\partial x_i(2 \rightarrow 0)$			–
$\partial\phi(2 \rightarrow 0)/\partial x_i(1 \rightarrow 0)$	(1, 0, 3)	(0, 1, 3)	(1, 1, 1)
$\partial\phi(3 \rightarrow 0)/\partial x_i(3 \rightarrow 2)$			–
$\partial\phi(3 \rightarrow 0)/\partial x_i(3 \rightarrow 1)$			–
$\partial\phi(3 \rightarrow 0)/\partial x_i(3 \rightarrow 0)$			(1, 1, 3)
$\partial\varphi(3 \rightarrow 0)/\partial x_i(2 \rightarrow 1)$			–
$\partial\varphi(3 \rightarrow 0)/\partial x_i(2 \rightarrow 0)$			(1, 1, 2)
$\partial\varphi(3 \rightarrow 0)/\partial x_i(1 \rightarrow 0)$	–	–	–

All critical state for every of components of the service system failure are shown in Table 5. The symbol "–" is in cell, if the critical states are absent for fixed component change. The grey cells agree with the situations where the indicated changes are not possible. The analysis of the exact boundary states in Table 5 shows conditions of the system failure. The system failure can be caused by break down of any system component (one of the service points or the infrastructure), but the degradation the system infrastructure has not influence for the system fault (changes the third component state x_3 from 3 to 2 and from 2 to 1). The probabilistic estimations of the critical states are shown in Table 6. According to these indices the system failure (performance level change from 3 to 0) depending on the third component fault is most possible. The probabilities of other failures of the system (performance level change from 2 to 0 and from 1 to 0) have large values in comparison with probabilities of other critical states. Therefore the third component (the system infrastructure) is most important for the functioning of the system in whole and the support of this component working state must be principal goal of this system maintenance. The index of the system failure depending of all possible changes of one of system components $p(x_i)$ has maximal value for the third component that indicates principal influence of this component to the system failure too.

Therefore the proposed indices for the estimation of the exact boundary states are useful for the analysis of the system availability and its functioning. These indices permit to indicate the system component with maximal influence to the fixed changes of the performance levels. These components will have priority in the maintenance plan.

Table 6. Probabilities for the estimation of critical states of the simple service system

	$p\begin{pmatrix}1\to0\\x_i\\1\to0\end{pmatrix}$	$p\begin{pmatrix}1\to0\\x_i\\2\to0\end{pmatrix}$	$p\begin{pmatrix}2\to0\\x_i\\3\to0\end{pmatrix}$	$p\begin{pmatrix}2\to0\\x_i\\1\to0\end{pmatrix}$	$p\begin{pmatrix}3\to0\\x_i\\2\to0\end{pmatrix}$	$p\begin{pmatrix}3\to0\\x_i\\3\to0\end{pmatrix}$	$p\begin{pmatrix}1\to0\\x_i\end{pmatrix}$	$p\begin{pmatrix}2\to0\\x_i\end{pmatrix}$	$p\begin{pmatrix}3\to0\\x_i\end{pmatrix}$	$p\begin{pmatrix}x_i\\1\to0\end{pmatrix}$	$p\begin{pmatrix}x_i\\2\to0\end{pmatrix}$	$p\begin{pmatrix}x_i\\3\to0\end{pmatrix}$	$p(x_i)$
x_1	0.098			0.014			0.098	0.014	0	0.112			0.112
x_2	0.168			0.024			0.168	0.024	0	0.192			0.192
x_3	0.228	0.038	–	0.336	–	0.056	0.266	0.336	0.056	0.564	0.038	0.056	0.658

5 Conclusion

The mathematical methods of MVL are used in reliability estimation of MSS. The mathematical background for application of mathematical methods of MVL for reliability analysis of MSS is considered in this paper. The correlation of the structure function (2) and MVL function are shown and proved by means of the conception of incompletely specified MVL function. This background allows using DPLD for analysis of MSS structure function.

In this paper the investigation of boundary values of the structure function of MSS and definition of MSS exact boundary states based on these valued are considered. Conception of exact boundary states is important for the examination of critical states/situation of the system functioning and availability. The analysis of probabilistic indices of exact boundary states allows indicating the system components with most influence and taking into account such components in the maintenance plan.

The analysis of MSS based on the exact boundary states has not limits for the numbers of components (n) and states for every component (m_i), and system performance levels (M) according to the theoretical background. But in real-world applications these numbers have important influence to the structure function dimension (number of structure function elements) that is calculated as:

$$N_{\text{structure function dimension}} = m_1 \times m_1 \times \ldots \times m_n$$

As a rule the number of system performance levels (M) and number of component states (m_i) are defined between two and seven. According to the investigation in papers [16–18] the Direct Partial Logical Derivatives is applicable for systems which have dimension less than ten millions elements. Therefore the proposed method can be useful for the MSS analysis with number of components under ten.

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