Output Regulation of Vaidyanathan 3-D Jerk Chaotic System

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Abstract This paper investigates the problem of output regulation of the Vaidyanathan 3-D jerk chaotic system (2014). Explicitly, state feedback control laws to regulate the output of the Vaidyanathan jerk chaotic system have been derived so as to track the constant reference signals as well as to track periodic reference signals. The control laws are derived using the regulator equations of C.I. Byrnes and A. Isidori (1990), who solved the problem of output regulation of nonlinear systems involving neutrally stable exosystem dynamics. The output regulation of the Vaidyanathan jerk chaotic system has important applications in Electrical and Communication Engineering. Numerical simulations using MATLAB are shown to illustrate the phase portraits of the Vaidyanathan jerk chaotic system and the output regulation results for the Vaidyanathan jerk chaotic system.

Keywords Chaos · Chaotic systems · Output regulation · Nonlinear control systems · Feedback stabilization

1 Introduction

Output regulation of control systems is one of the very important problems in control systems theory. Basically, the output regulation problem is to control a fixed linear or nonlinear plant in order to have its output tracking reference signals produced by some external generator (the exosystem).

For linear control systems, the output regulation problem has been solved by Francis and Wonham [12]. For nonlinear control systems, the output regulation problem has been solved by Byrnes and Isidori [5] generalizing the internal model principle obtained by Francis and Wonham [12]. Byrnes and Isidori [5] have made an important assumption in their work which demands that the exosystem dynam-

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ics generating reference and/or disturbance signals is a neutrally stable system (Lyapunov stable in both forward and backward time). The class of exosystem signals includes the important particular cases of constant reference signals as well as sinusoidal reference signals. Using Centre Manifold Theory [7], Byrnes and Isidori have derived regulator equations, which completely characterize the solution of the output regulation problem of nonlinear control systems.

The output regulation problem for linear and nonlinear control systems has been the focus of many studies in recent decades [48]. In [21], Mahmoud and Khalil obtained results on the asymptotic regulation of minimum phase nonlinear systems using output feedback. In [13], Fridman solved the output regulation problem for nonlinear control systems with delay using centre manifold theory [7]. In [11], Chen and Huang obtained results on the robust output regulation for output feedback systems with nonlinear exosystems. In [18], Liu and Huang obtained results on the global robust output regulation problem for lower triangular nonlinear systems with unknown control direction. In [88], Yang and Huang obtained new results on the global robust output regulation problem for nonlinear plants subject to nonlinear exosystems.

In [14], Immonen obtained results on the practical output regulation for bounded linear infinite-dimensional state space systems. In [23], Pavlov, Van de Wouw and Nijimeijer obtained results on the global nonlinear output regulation using convergence-based controller design. In [87], Xi and Ding obtained results on the global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. In [37], Serrani and Isidori obtained results on the global robust output regulation problem for a class of nonlinear systems.

In [39], Sundarapandian obtained results for the output regulation of the Lorenz attractor. In [52], Vaidyanathan obtained results for the output regulation of the unified chaotic system. In [51], Vaidyanathan derived results for the output regulation of the Arneodo-Coullet chaotic system. In [56], Vaidyanathan derived results for the output regulation of the Liu chaotic system.

Chaotic systems are defined as nonlinear dynamical systems which are sensitive to initial conditions, topologically mixing and with dense periodic orbits. Sensitivity to initial conditions of chaotic systems is popularly known as the *butterfly effect*. Small changes in an initial state will make a very large difference in the behavior of the system at future states. Chaotic behaviour was suspected well over hundred years ago in the study of three bodies problem by Henri Poincaré [4], but chaos was experimentally established by E.N. Lorenz [19] only a few decades ago in the study of 3-D weather models.

Some classical paradigms of 3-D chaotic systems in the literature are Rössler system [31], ACT system [1], Sprott systems [38], Chen system [9], Lü system [20], Liu system [17], Cai system [6], Chen-Lee system [10], Tigan system [49], etc.

Many new chaotic systems have been discovered in the recent years such as Zhou system [89], Zhu system [90], Li system [16], Wei-Yang system [86], Sundarapandian systems [40, 45], Vaidyanathan systems [58, 59, 61–64, 66, 68, 71, 82, 85], Pehlivan system [25], etc.

Synchronization of chaotic systems is a phenomenon that occurs when two or more chaotic systems are coupled or when a chaotic system drives another chaotic system. Because of the butterfly effect which causes exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, the synchronization of chaotic systems is a challenging research problem in the chaos literature [2, 3].

Major works on synchronization of chaotic systems deal with the complete synchronization of a pair of chaotic systems called the *master* and *slave* systems. The design goal of the complete synchronization is to apply the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically with time.

Pecora and Carroll pioneered the research on synchronization of chaotic systems with their seminal papers [8, 24]. The active control method [15, 32, 33, 44, 50, 73, 74, 77] is typically used when the system parameters are available for measurement. Adaptive control method [34–36, 41–43, 54, 60, 67, 72, 75, 76, 81, 84] is typically used when some or all the system parameters are not available for measurement and estimates for the uncertain parameters of the systems.

Backstepping control method [26–30, 47, 78, 83] is also used for the synchronization of chaotic systems, which is a recursive method for stabilizing the origin of a control system in strict-feedback form. Another popular method for the synchronization of chaotic systems is the sliding mode control method [46, 53, 55, 57, 65, 69, 70, 79, 80], which is a nonlinear control method that alters the dynamics of a nonlinear system by application of a discontinuous control signal that forces the system to "slide" along a cross-section of the system's normal behavior.

In this work, the output regulation problem for the Vaidyanathan jerk chaotic system [82] has been solved using the Byrnes-Isidori regulator equations [5] to derive the state feedback control laws for regulating the output of the Vaidyanathan jerk chaotic system for the important cases of constant reference signals (set-point signals) and periodic reference signals.

This work is organized as follows. In Sect. 2, a review of the solution of the output regulation for nonlinear control systems and Byrnes-Isidori regulator equations has been presented. In Sect. 3, a dynamic analysis of the Vaidyanathan jerk chaotic system [82] is detailed. In Sect. 4, output regulation problem for the Vaidyanathan jerk chaotic system is discussed and new results are derived. In Sect. 5, numerical simulations for the output regulation of the Vaidyanathan jerk chaotic system are detailed. Section 6 summarizes the main results obtained in this work.

2 Review of the Output Regulation for Nonlinear Control Systems

In this section, we consider a multi-variable nonlinear control system modelled by equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)\omega \tag{1}$$

$$\dot{\omega} = s(\omega) \tag{2}$$

$$e = h(x) - q(\omega) \tag{3}$$

Here, the differential equation (1) describes the *plant dynamics* with state x defined in a neighbourhood X of the origin of $\mathbf{R}^n$ and the input u takes values in $\mathbf{R}^m$ subject to the effect of a disturbance represented by the vector field $p(x)\omega$. The differential equation (2) describes an autonomous system, known as the *exosystem*, defined in a neighbourhood W of the origin of $\mathbf{R}^k$, which models the class of disturbance and reference signals taken into consideration. The Eq. (3) defines the error between the actual plant output $h(x) \in \mathbf{R}^p$ and a reference signal $q(\omega)$, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (1), (2) and the error Eq. (3), namely, f, g, p, s, h and q are C^1 mappings vanishing at the origin, i.e.

$$f(0) = 0, \;\; g(0) = 0, \;\; p(0) = 0, \;\; h(0) = 0 \;\text{ and }\; q(0) = 0.$$

Thus, for $u = 0$, the system (1), (2) has an equilibrium state $(x, \omega) = (0, 0)$ with zero error (3).

A *state feedback controller* for the composite system (1), (2) has the form

$$u = \alpha(x, \omega) \tag{4}$$

where α is a C^1 mapping defined on $X \times W$ such that $\alpha(0, 0) = 0$. Upon substitution of the feedback law (4) in the composite system (1), (2), we get the closed-loop system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x, \omega) + p(x)\omega \\ \dot{\omega} &= s(\omega) \end{aligned} \tag{5}$$

The purpose of designing the state feedback controller (4) is to achieve both *internal stability* and *output regulation*. Internal stability means that when the input is disconnected from (5) (i.e. when $\omega = 0$), the closed-loop system (5) has an exponentially stable equilibrium at $x = 0$. Output regulation means that for the closed-loop system (5), for all initial states $(x(0), \omega(0))$ sufficiently close to the origin, $e(t) \to 0$ asymptotically as $t \to \infty$. Formally, we can summarize the requirements as follows.

State Feedback Regulator Problem [5]:
Find, if possible, a state feedback control law $u = \alpha(x, \omega)$ such that

(OR1) *(Internal Stability)* The equilibrium $x = 0$ of the dynamics

$$\dot{x} = f(x) + g(x)\alpha(x, 0)$$

is locally asymptotically stable.

(OR2) *(Output Regulation)* There exists a neighbourhood $U \subset X \times W$ of $(x, \omega) = (0, 0)$ such that for each initial condition $(x(0), \omega(0)) \in U$, the solution $(x(t), \omega(t))$ of the closed-loop system (5) satisfies

$$\lim_{t\to\infty} [h(x(t)) - q(\omega(t))] = 0.$$

Byrnes and Isidori [5] have solved this problem under the following assumptions.

(H1) The exosystem dynamics $\dot{\omega} = s(\omega)$ is neutrally stable at $\omega = 0$, i.e. the system is Lyapunov stable in both forward and backward time at $\omega = 0$.

(H2) The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, i.e. if

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \quad \text{and} \quad B = \left[\frac{\partial g}{\partial x}\right]_{x=0},$$

then (A, B) is stabilizable, which means that we can find a gain matrix K so that $A + BK$ is Hurwitz. ■

Next, we recall the solution of the output regulation problem derived by Byrnes and Isidori [5].

Theorem 1 *[5] Under the hypotheses (H1) and (H2), the state feedback regulator problem is solvable if, and only if, there exist C^1 mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \phi(\omega)$ with $\phi(0) = 0$, both defined in a neighbourhod of $W^0 \subset W$ of $\omega = 0$ such that the following equations (called the* Byrnes-Isidori regulator equations*) are satisfied:*

(1) $\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\phi(\omega) + p(\pi(\omega))\omega$

(2) $h(\pi(\omega)) - q(\omega) = 0$

When the Byrnes-Isidori regulator equations (1) and (2) are satisfied, a control law solving the state feedback regulator problem is given by

$$u = \phi(\omega) + K[x - \pi(\omega)] \tag{6}$$

where K is any gain matrix such that $A + BK$ is Hurwitz. ■

3 Dynamic Analysis of the Vaidyanathan Jerk Chaotic System

The Vaidyanathan jerk chaotic system [82] is described by the 3-D dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 \end{aligned} \tag{7}$$

In (7), a and b are constant, positive parameters.

The system (7) exhibits a chaotic attractor when the parameter values are taken as

$$a = 0.4, \quad b = 0.8 \tag{8}$$

For numerical simulations, we take the initial conditions of the Vaidyanathan jerk system (7) as

$$x_1(0) = 0.8, \quad x_2(0) = 1.2, \quad x_3(0) = 0.5 \tag{9}$$

Figure 1 shows the 3-D phase portrait of the Vaidyanathan jerk chaotic system (7). Figures 2, 3, 4 show the 2-D projections of the Vaidyanathan jerk chaotic system (7) on the (x_1, x_2), (x_2, x_3) and (x_1, x_3) coordinate planes respectively.

Also, the Lyapunov exponents of the Vaidyanathan jerk chaotic system (7) for the parameter values (8) and the initial values (9) are numerically found as

$$L_1 = 0.0771, \quad L_2 = 0, \quad L_3 = -0.8791 \tag{10}$$

Since $L_1 + L_2 + L3 = -0.802 < 0$, the Vaidyanathan jerk chaotic system (7) is dissipative and the asymptotic motion of the Vaidyanathan jerk chaotic system (7) settles onto a strange attractor of the system.

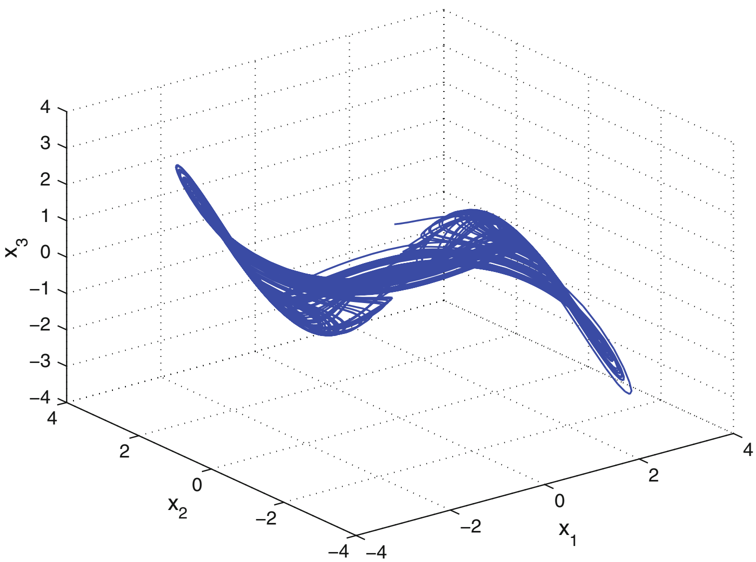

Fig. 1 3-D phase portrait of the Vaidyanathan jerk chaotic system

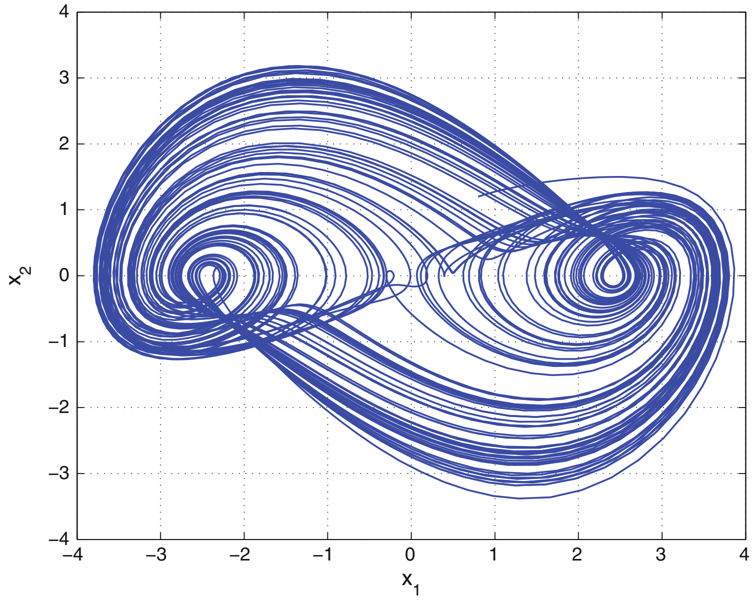

Fig. 2 2-D projection of the Vaidyanathan jerk chaotic system on the (x_1, x_2) plane

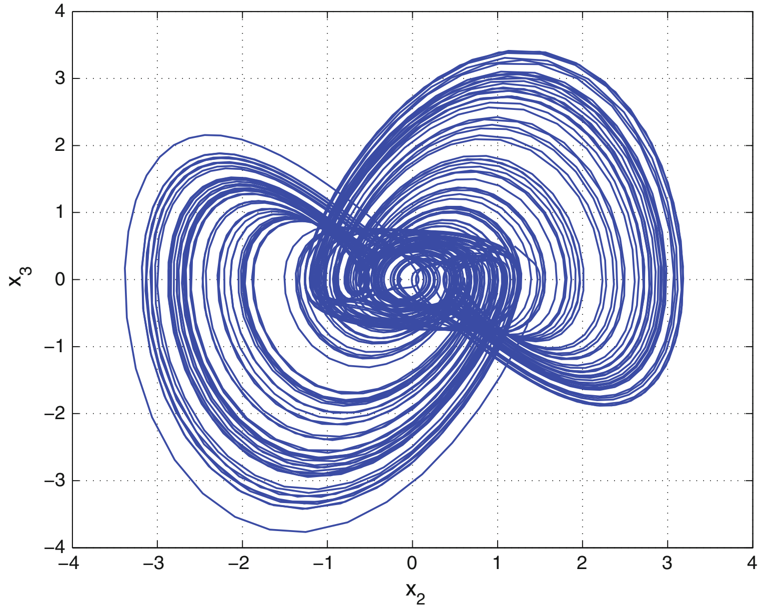

Fig. 3 2-D projection of the Vaidyanathan jerk chaotic system on the (x_2, x_3) plane

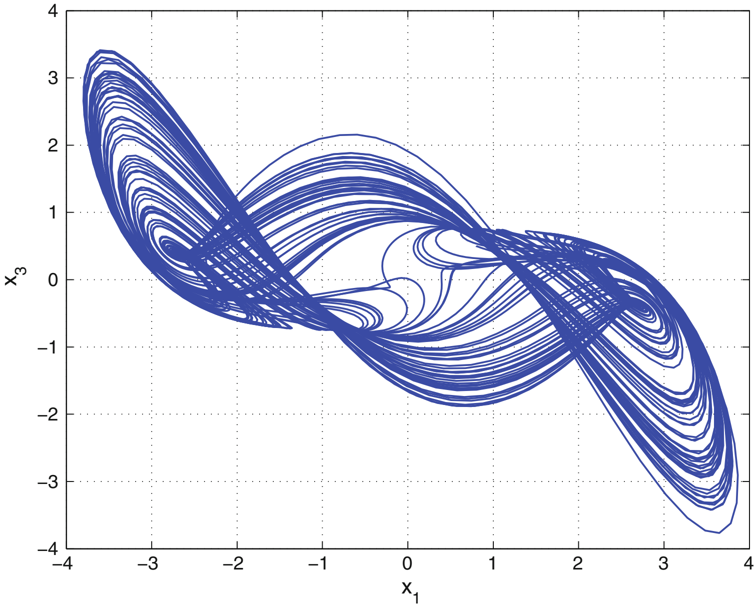

Fig. 4 2-D projection of the Vaidyanathan jerk chaotic system on the (x_1, x_3) plane

Also, the Kaplan-Yorke dimension of the Vaidyanathan jerk chaotic system (7) is calculated as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0877, \tag{11}$$

which is fractional.

4 Output Regulation of Vaidyanathan Jerk Chaotic System

The Vaidyanathan jerk chaotic system [82] is a novel chaotic system described by the dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \end{aligned} \tag{12}$$

where a, b are positive constants and u is an active feedback control.

In this work, we consider two important cases of output regulation for the Vaidyanathan jerk chaotic system [82]:

(I) Tracking of Constant Reference Signals
(II) Tracking of Periodic Reference Signals

4.1 *Tracking of Constant Reference Signals*

In this case, the exosystem is given by the scalar dynamics

$$\dot{\omega} = 0 \tag{13}$$

It is important to observe that the exosystem (13) is neutrally stable because the solutions of (13) are only constant trajectories, i.e.

$$\omega(t) \equiv \omega(0) = \omega_0 \quad \text{for all} \quad t$$

Thus, the assumption (H1) of Theorem 1 (Sect. 2) holds trivially.

Linearizing the dynamics of the Vaidyanathan jerk chaotic system (12) at the origin, we get the system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-a & -a & -b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{14}$$

Using Kalman's rank test for controllability [22], it can be easily seen that the pair (A, B) is completely controllable.

Since (A, B) is in Bush companion form, the characteristic equation of $A + BK$ is given by

$$\lambda^3 + (b - k_3)\lambda^2 + (a - k_2)\lambda + (a - k_1 - 1) = 0 \tag{15}$$

where $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$.

By the Routh's stability criterion [22], it can be easily shown that the closed-loop system matrix $A + BK$ is Hurwitz if and only if

$$k_1 < a - 1, \quad k_2 < a, \quad k_3 < b, \quad (b - k_3)(a - k_2) > a - k_1 - 1 \tag{16}$$

Thus, the assumption (H2) of Theorem 1 (Sect. 2) also holds.

Hence, Theorem 1 can be applied to solve the output regulation problem for the Vaidyanathan jerk chaotic system (12) for the tracking of constant reference signals (*set-point signals*).

4.1.1 Constant Tracking Problem for x_1

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ e &= x_1 - \omega \end{aligned} \tag{17}$$

By Theorem 1, the regulator equations of (17) are obtained as

$$\begin{cases} \pi_2(\omega) = 0 \\ \pi_3(\omega) = 0 \\ \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] - b\pi_3(\omega) + \phi(\omega) = 0 \\ \pi_1(\omega) - \omega = 0 \end{cases} \tag{18}$$

Solving the regulator equation (18), we get the unique solution

$$\begin{cases} \pi_1(\omega) = \omega \\ \pi_2(\omega) = 0 \\ \pi_3(\omega) = 0 \\ \phi(\omega) = a\sinh(\omega) - \omega \end{cases} \tag{19}$$

By Theorem 1, a state feedback control law solving the output regulation problem is given by

$$u = \phi(\omega) + K[x - \pi(\omega)] = a\sinh(\omega) - \omega + k_1(x_1 - \omega) + k_2x_2 + k_3x_3 \tag{20}$$

where k_1, k_2 and k_3 satisfy the inequalities (16).

4.1.2 Constant Tracking Problem for x_2

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ e &= x_2 - \omega \end{aligned} \tag{21}$$

By Theorem 1, the regulator equations of (21) are obtained as

$$\begin{cases} \pi_2(\omega) = 0 \\ \pi_3(\omega) = 0 \\ \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] - b\pi_3(\omega) + \phi(\omega) = 0 \\ \pi_2(\omega) - \omega = 0 \end{cases} \tag{22}$$

The first and fourth equations in (22) contradict each other.

Thus, the regulator equation (22) are not solvable.

Hence, by Theorem 1, we conclude that the output regulation problem is not solvable for this case.

4.1.3 Constant Tracking Problem for x_3

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ e &= x_3 - \omega \end{aligned} \tag{23}$$

By Theorem 1, the regulator equations of (23) are obtained as

$$\begin{cases} \pi_2(\omega) = 0 \\ \pi_3(\omega) = 0 \\ \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] - b\pi_3(\omega) + \phi(\omega) = 0 \\ \pi_3(\omega) - \omega = 0 \end{cases} \tag{24}$$

The second and fourth equations in (24) contradict each other.

Thus, the regulator equation (24) are not solvable.

Hence, by Theorem 1, we conclude that the output regulation problem is not solvable for this case.

4.2 *Tracking of Periodic Reference Signals*

In this case, the exosystem is given by the planar dynamics

$$\begin{aligned} \dot{\omega}_1 &= \nu\,\omega_2 \\ \dot{\omega}_2 &= -\nu\,\omega_1 \end{aligned} \tag{25}$$

where $\nu > 0$ is any fixed constant.

Clearly, the assumption (H1) (Theorem 1) holds. Also, as established in Sect. 4.1, the assumption (H2) of Theorem 1 also holds and that the closed-loop system matrix $A + BK$ will be Hurwitz if the constants k_1, k_2 and k_3 of the gain matrix K satisfy the inequalities (16).

Hence, Theorem 1 can be applied to solve the output regulation problem for the Vaidyanathan jerk chaotic system (12) for the tracking of periodic reference signals.

4.2.1 Periodic Tracking Problem for x_1

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ \dot{\omega}_1 &= \nu\,\omega_2 \\ \dot{\omega}_2 &= -\nu\,\omega_1 \\ e &= x_1 - \omega_1 \end{aligned} \tag{26}$$

By Theorem 1, the regulator equations of (26) are obtained as

$$\begin{cases} \frac{\partial \pi_1}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_1}{\partial \omega_2}(-\nu\omega_1) = \pi_2(\omega) \\ \frac{\partial \pi_2}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_2}{\partial \omega_2}(-\nu\omega_1) = \pi_3(\omega) \\ \frac{\partial \pi_3}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_3}{\partial \omega_2}(-\nu\omega_1) = \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] \\ \qquad\qquad -b\pi_3(\omega) + \phi(\omega) \\ \pi_1(\omega) - \omega_1 = 0 \end{cases} \tag{27}$$

Solving the regulator equation (27), we get the unique solution

$$\begin{cases} \pi_1(\omega) = \omega_1 \\ \pi_2(\omega) = \nu\omega_2 \\ \pi_3(\omega) = -\nu^2\omega_1 \end{cases} \tag{28}$$

and

$$\phi(\omega) = -\nu^3\omega_2 - \left(b\nu^2 + 1\right)\omega_1 + a[\sinh(\omega_1) + \sinh(\nu\omega_2)] \tag{29}$$

By Theorem 1, a state feedback control law solving the output regulation problem is given by

$$u = \phi(\omega) + K[x - \pi(\omega)] \tag{30}$$

where $\pi(\omega)$ is given by (28), $\phi(\omega)$ is given by (29) and k_1, k_2 and k_3 satisfy the inequalities (16).

4.2.2 Periodic Tracking Problem for x_2

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ \dot{\omega}_1 &= \nu\,\omega_2 \\ \dot{\omega}_2 &= -\nu\,\omega_1 \\ e &= x_2 - \omega_1 \end{aligned} \tag{31}$$

By Theorem 1, the regulator equations of (31) are obtained as

$$\begin{cases} \frac{\partial \pi_1}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_1}{\partial \omega_2}(-\nu\omega_1) = \pi_2(\omega) \\ \frac{\partial \pi_2}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_2}{\partial \omega_2}(-\nu\omega_1) = \pi_3(\omega) \\ \frac{\partial \pi_3}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_3}{\partial \omega_2}(-\nu\omega_1) = \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] \\ \qquad\qquad -b\pi_3(\omega) + \phi(\omega) \\ \pi_2(\omega) - \omega_1 = 0 \end{cases} \tag{32}$$

Solving the regulator equation (32), we get the unique solution

$$\begin{cases} \pi_1(\omega) = -\nu^{-1}\omega_2 \\ \pi_2(\omega) = \omega_1 \\ \pi_3(\omega) = \nu\omega_2 \end{cases} \tag{33}$$

and

$$\phi(\omega) = \nu^2\omega_1 + \left(\nu^{-1} + b\nu\right)\omega_2 + a\left[-\sinh\left(\nu^{-1}\omega_2\right) + \sinh(\omega_1)\right] \tag{34}$$

By Theorem 1, a state feedback control law solving the output regulation problem is given by

$$u = \phi(\omega) + K[x - \pi(\omega)] \tag{35}$$

where $\pi(\omega)$ is given by (33), $\phi(\omega)$ is given by (34) and k_1, k_2 and k_3 satisfy the inequalities (16).

4.2.3 Periodic Tracking Problem for x_3

Here, the tracking problem for the Vaidyanathan jerk chaotic system (12) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 + u \\ \dot{\omega}_1 &= \nu\,\omega_2 \\ \dot{\omega}_2 &= -\nu\,\omega_1 \\ e &= x_3 - \omega_1 \end{aligned} \tag{36}$$

By Theorem 1, the regulator equations of (36) are obtained as

$$\begin{cases} \frac{\partial \pi_1}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_1}{\partial \omega_2}(-\nu\omega_1) = \pi_2(\omega) \\ \frac{\partial \pi_2}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_2}{\partial \omega_2}(-\nu\omega_1) = \pi_3(\omega) \\ \frac{\partial \pi_3}{\partial \omega_1}(\nu\omega_2) + \frac{\partial \pi_3}{\partial \omega_2}(-\nu\omega_1) = \pi_1(\omega) - a[\sinh(\pi_1(\omega)) + \sinh(\pi_2(\omega))] \\ \qquad\qquad -b\pi_3(\omega) + \phi(\omega) \\ \pi_3(\omega) - \omega_1 = 0 \end{cases} \tag{37}$$

Solving the regulator equation (37), we get the unique solution

$$\begin{cases} \pi_1(\omega) = -\nu^{-2}\omega_1 \\ \pi_2(\omega) = -\nu^{-1}\omega_2 \\ \pi_3(\omega) = \quad \omega_1 \end{cases} \tag{38}$$

and

$$\phi(\omega) = \left(b + \nu^{-2}\right)\omega_1 + \nu\omega_2 - a\left[\sinh\left(\nu^{-2}\omega_1\right) + \sinh\left(\nu^{-1}\omega_2\right)\right] \tag{39}$$

By Theorem 1, a state feedback control law solving the output regulation problem is given by

$$u = \phi(\omega) + K[x - \pi(\omega)] \tag{40}$$

where $\pi(\omega)$ is given by (38), $\phi(\omega)$ is given by (39) and k_1, k_2 and k_3 satisfy the inequalities (16).

5 Numerical Simulations

For numerical simulations, we take the parameter values a and b so that the Vaidyanathan jerk system (12) is in the chaotic case, i.e.

$$a = 0.4, \quad b = 0.8 \tag{41}$$

For achieving the internal stability of the state feedback regulator problem, a gain matrix K which satisfies the inequalities (16) must be used.

With the choice

$$K = \left[\, k_1 \; k_2 \; k_3 \,\right] = \left[\, -125.6 \;\; -74.2 \;\; -14.2 \,\right],$$

the matrix $A + BK$ is Hurwitz with the eigenvalues $-5, -5, -5$.

In the periodic tracking output regulation problem, the value $\nu = 1$ is taken in the exosystem dynamics given by (25).

5.1 Tracking of Constant Reference Signals

5.1.1 Constant Tracking Problem for x_1

Here, the initial conditions are taken as

$$x_1(0) = 8.1, \quad x_2(0) = 5.4, \quad x_3(0) = 6.3, \quad \omega(0) = 2$$

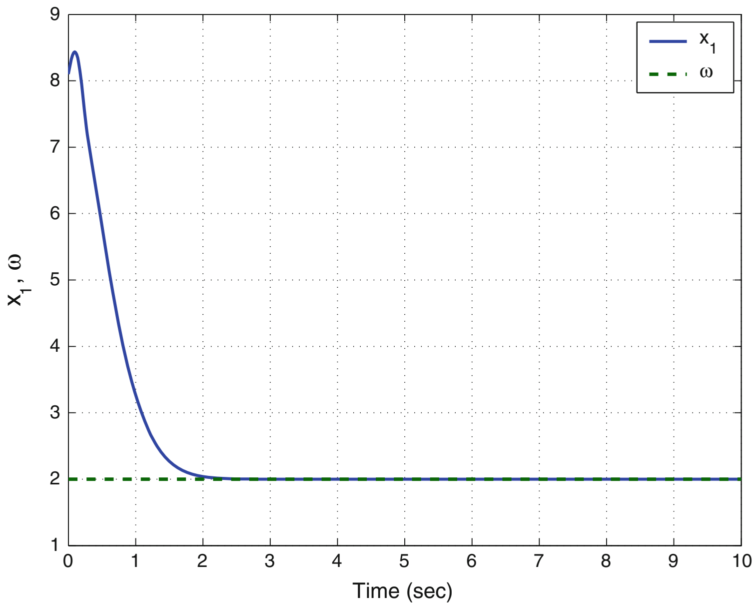

Fig. 5 Constant tracking of the state x_1

The simulation graph is depicted in Fig. 5 from which it is clear that the state trajectory $x_1(t)$ tracks the constant reference signal $\omega(t) \equiv 2$ in 3 s.

5.1.2 Constant Tracking Problem for x_2

As detailed in Sect. 4.1.2, the output regulation problem is not solvable for this case because the Byrnes-Isidori regulator equations do not admit any solution.

5.1.3 Constant Tracking Problem for x_3

As pointed out in Sect. 4.1.3, the output regulation problem is not solvable for this case because the Byrnes-Isidori regulator equations do not admit any solution.

5.2 Tracking of Periodic Reference Signals

5.2.1 Periodic Tracking Problem for x_1

Here, the initial conditions are taken as

$$x_1(0) = 5.1, \quad x_2(0) = 4.7, \quad x_3(0) = -2.5, \quad \omega_1(0) = 0, \omega_2(0) = 1$$

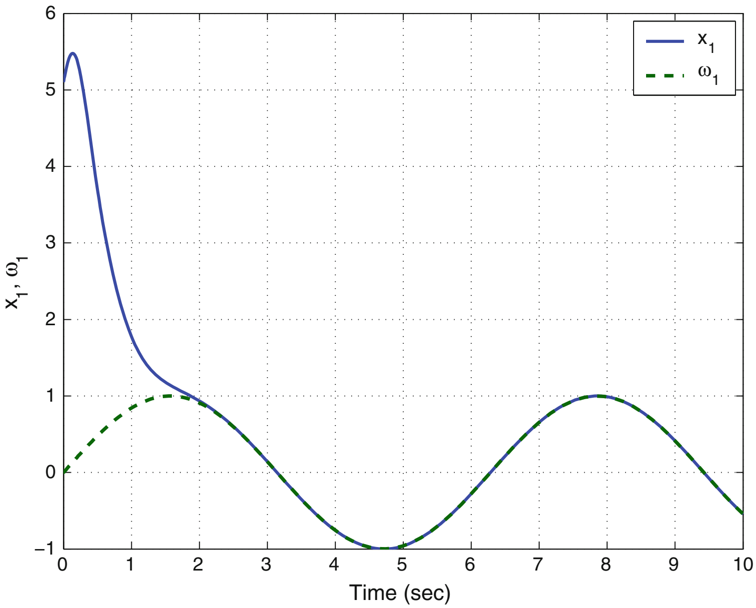

Fig. 6 Periodic tracking of the state x_1

Also, it is assumed that $\nu = 1$. The simulation graph is depicted in Fig. 6 from which it is clear that the state trajectory $x_1(t)$ tracks the periodic reference signal $\omega_1(t) = \sin t$ in 3 s.

5.2.2 Periodic Tracking Problem for x_2

Here, the initial conditions are taken as

$$x_1(0) = 8.1, \quad x_2(0) = 3.4, \quad x_3(0) = -2.7, \quad \omega_1(0) = 0, \omega_2(0) = 1$$

Also, it is assumed that $\nu = 1$. The simulation graph is depicted in Fig. 7 from which it is clear that the state trajectory $x_2(t)$ tracks the periodic reference signal $\omega_1(t) = \sin t$ in 3 s.

5.2.3 Periodic Tracking Problem for x_3

Here, the initial conditions are taken as

$$x_1(0) = 3.4, \quad x_2(0) = 2.5, \quad x_3(0) = -6.9, \quad \omega_1(0) = 0, \omega_2(0) = 1$$

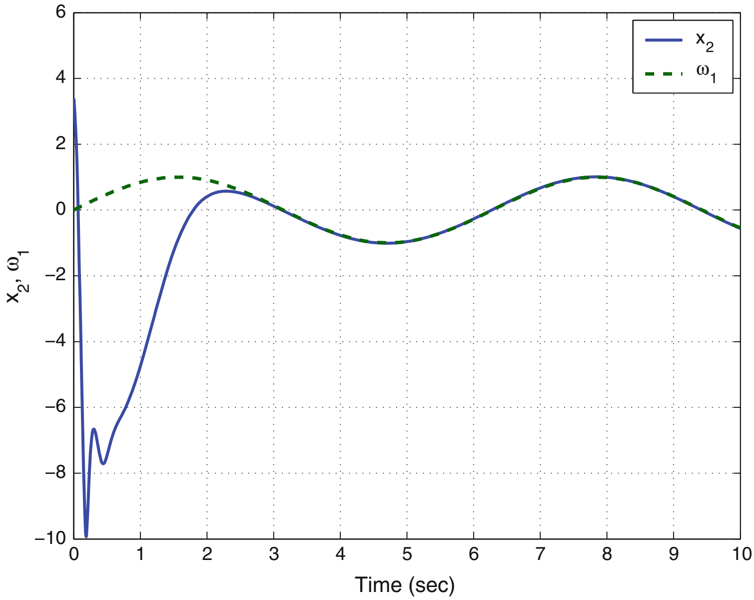

Fig. 7 Periodic tracking of the state x_2

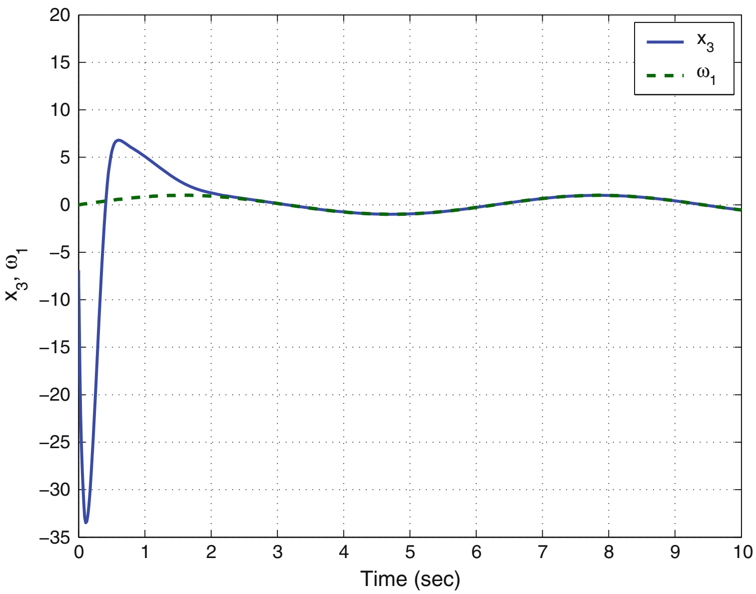

Fig. 8 Periodic tracking of the state x_3

Also, it is assumed that $\nu = 1$. The simulation graph is depicted in Fig. 8 from which it is clear that the state trajectory $x_3(t)$ tracks the periodic reference signal $\omega_1(t) = \sin t$ in 3 s.

6 Conclusions

Output regulation problem is one of the important problems in the control theory, which aims to control a fixed linear or nonlinear plant in order to have its output tracking reference signals produced by some external generator or the exosystem. Byrnes and Isidori (1990) solved the output regulation problem for a general class of nonlinear systems under some stability assumptions. In this work, the output regulation problem for the Vaidyanathan jerk chaotic system (2014)) has been studied in detail and a complete solution for the output regulation problem for the Vaidyanathan jerk chaotic system has been presented as well. Explicitly, using the Byrnes-Isidori regulator equations (1990), state feedback control laws for regulating the output of the Vaidyanathan jerk chaotic system have been derived. As tracking reference signals, constant and periodic reference signals have been considered and in each case, feedback control laws regulating the output of the Vaidyanathan chaotic system have been derived when the problem is solvable. Numerical simulations using MATLAB are shown to verify the results.

References

1. Arneodo A, Coullet P, Tresser C (1981) Possible new strange attractors with spiral structure. Commun Math Phys 79(4):573–576
2. Azar AT, Vaidyanathan S (2015a) Chaos modeling and control systems design, studies in computational intelligence, vol 581. Springer, Germany
3. Azar AT, Vaidyanathan S (2015b) Computational intelligence applications in modeling and control, studies in computational intelligence, vol 575. Springer, Germany
4. Barrow-Green J (1997) Poincaré and the three body problem. American mathematical society, providence
5. Byrnes CI, Isidori A (1990) Output regulation of nonlinear systems. IEEE Trans Autom Control 35(2):131–140
6. Cai G, Tan Z (2007) Chaos synchronization of a new chaotic system via nonlinear control. J Uncertain Syst 1(3):235–240
7. Carr J (1981) Appl Cent Manifold Theory. Springer, New York
8. Carroll TL, Pecora LM (1991) Synchronizing chaotic circuits. IEEE Trans Circuits Syst 38(4):453–456
9. Chen G, Ueta T (1999) Yet another chaotic attractor. Int J Bifurc Chaos 9(7):1465–1466
10. Chen HK, Lee CI (2004) Anti-control of chaos in rigid body motion. Chaos, Solitons and Fractals 21(4):957–965
11. Chen Z, Huang J (2005) Robust output regulation with nonlinear exosystems. Automatica 41(8):1447–1454
12. Francis BA, Wonham WM (1975) The internal model principle for linear multivariable regulators. Appl Math Optim 2(2):170–194
13. Fridman E (2003) Output regulation of nonlinear control systems with delay. Syst Control Lett 50:81–93
14. Immonen E (2007) Practical output regulation for bounded linear infinite-dimensional state space systems. Automatica 43(5):786–794
15. Karthikeyan R, Sundarapandian V (2014) Hybrid chaos synchronization of four-scroll systems via active control. J Electr Eng 65(2):97–103

16. Li D (2008) A three-scroll chaotic attractor. Phys Lett A 372(4):387–393
17. Liu C, Liu T, Liu L, Liu K (2004) A new chaotic attractor. Chaos, Solitions and Fractals 22(5):1031–1038
18. Liu L, Huang J (2008) Global robust output regulation of lower triangular systems with unknown control direction. Automatica 44(5):1278–1284
19. Lorenz EN (1963) Deterministic periodic flow. J Atmos Sci 20(2):130–141
20. Lü J, Chen G (2002) A new chaotic attractor coined. Int J Bifurc Chaos 12(3):659–661
21. Mahmoud NA, Khalil HK (1996) Asymptotic regulation of minimum phase nonlnear systems using output feedback. IEEE Trans Autom Control 41(10):1402–1412
22. Ogata K (2009) Modern Control Eng, 5th edn. Prentice Hall, New Jersey
23. Pavlov A, de Wouw NV, Nijimeijer H (2007) Global nonlinear output regulation: convergence based controller design. Automatica 43(3):456–463
24. Pecora LM, Carroll TL (1990) Synchronization in chaotic systems. Phys Rev Lett 64(8):821–824
25. Pehlivan I, Moroz IM, Vaidyanathan S (2014) Analysis, synchronization and circuit design of a novel butterfly attractor. J Sound Vib 333(20):5077–5096
26. Rasappan S, Vaidyanathan S (2012a) Global chaos synchronization of WINDMI and coullet chaotic systems by backstepping control. Far East J Math Sci 67(2):265–287
27. Rasappan S, Vaidyanathan S (2012b) Hybrid synchronization of n-scroll Chua and Lur'e chaotic systems via backstepping control with novel feedback. Arch Control Sci 22(3):343–365
28. Rasappan S, Vaidyanathan S (2012c) Synchronization of hyperchaotic Liu system via backstepping control with recursive feedback. Commun Comput Inf Sci 305:212–221
29. Rasappan S, Vaidyanathan S (2013) Hybrid synchronization of n-scroll chaotic Chua circuits using adaptive backstepping control design with recursive feedback. Malaysian J Math Sci 7(2):219–246
30. Rasappan S, Vaidyanathan S (2014) Global chaos synchronization of WINDMI and coullet chaotic systems using adaptive backstepping control design. Kyungpook Math J 54(1):293–320
31. Rössler OE (1976) An equation for continuous chaos. Phys Lett A 57(5):397–398
32. Sarasu P, Sundarapandian V (2011a) Active controller design for the generalized projective synchronization of four-scroll chaotic systems. Int J Syst Signal Control Eng Appl 4(2):26–33
33. Sarasu P, Sundarapandian V (2011b) The generalized projective synchronization of hyperchaotic Lorenz and hyperchaotic Qi systems via active control. Int J Soft Comput 6(5):216–223
34. Sarasu P, Sundarapandian V (2012a) Adaptive controller design for the generalized projective synchronization of 4-scroll systems. Int J Syst Signal Control Eng Appl 5(2):21–30
35. Sarasu P, Sundarapandian V (2012b) Generalized projective synchronization of three-scroll chaotic systems via adaptive control. Eur J Sci Res 72(4):504–522
36. Sarasu P, Sundarapandian V (2012c) Generalized projective synchronization of two-scroll systems via adaptive control. Int J Soft Comput 7(4):146–156
37. Serrani A, Isdori A (2000) Global robust output regulation for a class of nonlinear systems. Syst Control Lett 39(2):133–139
38. Sprott JC (1994) Some simple chaotic flows. Phys Rev E 50(2):647–650
39. Sundarapandian V (2010) Output regulation of the Lorenz attractor. Far East J Math Sci 42(2):289–299
40. Sundarapandian V (2013) Analysis and anti-synchronization of a novel chaotic system via active and adaptive controllers. J Eng Sci Technol Rev 6(4):45–52
41. Sundarapandian V, Karthikeyan R (2011a) Anti-synchronization of hyperchaotic Lorenz and hyperchaotic Chen systems by adaptive control. Int J Syst Signal Control Eng Appl 4(2):18–25
42. Sundarapandian V, Karthikeyan R (2011b) Anti-synchronization of Lü and Pan chaotic systems by adaptive nonlinear control. Eur J Sci Res 64(1):94–106
43. Sundarapandian V, Karthikeyan R (2012a) Adaptive anti-synchronization of uncertain Tigan and Li systems. J Eng Appl Sci 7(1):45–52
44. Sundarapandian V, Karthikeyan R (2012b) Hybrid synchronization of hyperchaotic Lorenz and hyperchaotic Chen systems via active control. J Eng Appl Sci 7(3):254–264

45. Sundarapandian V, Pehlivan I (2012) Analysis, control, synchronization, and circuit design of a novel chaotic system. Math Comput Model 55(7–8):1904–1915
46. Sundarapandian V, Sivaperumal S (2011) Sliding controller design of hybrid synchronization of four-wing chaotic systems. Int J Soft Comput 6(5):224–231
47. Suresh R, Sundarapandian V (2013) Global chaos synchronization of a family of n-scroll hyperchaotic Chua circuits using backstepping control with recursive feedback. Far East J Math Sci 73(1):73–95
48. Tarn TJ, Sanposh P, Cheng D, Zhang M (2005) Output regulation for nonlinear systems: some recent theoretical and experimental results. IEEE Trans Control Syst Technol 13(4):605–610
49. Tigan G, Opris D (2008) Analysis of a 3D chaotic system. Chaos, Solitons and Fractals 36:1315–1319
50. Vaidyanathan S (2011) Hybrid chaos synchronization of Liu and Lü systems by active nonlinear control. Commun Comput Inf Sci 204:1–10
51. Vaidyanathan S (2011b) Output regulation of the Arneodo-Coullet chaotic system. Commun Comput Inf Sci 133:98–107
52. Vaidyanathan S (2011c) Output regulation of the unified chaotic system. Commun Comput Inf Sci 198:1–9
53. Vaidyanathan S (2012a) Analysis and synchronization of the hyperchaotic Yujun systems via sliding mode control. Adv Intell Syst Comput 176:329–337
54. Vaidyanathan S (2012b) Anti-synchronization of Sprott-L and Sprott-M chaotic systems via adaptive control. Int J Control Theory Appl 5(1):41–59
55. Vaidyanathan S (2012c) Global chaos control of hyperchaotic Liu system via sliding control method. Int J Control Theory Appl 5(2):117–123
56. Vaidyanathan S (2012d) Output regulation of the Liu chaotic system. Appl Mech Mat 110–116:3982–3989
57. Vaidyanathan S (2012e) Sliding mode control based global chaos control of Liu-Liu-Liu-Su chaotic system. Int J Control Theory Appl 5(1):15–20
58. Vaidyanathan S (2013a) A new six-term 3-D chaotic system with an exponential nonlinearity. Far East J Math Sci 79(1):135–143
59. Vaidyanathan S (2013b) Analysis and adaptive synchronization of two novel chaotic systems with hyperbolic sinusoidal and cosinusoidal nonlinearity and unknown parameters. J Eng Sci Technol Rev 6(4):53–65
60. Vaidyanathan S (2013c) Analysis, control and synchronization of hyperchaotic Zhou system via adaptive control. Adv Intell Syst Comput 177:1–10
61. Vaidyanathan S (2014a) A new eight-term 3-D polynomial chaotic system with three quadratic nonlinearities. Far East J Math Sci 84(2):219–226
62. Vaidyanathan S (2014b) Analysis and adaptive synchronization of eight-term 3-D polynomial chaotic systems with three quadratic nonlinearities. Eur Phys J Spec Top 223(8):1519–1529
63. Vaidyanathan S (2014c) Analysis, control and synchronisation of a six-term novel chaotic system with three quadratic nonlinearities. Int J Model Identif Control 22(1):41–53
64. Vaidyanathan S (2014d) Generalized projective synchronisation of novel 3-D chaotic systems with an exponential non-linearity via active and adaptive control. Int J Model Identif Control 22(3):207–217
65. Vaidyanathan S (2014e) Global chaos synchronization of identical Li-Wu chaotic systems via sliding mode control. Int J Model Identif Control 22(2):170–177
66. Vaidyanathan S (2015) Analysis, properties and control of an eight-term 3-D chaotic system with an exponential nonlinearity. Int J Model Identif Control 23(2):164–172
67. Vaidyanathan S, Azar AT (2015a) Analysis and control of a 4-D novel hyperchaotic system. In: Azar AT, Vaidyanathan S (eds) Chaos Model Control Syst Des Stud Comput Intell, vol 581. Springer, Germany, pp 19–38
68. Vaidyanathan S, Azar AT (2015b) Analysis, control and synchronization of a nine-term 3-D novel chaotic system. In: Azar AT, Vaidyanathan S (eds) Chaos modelling and control systems design, studies in computational intelligence, vol 581. Springer, Germany, pp 19–38

69. Vaidyanathan S, Azar AT (2015c) Anti-synchronization of identical chaotic systems using sliding mode control and an application to Vaidhyanathan-Madhavan chaotic systems. Stud Comput Intell 576:527–547
70. Vaidyanathan S, Azar AT (2015d) Hybrid synchronization of identical chaotic systems using sliding mode control and an application to Vaidhyanathan chaotic systems. Stud Comput Intell 576:549–569
71. Vaidyanathan S, Madhavan K (2013) Analysis, adaptive control and synchronization of a seven-term novel 3-D chaotic system. Int J Control Theory Appl 6(2):121–137
72. Vaidyanathan S, Pakiriswamy S (2013) Generalized projective synchronization of six-term Sundarapandian chaotic systems by adaptive control. Int J Control Theory Appl 6(2):153–163
73. Vaidyanathan S, Rajagopal K (2011a) Anti-synchronization of Li and T chaotic systems by active nonlinear control. Commun Comput Inf Sci 198:175–184
74. Vaidyanathan S, Rajagopal K (2011b) Global chaos synchronization of hyperchaotic Pang and Wang systems by active nonlinear control. Commun Comput Inf Sci 204:84–93
75. Vaidyanathan S, Rajagopal K (2011c) Global chaos synchronization of Lü and Pan systems by adaptive nonlinear control. Commun Comput Inf Sci 205:193–202
76. Vaidyanathan S, Rajagopal K (2012) Global chaos synchronization of hyperchaotic Pang and hyperchaotic Wang systems via adaptive control. Int J Soft Comput 7(1):28–37
77. Vaidyanathan S, Rasappan S (2011) Global chaos synchronization of hyperchaotic Bao and Xu systems by active nonlinear control. Commun Comput Inf Sci 198:10–17
78. Vaidyanathan S, Rasappan S (2014) Global chaos synchronization of *n*-scroll Chua circuit and Lur'e system using backstepping control design with recursive feedback. Arabian J Sci Eng 39(4):3351–3364
79. Vaidyanathan S, Sampath S (2011) Global chaos synchronization of hyperchaotic Lorenz systems by sliding mode control. Commun Comput Inf Sci 205:156–164
80. Vaidyanathan S, Sampath S (2012) Anti-synchronization of four-wing chaotic systems via sliding mode control. Int J Autom Comput 9(3):274–279
81. Vaidyanathan S, Volos C, Pham VT (2014a) Hyperchaos, adaptive control and synchronization of a novel 5-D hyperchaotic system with three positive Lyapunov exponents and its SPICE implementation. Arch Control Sci 24(4):409–446
82. Vaidyanathan S, Volos C, Pham VT, Madhavan K, Idowu BA (2014b) Adaptive backstepping control, synchronization and circuit simulation of a 3-D novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities. Arch Control Sci 24(3):375–403
83. Vaidyanathan S, Idowu BA, Azar AT (2015a) Backstepping controller design for the global chaos synchronization of Sprott's jerk systems. Stud Comput Intell 581:39–58
84. Vaidyanathan S, Volos C, Pham VT, Madhavan K (2015b) Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system and its SPICE implementation. Arch Control Sci 25(1):5–28
85. Vaidyanathan S, Volos CK, Pham VT (2015c) Global chaos control of a novel nine-term chaotic system via sliding mode control. In: Azar AT, Zhu Q (eds) Advances and applications in sliding mode control systems, studies in computational intelligence, vol 576. Springer, Germany, pp 571–590
86. Wei Z, Yang Q (2010) Anti-control of Hopf bifurcation in the new chaotic system with two stable node-foci. Appl Math Comput 217(1):422–429
87. Xi Z, Ding Z (2007) Global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. Automatica 43(1):143–149
88. Yang X, Huang J (2012) New results on robust output regulation of nonlinear systems with a nonlinear exosystem. Int J Robust Nonlinear Control 22(15):1703–1719
89. Zhou W, Xu Y, Lu H, Pan L (2008) On dynamics analysis of a new chaotic attractor. Phys Lett A 372(36):5773–5777
90. Zhu C, Liu Y, Guo Y (2010) Theoretic and numerical study of a new chaotic system. Intell Inf Manag 2:104–109