Sliding Mode Control with State Derivative Feedback in Novel Reciprocal State Space Form

Yuan-Wei Tseng

Abstract This chapter introduces a novel reciprocal state space (RSS) system form. The concepts and the need of RSS form are comprehensively reviewed and explained. It shows that in RSS form, control design using state derivative related feedback is straightforward. Sliding mode control (SMC) is a nonlinear control design method and a highly active area of research. Finite-time convergence due to discontinuous control law, low sensitivity to plant parameter uncertainty and/or external perturbation, and greatly reduced-order modeling of plant dynamics are the main advantages of SMC. In the past, the majority of available SMC algorithms and the corresponding switching conditions involved only state related variables. In this chapter, the advantages of both RSS and SMC are combined to develop sliding mode control in RSS form so that state derivate related feedback can be systematically applied in SMC to handle wider range of control problems. To provide the theoretical foundation, stability analysis in RSS form is first reviewed. Next, novel switching function and approaching condition based on the derivative of sliding surface are proposed to carry out SMC design approach in RSS form with considerations of system uncertainty and disturbance. In addition, algorithm of finding upper bound of system uncertainty is developed for robustness analysis. To verify the proposed design algorithms, numerical examples are provided. Finally, conclusions are drawn.

Keywords Reciprocal state space (RSS) form · State derivate related feedback · Sliding mode control · Nonlinear control

1 Introduction

In recent years, robust control is one of the most popular topics in control area. One of the famous methods is the so-called sliding mode control (SMC) or variable structure control (VSC) [33, 35, 36] which is a nonlinear control and has been proven as an

Y.-W. Tseng (✉)
Department of Electrical Engineering, I- Shou University, Kaohsiung 84001, Taiwan, ROC
e-mail: yuanwei@isu.edu.tw

S. Vaidyanathan and C. Volos (eds.), *Advances and Applications in Nonlinear Control Systems*, Studies in Computational Intelligence 635,
DOI 10.1007/978-3-319-30169-3_9

effectively robust control technology with many practical applications. Through this chapter, the name of sliding mode control (SMC) is used for unification. The main idea of sliding mode control is to design a controller rendering the trajectory of states trapped on a predetermined sliding surface and remained on it thereafter. Sliding mode control utilizes a high-speed switching control law to drive the state trajectory staying on this sliding surface for all subsequence time such that the robust stability of the system is assured. In the present, sliding mode control is a highly active area of research. Finite-time convergence due to discontinuous control law, low sensitivity to plant parameter uncertainty and/or external perturbation, and greatly reduced-order modeling of plant dynamics are the main advantages of it. Therefore, based on SMC, many works in state space form have been developed [7, 11, 12, 20, 39].

The majority of available SMC algorithms for linear systems is developed in state space form and applies state feedback. However, in many applications, people face the problems that either cannot be handled in state space form or cannot directly apply state feedback in designs. More details are given in next section. To provide supplementary design algorithms of state derivative feedback in state space form, a direct state derivative feedback control scheme was developed in "Reciprocal state space" (RSS) form [21–26] by the author of this chapter.

In this chapter, the algorithms of SMC design utilizing state derivative feedback in RSS form are introduced [27, 28, 40]. The main purpose of this chapter is to combine the advantages of both RSS and SMC so that state derivate feedback can be systematically applied in SMC designs to handle wider range of control problems.

The rest of chapter is organized as follows. "Reciprocal state space" (RSS) form is introduced and reviewed in Sect. 2. In Sect. 3, the design approach of sliding mode control with state derivative feedback in RSS form is described and the method for finding the upper bound of system uncertainty in RSS form is also developed. The contribution of this chapter, discussion and suggested future research are given in the conclusion section.

2 Reciprocal State Space (RSS) Form and State Derivative Feedback Control Designs

The concepts and the needs of RSS form are first comprehensively reviewed and explained in this section as follows.

In general, a dynamic linear continuous time invariant system using state variables with physical meanings can be naturally expressed in the following equation under the names of generalized state space form [38] or descriptor form [42] or singular system form [2].

$$E\dot{x} = Fx + Nu \tag{1}$$

where $x_{n\times 1}$ and $u_{m\times 1}$ are state vector and control vector, respectively, and $E_{n\times n}$, $F_{n\times n}$ and $N_{n\times m}$ are known constant system matrices. Controllability and observability of generalized state space systems have been investigated in [2, 42]. The following is the characteristic equation of open loop generalized state space system in (1).

$$\det(sE - F) = 0 \tag{2}$$

The degree r of characteristic equation in (2) is the number of system's finite eigenvalues while $n - r$ is the number of system's eigenvalues at infinity [5].
If E in (1) is nonsingular, the system has no eigenvalue at infinity but can have zero eigenvalues. Such system can be expressed in the following standard state space system form:

$$\dot{x} = E^{-1}Fx + E^{-1}Nu = \bar{A}x + \bar{B}u. \tag{3}$$

For state space system, state derivative vector can be explicitly expressed in terms of state vector and control input vector. Most of control algorithms developed for state space systems are related to state feedback such as full state feedback, state related output feedback and estimated state feedback when estimators are implemented. However, in many applications, the sensors directly measure state derivatives rather than states. For instance, accelerometers [9] in micro and nano-electro-mechanical systems (M/NEMS) and structural applications [10, 26] are such cases because acceleration signals can only be modeled as state derivatives [6, 10, 26]. Consequently, abundant control algorithms with state related feedback developed in standard state space form cannot be readily applicable in this situation. Additional integrators which may increase the cost and complexity of the implementation are needed. Mathematically speaking, state derivative related feedback designs cannot be carried out as straightforward as state related feedback for systems expressed in standard state space form. For example, if we apply the following full state derivative feedback control law

$$u = -K\dot{x} \tag{4}$$

to the state space system in (3), the closed loop system becomes

$$\dot{x} = (I + \bar{B}K)^{-1}\bar{A}x \tag{5}$$

In (5), since gain K is inside an inverse matrix $(I + \bar{B}K)^{-1}$ which is further coupled with the open loop system matrix $\bar{A}$ by multiplication, it is obvious that advanced mathematics is needed to design gain K in (4). Therefore, in the past, the developed algorithms of state derivative related feedback for systems in state space form were very few and rarely used to control the system alone [6]. In a word, standard state space system in (1) is the best system form for open loop systems without poles at infinity in designing state related feedback control algorithms. However, standard state space system is not the most suitable form to develop state derivative related feedback control algorithms and cannot handle the systems with open loop poles at infinity.

If E in (1) is singular, the system has poles at infinity and is called generalized state space system. In the past, the majority of control designs for system with poles at infinity were directly developed in the generalized state space form in (1).

Extensive applications of generalized state space systems arise in many areas of engineering such as electrical networks [16], aerospace systems [1], smart structures [26, 41] and chemical processes [18]. Generalized state space systems also exist in other areas such as the dynamic Leontief model for economic production sectors [15] and biological complex systems [14]. A comprehensive review is available in Yeh et al. [41]. In this paper, generalized state space system is used as the name to represent such systems. In previous studies, generalized state space systems are further categorized as impulse-free ones [3] and with impulse mode ones in analysis. To explain that, singular value decomposition (SVD) is performed on the original generalized state space system. This transfers the original system to the following form.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} u \tag{6}$$

where I_r is an $r \times r$ identity matrix.

When F_{22}^{-1} in (6) exists, the system is impulse free, one can further obtain

$$q_2 = -F_{22}^{-1} F_{21} q_1 - F_{22}^{-1} N_2 u \tag{7}$$

Substituting (7) to (6), we have

$$\dot{q}_1 = (F_{11} - F_{12} F_{22}^{-1} F_{21}) q_1 + (N_1 - F_{12} F_{22}^{-1} N_2) u \tag{8}$$

To q_1, (8) is a standard state space system, if it is controllable, one can design a state feedback control law $u = -kq_1$ to control (8). Consequently, q_2 is stabilized through the coupling equation in (7). Therefore, control designs for impulse-free generalized state space systems can be handled and have been an active area in research. Obviously, applying state feedback methods only can control part of the states while the rest of states are just stabilized for impulse-free generalized state space systems. Therefore, the closed loop performance is limited.

For impulse-free generalized state space systems, the available control design algorithms which are usually carried out in augmented systems and require feedbacks of both state and state derivative variables [3, 4, 13, 19, 37] are much more complex than algorithms for the standard state space systems. Consequently, there are difficulties for engineers without strong mathematical background to apply those sophisticated control algorithms.

When F_{22}^{-1} in (6) does not exist, the generalized state space system has impulse mode. In this case, further investigations of impulse controllable and the impulse mode elimination [3] have to be analyzed in control designs. Therefore, this kind of generalized state space system is considered to be difficult in control designs.

As mention before, when the state derivative coefficient matrix E in (1) is nonsingular, the system can be expressed in standard state space form in (3). If the system is controllable, applying state feedback alone is sufficient to control the system. Similarly, it is natural to ask if applying state derivative feedback alone is sufficient to control the system when the state coefficient matrix F in (1) is nonsingular. To answer this question and to provide supplementary design algorithms of state derivative feedback, a direct state derivative feedback control scheme was developed using the "Reciprocal State Space" (RSS) methodology by the author of this chapter as follows.

$$x = F^{-1}E\dot{x} - F^{-1}Nu = A\dot{x} + Bu \tag{9}$$

For above reciprocal state space (RSS) systems, state vector can be explicitly expressed in terms of state derivative vector and control input vector. The controllability and observability analyses for system in RSS form have been investigated in Tseng et al. and Tseng [24, 25]. It shows that they turn out to be the same as their counterparts in state space form. After apply full state derivative feedback control law in (4), the closed loop system becomes

$$x = (A - BK)\,\dot{x} = A_c\dot{x} \tag{10}$$

The concept of RSS is based on a fact that for a nonsingular matrix, the eigenvalues of its inverse matrix must be the reciprocals of its eigenvalues. Therefore, the eigenvalues of A_c in (10) are the reciprocals of the closed loop system poles. To address this nature, the name of reciprocal state space form was given. If state derivative feedback gain K can be designed such that real parts of all eigenvalues of A_c in (10) are strictly negative, the closed loop system in RSS form in (10) can achieve globally asymptotically stable. When a controllable system has no open loop pole at zero, it can be expressed in RSS form to carry out state derive related feedback control designs.

It also shows that state derivative feedback designs can be carried out as straightforward in RSS form as state feedback designs in standard state space form in pole placement, eigenstructure assignment, and linear quadratic regulator (LQR) designs [21–26].

The following is an example for quick understanding why expressing system in RSS form and applying state derivative feedback can easily accomplish control designs for some systems that were once thought difficult to be controlled. For the following generalized state space system with impulse mode [13], its state coefficient matrix is invertible. Therefore, the open loop system has no open loop pole at zero and the system can be expressed in RSS form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0.5 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

Suppose that we want to place the closed loop poles at -2, -2.5, -5. Using state feedback control laws in generalized state space form cannot place all of the desired closed loop poles. However, one can first express the system in RSS form as follows.

$$x = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -1 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} u = A\dot{x} + Bu$$

Then apply the state derivative feedback law $u = -K\dot{x}$ to assign -0.5, -0.4 and -0.2 (the reciprocals of -2, -2.5 and -5, respectively) as the eigenvalues of matrix $(A - BK)$. Using "place" command of *Matlab*, one can easily get $K = \begin{bmatrix} -1.63 & -0.2 & 0.02 \end{bmatrix}$. Therefore, for the systems without open loop pole at zero, including difficult systems to be controlled such as generalized state space systems with impulse mode in this example, they can be expressed in RSS form in (9) and properly controlled by applying state derivative feedback alone. Usually, handling the same problem in generalized state space system form, both state feedback and state derivative feedback are needed [13].

Put RSS form into consideration, to streamline the design processes and keep the controller as compact as possible, the following control design procedure is suggested: For an open loop system, if it has no pole at infinity, one can express the system in state space form and apply state related feedback to control it. If it has poles at infinity but has no pole at zero, one can express the system in RSS form and apply state derivative related feedback to control it. If it has neither pole at infinity nor pole at zero, based on the type of available sensors (state related sensors or state derivative related sensors), one can make choice between state space form and RSS form to carry out control design. Generalized state space system form and control laws applying both state feedback and state derivative feedback might be considered as the last resort to handle the system with poles at both infinity and zero. In a nutshell, RSS form fills in the gap between standard state space system and generalized state space system and provides additional flexibility in control designs.

3 Sliding Mode Control with State Derivative Feedback in Reciprocal State Space Form

This section explains how sliding mode control is carried out in novel reciprocal state space (RSS) form with state derivative feedback.

Beginning with Lyapunov stability analysis in RSS form in Sects. 3.1 and 3.2 is an introduction to the proposed novel approach condition suitable for systems in RSS form. SMC design approach for a simple nominal system in RSS form and numerical examples to verify the proposed novel approach condition are presented in Sect. 3.3. Section 3.4 explains the process of finding the upper bound of system uncertainty and SMC design approach for RSS systems with both uncertainty and disturbance. Numerical example is also provided to verify the proposed methods.

3.1 Lyapunov Stability Analysis in RSS Form

Since Lyapunov stability is the fundamental of sliding mode control, in this subsection Lyapunov stability analysis in RSS form is presented.

For a linear time invariant system, it is globally asymptotically stable if the real parts of all system poles are strictly negative. Therefore, such system must have no pole at infinity or pole at zero. Consequently, a globally asymptotically stable system can be expressed in both state space form and RSS form as follows.

$$\dot{x} = \bar{A}x \tag{11}$$

$$x = A\dot{x} \tag{12}$$

where $\bar{A} = A^{-1}$ and both A and $\bar{A}$ are nonsingular. Furthermore, the eigenvalues of A are the reciprocals of the eigenvalues of $\bar{A}$ which are the system poles. If the real parts of all eigenvalues of $\bar{A}$ are strictly negative, so are all eigenvalues of A. Based on the above discussion, the following Lyapunov equation can also test the stability of RSS systems in (12).

$$PA + A^T P = -Q \tag{13}$$

The solution of P in Lyapunov equation (13) must be symmetric positive definite to ensure RSS system matrix A is globally asymptotically stable when a symmetric and positive matrix Q is used.

3.2 Novel Approaching Condition for SMC Designs in RSS Form

In general, design of sliding mode control consists of two parts. The first part involves the selection of an appropriate sliding surface and the second part is the design of a controller to meet the approaching condition. To provide the fundamentals of SMC, approaching condition and sliding mode are briefly reviewed as follows.

Approaching condition can force the system toward the predetermined sliding surface $s(t)$ which can stabilize the system (some studies call this “reaching condition” or “hitting condition”), and we usually consider it as follows [8, 33].

$$s^T(t) \cdot \dot{s}(t) < 0 \tag{14}$$

When the system in the predetermined sliding surface, the system will remain in the neighborhood of sliding surface therefore slip toward the target with any external disturbance.

There is a simple SMC method’s moving trajectory shown in Fig. 1. From $t=0$, the state x starts to approach the sliding surface $s = 0$ and lands on the sliding surface at

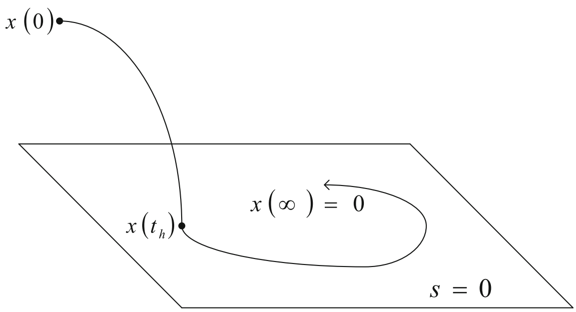

Fig. 1 The state trajectory of SMC

the finite time t_h. When $t > t_h$, this state remains on the sliding surface, and moves toward the equilibrium point $x = 0$ (i.e. $x(\infty) \to 0$).

According to the matrix sizes specified in (1), suppose that the sliding surface $s(t)$ is selected by

$$s(t) = Cx(t) = 0 \tag{15}$$

where $s \in \Re^{m\times 1}$ and $c \in \Re^{m\times n}$.

Approaching condition is briefly explained as follows. Define a Lyapunov function candidate based on the sliding surface $s(t)$ as follows.

$$V = \frac{1}{2}s^T(t)s(t) \tag{16}$$

The derivative of V with respect to time becomes

$$\dot{V} = s^T(t) \cdot \dot{s}(t) \tag{17}$$

For SMC designs in RSS form with state derivative related feedback laws, given a positive constant α, the following novel approaching condition is proposed.

$$\dot{V} = s^T(t) \cdot \dot{s}(t) < -\alpha \|\dot{s}\| < 0 \tag{18}$$

where $\| \|$ denotes norm in this chapter.

Detailed discussion will be given in the following subsections.

3.3 Sliding Model Control Design for Nominal System in RSS Form

In this subsection, sliding mode control design for nominal system in RSS form without any system uncertainty is presented for readers to easily understand the fundamental of SMC design approach carried out in RSS form. We first consider the RSS system described as follows:

$$x\left(t\right)=A\dot{x}\left(t\right)+Bu\left(t\right)+d\left(t\right) \tag{19}$$

where $x\left(t\right)\in\Re^{n}$, $u\left(t\right)\in\Re^{m}$ and $d\left(t\right)\in\Re^{n\times1}$ are states, control inputs, and external disturbance respectively. Assuming that the pair (A,B) is known and their dimensions are $A\in\Re^{n\times n}$ and $B\in\Re^{n\times m}$.

The following assumptions are applied in this chapter.

- The nominal RSS linear system is unstable.
- The pair (A,B) is controllable; $Rank\left[\,B\ \ AB\ \ A^{2}B\ \cdots\ A^{n-1}B\,\right]=n$.
- The $d(t)$ is a matched external disturbance as follows.

$$d\left(t\right)=Bd_{r}\left(t\right) \tag{20}$$

where $d_{r}\left(t\right)\in\Re^{m\times1}$ and has an upper bound δ such that $\|d_{r}\left(t\right)\|\leq\delta$.

So, (19) can be rewritten as

$$x\left(t\right)=A\dot{x}\left(t\right)+B\left(u\left(t\right)+d_{r}\left(t\right)\right) \tag{21}$$

The sliding face to be selected is

$$s\left(t\right)=Cx\left(t\right)=0 \tag{22}$$

3.3.1 Selecting Sliding Surface with Modified Transfer Matrix Method in RSS Form

In this subsection, we present a method to select a sliding surface for developing a sliding mode controller for the system in RSS form (19). The proposed method is modified from the popular transfer matrix method [34].

If matrix B is partitioned into

$$B=\begin{bmatrix}B_{1}\\B_{2}\end{bmatrix}, \tag{23}$$

where B_1 is $(n-m)\times m$ and B_2 is $m\times m$.

One can define the following transfer matrix

$$T=\begin{bmatrix}I_{(n-m)\times(n-m)} & -B_{1}B_{2}^{-1}\\0_{m\times(n-m)} & I_{m\times m}\end{bmatrix}, \tag{24}$$

such that $T\cdot B=\left[\,0\ \ B_{2}\,\right]^{T}$.

Please note that for a controllable system, one can always obtain a B matrix with an invertible sub-matrix B_2 by properly define the state variables and consequently obtain T.

Applying the following transfer,

$$q = Tx \tag{25}$$

(21) is transferred to

$$q_1 = A_{11}\dot{q}_1 + A_{12}\dot{q}_2 \tag{26}$$

$$q_2 = A_{21}\dot{q}_1 + A_{22}\dot{q}_2 + B_2u + B_2 \cdot d_r \tag{27}$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \begin{bmatrix} R^{n-m} \\ R^m \end{bmatrix}, TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{28}$$

Now, the sliding surface can be expressed as follows.

$$s = Cx = CT^{-1}q = \begin{bmatrix} S_1 & S_2 \end{bmatrix} q = \begin{bmatrix} S_2k & S_2 \end{bmatrix} q = S_2 \begin{bmatrix} k & I_m \end{bmatrix} q = 0 \tag{29}$$

where $S_1 \in \Re^{m\times(n-m)}$, $S_2 \in \Re^{m\times m}$ and $k \in \Re^{m\times(n-m)}$.

In (29), S_2 can be considered as any square matrix, and if we select S_2 as an identity matrix I_m, we have sliding surface as follows.

$$s = \begin{bmatrix} k & I_m \end{bmatrix} q = kq_1 + q_2 = \begin{bmatrix} k & I_m \end{bmatrix} Tx = 0. \tag{30}$$

Solving for q_2 with (30), we have

$$q_2 = -kq_1 \tag{31}$$

Taking derivative of both sides of (31), we obtain

$$\dot{q}_2 = -k\dot{q}_1 \tag{32}$$

Substituting (32) into (26), we have:

$$q_1 = (A_{11} - A_{12}k)\,\dot{q}_1 \tag{33}$$

If (A_{11}, A_{12}) is controllable, designing k in (33) is just a pole placement problem with full state-derivative feedback in RSS framework. Note that k in (33) should be designed such that the eigenvalues of $(A_{11} - A_{12}k)$ are equal to the reciprocal of the desired closed loop poles. After k is designed, the sliding surface in (30) is obtained.

3.3.2 Sliding Mode Control Design in RSS Form

This subsection introduces a SMC control law to make the approaching condition ($s^T(t) \cdot \dot{s}(t) < 0$) happen so that the system in (21) is guaranteed to reach and maintain on the sliding surface consequently.

After sliding surface is selected, we have to find the equivalent control $u_{eq}(t)$ which is the control law to let the system operates in the sliding mode.
Substituting (19) into (22), we have

$$s(t) = Cx(t) = C[A\dot{x}(t) + Bu(t) + d(t)] = 0 \tag{34}$$

If we let $u(t) = u_{eq}(t)$ in (34), the equivalent control can be found as

$$u_{eq}(t) = -(CB)^{-1}[CA\dot{x}(t) + Cd(t)] \tag{35}$$

Here u_{eq} is related to state derivative $\dot{x}$. Therefore, state derivative signals can be directly used in SMC design. Physically, the equivalent control $u_{eq}(t)$ cannot obtain the sliding motion if the initial state is not on the sliding surface. An ideal control law for the RSS system in (19) to generate the approaching condition of sliding mode is proposed as follows.

$$u(t) = -(CB)^{-1}CA\dot{x}(t) - (CB)^{-1} \cdot (\gamma + \alpha) \cdot sign(\dot{s}(t)) \tag{36}$$

where γ and α are all positive scalars such that $\|\gamma\| = \|C\| \cdot \|B\| \cdot \delta > \|CBd_r(t)\|$ and $\alpha > 0$.

The matrix CB is nonsingular and $sign(\dot{s}_i)$ is a novel switching function proposed as follows.

$$sign(\dot{s}_i) = \begin{cases} 1 & \dot{s}_i > 0 \\ 0 & \dot{s}_i = 0 \\ -1 & \dot{s}_i < 0 \end{cases} \tag{37}$$

Note that $sign(\dot{s}_i)$ is not a function of the sliding surface, but a function of the derivative of the sliding surface.

Proof Substituting (19) and (36) into sliding surface (22), we get the following result.

$$\begin{aligned} &s(t) = \\ &\{CA\dot{x}(t) + CB[-(CB)^{-1}CA\dot{x}(t) - (CB)^{-1}(\gamma + \alpha) \cdot sign(\dot{s}(t))] + Cd(t)\} \end{aligned} \tag{38}$$

Taking transposes and multiplying $\dot{s}(t)$ on both sides of (38) to get the equation of approaching condition, we have

$$
\begin{aligned}
s^T(t)\cdot\dot{s}(t) &= \{CA\dot{x}(t) + CB[-(CB)^{-1}CA\dot{x}(t) - (CB)^{-1}(\gamma+\alpha)\cdot sign(\dot{s}(t))] \\
&\quad + CBd_r(t)\}^T\dot{s}(t) \\
&= \left[(CBd_r)^T\dot{s}(t) - (\gamma+\alpha)^T\cdot\|\dot{s}(t)\|\right] \\
&= -\alpha\cdot\|\dot{s}(t)\| - \gamma\cdot\|\dot{s}\|\left(1 - \frac{(CBd_r)^T\dot{s}(t)}{\gamma\cdot\|\dot{s}(t)\|}\right)
\end{aligned} \tag{39}
$$

Since $-1 < \frac{(CBd_r)^T\dot{s}(t)}{\gamma\cdot\|\dot{s}(t)\|} < 1$, consequently, $\left(1 - \frac{(CBd_r)^T\dot{s}(t)}{\gamma\cdot\|\dot{s}(t)\|}\right) > 0$. We can conclude the following result.

$$
s^T(t)\cdot\dot{s}(t) = -\alpha\cdot\|\dot{s}(t)\| - \gamma\cdot\|\dot{s}(t)\|\left(1 - \frac{(CBd_r)^T\dot{s}(t)}{\gamma\cdot\|\dot{s}(t)\|}\right) < -\alpha\cdot\|\dot{s}(t)\| < 0 \tag{40}
$$

Therefore, applying the ideal controller in (36), the approaching condition ($s^T(t)\cdot\dot{s}(t) < 0$) holds. Consequently, the motion in the sliding mode is asymptotically stable. However, the ideal controller in (36) which using "*sign*" function may cause "Chattering Phenomenon". To avoid this problem, "*sign*" function is replaced by a novel "*sat*" saturation function in the modified control law given as follows.

$$
u(t) := -(CB)^{-1}CA\dot{x}(t) - (CB)^{-1}(\gamma+\alpha)sat(\dot{s}(t), \varepsilon) \tag{41}
$$

where *"sat"* is a novel saturation function to handle the switching as follows.

$$
sat(\dot{s}_i, \varepsilon) = \begin{cases} 1 & \dot{s}_i > \varepsilon \\ \frac{\dot{s}_i}{\varepsilon} & |\dot{s}_i| \le \varepsilon \\ -1 & \dot{s}_i < -\varepsilon \end{cases} = \begin{cases} sign(\dot{s}_i) & |\dot{s}_i| > \varepsilon \\ \frac{\dot{s}_i}{\varepsilon} & |\dot{s}_i| \le \varepsilon \end{cases} \tag{42}
$$

Here ε is a small positive value as the bound of the differential sliding surface $\dot{s}$.

$$
|\dot{s}| \le \varepsilon \tag{43}
$$

Although the control law (41) cannot completely eliminate the external disturbance, it still can reduce the influence of the external disturbance and can ensure the convergence of states in a boundary layer. It is still worth to avoid "Chattering Phenomenon" by paying the price of losing small accuracy.

One may wonder that if $\dot{s}$ is just bounded inside the differential sliding layer $|\dot{s}| \le \varepsilon$, can the amplitude of the sliding surface s keeps increasing as time goes by and finally become diverged? The answer is negative because when $\dot{s} = C\dot{x}$ is bounded, so is $\dot{x}$ due to the fact that C is a constant matrix. When $\dot{x}$ is bounded, from the system equation in (19) and controller in (41), x which can be expressed in term of $\dot{x}$ must be bounded, too. Consequently, $s = Cx$ must also be bounded. Similarly, through the system constraint in (19), when the approaching condition does not happen inside the differential sliding layer of $|\dot{s}| \le \varepsilon$, both $|s|$ and $|x|$ will be increased, so will $|\dot{s}|$ and $|\dot{x}|$. When $\dot{s}$ finally reach to the condition of $|\dot{s}| > \varepsilon$, the controller will switch to (36) to push the system back to the differential sliding layer of $|\dot{s}| \leq \varepsilon$. In this manner, the controller can keep the $\dot{s}$ inside the differential sliding layer of $|\dot{s}| \leq \varepsilon$ in steady state. Consequently, through the system constraint in (19), s and x should also be bounded in steady state.

Numerical Example 1

Theoretically, if no external disturbance is considered in (19), the system should be driven toward the sliding surface and stuck on it when SMC law is applied. In the other word, one should obtain $s(\infty) \to 0$ in simulation. The following is an example to verify that the proposed SMC algorithm can achieve $s(\infty) \to 0$ for RSS systems without external disturbance. The system matrices are given as follows.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since A is singular, SMC design cannot be directly carried out in standard state space form. The initial condition is given as:

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The first step is to select the sliding surface by applying the presented transfer matrix method. If pole at -2 is selected, the corresponding sliding surface is found to be $s = \begin{bmatrix} 1.5 & 1 \end{bmatrix} x$. The second step is to design the controller in (41). The following parameter are used in the simulation $\varepsilon = 0.5$, $\gamma = 0$, and $\alpha = 5$.

Figures 2, 3 and 4 show the time responses of states, sliding surface, and control effort, respectively. In Fig. 2, we find that the trajectories of $x_1(t)$ and $x_2(t)$ are asymptotically stable. In Fig. 3, the sliding surface response does converge to zero when no external disturbance is considered. Therefore, the proposed approach condition in (18) and control law in (41) are successfully verified.

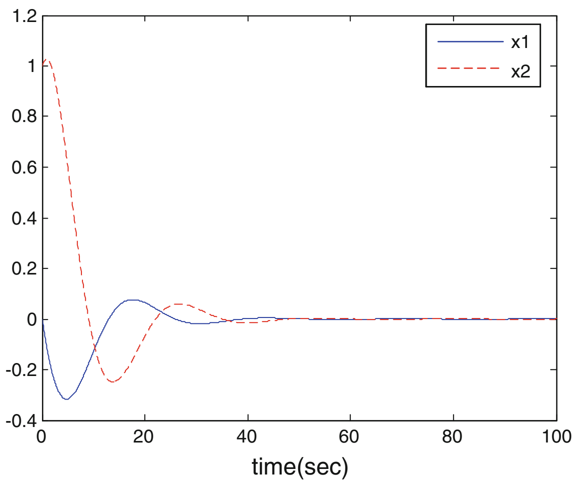

Fig. 2 The state responses of numerical example 1

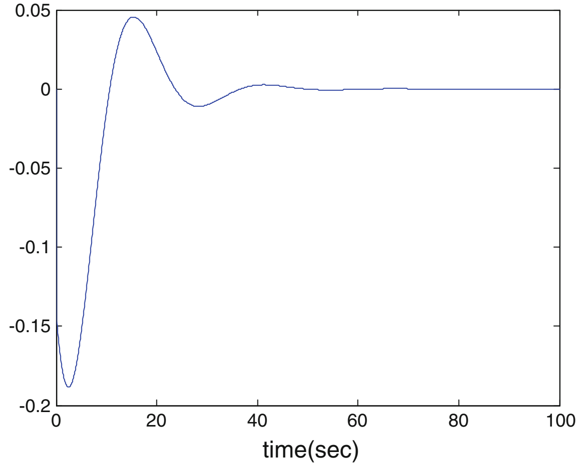

Fig. 3 The sliding surface response of numerical example 1

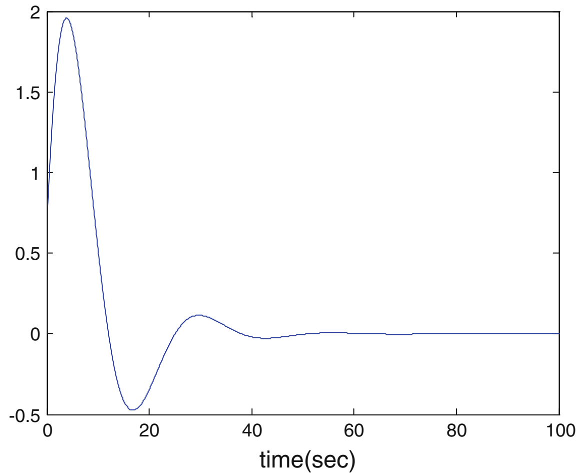

Fig. 4 The control effort of numerical example 1

From the simulation results, we conclude that the controller design in (41) as well as the proposed novel saturation switching function in (42) dose work effectively for the RSS system in (19). It also shows that SMC design can directly utilize state derivative feedback if the design is carried out in RSS framework. Furthermore, for generalized state space system in (1), if matrix E is singular but matrix F is nonsingular, it can be expressed in RSS framework to directly carry out SMC design.

Numerical Example 2

Here is another example with disturbance to verify the proposed SMC algorithm. Consider a dynamic RSS system in (19) with following parameters:

$$A = \begin{bmatrix} 1 & -0.5 & 0.25 \\ 0 & 0.5 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} -0.25 \\ 0.25 \\ -0.5 \end{bmatrix}, d_r(t) = 0.2 \sin(0.3333t)$$

and the initial condition is given as: $x_0 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}$.

Using (24), the transpose matrix is constructed as follows.

$$T = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & -2 \end{bmatrix}.$$

The first step is to select the sliding surface, and the transfer matrix method is applied. If poles at -5 and -2.5 are selected, the obtained sliding surface is $s = \begin{bmatrix} -84 & -180 & 1 \end{bmatrix} x$. The second step is to design the controller given in (41). $\varepsilon = 0.5$, $\gamma = 2$ and $\alpha = 4$ are used in the simulation.

Figures 5, 6 and 7 show the time responses of states $x_1(t) - x_3(t)$, sliding surface, and control effort, respectively. As expected, in Fig. 5, under the influence of disturbance, we find that the trajectories of states are still bounded, so is the sliding surface response in Fig. 6. Therefore, the controller designed in (41) for the RSS system (19) indeed works effectively.

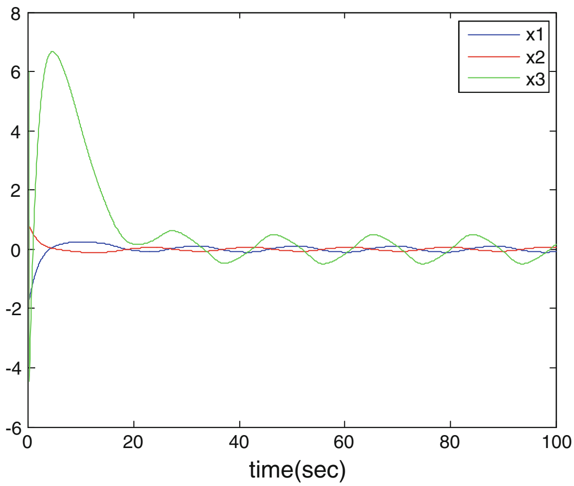

Fig. 5 The state responses of numerical example 2

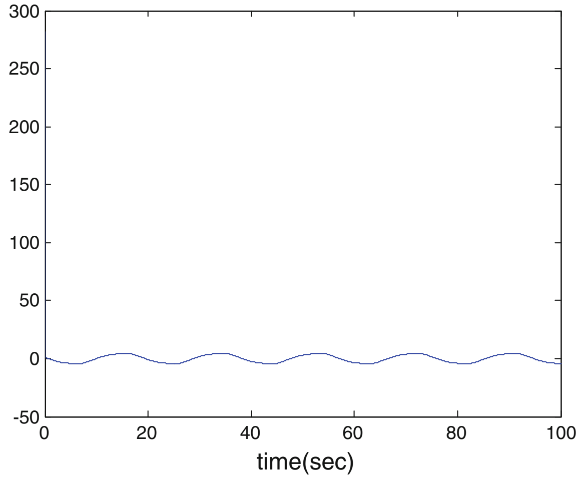

Fig. 6 Time response of the sliding surface of numerical example 2

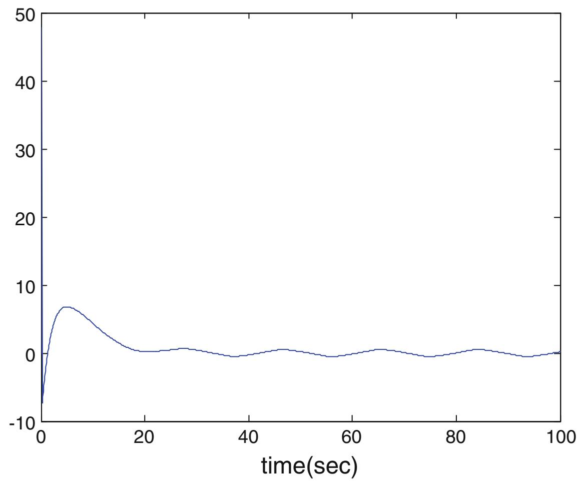

Fig. 7 Time response of control effort of numerical example 2

3.4 Sliding Model Control Design for a System with System Uncertainty and Disturbance in RSS Form

In this subsection, SMC design procedure for more realistic systems with system uncertainty and external disturbance in RSS form is presented. Consider the following RSS system:

$$x(t) = [A + \Delta A(t)]\,\dot{x}(t) + Bu(t) + d(t) \tag{44}$$

where $x(t) \in \Re^n$, $\Delta A(t) \in \Re^{n\times n}$, $u(t) \in R^m$ and $d(t) \in \Re^{n\times 1}$ are states, mismatched uncertainty, control inputs, and external disturbance respectively. Assuming that the nominal RSS linear system pair (A, B) is known and matrix dimensions are

$A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$. The external disturbance is a matched one and is defined as

$$d(t) = Bd_r(t) \tag{45}$$

where $d_r(t) \in \Re^{m\times 1}$.

Therefore, applying (45), (44) can be rewritten as

$$x(t) = [A + \Delta A(t)]\,\dot{x}(t) + B[u(t) + d_r(t)] \tag{46}$$

We assume that positive scalars, δ_A and δ_d, are the upper bounds of the uncertainty and the external disturbance, respectively.

$$\|\Delta A(t)\| \le \delta_A, \text{ and } \|d_r(t)\| \le \delta_d \tag{47}$$

3.4.1 Sufficient Condition for Finding the Upper Bound of System Uncertainty to Guarantee the Stability in Sliding Surface

In this subsection, we will provide a sufficient condition to determine the upper bound of uncertainty ΔA so that the stability in sliding surface is still guaranteed.

Like we mentioned in subsection 3.3.1, we can find the transfer matrix T in (25) such that $TB = \begin{bmatrix} 0 & B_2 \end{bmatrix}^T$.

Then (44) is transferred to the following equations.

$$q_1 = (A_{11} + \Delta A_{11})\,\dot{q}_1 + (A_{12} + \Delta A_{12})\,\dot{q}_2 \tag{48}$$

$$q_2 = (A_{21} + \Delta A_{21})\,\dot{q}_1 + (A_{22} + \Delta A_{22})\,\dot{q}_2 + B_2(u + d) \tag{49}$$

where every matrix with appropriate dimensions and B_2 is nonsingular.

We may neglect the uncertainty ΔA and disturbance d in (44) and apply the method in subsection 3.3.1 to design the sliding surface.

$$s = \begin{bmatrix} k & I_m \end{bmatrix} q = kq_1 + q_2 = \begin{bmatrix} k & I_m \end{bmatrix} Tx = 0 \tag{50}$$

Consequently, the derivative of sliding surface in (50) with respect to time can also be transferred to

$$\dot{s} = k\dot{q}_1 + \dot{q}_2 = 0 \tag{51}$$

Consequently, we have

$$\dot{q}_2 = -k\dot{q}_1 \tag{52}$$

Then substituting (52) into (48), we have

$$q_1 = A_e\dot{q}_1 + \Delta A_e\dot{q}_1 \tag{53}$$

where $A_e = A_{11} - A_{12}k$ and $\Delta A_e = \Delta A_{11} - \Delta A_{12}k$.

It has been proved that we also can apply Lyapunov equation to test the stability of a RSS system [24]. Based on that, the following two theorems are introduced for determining the upper bound of system uncertainty.

Theorem 1 *Assuming that A_e is a stable matrix and the time-varying uncertainty matrix ΔA_e in (53) is in a bounded value ζ such that $\|\Delta A_e\| < \zeta$, we have*

$$\|\Delta A_e\| < \zeta = \frac{\min\{\eta_i\}}{2\lambda_{\max}(P_e)}, \quad i = 1, 2, \ldots, (n-m)\,. \tag{54}$$

where η_i are all eigenvalues in a selected positive symmetric definite matrix Q_e while P_e is a positive symmetric definite matrix solved from the following Lyapunov equation.

$$A_e^T P_e + P_e A_e = -Q_e \tag{55}$$

Proof Define the Lyapunov functional:

$$V = q_1^T P_e q_1 \tag{56}$$

where P_e is symmetric positive definite matrix. It can be easily verified that V is a positive function. The time derivative of V along the trajectory of the system (53) is expressed as

$$\begin{aligned}\dot{V} &= \dot{q}_1^T P_e q_1 + q_1^T P_e \dot{q}_1 \\ &= \dot{q}_1^T P_e \left[A_e \dot{q}_1 + \Delta A_e \dot{q}_1\right] + \left[A_e \dot{q}_1 + \Delta A_e \dot{q}_1\right]^T P_e \dot{q}_1 \\ &= \dot{q}_1^T P_e A_e \dot{q}_1 + \dot{q}_1^T P_e \Delta A_e \dot{q}_1 + \dot{q}_1^T A_e^T P_e \dot{q}_1 + \dot{q}_1^T \Delta A_e^T P_e \dot{q}_1 \\ &= \dot{q}_1^T [P_e A_e + A_e^T P_e] \dot{q}_1 + 2\dot{q}_1^T P_e \Delta A_e \dot{q}_1\end{aligned} \tag{57}$$

Then, substituting (55) into (57), one obtains

$$\dot{V} = \dot{q}_1^T \left[-Q_e\right] \dot{q}_1 + 2\dot{q}_1^T P_e \Delta A_e \dot{q}_1 \tag{58}$$

From (58), when the following condition holds, one can conclude that $\dot{V} < 0$.

$$\dot{q}_1^T Q_e \dot{q}_1 > 2\dot{q}_1^T P_e \Delta A_e \dot{q}_1 \tag{59}$$

By Rayleigh principle, the lower bound of $\dot{q}_1^T Q_e \dot{q}_1$ in (59) can be obtained as follows.

$$\dot{q}_1^T Q_e \dot{q}_1 \geq \lambda_{\min}(Q_e) \dot{q}_1^T \dot{q}_1 = \lambda_{\min}(Q_e) \left\|\dot{q}_1\right\|^2 = \min\{\eta_i\} \left\|\dot{q}_1\right\|^2 \tag{60}$$

The following inequality can also be obtained

$$2\dot{q}_1^T P_e \Delta A_e \dot{q}_1 \leq 2\left\|\Delta A_e\right\| \lambda_{\max}(P_e) \dot{q}_1^T \dot{q}_1 = 2\left\|\Delta A_e\right\| \lambda_{\max}(P_e) \left\|\dot{q}_1\right\|^2 \tag{61}$$

From (60), (61) and (59), if we have

$$2\,\|\Delta A_e\|\,\lambda_{\max}(P_e)\,\|\dot{q}_1\|^2 < \min\{\eta_i\}\,\|\dot{q}_1\|^2 \tag{62}$$

and consequently,

$$\|\Delta A_e\| < \frac{\min\{\eta_i\}}{2\lambda_{\max}(P_e)} = \zeta, \quad i = 1, 2, \ldots, (n-m), \tag{63}$$

it implies that (58) is negative, namely, $\dot{V} < 0$ for $t \geq 0$. Consequently, the system with mismatched time-varying uncertainty ΔA_e in (44) in the sliding surface is asymptotically stable. Next, we have to provide another condition to find the upper bound of the mismatched uncertainty ΔA.

Theorem 2 *Let the transform matrix T in (24) be partitioned as*

$$T = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} R_1 & R_2 \end{bmatrix} \tag{64}$$

where $L_1 \in \Re^{(n-m)\times n}$, $L_2 \in \Re^{m\times n}$, $R_1 \in \Re^{n\times(n-m)}$, and $R_2 \in \Re^{n\times m}$.
If the following condition holds,

$$\|\Delta A\| \leq \frac{\min\{\eta_i\}}{2\left(\|L_1\|\cdot\|R_1\| + \|L_1\|\cdot\|R_2\|\cdot\|k\|\right)\lambda_{\max}(P_e)} \tag{65}$$

where P_e and Q_e are defined in (55) in Theorem 1, the RSS system with mismatched uncertainty ΔA in (44) is stable on the sliding surface.

Proof Since the transform matrix T in (24) can be partitioned as $T = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$, the uncertain matrix ΔA_{11} and ΔA_{12} in (48) can be expressed as

$$\Delta A_{11} = L_1 \Delta A R_1 \text{ and } \Delta A_{12} = L_1 \Delta A R_2. \tag{66}$$

So the uncertainty ΔA_e given in (53) can be rewritten as:

$$\Delta A_e = L_1 \Delta A R_1 - L_1 \Delta A R_2 k \tag{67}$$

Taking the norm of (67), one can obtain the following inequality.

$$\begin{aligned} \|\Delta A_e\| &\leq \|L_1 \Delta A R_1\| + \|L_1 \Delta A R_2 F C_1\| \leq \|\Delta A\|\left(\|L_1\|\cdot\|R_1\| + \|L_1\|\cdot\|R_2\|\cdot\|k\|\right) \\ &< \frac{\min\{\eta_i\}}{2\lambda_{\max}(P_e)} \end{aligned} \tag{68}$$

Consequently, the upper bound of ΔA is obtained as follow.

$$\|\Delta A\| < \frac{\min\{\eta_i\}}{2\left(\|L_1\| \cdot \|R_1\| + \|L_1\| \cdot \|R_2\| \cdot \|k\|\right)\lambda_{\max}\left(P_e\right)}, \quad i = 1, 2, \ldots, (n-m) \tag{69}$$

This concludes our proof.

From the above proof, it is clear to find that if both (54) in Theorems 1 and (65) in Theorem 2 hold, the system with the mismatched uncertainty ΔA is stable in the sliding mode. Since the presented methodology is a sufficient condition for determining the upper bound of system uncertainty, if (54) and (65) do not hold, it does not mean that the system will definitely become unstable. Above procedure is analogous to that in [17].

Remark In this remark, the procedure of finding the upper bound of the uncertainty ΔA is summarized as follows.

Step 1: Select a T such that (44) is transferred to (48) and (49).

Step 2: Neglect the uncertainty ΔA and disturbance d in (48) and (49), then select a sliding surface with the method introduced in this chapter.

Step 3: Calculate A_e in (53).

Step 4: Select a positive symmetric definite matrix Q_e, then calculate P_e using (55).

Step 5: Calculate the upper bound of ΔA_e in Theorem 1 by calculating the minimum eigenvalue of Q_e and the maximum eigenvalue of P_e.

Step 6: Finding L_1, L_2, R_1, and R_2 from T and T^{-1} in (64), then calculate the upper bound of ΔA in Theorem 2.

3.4.2 Design the SMC Controller for System with System Uncertainty and External Disturbance

When the system operates in the sliding mode, it meets the approaching condition. Applying (46), the sliding surface becomes

$$s(t) = Cx = C\left[\left(A + \Delta A(t)\right)\dot{x}(t) + B\left(u(t) + d_r(t)\right)\right] = 0. \tag{70}$$

If we choose $u(t) = u_{eq}(t)$ in (70), the equivalent control is found as

$$u_{eq}(t) = -(CB)^{-1}\left(CA\dot{x}(t) + C\Delta A\dot{x}(t) + CBd_r(t)\right) \tag{71}$$

Physically, the equivalent control $u_{eq}(t)$ cannot obtain the sliding motion if the initial state is not in the sliding surface. The SMC control law for the dynamic system in (44) must satisfy the approaching condition of sliding mode. Based on (71), the SMC control law is selected as

$$u(t) := -(CB)^{-1}\left(CA\dot{x}(t)\right) - (CB)^{-1}\left(\|\delta\dot{x}(t)\| + \gamma + \alpha\right) \cdot sat\left(\dot{s}(t), \varepsilon\right) \tag{72}$$

where δ, γ, and α are positive scalars such that $\delta = \|C\| \cdot \delta_A > \|C\Delta A(t)\|$, $\gamma = \|C\| \cdot \|B\| \cdot \delta_d > \|CBd_r(t)\|$, and $\alpha > 0$, respectively. Moreover, ε is a small positive value, and sat is a saturation function of the derivative of sliding surface $\dot{s}$ and is used to handle the switching and is described as

$$sat(\dot{s}_i, \varepsilon) = \begin{cases} 1 & \dot{s}_i > \varepsilon \\ \frac{\dot{s}_i}{\varepsilon} & |\dot{s}_i| \leq \varepsilon \\ -1 & \dot{s}_i < -\varepsilon \end{cases} = \begin{cases} sign\,(\dot{s}_i) & |\dot{s}_i| > \varepsilon \\ \frac{\dot{s}_i}{\varepsilon} & |\dot{s}_i| \leq \varepsilon \end{cases} \tag{73}$$

As mention in Sect. 3.3, the control law in (72) which uses saturation function cannot completely eliminate the external disturbance, but it can reduce the influence of the external disturbance so that the states are bounded.

At first, we consider the controller as follow.

$$u\,(t) = -\,(CB)^{-1}\,(CA\dot{x}\,(t)) - (CB)^{-1}\,(\|\delta\dot{x}\,(t)\| + \gamma + \alpha) \cdot sign\,(\dot{s}\,(t)) \tag{74}$$

where $sign$ is a function of $\dot{s}$ and is described as

$$sign(\dot{s}_i) = \begin{cases} 1 & \dot{s}_i > 0 \\ 0 & \dot{s}_i = 0 \\ -1 & \dot{s}_i < 0 \end{cases} \quad i = 1..m \tag{75}$$

Then, substituting (46) and (74) into sliding surface $s\,(t)$, we have

$$s\,(t) = \{C\Delta A\dot{x}\,(t) + CBd_r\,(t) - (\|\delta\dot{x}\,(t)\| + \gamma + \alpha) \cdot sign\,(\dot{s}\,(t))\} \tag{76}$$

Applying (76), approaching condition becomes

$$s^T\,(t) \cdot \dot{s}\,(t) = \left[(C\Delta A\dot{x}\,(t) + CBd_r\,(t))^T\,\dot{s} - (\|\delta\dot{x}\,(t)\| + \gamma + \alpha)^T \cdot \|\dot{s}\,(t)\|\right]$$

$$= -\alpha \cdot \|\dot{s}\,(t)\| - (\|\delta\dot{x}\,(t)\| + \gamma) \cdot \|\dot{s}\,(t)\| \left(1 - \frac{(C\Delta A\dot{x}\,(t) + CBd_r\,(t))^T \cdot \dot{s}\,(t)}{(\|\delta\dot{x}\,(t)\| + \gamma) \cdot \|\dot{s}\,(t)\|}\right) \tag{77}$$

Since $-1 < \frac{(C\Delta A\dot{x}(t)+CBd_r(t))^T \cdot \dot{s}(t)}{(\|\delta\dot{x}(t)\|+\gamma)\cdot\|\dot{s}(t)\|} < 1$, we have

$$s^T\,(t) \cdot \dot{s}\,(t) = -\alpha \cdot \|\dot{s}\,(t)\| - (\|\delta\dot{x}\,(t)\| + \gamma)^T \cdot \|\dot{s}\,(t)\| \left(1 - \frac{(C\Delta A\dot{x}\,(t) + Cd\,(t))^T \cdot \dot{s}\,(t)}{(\|\delta\dot{x}\,(t)\| + \gamma) \cdot \|\dot{s}\,(t)\|}\right) \tag{78}$$

$$< -\alpha \cdot \|\dot{s}\,(t)\| < 0$$

Thus, the approaching condition ($s^T\,(t) \cdot \dot{s}\,(t) < 0$) satisfies the Lyapunov stability theorem. Consequently, the motion in the sliding mode is asymptotically stable. Since the controller in (74) may cause “Chattering Phenomenon”, “$sign$” function is replaced by “sat” function in the applied control law given in (72).

Numerical Example 3
Consider a dynamic RSS system (44) with following parameters:

$$A = \begin{bmatrix} -0.0104 & -0.0583 & 0.1945 \\ 0.9971 & 0.0162 & 0.07715 \\ 0 & 0 & -0.1499 \end{bmatrix}, \quad \Delta A\,(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.001\sin(t) & 0.0013\sin(t) & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.7114 \\ -0.1969 \\ 1 \end{bmatrix}, \quad d\,(t) = \begin{bmatrix} -0.3557 \\ -0.0984 \\ 0.5 \end{bmatrix} \cos(2t) = \begin{bmatrix} -0.7114 \\ -0.1969 \\ 1 \end{bmatrix} \times 0.5\cos(2t) = Bd_r(t).$$

The initial condition is given as:

$$x_0 = \begin{bmatrix} -0.5 \\ 0 \\ 10 \end{bmatrix}$$

With above given matrices, the transfer matrix is then constructed as follows.

$$T = \begin{bmatrix} 1 & 0 & 0.7114 \\ 0 & 1 & 0.1969 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, L_1, L_2, R_1, and R_2 in (64) are then found from T.
If the poles of A_e in (53) are selected at -0.1 and -0.3, the sliding surface is selected as $s = \begin{bmatrix} 0.6179 & -0.5112 & 1 \end{bmatrix} x$.
Calculating A_e in (53), selecting $Q_e = \begin{bmatrix} 7 & 0 \\ 0 & 8 \end{bmatrix}$ and then solving P_e from (55), we have

$$P_e = \begin{bmatrix} 700.5910 & 35.5079 \\ 35.5079 & 11.9477 \end{bmatrix}.$$

Consequently, the upper bound of uncertainty $\|\Delta A\|$ is calculated using (65) and is obtained as 0.002. Since the spectral norm of given $\Delta A(t)$ is less than 0.0016, the system with $\Delta A(t)$ as uncertainty still can guarantee the stability.

The following parameters are used for controller in (72):
$\varepsilon = 0.5$, $\alpha = 2$, $\delta_A = 0.0025$ and $\delta_d = 0.25$.

In Fig. 8, we can find that the simulation trajectories of states are bounded in a boundary layer. Therefore, the proposed controller works effectively as expected.

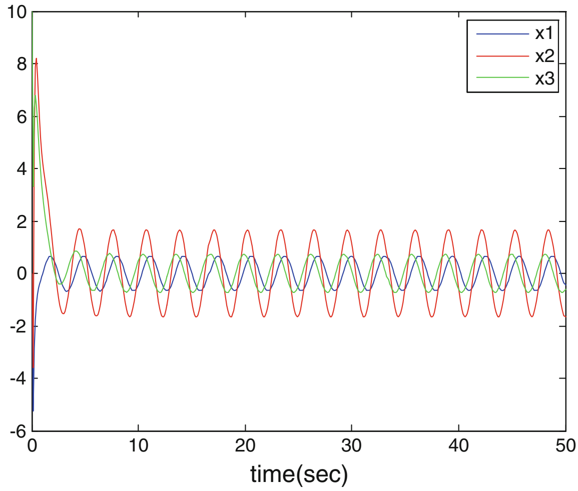

Fig. 8 State responses of example 3

4 Conclusion

In this chapter, the needs for reciprocal state space (RSS) form and state derivative related feedback control designs have been addressed. The fundamentals of state derivative feedback design in RSS form have been introduced. Basically, for controllable time invariant systems with no open loop poles at zero, the systems can be expressed in RSS form. The main advantage of RSS form is that state derivative feedback control designs can be systematically carried out in this form. Some once though tough systems to be controlled such as generalized state space system with impulse modes, can be fully controlled if it can be expressed in RSS form and apply state derivative feedback as shown in this chapter.

To better handle systems with external disturbance and system uncertainty with state derivative feedback control designs, novel sliding mode control design approach with state derivative feedback in RSS form is then presented. For systems in RSS form, nontraditional switching function utilizing the derivative of sliding surface is proposed and proven to satisfy the approaching condition of sliding mode. In addition, algorithm of finding upper bound of system uncertainty has been developed for robustness analysis. Simulation results successfully verify the proposed algorithms. State derivative output feedback algorithm for SMC design in RSS form has also been reported by author [31]. Our derivation is basically parallel to that for systems in standard state space form. Experienced engineers or researchers can be quickly familiar with the proposed design methods.

The contribution of this chapter is to provide SMC design approach by applying direct state derivative feedback in nontraditional RSS form so that people can handle more control problems without too much of mathematical overhead.

The future directions of research are suggested as follows:

- *Considering nonlinear constraint in control input*: In implementation, actuators that generate control inputs have various operating limitations such as saturation and dead zone [29]. People should develop algorithms that put actuator limits into consideration for more realistic considerations in design.
- *Using state derivative space (SDS) form in control design for nonlinear system:* State derivative space (SDS) form [30, 32] is a more general system form which can handle nonlinear systems. RSS form is a linear time invariant case of SDS form. SDS form is described as follows.

$$x = f(\dot{x}, u, t) \tag{79}$$

People may consider carrying out control design in SDS form with state derivative related feedback for some nonlinear systems. Author is working on it.

References

1. Brenan KE (1986) Numerical solution of trajectory prescribed path control problems by backward differentiation formulas. IEEE Trans Autom Control 31:266–269
2. Cobb D (1984) Controllability, observability and duality in singular systems. IEEE Trans Autom Control 29:1076–1082
3. Cobb D (2006) State feedback impulse elimination for singular systems over a hermite domain. SIAM J Control Optim 44:2189–2209
4. Cobb D (2010) Eigenvalue conditions for convergence of singularly perturbed matrix exponential functions. SIAM J Control Optim 48:4327–4351
5. Fahmy MM, O'Reilly YJ (1989) Parametric eigenstructure assignment for continuous-time descriptor systems. Int J Control 49:129–143
6. Fallah S, Khajepour A, Fidan B, Chen SK, Litkouhi B (2013) Vehicle optimal torque vectoring using state-derivative Feedback and linear matrix inequality. IEEE Trans Veh Technol 62: 1540–1552
7. Guo BZ, Jin FF (2013) Sliding mode and active disturbance rejection control to stabilization of one-dimensional anti-stable wave equations subject to disturbance in boundary input. IEEE Trans Autom Control 58:1269–1274
8. Itkis U (1976) Control systems of variable structure. Wiley, New York
9. Kulah H, Chae J, Yazdi N, Najafi K (2006) Noise analysis and characterization of a S-D capacitive silicon micro-accelerometers. IEEE J Solid-State Circuits 41:352–361
10. Kwak SK, Washington G (2001) A broadband vibration control using passive circuits in the matrix second order form. SPIE Proc Smart Struct Mater 2001(4326):14–19
11. Li H, Yu J, Hilton C, Liu H (2013) Adaptive sliding-mode control for nonlinear active suspension vehicle systems using T-S fuzzy approach. IEEE Trans Ind Electron 60:3328–3338
12. Lin FJ, Kuo LC (1997) Driving circuit for the ultrasonic motor servo drive with variable structure adaptive model-following control. IEE Proc Electr Power Appl 144:199–206
13. Liu D, Zhang G, Xie Y (2009) Guaranteed cost control for a class of descriptor systems with uncertainty. Int J Inf Syst 5:430–435
14. Liu P, Zhang Q, Yang X, Yang L (2008) Passivity and optimal control of descriptor biological complex systems. Joint Spec issue IEEE Trans Circuits Syst IEEE Trans Autom Control 53(1):122–125
15. Luenberger DG, Arbel A (1977) Singular dynamic leontief systems. Econometrica 45:991–995

16. Newcomb RW (1981) The semistate description of nonlinear time variable circuits. IEEE Trans Circuits Syst 28:28–33
17. Pai MC, Sinha A (2006) Sliding mode output feedback control of time-varying mismatched uncertain systems. IEEE international conference on systems, man, and cybernetics, pp 1355–1360
18. Pantelides CC (1988) The consistent initialization of differential-algebraic systems. SIAM J Sci Stat Comput 9:213–231
19. Saadni SM, Chaabane M, Mehdi D (2006) Robust stability and stabilization of a class of singular systems with multiple time-varying delay. Asian J Control 8:1–11
20. Shyu KK, Liu CY (1996) Variable structure controller design for robust tracking and model following. J Guid Control Dyn 19:1395–1397
21. Tseng YW (1997) Control design of linear dynamic system with matrix differential equations for aerospace applications. Ph.D. Dissertation, Department of Aerospace Engineering, The Ohio State University, Columbus, Ohio, USA
22. Tseng YW, Yedavalli RK (1997) Vibration control of a wing box via reciprocal state space framework. In: Proceedings of the IEEE conference on control applications. Hartford, Connecticut, USA, pp 854–89
23. Tseng YW, Yedavalli RK (1998) Control design via generalized state space system with state derivative measurement and reciprocal state space framework. In: Proceedings of the 1998 American control conference, Philadelphia, PA, USA, pp 2520–2521
24. Tseng YW, Kwak SK, Yedavalli RK (2003) Stability, controllability and observability criteria for the reciprocal state space framework. Proceedings of the American control conference, pp 5093–5097
25. Tseng YW (2008) Control designs of singular systems expressed in reciprocal state space framework with state derivative feedback. Int J Control Theory Appl 1:55–67
26. Tseng YW (2009) Vibration control of piezoelectric smart plate using estimated state derivatives feedback in reciprocal state space framework. Int J Control Theory Appl 2:61–71
27. Tseng YW, Wang YN (2009) Variable structure control in reciprocal state space framework. (2009) CACS international automatic control conference. Taipei, Taiwan
28. Tseng YW, Wang YN (2010) Variable structure control design and stability analysis in reciprocal state space framework. In: Proceedings of the 2nd international conference on computer and automation engineering (ICCAE) conference, vol 4, Singapore, pp 380–384,
29. Tseng YW, Wang YN (2012) Model following variable structure control design in reciprocal state space framework with dead-zone nonlinearity and lumped uncertainty. In: Proceedings of the IEEE conference on industrial electronics and applications, Singapore, pp 1627–1632
30. Tseng YW (2012) Optimal control in novel state derivative space system. In: Proceedings of the IEEE conference on industrial electronics and application, pp 371–376
31. Tseng YW, Wang YN (2013) Sliding mode control with state derivative output feedback in reciprocal state space form. special issue of switched dynamics with its applications, abstract and applied analysis, p 12. Article ID 590524. http://dx.doi.org/10.1155/2013/590524
32. Tseng YW, Hsieh JG (2013) Optimal control for a family of systems in novel state derivative space form with experiment in a double inverted pendulum system. Special issue of switched dynamics with its applications, Abstract and applied analysis. p 8, Article ID 715026. http://dx.doi.org/10.1155/2013/715026
33. Utkin VI (1977) Variable structure systems with sliding modes. IEEE Trans Autom Control 22:212–222
34. Utkin VI, Young KD (1979) Method for constructing discontinuity planes in multidimensional variable systems. Autom Remote Control 39:1466–1470
35. Utkin VI (1992) Sliding modes in control optimization. Springer, Berlin
36. Utkin VI, Guldner J, Shi J (1999) Sliding mode control in electromechanical systems. Taylor & Francis, Philadelphia
37. Varga A (2000) Robust pole assignment for descriptor systems. In: Proceeding of mathematical theory of networks and systems

38. Verghese GC, Levy BC, Kailath T (1981) A Generalized state-space for singular systems. IEEE Trans Autom Control 26:11–831
39. Wang N, Xu W, Chen F (2009) Output feedback variable structure control of uncertain linear systems in the presence of actuator dynamics. Asian J Control 11:420–426
40. Wang YN (2009) Variable structure control in reciprocal state space framework, M.S. thesis, advisor: Tseng YW Department of Electrical Engineering, I- Shou University, Kaohsiung, Taiwan, ROC
41. Yeh FB, Huang HN (2000) H infinity state feedback control of smart beam-plates via the descriptor system approach. Tunghai Sci 2:21–42
42. Yip EL, Sincovec RF (1981) Solvability, controllability and observability of continuous descriptor systems. IEEE Trans Autom Control 26:702–707