Chapter 5
Generalized Inverses: Quaternions

5.1 Introduction

A quaternion algebra $\mathbb{H}$ was discovered by Sir Rowan Hamilton in 1843, which is a four-dimensional non-commutative algebra over real number field $\mathbb{R}$ with canonical basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the conditions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

so that one has

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \text{ and } \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Any element $\alpha \in \mathbb{H}$ can be written in a unique way: $\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where a, b, c, and d are real numbers, i.e., $\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$. The conjugate of α is defined as $\bar{\alpha} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$, and the norm $|\alpha|$ is given by $|\alpha| = \sqrt{\alpha\bar{\alpha}}$. It is well-known that $\mathbb{H}$ is a skew field (or called a division ring).

The study of polynomials with quaternion coefficients may go back to Niven [79, 80] in the early 1940s. In these two seminal papers, Niven established the "Fundamental Theorem of Algebras" for quaternions, that is, $x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m = 0$ $(a_m \neq 0)$ with coefficients in a division ring D has a solution in D if and only if the centre C of D is a real-closed field and D is the algebra of real quaternions over C. Furthermore, Niven proved that there may be infinitely many roots or a finite number, but in the latter case there are at most $(2m - 1)^2$, which shows the essential difference between the polynomials over division rings and over commutative fields.

Unlike the polynomials over commutative fields, there are several forms of quaternion polynomials depending on the positions of coefficients due to the non-commutativity of $\mathbb{H}$. For example, regular quaternion polynomials in [22] and

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quaternion simple polynomials in [81]. Some properties of these polynomials have been discussed (see, for example, [63, 86]). In this chapter, we will use the following Definition 5.1.1, which places the coefficients on the left side of a variable x:

Definition 5.1.1. A quaternion polynomial $f(x)$ over $\mathbb{H}$ is defined as

$$f(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{H}, a_n \neq 0, \ i = 0, \ldots, n,$$

where x commutes with each element in $\mathbb{H}$.

The set of all quaternion polynomials in x is denoted by $\mathbb{H}[x]$. The degrees, leading terms, and leading coefficients are defined in a natural way. It is well-known that $\mathbb{H}[x]$ becomes a non-commutative domain under the usual polynomial operations.

The quaternion polynomials and matrices with quaternion polynomial entries have been widely studied with many applications in the past decades. For example, in [145], the Fast Fourier Transform for the product of two quaternion polynomial has been discussed with the complexity analysis. In [22], they studied Gröbner basis theory for the ring of quaternion polynomials and explored how to compute the module syzygy. Smith-McMillan forms of quaternion polynomial matrices are defined and some applications to dynamical systems are given in [87]. Some properties of Ore matrices can be found in [37, 144].

For matrices over commutative rings, it is well-known that the various generalized inverses have been defined and explored for many years (see, for example, [10, 85]). This motivates us to consider the generalized inverses questions for quaternion polynomial matrices. The numerical computations for generalized inverses have been discussed for a long time. We will use the symbolic computational methods which have attracted more and more attentions recently, for example, [42, 45, 46, 89].

The structure of this chapter is as follows. In Sect. 5.2, we discuss $\{1\}$-inverses of the quaternion polynomial matrices and present an algorithm to determine the existence of $\{1\}$-inverses. Using one-sided greatest common divisors of quaternion polynomials, we develop an efficient algorithm to compute $\{1\}$-inverses if they exist. In Sect. 5.3, we give the definition of the Moore–Penrose inverse for quaternion polynomial matrices and discuss some basic properties. This includes a necessary and sufficient condition for the existence of the Moore–Penrose inverse. In Sect. 5.4, the well-knownv Leverrier–Faddeev algorithm is extended to quaternion polynomial matrices by using generalized characteristic polynomials. Finally, we discuss the interpolation problems for quaternion polynomials and give an efficient algorithm to compute the Moore–Penrose inverse in Sect. 5.5. We have implemented our algorithms in Maple and some examples are also given in Sect. 5.6.

5.2 {1}-Inverses of Quaternion Matrices

In this section, we first discuss some properties of {1}-inverses of quaternion polynomial matrices, which will be used to formulate an algorithm for finding {1}-inverses for quaternion polynomial matrices. Let $\mathbb{H}[x]^{m\times n}$ be the set of all $m \times n$ matrices over $\mathbb{H}[x]$. Recall that $A \in \mathbb{H}[x]^{m\times n}$ has a {1}-inverse $G \in \mathbb{H}[x]^{n\times m}$ if $AGA = A$.

The main technical idea here is to use a well-known result: $\mathbb{H}[x]$ is a non-commutative principal ideal domain. From this point, we can define one-sided greatest common divisors and least common multiples of quaternion polynomials as follows:

Let $f, g \in \mathbb{H}[x]\setminus\{0\}$. A *greatest common right divisor (GCRD)* of f and g, written $\text{gcrd}(f, g)$, is a nonzero monic $d \in \mathbb{H}[x]$ such that

(a) d is a common right divisor of f and g, namely $f = f_1 d, g = g_1 d$ for some $f_1, g_1 \in \mathbb{H}[x]$;
(b) If $d_1 \in \mathbb{H}[x]$ is a common right divisor of f and g, then d_1 is a right divisor of d.

A *least common right multiple (LCRM)* of f and g, written $\text{lcrm}(f, g)$, is a nonzero monic $s \in \mathbb{H}[x]$ such that

(1) s is a common right multiple of f and g, namely $s = ff_1 = gg_1$ for some $f_1, g_1 \in \mathbb{H}[x]$,
(2) If $s_1 \in \mathbb{H}[x]$ is a common right multiple of f and g, then s_1 is a right multiple of s.

It is easy to prove that GCRD and LCRM are unique. The *greatest common left divisor (GCLD)* and the *least common left multiple* (LCLM) of f and g are defined correspondingly. The following two lemmas can be proved by using the properties of one-sided principal ideals.

Lemma 5.2.1. *Let $a_1, a_2, \ldots, a_n, d \in \mathbb{H}[x]$ and d be monic. The following statements are equivalent:*

(i) $\mathbb{H}[x]a_1 + \mathbb{H}[x]a_2 + \cdots + \mathbb{H}[x]a_n = \mathbb{H}[x]d.$
(ii) $d = \text{gcrd}(a_1, a_2, \ldots, a_n).$

Lemma 5.2.2. *Let $a_1, a_2, \ldots, a_n, s \in \mathbb{H}[x]$ and s be monic. The following statements are equivalent:*

(i) $a_1\mathbb{H}[x] \bigcap a_2\mathbb{H}[x] \bigcap \cdots \bigcap a_n\mathbb{H}[x] = s\mathbb{H}[x].$
(ii) $s = \text{lclm}(a_1, a_2, \ldots, a_n).$

There are several ways to compute the GCRD and LCLM (see, for example, [22]). Here we use the following algorithm that is analogous to the traditional extended Euclidean algorithm for commutative Euclidean domain ([135], Algorithm 3.6). For $f = qg + r$, we denote $q := f \text{ quo}_l\, g$ the left quotient of the division of f by g.

Algorithm 1 Extended Euclidean Algorithm (EEA)

Input $f, g \in \mathbb{H}[x]$, where $\deg(f) = n$, $\deg(g) = m$, $m \le n$, $m, n \in \mathbb{N}$.
Output $k \in \mathbb{N}$, $r_i, s_i, t_i \in \mathbb{H}[x]$ for $0 \le i \le k+1$, and $q_i \in \mathbb{H}[x]$ for $1 \le i \le k$, as computed below.

1: $r_0 \leftarrow f$, $s_0 \leftarrow 1$, $t_0 \leftarrow 0$, $r_1 \leftarrow g$, $s_1 \leftarrow 0$, $t_1 \leftarrow 1$
2: $i \leftarrow 1$
3: **while** $r_i \neq 0$ **do**
  $q_i \leftarrow r_{i-1} \operatorname{quo_l} r_i$, $r_{i+1} \leftarrow r_{i-1} - q_i r_i$,
  $s_{i+1} \leftarrow s_{i-1} - q_i s_i$, $t_{i+1} \leftarrow t_{i-1} - q_i t_i$, $i \leftarrow i+1$.
4: **end while**
5: $k \leftarrow i-1$
6: **return** k, r_i, s_i, t_i for $0 \le i \le k+1$, and q_i for $1 \le i \le k$.

The correctness of the above algorithm follows the strictly decreasing degrees: $\deg(r_1) > \deg(r_2) > \cdots > \deg(r_k) \ge 0$. Next, we shall verify that above algorithm also produces some one-sided greatest common divisors and least common multiples in quaternion polynomial case, which we shall use later.

Lemma 5.2.3. *Let r_i, s_i, t_i for $0 \le i \le k+1$ and q_i for $1 \le i \le k$ be as in Algorithm 1. Consider the matrices*

$$R_0 = \begin{bmatrix} s_0 & t_0 \\ s_1 & t_1 \end{bmatrix}, \; Q_i = \begin{bmatrix} 0 & 1 \\ 1 & -q_i \end{bmatrix} \text{ for } 1 \le i \le k$$

in $\mathbb{M}_{2\times 2}(\mathbb{H}[x])$, and $R_i = Q_i \cdots Q_1 R_0$ for $0 \le i \le k$. Then

(a) $R_i \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}$.
(b) $R_i = \begin{bmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{bmatrix}$.
(c) $s_i f + t_i g = r_i$ *for all* $1 \le i \le k+1$.
(d) $\operatorname{gcrd}(f, g) = r_k$.
(e) $\operatorname{lclm}(f, g) = s_{k+1} f = t_{k+1} g$.

Proof. (a) and (b) can be proved by mathematical induction on i and the relation $R_i = Q_i R_{i-1}$. (c) follows directly from (a).

To prove (d), from assumptions and (a)–(c), we have

$$\begin{bmatrix} r_k \\ 0 \end{bmatrix} = R_k \begin{bmatrix} f \\ g \end{bmatrix} = Q_k \cdots Q_1 R_0 \begin{bmatrix} f \\ g \end{bmatrix} = Q_k \cdots Q_1 \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = Q_k \cdots Q_1 \begin{bmatrix} f \\ g \end{bmatrix}.$$

Note that for each $i \in \{1, \ldots, k\}$, Q_i has an invertible $Q_i^{-1} = \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{H}[x]$.

Hence

$$\begin{bmatrix} f \\ g \end{bmatrix} = Q_1^{-1} \cdots Q_k^{-1} \begin{bmatrix} r_k \\ 0 \end{bmatrix},$$

which implies that r_k is a common right divisor of f and g. On the other hand, by (c), $r_k = s_k f + t_k g$ implies that any common right divisor of f and g is also a right divisor of r_k. Therefore, $\mathrm{gcrd}(f, g) = r_k$.

Finally, since $0 = r_{k+1} = s_{k+1} f + t_{k+1} g$, we have $h = s_{k+1} f = -t_{k+1} g$ is a left common multiple of f and g. Meanwhile, $\deg(h) = \deg(f) + \deg(g) - \deg(\mathrm{gcrd}(f, g))$. Thus, $h = \mathrm{lclm}(f, g)$. □

Our purpose is to use one-sided greatest common divisors and least common multiples to compute {1}-inverses of quaternion polynomial matrices. Our algorithm is based on recursively computing GCRDs and LCLMs. The following result is well-known in commutative case.

Theorem 5.2.4. *Let* $A = \begin{bmatrix} a & \vec{b} \\ \vec{0} & B \end{bmatrix} \in \mathbb{H}[x]^{(m+1)\times(n+1)}$ *with* $0 \neq a \in \mathbb{H}[x]$, $\vec{b} = \begin{bmatrix} b_1 \cdots b_n \end{bmatrix} \in \mathbb{H}[x]^{1\times n}$ *and* $B \in \mathbb{H}[x]^{m\times n}$.

(a) If A *has a* {1}*-inverse over* $\mathbb{H}[x]$*, then* $\mathrm{gcld}(a, b_1, \ldots, b_n) = 1$.

(b) Suppose $A = \begin{bmatrix} a & \vec{0} \\ \vec{0} & B \end{bmatrix}$. *If* A *has a* {1}*-inverse over* $\mathbb{H}[x]$*, then* $a \in \mathbb{H}$ *and* B *has a* {1}*-inverse over* $\mathbb{H}[x]$.

Proof. Let $G = \begin{bmatrix} g & \vec{h} \\ \vec{k} & H \end{bmatrix}$ be a {1}-inverse of A, where $g \in \mathbb{H}$, $\vec{h} = [h_1, \ldots, h_n] \in \mathbb{H}[x]^{1\times n}$, $\vec{k} = \begin{bmatrix} k_1 \cdots k_m \end{bmatrix}^T \in \mathbb{H}[x]^{m\times 1}$ and $H \in \mathbb{H}[x]^{n\times m}$.

Since $A = AGA$, we have

$$\begin{bmatrix} a & \vec{b} \\ \vec{0} & B \end{bmatrix} = \begin{bmatrix} a & \vec{b} \\ \vec{0} & B \end{bmatrix} \begin{bmatrix} g & \vec{h} \\ \vec{k} & H \end{bmatrix} \begin{bmatrix} a & \vec{b} \\ \vec{0} & B \end{bmatrix} = \begin{bmatrix} aga + \vec{b}\vec{k}a & * \\ * & B\vec{k}\vec{b} + BHB \end{bmatrix}. \tag{5.1}$$

Then $aga + \vec{b}\vec{k}a = a$, and thus $(ag + \vec{b}\vec{k} - 1)a = 0$. Since $\mathbb{H}[x]$ is a principal ideal domain, we have $ag + \vec{b}\vec{k} - 1 = 0$, i.e., $ag + b_1k_1 + \cdots + b_nk_n = 1$. Therefore, $\mathrm{gcld}(a, b_1, \ldots, b_n) = 1$.

To prove (b), let $\vec{b} = 0$ in (5.1). Comparing the correspondent entries of matrices on both sides, we have $aga = a$ and $BHB = B$. Hence B has a {1}-inverse H over $\mathbb{H}[x]$. $aga = a$ implies that $(ag - 1)a = 0$, and either $a = 0$ or $ag = 1$ since $\mathbb{H}[x]$ is a domain. Note that both a and g are quaternion polynomials. Therefore $a \in \mathbb{H}$. □

Corollary 5.2.5. *Let* $A = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix} \in \mathbb{H}[x]^{(m+1)\times(n+1)}$. *Then A has a* $\{1\}$*-inverse over* $\mathbb{H}[x]$ *if and only if B a* $\{1\}$*-inverse over* $\mathbb{H}[x]$. *Moreover, if* $C \in \mathbb{H}[x]^{n\times m}$ *is a* $\{1\}$*-inverse of B, then* $\begin{bmatrix} 1 & \vec{0} \\ \vec{0} & C \end{bmatrix}$ *is a* $\{1\}$*-inverse of A over* $\mathbb{H}[x]$.

Proof. If A has a $\{1\}$-inverse over $\mathbb{H}[x]$, then by Theorem 5.2.4(b), B has a $\{1\}$-inverse over $\mathbb{H}[x]$. Conversely, suppose $C \in \mathbb{H}[x]^{n\times m}$ is a $\{1\}$-inverse of B, that is, $BCB = B$. Let $G = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & C \end{bmatrix}$. Then

$$AGA = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & BCB \end{bmatrix} = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix} = A,$$

and so G is a $\{1\}$-inverse of A over $\mathbb{H}[x]$. Therefore, A has a $\{1\}$-inverse over $\mathbb{H}[x]$. □

Next we shall discuss the row and column transformations of quaternion polynomial matrices. Unlike matrices over fields, we cannot use the usual three elementary row (column) transformations freely since $\mathbb{H}[x]$ is a non-commutative domain, not a field. In the following, we will show how to use one-sided greatest common divisors and least common multiples to make row (column) transformations.

Lemma 5.2.6. *Let* $E, E_1 \in \mathbb{H}[x]^{n\times n}$. *Then* $EE_1 = I$ *implies* $E_1E = I$.

Proof. Since $\mathbb{H}[x]$ is a principal ideal domain and Noetherian, we know that $\mathbb{H}[x]$ is stably finite by Proposition 1.13 in [62]. Hence E_1 is also a left inverse of E. □

Lemma 5.2.7. *Let* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{H}[x]^{2\times 2}$, $g_R = \mathrm{gcrd}(a_{11}, a_{21})$ *and* $g_L = \mathrm{gcld}(a_{11}, a_{12})$. *Then there exist invertible matrices* $E, F \in \mathbb{H}[x]^{2\times 2}$, *such that*

$$EA = \begin{bmatrix} g_R & * \\ 0 & * \end{bmatrix}, \qquad AF = \begin{bmatrix} g_L & 0 \\ * & * \end{bmatrix},$$

where each $*$ *stands for some element in* $\mathbb{H}[x]$.

Proof. To design our algorithms, the following proof is constructive. If only one of a_{11} and a_{21} is equal to zero, then we just simply switch two rows. If both of a_{11} and a_{21} are equal to zero, then we will do nothing. Now we assume that both of a_{11} and a_{21} are nonzero. Using the Extended Euclidean Algorithm 1 and Lemmas 5.2.3, we can calculate $s, t, k, l \in \mathbb{H}[x]$, such that

$$sa_{11} + ta_{21} = g_R, \qquad \mathrm{lclm}(a_{11}, a_{21}) = ka_{11} = la_{21}. \tag{5.2}$$

Compute $b_{11}, b_{21} \in \mathbb{H}[x]$ such that $a_{11} = b_{11}g_R$, $a_{21} = b_{21}g_R$. Then $(sb_{11} + tb_{21} - 1)g_R = 0$, and $(kb_{11} - lb_{21})g_R = 0$. Since $\mathbb{H}[x]$ is a domain and $g_R \neq 0$, we have

$$sb_{11} + tb_{21} = 1 \qquad \text{and} \qquad kb_{11} - lb_{21} = 0. \tag{5.3}$$

Again, by (5.2), $\mathrm{gcld}(k, l) = 1$, and we can use the Extended Euclidean Algorithm 1 to find $p, q \in \mathbb{H}[x]$ such that

$$kp - lq = 1. \tag{5.4}$$

Now set

$$E = \begin{bmatrix} s & t \\ k & -l \end{bmatrix}, \qquad E_1 = \begin{bmatrix} b_{11} & p - b_{11}sp - b_{11}tq \\ b_{21} & q - b_{21}sp - b_{21}tq \end{bmatrix}.$$

Then, by (5.2)–(5.4),

$$EA = \begin{bmatrix} s & t \\ k & -l \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} sa_{11} + ta_{11} & * \\ ka_{11} - la_{21} & * \end{bmatrix} = \begin{bmatrix} g_R & * \\ 0 & * \end{bmatrix}$$

and

$$EE_1 = \begin{bmatrix} s & t \\ k & -l \end{bmatrix} \begin{bmatrix} b_{11} & p - b_{11}sp - b_{11}tq \\ b_{21} & q - b_{21}sp - b_{21}tq \end{bmatrix}$$

$$= \begin{bmatrix} sb_{11} + tb_{21} & s(p - b_{11}sp - b_{11}tq) + t(q - b_{21}sp - b_{21}tq) \\ kb_{11} - lb_{21} & k(p - b_{11}sp - b_{11}tq) - l(q - b_{21}sp - b_{21}tq) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & sp - (sb_{11} + tb_{21})sp - (sb_{11} + tb_{21})tq + tq \\ 0 & kp - lq - (kb_{11} - lb_{21})sp - (kb_{11} - lb_{21})tq \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, E_1 is a right inverse of E over $\mathbb{H}[x]$. By Lemma 5.2.6, E_1 is also a left inverse of E.

The construction for F can be done in a similar way. □

Next we generalize this kinds of row/column transformations determined by one-sided greatest common divisors and least common multiplies to matrices with arbitrary sizes.

Theorem 5.2.8. *Let* $A = (a_{ij}) \in \mathbb{H}[x]^{m\times n}$. *Then we can compute two invertible matrices,* $E \in \mathbb{H}[x]^{m\times m}$ *and* $F \in \mathbb{H}[x]^{n\times n}$, *such that*

$$EA = \begin{bmatrix} g_R & * \\ \vec{0} & * \end{bmatrix}, \qquad AF = \begin{bmatrix} g_L & \vec{0} \\ * & * \end{bmatrix},$$

where $g_R = \mathrm{gcrd}(a_{11}, \ldots, a_{m1})$, $g_L = \mathrm{gcld}(a_{11}, \ldots, a_{1n})$, *and each* $*$ *stands for some matrix with suitable size over* $\mathbb{H}[x]$.

Proof. Using Lemma 5.2.7, we can compute an invertible matrix $E_1 \in \mathbb{H}[x]^{2\times 2}$ such that

$$\begin{bmatrix} E_1 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix} A = \begin{bmatrix} \mathrm{gcrd}(a_{11}, a_{21}) & * & \cdots & * \\ 0 & * & \cdots & * \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

It is easy to see that $\begin{bmatrix} E_1 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix}$ is invertible over $\mathbb{H}[x]$. If $a_{31} = 0$, we will go to a_{41}. Otherwise, interchange row 2 and row 3 by multiplying an elementary matrix M on the left side. Then applying Lemma 5.2.7 to the 2×2-matrix on the upper left corner to compute an invertible matrix $E_2 \in \mathbb{H}[x]^{2\times 2}$ such that

$$\begin{bmatrix} E_2 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix} M \begin{bmatrix} E_1 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix} A = \begin{bmatrix} \mathrm{gcrd}(a_{11}, a_{21}, a_{31}) & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ a_{41} & a_{42} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Again, it is easy to verify that $\begin{bmatrix} E_2 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix} M \begin{bmatrix} E_1 & \vec{0} \\ \vec{0} & I_{m-2} \end{bmatrix}$ is invertible over $\mathbb{H}[x]$. Continuing on the same process, we can obtain an invertible matrix $E \in \mathbb{H}[x]^{m\times m}$, such that $EA = \begin{bmatrix} g_R & * \\ \vec{0} & * \end{bmatrix}$.

The construction of the matrix F can be done in a similar way. □

Based on the above results, we design an algorithm for computing a $\{1\}$-inverse of a given matrix over the quaternion polynomial ring $\mathbb{H}[x]$.

Algorithm 2 Computing a {1}-inverse of a given matrix over $\mathbb{H}[x]$

Input $A = (a_{ij}) \in \mathbb{H}[x]^{m\times n}$.

Output $\begin{cases} \text{a } \{1\}\text{-inverse of } G \in \mathbb{H}[x]^{n\times m} \text{ such that } AGA = A, \\ \text{"no } \{1\}\text{-inverse exist.", otherwise} \end{cases}$

1: Computing $g_1 \leftarrow \text{gcrd}(a_{11}, a_{21}, \ldots, a_{m1})$

2: Computing an invertible matrices $E \in \mathbb{H}[x]^{m\times m}$ such that

$EA = \begin{bmatrix} g_1 & \vec{b} \\ \vec{0} & * \end{bmatrix}$, where $\vec{b} = \begin{bmatrix} b_1 & \cdots & b_{n-1} \end{bmatrix}$.

3: Computing $g_2 \leftarrow \text{gcld}(g_1, b_1, \ldots, b_{n-1})$ and an invertible matrix $F \in \mathbb{H}[x]^{n\times n}$ such that $(EA)F = \begin{bmatrix} g_2 & \vec{0} \\ * & B \end{bmatrix}$,

4: **if** $g_2 \neq 1$ **then return** "no {1}-inverse exist."

5: **else** use usual column transformations and computing an invertible matrix $M \in \mathbb{H}[x]^{m\times m}$ such that

$M((EA)F) = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix}$

Recursively call Algorithm 2 to determine (compute) if B has a $\{1\}$-inverse. If find a $\{1\}$-inverse H of B over $\mathbb{H}[x]$,

return $G \leftarrow F \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & H \end{bmatrix} ME$

6: **end if**

Theorem 5.2.9. *Algorithm 2 is correct.*

Proof. Note that

$$MEAF = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix}, \qquad G = F \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & H \end{bmatrix} ME, \qquad BHB = B.$$

We have

$$\begin{aligned}(MEA)G(AF) &= MEAF \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & H \end{bmatrix} MEAF \\ &= \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix} \begin{bmatrix} 1 & \vec{0} \\ 0_{(n-1)\times 1} & H \end{bmatrix} \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix} \\ &= \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & BHB \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & B \end{bmatrix}$$

$$= MEAF.$$

Since E, F, and M are invertible over $\mathbb{H}[x]$ (Theorem 5.2.8), we have $AGA = A$, which completes the proof. □

5.3 The Moore–Penrose Inverse

It is well-known that the Moore–Penrose inverse is the most famous generalized inverse with numerous applications. In the following sections, we discuss the Moore–Penrose inverse for matrices over $\mathbb{H}[x]$.

The conjugate of $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{H}[x]$ is defined as $\bar{f}(x) = \bar{a}_n x^n + \cdots + \bar{a}_0$. For $A \in \mathbb{H}[x]^{m\times n}$, the conjugate $\overline{A}$ of A is defined as $\overline{A} = (\overline{A_{ij}})$. Moreover, $A^T, A^* \in \mathbb{H}[x]^{n\times m}$ denote the transpose and the conjugate transpose of A, respectively. More properties can be found in, for example, [87, 88].

Lemma 5.3.1 ([88]). *Let $f, g \in \mathbb{H}[x]$. Then (i) $\overline{fg} = \bar{g}\bar{f}$ (ii) $f\bar{f} = \bar{f}f \in \mathbb{R}[x]$ (iii) If $fg \in \mathbb{R}[x]$, then $fg = gf$.*

Definition 5.3.2. A matrix in $\mathbb{H}[x]^{n\times m}$ is called a Moore–Penrose inverse of $A \in \mathbb{H}[x]^{m\times n}$ if it is a solution of the following system of equations:

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA.$$

It is easy to prove that if there is a solution, then it is unique. As usual, we denote the Moore–Penrose inverse of A as $A^\dagger$. Using similar methods in commutative case, we can get some properties for quaternion polynomial matrices, for example:

Proposition 5.3.3. *Let $A \in \mathbb{H}[x]^{m\times n}$ with $A^\dagger$. Then*

(i) $(A^*)^\dagger = \left(A^\dagger\right)^*$, $A^\dagger \left(A^\dagger\right)^* A^* = A^\dagger = A^* \left(A^\dagger\right)^* A^\dagger$ *and* $A^\dagger A A^* = A^* = A^* A A^\dagger$.

(ii) Let $U \in \mathbb{H}^{m\times m}$ is a unitary matrix, that is, $UU^* = U^*U = I_m$. Then $(UA)^\dagger = A^\dagger U^*$.

Lemma 5.3.4. *If $E \in \mathbb{H}[x]^{m\times m}$ and satisfies $E = E^2 = E^*$, then $E \in \mathbb{H}^{m\times m}$.*

Proof. Let $f_1, \ldots, f_m$ be the entries on the first row of E. From $E = E^*$, without loss of generality, we may assume that $f_1 = \overline{f_1} \neq 0$. Then by $E = E^2$, we have

$$f_1 = f_1\overline{f_1} + \sum_{i=2}^{m} f_i\overline{f_i} = f_1^2 + \sum_{i=2}^{m} f_i\overline{f_i}.$$

Since $f_1 = \overline{f_1}$, the leading coefficient of f_1^2 is a positive real number. Note that the leading coefficient of $\sum_{i=2}^{m} f_i\overline{f_i}$ is also a positive real number. Thus,

$$\deg\left(f_1^2\right) \geq \deg f_1 = \deg\left(f_1^2 + \sum_{i=2}^{m} f_i\overline{f_i}\right)$$
$$= \max\left\{\deg\left(f_1^2\right), \deg\left(\sum_{i=2}^{m} f_i\overline{f_i}\right)\right\} \geq \deg\left(f_1^2\right).$$

This shows that $f_1 \in \mathbb{H}$. Furthermore, $0 = \deg f_1 = \deg\left(\sum_{i=2}^{m} f_i\overline{f_i}\right)$ and the leading coefficients of $\{f_i\bar{f_i}\}$ $(f_i \neq 0)$ are positive reals imply that $f_i \in \mathbb{H}$ for all $1 \leq i \leq m$. The same discussion can be done for the other rows of E. Therefore, $E \in \mathbb{H}^{m\times m}$. □

Note that we require that $A^\dagger$ must be in $\mathbb{H}[x]^{n\times m}$. Therefore unlike matrices over fields or skew fields, the Moore–Penrose inverses for some quaternion polynomial matrices might not exist. Clearly, $A^\dagger$ must be a $\{1\}$-inverse of A. Thus algorithms in Sect. 5.2 provide a way to check that $A^\dagger$ doesn't exist. In general, we don't have efficient algorithms to verify the existence of $A^\dagger$.

Next we will give conditions for quaternion polynomial matrices to have Moore–Penrose inverses. But the proofs are non-constructive.

It is easy to see that $A \in \mathbb{H}^{m\times n}[x]$ induces an additive homomorphism from $\mathbb{H}[x]^{n\times 1}$ to $\mathbb{H}[x]^{m\times 1}$, that is, for all $P, Q \in \mathbb{H}[x]^{n\times 1}$, $A(P+Q) = AP + AQ \in \mathbb{H}[x]^{m\times 1}$. By the definition of Moore–Penrose inverses and Proposition 5.3.3, it is easy to prove the following lemma:

Lemma 5.3.5. *Let $A \in \mathbb{H}[x]^{m\times n}$ such that $A^\dagger$ exists. Considering A as a homomorphism from $\mathbb{H}[x]^{n\times 1}$ to $\mathbb{H}[x]^{m\times 1}$, one has $Image(A) = Image(AA^*) = Image(AA^\dagger)$ and $Image(A^*) = Image(A^*A) = Image(A^\dagger A)$.*

It is well-known that there are two types of eigenvalues for a given quaternion matrix $A_{m\times n}$: right eigenvalues and left eigenvalues, since $\mathbb{H}$ is a non-commutative domain. Right eigenvalues have been studied extensively (see, for example, [4, 14, 64]). We shall work with right eigenvalues towards our main result, that is, find a nonzero vector $\vec{x} \in \mathbb{H}^{n\times 1}$ and a $\lambda \in \mathbb{H}$ such that $A\vec{x} = \vec{x}\lambda$. For simplicity, we shall just use the term "eigenvalue" instead of right eigenvalue from now on. The following result is well-known and very useful.

Lemma 5.3.6 ([141]). *$A \in \mathbb{H}^{m\times m}$ is hermitian, that is, $A = A^*$, if and only if there exists a unitary matrix $U \in \mathbb{H}^{m\times m}$ such that $U^*AU = diag(\lambda_1, \ldots, \lambda_m)$, where λ_i are the eigenvalues of A.*

Now we are ready to give conditions that quaternion polynomial matrices must satisfy in order to have Moore–Penrose inverses. The following theorem is well-known in some cases, see, for example, [10, 91]. Here is an analogue for quaternion polynomial matrices.

Theorem 5.3.7. *Let* $A \in \mathbb{H}[x]^{m\times n}$. *Then* $A^\dagger$ *exists if and only if* $A = U\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$ *with* $U \in \mathbb{H}^{m\times m}$ *unitary and* $A_1A_1^* + A_2A_2^*$ *a unit in* $\mathbb{H}[x]^{r\times r}$ *with* $r \le \min\{m, n\}$. *Moreover,*

$$A^\dagger = \begin{pmatrix} A_1^*\left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \\ A_2^*\left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \end{pmatrix} U^*.$$

Proof. ($\Longrightarrow$) If A has the Moore–Penrose inverse $A^\dagger$, then

$$AA^\dagger = (AA^\dagger A)A^\dagger = \left(AA^\dagger\right)^2 = \left(AA^\dagger\right)^*.$$

By Lemma 5.3.4, $AA^\dagger \in \mathbb{H}^{m\times m}$. $AA^\dagger$ is hermitian and hence, by Lemma 5.3.6, there exists a unitary matrix $U \in \mathbb{H}^{m\times m}$ such that $U^*AA^\dagger U = D$ where D is diagonal. Since

$$D^2 = (U^*AA^\dagger U)(U^*AA^\dagger U) = U^*AA^\dagger AA^\dagger U = U^*AA^\dagger U = D,$$

the diagonal entries of D are either 1 or 0. Therefore, we can rearrange the rows and columns of U so that $D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ with $r \le \min\{m, n\}$.

Set $B = U^*A$. By Lemma 5.3.3, B has its own generalized inverse $B^\dagger$ and $BB^\dagger = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Write B as a blocked matrix form, that is, $B = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, for some arbitrary quaternion polynomial matrices $A_1 \in \mathbb{H}[x]^{r\times r}$, $A_2 \in \mathbb{H}[x]^{r\times(n-r)}$, $A_3 \in \mathbb{H}[x]^{(m-r)\times r}$, and $A_4 \in \mathbb{H}[x]^{(m-r)\times(n-r)}$. Since $B = BB^\dagger B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$, we must have $A_3 = 0, A_4 = 0$, and thus $BB^* = \begin{pmatrix} A_1A_1^* + A_2A_2^* & 0 \\ 0 & 0 \end{pmatrix}$. Similarly, $B^\dagger = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix}$ for some B_1 and B_2. By Lemma 5.3.5,

$$\text{Image}(BB^*) = \text{Image}(B) = \text{Image}(BB^\dagger) = \text{Image}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies the surjectivity of $A_1A_1^* + A_2A_2^*$ on $\mathbb{H}[x]^{r\times 1}$. Therefore, $A_1A_1^* + A_2A_2^*$ is a unit in $\mathbb{H}[x]^{r\times r}$ and

$$A = UB = U\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}.$$

Next, we have that:

$$B^\dagger = B^\dagger \left(B^\dagger\right)^* B^* = B^\dagger \left(B^*\right)^\dagger B^* = B^* \left(BB^*\right)^\dagger$$
$$= \begin{pmatrix} A_1^* & 0 \\ A_2^* & 0 \end{pmatrix} \begin{pmatrix} \left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^* \left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \\ A_2^* \left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \end{pmatrix},$$

which gives

$$A^\dagger = \begin{pmatrix} A_1^* \left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \\ A_2^* \left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \end{pmatrix} U^*.$$

($\Longleftarrow$) The converse can be proved by direct computation. □

5.4 Leverrier–Faddeev Algorithm

There are many algorithms for computing the Moore–Penrose inverse. In [26], Faddeev provided an algorithm to compute the characteristic polynomial of an $n \times n$ matrix over a field, which is a modification of a method of Levereier (1840). This algorithm is not computational efficiency. But the proof is constructive in a rather clear way. Now the Leverrier–Faddeev algorithm is one of the classical methods that has been used to compute the Moore–Penrose inverse. We refer the reader to [5, 23, 26, 51, 127] for more details.

For a given quaternion polynomial matrix A, our trick is to use a square matrix AA^* instead of A. First we define the characteristic polynomial for quaternion polynomial matrix A by using AA^*, and prove that the coefficients of this characteristic polynomial are reals. Then we show that Leverrier–Faddeev algorithm works very well for quaternion polynomial matrices.

Lemma 5.4.1. *Let $A \in \mathbb{H}[x]^{m\times n}$. Then the eigenvalues of AA^* are real.*

Proof. Let $B = AA^*$ and $\lambda \in \mathbb{H}$ be an eigenvalue of B with corresponding eigenvector $0 \neq \vec{x} = \left(x_1 \cdots x_m\right)^T \in \mathbb{H}[x]^{m\times 1}$ such that $B\vec{x} = \vec{x}\lambda$. Then $\vec{x}^*B\vec{x} = \vec{x}^*\vec{x}\lambda$. Note that $B = B^*$. We have that $\vec{x}^*B\vec{x} = \lambda^*\vec{x}^*\vec{x}$, and thus

$$\vec{x}^*\vec{x}\lambda = \lambda^*\vec{x}^*\vec{x} = \left(\vec{x}^*\vec{x}\lambda\right)^*,$$

that is,

$$(\bar{x}_1, \cdots \bar{x}_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \lambda = \left(\sum_{i=1}^m \bar{x}_i x_i\right)\lambda = \left(\left(\sum_{i=1}^m \bar{x}_i x_i\right)\lambda\right)^*$$
$$= \lambda^* \left(\sum_{i=1}^m \bar{x}_i x_i\right)^* = \lambda^* \left(\sum_{i=1}^m \bar{x}_i x_i\right).$$

By Lemma 5.3.1, $0 \neq \sum_{i=1}^{m} \bar{x}_i x_i \in \mathbb{R}[x]$. The above equation gives $\lambda = \lambda^*$, which implies $\lambda \in \mathbb{R}$. □

The Cayley–Hamilton theorem for quaternion matrices has been extensively studied. A survey can be found in [141]. For $A \in \mathbb{H}[x]^{m\times n}$, if $A = P + Q\mathbf{j}$ with $P, Q \in \mathbb{C}[x]^{m\times n}$, then the complex adjoint of A is defined as

$$\chi_A = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} \in \mathbb{C}[x]^{2m\times 2n}.$$

Next, we define the characteristic polynomial for a quaternion polynomial matrix.

Definition 5.4.2. For $A \in \mathbb{H}[x]^{m\times n}$, let $B = AA^*$ and χ_B be its complex adjoint. Then $f_B(\lambda) = \det(\lambda I_{2m} - \chi_B)$ is called the characteristic polynomial of A.

Remark 5.4.3. By Lemma 5.4.1, λ can be assumed to be a real indeterminate that enjoys the following: $\lambda = \overline{\lambda}$ and λ commutes element-wise with $\mathbb{H}[x]$.

Theorem 5.4.4. *Let $A \in \mathbb{H}[x]^{m\times n}$ and $B = AA^*$. Then $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (\mathbb{R}[x])[\lambda]$, that is, a polynomial in one determinate λ over polynomial ring $\mathbb{R}[x]$.*

Proof. We first show that $f_B(\lambda) \in (\mathbb{R}[x])[\lambda]$. Note that $B = AA^*$. We have

$$\det\left((\lambda I_{2m} - \chi_B)^T\right) = \det(\lambda I_{2m} - \chi_B) = \det\left((\lambda I_{2m} - \chi_B)^*\right),$$

and thus

$$\det(\lambda I_{2m} - \chi_B) = \det\overline{(\lambda I_{2m} - \chi_B)} = \overline{\det(\lambda I_{2m} - \chi_B)}.$$

Therefore

$$\det(\lambda I_{2m} - \chi_B) = f_B(\lambda) \in (\mathbb{R}[x])[\lambda]. \tag{5.5}$$

Next, we show that $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (\mathbb{C}[x])[\lambda]$. Let $B = P + Q\mathbf{j}$. It is easy to check that $P^T = \overline{P}$ and $Q = -Q^T$. Therefore,

$$\chi_B = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} P & -Q^T \\ -\overline{Q} & P^T \end{pmatrix} \implies \lambda I_{2m} - \chi_B = \begin{pmatrix} \lambda I_m - P & Q^T \\ \overline{Q} & \lambda I_m - P^T \end{pmatrix}.$$

Next, we have

$$\begin{pmatrix} I_m & -I_m \\ 0 & I_m \end{pmatrix}\begin{pmatrix} I_m & 0 \\ I_m & I_m \end{pmatrix}\begin{pmatrix} I_m & -I_m \\ 0 & I_m \end{pmatrix}\begin{pmatrix} \lambda I_m - P & Q^T \\ \overline{Q} & \lambda I_m - P^T \end{pmatrix} = \begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q^T \end{pmatrix}.$$

Therefore,

$$f_B(\lambda) = \det\begin{pmatrix} \lambda I_m - P & Q^T \\ \overline{Q} & \lambda I_m - P^T \end{pmatrix} = \det\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q^T \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q^T \end{pmatrix}^T = -\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix},$$

which implies that $\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q^T \end{pmatrix}$ is skew-symmetric. By [74], the determinant of $\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q^T \end{pmatrix}$, also called its Pfaffian, can be written as the square of a polynomial in its entries. Therefore, $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (\mathbb{C}[x])[\lambda]$, a polynomial in one determinate λ over the polynomial ring $\mathbb{C}[x]$.

Finally we show that $g(\lambda) \in (\mathbb{R}[x])[\lambda]$. Suppose otherwise. Then $g(\lambda) = a(\lambda) + b(\lambda)\mathbf{i}$ where $a(\lambda)$ and $b(\lambda) \in (\mathbb{R}[x])[\lambda]$ with $b(\lambda) \neq 0$. By (5.5), $g(\lambda)^2 = a(\lambda)^2 - b(\lambda)^2 + 2a(\lambda)b(\lambda)\mathbf{i} \in (\mathbb{R}[x])[\lambda]$. Hence $a(\lambda) = 0$ and $f_B(\lambda) = g(\lambda)^2 = (b(\lambda)\mathbf{i})^2 = -b(\lambda)^2$ where $b(\lambda) \in (\mathbb{R}[x])[\lambda]$. For a fixed $x \in \mathbb{R}$, let $\lambda' \in \mathbb{R}$ be large enough such that $\lambda' I_{2m} - \chi_B \in \mathbb{C}^{2m\times 2m}$ is diagonally dominant with nonnegative diagonal entries and that $(b(x))(\lambda') \neq 0$. Since $\lambda' I_{2m} - \chi_B$ is also hermitian, $\lambda' I_{2m} - \chi_B$ is positive definite [44]. But $\det(\lambda' I_{2m} - \chi_B) = -((b(x))(\lambda'))^2 < 0$, a contradiction. Therefore, $b(x) = 0$ and thus $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (\mathbb{R}[x])[\lambda]$. □

Corollary 5.4.5. *Let $A \in \mathbb{H}[x]^{m\times n}$, $B = AA^*$, and $f_B(\lambda) = g(\lambda)^2$. Then $g(B) = 0$. $g(\lambda)$ is said to be the generalized characteristic polynomial of A.*

Proof. Note that $g(\lambda) \in (\mathbb{R}[x])[\lambda]$ by Theorem 5.4.4. Then $\chi_{g(B)} = g(\chi_B)$. Next, $f_B(\chi_B) = 0$ by the Cayley–Hamilton theorem for complex polynomial matrices [44]. Therefore $g(\chi_B) = 0$, and $0 = g(\chi_B) = \chi_{g(B)}$, that is, $g(B) = 0$. □

From the definition, it is easy to check the following lemma, which have analogues in the complex case.

Lemma 5.4.6. *Let $A \in \mathbb{H}[x]^{m\times n}$ such that $A^\dagger$ exists. Set $B = AA^*$. Then*

(i) $B^\dagger = (A^*)^\dagger A^\dagger$ *and* $B^\dagger B = AA^\dagger$.
(ii) $B^\dagger B = BB^\dagger$ *and* $(B^\dagger B)^2 = B^\dagger B$.
(iii) $\left(B^\dagger\right)^k = \left(B^k\right)^\dagger$ *and* $(B^{n-k})^\dagger B^{n-k} = B^\dagger B$, *for any* $k \in \mathbb{N}$.

The following result is well-known for quaternion matrices and it is easy to check that the result also holds for quaternion polynomials.

Lemma 5.4.7. *Let $A \in \mathbb{H}[x]^{m\times n}$, $B \in \mathbb{H}[x]^{p\times q}$, and $C \in \mathbb{H}[x]^{m\times q}$. If $A^\dagger$ and $B^\dagger$ both exist, then the quaternion polynomial matrix equation $AXB = C$ has a solution in $\mathbb{H}[x]^{n\times p}$ if and only if $AA^\dagger CB^\dagger B = C$, in which case the general solution is*

$$X = A^{\dagger}CB^{\dagger} + Y - A^{\dagger}AYBB^{\dagger},$$

where $Y \in \mathbb{H}[x]^{n\times p}$ *is arbitrary.*

Theorem 5.4.8. *Let* $A \in \mathbb{H}[x]^{m\times n}$ *such that* $A^{\dagger}$ *exists and* $B = AA^*$. *Suppose that the generalized characteristic polynomial of A is*

$$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m,$$

where $a_i \in \mathbb{R}[x]$. *If k is the largest integer such that* $a_k \neq 0$, *then the Moore–Penrose inverse of A is given by*

$$A^{\dagger} = -\frac{1}{a_k}A^*\left[B^{k-1} + a_1B^{k-2} + \cdots + a_{k-1}I\right].$$

If $a_i = 0$ *for all* $1 \le i \le m$, *then* $A^{\dagger} = 0$.

Proof. The proof is similar to the complex case in [23] by using Corollary 5.4.5, Lemmas 5.4.6, 5.4.7, and 5.3.3. □

From the above theorem, we can find the Moore–Penrose inverse $A^{\dagger}$ of A by computing its generalized characteristic polynomial. Fadeev [27] modified Leverrier's method and gave an algorithm to compute $\{a_i\}$ without computing $g(\lambda)$. Next, we extend this algorithm to quaternion polynomial matrices.

Lemma 5.4.9. *Let* $A \in \mathbb{H}[x]^{m\times n}$ *such that* $A^{\dagger}$ *exists and set* $B = AA^*$. *Then for* $1 \le k \le m$,

$$tr\left[\left(B^k + a_1B^{k-1} + \cdots + a_{k-1}B\right)\right] = -ka_k,$$

where the a_i'*s arise from the following generalized characteristic polynomial of A:*

$$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m \in (\mathbb{R}(\lambda))[x].$$

Proof. Let $Y = yI$ where $y \in \mathbb{R}$. We can write $g(Y)$ as:

$$\begin{aligned} g(Y) &= g(Y) - g(B) \\ &= (Y - B)\left[Y^{m-1} + (B + a_1I)\,Y^{m-2} + \cdots \;+ \left(B^{m-1} + a_1B^{m-2} + \cdots + a_mI\right)\right]. \end{aligned}$$

As long as y is not an eigenvalue of B, $yI - B = Y - B$ is nonsingular, so we can write

$$\begin{aligned} (Y - B)^{-1}g(Y) =& Y^{m-1} + (B + a_1I)\,Y^{m-2} + \left(B^2 + a_1B + a_2I\right)Y^{m-3} + \cdots \\ &+ \left(B^{m-1} + a_1B^{m-2} + \cdots + a_mI\right). \end{aligned}$$

Taking the traces gives

$$\begin{aligned}&\operatorname{tr}\left[(Y-B)^{-1}g(Y)\right]\\&=my^{m-1}+\operatorname{tr}\left[(B+a_1I)\right]y^{m-2}+\cdots+\operatorname{tr}\left(B^{m-1}+a_1B^{m-2}+\cdots+a_mI\right).\end{aligned}$$

Let $C=(Y-B)^{-1}g(Y)$. Since $g(Y)=g(yI)=g(y)I$, we have $C=g(y)(Y-B)^{-1}$. Therefore,

$$\operatorname{tr}(C)=g(y)\operatorname{tr}\left[(Y-B)^{-1}\right].$$

Let $\lambda_1,\cdots,\lambda_{m'}$, where $m'\le m$, be all the nonzero eigenvalues of B. $\operatorname{tr}\left[(Y-B)^{-1}\right]$ is the sum of the eigenvalues of $(Y-B)^{-1}$. We will show that these eigenvalues are the fractions $\frac{1}{y-\lambda_1},\cdots,\frac{1}{y-\lambda_{m'}}$.

Let ζ be an eigenvalue of $(Y-B)^{-1}$ with corresponding eigenvector $\vec{v}$ such that:

$$(Y-B)^{-1}\vec{v}=\vec{v}\zeta.$$

ζ is real by Lemma 5.4.1, and hence

$$(Y-B)\vec{v}=\vec{v}\frac{1}{\zeta}\implies B\vec{v}=\vec{v}\left(y-\frac{1}{\zeta}\right).$$

Therefore, $y-\frac{1}{\zeta}=\lambda_i$ implies $\zeta=\frac{1}{y-\lambda_i}$ for some $1\le i\le m'$.

Since $g(y)=(y-\lambda_1)(y-\lambda_2)\cdots(y-\lambda_{m'})$, we have that the first derivative $g'(y)=g(y)\left(\frac{1}{y-\lambda_1}+\cdots+\frac{1}{y-\lambda_{m'}}\right)$ and $\operatorname{tr}(C)=g'(y)$. On the other hand, the derivative of g is also equal to:

$$g'(y)=my^{m-1}+a_1(m-1)y^{m-2}+\cdots+a_{m-1}.$$

Therefore,

$$\begin{aligned}&my^{m-1}+a_1(m-1)y^{m-2}+\cdots+a_{m-1}\\&=my^{m-1}+\operatorname{tr}(B+a_1I)y^{m-2}+\cdots+\operatorname{tr}\left(B^{m-1}+a_1B^{m-2}+\cdots+a_mI\right).\end{aligned}$$

Comparing the coefficient of y^{m-k-1} on both sides, we obtain

$$\begin{aligned}a_k(m-k)&=\operatorname{tr}\left(B^k+a_1B^{k-1}+\cdots+a_{k-1}B+a_kI\right)\\&=\operatorname{tr}\left(B^k+a_1B^{k-1}+\cdots+a_{k-1}B\right)+\operatorname{tr}(a_kI),\end{aligned}$$

and so

$$-ka_k = \operatorname{tr}\left(B^k + a_1 B^{k-1} + \cdots + a_{k-1} B\right).$$

□

Now the question is changed to find the coefficients of the generalized characteristic polynomial in order to compute the Moore–Penrose inverse. Next, we present the Leverrier–Faddeev algorithm for finding Moore–Penrose inverses of quaternion polynomial matrices by recursively computing traces.

Proposition 5.4.10. *Let $A \in \mathbb{H}[x]^{m\times n}$ such that $A^\dagger$ exists and $B = AA^*$. Suppose that the generalized characteristic polynomial of A is*

$$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m,$$

where $a_i \in \mathbb{R}[x]$. Define $a_0 = 1$. If p is the largest integer such that $a_p \neq 0$ and we construct the sequence $A_0, \cdots, A_p$ as follows:

$$\begin{array}{llllll}
A_0 & = 0 & -1 & = q_0 & B_0 & = I \\
A_1 & = AA^*B_0 & \frac{trA_1}{1} & = q_1 & B_1 & = A_1 - q_1 I \\
\vdots & & \vdots & & \vdots & \\
A_{p-1} & = AA^*B_{p-2} & \frac{trA_{p-1}}{p-1} & = q_{p-1} & B_{p-1} & = A_{p-1} - q_{p-1} I \\
A_p & = AA^*B_{p-1} & \frac{trA_p}{p} & = q_p & B_p & = A_p - q_p I
\end{array}$$

then $q_i(x) = -a_i(x)$, $i = 0, \cdots, p$.

Proof. We will show $q_i(x) = -a_i(x)$ by mathematical induction. By the definition, clearly $q_0 = -a_0$ holds.

Now we assume that $q_i(x) = -a_i(x)$ holds for all $0 \le i \le k-1$. Then

$$\begin{aligned}
A_k =& AA^*B_{k-1} \\
=& BB_{k-1} \\
=& B(A_{k-1} - q_{k-1}I) \\
=& B(B(A_{k-2} - q_{k-2}I) - q_{k-1}I) \\
& \cdots \\
=& B^k - q_1B^{k-1} - q_2B^{k-2} - \cdots - q_{k-1}B \\
=& B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B.
\end{aligned}$$

and thus

$$\operatorname{tr}(A_k) = \operatorname{tr}\left(B^k + a_1B^{k-1} + \cdots + a_{k-1}B\right),$$

which, by Lemma 5.4.9, is equal to $-ka_k$. So $q_k = \frac{\operatorname{tr} A_k}{k} = -a_k$. Therefore, $q_i(x) = -a_i(x)$ for all $p \geq i \geq 0$. □

Now, combining Theorem 5.4.8 and Proposition 5.4.10, we have the following algorithm to compute the Moore–Penrose inverse:

Algorithm 3 Leverrier–Faddeev algorithm for quaternion polynomial matrices

Input $A \in \mathbb{H}[x]^{m\times n}$
Output The Moore–Penrose inverse $A^\dagger$ of A in $\mathbb{H}[x]^{n\times m}$ if exists
1: $B_0 \leftarrow I_m,\ a_0 \leftarrow 1$
2: for $i = 1, \ldots, m$ do
$A_i \leftarrow AA^*B_{i-1},\ a_i \leftarrow -\frac{\operatorname{tr} A_i}{i},\ B_i \leftarrow A_i + a_i I_m$
3: Find the maximal index p such that $a_p \neq 0$.
4: Return $A^\dagger = \begin{cases} -\frac{1}{a_p}A^*B_{p-1}, & p > 0, \\ 0, & p = 0. \end{cases}$

Note that we have to compute many matrix products in Algorithm 3, which means that Leverrier–Faddeev method is not efficient. In the next section, we will present a more efficient way by combining Theorem 5.4.8 and interpolation methods.

5.5 Finding Moore–Penrose Inverses by Interpolation

Interpolation is an efficient method in many computational questions over commutative fields. In non-commutative case like $\mathbb{H}$, the situation becomes very complicated since some basic properties fail, for example, for a given quaternion polynomial, there might have infinite roots and infinite factors. To overcome this difficulty, we choose the interpolation at data points of real numbers and present an efficient method to obtain the Moore–Penrose inverse of a quaternion polynomial matrix.

Recently, there are few papers regarding non-commutative interpolations and applications (see, for example, [46, 67, 143]). Lets recall some important concepts and properties. An element $r \in \mathbb{H}$ is a root of a nonzero polynomial $f = a_n x^n + \cdots + a_0 \in \mathbb{H}[x]$ if $a_n r^n + \cdots + a_0 = 0$. Since $\mathbb{H}$ is a principal idea domain, using Euclidean Algorithm, it is easy to see that $f(r) = 0$ if and only if $x - r$ is a right divisor of f. The set of polynomials in $\mathbb{H}[x]$ having r as a root is the left ideal $\mathbb{H}[x] \cdot (x - r)$. It is worth mentioning that the evaluations of quaternion polynomials are quite different from the commutative case. It is defined as following: let f, g and $h \in \mathbb{H}[x]$, $f = gh$ and $r \in \mathbb{H}$. If $h(r) = 0$, then $f(r) = 0$. Otherwise, set $\beta = h(r) \neq 0$. Then the evaluation of $f(x)$ at $x = r$ is

$$f(r) = g\left(\beta r \beta^{-1}\right) h(r). \tag{5.6}$$

In particular, if r is a root of f but not of h, then $\beta r \beta^{-1}$ is a root of g. We refer the reader to [63] for more details.

Although a quaternion could have infinite many roots, in [38], it is proved that if $f \in \mathbb{H}[x]$ is of degree n, then the roots of f lie in at most n conjugacy classes of $\mathbb{H}$.

It is well-known that Newton's interpolation and Lagrange's interpolation play important roles in studying polynomials over fields. Unfortunately, one cannot get similar nice formulas in quaternion case. Fortunately, we can still compute a quaternion polynomial from a given set of pairs of quaternions.

Lemma 5.5.1. *Let $c_1, \ldots, c_n$ be n pairwise non-conjugate elements of $\mathbb{H}$. Then there is a unique monic polynomial $g_n \in \mathbb{H}[x]$ of degree n such that $g_n(c_1) = \cdots = g_n(c_n) = 0$. Moreover, $c_1, \ldots, c_n$ are the only roots (up to conjugacy classes) of g_n in $\mathbb{H}$.*

Proof. We first show the existence of g_n for all $n \geq 1$ by mathematical induction. For $n = 1$, it is trivially true as $g_1 = x - c_1$.

Suppose the claim holds for all $1 \leq n \leq k - 1$. Let $c_1, \cdots, c_k \in \mathbb{H}$ be pairwise non-conjugate. Invoking the inductive hypothesis, there exists a monic polynomial g_{k-1} of degree $k - 1$ with $c_2, \cdots, c_k$ as its only roots (up to conjugacy classes), that is, $g_{k-1}(c_1) \neq 0$. Construct g_k as follows:

$$g_k(x) = \left(x - g_{k-1}(c_1)\, c_1 g_{k-1}(c_1)^{-1}\right) \cdot g_{k-1}(x).$$

By Eq. (5.6), $g_k(c_1) = 0$. Thus, the claim holds for k. Therefore this claim holds for all $n \geq 1$.

We next show that g_n is unique. For a fixed n, let $g \neq g_n$ be a monic polynomial of degree n such that $g(c_1) = \cdots = g(c_n) = 0$, too. Then $\deg(g_n - g) \leq n - 1$ but $g_n - g$ has roots $c_1, \ldots, c_n$ which lie in n different conjugacy classes of $\mathbb{H}$, a contradiction. Therefore, g_n is unique for all $n \geq 1$. □

Proposition 5.5.2. *Let $c_1, \ldots, c_{n+1} \in \mathbb{H}$ be pairwise non-conjugate and let $d_1, \ldots, d_{n+1} \in \mathbb{H}$. Then there exists a unique lowest degree polynomial $f \in \mathbb{H}[x]$, of degree $p \leq n$, such that $f(c_i) = d_i$ for all $1 \leq i \leq n + 1$.*

Proof. For any $1 \leq s \leq n + 1$, let $S = \{1, \cdots, n + 1\} \setminus \{s\}$. By Lemma 5.5.1, we can find a unique monic $h_S \in \mathbb{H}[x]$ of degree n such that $h_S(c_i) = 0,\ i \in S$ and that $\{c_i \mid i \in S\}$ are the only roots (up to conjugacy class) of h_S in $\mathbb{H}$. Then $h_S(c_s) \neq 0$, and thus we can construct a quaternion polynomial g_S of degree n such that

$$g_S(c_\alpha) = \begin{cases} 0 & \alpha \in S, \\ 1 & \alpha = s, \end{cases}$$

as follows:

$$g_S(x) = h_S(c_s)^{-1} h_S(x).$$

Furthermore, we construct a quaternion polynomial f of degree at most n such that $f(c_i) = d_i$ for all $1 \le i \le n+1$ as follows:

$$f = \sum_{s=1}^{n+1} d_s g_S.$$

Finally, we show that f is unique. Suppose we have $f_1 \in \mathbb{H}[x]$ of degree $p_1 \le n$ such that $f_1 \ne f$ and that $f_1(c_i) = d_i$ for all $1 \le i \le n+1$, too. Then $f - f_1 \ne 0$ is of degree at most n. But $f - f_1$ has roots $c_1, \ldots, c_{n+1}$ which lie in $n+1$ conjugacy classes of $\mathbb{H}$, a contradiction. Therefore, f is unique. □

From above proof, we can see that it is impossible to construct the so-called Newton divided difference formula for quaternion polynomials. Next we extend the interpolation to quaternion polynomial matrices. Recall that the degree of a given $A \in \mathbb{H}[x]^{m\times n}$ is defined as

$$\deg A = \max\left\{\deg\left(A_{ij}\right) \mid 1 \le i \le m, 1 \le j \le n\right\}.$$

The following lemma estimates the upper bound of the degree of its Moore–Penrose inverse $A^\dagger$ (if it exists).

Lemma 5.5.3. *Let $A \in \mathbb{H}[x]^{m\times n}$ such that $A^\dagger$ exists. Then*

$$\deg A^\dagger \le (2m-1)\deg A.$$

Proof. By Theorem 5.4.8,

$$\deg A^\dagger \le \deg\left(A^*\left(AA^*\right)^{m-1}\right) \le \deg\left(A^{2m-1}\right) \le (2m-1)\deg A.$$

□

For $A = (A_{ij}) \in \mathbb{H}[x]$ and $c \in \mathbb{H}$, the evaluation of A at c can be defined as entrywise in a common sense, that is, $A(c) = (A_{ij}(c))$. One has to pay an attention that the evaluations of quaternion polynomials have some special rules as we explained at the beginning of this section.

Proposition 5.5.4. *Let $c_1, \cdots, c_{k+1} \in \mathbb{H}$ be pairwise non-conjugate and let $A_1, \cdots, A_{k+1} \in \mathbb{H}^{n\times m}$. Then there is a unique lowest degree matrix $A \in \mathbb{H}[x]^{n\times m}$ of degree $p \le k$, such that $A(c_i) = A_i$ for all $1 \le i \le k+1$.*

Proof. For any $1 \le n_1 \le n$ and $1 \le m_1 \le m$, by Proposition 5.5.2, there is a lowest degree polynomial $A_{n_1 m_1}(x)$ determined by the values $c_1, \cdots, c_{k+1}$ and $(A_1)_{n_1 m_1}, \ldots, (A_{k+1})_{n_1 m_1}$. In fact, for any $1 \le s \le k+1$, let $S = \{1, \cdots, k+1\} \setminus \{s\}$. Then

$$A_{n_1 m_1}(x) = \sum_{s=1}^{k+1} (A_s)_{n_1 m_1} g_S(x),$$

where $g_S(c_\alpha) = \begin{cases} 0 & \alpha \in S, \\ 1 & \alpha = s \end{cases}$. Since n_1 and m_1 are chosen randomly, the lowest degree matrix A that satisfies $A(c_i) = A_i$ for all $1 \le i \le k+1$ is determined by $A = (\sum_{s=1}^{k+1} A_s g_S)$.

Next we show that A is unique. Suppose $C \neq A$ of degree $p' \le p$ also satisfies $C(c_i) = A_i$ for all $1 \le i \le k+1$. Then for some $1 \le n_2 \le n$ and $1 \le m_2 \le m$, $(A-C)_{n_2 m_2} \neq 0$. But $(A-C)_{n_2 m_2}$, of degree at most $p \le k$, has roots $c_1, \ldots, c_{k+1}$ which lie in $k+1$ conjugacy classes of $\mathbb{H}$, a contradiction. Therefore, A is unique. □

Let $A \in \mathbb{H}[x]^{m \times n}$ such that $A^\dagger$ exists, and set $B = AA^*$. Let p be the largest integer such that $a_p \neq 0$. We can construct the sequence $A_0, \ldots, A_p$ as in Proposition 5.4.10. The next theorem presents the interpolation version of Leverrier–Faddev algorithm.

Theorem 5.5.5. *In the above setting, let* $k = (2m-1)\deg A$ *and* $c_1, \ldots, c_{k+1} \in \mathbb{R}$ *be* $k+1$ *distinct real numbers such that* $q_p(c_s) \neq 0$ *for any* $1 \le s \le k+1$. *Let* $S = \{1, \cdots, k+1\} \setminus \{s\}$. *Then*

$$A^\dagger = \sum_{s=1}^{k+1} A(c_s)^\dagger g_S$$

where

$$A(c_s)^\dagger = \frac{1}{q_p(c_s)} A(c_s)^* \left[B(c_s)^{p-1} - q_1(c_s) B(c_s)^{p-2} - \cdots - q_{p-1}(c_s) I \right]$$

and

$$g_S(c_\alpha) = \begin{cases} 0 & \alpha \in S, \\ 1 & \alpha = s. \end{cases}$$

Proof. It follows from Theorem 5.4.8, Propositions 5.4.10 and 5.5.4. □

The upper bound of degrees of $A^\dagger$ in Lemma 5.5.3 is not sharp. In fact, in many questions, one only needs to pick up a few real points. (see Example 5.6.1)

5.6 Implementations and Examples

The calculations of quaternions are very complicated and time-consuming. It is almost impossible to do some calculations for quaternion polynomials and quaternion polynomial matrices even for a small sized matrices by hand. There are only few quaternion packages in the computer algebra system Maple. But none of these has commands for quaternion polynomials and quaternion polynomial matrices. In [47], we developed a Maple package which includes all basic operations for quaternion polynomials and quaternion polynomial matrices. In particular, all the algorithms in this chapter were implemented. We give the following illustrative example:

Example 5.6.1. Let us consider the problem of determining the Moore–Penrose inverse of the following quaternion polynomial matrix:

$$A = \begin{pmatrix} 14x + 14 + 76\mathbf{i} + 70\mathbf{j} + 56\mathbf{k} & 56 - 28\mathbf{i} - 70\mathbf{j} + 70\mathbf{k} & 28\mathbf{j} - 56\mathbf{k} & 14x - 56 - 8\mathbf{i} - 14\mathbf{j} - 56\mathbf{k} \\ -2x - 2 - 43\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} & -8 + 4\mathbf{i} + 10\mathbf{j} - 10\mathbf{k} & -4\mathbf{j} + 8\mathbf{k} & -2x + 8 - 31\mathbf{i} + 2\mathbf{j} + 8\mathbf{k} \\ -3x - 3 + 3\mathbf{i} - 15\mathbf{j} - 12\mathbf{k} & -12 + 6\mathbf{i} + 15\mathbf{j} - 15\mathbf{k} & -6\mathbf{j} + 12\mathbf{k} & -3x + 12 + 21\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} \\ -4x - 4 + 4\mathbf{i} - 20\mathbf{j} - 16\mathbf{k} & -16 + 8\mathbf{i} + 20\mathbf{j} - 20\mathbf{k} & -8\mathbf{j} + 16\mathbf{k} & -4x + 16 + 28\mathbf{i} + 4\mathbf{j} + 16\mathbf{k} \end{pmatrix} \in \mathbb{H}^{4\times 4}[x].$$

From Lemma 5.5.3, we know that the upper bound of the degree of $A^\dagger$ is less than $(2m - 1)\deg A = (2 \times 4 - 1) \cdot 1 = 7$. In practice, we don't need to start from the upper bound. For this example, we may guess $\deg A^\dagger = 2$, and choose $c_1 = 0$ and $c_2 = 1$. Then using our Maple package, it is easy to do the following calculations:

$$A(c_1) = \begin{pmatrix} 14 + 76\mathbf{i} + 70\mathbf{j} + 56\mathbf{k} & 56 - 28\mathbf{i} - 70\mathbf{j} + 70\mathbf{k} & 28\mathbf{j} - 56\mathbf{k} & -56 - 8\mathbf{i} - 14\mathbf{j} - 56\mathbf{k} \\ -2 - 43\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} & -8 + 4\mathbf{i} + 10\mathbf{j} - 10\mathbf{k} & -4\mathbf{j} + 8\mathbf{k} & 8 - 31\mathbf{i} + 2\mathbf{j} + 8\mathbf{k} \\ -3 + 3\mathbf{i} - 15\mathbf{j} - 12\mathbf{k} & -12 + 6\mathbf{i} + 15\mathbf{j} - 15\mathbf{k} & -6\mathbf{j} + 12\mathbf{k} & 12 + 21\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} \\ -4 + 4\mathbf{i} - 20\mathbf{j} - 16\mathbf{k} & -16 + 8\mathbf{i} + 20\mathbf{j} - 20\mathbf{k} & -8\mathbf{j} + 16\mathbf{k} & 16 + 28\mathbf{i} + 4\mathbf{j} + 16\mathbf{k} \end{pmatrix}$$

and

$$A(c_2) = \begin{pmatrix} 28 + 76\mathbf{i} + 70\mathbf{j} + 56\mathbf{k} & 56 - 28\mathbf{i} - 70\mathbf{j} + 70\mathbf{k} & 28\mathbf{j} - 56\mathbf{k} & -42 - 8\mathbf{i} - 14\mathbf{j} - 56\mathbf{k} \\ -4 - 43\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} & -8 + 4\mathbf{i} + 10\mathbf{j} - 10\mathbf{k} & -4\mathbf{j} + 8\mathbf{k} & 6 - 31\mathbf{i} + 2\mathbf{j} + 8\mathbf{k} \\ -6 + 3\mathbf{i} - 15\mathbf{j} - 12\mathbf{k} & -12 + 6\mathbf{i} + 15\mathbf{j} - 15\mathbf{k} & -6\mathbf{j} + 12\mathbf{k} & 9 + 21\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} \\ -8 + 4\mathbf{i} - 20\mathbf{j} - 16\mathbf{k} & -16 + 8\mathbf{i} + 20\mathbf{j} - 20\mathbf{k} & -8\mathbf{j} + 16\mathbf{k} & 12 + 28\mathbf{i} + 4\mathbf{j} + 16\mathbf{k} \end{pmatrix}.$$

By the algorithm stated in Theorem 5.5.5, we calculate and obtain

$$A(c_1)^\dagger = A(0)^\dagger = \frac{1}{230175} \times$$
$$\begin{pmatrix} 140 - 560\mathbf{i} - 228\mathbf{j} - 342\mathbf{k} & 355 + 1730\mathbf{i} - 96\mathbf{j} + 81\mathbf{k} & -255 - 870\mathbf{i} + 126\mathbf{j} + 54\mathbf{k} & -340 - 1160\mathbf{i} + 168\mathbf{j} + 72\mathbf{k} \\ 276 + 88\mathbf{i} + 426\mathbf{j} - 382\mathbf{k} & 282 + 416\mathbf{i} - 93\mathbf{j} - 149\mathbf{k} & -252 - 276\mathbf{i} - 72\mathbf{j} + 204\mathbf{k} & -336 - 368\mathbf{i} - 96\mathbf{j} + 272\mathbf{k} \\ 32 + 16\mathbf{i} - 176\mathbf{j} + 292\mathbf{k} & -176 - 88\mathbf{i} + 68\mathbf{j} + 194\mathbf{k} & 96 + 48\mathbf{i} + 12\mathbf{j} - 204\mathbf{k} & 128 + 64\mathbf{i} + 16\mathbf{j} - 272\mathbf{k} \\ -140 - 122\mathbf{i} + 228\mathbf{j} + 342\mathbf{k} & -355 + 2021\mathbf{i} + 96\mathbf{j} - 81\mathbf{k} & 255 - 1176\mathbf{i} - 126\mathbf{j} - 54\mathbf{k} & 340 - 1568\mathbf{i} - 168\mathbf{j} - 72\mathbf{k} \end{pmatrix}$$

and

$$A(c_2)^\dagger = A(1)^\dagger = \frac{1}{230175} \times$$
$$\begin{pmatrix} 152 - 550\mathbf{i} - 244\mathbf{j} - 330\mathbf{k} & 289 + 1675\mathbf{i} - 8\mathbf{j} + 15\mathbf{k} & -219 - 840\mathbf{i} + 78\mathbf{j} + 90\mathbf{k} & -292 - 1120\mathbf{i} + 104\mathbf{j} + 120\mathbf{k} \\ 268 + 104\mathbf{i} + 406\mathbf{j} - 402\mathbf{k} & 326 + 328\mathbf{i} + 17\mathbf{j} - 39\mathbf{k} & -276 - 228\mathbf{i} - 132\mathbf{j} + 144\mathbf{k} & -368 - 304\mathbf{i} - 176\mathbf{j} + 192\mathbf{k} \\ 32 + 16\mathbf{i} - 160\mathbf{j} + 300\mathbf{k} & -176 - 88\mathbf{i} - 20\mathbf{j} + 150\mathbf{k} & 96 + 48\mathbf{i} + 60\mathbf{j} - 180\mathbf{k} & 128 + 64\mathbf{i} + 80\mathbf{j} - 240\mathbf{k} \\ -152 - 132\mathbf{i} + 244\mathbf{j} + 330\mathbf{k} & -289 + 2076\mathbf{i} + 8\mathbf{j} - 15\mathbf{k} & 219 - 1206\mathbf{i} - 78\mathbf{j} - 90\mathbf{k} & 292 - 1608\mathbf{i} - 104\mathbf{j} - 120\mathbf{k} \end{pmatrix}.$$

By Theorem 5.5.5, we have

$$\sum_{s=1}^{2} A\,(c_s)^{\dagger}\, g_S = A\,(0)^{\dagger}\,(1-x) + A\,(1)^{\dagger}\,x$$

$$= \frac{1}{230175}\times$$

$$\left(\begin{array}{cc} (12+10\mathbf{i}-16\mathbf{j}+12\mathbf{k})\,x+140-560\mathbf{i}-228\mathbf{j}-342\mathbf{k} & (-66-55\mathbf{i}+88\mathbf{j}-66\mathbf{k})\,x+355+1730\mathbf{i}-96\mathbf{j}+81\mathbf{k} \\ (-8+16\mathbf{i}-20\mathbf{j}-20\mathbf{k})\,x+276+88\mathbf{i}+426\mathbf{j}-382\mathbf{k} & (44-88\mathbf{i}+110\mathbf{j}+110\mathbf{k})\,x+282+416\mathbf{i}-93\mathbf{j}-149\mathbf{k} \\ (16\mathbf{j}+8\mathbf{k})\,x+32+16\mathbf{i}-176\mathbf{j}+292\mathbf{k} & (-88\mathbf{j}-44\mathbf{k})\,x-176-88\mathbf{i}+68\mathbf{j}+194\mathbf{k} \\ (-12-10\mathbf{i}+16\mathbf{j}-12\mathbf{k})\,x-140-122\mathbf{i}+228\mathbf{j}-342\mathbf{k} & (66+55\mathbf{i}-88\mathbf{j}+66\mathbf{k})\,x-355+2021\mathbf{i}+96\mathbf{j}-81\mathbf{k} \end{array}\right.$$

$$\left.\begin{array}{cc} (36+30\mathbf{i}-48\mathbf{j}+36\mathbf{k})\,x-255-870\mathbf{i}+126\mathbf{j}+54\mathbf{k} & (48+40\mathbf{i}-64\mathbf{j}+48\mathbf{k})\,x-340-1160\mathbf{i}+168\mathbf{j}+72\mathbf{k} \\ (-24+48\mathbf{i}-60\mathbf{j}-60\mathbf{k})\,x-252-276\mathbf{i}-72\mathbf{j}+204\mathbf{k} & (-32+64\mathbf{i}-80\mathbf{j}+80\mathbf{k})\,x-366-368\mathbf{i}-96\mathbf{j}+272\mathbf{k} \\ (48\mathbf{j}+24\mathbf{k})\,x+96+48\mathbf{i}+12\mathbf{j}-204\mathbf{k} & (64\mathbf{j}+32\mathbf{k})\,x+128+64\mathbf{i}+16\mathbf{j}-272\mathbf{k} \\ (-36-30\mathbf{i}+48\mathbf{j}-36\mathbf{k})\,x+255-1176\mathbf{i}-126\mathbf{j}-54\mathbf{k} & (-48-40\mathbf{i}+64\mathbf{j}-48\mathbf{k})\,x+340-1568\mathbf{i}-168\mathbf{j}-72\mathbf{k} \end{array}\right).$$

It is easy to verify that $\sum_{s=1}^{2} A\,(c_s)^{\dagger}\, g_S$ satisfies the four defining relations of the Moore–Penrose inverse. Therefore it is the Moore–Penrose inverse of A.