Chapter 1
Introduction

In this chapter, first we provide a brief overview of some of the advances that have been made recently in the theory of infinite matrices and their applications. Then we include a summary of the contents of each chapter.

Infinite matrices have applications in many branches of classical mathematics such as infinite quadratic forms, integral equations, and differential equations. As illustrated in Chap. 8, this topic has applications in other sciences besides mathematics, as well. A review of some of the topics of this monograph was recently undertaken by Shivakumar and Sivakumar [111]. In this monograph, apart from elaborating on some of the topics that are discussed in that review, we include other interesting topics such as quaternion matrices and infinite dimensional extensions of certain positivity classes of matrices.

Gaussian elimination, the familiar method for solving systems of finitely many linear equations in finitely many unknowns, has been around for over two hundred years. Unlike the matrix methods, matrices were not used in the early formulations of Gaussian elimination, until the mid-twentieth century.

The word "matrix" was coined by James Sylvester in 1850. Roughly speaking, a matrix over a field F is a two fold table of scalars each of which is a member of F. A simple example of an infinite matrix is the matrix representing the derivative operator which acts on the Taylor series of a function. In the seventeenth and eighteenth centuries, infinite matrices arose from attempts to solve differential equations using series methods. This method would lead to a system of infinitely many linear equations in infinitely many unknowns. Therefore, a main source of infinite matrices is solutions of differential equations.

As mentioned in the preface, a general theory for infinite matrices originated with Henri Poincare in 1844. Infinite determinants were first introduced into analysis in 1886 in the discussion of the well-known Hill's equation. By 1893, Helge Von Koch established standard theorems on infinite matrices. In 1906, David Hilbert attracted the attention of other mathematicians to the subject by solving a Fredholm integral equation using infinite matrices. Since then, many theorems, fundamental

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to the theory of operators on function spaces were discovered although they were expressed in special matrix terms. In 1929 John Von Neumann showed that an abstract approach is more powerful and preferable rather than using infinite matrices as a tool to the study of operators. Hence, the modern operator theory stems from the theory of infinite matrices. Despite this, the infinite matrix theory remains a subject of interest for its numerous and natural appearances in mathematics as well as in other sciences. For example, in mathematical formulation of physical problems and their solutions, infinite matrices appear more naturally than finite matrices. Some of the recent applications include flow of sap in trees, leakage of electricity in coaxial cables (attenuation problem), cholesterol problem in arteries, and simultaneous flow of oil and gas.

Let us summarize the contents of this monograph. In Chap. 2, we consider the notion of diagonal dominance of complex matrices and discuss the various results that guarantee invertibility of such matrices. Possibly reducible matrices satisfying a chain condition are discussed. Recent results on specially structured matrices like tridiagonal matrices and matrices satisfying certain sign patterns are reviewed.

Given an infinite matrix $A = (a_{i,j})$, $i, j \in \mathbb{N}$, a space X of infinite sequences, and $x = (x_i)$, $i \in \mathbb{N}$, we define Ax by $(Ax)_i = \sum_{j=1}^{\infty} a_{ij}x_j$ provided this series converges for each $i \in \mathbb{N}$, and define the domain of A as $\{x \in X : Ax \text{ exists and } Ax \in X\}$. We define an eigenvalue of A to be any scalar λ for which $Ax = \lambda x$ for some nonzero x in the domain of A. A matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \text{ for all } i.$$

Let $A = (a_{ij})$, $i, j \in \mathbb{N}$ be an infinite matrix. The linear differential system

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{\infty} a_{ij}x_j(t) + f_i(t),\ t \geq 0,\ x_i(0) = y_i,\ i = 1, 2, \ldots,$$

is of considerable theoretical and applied interest. In particular, such systems occur frequently in the theory of stochastic processes, perturbation theory of quantum mechanics, degradation of polynomials, infinite ladder network theory, etc. Arley and Brochsenius [3], Bellman [7], and Shaw [100] are the notable mathematicians who have studied this problem. In particular, if A is a bounded operator on l^1, then the convergence of a truncated system has been established. For instance, Shivakumar, Chew and Williams [118] provide an explicit error bound for such a truncation.

Diagonally dominant infinite matrices occur in solutions of elliptic partial differential equations as well as solutions of second order linear differential equations. The eigenvalue problem for particular infinite matrices including diagonally dominant matrices is studied by Shivakumar, Williams and Rudraiah [119]. A discussion of these topics is presented in Chap. 3.

In Chap. 4, first we provide a brief review of generalized inverses of matrices with real or complex entries followed by a discussion on the Moore–Penrose inverses of operators between Hilbert spaces. Certain non-uniqueness results for generalized inverses of infinite matrices are reviewed later.

Chapter 5 considers the case of generalized inverses of matrices whose entries are quaternionic polynomials. After a discussion on the theoretical aspects, some algorithmic approaches are proposed. The contents report recent results in this area.

A vast literature exists for M-matrices for which more than fifty characterizations are given. Relatively little is known for the case of infinite dimensional spaces. Chapter 6 presents a review of M-operator results obtained recently, including some results on extensions of two other matrix classes quite well known in the theory of linear complementarity problems.

Infinite linear programming problems are linear optimization problems where, in general, there are infinitely many (possibly uncountable) variables and constraints related linearly. There are many problems arising from the real world situations that can be modelled as infinite linear programs. Some examples include the bottleneck problem of Bellman in economics, infinite games, and continuous network flow problems [2]. A finite dimensional approximation scheme for semi-infinite linear programming problems is presented in Chap. 7, where an application to obtaining approximate solutions to doubly infinite programs is considered.

The importance of eigenvalue problems concerning the Laplacian is well documented in classical and modern literature. Finding the eigenvalues for various geometries of the domains has posed many challenges for solution methods, which have included infinite systems of algebraic equations, asymptotic methods, integral equations, and finite element methods. The eigenvalue problems of the Laplacian is represented by Helmholtz equations, Telegraph equations, or the equations of the vibrating membrane and is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0 \text{ in } D, \text{ and } u = 0 \text{ on } \partial D,$$

where D is a plane region bounded by smooth curve ∂D. The eigenvalues k_n and corresponding eigenfunctions u_n describe the natural modes of vibration of a membrane. The eigenvalues of the Laplacian provide an explanation for the various cases when the shape of a drum cannot be determined just by knowing its eigenvalues. To hear the shape of a drum is to infer information about the shape of the drumhead from the sound it makes, i.e., from the list of basic harmonics, via the use of mathematical theory. In 1964, John Milnor with the help of a result of Ernst Witt showed that there exist two Riemannian flat tori of dimension 16 with the same eigenvalues but different shapes. However, the problem in two dimensions remained open until 1992, when Gordon, Webb, and Wolpert showed the existence of a pair of regions in the plane with different shapes but identical eigenvalues. The regions are non-convex polygons. Chapter 8 provides more information on this intriguing problem, among other interesting applications.