Chapter 44
Developments in the Prediction of Full Field Dynamics in the Nonlinear Forced Response of Reduced Order System Models

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Abstract Recent developments in the efficient modeling and reduction of finite element models (FEM) have been shown to provide a viable alternative to current approaches in the nonlinear simulation of computationally expensive detailed models of structural systems. This paper presents a summary of results in the prediction of nonlinear response from highly reduced order models interconnected with discrete nonlinear connections. The nonlinear response is approximated by piecewise linear solutions with reduction methodologies employed for the efficient computation; expansion is performed to retain system level response at a full set of FEM degrees of freedom and allows the calculation of stress and strain to ensure the practical usefulness of the techniques. Several cases will be shown here to illustrate the accuracy and efficiency of these methodologies in models with multiple components/cascaded configurations.

Keywords Forced nonlinear response • Reduced order modeling

44.1 Introduction

With the advent of the supercomputer and the wide availability of modern computing, finite element models for typical engineering applications have reached orders beyond millions of degrees of freedom (DOF). Such complex models carry a large cost and lead to a significant amount of time and computational resources. Moreover, better approximations of the physical phenomena present in the behavior of the structure require the modeling of nonlinear effects which in turns precipitates the need for nonlinear solvers. However, these complex systems are often times made up of multiple components whose overall behavior is linear and that only induce nonlinear response due to interactions of contact regions or connecting elements. A nonlinear solution scheme for these types of models is inefficient considering that the effect of the nonlinearity is limited to a small set of discrete point contact locations in the structure and its influence on the response is limited to a few lapses in time in comparison to the full time block of the system response.

With such large complex models, reduction methodologies are currently employed to efficiently decrease the set of active degrees of freedom (ADOF) down to a more manageable size and allow the accurate computation of response at the smaller set of FEM points. In particular, reduction and expansion techniques have been shown to produce comparable results to the full scale model for a family of nonlinear problems consisting of linear components interconnected with nonlinear elements. In this setting, the model and solution rely on the approximation of the nonlinear system response as the piecewise linear combination of all possible configurations in the structure.

The work presented here is a summary of recent developments and applications of efficient reduced order modeling and expansion techniques. The work by Thibault [1–3] and Marinone [4–6] showed that system level response can be accurately and efficiently calculated for highly reduced system models. Moreover, Pingle [7–10] and Carr [11–14] demonstrated that the expansion of such systems can be used for the prediction of full field results as well as stress and strain from limited sets of data. Harvie [15–17] extended this work to include the application of expansion to nonlinear systems made out of linear components but subjected to nonlinear interactions. In that work, the expansion was greatly simplified by using the unconnected component modes to expand the response of the reduced system model as developed by Nonis [18–20]. In addition, Obando [21–24] showed that these techniques can not only be used for the prediction of the dynamics of the full system but also the characterization of complicated subcomponent configurations (such as appended/ancillary subsystem components). The approach was shown to accurately predict the response at all DOF using embedded subcomponent information even in cases where no DOF were placed on the appended subcomponent and/or the connecting locations.

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The novel part of the expansion and prediction at all DOF is that the response is essentially being extrapolated from the embedded information in the reduction process by using a linear combination of the mode shapes of the structure as building blocks to all possible configurations in the system.

A summary of the salient results of these efficient reduced order modeling techniques is presented here. The methodology is applied to cantilever and multiple beam assemblies.

44.2 Theoretical Background

The techniques summarized here rely on the use of the mode shapes (eigenvectors) of the system as building blocks of all possible dynamic behavior of the structure and utilize well-established modal modeling (SDM [25]) and reduction techniques (Guyan Condensation [26] and SEREP [27]). An overview of the relevant theory of reduced order modeling and forced response is presented with references for more in depth treatment of each topic.

44.2.1 Equations of Motion for Multiple DOF System

The general equation of motion for a multiple degree of freedom system written in matrix form is

$$[M_1]\{\ddot{x}\} + [C_1]\{\dot{x}\} + [K_1]\{x\} = \{F(t)\} \tag{44.1}$$

Assuming proportional damping, the eigensolution is obtained from

$$[[K_1] - \lambda [M_1]]\{x\} = \{0\} \tag{44.2}$$

The eigensolution yields the eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the system. The eigenvectors are arranged in column fashion to form the modal matrix $[\mathrm{U}_1]$. Often times, only a subset of modes is included in the modal matrix to save on computation time and due to the fact that only certain modes actually contribute to the response. Exclusion of modes results in truncation error which can be serious if key modes are excluded. Truncation error will be discussed in further detail in the structural dynamic modification section.

The physical system can be transformed to modal space using the modal matrix as

$$[U_1]^T[M_1][U_1]\{\ddot{p}_1\} + [U_1]^T[C_1][U_1]\{\dot{p}_1\} + [U_1]^T[K_1][U_1]\{p_1\} = [U_1]^T\{F(t)\} \tag{44.3}$$

Scaling to unit modal mass yields

$$\begin{bmatrix} \ddots & & \\ & I_1 & \\ & & \ddots \end{bmatrix}\{\ddot{p}_1\} + \begin{bmatrix} \ddots & & \\ & 2\zeta\omega_n & \\ & & \ddots \end{bmatrix}\{\dot{p}_1\} + \begin{bmatrix} \ddots & & \\ & \Omega_1^2 & \\ & & \ddots \end{bmatrix}\{p_1\} = \left[U_1^n\right]^T\{F(t)\} \tag{44.4}$$

where $[\mathrm{I}_1]$ is the diagonal identity matrix, $[\Omega_1{}^2]$ is the diagonal natural frequency matrix and $[2\zeta\omega_n]$ is the diagonal damping matrix (assuming proportional damping). More detailed information on the equation development is contained in [25].

44.2.2 System Modeling and Mode Contribution

The approach used for the prediction of the nonlinear system response uses concepts from system modeling. In order to create multi-component models, various techniques are available for the coupling of several component models into a single system model. These system modeling techniques are used to define the various states that the system will undergo when the different nonlinear contact connections occur. The system modeling can be performed in physical space, modal space, or a combination of both physical and modal space. Consider two beams that are completely independent of one another, as illustrated in Fig. 44.1.

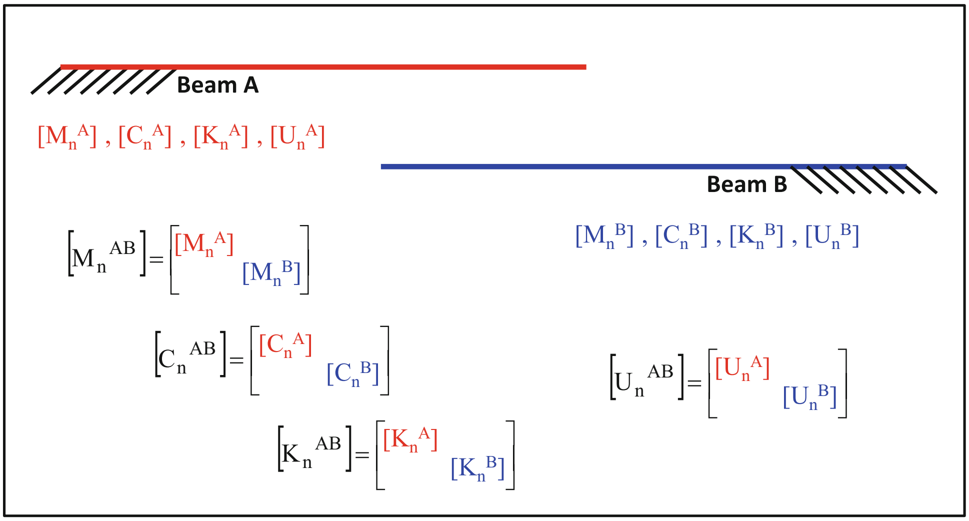


Fig. 44.1 Sample components arranged into common matrix space. The mass, stiffness, and damping matrices of each component are assembled for the combined system as well as the resulting mode shapes (U_n) of the two beam structure

The two beams shown in Fig. 44.1 are completely uncoupled and will respond independent of one another when excited. A system model of the uncoupled components is generated by simply writing the variables in common matrix space, as shown in the diagram. To generate a coupled system model, specific coupling terms must be introduced at the desired locations. To include the spring(s) in the system modeling, either a modal or physical approach can be employed. The modal approach involves using Structural Dynamic Modification (SDM) and Component Mode Synthesis (CMS). The physical approach involves using a physical tie matrix to couple the beams. Both approaches involve the use of a mode contribution matrix to determine the appropriate number of component modes that contribute to the system modes. For the results presented here, physical system modeling techniques were used to generate databases for the various configurations.

44.2.2.1 Physical Space System Modeling

To form a physical system model, the mass and stiffness matrices of each component (A and B) are assembled in stacked form into the system mass and stiffness matrices. In physical space, these are coupled with a stiffness tie matrix; a mass tie can also be included if desired but not included in this work.

$$\begin{bmatrix} [M^A] & \\ & [M^B] \end{bmatrix} \{\ddot{x}\} + \left[\begin{bmatrix} [K^A] & \\ & [K^B] \end{bmatrix} + K_{TIE} \right] \{x\} = \{F\} \tag{44.5}$$

This can be cast in a modal space representation as

$$\begin{aligned} &\left[\begin{bmatrix} \left[\overline{M}^A \right] & \\ & \left[\overline{M}^B \right] \end{bmatrix} \right] \begin{Bmatrix} \{\ddot{p}^A\} \\ \{\ddot{p}^B\} \end{Bmatrix} + \begin{bmatrix} \left[\overline{K}^A \right] & \\ & \left[\overline{K}^B \right] \end{bmatrix} \begin{Bmatrix} \{p^A\} \\ \{p^B\} \end{Bmatrix} \\ &+ \left[[U]^T [\Delta K] [U] \right] \begin{Bmatrix} \{p^A\} \\ \{p^B\} \end{Bmatrix} = \{0\} \end{aligned} \tag{44.6}$$

where $\left[\overline{M} \right]$ and $\left[\overline{K} \right]$ are diagonal matrices and with the mode shapes of each component stacked as

$$[\mathrm{U}] = \begin{bmatrix} [\mathrm{U}^A] & \\ & [\mathrm{U}^B] \end{bmatrix} \tag{44.7}$$

Equation (44.5) is a general equation of motion in physical space; Eq. (44.6) is the modal space equation used for the eigensolution.

44.2.2.2 Structural Dynamic Modification

Structural Dynamic Modification (SDM) is a technique that uses the original mode shapes and natural frequencies of a system to estimate the dynamic characteristics due to changes in the mass and/or stiffness of the system. First, the change of mass and stiffness are transformed to modal space as shown

$$\left[\Delta\overline{M}_{12}\right] = [U_1]^T\,[\Delta M_{12}]\,[U_1] \tag{44.8}$$

$$\left[\ \Delta\overline{K}_{12}\right] \ = [U_1]^T\ [\ \Delta K_{12}]\ [U_1] \tag{44.9}$$

The modal space mass and stiffness changes are added to the original modal space equations as seem in Eq. (44.6) to obtain

$$\left[\begin{bmatrix}\ddots & & \\ & \overline{M}_1 & \\ & & \ddots\end{bmatrix} + \left[\Delta\overline{M}_{12}\right]\right]\{\ddot{p}_1\} \quad + \left[\begin{bmatrix}\ddots & & \\ & \overline{K}_1 & \\ & & \ddots\end{bmatrix} + \left[\Delta\overline{K}_{12}\right]\right]\{p_1\} = [0] \tag{44.10}$$

The eigensolution of the modified modal space model is computed and the resulting eigenvalues are the new frequencies of the system. The resulting eigenvector matrix is the [U_{12}] matrix, which is used to transform the original modes to the new modes as indicated by

$$[U_2] = [U_1]\,[U_{12}] \tag{44.11}$$

The new mode shapes are [U_2]. The new mode shapes are formed from linear combinations of the original mode shapes. The [U_{12}] matrix shows how much each of the [U_1] modes contributes to forming the new modes. Figure 44.2 shows the formation of the new mode shapes as seen on Eq. (44.11). Harvie [15] and Avitabile [25] has additional information on SDM, mode superposition and participation.

The mode contribution matrix is used to determine which original mode shapes are necessary for the accurate reconstruction of each of the desired final mode shapes. If a dynamic response involves multiple system states, then the [U_{12}] matrix must be computed for each configuration to determine the number of original system modes to appropriately span the space of the solution and avoid truncation.

44.2.3 *General Reduction and Expansion Methodology*

Model reduction is a tool used to reduce the number of degrees of freedom (DOF) in order to minimize the required computation time of an analytical model, while attempting to preserve the full DOF dynamic characteristics. The relationship between the full space and reduced space model can be written as

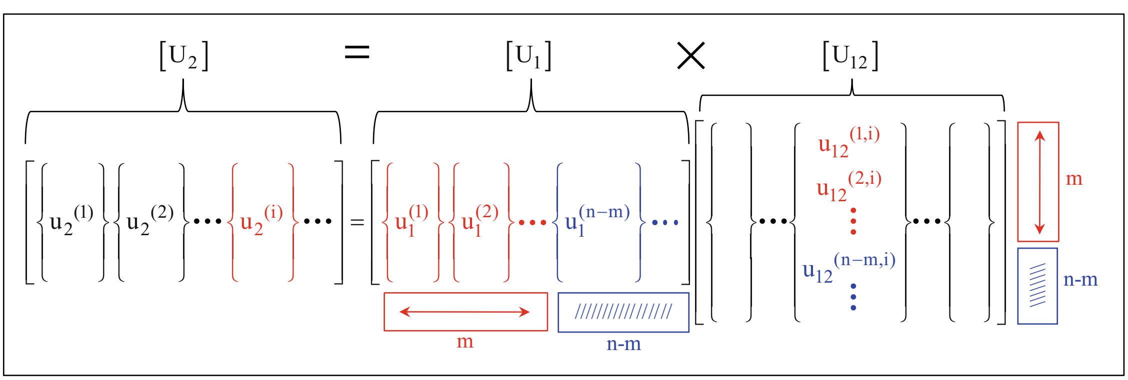


Fig. 44.2 Structural Dynamic Modification, mode contribution identified using [U_{12}]

$$\{X_n\} = \begin{Bmatrix} X_a \\ X_d \end{Bmatrix} = [T]\{X_a\} \tag{44.12}$$

where subscript ‘n’ signifies the full set of DOF (NDOF), ‘a’ signifies the reduced set of DOF (ADOF) and ‘d’ is the deleted DOF (those DOF not used during the reduced computation process). The transformation matrix [T] relates the full set of NDOF to the reduced set of ADOF. The transformation matrix is used to reduce the mass and stiffness matrices as

$$[M_a] = [T]^T [M_n][T] \quad \text{and} \quad [K_a] = [T]^T [K_n][T] \tag{44.13}$$

The eigensolution of these ‘a’ set mass and stiffness matrices are the modes of the reduced model. These modes can be expanded back to full space using the transformation matrix

$$[U_n] = [T][U_a] \tag{44.14}$$

If an optimal ‘a’ set is not selected when using methods such as Guyan Condensation [26] or Improved Reduced System Technique [28], the reduced model may not perfectly preserve the dynamics of the full space model. If System Equivalent Reduction Expansion Process (SEREP) [27] is used, the dynamics of selected modes will be perfectly preserved regardless of the ‘a’ set selected as long as the matrix is formed from a linearly independent set of vectors.

44.2.3.1 Guyan Reduction

Guyan Condensation is a stiffness based approach with long tradition of use in modal analysis [26]. The Guyan transformation matrix is calculated as

$$[T_S] = \begin{bmatrix} [I] \\ [t_s] \end{bmatrix} = \begin{bmatrix} [I] \\ -[K_{dd}]^{-1}[K_{da}] \end{bmatrix} \tag{44.15}$$

where a represents the active DOF and d represents the embedded or deleted DOF. Due to the formulation of the Guyan reduction process, the reduced model error increases for the higher preserved modes and the selection of DOF location has a large influence on the accuracy of the preserved eigenvalues.

44.2.3.2 SEREP

The SEREP modal transformation relies on the partitioning of the modal equations representing the system using selected DOFs and modes to obtain a reduced model that perfectly preserves the eigenvalues and eigenvectors of interest [27]. The SEREP technique utilizes the mode shapes from a full finite element solution to map to the limited set of active DOF. SEREP is not performed to achieve efficiency in the solution but rather is intended to perform an accurate mapping matrix for the transformation. The SEREP transformation matrix is formed using a subset of modes at full space and reduced space as

$$[T_U] = [U_n][U_a]^g \tag{44.16}$$

where $[U_a]^g$ is the generalized inverse and $[T_U]$ is the SEREP transformation matrix. When the SEREP transformation matrix is used for model reduction/expansion as outlined in the previous section, the reduced model perfectly preserves the full space dynamics of the modes in $[U_n]$ [27].

44.2.3.3 KM_AMI

A more recent technique has been developed that utilizes Guyan Reduction (or any reduced matrix for that matter) along with direct updating of the reduced system matrices with the full space modal vectors as targets for the updating process [29, 30]. This reduction technique also overcomes some of the rank problems associated with SEREP and provides a reduced set of ADOF that retain all the eigenvalues and eigenvectors of the full system matrices. The Guyan reduced mass and stiffness matrices are updated using

$$[M_I] = [M_S] + [V]^T \left[[I] - \left[\overline{M}_S\right] \right] [V] \tag{44.17}$$

and

$$[K_I] = [K_S] + [V]^T \left[\left[\Omega^2_{REF}\right] + \left[\overline{K}_S\right] \right] [V] - [[K_S] [U_{REF}] [V]] - [[K_S] [U_{REF}] [V]]^T \tag{44.18}$$

with

$$[V] = \left[\overline{M}_S\right]^{-1} [U_{REF}]^T [M_S] \tag{44.19}$$

44.2.4 Expansion of System Modes from Uncoupled Component Modes

Expansion is generally used for providing full N-space mode shape information extracted from limited a-space information. The expansion to full space in this work is based on recent work by Nonis [18–20] showing that full N-space mode shape information for an assembled system model can be obtained using the expansion matrices from the uncoupled, unconnected, original component modes of each component. Figure 44.3 shows the entire expansion process schematically to further describe the overall procedure. The uncoupled components (red and blue beams where each circle identifies a DOF in the model) are reduced and assembled into a coupled system model. The mode shapes of the coupled two beam system are obtained from the eigensolution but only for the limited set of DOF available in the model. In order to obtain mode shapes with the full resolution of the full space model, expansion is performed using the transformation matrices (T_u^A and T_u^B) of the original uncoupled components that are available from the reduction process. Nonis and Avitabile [20] further details the expansion process and considerations for modes included.

While the systems studied here undergo several configurations, the expansion can be performed with a single transformation matrix obtained from the original unconnected unmodified systems. As long as the modes retained during the reduction process span the space of the full space solution of the system, the transformation matrix will produce accurate expanded predictions of the multi-configuration response.

Using a single transformation matrix of the unconnected individual components/systems is much more efficient since the full space modes of the connected system are not computed.

44.2.5 Expansion of Reduced Order Real Time Response

Chipman [31, 32] and others [9, 12–14] showed that a transformation matrix can be used to expand not only mode shapes, but dynamic time response data. Therefore the same principles presented above can be extended to the expansion of coupled response using uncoupled component information. The response of the coupled system at reduced space can be expanded to full space using the transformation matrices of the individual, unconnected beams systems because the dynamic characteristics of the system are directly related to the dynamic characteristics of the uncoupled components. If the $[U_{12}]$ matrix is evaluated to include enough modal information in the reduced model to accurately represent the system with the coupling elements present, then the time response of the coupled system is accurately expanded to full space using information from the uncoupled components.

In this paper, nonlinear systems are analyzed where several possible configurations can exist during response. If all possible configurations are made up of linear combinations of the component mode shapes, then the expansion of nonlinear response of a reduced model can be expanded using the original transformation matrix regardless of the configurations encountered. Figures 44.4, 44.5, and 44.6 show the process for determining full space dynamic response of a nonlinear system using the original transformation matrices. The full space linear subcomponents are assembled into System 1 and System 2 and reduced to a smaller set of ADOF (red and blue points in Fig. 44.4a). Nonlinear contacts in green (Fig. 44.4b) indicate when the spring comes in contact with the structure and thus changes from one configuration to another. Mode contribution is an essential part of this step as the necessary number of modes must be determined from all possible configurations of the system.

Although several configurations exist within the time block shown, all modified states can be generated based on the mode shapes of the original system. Therefore only the transformation matrices of the original, uncoupled components are necessary to expand the nonlinear dynamic response of the system regardless of the configurations encountered. In Fig. 44.5,

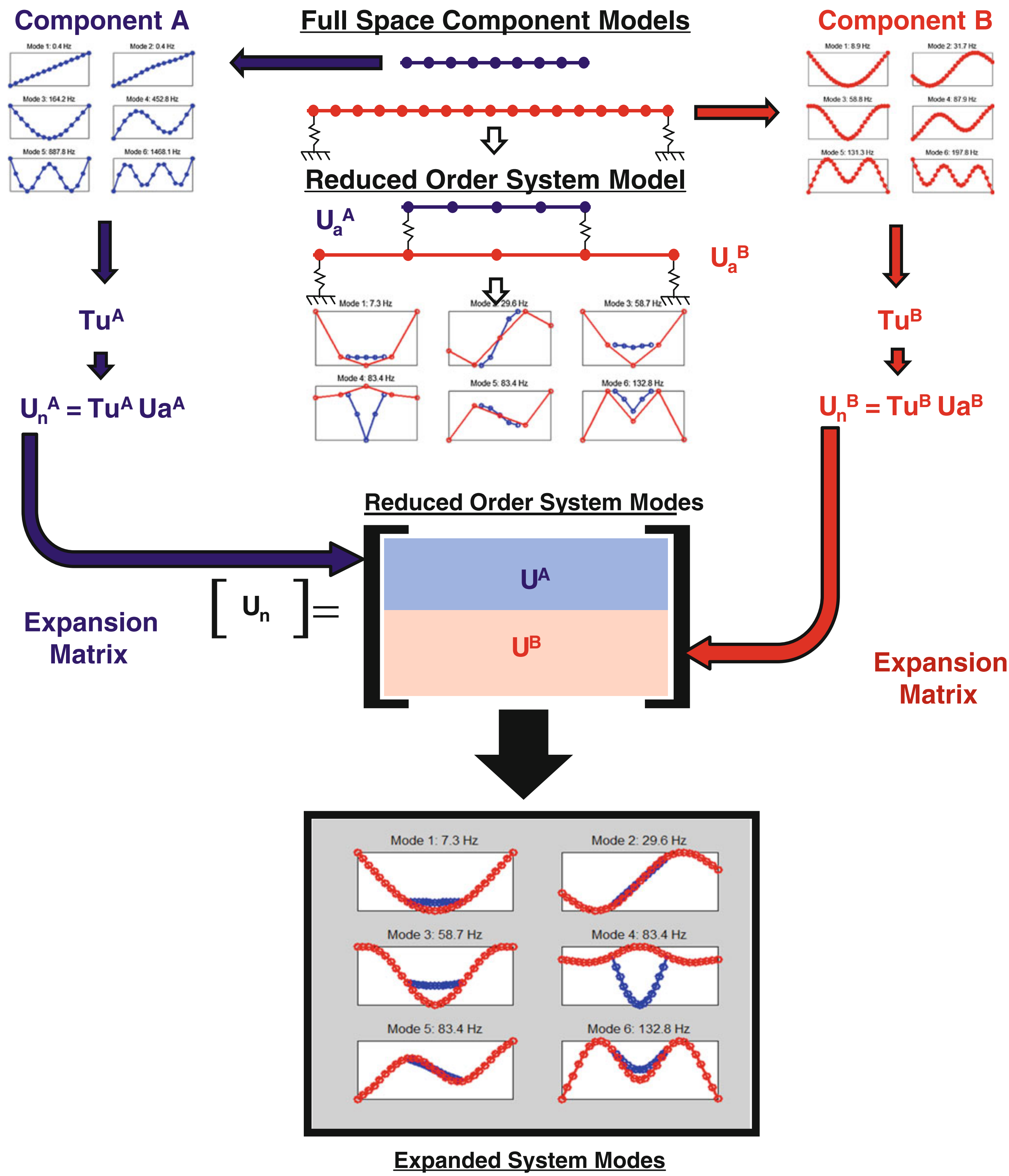


Fig. 44.3 Overall expansion process schematic using the transformation matrices (T_u^A and T_u^B) from unconnected system components. The reduced order model mode shapes are expanded to obtain full resolution mode shapes of the two beam system, where the *red* and *blue circles* indicates DOF available in each of the models

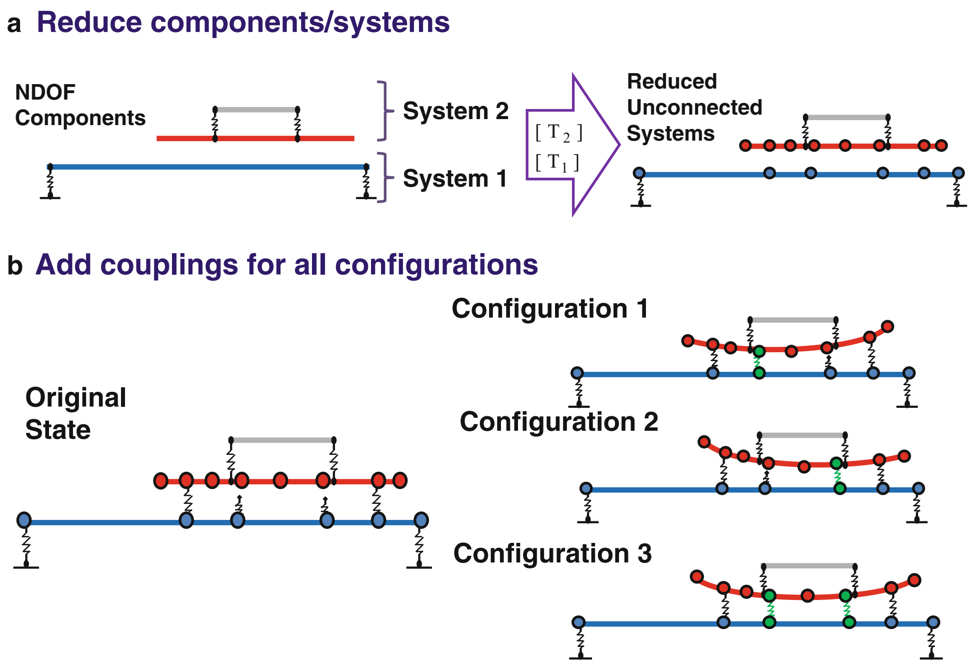


Fig. 44.4 (**a**) Assembly of reduced order model from uncoupled full space subcomponents and (**b**) computation of relevant system configurations due to nonlinear interactions of the gap-springs (in *green*)

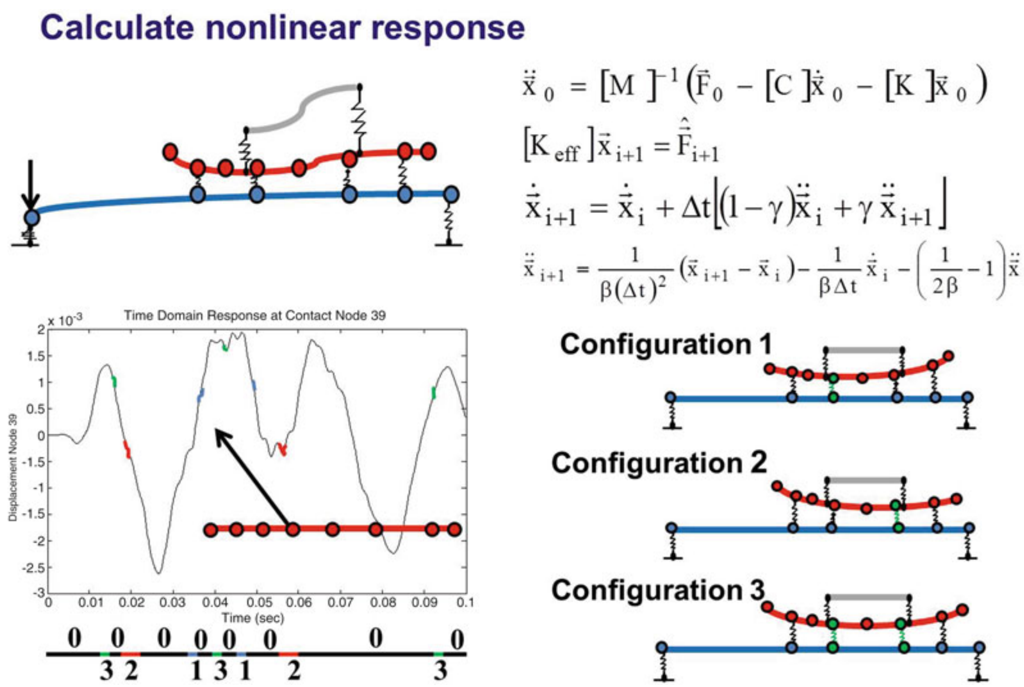


Fig. 44.5 Calculation of nonlinear response of three beam system from the linear combination of all configurations of the structure (indicated from the contact interactions of the *green* gap-spring as seen on the time response) and obtained using Newmark direct integration of the equations of motion

the response of the nonlinear reduced order model is obtained as the piecewise combination of all linear states (configurations 0–3) and calculated from the direct integration of the equations of motion of the system.

The response is obtained for the selected ADOF present in the reduced model (blue and red dots in Figs. 44.4, 44.5, and 44.6). Expansion is performed using the same projection mode shape vectors obtained from the reduction process of the unassembled unreduced subcomponents as seen in Fig. 44.6a.

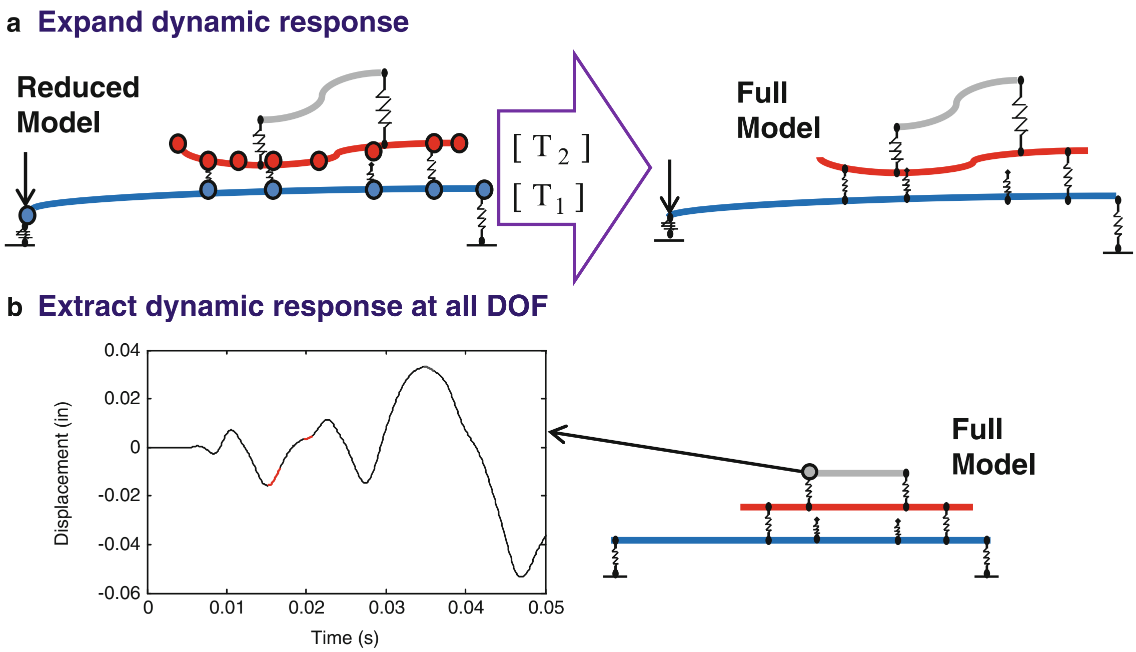


Fig. 44.6 Prediction of subcomponent dynamics from expansion of reduced order model time response using uncoupled subsystem component transformation matrices

The full field dynamics at all DOF in the model are obtained from the expansion process and the response at the subcomponent level obtained (as in Fig. 44.6b) despite the fact that no ADOF where placed there during the reduction process.

44.2.6 *Timer Response Correlation Tools*

In order to quantitatively compare two different time solutions, two correlation tools were employed: The Modal Assurance Criterion (MAC) and the Time Response Assurance Criterion (TRAC).

44.2.6.1 Modal Assurance Criteria (MAC)

The Modal Assurance Criterion (MAC) [33] is widely used as a vector correlation tool. In this work, the MAC was used to correlate all DOF at a single instance in time. The MAC is written as

$$MAC_{ij} = \frac{\left[\{X1_i\}^T \ \{X2_j\}\right]^2}{\left[\{X1_i\}^T \ \{X1_i\}\right] \left[\{X2_j\}^T \ \{X2_j\}\right]} \tag{44.20}$$

where X1 and X2 are displacement vectors. MAC values close to 1.0 indicate strong similarity between vectors, whereas values close to 0.0 indicate minimal or no similarity.

44.2.6.2 Time Response Assurance Criteria (TRAC)

The Time Response Assurance Criterion (TRAC) [34] quantifies the similarity between a single DOF across all instances in time. The TRAC is written as

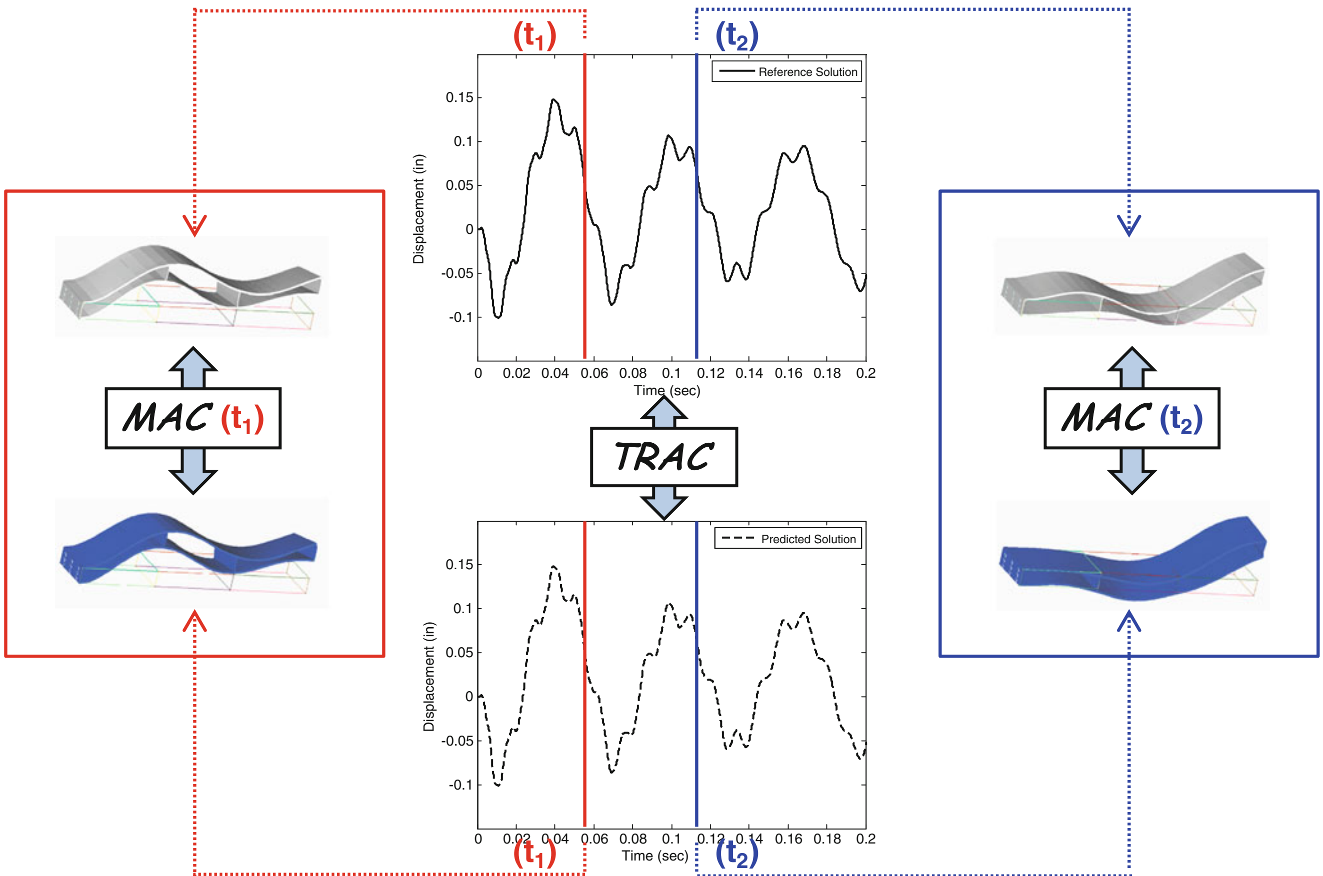


Fig. 44.7 Physical interpretation of MAC and TRAC. MAC is a vector to vector correlation taken at an instance in time t_i, while TRAC compares the response for its full duration at each DOF in the model. In the figure, the shape of two models (*blue* and *grey*) are correlated at a particular time step and the TRAC is computed for the displacement vector at one DOF

$$TRAC_{ji} = \frac{\left[\left\{X1_j(t)\right\}^T \left\{X2_i(t)\right\}\right]^2}{\left[\left\{X1_j(t)\right\}^T \left\{X1_j(t)\right\}\right]\left[\left\{X2_i(t)\right\}^T \left\{X2_i(t)\right\}\right]} \tag{44.21}$$

where X1 and X2 are time response vectors for a particular DOF. TRAC values close to 1.0 indicate strong similarity between vectors, while values close to 0.0 indicate minimal or no similarity.

The MAC is calculated between the shapes of the full space reference solution and estimated solution obtained from the reduced order model at each time step. Similarly, the TRAC is used to compare the time response from the reduced order model to the time response from the full space finite element solution at each degree of freedom. A diagram detailing the two correlation techniques is shown in Fig. 44.7.

44.2.7 Numerical Computations

The computation of the time response developed in this work is based on the Equivalent Reduced Model Technique (ERMT), a technique developed by Avitabile and Thibault [1, 35]. This technique uses an exact reduced model representation for the calculation of the system response. Newmark integration technique [36–39] is used to perform the direct integration of the equations of motion for the ERMT solution process due to similarity with the HHT (Hilber–Hughes–Taylor [38, 39]) method commonly used in FEA software. From the known initial conditions for displacement and velocity, the initial acceleration vector is computed using the equation of motion and the applied forces as

$$\ddot{\vec{x}}_0 = [M]^{-1}\left(\vec{F}_0 - [C]\ \dot{\vec{x}}_0 - [K]\ \vec{x}_0\right) \tag{44.22}$$

Choosing an appropriate Δt, α, and β, the displacement vector is

$$\begin{aligned}\vec{x}_{i+1} = &\left[\tfrac{1}{\alpha(\Delta t)^2}[M] + \tfrac{\beta}{\alpha(\Delta t)}[C] + [K]\right]^{-1} \\ &\left\{\vec{F}_{i+1} + [M]\ \left(\left(\tfrac{1}{\alpha(\Delta t)^2}\right)\ \vec{x}_i + \left(\tfrac{1}{\alpha(\Delta t)}\right)\dot{\vec{x}}_i + \left(\tfrac{1}{2\alpha} - 1\right)\ \ddot{\vec{x}}_i\right)\right. \\ &\left.\quad + [C]\ \left(\left(\tfrac{\beta}{\alpha(\Delta t)}\right)\ \vec{x}_i + \left(\tfrac{\beta}{\alpha} - 1\right)\ \dot{\vec{x}}_i + \left(\tfrac{\beta}{\alpha} - 2\right)\ \tfrac{\Delta t}{2}\ \ddot{\vec{x}}_i\right)\right\}\end{aligned} \tag{44.23}$$

The values chosen for α and β were ¼ and ½, respectively. This assumes constant acceleration and the integration process is unconditionally stable, where a reasonable solution will always be reached regardless of the time step used. However, the time step should be chosen such that the highest frequency involved in the system response can be characterized properly to avoid numerical damping in the solution. Following the displacement vector calculation, the acceleration and velocity vectors are computed for the next time step using

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + (1-\beta)\,\Delta t\,\ddot{\vec{x}}_i + \beta\Delta t\,\ddot{\vec{x}}_{i+1} \tag{44.24}$$

$$\ddot{\vec{x}}_{i+1} = \frac{1}{\alpha(\Delta t)^2}\left(\vec{x}_{i+1} - \vec{x}_i\right) - \frac{1}{\alpha\Delta t}\ \dot{\vec{x}}_i - \left(\frac{1}{2\alpha} - 1\right)\ \ddot{\vec{x}}_i \tag{44.25}$$

This process is repeated at each time step for the duration of the time response solution desired. Additional information on the Newmark formulation can be found in [37, 39].

44.3 Analytical Case Studies

The approach taken in this work utilizes detailed linear finite element models, transformed to a smaller reduced order model, and addresses nonlinear phenomena such as contact with discrete elemental connections for these reduced order models. Figure 44.8 shows a summary of the efficient modeling approaches implemented in this work. The left side of the figure shows the traditional modeling approach using a costly nonlinear solution scheme for the full size model. The efficient modeling approaches are shown on the right side of Fig. 44.8, where the reduced system is modeled through linear approximations in physical space (Equivalent Reduced Model Technique [1]) or modal space (Modal Modification Response Technique [4]) and then solved by direct integration of the equations of motion. Expansion is then performed to return to the full field solution and compute displacement/strain at all DOF. On the other hand, the traditional approach (on the left side of Fig. 44.8) requires the discretization of the structure, calculation of all relevant loads and boundary conditions as well as the appropriate modeling of the nonlinear behavior of the structure before a costly nonlinear numerical methodology can be implemented to solve the full space problem usually stated through a set of partial differential equations. The end result of the traditional approaches is the same type of full field displacement/strain calculated in the reduced methodology but computed through a much more lengthy and costly process. The approximation of loads as well as the simplified modeling of the nonlinear behavior involved in the traditional modeling approach can lead to significant errors without any gains through the use of large computationally costly models. In contrast, an efficient reduced order modeling approach bypasses these typical limitations and allows for modification/tuning and quick testing of many configurations due to the fast solution times in the techniques.

The reduced order modeling techniques are identified as:

Equivalent Reduced Model Technique (ERMT) utilizes highly accurate model reduction schemes (SEREP, HYBRID and KM_AMI) to formulate extremely reduced models to form dynamic descriptions of linear components which are then interconnected with highly nonlinear connection elements for the dynamic response evaluation using direct integration techniques.

Modal Modification Response Technique (MMRT) utilizes the Structural Dynamic Modification process and mode superposition method to develop modal representations that are interconnected with highly nonlinear connection elements and solve using direct integration of modal based equations.

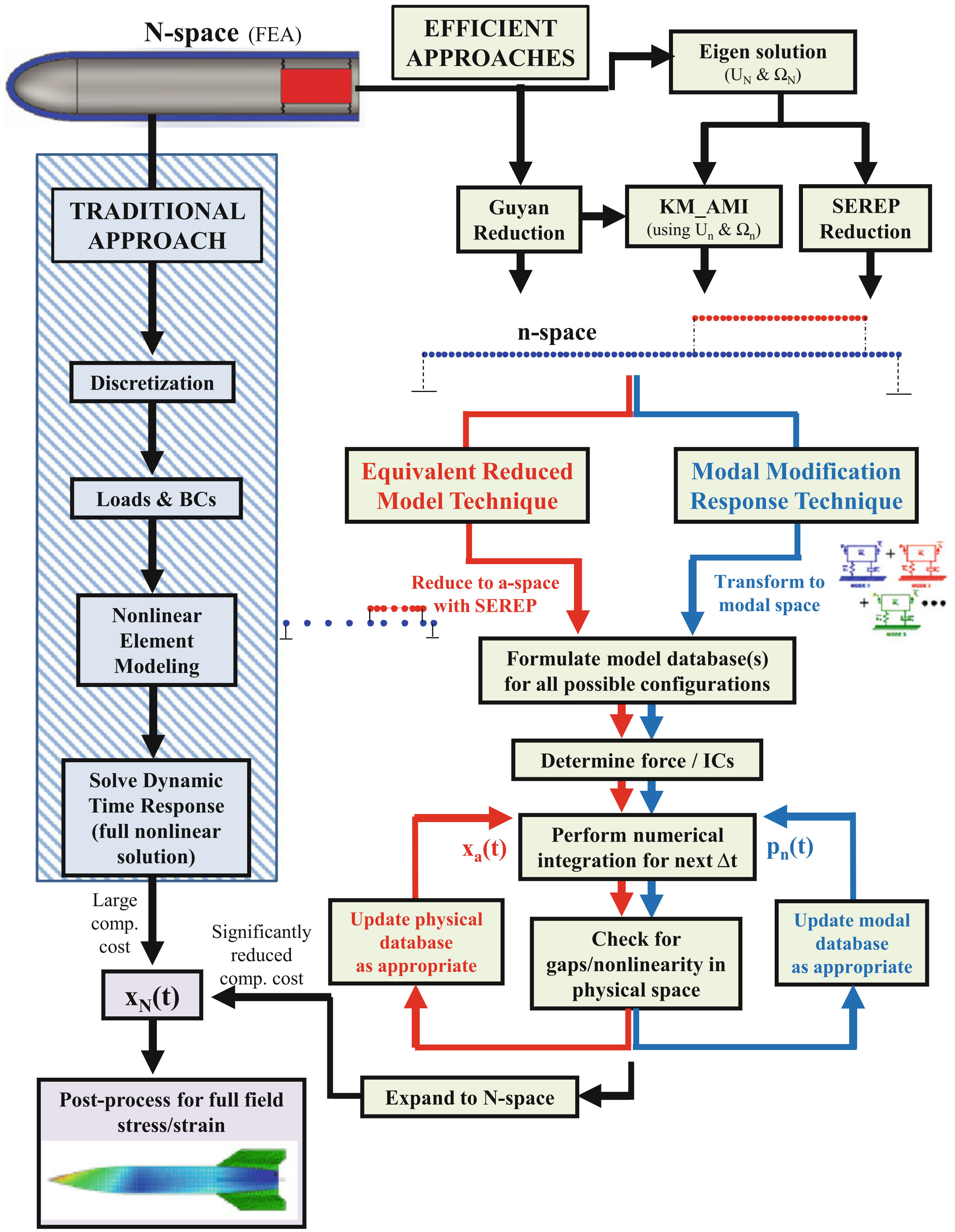


Fig. 44.8 Traditional modeling of structural systems compared to efficient modelling approaches. *Left side* shows traditional approach of full space model using a nonlinear solver, while the efficient approaches (on the *right side*) use a piecewise linear formulation at reduced space and performs expansion to obtain full field results [15]

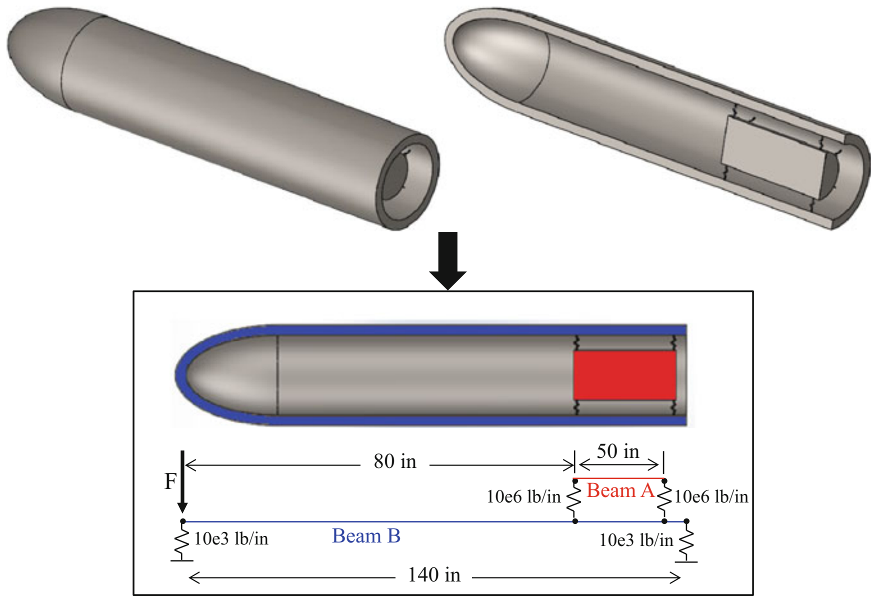


Fig. 44.9 Physical representation of two-beam system

Table 44.1 Overall comparison of results for all cases studied [15]

Case	Model	# of DOF	Solution time (s)	Average MAC	Average TRAC
Linear	Full space	1904	740.83	1.0000	
	Reduced	13	0.23	0.9960	0.9997
		24	0.29	0.9998	1.0000
Soft contacts	Full space	1904	741.95	1.0000	
	Reduced	13	0.23	0.9960	0.9997
Hard contacts	Full space	1904	740.18	1.0000	
	Reduced	13	0.23	0.9892	0.9918
		24	0.28	0.9998	0.9999

A variety of analytical cases were studied using both ERMT and MMRT to show the use of these efficient reduced modeling techniques for the prediction of full field dynamics. Only the case analyzed using a physical space formulation (ERMT) will be discussed. The model consists of a simple two-beam system as illustrated in Fig. 44.9. This case was used to demonstrate the proposed approach for calculating full field dynamics at a fraction of the full space computational cost; additionally one of the beam components has an ancillary subcomponent/appendage that is dynamically active during the response of the structure. Many other models have been studied with results in the references to fully demonstrate the approach.

The analytical cases were studied first in order to validate the methodology for the proposed techniques [2, 3, 5, 16, 17, 30]. For all of the cases shown, the efficient models were found to be accurate in comparison to the full space solution, while drastically reducing the computation time required. Table 44.1 summarizes the results obtained for both techniques in comparison to the full space solution [15]. The average modal assurance criterion (MAC) and time response assurance criterion (TRAC) are two time correlation tools used in this work and are included; values approaching 1 indicate a high level of similarity, while values close to 0 indicate a minimal level of correlation.

The full space beam model contains nodes with 0.2 inch spacing on each beam. Each node contains a shear DOF and a rotational DOF to capture planar beam bending only. Discrete nonlinearities are introduced to the system as a gap-spring contact. The stiffness of the spring contact is set to a predefined stiffness value when the specified gap distance is closed or set to zero when the gap distance is open and the beams are not in contact. The nonlinear cases have two contact locations between the beams, as shown in Fig. 44.10 [15]. The contact locations were chosen so that both gap-springs could engage during the response. The three possible configurations that the beams can encounter with the springs engaged are also shown in Fig. 44.10. For the cases shown in the table, a frequency band-limited analytical force pulse was used to excite a frequency range of roughly 400 Hz; the frequency range excited by the force pulse includes roughly 13 modes in all configurations. The use of such a force pulse allows for minimal excitation of higher modes and controls the number of modes that substantially participate in the system response due to the impulse.

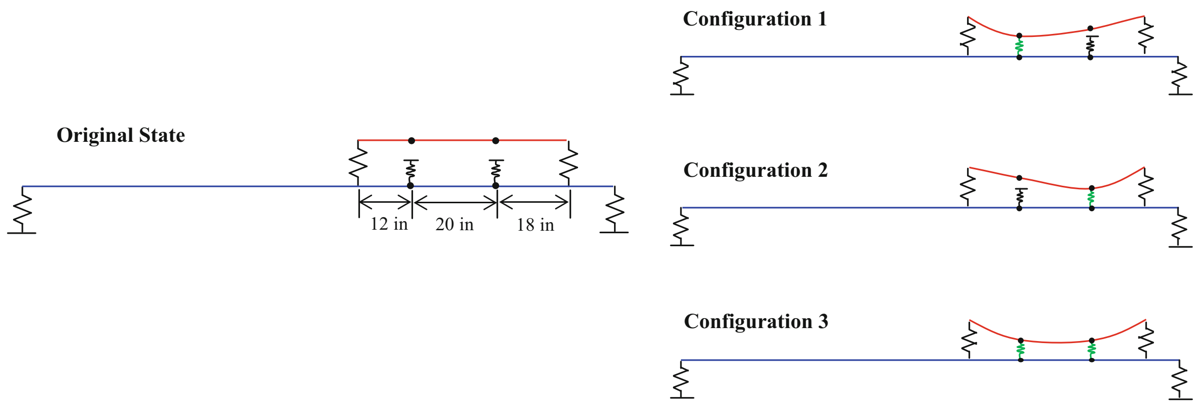


Fig. 44.10 Contact locations and configurations for nonlinear cases [15]

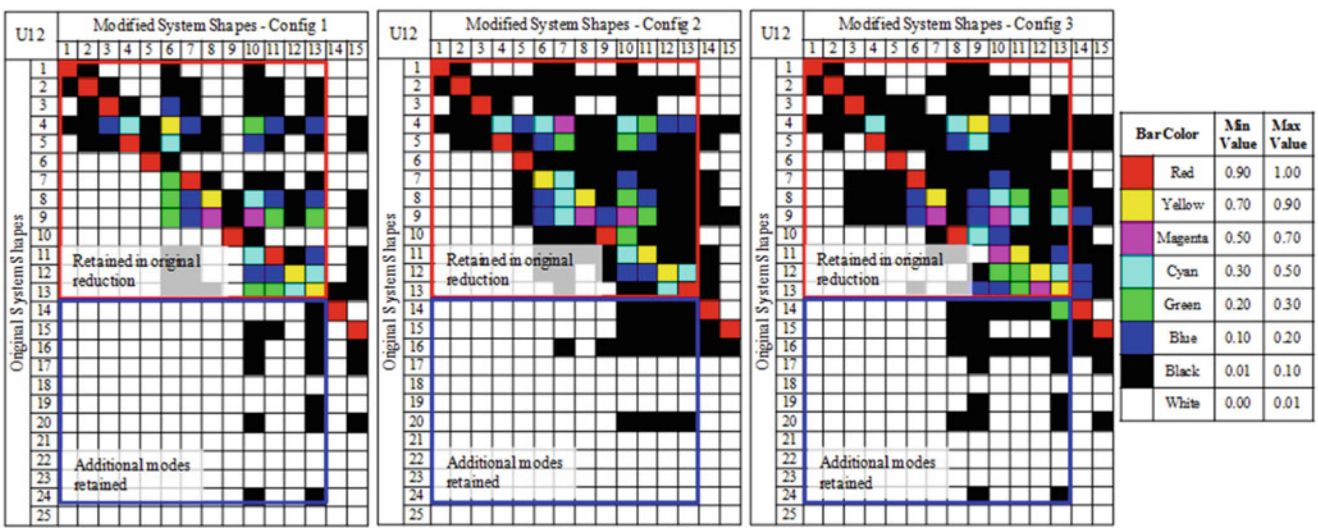


Fig. 44.11 Mode contribution matrix [U12] for all configurations [15]

For this study, there were a couple of considerations needed to obtain the accurate response. The first consideration involves the number of modes or degrees of freedom needed in the efficient model. Both techniques can suffer from the effect of truncation, which refers to the error caused by reducing the number of modes or degrees of freedom. Truncation error can be mitigated by including a sufficient number of modes or degrees of freedom such that all of the modes with significant energy are included in the response.

The second consideration involved forming the modified system database of mode shapes from the original mode shapes of the uncoupled components. Truncation error is again significant when assembling reduced order components or component mode shapes together. In order to reduce truncation error, the mode contribution matrix or $[U_{12}]$ matrix is used to determine the necessary modes required to form the final system assembly. This matrix contains the scaling coefficients necessary to form the final modal vectors from the linear combination of the starting modal vectors and is more easily visualized as plotted using a color bar chart as in Fig. 44.11.

44.3.1 Nonlinear Solution with Reduced Model from Beam/Line Elements

Using 24 modes and ADOF produces a proper representation for this system. The use of fewer modes leads to inaccuracies as the results of this case show; more detailed discussion on these cases is presented in [15]. The full space displacements were determined using both a full space solution and the proposed efficient technique. Comparisons of the computation time and accuracy for this case are shown in Table 44.1 (Hard Contact Case). The solution time was reduced significantly by utilizing

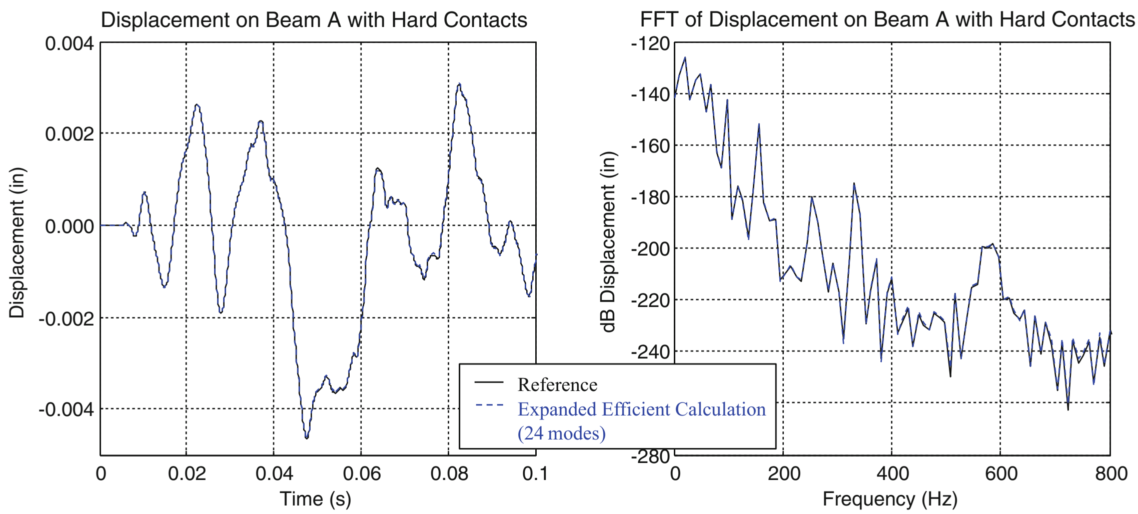


Fig. 44.12 Time response and corresponding FFT on Beam A (*top beam*) for hard contact case with 24 modes [15, 16]

such a smaller model to solve for the response. For this case, very high accuracy was obtained using an extremely reduced model. The displacement results for the full and predicted displacement are shown in the time and frequency domains in Fig. 44.12. The comparison of displacement overlays nearly perfectly in both domains between the two response calculations. The frequency range of the original system modes included in the reduction extends over 1300 Hz, so the energy distribution of the response is accurately captured in the frequency domain.

The displacement is also shown in Fig. 44.13 with the corresponding dynamic strain at the location of interest [16]. Both the displacement and strain are predicted accurately using the reduced order model implemented in this case. The model used in the efficient technique is able to better predict the higher order curvature in the dynamic displacement and strain response because more modes were included in the reduction. While the efficient calculation of the dynamic displacement matches nearly perfectly to the reference solution, some high frequency content is present in the reference strain solution that is not captured using the efficient calculation. The slight differences in the strain calculations can be further reduced by including even more modes in the reduced order model. The strain and displacement results calculated matched well those of the full space model for this case, yet the solution time for the reduced order model is substantially lower than the time required for a full space calculation.

The full solution took over 12 minutes to compute using the large beam/line element model, but accurate strain and displacement data could be obtained in less than a second without compromising accuracy. While the hard contact springs in the system causes the necessity for additional modes to be retained in a reduced order model, the full space model can still be substantially reduced to retain only the dynamic characteristics that are necessary to the response.

44.3.2 Expansion of Beam/Line Elements to a Full 3D Finite Element Model

The results obtained above clearly show the efficiency of the proposed technique. The computational savings are significant without any appreciable loss in accuracy. However, the expansion process can also be used to further expand the results from the beam/line element model to a full 3D model. Using an alternate model reduction expansion process proposed by Truong [40], these beam element results can be expanded to a much larger space/size model without any loss in accuracy. To illustrate this, a 3D model of the beam line element was created along with an appropriate mapping to the neutral axis of the beam where no geometry exists [17, 40]. Special constraint elements are utilized to form an additional transformation matrix to map from the beam neutral axis to the surface of the 3D model; the details of this work can be found in [40]. Figure 44.14 shows the model reduction from a 3D finite element model to a line element model.

For the full 3D model, the N-space was set at 7616 nodes with six DOF per node for a total of 45,696 DOF. The results obtained from the expansion process have been shown to be well correlated to the expected results as seen in Fig. 44.15, which shows the results for just one time step.

Figure 44.16 shows the time response and predicted strain for multiple steps of the system response obtained from the expansion of the reduced line model.

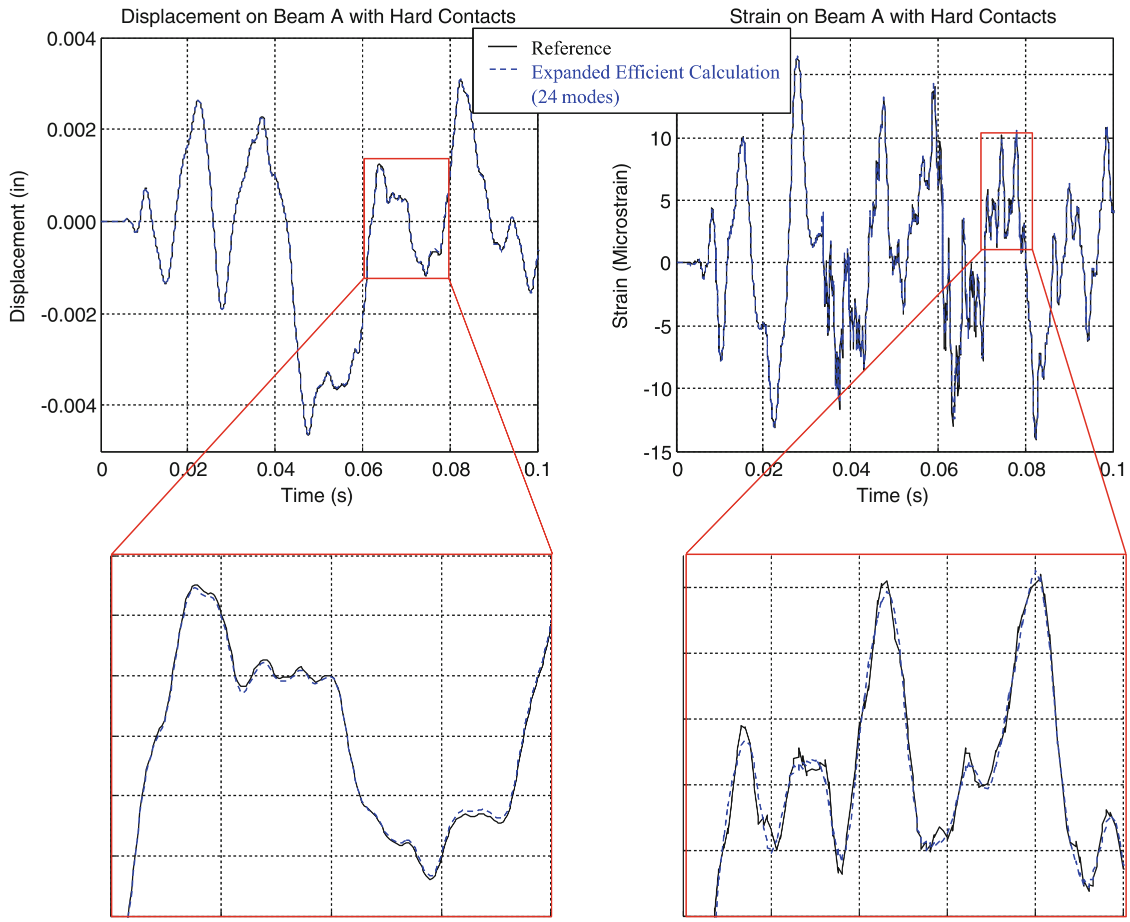


Fig. 44.13 Displacement and corresponding strain on Beam A (*top beam*) for hard contact case with 24 modes [16]

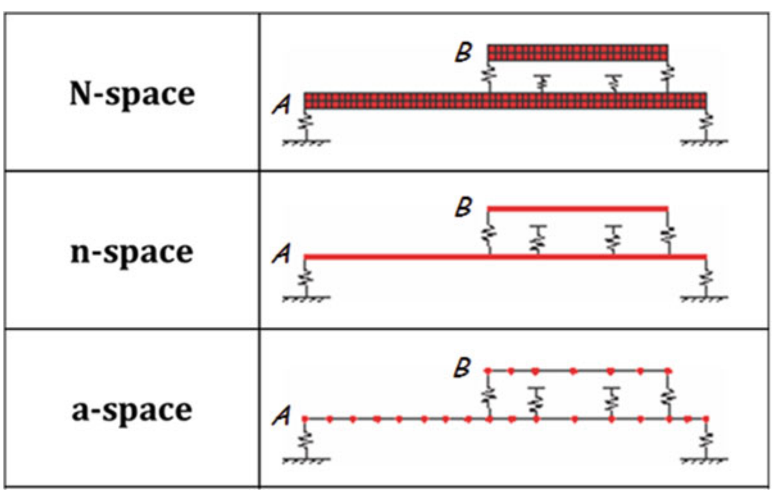


Fig. 44.14 Model reduction of full 3D FEM of two-beam system with nonlinear contacts

44.3.3 Ancillary Subcomponent Response from Embedded Reduced Order Model

Multi-component structural systems were addressed in the context of retaining embedded structural information of ancillary subcomponents for the calculation of nonlinear reduced order model time response. For linear forced response, expansion was shown to return a precise approximation of the ancillary subcomponent even in cases where the reduction does not

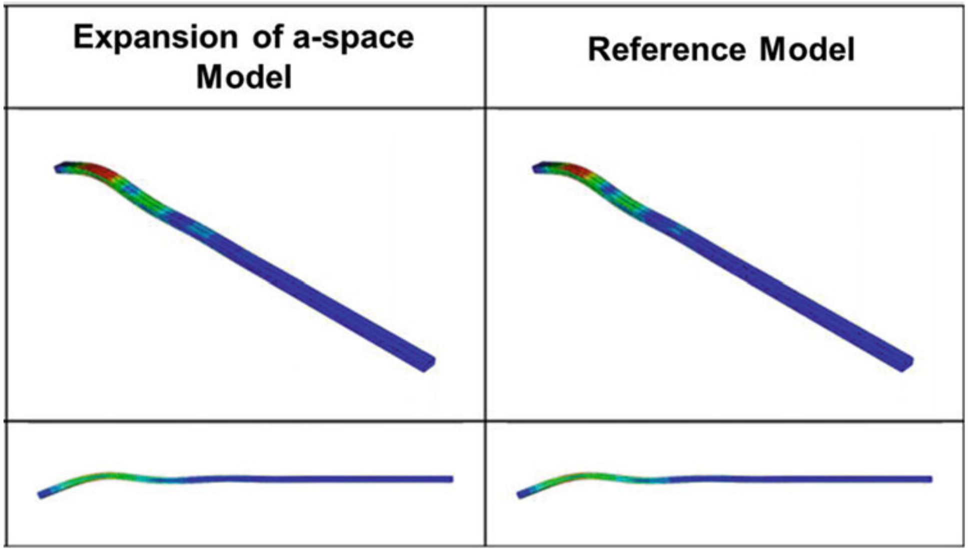


Fig. 44.15 Dynamic strain on Beam B for one time step of hard contact case from 3D model reduction

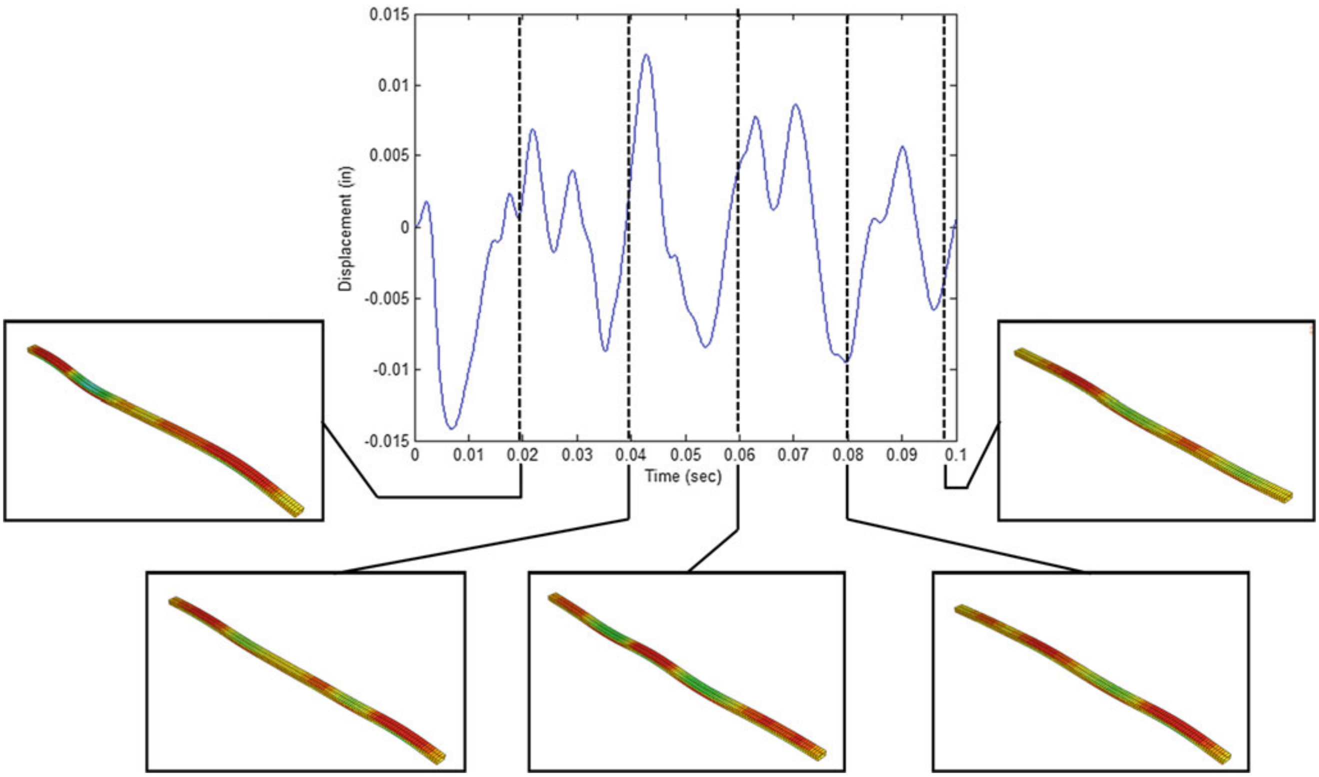


Fig. 44.16 Predicted dynamic strain on Beam B from expansion of reduced line model to full 3D FEM at multiple time steps

include active DOF at that subcomponent level [21]. The model was developed as a natural extension of the nonlinear worked discussed in the last cases. A full space finite element model consisting of two systems, one of which contains a dynamically active ancillary subcomponent, were reduced to a smaller set of degrees freedom and used for the prediction of the forced time response of the system as seen in Fig. 44.17. Gap-spring contact elements were introduced to generate nonlinear response between the two systems. The reduced order model (with embedded ancillary subcomponent information)

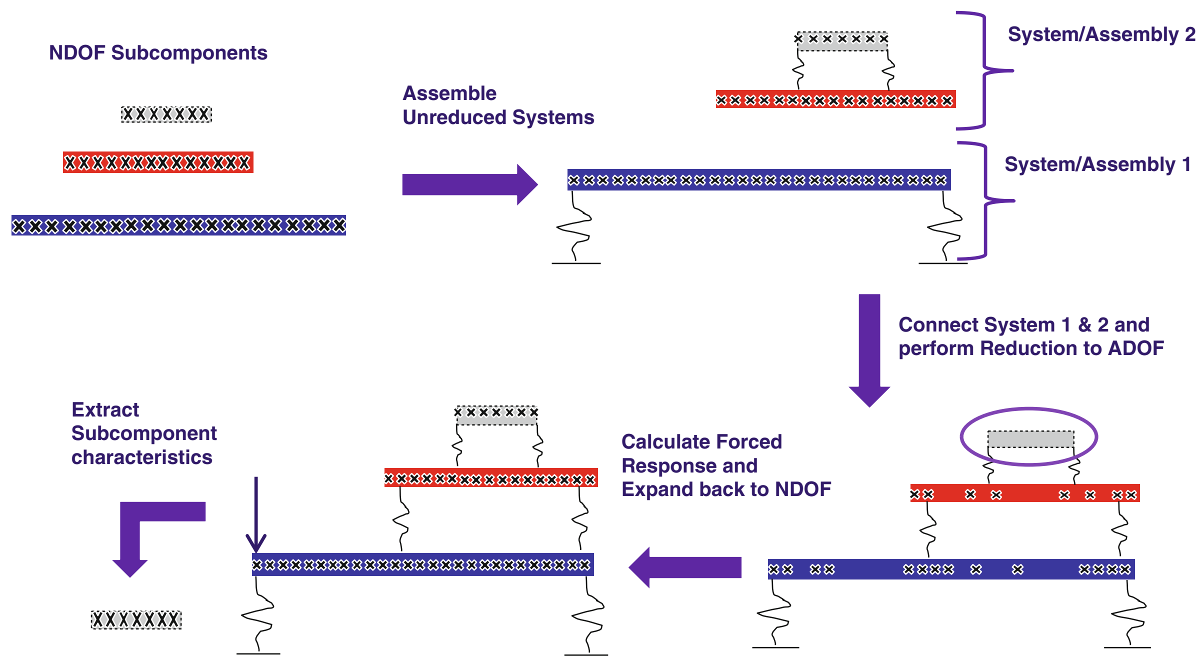
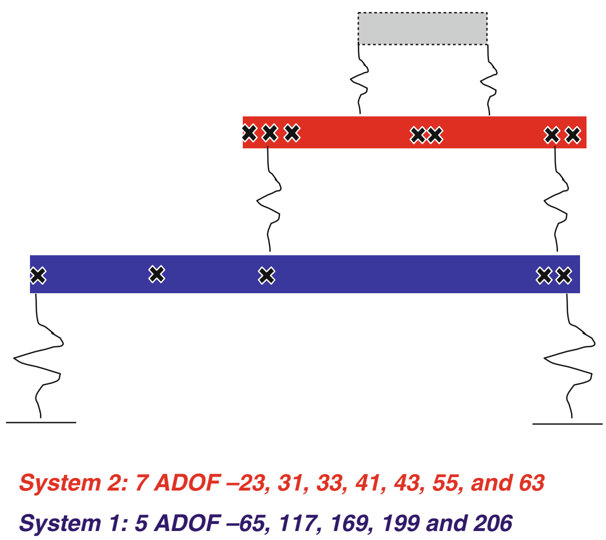


Fig. 44.17 Sequence for the development of reduced system response models



	Reference System Freq. (Hz)	SEREP Reduced System Freq. (Hz)	% Difference
Mode 1	16.8	16.8	0%
Mode 2	37.5	37.5	0%
Mode 3	68.1	68.2	0%
Mode 4	84.6	84.6	0%
Mode 5	102.1	102.1	0%
Mode 6	129.1	130.2	1%
Mode 7	210.0	211.5	1%
Mode 8	282.0	283.2	0%
Mode 9	343.0	351.2	2%
Mode 10	396.2	396.6	0%
Mode 11	477.3	2611.8	447%
Mode 12	645.3	3023.1	369%

Fig. 44.18 Comparison of SEREP reduced order model (12 DOF) frequencies with respect to (206 DOF) reference solution. "x" indicates relative location of nodes in FEM

was then used to calculate the response at ADOF and then to expand back to the full space finite element model and extract the predicted forced response of the ancillary subcomponent. The models studied were similar to those of Figs. 44.9 and 44.10 except that now the top beam has an appendage/ancillary beam attached and active during the response of the system. The results for the linear and the nonlinear hard and soft contact cases will be discussed here but further information can be found in [21, 23, 24].

In the linear case, SEREP model will be discussed. A reduced model was computed such that no DOF from the ancillary subcomponent (grey beam at the top of the red beam) was preserved in the reduction process. The model is shown in Fig. 44.18 and a comparison of the response as well as correlation results of the expansion process in Fig. 44.19.

The SEREP reduction and expansion process resulted in high correlation with full space model using only a very small fraction of the DOFs. Furthermore, the omission of the connecting DOF for the ancillary subcomponent did not yield any

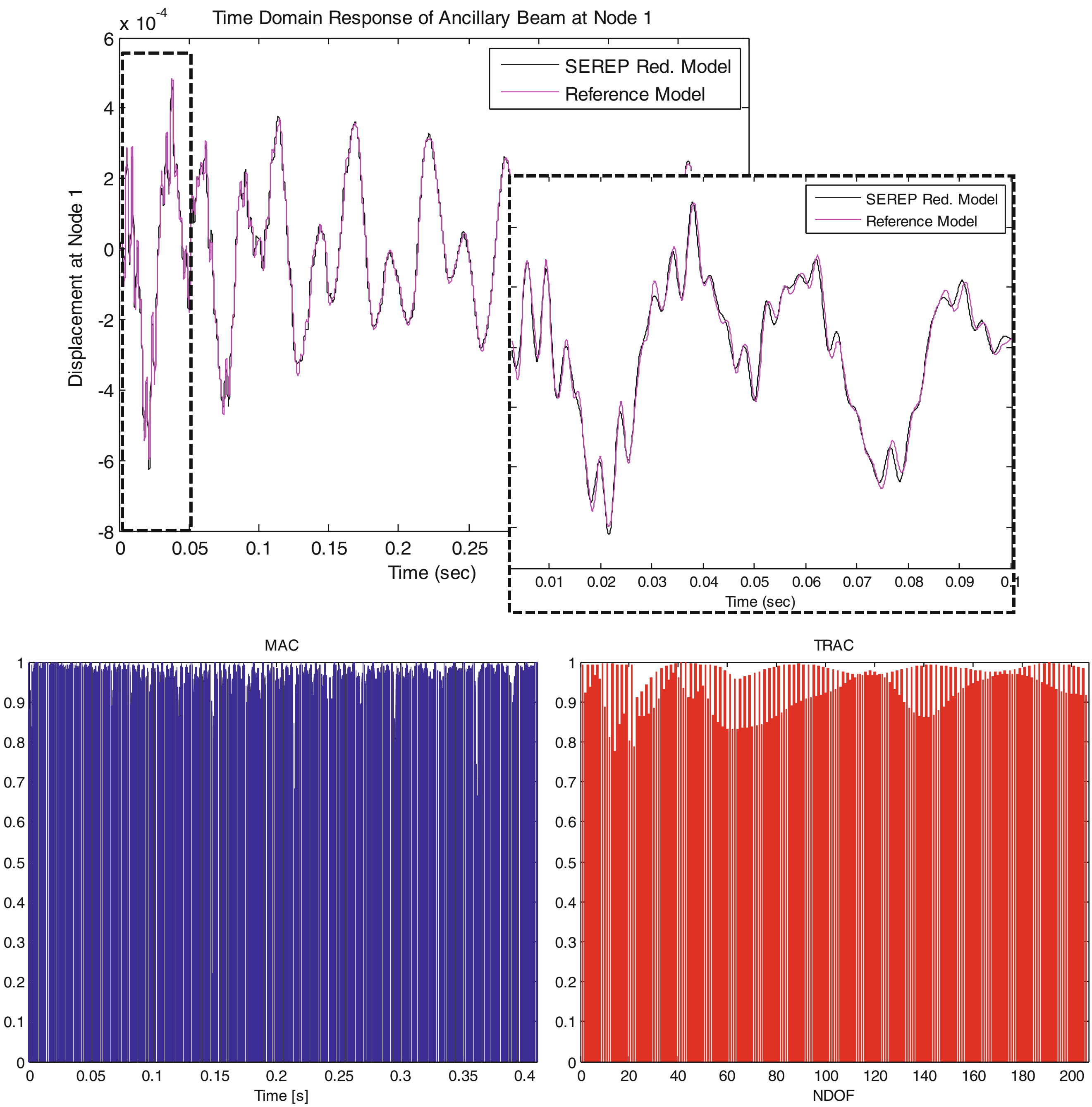


Fig. 44.19 *Top*: Comparison of time response at node 1 of the ancillary subcomponent from the expansion of a 12DOF SEREP reduced model versus the 206 DOF full space reference solution (in *magenta*). Zoomed in region shows the response for the first 0.1 s. *Bottom*: MAC and TRAC bar plots showing the correlation of the expanded SEREP reduced model to the reference model

additional error. Addition of modes beyond the 12 modes indicated in the $[U_{12}]$ matrix in the SEREP reduced model showed large improvement from the resulting expanded model response as illustrated in Fig. 44.20 where expansion is used on a 17 mode and DOF reduced model. When the reduction process is successful (as it was with the SEREP and KM_AMI models), the modes selected span the space of the system response. Modes beyond the 12 indicated by the $[U_{12}]$ smooth the approximation of the system response and further addition of modes results in better results until the reduced 'a' space model approaches the full 'N' space solution and hence spans the whole space of the full assembled system response.

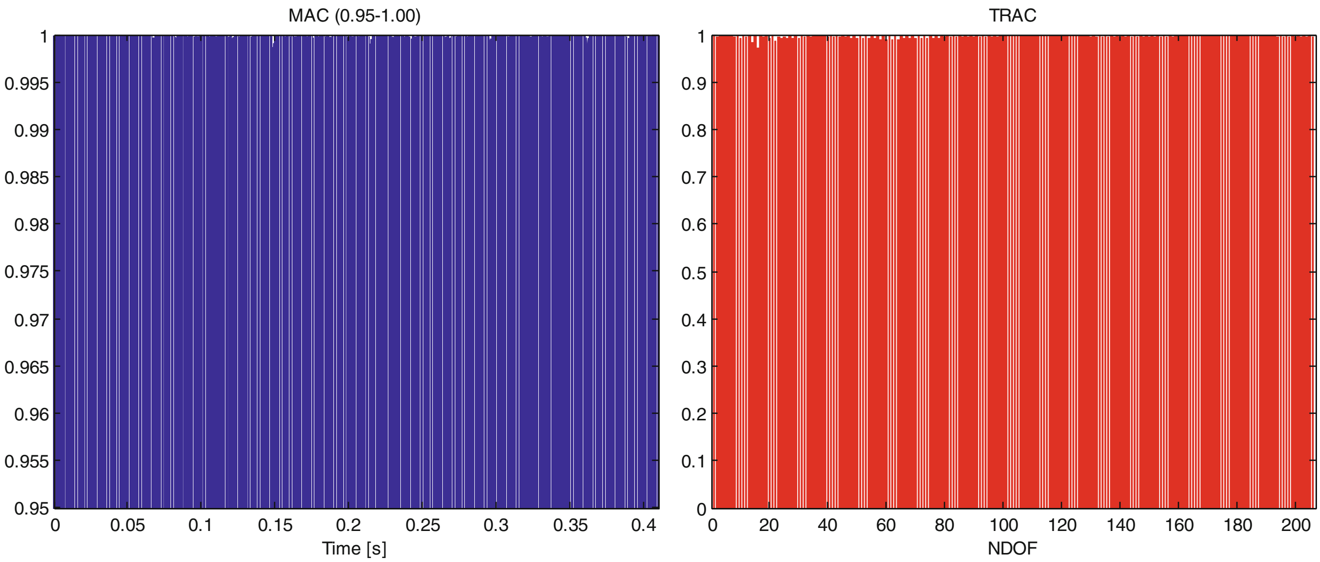


Fig. 44.20 MAC and TRAC plots showing correlation of the expanded 17 DOF SEREP reduced model to reference model. MAC y-axis is showing values from 0.95 to 1.0

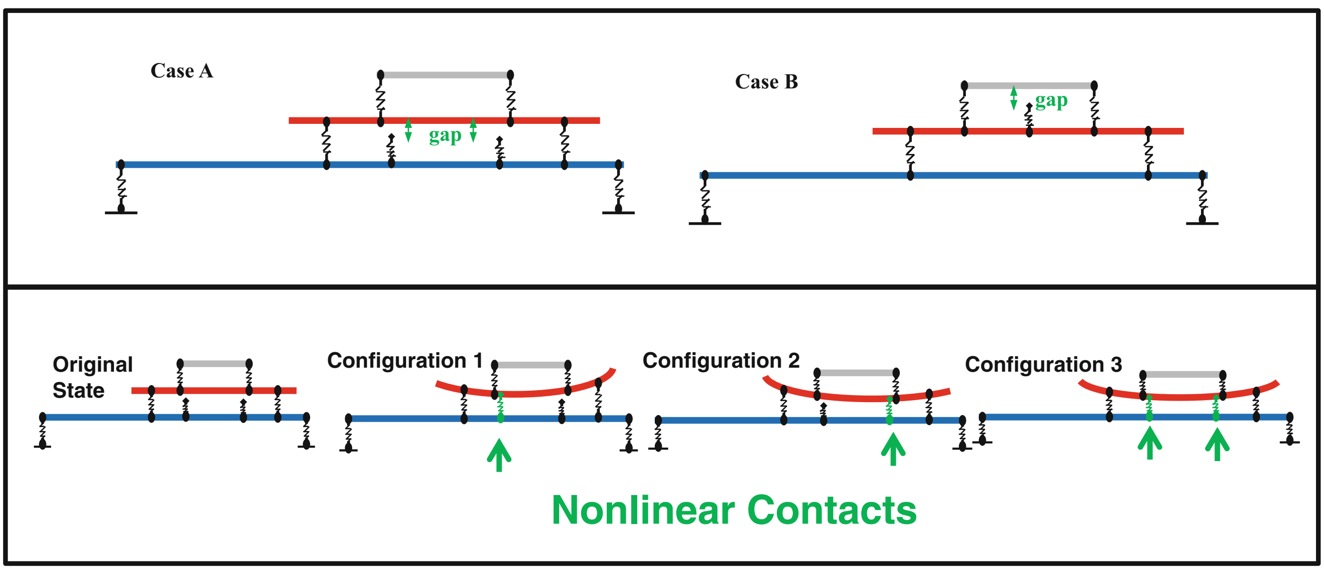


Fig. 44.21 Nonlinear gap-spring contact cases analyzed. The nonlinear configurations for Case A are shown. Each case indicates the closing or opening of the gap springs (in *green* when closed)

The following models expand the reduced order modeling of the three beam system to include nonlinear response. Gap-spring contact elements are introduced to generate nonlinear response between the two systems as done in the first example shown here. However, the possibility of nonlinear contacts is introduced at the subcomponent level as shown in Fig. 44.21. For the first nonlinear case, four different configurations are possible (initial configuration, one configuration for each spring coming in contact, and one for both springs in contact) as seen in Fig. 44.21. Using the calculated $[U_{12}]$ for all configurations, the necessary number of component modes were determined in order to properly characterize the system; using the force described above, the system modes must be able to characterize the response over a 250 Hz frequency span. While this is the initial frequency range of interest for any structural dynamic study, there also needs to be consideration for the nonlinear contacts which occur that may require a frequency range beyond that initially determined from the applied forcing function.

In these nonlinear cases, the location of the nonlinearity can have a significant effect on the accuracy of the prediction and on the number of modes (and DOF) required in the reduction/expansion process. SEREP reduction was used to reduce the active DOF of the system to an ‘a’ set not including DOFs on the ancillary beam. The forced responses of the reduced ADOF

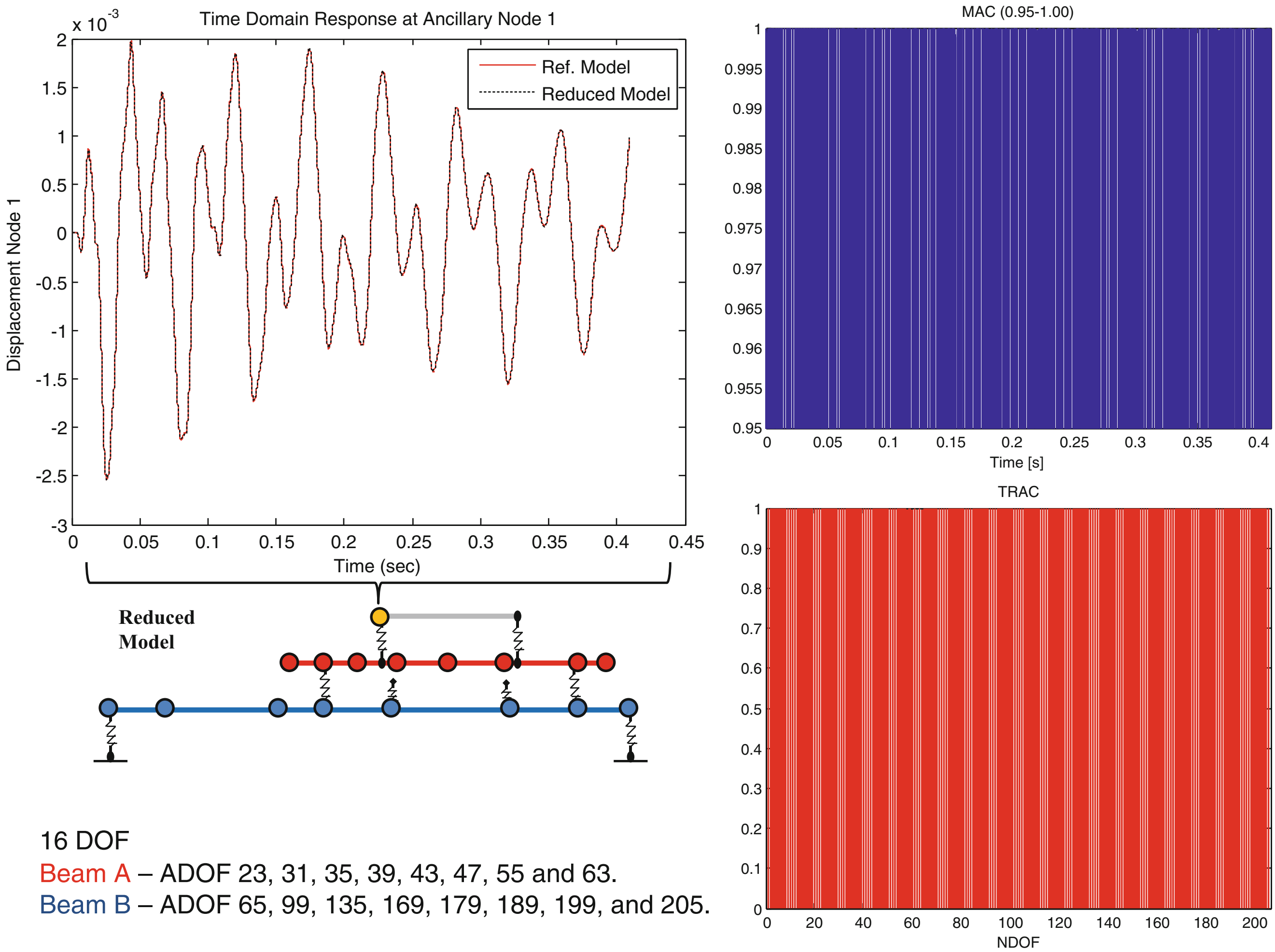


Fig. 44.22 Predicted response at ancillary node 1 (in *yellow*) from 16 DOF reduced model. MAC from 0.95 to 1.0 (*top right*) and TRAC (*bottom right*) correlation of models

systems were computed. The dynamic characteristics of the ancillary subcomponent were then extracted using the system information available from the reduction process. A SEREP reduced model using 16 modes was created. Figure 44.22 shows the comparison of the predicted response of the expanded model and the reference model.

Using 16 modes allows for the accurate prediction of up to 8 modes (which spans approximately 300 Hz) of the three-beam assembled system. Figure 44.23 shows the comparison of the FFT of the displacement at the left spring contact location.

The FFT of the displacement at the left contact location in Fig. 44.23 shows no significant effect of mode truncation of the higher order modes and the response is accurate approximation of the full NDOF solution. Note that no ADOF has been placed at the ancillary subcomponent or at the DOFs that connect the ancillary beam to the top beam. Nevertheless, the embedded information in the reduction process has successfully allowed the prediction of the time response at all NDOF.

The effect of higher order modes can be exacerbated if the type of contact is a hard impact, thus producing a narrow time pulse that translates into a high order pulse in the frequency domain. The next case will explore a hard contact spring acting at two locations of the structure. A comparison of the two types of contacts, soft and hard, can be seen in Fig. 44.24.

Nonlinear contacts of 10,000 lb/in were implemented instead of the soft contacts of 100 lb/in previously used. Because the hard contact excites frequencies in the range of 700 Hz, the 16 mode model previously used cannot give the best approximation of the response of the system. The selection of modes preserved in the reduction must form a linearly independent set of vectors spanning the space of the full response of the system. In other words, the selected projection vectors in the transformation matrix must be able to approximate any other vector in the space as linear combinations of the mode shape vectors preserved in the reduced space. A 21 modes and DOF model was created to better approximate the hard contact nonlinear response of the system. Figure 44.25 shows the correlation and response for the 21 DOF model and the reference model.

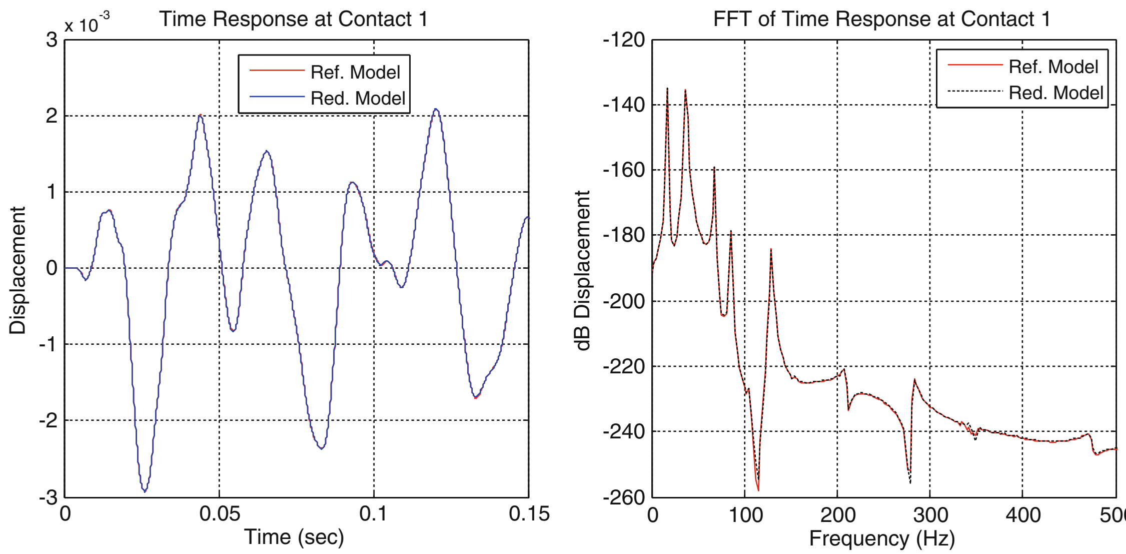


Fig. 44.23 Comparison of response (*left*) and FFT (*right*) of reference model and 16 mode reduced order model at spring contact location 1

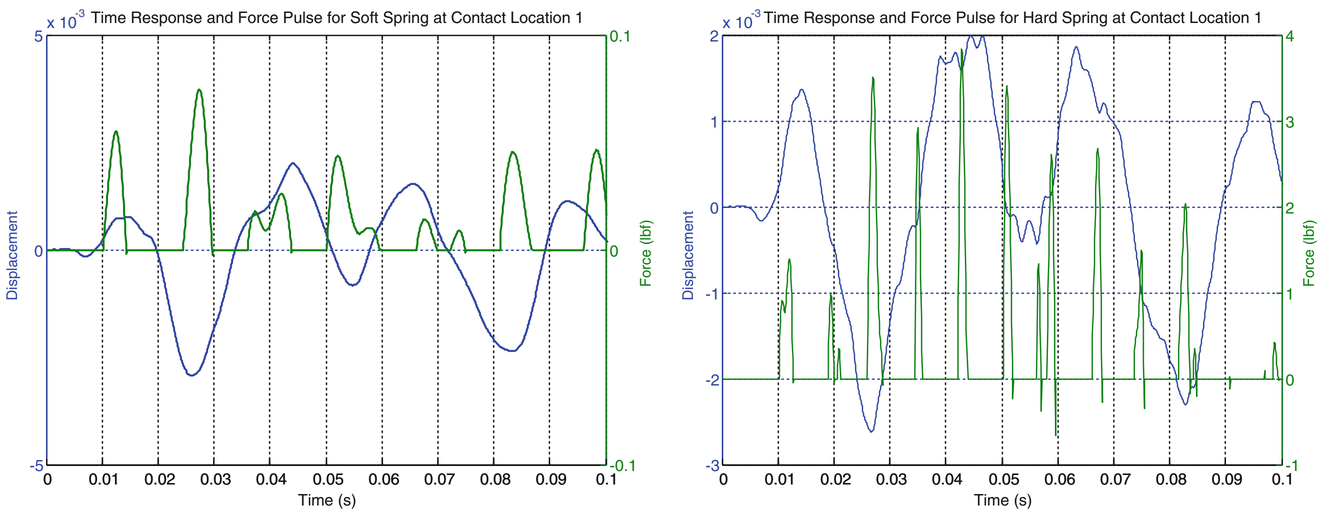


Fig. 44.24 Comparison of displacement and force magnitude for soft (*left*) and hard (*right*) type of nonlinear contacts at DOF 37 of the three beam system at contact location 1 (*left gap spring*)

The modes used in the reduced model resulted in an accurate approximation of the response of the NDOF system. However, the effects of mode truncation could not be completely mitigated. Caution must be exercised when simply adding modes to the reduced order model since linear independence of the vectors formed in the expansion process is very important. The reduced space matrices can become ill-conditioned for certain choice and number of DOF and the predicted response is then subjected to high levels of numerical error. Figure 44.26 shows that the FFT and response at the contact location 1 is reasonably approximated by the reduced model of 21 modes.

The cases considered thus far have shown that full field results can be obtained from reduced order models with subcomponent interactions from the embedded information preserved in the reduction process. Using the necessary number of modes in the reduced model to span the space of all modes of interest allows the response at the ancillary and any other DOF to be predicted accurately. The $[U_{12}]$ contribution matrix and the effect of the higher order modes from nonlinear interactions must be taken into account as well as the linear independence and well-determined behavior of the reduced matrices in order to obtain a good approximation of the dynamic characteristics of the system. Moreover, when proper precautions are taken to reduce the N space system to 'a' space, no ADOF are needed in the dynamically active ancillary subcomponent.

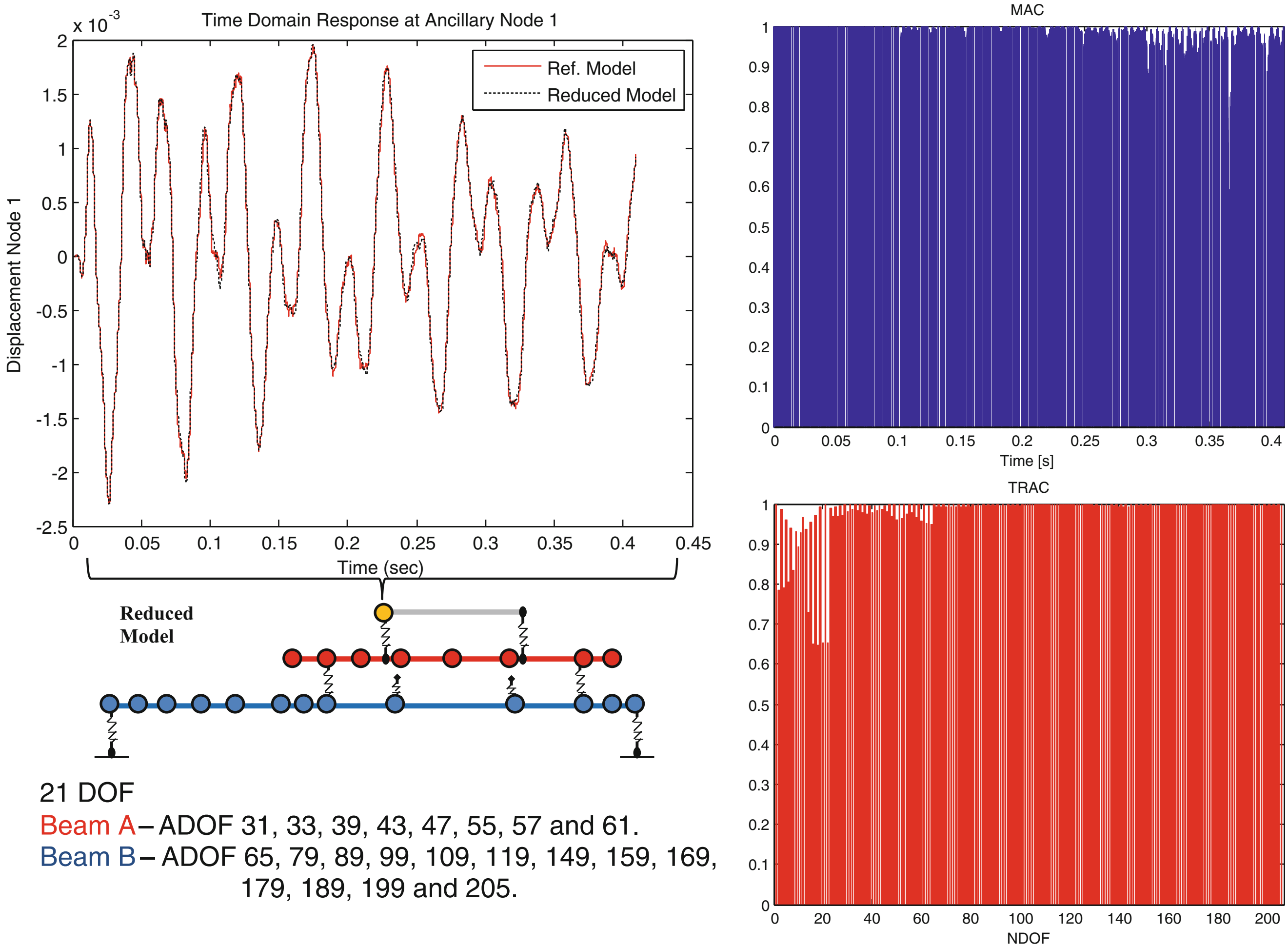


Fig. 44.25 Predicted response at ancillary node 1 from expansion of 21 DOF reduced order model with hard contacts. MAC (*top right*) and TRAC (*bottom right*) correlation of models

The next case explores the natural extension of this work to include a nonlinear contact occurring at the ancillary subcomponent. The nonlinear contact of 100 lb/in was placed at Ancillary but no ADOF were selected on the ancillary subcomponent or the connecting DOF between the red beam and the ancillary. In order to check for a change of configuration during the time response of the reduced model, the algorithm used the expansion of the displacement at time t_i for the contact DOF and then continued to the next time step t_{i+1} making the necessary adjustments if the spring came in contact with the ancillary subcomponent. Only the response at the contact DOF was expanded throughout the response since no other DOF are needed during the computation of the reduced model response. The response at 'a' space was expanded to NDOF using the transformation matrices (mode shape projection vectors) from the reduction process. Figure 44.27 shows a comparison of the response at DOF 1 of the ancillary as well as the correlation with the reference model.

Figure 44.28 shows a comparison of the response and its FFT at the contact location. In the previous case, the contact point was included in the active set of DOF. In this case, the contact point was not included. The results obtained were not as accurate and well behaved as in the previous case. This may be due to many different factors such as numerical issues, rank problems, need for more modes, and many other factors which need to be addressed in future work.

Additional models of the SEREP reduced case where more modes were included in the reduction process (beyond 18 modes) lead to issues with the condition number of the reduced matrices. Furthermore, as the goal is to avoid placing ADOF at the ancillary subcomponent in order to gage the accuracy of the technique in reconstructing the subcomponent response from embedded information, there was limited number of choices for DOF locations in the new reduced models.

Guyan reduced models with KM_AMI improvement provided a way to create larger reduced models without the possible rank deficiencies of the SEREP reduced models. A 25 mode KM_AMI model is discussed next.

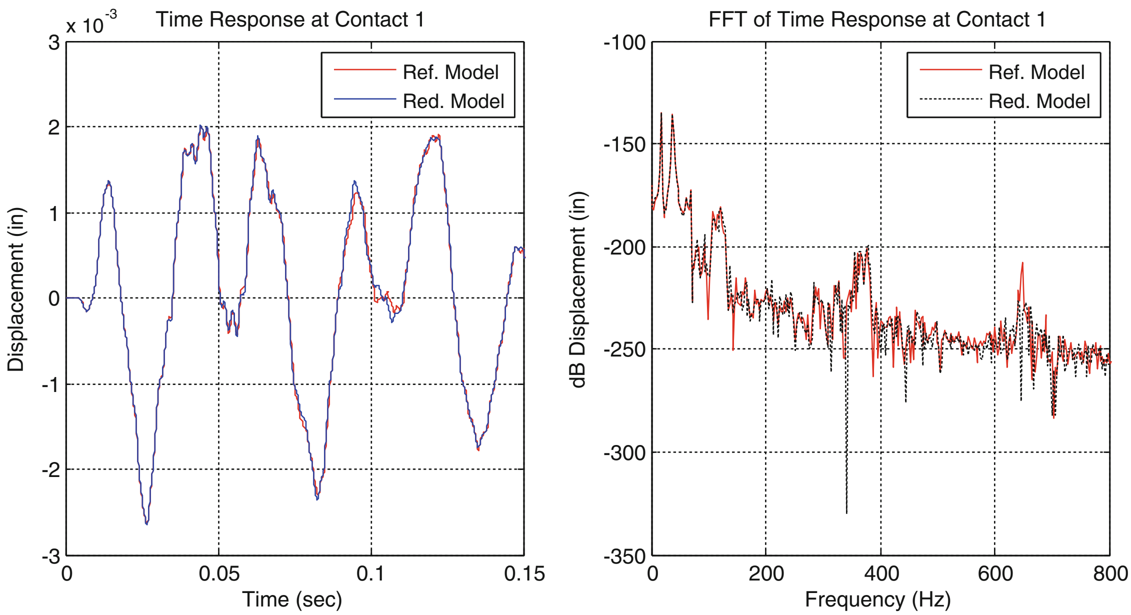


Fig. 44.26 Comparison of response (*left*) and FFT (*right*) of reference (in *red*) and expansion of 21 DOF reduced order model at contact location 1 (*left gap spring*)

Figure 44.29 shows a comparison of the predicted response at the ancillary DOF 1 from this model and the correlation plots with the full space reference model. The addition of modes resulted in a noticeable improvement in the quality of the predicted response. Note that the locations of low correlation for TRAC occur at the rotary DOF and not at the translational DOF.

Figure 44.30 shows a comparison of the FFT and response at the contact location. The addition of modes is shown to better characterize the influence of the higher order modes present due to the nonlinear contacts but the effects of mode truncation cannot be completely eliminated.

Additional cases were studied involving hard and soft. When enough modes of the unconnected components are preserved such that they form a linearly independent set of vectors spanning the space of the full response, the response was accurately predicted at all DOF of the system. The prediction at the ancillary subcomponent was successfully performed even when no active DOF were preserved in the reduced model. These studies are further explained in [21, 23, 24].

44.4 Conclusion

Understanding of the linear reduction/expansion methodology showed the advantage of using reduced models for full field prediction of systems that may undergo localized nonlinear response due to component contact or coupling interactions. For these types of problems, the presence of highly nonlinear coupling elements such as hard contacts, isolation mounts, gap springs, bilinear springs, etc. is predicted with a piecewise linear approximation providing the dynamic response of the structure as a superposition of all possible configurations of the system.

For all cases considered, linear and nonlinear, high correlation values were obtained between the reference model and the predicted expanded model response. For such models, the computation time was greatly reduced as well as the number of DOF in the model. The scalability of the expansion process was previously demonstrated by other researchers as well as the application of these methodologies for the prediction of strain.

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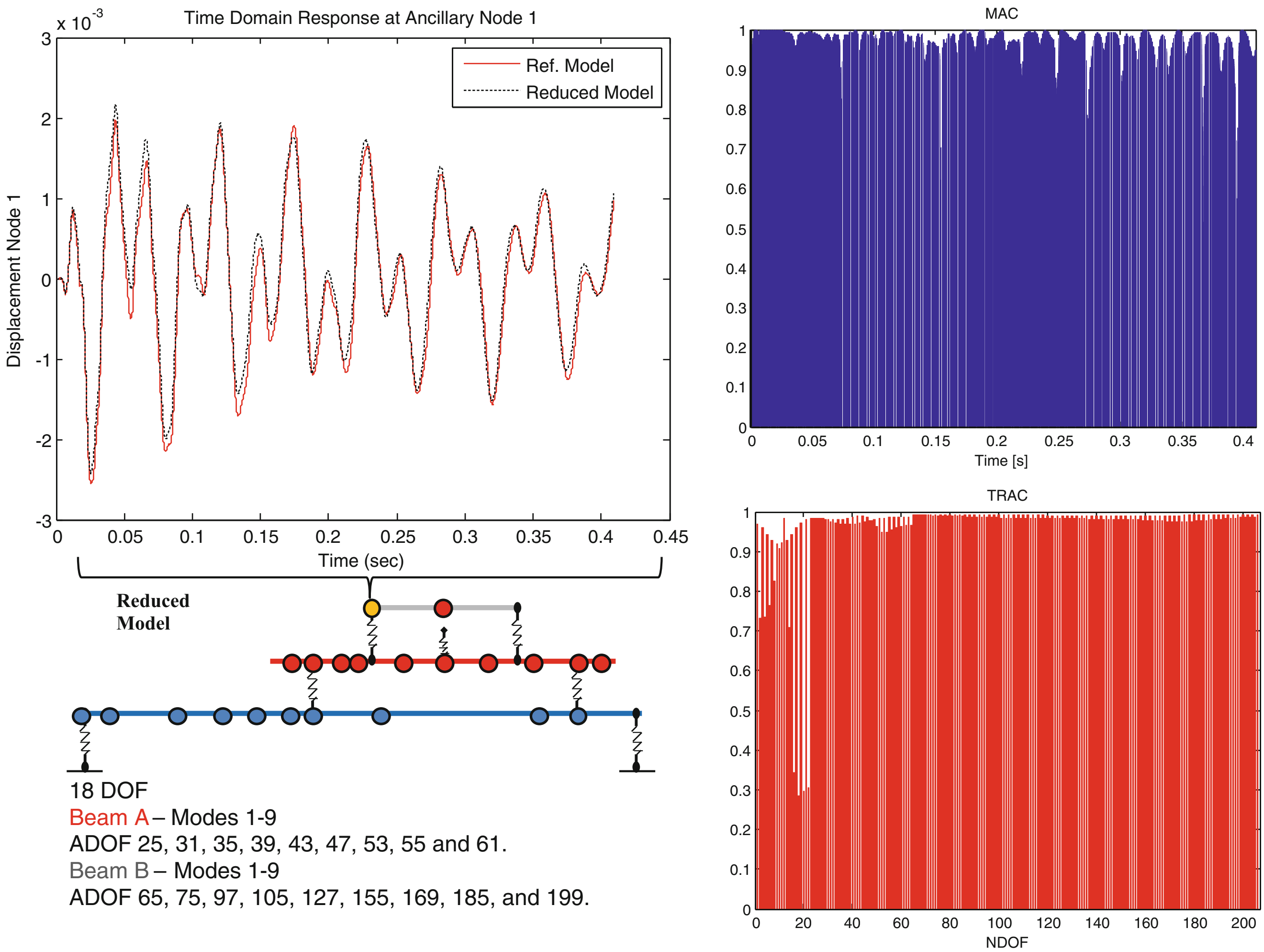

Fig. 44.27 Predicted response at ancillary node 1 from 18 DOF reduced model with soft contact. MAC (*top right*) and TRAC (*bottom right*) correlation of models

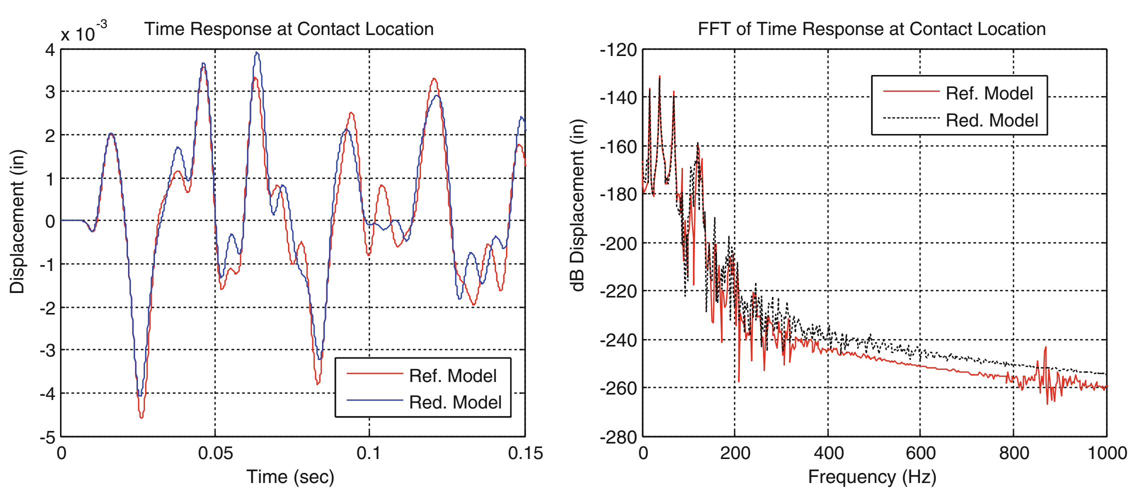

Fig. 44.28 Comparison of time response (*left*) and FFT (*right*) of reference model and expansion of 18 DOF reduced order model at contact location. Reference model is shown in *red*

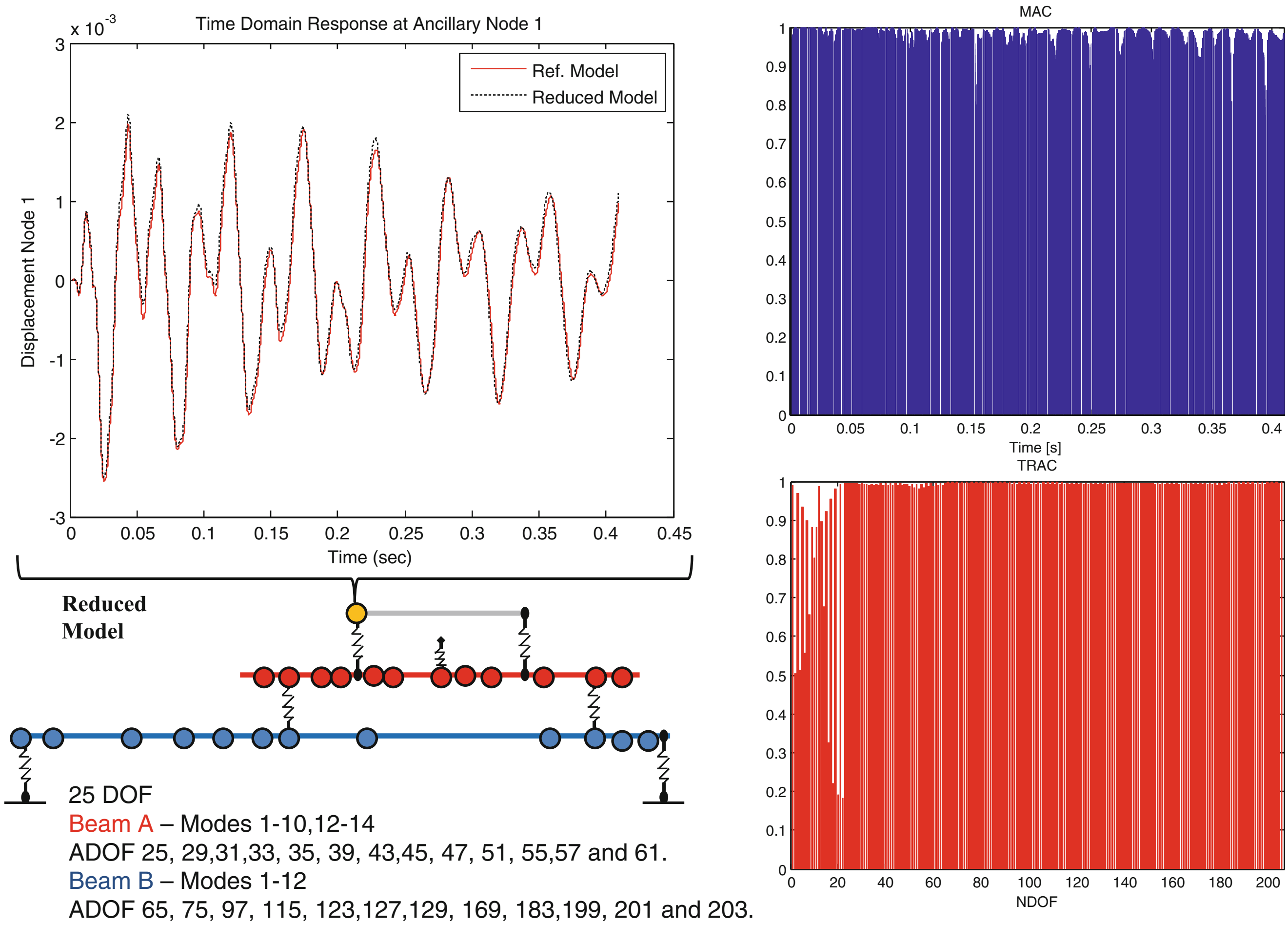

Fig. 44.29 Predicted response at ancillary DOF 1 from expansion of 25 DOF KM_AMI improved Guyan reduced model with soft contact. MAC (*top right*) and TRAC (*bottom right*) correlation of models

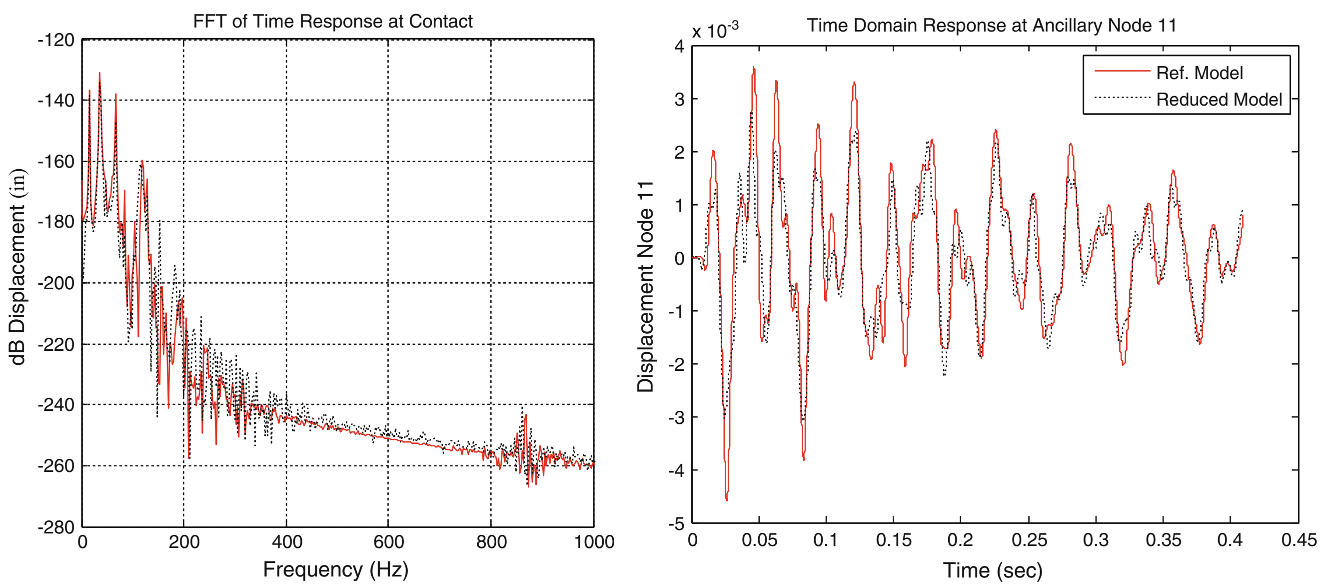

Fig. 44.30 Comparison of FFT (*left*) and time response (*right*) of reference model and expansion of 25 DOF reduced order model at contact location. Reference model is shown in *red*

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