Design of a Controller of Switched Nonlinear Systems Based on Multiple Lyapunov Functions

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Abstract In this work, we focus on the stabilization issue of a class of non-minimum phase switched nonlinear systems where the internal dynamics of each mode may be unstable and uncontrollable. We develop a hybrid nonlinear control technique based on the coupling between bounded nonlinear feedback controllers and the switching laws designed to stabilize the transitions between the stability regions associated to each modes arising from the limitations imposed by the input constraints. The key feature of the proposed approach is based on the formalism of the input–output feedback linearization. The performed developments largely rely on Hauser's approximation and multiple Lyapunov functions. In summary, the synthesized controllers can guarantee the stability of individual modes while switching law that will be generating ensures overall system stability. The differences between the switching strategies, and their implications on the switching logic, are discussed. A non-minimum phase Continuously Stirred Tank Reactor (CSTR) illustrates the efficiency of the proposed approach

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1 Introduction

There are numerous examples of chemical processes include both continuous dynamics and discrete events. Continuous behavior of a process caused by some factors, such as momentum, mass, and energy conservation, can be modeled more conveniently as discrete events by continuous-time differential equations.

However discrete phenomena is ubiquitously multifaceted and can originate from physical constraints or manufacturing distinct phases such as discontinuous actuators, phase changes, flow reversals, shocks, and transitions; the use of measurement sensors and control actuators with discrete settings/positions or filling/emptying a reactor [7, 14].

The overall process behavior in all of these instances is characterized by the interaction of continuous and discrete dynamics that they cannot be decoupled effectively, and is modeled by hybrid systems.

A class of hybrid systems that have drawn considerable attention in the past decade is the class of switched systems which consists of a family of subsystems and a switching signal, which defines a specific subsystem that is active at each instant of time. For a survey on switched systems we refer to [6, 10, 11, 18, 21, 23, 24].

The study of stability analysis and stabilization of switched systems is an important problem that has been the subject of significant research works in control theory [1–4, 20, 22, 28, 29, 32, 35]. In this framework one of these problems is the stabilization of non-minimum phase nonlinear switched systems.

This system control, however, is a delicate task. Some contributions have been devoted to non-minimum phase switched nonlinear systems where each nonlinear mode may be non-minimum phase. In [34], $H\infty$ control goal is achieved for a class of non-minimum phase cascade switched nonlinear systems where the internal dynamics of each mode is assumed to be asymptotically stabilizable. Output tracking of non-minimum phase switched nonlinear systems has been considered in [26], where an approximated minimum phase model is utilized. The same problem is also investigated in [8] by means of an inversion based control strategy. In [33], a switching control methodology for non-minimum phase nonlinear switched systems with the control law which has singularities was developed. However, it is well known that the stabilization of non-minimum phase nonlinear systems is quite difficult, and is even impossible to achieve if the unstable zero dynamics is uncontrollable.

Motivated by these considerations, we present in this work a nonlinear control methodology for a class of non-minimum phase nonlinear switched systems with input constraints.

The main feature of the proposed approach is not only to synthesize the bounded nonlinear feedback controllers of the individual subsystems, but also to design an appropriate switching scheme that organizes the transitions between the different non-minimum phase modes and keeps all the system stable. The controller synthesis procedure yields also an explicit characterization that coupling the switching strategy and the stability regions associated for each mode arising from the limitations imposed by the input constraints.

The proposed method involves the integration of Input–output feedback linearization control and Hauser's approximation [17] in the particular case where the relative degree coincides with the system order.

To this end, we Use Multiple Lyapunov Functions [13, 36], one for each mode, a family of feedback controllers was synthesized for the individual closed-loop approximate modes and has provided an explicit characterization of the corresponding stability regions in terms of the input constraints. Then for the synthesis of a family of feedback controllers that enforce the desired stability and performance properties within each individual dynamical mode in the presence of input constraints. Finally we derive a set of switching rules that organize stabilizing transitions between the output feedback stability regions of the non-minimum phase modes.

The outline of this chapter is as follows: In Sect. 2, we introduce a motivating example. Section 3 provides the system description and preliminaries. The problem formulated is solved in Sect. 4. The proposed method is successfully applied to the switched exothermic chemical reactor [27] example in spite of to the fact that it is a nonlinear non-minimum phase system and that it is also characterized by a dynamic that leads to the instability of the dynamic of zero. Finally, a conclusion is drawn in Sect. 6.

2 Motivating Example: A Continuously Stirred Tank Reactor with Two Modes

Chemical reactors are known to be ones of the most important plants in chemical industry. The process in the reactor is usually exhibit complex behavior, so it is necessary to control their operation. In recent years, various nonlinear design tools have been proposed to provide global stabilization [5, 15, 16, 30]. One of the major control problems which has attracted the attention of researchers for a long time deal with the temperature regulation under input constraints of exothermic irreversible reaction in a continuously stirred tank reactor (CSTR).

Consider a general class of a constant-volume, non-isothermal CSTR system with a hybrid behavior, in which the reaction $A \rightarrow B$ takes place in the liquid phase. The reactor has two inlet streams: the first continuously feeds pure A at flow rate $F_r = 0.45\,\mathrm{m}^3/\mathrm{min}$, concentration $C_{A_0} = 12\,\mathrm{kmol\,m}^{-3}$ and $T_{A_0} = 300\,\mathrm{K}$, while the second can be turned on or off (by means of an on/off valve) during reactor operation. When turned on, the second stream feeds pure A at flow rate $F_r^* = 0.7\,\mathrm{m}^3/\mathrm{min}$, concentration $C_{A_0}^* = 14\,\mathrm{kmol\,m}^{-3}$ and $T_{A_0}^* = 320\,\mathrm{K}$. For the parameters given in Table 1 under standard modeling assumptions, the mathematical model of the process takes the following form:

Table 1 Parameter values of the non-isothermal reactor

Parameter	Value	Unit
Z_r	Reaction rate pre-exponential factor	$5 \times 10^8\, S^{-1}$
V_0	Reactor volume	$0.1\, m^3$
E_a	Activation energy	49.884 kJ/mol
R_r	Ideal gas constant	8.31×10^{-3} kJ/mol °C
C_{A_0}	Inlet concentration of reactant A	$12\, kmol/m^{-3}$
γ	Reactor model parameter	3.9

$$\begin{cases} \dfrac{dC_A}{dt} = -Z_r\, exp\left(-\dfrac{E_a}{R_r T}\right) C_A + \left(C_{A_0} - C_A\right) \dfrac{F_r}{V_0} + (i-1)\left(C_{A_0}^{*} - C_A\right) \dfrac{F_r^{*}}{V_0} \\ \dfrac{dT}{dt} = \gamma_r Z_r\, exp\left(-\dfrac{E_a}{R_r T}\right) C_A + \left(T_{A_0} - T\right) \dfrac{F_r}{V_0} + (i-1)\left(T_{A_0}^{*} - T\right) \dfrac{F_r^{*}}{V_0} + Q_r \end{cases} \tag{1}$$

If the variable i is equal to *1* then the second inlet stream is turned off and it is turned on when i has the value of *2*. Initially, it is assumed that $i = 1$. The control objectives are to stabilize the reactor temperature at the unstable steady state of mode *1* ($x_e = 302.0\,\text{K}$), and to maintain this temperature at this steady state when the reactor switches to mode *2* subject to the constraint $|Q_r| \leq 10 \times 10^{-2}\,\text{K}\,\text{s}^{-1}$.

3 System Description and Preliminaries

We consider a class of single-input single-output switched nonlinear systems of the following state-space equation:

$$\begin{cases} \dot{x} = f_i(x) + g_i(x)\, u_i \\ y = h(x) \\ i \in I = \{1, 2, \ldots, N\} \end{cases} \tag{2}$$

where $x = [x_1 \ldots x_n]^T \in \Re^n$ denote the vector of continuous state variables, $u_i = \left[u_i^1 \ \ldots \ u_i^m\right]^T$ is the vector of manipulated inputs taking values in a nonempty compact convex subset $U = \left\{u_i \in \Re^m : \|u_i\| \leq u_i^{max}\right\}$ with $u_{\max}^i \geq 0$ denotes the bound on the manipulated inputs, the notation $\|\cdot\|$ will be used to denote the standard Euclidean norm of a vector u_i. The nonlinear vector functions $f_i(.)$, $g_i(.)$ and the scalar function $h(x)$ are assumed to be sufficiently smooth which gives rise to the switched nonlinear system (2). The index i represent a discrete state that takes values in a finite index set I which specifies the active subsystem. The number N of the switches is finite on every bounded time interval. Throughout the paper, we use the notations t_i^k

and t_i^{k+1} to denote the tth times that the ith subsystem is switched in and out. We can assume, in the rest of the study, that the continuous state of the ith active mode evolves according to the state equation and the output equation governed for each $t_i^k < t < t_i^{k+1}$.

In order to provide the necessary background for our main results in Sect. 3, we will briefly review in the remainder of this section the stability properties of the system viewed as a finite collection of continuous-time nonlinear systems with discrete events that direct the transition between them. One of the main tools for stability analysis of switched systems is Multiple Lyapunov Functions (MLFs). In fact, its principle lies in the use of a family of functions named pseudo-Lyapunov functions $\{V_i : i \in I\}$ associated with each field of vectors $\dot{x} = f_i(x)$, to demonstrate stability.

Definition 1 ([19] *Pseudo-Lyapunov function*) A pseudo Lyapunov function for the system (2), around an operating point in a stability region of the space ($x_n \in \Omega_i \subset \Re^n$) is a real-valued function $V_i(x)$ defined in a region Ω_i satisfying the following conditions:

- Positive definite: $V_i(x_n) = 0$ and $V_i(x) > 0$ for $x_n \neq x \in \Omega_i$
- Derivative defined non-positive: for all x included in the stability region Ω_i

$$\frac{dV(x)}{dt} = (\partial V_i(x)/\partial x)\left[f_i(x) + g_i(x)u_i\right] \leq 0 \tag{3}$$

We can, then, write the following result proving the sufficient conditions for stability:

Theorem 1 *([9, 11]) Suppose that $\cup\Omega_i = \Re^n$ and each vector field f_i has an associated Lyapunov like function V_i in the region Ω_i, neighborhood x_n.*

For the N switched nonlinear system (1), with $u_i \equiv 0, i \in I$, the switching sequence can take the value of i only if $x \in \Omega_i$, then the value of V_i decreases on each interval when the ith subsystem is active, more specifically

$$V_i\left(x\left(t_i^k\right)\right) \leq V_i\left(x\left(t_i^{k-1}\right)\right) \tag{4}$$

We pose t_i^k the kth switching instant for the sequence. Then, the adjacent of the operating point x_e of the system (2), is Lyapunov stable.

As shown above in Theorem 1, The Multiple Lyapunov Function approach, usually one for each of the individual subsystems being switched, can be used to determine the stability of switched systems without input signals; such that if for every i the value of V_i, at the end of each such interval exceeds the value at the end of the next interval on which the ith subsystem is active, the switched system can be shown to be asymptotically stable. However it cannot inquire about the existence of a stabilizing feedback law for the switched control system (2). Here we introduce the notion of control Lyapunov function for feedback controller synthesis. The idea is to expect the MLFs method to play an important role for designing the feedback controllers.

Referring to the system (2), the concept of Control Lyapunov Function (CLF) introducing as follows:

Definition 2 ([31] *Control Lyapunov Function*) A smooth, proper, and Positive-Definite function $V : \Re^n \to \Re_+$ is called a CLF for a nonlinear control system of the form $\dot{x} = f(x) + g(x)u$ when there is an admissible value $u^1, \ldots, u^m$ for the controls such that:

$$inf \left\{ L_f V + L_{g_1} V u^1 + \cdots + L_{g_m} V u^m \right\} < 0 \tag{5}$$

where $L_f V = [\partial V / \partial x] f(x)$, g_k is the kth column of the matrix g.

We can generalize the Definition 1 to a switched nonlinear system as shown in this assumption:

Assumption 1 For every $i \in I = \{i = 1, \ldots, N\}$, a Control Lyapunov Function, V_i, exists for system (2).

By Assumption 1; if we can find a family of CLF for the switched System (2), one for each subsystem, then for the solution of (2) we can derive a control signal u such that family of CLF monotonically decreases.

4 Main Results

4.1 Problem Formulation

In order to clear presentation of the main results of this paper, we will start in this section by reviewing the state feedback control problem.

Consider the class of nonlinear systems that has been represented by Eq. (2). We need to assume that for all $i \in I$, there exists an integer r (this assumption is made only to simplify notations and can be readily relaxed to allow a different relative degree r_i for each subsystem) and a set of coordinates (see [19] for a detailed treatment of feedback linearizable nonlinear systems)

$$\Phi_i(x) = \begin{bmatrix} \Phi_{i,1}(x) \\ \Phi_{i,2}(x) \\ \vdots \\ \Phi_{i,r_i}(x) \\ \Phi_{i,r_i+1}(x) \\ \vdots \\ \Phi_{i,n}(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_{f_i} h(x) \\ \vdots \\ L_{f_i}^{r_i} h(x) \\ \chi_{i,1}(x) \\ \vdots \\ \chi_{i,n-r_i}(x) \end{bmatrix} \tag{6}$$

where $\chi_{i,1}(x), \ldots, \chi_{i,n-r_i}(x)$ are nonlinear scalar functions of x.

The coordinate change $\phi_i(x)$ allows transforming the subsystem of Eq. (1) into a partially linear form such that the system takes the form:

$$\begin{cases} \begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{r_i-1} = \xi_{r_i} \\ \dot{\xi}_{r_i} = L_{f_i}^{r_i} h(x) + L_{g_i} L_{f_i}^{r_i-1} h(x) u_i \\ \dot{\eta}_{i,1} = Q_{i,1}(x) \\ \vdots \\ \dot{\eta}_{i,n-r_i} = Q_{i,n-r_i}(x) \end{cases} \\ y = \xi_1 \end{cases} \tag{7}$$

where $L_{g_i} L_{f_i}^{r-1} h(x) \neq 0$ for all $x \in \Re^n$, $i \in I$ and $Q_{1,i}(x), \ldots, Q_{(n-r),i}(x)$ are nonlinear functions of their arguments describing the evolution of the inverse dynamics of the ith mode.

4.2 Theory and Design

In this section, we present a technique that combines the multiple Lyapunov functions and Hauser's [17] approximation to develop a nonlinear control strategy for the stabilization of a switched nonlinear system where each mode may be non-minimum phase. The key component of this methodology is to use a family of control Lyapunov functions, one for each subsystem, to:

1. Synthesize the bounded nonlinear feedback controllers of the individual subsystems.
2. Design an appropriate switching scheme that organizes the transitions between the different modes and keeps all the system stable.

Owing to the presence of the unstable zero dynamics, the problem becomes more challenging not only in the synthesizes of the control laws but also in the design of an appropriate switching scheme that guarantees stability in the presence of non-minimum phase modes. To present the solution, we will first define the notion of robust relative degree.

Consider the system (2), we assume that $x = x_e$ is an equilibrium point, that is $f_i(x_e) = 0$, and without loss of generality we assume that $h(x_e) = 0$.

If we will also assume the following "controllability" rank condition: *Rank* $\left\{g_i, ad(f_i g_i), \ldots, ad^{n-1}(f_i g_i)\right\} = n$ for each mode $i \in I = \{i = 1, \ldots, N\}$ at $x = x_e$, we will impose the following assumptions on the system (2).

Assumption 2 The nonlinear system (7) has a robust relative degree γ_i, for all for each mode $i (i \in I)$, in the neighborhood of x_e if there is a set of smooth functions $\hat{\Phi}_{i,j}(x)$ as the following one:

$$\begin{cases} y = h(x) = \hat{\Phi}_{i,1}(x) + \psi_{i,0}(x, u_i) \\ y^{(j)} = \hat{\Phi}_{i,j+1}(x) + \psi_{i,j}(x, u_i) \\ \vdots \\ y^{(\gamma_i - 1)} = \hat{\Phi}_{i,\gamma_i}(x) + \psi_{i,\gamma_i - 1}(x, u_i) \\ y^{(\gamma_i)} = L_{f_i}^{\gamma_i} h(x) + L_{g_i} L_{f_i}^{\gamma_i} h(x) u_i \\ i = 1, \ldots, N \\ j = 0, \ldots, \gamma_i \end{cases} \tag{8}$$

where the functions $\psi_{i,j}(x, u_i)$, are sums of terms $O(x)^2$, $O(x, u_i)$, or $O(u_i)^2$, $L_{f_i}^{\gamma_i} h(x)$ and $L_{g_i} L_{f_i}^{\gamma_i} h(x)$ are smooth, and $L_{g_i} L_{f_i}^{\gamma_i} h(x_e) \neq 0$.

Let us note that that a function $\delta(x)$ is $O(x)^n$ if $\lim_{|x| \to 0} (|\delta(x)| / |x|^n) \neq 0$. Moreover, the functions known as $O(x)^0$ will be indicated by $O(1)$.

The determination of the robust relative degree γ_i of a nonlinear system shows that the latter arises in a way similar to the case of the classic relative degree r_i. Indeed, one also obtains: $\gamma_i < n$.

The study of the properties of the approximately linearized system on a parameterized family of operating envelopes can be defined as follows:

Definition 1 For all $i \in I$, we call $B_{\varepsilon_i}^i \subset \Re^n$ for some $\varepsilon_i > 0$, a family of operating envelopes provided that $B_{\delta_i}^i \subset B_{\varepsilon_i}^i$, whenever $\delta_i < \varepsilon_i$ and $\sup \left\{ \delta : B_{\delta_i}^i \subset B_{\varepsilon_i}^i \right\} = \varepsilon_i$ where $B_{\delta_i}^i$, is a ball of radius δ_i centered at the origin.

Then, for the approximation in a larger region, we will impose following assumption

Assumption 3 For all $i \in I$, a function $\psi_i : \Re^n \times \Re \to \Re$ is said to be of uniformly high order on $B_{\varepsilon_i}^i \times B_{\sigma_i}^i$ if for some $\varepsilon_i > 0$, $\sigma_i > 0$ there exists a monotone increasing function of ε_i, $\lambda_i(\varepsilon_i)$, such that:

$$\begin{cases} |\psi_i(x, u_i)| \leq \varepsilon_i \lambda_i(\varepsilon_i)(|x| + |u_i|) \\ \quad \forall x \in B_{\varepsilon_i}^i, \forall u_i \in B_{\sigma_i}^i \end{cases} \tag{9}$$

where $B_{\varepsilon_i}^i$ is a ball of radius ε_i centered at x_e, and $B_{\varepsilon_i}^i$ is a ball of radius σ_i centered at the origin.

Now, we return to the original problem. We assume that system (1) has robust relative degree γ_i. Adopting the notation of [17], we define new coordinates ξ with $\xi_j = \hat{\Phi}_{i,j}(x)$, $j = 0, \ldots, \gamma_i$. Thus, we obtain the new representation of the system (1) which is written in mixed ξ and x coordinates as follows:

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \psi_{i,1}(x, u_i) \\ \vdots \\ \dot{\xi}_{\gamma_i - 1} = \xi_{\gamma_i} + \psi_{i,\gamma_i - 1}(x, u_i) \\ \dot{\xi}_{\gamma_i} = L_{f_i}^{\gamma_i} h(x) + L_{g_i} L_{f_i}^{\gamma_i - 1} h(x) u_i \end{cases} \tag{10}$$

Consider the switched nonlinear system of Eq. (10). Our Objective now is twofold. The first is to synthesize an output feedback controller from where the requested closed-loop properties for each mode Then the second objective is to design an appropriate set of switching scheme that organizes the transitions between constituent modes and their respective controllers and keeps all the system stable.

In order to proceed with the controller synthesis task, we will impose the following assumption on the on the process of Eq. (10). This assumption allows constructing bounded controls using the Lyapunov function [12, 25].

Assumption 4 For each $i \in I$, there exists a family of N bounded nonlinear state feedback controllers of the form:

$$u_i = -k_i\left(x\right)\left(L_{g_i} V_i\left(x\right)\right)^T, i = 1, \ldots, N \tag{11}$$

where V_i is a CLF for the ith mode and $L_{g_i} V_i$ is the Lie derivatives of the control Lyapunov function V_i for the ith mode along the column vectors of the matrix g_i.

Theorem 2 that follows provides the explicit synthesis formula for the desired bounded nonlinear state feedback controllers and states precise switching conditions that guarantee closed-loop stability.

Theorem 2 *Consider the switched nonlinear system (10), for which a family of CLFs V_i, $i = 1, \ldots, N$ has been founds, using each control Lyapunov function, we construct the following family of bounded nonlinear feedback controllers:*

$$u_i = -k_i\left(x, u_i^{max}\right)\left(L_{g_i} V_i\left(x\right)\right)^T, i = 1, \ldots, N \tag{12}$$

where

$$k_i\left(V_i, u_i^{\max}\right) = \begin{cases} \dfrac{\beta_i(x) + \left(\beta_i^2(x) + \left(u_i^{\max}\left\|\left(\left(L_{g_i} V_i\right)^T(x)\right)\right\|\right)^4\right)^{\frac{1}{2}}}{\left\|\left(\left(L_{g_i} V_i\right)^T(x)\right)\right\|^2\left(1 + \left(1 + \left(u_i^{\max}\left\|\left(\left(L_{g_i} V_i\right)^T(x)\right)\right\|\right)^2\right)^{\frac{1}{2}}\right)} & \left(L_{g_i} V_i\right)^T(x) \neq 0 \\ 0 & \left(L_{g_i} V_i\right)^T(x) = 0 \end{cases}$$

with $\beta_i(x) = L_{f_i} V_i\left(x\right) + \rho_i V_i\left(x\right), \rho_i > 0.$

Let $\Upsilon_i\left(u_i^{max}\right)$ be the largest set of x, containing the origin, such that $\beta_i(x) \leq u_i^{max}\left\|\left(L_{g_i} V_i\left(x\right)\right)^T\right\|$. Also, let $\Omega_i^*\left(u_i^{max}\right) := \left\{x \in \Re^n : V_i\left(x\right) \leq \varsigma_{x,i}\right\}$ be a level set of V_i, completely contained in Υ_i, for some $\varsigma_{x,i} > 0$, and assume, without loss of generality, that $x\left(0\right) \in \Omega_i^*\left(u_i^{max}\right)$ for some $i \in I$. If, at any given time, T, the following conditions hold:

$$\begin{cases} x\left(T\right) \in \Omega_l^*\left(u_l^{max}\right) \\ V_l\left(x\left(T\right)\right) < V_l\left(x\left(t_{l*}\right)\right) \end{cases} \tag{13}$$

For some $l \in I, l \neq i$, where $t_{l*} < T$ is the time when the lth subsystem was last switched in, i.e., for $t \geq T^{+}$, guarantees that the origin of the switched closed-loop system is asymptotically stable.

The stability requirement of Theorem 2, on the other hand, the behavior is globally input–output linearized according to the previous design as it is allowed to synthesize controllers ensuring the stability of the closed loop system. For this reason, we adopt the following notation $e_k = \xi_k - x_e^k = \left[e_1 \; e_2 \; \ldots \; e_k \right]^T$, $\bar{x}_e = \left[x_e \; x_e^{(1)} \; \ldots \; x_e^{(\gamma_i - 1)} \right]^T$, where $\bar{x}_e^k$, kth time derivative of the reference input x_e which is assumed to be a smooth function of time. Consequently, one may prove that the ξ-subsystem of Eq. (10) will be equivalent to the following more compact form:

$$\dot{e} = \bar{f}_i\left(e, \bar{x}_e\right) + \bar{g}_i\left(e, \bar{x}_e\right) u_i, i = 1, \ldots, N \tag{14}$$

where $\bar{f}_i\left(e, \bar{x}_e\right) = A_i e + b_i L_{f_i}^{\gamma_i} h\left(x\right)$, $\bar{g}_i\left(e, \bar{x}_e\right) = b_i L_{g_i} L_{f_i}^{\gamma_i - 1} h\left(x\right)$ are $\gamma_i \times 1$ vector functions, and

$$A_i = \begin{bmatrix} 0 \; 1 \; 0 \ldots 0 \\ 0 \; 0 \; 1 \ldots 0 \\ \vdots \qquad \quad \vdots \\ 0 \; 0 \; 0 \ldots 0 \\ 0 \; 0 \; 0 \ldots 0 \end{bmatrix}, \; \mathrm{b}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

are $\gamma_i \times \gamma_i$ matrix and $\gamma_i \times 1$ vector, respectively.

We use the above normal form to construct a control Lyapunov Function for each mode of the switched system. A sufficient condition to construct CLF is provided in the following theorem.

Theorem 3 *Consider the system (2) with the form (14), a simple choice for a Control Lyapunov Function is a quadratic function:*

$$\overline{V_i} = e^T P_i e \tag{16}$$

where P_i is a positive definite matrix chosen so that $A_i^T P_i + P_i A_i - P_i b_i b_i^T P_i < 0$.

We must note that the Lyapunov functions $\bar{V}_i$ used in designing the controllers are equal to the Lyapunov functions V_i used in implementing the switching rules because the robust relative degree γ_i is equal to order n of the system.

Using these quadratic CLFs, a controller can be designed for each mode using (12) applied to the system (14). By means of a standard Lyapunov argument, it can be shown that each controller asymptotically stabilizes the e-states in each mode. This result with the Assumptions 3 and 4 can then show that the closed-loop system (14), for each individual mode, is asymptotically stable.

5 Results and Discussion

In this section, we show the applicability and effectiveness of our approach on the CSTR example illustrating the main results of the paper. Let's revisit the CSTR system (1) presented in Sect. 2.

Defining $x = \left[x_1 \ x_2 \right] = \left[C_A \ T \right]$, $u = [Q_r]$, and $y = [T]$. The model of CSTR (1) can be written under the same form of system (2). Hence, we have:

$$f_i(x) = \begin{bmatrix} -Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right) x_1 + \left(C_{A_0} - x_1\right) \frac{F_r}{V_0} + (i-1)\left(C^*_{A_0} - x_1\right) \frac{F^*_r}{V_0} \\ \gamma_r Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right) x_1 + \left(T_{A_0} - x_2\right) \frac{F_r}{V_0} + (i-1)\left(T^*_{A_0} - x_2\right) \frac{F^*_r}{V_0} \end{bmatrix},$$

$$g_i(x) = \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} \text{ and } h(x) = x_2$$

We apply the approach presented in Sect. 3, the system given by (1) satisfying the Assumptions 2 and 3 is transformed by the two following modes:

- Mode 1:

$$\begin{cases} \dot{\xi}_1 = \underbrace{\gamma_r Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right) x_1 + \left(T_{A_0} - x_2\right) \frac{F_r}{V_0}}_{\xi_2} + \underbrace{u_1}_{\psi_{1,1}(x,u_1)} \\ \dot{\xi}_2 = L^2_{f_1} h(x) + L_{g_1} L_{f_1} h(x) u_1 \end{cases} \tag{17}$$

where $\begin{cases} L^2_{f_1} h(x) = \left[\gamma_r Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right)\right] \times \left[Z_r x_1 \ exp\left(-\frac{E_a}{R_r x_2}\right) + \left(C_{A_0} - x_1\right) \frac{F_r}{V_0}\right] \\ + \left[\left(\frac{\gamma_r Z_r E_a x_1}{R_r x_2^2}\right) \ exp\left(-\frac{E_a}{R_r x_2}\right) + \frac{F_r}{V_0}\right] \times \left[\gamma_r Z_r x_1 \ exp\left(-\frac{E_a}{R_r x_2}\right) - \left(T_{A_0} - x_2\right) \frac{F_r}{V_0}\right] \end{cases}$

and $L_{g_1} L_{f_1} h(x) = \left(\frac{\gamma_r Z_r E_a x_1}{R_r x_2^2}\right) \ exp\left(-\frac{E_a}{R_r x_2}\right) + \frac{F_r}{V_0}$

- Mode 2:

$$\begin{cases} \dot{\xi}_1 = \underbrace{\gamma_r Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right) x_1 + \left(T_{A_0} - x_2\right) \frac{F_r}{V_0} + \left(T^*_{A_0} - x_2\right) \frac{F^*_r}{V_0}}_{\xi_2} + \underbrace{u_2}_{\psi_{2,1}(x,u_2)} \\ \dot{\xi}_2 = L^2_{f_2} h(x) + L_{g_2} L_{f_2} h(x) u_2 \end{cases} \tag{18}$$

where $\begin{cases} L^2_{f_1} h(x) = \left[\gamma_r Z_r \ exp\left(-\frac{E_a}{R_r x_2}\right)\right] \times \left[Z_r x_1 \ exp\left(-\frac{E_a}{R_r x_2}\right) + \left(C_{A_0} - x_1\right) \frac{F_r}{V_0} + \left(C^*_{A_0} - x_1\right) \frac{F^*_r}{V_0}\right] \\ + \left[\left(\frac{\gamma_r Z_r E_a x_1}{R_r x_2^2}\right) exp\left(-\frac{E_a}{R_r x_2}\right) + \frac{F_r}{V_0} + \frac{F^*_r}{V_0}\right] \times \left[\gamma_r Z_r x_1 \ exp\left(-\frac{E_a}{R_r x_2}\right) - \left(T_{A_0} - x_2\right) \frac{F_r}{V_0} + \left(T^*_{A_0} - x_2\right) \frac{F^*_r}{V_0}\right] \end{cases}$

and

$$L_{g_2}L_{f_2}h\,(x) = \left(\frac{\gamma_r Z_r E_a x_1}{R_r x_2^2}\right)\, exp\left(-\frac{E_a}{R_r x_2}\right) + \frac{F_r}{V_0} + \frac{F_r^*}{V_0}$$

with $e = \left[\, e_1 = \xi_1 - x_e \;\; e_2 = \xi_2 - \dot{x}_e \,\right]^T$, a scalar system under the same form of system (14), describing the approximate input–output dynamics, can be obtained for controller design:

- For mode 1:

$$\dot{e} = \bar{f}_1\,(e, \bar{x}_e) + \bar{g}_1\,(e, \bar{x}_e)\,u_1 \tag{19}$$

where $\bar{f}_1\,(e, \bar{x}_e) = A_1 e + b_1 L_{f_1}^2 h(x)$, $\bar{g}_i\,(e, \bar{x}_e) = b_1 L_{g_1} L_{f_1} h(x)$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathrm{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- For mode 2:

$$\dot{e} = \bar{f}_2\,(e, \bar{x}_e) + \bar{g}_2\,(e, \bar{x}_e)\,u_2 \tag{20}$$

where $\bar{f}_2\,(e, \bar{x}_e) = A_2 e + b_2 L_{f_2}^2 h(x)$, $\bar{g}_2\,(e, \bar{x}_e) = b_2 L_{g_2} L_{f_2} h(x)$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For each mode $i\,(i = 1, 2)$ the relative degree $\gamma_i = 2$, then the choice $V_i = \bar{V}_i$ is sufficient.

We construct the controllers and for each mode $i\,(i = 1, 2)$ under the same form of Eq. (14) satisfying the Theorem 3, we choose the following quadratic Lyapunov functions:

- $V_1 = \bar{V}_1$ for mode 1:

$$V_1 = \bar{V}_1 = \frac{1}{2}c_1 e_1^2 + \frac{1}{2}c_2 e_2^2 \tag{21}$$

- $V_2 = \bar{V}_2$ for mode 2:

$$V_2 = \bar{V}_2 = \frac{1}{2}c_3 e_1^2 + \frac{1}{2}c_4 e_2^2 \tag{22}$$

The stabilizing controller u_1 is:

$$u_1 = -\left(L_{\bar{g}_1}\bar{V}_1\right) \times \left(\frac{L_{\bar{f}_1}\bar{V}_1 + 1.2V_1 + \left(\left(L_{\bar{f}_1}\bar{V}_1 + 1.2\bar{V}_1\right)^2 + \left(u_1^{max} L_{\bar{g}_1}\bar{V}_1\right)^4\right)^{\frac{1}{2}}}{\left(L_{\bar{g}_1}\bar{V}_1\right)^2 \left(1 + \left(u_1^{max} L_{\bar{g}_1}\bar{V}_1\right)^2\right)^{\frac{1}{2}}}\right) \tag{23}$$

The stabilizing controller u_2 is:

$$u_2 = -\left(L_{\bar{g}_2}\bar{V}_2\right) \times \left(\frac{L_{\bar{f}_2}\bar{V}_2 + 2.3V_2 + \left(\left(L_{\bar{f}_2}\bar{V}_2 + 2.3\bar{V}_2\right)^2 + \left(u_2^{max}L_{\bar{g}_2}\bar{V}_2\right)^4\right)^{\frac{1}{2}}}{\left(L_{\bar{g}_2}\bar{V}_2\right)^2\left(1 + \left(u_2^{max}L_{\bar{g}_2}\bar{V}_2\right)^2\right)^{\frac{1}{2}}} \right) \quad (24)$$

In order to validate the performance of the proposed approach, we have performed the simulations on Matlab.

A first simulation study is shown in Figs. 1 and 2. In these figures (solid lines) we respectively represent the evolution of the reaction temperature and the evolution of the control variable when the reactor is initialized at $x(0) = x_0 = \left[\,13\,\text{kmol}\,\text{m}^{-3}\;\; 293\,\text{K}\,\right]$ and is operating in mode1 for all times (without switching). We observe that for this mode the controller successfully stabilizes the reactor temperature at the desired steady-state ($x_e = 302.0\,\text{K}$).

Figures 1 and 2 (dashed lines) depict the result when the reactor (initialized at x_0 within) switches to mode 2 at a randomly chosen time t = *1.1* min. It is clear that in this case the controller is unable to stabilize the temperature at the desired steady-state. The reason is the fact that at $t = 1.1$ min, the state of the system lies

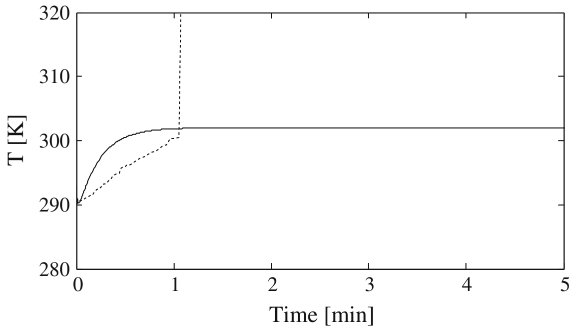

Fig. 1 Evolution of the reactor temperature when the reactor is initialized and operates in mode 1 (*solid*), when the reactor switches to mode 2 at t = 1.1 min (*dashed*)

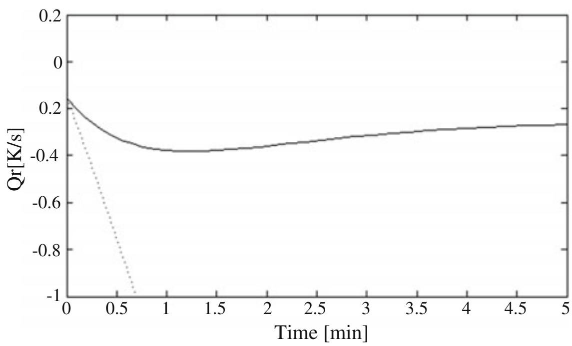

Fig. 2 Evolution of the controller when the reactor is initialized and operates in mode 1 (*solid*), when the reactor switches to mode 2 at *t =1.1* min (*dashed*)

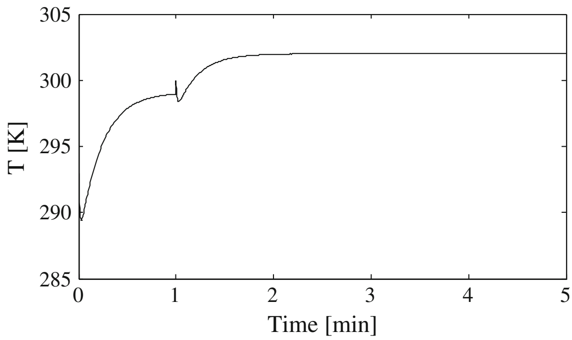

Fig. 3 Evolution of the reactor temperature while applying the Theorem 2

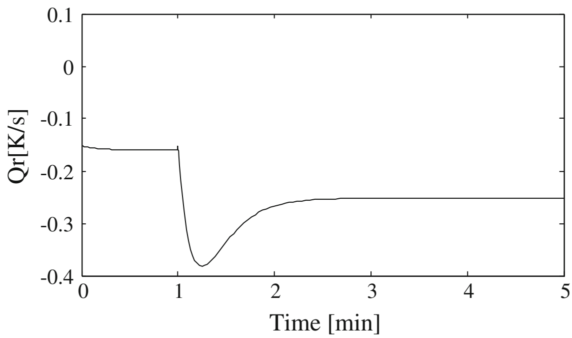

Fig. 4 Evolution of the controller while applying the Theorem 2

outside the stability region of mode 2 and, therefore, the available control action is insufficient to achieve stabilization.

To avoid this instability, we used the switching scheme proposed in Theorem 2 in a second study. The simulation results representing the evolution of the reaction temperature and the evolution of the control variable are respectively given by Figs. 3 and 4. It appears in these figures that the controllers successfully drive the reactor temperature to the desired steady-state ($x_e = 302.0\,\text{K}$) and maintain it there with the available control action.

6 Conclusions

In this chapter, we have considered the global stabilization problem of a class of non-minimum phase switched nonlinear systems where the global stabilization problem of individual subsystems is not assumed to be solvable when applying the formalism of the input–output feedback linearization.

Based on the MLFs method and the Hauser's approximation, we have designed state feedback controllers of subsystems and constructed a switching law, which guarantees global asymptotic stability of the corresponding closed-loop system.

The main idea is the coupling between the switching strategy and the stability regions arising from the limitations imposed by the input constraints. A set of switching rules is designed to stabilize the transitions between the stability regions associated for each mode. We demonstrated the efficiency of the proposed approach through a non-minimum phase CSTR example.

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