Chapter 6
Force Limited Vibration Using the Apparent Mass Method

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Abstract Force Limited Vibration is a technique that reduces the severity of over-tests during vibration testing by limiting the acceleration input based on the maximum expected in-service interface force. In this paper, the sources of over testing are reviewed and a new method called the Apparent Mass method for deriving the maximum in-service interface force is presented. In this method, the maximum in-service interface force over a given bandwidth is predicted using the full three dimensional linear-elastic theory supplemented by the necessary conditions for a representative vibration test, the assumption that the test article and its in-service mounting structure are lightly damped, and that the in-service forces exciting the mounting structure are broadband. The resulting equation is shown by numerical example to produce significant notching without producing an under-test situation, even for extremely light test articles.

Keywords Interface • Force • Limiting • Vibration • Apparent mass

Nomenclature

A	Acceleration
c	Damping
E	Young's modulus of elasticity
f_{nn}	Natural frequency nn of coupled system
f_o	Notched mode of unit under test
F	Force
$F_{\text{int\_total\_max\_X}}$	Maximum total interface force in the X direction
H	Accelerance matrix
k	Stiffness
K	Stiffness matrix
m	Mass
m_{int}	Mass of nodes at the interface
M	Mass matrix
M_{app}	Apparent Mass matrix
M_{app_total}	Total Apparent Mass in the X direction
$M_{eff\_n,X}$	Effective Mass of mode n in the X direction
$M_{\text{int\_app}}$	Interface Apparent Mass matrix
$S_{\ddot{x}\ddot{x}\text{\_test}}$	Test acceleration PSD in the X direction
$S_{ff\text{\_int\_total\_max\_X}}$	Maximum total interface force PSD in the X direction
u	Displacement
$\ddot{X}$	Acceleration in the X direction
υ	Poisson's ratio
ρ	Density

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6.1 Introduction

Force Limiting Vibration (FLV) testing is a method of notching the acceleration input during a vibration test to avoid severely over testing the Unit Under Test (UUT). In FLV, the total interface force is measured in situ and the input acceleration is notched automatically in real time to prevent the total interface force from exceeding a prescribed value. Consequently, the method requires the frequency specification of both acceleration and interface force.

This paper aims to provide a method for determining force limits for use in FLV by examining the interface forces predicted by linear Frequency Response Function (FRF) theory between two arbitrary bodies in three dimensions.

6.2 Roots of the Over-Test Problem

The vibration over-test phenomenon has been documented for about 60 years [1] and is the reason why the input acceleration is often notched (i.e. reduced) at and near the main natural frequencies of the UUT during vibration testing. There are two possible causes of an over-test and it is important to understand how they both contribute to an over-test situation.

The first cause is an interface impedance difference between the in-service support structure and the test support structure. In service, the UUT can push back on its interface to reduce the acceleration at its interface. During testing however, the shaker control system simply pushes harder to compensate for any pushback from the UUT in order to meet the programmed test spectrum. The amount of push–back is largest at frequencies where the UUT has modes with high Effective Masses and low damping, and it is these modes that we traditionally wish to notch when using FLV.

The second cause is the enveloping process. The test acceleration can be based on one of two cases. In the first case, the test specification is an envelope of the measured or predicted interface acceleration based on the coupled structure response, i.e. the UUT is attached to its in-service supporting structure. In the second case, the test specification is an envelope of the measured or predicted interface acceleration based on the un-coupled supporting structure response, i.e. the UUT is not attached to its intended supporting structure, and the acceleration at the intended interface is used as a basis for the test specification. An enveloping process is necessary to produce a simplified test spectrum that accounts for possible variations. Regardless of which case is used to derive the test specification, the enveloping process does a good job enveloping the peaks accelerations of the data but significantly over-estimates the acceleration at the valleys. Moreover, because of the vibration absorber effect, some these valleys correspond to the natural frequencies of the UUT in its test configuration.

As an example, consider Figs. 6.1 and 6.2. The lower portions of Figs. 6.1 and 6.2 show the apparent mass of two UUT, each with the same natural frequencies of 100 and 263 Hz but with different masses. The difference between the Test Specification and the Coupled Data in the upper portions of these figures is the amount of over-testing present. Note that the over-test is typically large at the natural frequencies of the UUT because of the vibration absorber effect. The un-coupled data represents the interface acceleration as if the support structure was unaffected by the presence of the UUT, meaning no impedance mismatch just like the test configuration. The systems in Figs. 6.1 and 6.2 were chosen to demonstrate a situation when both the enveloping process and the impedance difference contribute to the over-test (at 100 Hz in Fig. 6.1), when only the impedance difference contributes to the over-test (263 Hz in Fig. 6.1), when the over-test is almost all due to the enveloping process (100 Hz in Fig. 6.2), and when essentially no over-test is present (263 Hz in Fig. 6.2).

In these examples, the UUT first natural frequency of 100 Hz has been chosen to be very different from any of the natural frequencies of the un-coupled supporting structure of 69, 175, 263, and 362 Hz. This can lead to large over-testing from the enveloping process, even if the UUT has a very small mass compared to its supporting structure. Conversely, the UUT second natural frequency of 263 Hz has been chosen to correspond to one of the natural frequencies of the un-coupled supporting structure. In this instance, the cause the over-test is typically dominated by the impedance difference such that the UUT mass compared to its mounting structure becomes an important factor.

6.3 Force Limited Testing

The FLV notching method automatically adjusts the test input to limit the interface force to a level that is representative of the maximum anticipated in-service interface force. By Newton's second law, the sum of the applied forces on any flexible or rigid body is directly proportional to the acceleration of its instantaneous center of mass. Thus, FLV effectively limits the maximum acceleration of the center of mass during testing. This cannot be accomplished by placing an accelerometer at, or

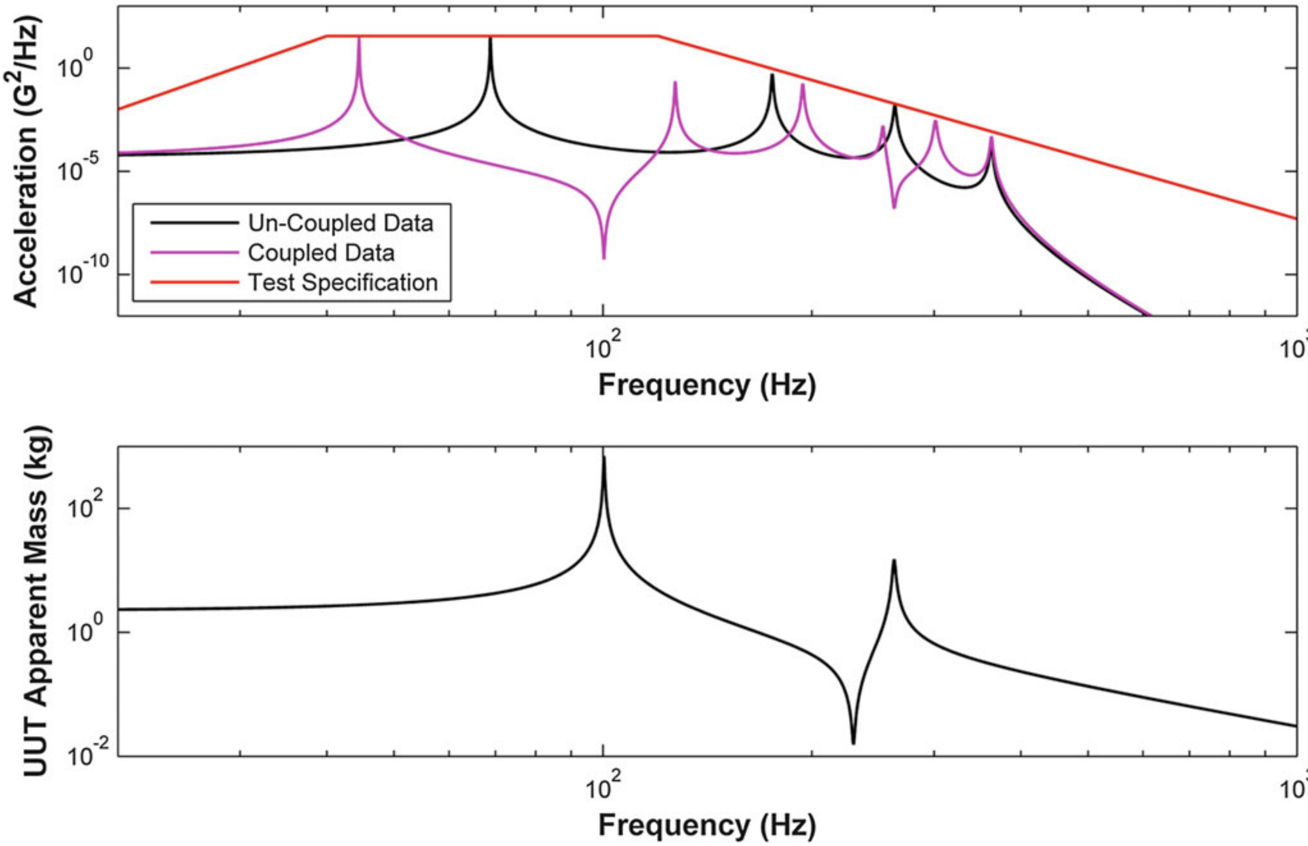

Fig. 6.1 Vibration Over-Testing Example, UUT to Supporting Structure Mass Ratio = 0.5

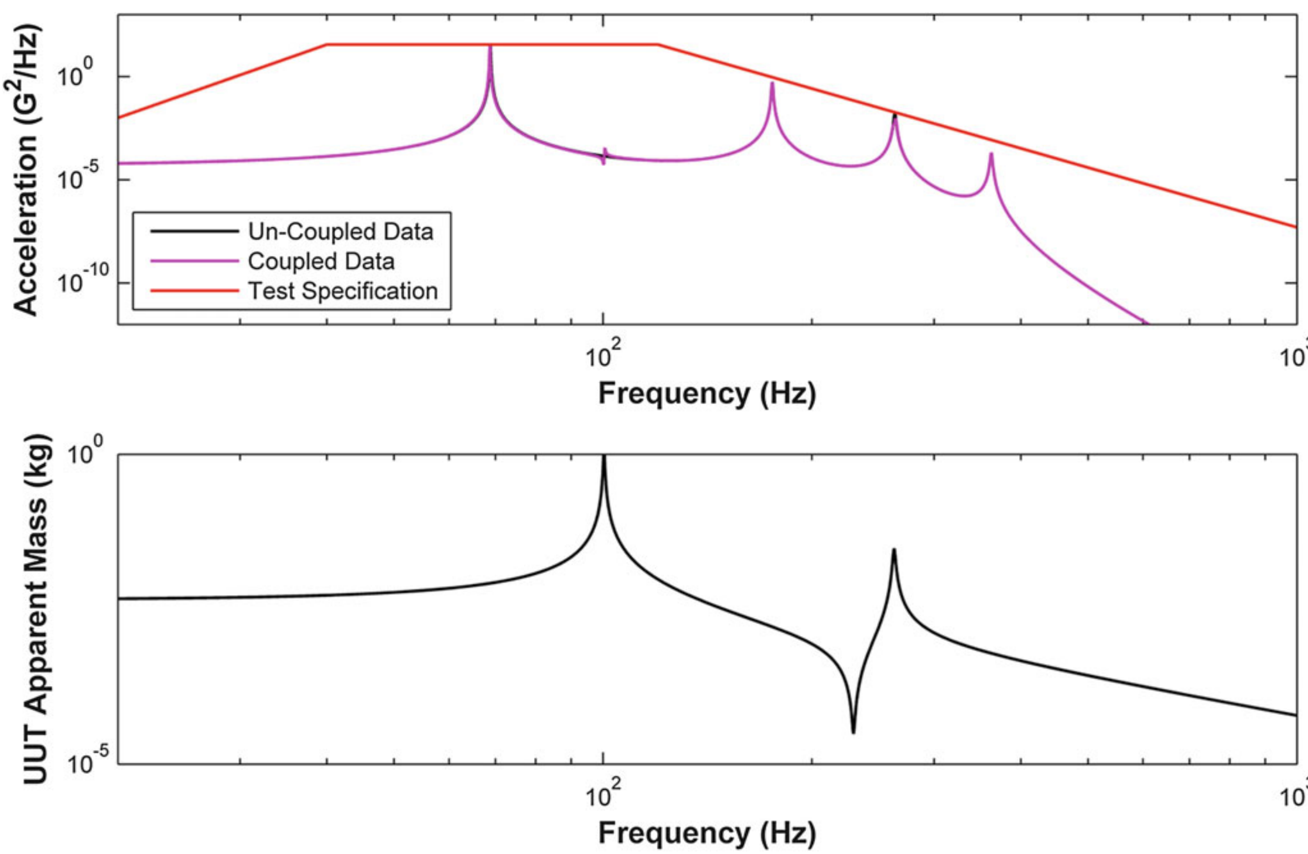

Fig. 6.2 Vibration Over-Testing Example, UUT to Supporting Structure Mass Ratio = 0.001

near, the center of mass because the instantaneous center of mass moves when the structure deforms. Even if the movement is small, the center of mass can have very large accelerations because the acceleration is proportional to both the displacement and the square of the (circular) excitation frequency.

The amount of over testing can be significant. For instance, in-situ acceleration Power Spectral Density (PSD) measurements during launch of the Advanced Composition Explorer spacecraft showed that un-notched random spectral vibration test levels were about 20 dB (~100 times) higher than had occurred during flight near three of its instruments [2]. Similar measurements for a sounding rocket also demonstrated that random spectral vibration test levels were about 20 dB (~100 times) higher during un-notched testing than had occurred during launch at the main natural frequency of the UUT [3]. Smallwood has even suggested that the over-test in acceleration PSD test level can be in practice as much as 30 dB (~1000 times) higher than the in-field level [4].

Force limiting can often provide significant notching even when the force limit is grossly over-estimated. For example, consider reference [1] where the FLV testing on the SVF-2 experiment is described. The PSD force limits used during testing produced a notch of 16 dB, even when the PSD force limit was 10 times greater than the actual force PSD measured in-flight. This is a very important feature of FLV. It means that approximate methods can be used to estimate the maximum interface force and is why there are several successful, but approximate, methods available [1] to estimate the interface force limit and why each method can produce significant reduction in over-testing even if they don't predict the same maximum in-service interface force.

6.4 Linear Dynamic Equations of Motion

The equation of motion of a flexible structure can be derived by considering the force distribution on the structure, Newton's second law applied to differential elements on the structure, the constitutive relationship of the material (i.e. stress-strain curve), and enforcing compatibility such that the material deforms in such a way as to not leave gaps or overlap with itself. The resulting equation is a non-linear differential equation of motion in three dimensions applicable at any instant for any arbitrary force (sine, random, shock, etc.). For example, the resulting equation for a homogeneous isotropic material is given by the Navier elasto-dynamic wave equation [5] as

$$\frac{E}{2(1+\upsilon)}\nabla^2\vec{u}+\left(\frac{\nu E}{(1+\upsilon)(1-2\upsilon)}+\frac{E}{2(1+\upsilon)}\right)\nabla(\nabla\cdot\vec{u})+\vec{F}=\rho\ddot{\vec{u}} \tag{6.1}$$

Note that moments are not part of the general equation of motion, as you cannot apply a moment on a differential element. In general, Eq. (6.1) is difficult to solve, so in practice one typically must resort to approximate methods like Finite Element Analysis (FEA) to solve for the motion of a structure.

Linear FEA transforms the non-linear differential equation Eq. (6.1) into a set of linear discrete differential equations of motion by assuming small strains, small displacements, and linear elastic material behavior. The result is a matrix equation of the form

$$[M]\{\ddot{U}\}+[K]\{U\}=\{F\} \tag{6.2}$$

Damping can be added to the FEA formulation. Like the general differential element upon which it is based, the nodes of a general 3D finite element cannot support rotations or moments at nodes. Node rotations and moments are concepts used to simplify the general 3D element into simpler elements such as plates and beams. Barring finite precision mathematical issues, the FEA formulation will converge to the exact differential equation as the number of nodes is increased. Thus, in principle, the FEA formulation is exact to any desired precision. It is with this idealized FEA formulation, and all associated assumptions, upon which the Apparent Mass method presented in this paper is based on. It is also with this linearized formulation that Frequency Response Function (FRF) theory can be applied.

6.5 Definitions Used for Apparent Masses

FRF theory expresses quantities that relate the steady state outputs in terms of inputs. One of these quantities is the Apparent Mass that expresses the forces in terms of accelerations. There are four different but related types of quantities that will be discussed that can be associated with the term Apparent Mass. For clarity, these will be labelled: Apparent Mass, Interface Apparent Mass, Test (Interface) Apparent Mass, and Total (Test Interface) Apparent Mass. The words in parentheses will be typically omitted for brevity but are included here to show the hierarchy of the concepts.

In FRF theory, the Apparent Mass is defined as the inverse of the Accelerance FRF matrix such that

$$\{F\}_{3Nx1} = \lfloor M_{app} \rfloor_{3Nx3N} \{A\}_{3Nx1} \tag{6.3}$$

Each matrix element of the Apparent Mass is a complex valued frequency-dependent quantity. The Apparent Mass gives the relationship between the forces at ANY node given the acceleration of EVERY node on the body. It is a 3N × 3N square matrix, where N is the number of nodes representing the body and the factor of 3 arises from the three translational Degrees-Of-Freedom (DOF) associated with translational movement in three dimensions.

The Accelerance matrix can be written in terms of p interface points and r non-interface points as

$$\left\{ \begin{array}{c} \{A_{\text{int}}\}_{3px1} \\ \{A_{non-\text{int}}\}_{3rx1} \end{array} \right\}_{3Nx1} = \begin{bmatrix} [H_{11}]_{3px3p} & [H_{12}]_{3px3r} \\ [H_{21}]_{3rx3p} & [H_{22}]_{3rx3r} \end{bmatrix}_{3Nx3N} \left\{ \begin{array}{c} \{F_{\text{int}}\}_{3px1} \\ \{F_{\text{non}-\text{int}}\}_{3rx1} \end{array} \right\}_{3Nx1} \tag{6.4}$$

Assuming only interface forces are present, i.e. $\{F_{\text{non-int}}\} = 0$, leads to the relationship

$$\{A_{\text{int}}\}_{3px1} = [H_{11}]_{3px3p} \{F_{\text{int}}\}_{3px1} \tag{6.5}$$

The Interface Apparent Mass is defined in terms of the inverse of one of the Accelerance sub-matrices as

$$\left[M_{\text{int\_app}}\right]_{3px3p} = [H_{11}]^{-1}_{3px3p} \tag{6.6}$$

such that the interface forces can be described in terms of interface accelerations as

$$\{F_{\text{int}}\}_{3px1} = \lfloor M_{\text{int\_app}} \rfloor_{3px3p} \{A_{\text{int}}\}_{3px1} \tag{6.7}$$

Unlike the Apparent Mass, the Interface Apparent Mass matrix gives the relationship between the forces on INTERFACE nodes given accelerations of the INTERFACE nodes when ONLY INTERFACE FORCES are present. The matrix is a 3p × 3p square matrix where p is the number of nodes on the interface and typically p ≪N. Note that p must consist of at least N = 3 non-collinear points in order to have a stable interface. Although the elements in the Interface Apparent Mass matrix are different than the elements in the Apparent Mass (i.e. the Interface Apparent Mass is not a sub-matrix of the Apparent Mass matrix), the Apparent Mass and the Interface Apparent Mass provide identical answers for the interface force when only interface forces are acting on the body.

The Test (Interface) Apparent Mass is the same as the Interface Apparent mass but with the added two conditions that the interface is perfectly rigid, such that we can set N = 3, and moves in a single translational motion. These restrictions are idealized versions of the interface during testing. For the X direction, the forces at the interface reduce to

$$\left\{ \begin{array}{c} \{F_{\text{int\_}X}\}_{3x1} \\ \{F_{\text{int\_}Y}\}_{3x1} \\ \{F_{\text{int\_}Z}\}_{3x1} \end{array} \right\}_{9x1} = \begin{bmatrix} \left[M_{\text{int\_app\_}XX}\right]_{3x3} & \left[M_{\text{int\_app\_}XY}\right]_{3x3} & \left[M_{\text{int\_app\_}XZ}\right]_{3x3} \\ \left[M_{\text{int\_app\_}YX}\right]_{3x3} & \left[M_{\text{int\_app\_}YY}\right]_{3x3} & \left[M_{\text{int\_app\_}YZ}\right]_{3x3} \\ \left[M_{\text{int\_app\_}ZX}\right]_{3x3} & \left[M_{\text{int\_app\_}ZY}\right]_{3x3} & \left[M_{\text{int\_app\_}ZZ}\right]_{3x3} \end{bmatrix}_{9x9} \left\{ \begin{array}{c} \{1\}_{3x1} \\ \{0\}_{3x1} \\ \{0\}_{3x1} \end{array} \right\}_{9x1} \cdot \ddot{X}_{\text{int\_}X} \tag{6.8}$$

which, for the force in the X direction, reduces further to

$$\{F_{\text{int\_}X}\}_{3x1} = \left(\left[M_{\text{int\_app\_}XX}\right]_{3x3} \{1\}_{3x1}\right)_{3\text{x1, test\_app\_X}} \cdot \ddot{X}_{\text{int\_}X} \tag{6.9}$$

The Test Apparent Mass is a 3 $\times$ 1 matrix because the rigid interface can be expressed with any 3 non-collinear nodes on the interface.

In FLV, the ratio of the total interface force divided by the rigid interface acceleration is of interest. This quantity is a non-matrix FRF that can be expressed as

$$F_{\text{int\_}X} = \{1\}_{1x3}\left[M_{\text{int\_app\_}XX}\right]_{3x3}\{1\}_{3x1} \cdot \ddot{X}_{\text{int\_}X} \tag{6.10}$$

Total (Test Interface) Apparent Mass in the X direction is a non-matrix quantity that is defined as the value

$$M_{app\_total\_X} = \{1\}_{1x3}\left\lfloor M_{\text{int\_app\_}XX}\right\rfloor_{3x3}\{1\}_{3x1} \tag{6.11}$$

The Total Apparent Mass is the FRF that is measured during testing and will be used by the Apparent Mass FLV method. It gives the relationship between the total interface force given the interface acceleration when only interface forces are present, the interface is perfectly rigid, and the interface is restricted to move in a single translational direction.

The Total Apparent Mass has the special property that it can be expressed as a modal sum of Effective Masses and the Single Degree-of-Freedom (SDOF) force transmissibility functions (T_{SDOF_n}) as

$$M_{app\_total\_X}(f) = m_{\text{int}} + \sum_{n=1}^{r} M_{eff\_n,x} \cdot T_{SDOF\_n}(f) \tag{6.12}$$

The mass of the interface nodes, m_{int}, is often omitted in texts since it can be arbitrarily small, but is included here for completeness of expressing the total force on the interface. The Effective Masses are the mass participation factors, as calculated by FEA programs, multiplied by the total mass. The Effective Masses are a function of the mass matrix, a given mode shape Φ_n, and the rigid body mode shape Ψ when the interface is restricted to move in the single direction considered. For a diagonal mass matrix, the Effective Mass of mode n in direction X can be expressed as [6]

$$M_{eff\_n,X} = \frac{\left(\left(\{\Phi_n\}_{3rx1}\right)^T [M_{UUT}]_{3rx3r}\{\Psi_x\}_{3rx1}\right)^2_{1x1}}{\left(\left(\{\Phi_n\}_{3rx1}\right)^T [M_{UUT}]_{3rx3r}\{\Phi_n\}_{3rx1}\right)_{1x1}} \tag{6.13}$$

The Effective Mass concept can be derived using a single interface node with 6 DOF, 3 translational and 3 rotational [6]. This derivation uses the same three restrictions that we used so far, namely that only interface forces are present, that the interface is perfectly rigid, and that the movement only occurs in one direction at a time. With the 6 DOF of the interface node, one can derive 6 Effective Masses for each mode of the UUT. Although it is tempting to place these in a single 6 $\times$ 6 Effective Mass matrix, this practice should be avoided as it gives the impression that you can linearly combine the interface accelerations from different directions to get the correct resulting interface forces. This is not physically correct as it ignores cross-axis force contributions and non-linear rotational effects that arise from combining rotational and translational interface motion. Consequently, the Effective Masses will be written as a non-matrix quantity with the applicable direction indicated with a subscript.

6.6 Interface Forces

Consider the two previously discussed cases for deriving a test specification. In the first case, the test acceleration is based on enveloping the predicted or measured in-service acceleration at the base of the UUT when the UUT is attached (coupled) to its supporting structure. If only interface forces act on the UUT, the exact interface force at each attachment point p predicted by linear theory is given directly by the unit's Interface Apparent Mass as

$$\{F_{\text{int}}(f)\}_{3px1} = \left\lfloor M_{\text{app\_int\_UUT}}(f)\right\rfloor_{3px3p}\{A_{\text{int}}(f)\}_{3px1} \tag{6.14}$$

Applying the conditions that the interface is perfectly rigid and that the interface moves in a single translational direction, the exact total interface force for the X direction can be written in terms of the UUT's Total Apparent Mass as

$$F_{\text{int\_total\_X}}(f) = M_{\text{app\_total\_UUT}}(f) \cdot \ddot{X}_{\text{int}}(f) \tag{6.15}$$

In the second case, the test acceleration is based on enveloping the predicted or measured in-service acceleration of the supporting structure (base) at the intended interface to the UUT but without the presence of the UUT (i.e. un-coupled). If in service only interface forces act on the UUT, the exact interface force at each attachment point p is given by a combination of the supporting structure (base) and the UUT Interface Apparent Mass as

$$\{F_{\text{int}}(f)\}_{3px1} = \left[\left[M_{\text{app\_int\_UUT}}(f) \right]^{-1}_{3px3p} + \left[M_{\text{app\_int\_Base}}(f) \right]^{-1}_{3px3p} \right]^{-1}_{3px3p} \{A_{\text{int\_un-coupled}}(f)\}_{3px1} \tag{6.16}$$

Equation (6.16) is the general 3D elastic version of Norton's electrical circuit theorem. As its derivation is long, it could not be included in this paper. The only additional assumption needed to arrive at Eq. (6.16) was that the supporting structure's FRF obeyed reciprocity. As per the definition of Interface Apparent Mass, the supporting structure's Interface Apparent Mass must be computed at the UUT interface by assuming that only interface forces are present on the base. Interestingly, it is the Interface Apparent Mass of the supporting structure, which is based on having only interface forces acting on it, that is required in order to correctly predict the interface forces when non-interface forces are applied to the supporting structure. This was noticed in the derivation and has been verified for a numerical example.

Applying the conditions that the interface is perfectly rigid and that the interface moves in a single translational direction, the exact total interface force for the X direction can be written in terms of the structures' Interface Apparent Mass as

$$F_{\text{int\_total\_X}}(f) = \{1\}_{1x3p} \left[\left[M_{\text{app\_int\_UUT}}(f) \right]^{-1}_{3px3p} + \left[M_{\text{app\_int\_Base}}(f) \right]^{-1}_{3px3p} \right]^{-1}_{3px3p} \{1\}_{3px1} \cdot \ddot{X}_{\text{int\_un-coupled}}(f) \tag{6.17}$$

Results from Eqs. (6.15) and (6.17) give identical answers for the total interface force, even if one is based on Total Apparent Mass while the other is based on Interface Apparent Mass. The relationship between coupled and un-coupled interface accelerations can be obtained by equating Eqs. (6.14) and (6.16) to each other.

6.7 Conditions for a Representative Test

To properly evaluate the UUT's suitability for its service environment, the vibration test needs to be representative of the in-service environment. Although this statement may seem an obvious criterion for a valid vibration test, there are several assumptions inherent in it that may not be obvious but are nonetheless important and worth noting. A representative test is taken to mean that the response and stresses induced in the UUT during the test will be equal to or greater than, at every frequency and every point, to what it will experience in service. For this to occur, the following five conditions must be true or, at the very least, accepted as approximately true.

Condition 1: In-Service, only interface forces act on the UUT. Any forces, such as acoustic forces or body accelerations, acting directly on the UUT while in service will excite the mode shapes in a combination that cannot be replicated with any amount of fixed based excitation. The forces must therefore be negligible compared to the forces occurring at the interface.

Condition 2: In-service, all points on the interface move with the same direction, magnitude, and phase. In order to efficiently transmit the shaker force to the UUT interface, the structure between where the shaker force is generated and where the interface with the UUT begins (e.g. armature, slip plates, interface plates, etc.) needs to be as rigid as practically possible and be restricted to move only in the same linear direction of the shaker. The in-service configuration must also match this necessary test condition in order for the test response of the unit to be similar to the in-service response. This condition is fulfilled at low frequencies, where the UUT and its base move essentially as a rigid body, but becomes invalid beyond a certain frequency that will be referred to the maximum test frequency. Since the test is no longer representative beyond the maximum test frequency, any form of notching above this frequency becomes a questionable practice.

Condition 3: In service, the interface moves in the same single linear direction as during testing. Most vibration tests are performed with a single shaker. This limits the control of the interface motion to a single, usually linear, direction. In service, however, the interface is typically moving in several directions simultaneously and can rotate. Although the cross-axis movement and interface rotations occurring in-service will excite additional modes that are not excited during testing, we are forced to accept condition 3 as valid and carry on with testing because we cannot do otherwise with a single shaker. One way to look at the situation is that the test simulates the hypothetical ideal in-service situation where there are additional forces exciting the UUT's supporting structure such that the interface moves in a single direction.

Condition 4: The dynamic structural properties of the UUT are the same during testing as in service. The dynamics of the UUT must be the same, or nearly the same, during testing as in-service in order for the test to be representative. This can be an issue if the structural properties are time variant, temperature variant, or if a similar but different unit is tested (e.g. manufacturing tolerances). Condition 4 also entails the assumption that the boundary conditions during testing are representative of the boundary conditions in-service. In practice, this means that seemingly unimportant issues like the amount of bolt torque present can actually modify the structural response and degrade the representativeness of the test.

Condition 5: The test input specification envelops the peaks of the coupled response. For the test to induce equal or greater damage potential as in service, the test acceleration input must envelope the peak accelerations of the coupled in-service structure and not necessarily the un-coupled structure.

6.8 Apparent Mass Method

When the five test conditions are fulfilled, one can consider using either the interface force or the interface motion as the control to correctly reproduce the in-service response of the UUT during testing [7]. Conversely, this means that if the test is not representative, then neither the interface forces nor the interface accelerations are representative.

The Apparent Mass method takes the Apparent Mass FRF given by Eq. (6.3) and applies the necessary test conditions for a representative test to get a formula describing the in-service interface forces. Conditions 1 was the same condition used to define the Interface Apparent Mass. Conditions 1 to through 3 were the same conditions used to define the Test Apparent Mass and the Total Apparent Mass. Since both the Interface Apparent Mass and the Total Apparent Mass are compatible with the necessary test conditions, we can use either Eq. (6.15) or Eq. (6.17) to describe the in-service interface force, as both describe the same interface force. The Apparent Mass method uses the simpler Eq. (6.15) as its key equation to estimate the maximum in-service interface force.

The maximum interface force over any given bandwidth can be found by evaluating Eq. (6.15) at a single frequency. Therefore, the three variables we need to determine are the frequency at which the in-service maximum interface force occurs, the value of the Total Interface Apparent Mass at that frequency, and the value of the in-service acceleration at that frequency.

The Total Interface Apparent Mass can be measured directly at the start of testing using a low level sine or random survey as the ratio of the total interface force to the average interface acceleration, both in the excitation direction. Although we do not usually know the value of the in-service acceleration, we do have the test spectrum, and if the test is representative, then test condition 5 implies that

$$A_{\text{test}}(f) \geq A_{\text{int\_coupled}}(f) \tag{6.18}$$

Since at any given frequency the interface force is proportional to the interface acceleration, using Eq. (6.18) in Eq. (6.15) will always produce equal or higher force limits than if the actual coupled acceleration was used.

The only remaining variable left to determine is the frequency where the maximum in-service interface force occurs. In principle, this frequency can be any frequency. For example, if the supporting structure was excited using a single sinusoidal force of arbitrary frequency, the maximum interface force would occur at that frequency. However, if the coupled structure is lightly damped and is excited by a reasonably constant and broadband excitation, one can argue that the maximum response, and therefore maximum interface force, would occur at one of the coupled system's natural frequencies. Assuredly, if the structure had no damping, this would be the case for any form of broadband excitation because the steady state response at the coupled natural frequencies would go to infinity. Thus, to ensure that the maximum interface force occurs at one of the coupled natural frequencies, we will limit the applicability of the Apparent Mass method to structures that are lightly damped and excited by broadband excitation, at least in the vicinity of f_o, where f_o is the UUT's natural frequency we wish to notch.

Given that the maximum interface force occurs at one of the coupled natural frequencies, we only need to evaluate Eq. (6.15) at the coupled natural frequencies occurring over a bandwidth extending over the Total Apparent Mass response of the mode or modes we wish to notch. For the X direction, this can be written in equation form as

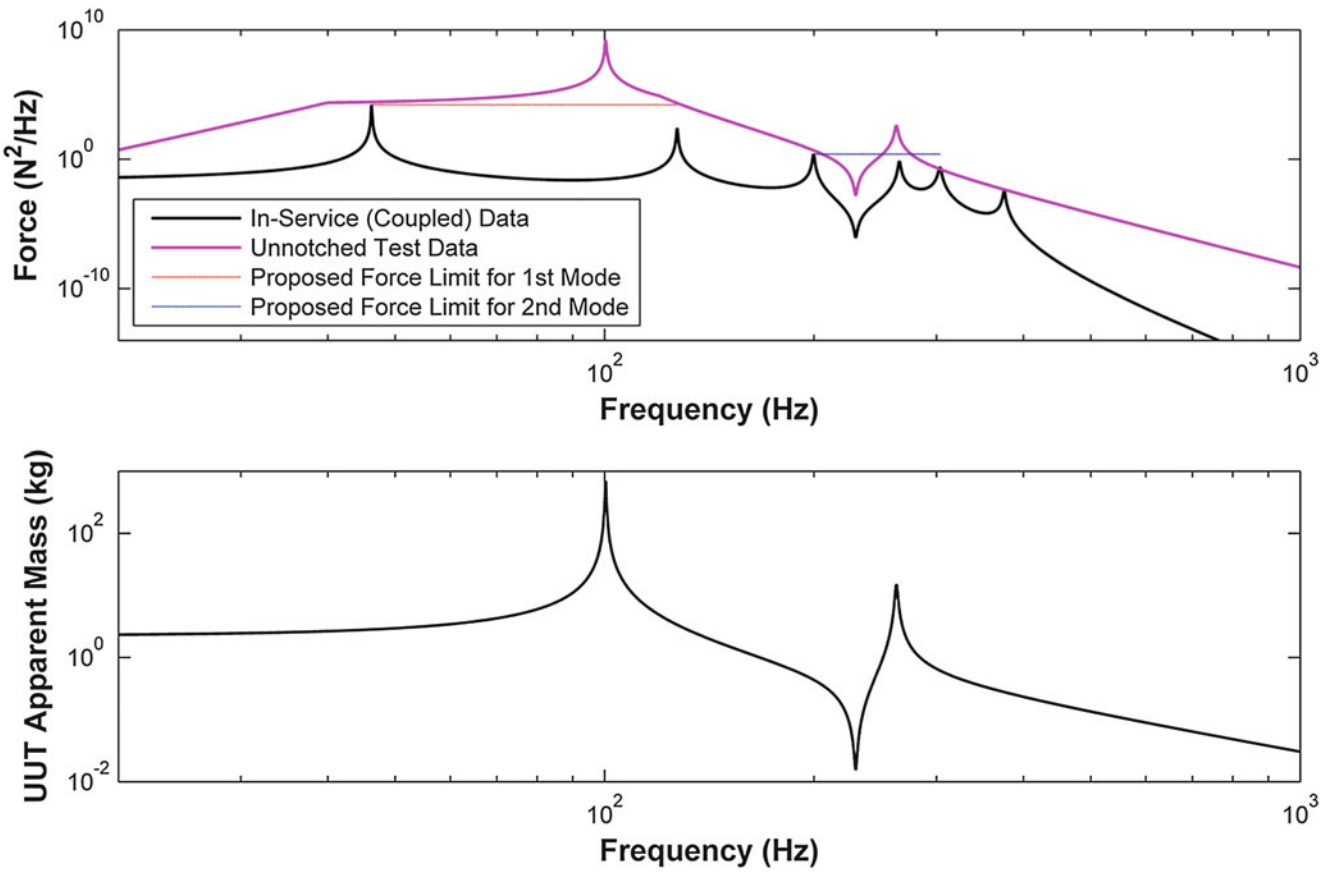

Fig. 6.3 Example of In-Service Interface Forces and Proposed Limits at Different Frequencies

$$F_{\text{int\_total\_max\_X}} = MAX\left[M_{\text{app\_total\_X}}(f_{nn}) \cdot \ddot{X}_{\text{test}}(f_{nn})\right]_{\text{set of } f_{nn} \text{ covering } f_o} \tag{6.19}$$

An example of this procedure is given in Fig. 6.3. Since the first mode of the UUT falls completely between the first and second modes of the coupled structure, the force limit for this mode is taken as the maximum result of Eq. (6.15) evaluated at these two modes and applied over the bandwidth defined by these two modes. The second mode of the UUT is very close to the 4th coupled mode and is better covered by considering the 3rd, 4th, and 5th modes of the coupled structure. The force limit for this mode is taken as the maximum result of Eq. (6.15) evaluated at these three modes and applied over the bandwidth defined by these three modes. In this way, the Apparent Mass method can be applied repeatedly to different modes that we wish to notch. Alternatively, we could have simply used all coupled modes given in the test bandwidth and take the maximum result of Eq. (6.15) evaluated at all these modes to define a single maximum force limit for the entire test.

While the total solution incorporates both the transient and the steady state parts, the solution given by FRF theory is only the steady-state part of the solution. This means that the Apparent Mass concept cannot be applied to transient force excitations such as shock. However, the theory can deal with both periodic excitations and random excitations. Although the ever changing random excitation is characterized more as transient than steady-state, if the random excitation can be described with a time-independent probability distribution (stationary) and if the average of any sample is representative of the average of the entire signal (ergodic), then FRF theory can predict the random responses in a probabilistic sense in terms of the Power Spectral Density (PSD). For random vibration, Eq. (6.19) becomes [6]

$$S_{ff\text{\_int\_total\_max\_X}} = MAX\left[\left|M_{\text{app\_total\_X}}(f_{nn})\right|^2 \cdot S_{\ddot{x}\ddot{x}\text{\_test}}(f_{nn})\right]_{\text{set of } f_{nn} \text{ covering } f_o} \tag{6.20}$$

Similar equations can be derived for the Y and Z directions. The force limits expressed by Eqs. (6.19) and (6.20) are based on the test acceleration. As such, the force limits derived by these equations have the same conservatism, or lack thereof, as the input acceleration.

6.9 Practical Application of the Apparent Mass Method

When the interface loads of a coupled load analysis is available, those loads should be used as the interface force limits. When such analysis is unavailable, the Apparent Mass method can be used to get a theoretically exact interface force limit, given the assumptions used in its derivation. One of the main benefits of the Apparent Mass Method is its ease of use, since only a minimum amount of information is required to correctly predict a maximum value of the interface force.

Since the Total Apparent Mass can be measured directly at the time of testing, the key variables needed when using the Apparent Mass method are the coupled natural frequencies. The coupled natural frequencies can be obtained using a coupled FEA modal analysis. This is an important feature of the apparent mass method as it is simpler to perform a single analysis and report the natural frequencies of the coupled system to all sub-components than to perform a full coupled loads analysis to calculate the maximum interface forces for every sub-component interface over numerous launch load cases.

However, there are times when the fully coupled modal analysis is lacking and one needs to resort to more approximate methods. For theoretically exact results, we need to evaluate the Total Apparent Mass of the UUT at coupled frequencies at either side of f_o. However, it will often be at the lowest coupled natural frequency closest to f_o, where the maximum force occurs. This assumption is included implicitly in the Semi-Empirical Method [1], where the force limit is always larger at frequencies lower than f_o. Using the same assumption, one can simplify the Eq. (6.19) or Eq. (6.20) and base the force limit on the lower frequency. This method is appealing because it is often easier to find or estimate the lower shift in natural frequency when the exact data is unavailable.

This lower mode assumption can be used in conjunction with defining the mode with the largest Effective Mass as f_o in order to derive an equivalent value for the C^2 factor employed in the Semi-Empirical Method [1]. This provides a quick and easy sanity check for the chosen value of C^2 (Figs. 6.4, 6.5, 6.6, and 6.7).

One way to estimate the shifted lower frequency is to employ a simplified FEA model of the support structure. One only needs to model the immediate support structure, such as the panel to which the UUT will be attached to in service, in order to get an estimate of the minimum frequency shift from f_o to $f_{nn\_before\_f0}$. In this method, the nodes at the edges of the simplified support model would be modelled as fixed in all possible degrees of freedom. This is equivalent to assuming that the un-modelled portion of the support structure is perfectly rigid. This always produces smaller frequency shifts than actually present because any added flexibility in the support structure will result in lower values of $f_{nn\_before\_f0}$. The region that one needs to model is arbitrary, with more modelling of the support structure giving more accurate results. As seen if Fig. 6.8, even small frequency shifts can provide significant reduction in interface force.

It is true that the interface force limit given by Eqs. (6.18) and (6.20) may not match the actual in-service maximum interface force, but in such cases the test is not representative and the acceleration will not match either. Even so, since large overestimates of the force limit can provide significant notching, Eqs. (6.19) and (6.20) can still be use to define a maximum value for the force limit. A comfort factor can then be applied e.g. x2 to calculate the force limit used in testing, For example, if the force limits prediction would produce an extreme notch, the force limit may be increased to give at least a 6 dB resonance response, as recommended in NASA 70004C [1].

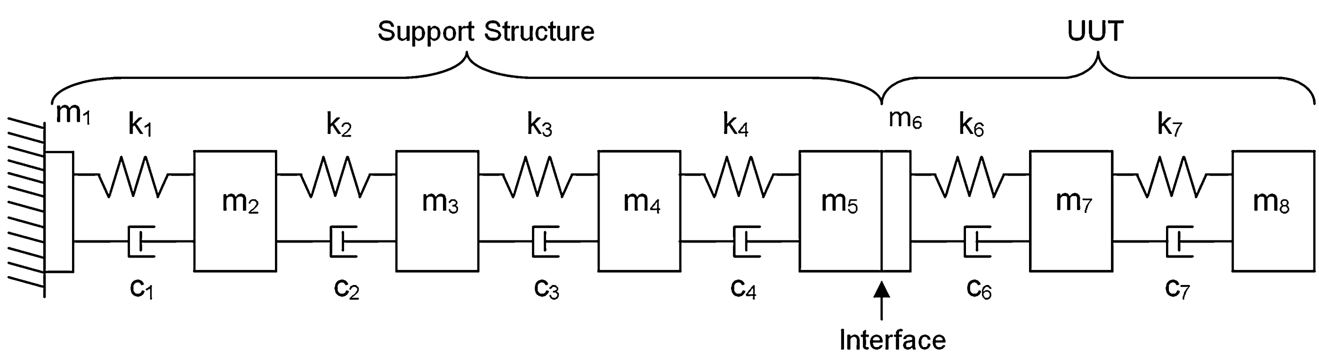

Fig. 6.4 Numerical Model in Coupled Configuration

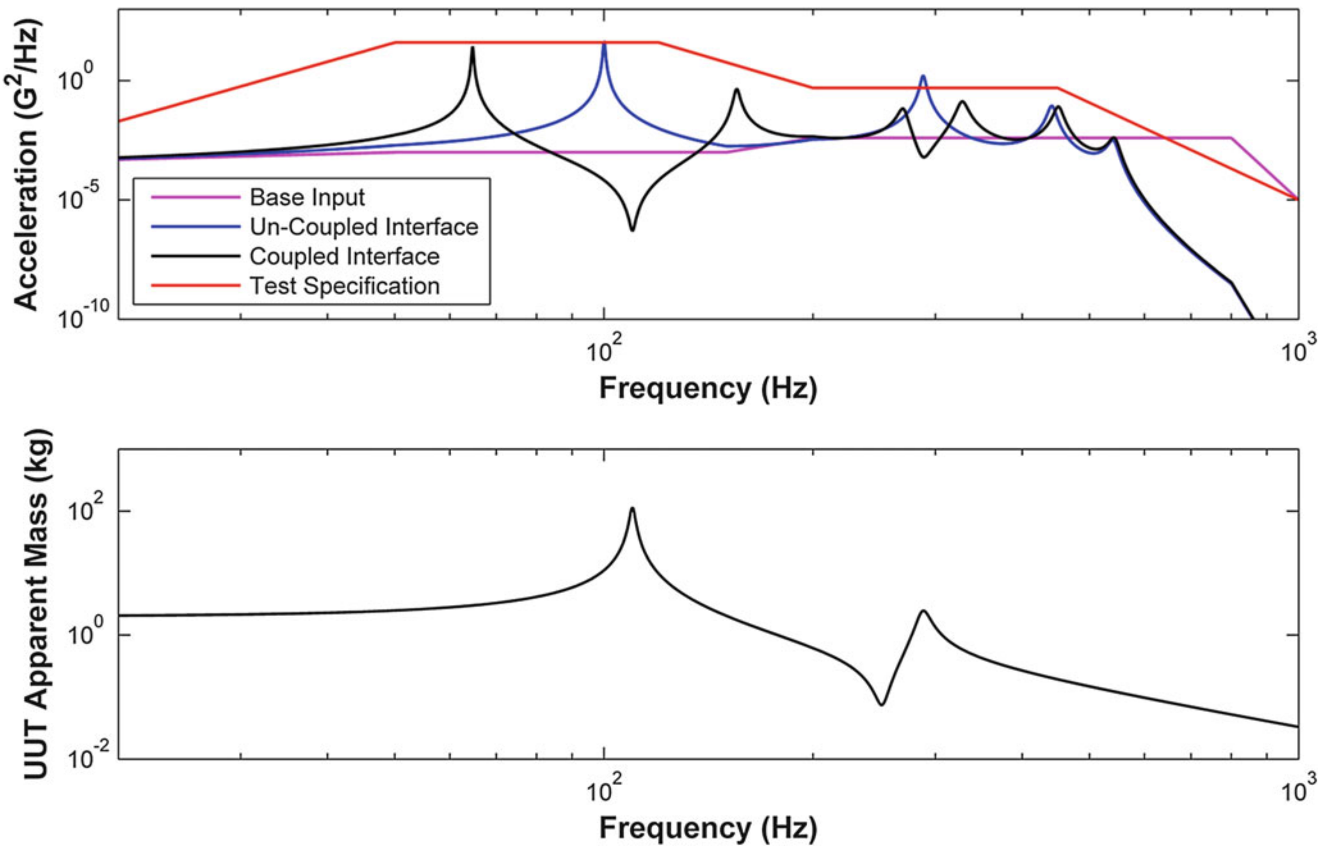

Fig. 6.5 Accelerations and UUT Apparent Mass, Case 1 (Load to Source Mass Ratio is 0.5)

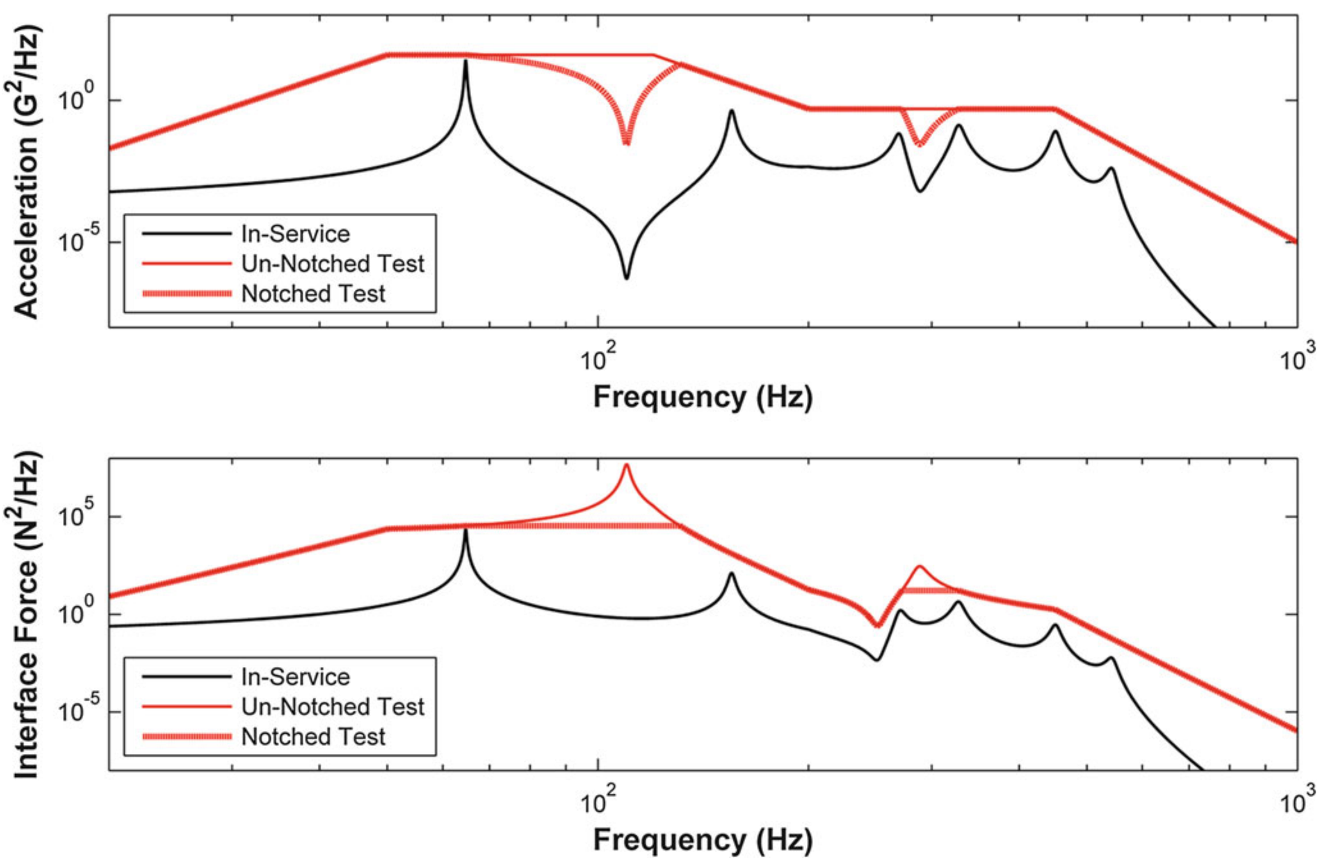

Fig. 6.6 Accelerations and Interface Forces, Case 1 (Load to Source Mass Ratio is 0.5)

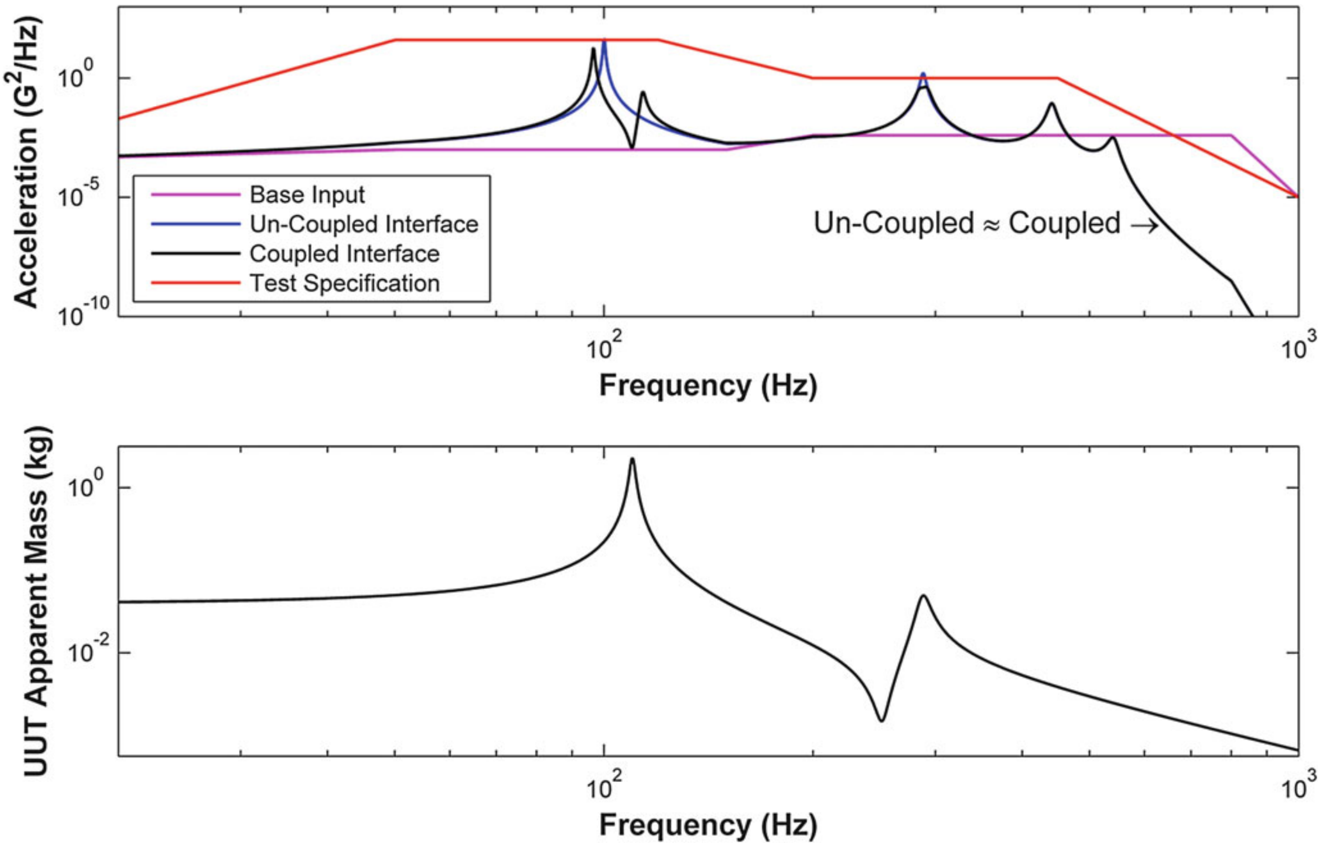

Fig. 6.7 Accelerations and UUT Apparent Mass, Case 2 (Load to Source Mass Ratio is 0.01)

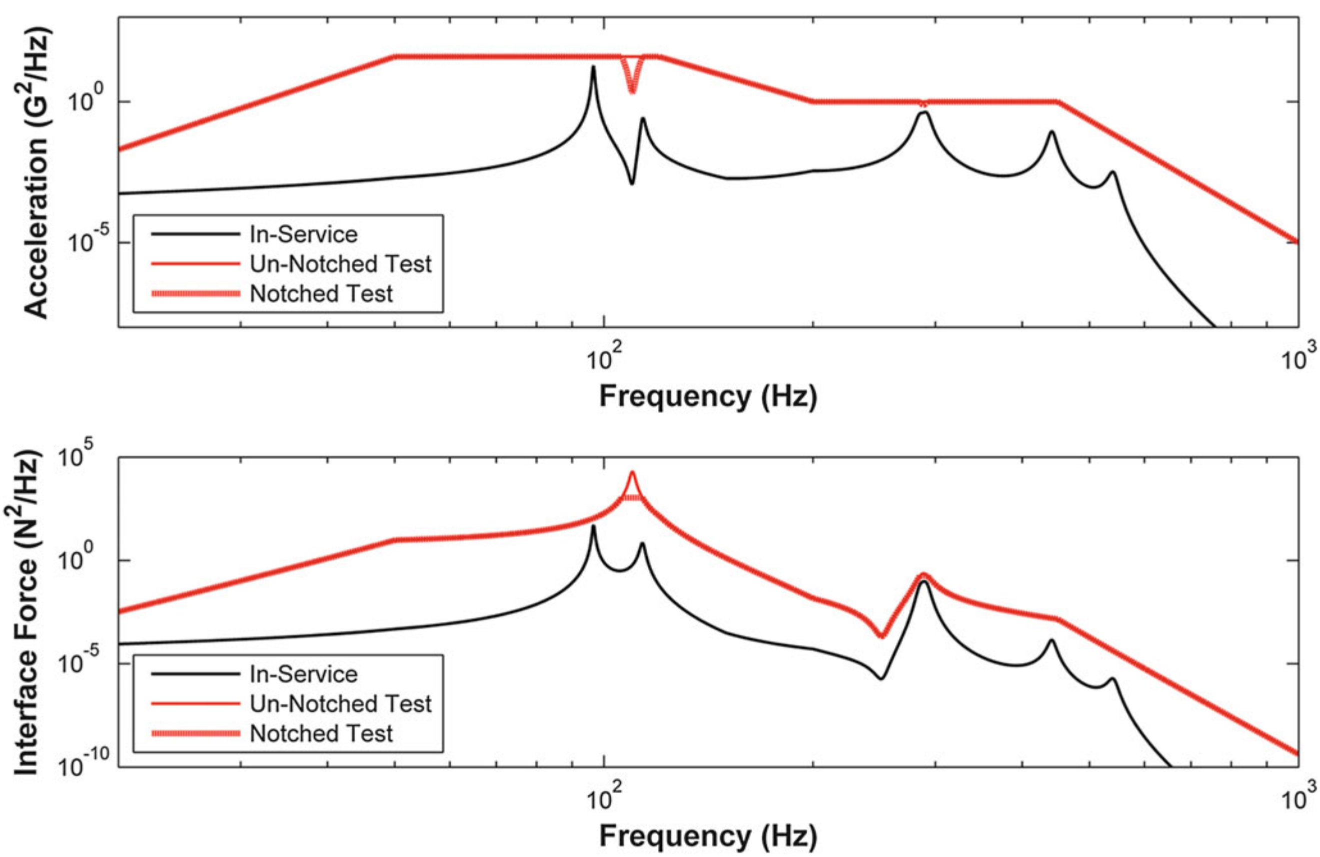

Fig. 6.8 Accelerations and Interface Forces, Case 2 (Load to Source Mass Ratio is 0.01)

6.10 Numerical Examples

To demonstrate the merits of the Apparent Mass method, and to show specifically that one can perform force limiting using the Apparent Mass method when the UUT mass is either significant or very small compared to its support structure, the results of two numerical examples are presented. Both cases use the same configuration shown in Fig. 6.4 but have different values for the masses (m), stiffness (k), and damping (c) as described in Table 6.1. The model could represent an instrument (UUT) mounted on a spacecraft (Support Structure) installed in a rocket. The rocket imparts random vibration at the base of the spacecraft. This base is represented by mass m_1 in Fig. 6.4.

In both cases, the base of the Support Structure is excited with identical acceleration PSD. This base input is broadband as it includes frequencies from 20 to 1000 Hz. From the resulting acceleration at the UUT interface (mass m_5), a test PSD that envelops the coupled structure interface acceleration is derived. The base PSD, along with both test PSDs used in the analyses, are described in Table 6.2 and in the top portion of Figs. 6.5 and 6.7. The final results are shown in Figs. 6.5, 6.6, 6.7, and 6.8.

In both numerical examples, the UUT natural frequencies are at 110 and 287.7 Hz, even though the mass of the UUT is changed from 2 kg in case 1 to 0.04 kg in case 2. The natural frequencies of the un-coupled support structure, meaning when the UUT is not attached, are 100, 288.1, 441, and 541 Hz for both cases. Thus, the natural frequencies of the UUT are very close to some of the un-coupled natural frequencies of the support structure. The support structure has a mass of 4 kg for both cases giving a UUT to supporting structure mass ratio of 0.5 for case 1 and 0.01 for case 2.

The Apparent Mass method was used with Eq. (6.20) in the calculation of the force limits for both numerical examples. Note that in both cases, as required for a representative test, the test input acceleration PSD enveloped the coupled acceleration PSD at the interface, even if the acceleration at m_5 of the un-coupled structure was not fully enveloped.

The results for the first case are shown in Figs. 6.5 and 6.6. In this case, the Apparent Mass FLV formulation produced a notch of almost 73 dB for the first mode and almost 29 dB for the second mode. Because the mass ratio is close to unity and because the UUT natural frequencies that were notched are close to some of the coupled natural frequencies, the un-notched over-test in this case is mainly due to the impedance differences.

The results for the second case are shown in Figs. 6.7 and 6.8. In this case, the Apparent Mass FLV formulation produced a notch of over 29 dB for the first mode and a small notch of almost 2.4 dB for the second mode. Because of the small mass ratio involved, the impedance mismatch is practically negligible and one might have expected that no notching be produced. The over-test in this case is mainly due to the enveloping process. This case was chosen to show that the Apparent Mass method of FLV can notch a UUT with a small mass and reduce over-testing arising from the enveloping process.

In both cases, the notches produced conservative interface acceleration in that the notched acceleration inputs were equal to or greater than the coupled in-service responses. Thus, Apparent Mass FLV did not produce any under-test situation even with the large notches that were created. The large depth of the notches without resulting in under-testing suggests that in practice, a large over-estimate of the force limits can be used and still end up with significant notching.

Table 6.1 Model parameters

Case	Component	Mass	Stiffness	Damping
1	Source	m_1 = 0 kg, m_2 to m_5 = 1 kg	k_1 to k_4 = 3.276E6 N/m	c_1 to c_4 = 30 N s/m
	Load	m_6 = 0 kg, m_7 and m_8 = 1 kg	k_6 and k_7 = 1.248E6 N/m	c_6 and c_7 = 30 N s/m
2	Source	m_1 = 0 kg, m_2 to m_5 = 1 kg	k_1 to k_4 = 3.276E6 N/m	c_1 to c_4 = 30 N s/m
	Load	m_6 = 0 kg, m_7 and m_8 = 0.02 kg	k_6 and k_7 = 24960 N/m	c_6 and c_7 = 0.6 N s/m

Table 6.2 Test input acceleration PSD applied at UUT interface

Base input at m_1						
Frequency	20 Hz	50 Hz	150 Hz	200 Hz	800 Hz	1000 Hz
Base PSD	0.0005 G^2/Hz	0.001 G^2/Hz	0.001 G^2/Hz	0.004 G^2/Hz	0.004 G^2/Hz	0.0001 G^2/Hz
Case 1 Test PSD at m_6						
Frequency	20 Hz	50 Hz	120 Hz	200 Hz	400 Hz	1000 Hz
Test input PSD Case 1	0.02 G^2/Hz	40 G^2/Hz	40 G^2/Hz	1 G^2/Hz	1 G^2/Hz	0.00001 G^2/Hz
Case 2 Test PSD at m_6						
Frequency	20 Hz	80 Hz	120 Hz	200 Hz	400 Hz	1000 Hz
Test input PSD Case 2	0.02 G^2/Hz	60 G^2/Hz	60 G^2/Hz	5 G^2/Hz	5 G^2/Hz	0.00001 G^2/Hz

6.11 Conclusion

The over-testing problem in vibration testing was shown to arise from two sources: the enveloping process and the interface impedance differences between the in-service and test configurations. The Apparent Mass method of performing FLV was derived from first principles and the reduction of the equations of motion to a linear system. The resulting equations were then supplemented with the same assumptions required for a representative test, along with the assumption that the maximum force response occurs at one of the natural frequencies of the coupled structure. The Apparent Mass method can reduce over-testing arising from both the interface impedance difference and the enveloping process. As such, the method is applicable to a UUT of any mass. In addition, since the method uses vibration theory to predict the maximum interface force over a given bandwidth defined by a set of coupled natural frequencies, the force limits produced will never lead to an under-test situation as long as the assumptions used to develop the equations are valid. Equations (6.19) and (6.20) are the key equations of the Apparent Mass method. These equations have been derived with the following assumptions:

- The vibration test is representative
  - Only interface forces act on the unit
  - In service, all points on the interface move with same magnitude and phase
  - In service, the interface moves in a single direction
  - The dynamics of the test structure are the same during testing as in service
  - The test specification envelops the acceleration of the coupled interface acceleration
- The coupled structure behaves linearly
  - Small strains
  - Small displacements
  - Linear material behavior
- In-service interface force peaks occur at the natural frequency of the coupled structure
  - Excitation is broadband and somewhat constant near the frequency we wish to notch
  - Coupled structure is lightly damped

By numerical examples, the method was shown produce significant notches without resulting in under-testing. This was true, even when the mass of the UUT was very small compared to the mass of its in-service supporting structure. One of the main benefits of the method is its ease of use, requiring a minimal amount of information. The method can be used as presented or in conjunction with other methods, especially the Semi-Empirical method, to validate a particular choice of force limit.

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