Chapter 6
Lorentz Spaces

Abstract The spaces considered in the previous chapters are one-parameter dependent. We now study the so-called Lorentz spaces which are a scale of function spaces which depend now on two parameters. Our first task therefore will be to define the Lorentz spaces and derive some of their properties, like completeness, separability, normability, duality among other topics, e.g., Hölder's type inequality, Lorentz sequence spaces, and the spaces $L\exp$ and $L\log L$, which were introduced by Zygmund and Titchmarsh.

6.1 Lorentz Spaces

We start to recall that, by Remark 4.14, we can calculate the Lebesgue norm of a function using the notion of nonincreasing rearrangement in the following way

$$\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}\mu(x) = \int_0^\infty f^*(t)^p \,\mathrm{d}t = \int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^p \frac{\mathrm{d}t}{t}.$$

Using the right-hand side representation we can try to replace the power p by q to obtain some kind of generalization. It turns out that this indeed will produce a new scale of function spaces which have the Lebesgue spaces as a special case.

Definition 6.1. Let $(X, \mathscr{A}, \mu)$ be a measure space. For any $f \in \mathfrak{F}(X, \mathscr{A})$ and any two extended real numbers p and q in the set $[1, \infty]$ put

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$$\left\| f \mid L_{(p,q)} \right\| = \|f\|_{(p,q)} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases} \tag{6.1}$$

The functionals $\|\cdot\|_{(p,q)}$ are thus extended nonnegative valued functions on $\mathfrak{F}(X,\mathscr{A})$ and the Lorentz space will be defined in terms of these functions just as the Lebesgue spaces were defined in terms of the functional $\|\cdot\|_p$. ⊘

In view of Remark 4.14 (ii), it is obvious that

$$\|f\|_p = \|f\|_{(p,p)}, \tag{6.2}$$

for any function $f \in \mathfrak{F}(X,\mathscr{A})$ and any $p \in [1,\infty)$ and this equality also holds for $p = \infty$. An easy calculation, using the fact that

$$\chi_A^*(t) = \chi_{[0,\mu(A)]}(t),$$

will show that

$$\|\chi_A\|_{(p,q)} = \begin{cases} \left(\frac{p}{q}\right)^{1/q} \left(\mu(A)\right)^{1/p}, & 1 \leq p,q < \infty \\ \infty, & p = \infty, q < \infty \\ \left(\mu(A)\right)^{1/p}, & 1 \leq p < \infty, q = \infty \\ 1, & p = q = \infty, \end{cases}$$

for any set $A \in \mathscr{A}$ for which $0 < \mu(A) < \infty$.

We are now in conditions to introduce the Lorentz spaces

Definition 6.2. For any measure space $(X,\mathscr{A},\mu)$ and any two extended real numbers p and q in the interval $[0,\infty]$ the set

$$\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu) = \left\{ f \in \mathfrak{F}(X,\mathscr{A}) : \|f\|_{(p,q)} < \infty \right\},$$

is called the *pre-Lorentz spaces* associated with $(X,\mathscr{A},\mu)$. ⊘

The space $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$ with $1 \leq q < \infty$ and $p = \infty$ will not be of any interest. In fact, if f is a function in $\mathfrak{F}(X,\mathscr{A})$ with the property that $\|f\|_{(\infty,q)} < \infty$ for some $q \in [1,\infty)$, then

$$\int_0^\infty [f^*(t)]^q \frac{\mathrm{d}t}{t} \geq \int_0^s [f^*(t)]^q \frac{\mathrm{d}t}{t} \geq [f^*(s)]^q \int_0^s \frac{\mathrm{d}t}{t},$$

since $f^*(t) \geq f^*(s)$ whenever $0 \leq t \leq s$ and therefore $f^*(s) = 0$ for all $s > 0$. Thus $f = 0$ μ-a.e. by (6.2). For this reason we have $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu) = \{0\}$ for every $0 < q < \infty$.

Theorem 6.3 *Let $(X, \mathscr{A}, \mu)$ be a measure space and let p, q and r be three extended real numbers satisfying $0 < p \leq \infty$ and $0 < q < r \leq \infty$. Then*

(a) $\|f\|_{(p,r)} \leq \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{r}} \|f\|_{(p,q)}$ *for any $f \in \mathfrak{F}(X, \mathscr{A})$.*
(b) $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu) \subseteq \mathbf{L}_{(p,r)}(X, \mathscr{A}, \mu)$.

Proof. It is clearly sufficient to prove (i). Indeed for $r = \infty$, we have

$$\begin{aligned} \|f\|_{(p,q)}^q &= \int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \\ &\geq \int_0^s \left(t^{1/p} f^*(s)\right)^q \frac{\mathrm{d}t}{t} \\ &= \left(\frac{p}{q}\right) s^{\frac{q}{p}} \left(f^*(s)\right)^q, \end{aligned}$$

for any $s > 0$ thus

$$\|f\|_{(p,\infty)} = \sup_{t>0} t^{1/p} f^*(t) \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_{(p,q)}. \tag{6.3}$$

On the other hand, if $1 \leq q < r < \infty$, then

$$\begin{aligned} \|f\|_{(p,r)}^r &= \int_0^\infty \left(t^{1/p} f^*(t)\right)^{r-q} \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \\ &\leq \|f\|_{(p,\infty)}^{r-q} \int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \\ &\leq \left(\frac{q}{p}\right)^{\frac{r-q}{q}} \|f\|_{(p,q)}^{r-q} \|f\|_{(p,q)}^q \\ &= \left(\frac{q}{p}\right)^{\frac{r-q}{q}} \|f\|_{(p,q)}^r, \end{aligned}$$

by definition of $\|\cdot\|_{(p,\infty)}$ and (6.3), and this completes the proof of Theorem 6.3. □

A natural interrogation is to ask if the functional $\|\cdot\|_{(p,q)}$ defined in (6.1) is a norm on $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$. The following result gives us some light on this regard.

Theorem 6.4. *If $(X, \mathscr{A}, \mu)$ is a measure space and if $p \in [1, \infty]$. Then*

(a) $\|f+g\|_{(p,q)} \leq 2^{1/p}\left(\|f\|_{(p,q)} + \|g\|_{(p,q)}\right)$ *for any two functions f, $g \in \mathfrak{F}(X, \mathscr{A})$.*

(b) $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$ *is a vector space.*

(c) If f *is a function in* $\mathfrak{F}(X,\mathscr{A})$*, then* $\|f\|_{(p,q)}=0$ *if and only if* $f=0$ μ*-a.e.*

Proof. (a) Assume that f and g are two functions in $\mathfrak{F}(X,\mathscr{A})$. The particular case of theorem1.2.3 implies that

$$\begin{aligned}\|f+g\|_{(p,\infty)} &= \sup_{t>0} t^{1/p}(f+g)^*(t)\\ &\le \sup_{t>0} t^{1/p}\Big[f^*(t/2)+g^*(t/2)\Big]\\ &\le 2^{1/p}\Big(\|f\|_{(p,\infty)}+\|g\|_{(p,\infty)}\Big)\end{aligned}$$

and, if $q<\infty$, that

$$\begin{aligned}\|f+g\|_{(p,q)} &= \left(\int_0^\infty \Big(t^{1/p}(f+g)^*(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\\ &\le \left(\int_0^\infty \Big[t^{1/p}\Big(f^*(t/2)+g^*(t/2)\Big)\Big]^q \frac{\mathrm{d}t}{t}\right)^{1/q}.\end{aligned}$$

By Minkowski's inequality we have

$$\begin{aligned}\|f+g\|_{(p,q)} &\le \left(\int_0^\infty \Big(t^{1/p}f^*(t/2)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q} + \left(\int_0^\infty \Big(t^{1/p}g^*(t/2)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\\ &= 2^{1/p}\left[\left(\int_0^\infty \Big(t^{1/p}f^*(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q} + \left(\int_0^\infty \Big(t^{1/p}g^*(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right]\\ &= 2^{1/p}\Big(\|f\|_{(p,q)}+\|g\|_{(p,q)}\Big).\end{aligned}$$

(b) Item (a) implies that if f and g belong to $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$, then so does $f+g$. Since it is easy to see that $\|cf\|_{(p,q)}=|c|\|f\|_{(p,q)}$ for any $c\in\mathbb{R}$ and any $f\in\mathfrak{F}(X,\mathscr{A})$ this shows that $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$ is a vector space.

(c) Suppose that $\|f\|_{(p,q)}=0$, then

$$0 = \int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \geq \int_0^s \left(t^{1/p} f^*(s)\right)^q \frac{\mathrm{d}t}{t}$$
$$= \left(\frac{p}{q}\right) s^{q/p} \left(f^*(s)\right)^q \geq 0$$

hence $f^*(s) = 0$ for all $s > 0$, from this and Remark 4.14 (ii) we have $\int_X |f| \, \mathrm{d}\mu = \int_0^\infty f^*(s) \, \mathrm{d}s = 0$ which implies $f = 0$ μ-a.e.

□

Part (a) of this theorem leaves unanswered the question of whether $\|\cdot\|_{(p,q)}$ is a norm on $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$. It turns out that $\|\cdot\|_{(p,q)}$ is a norm on $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$ if $1 \leq q \leq p < \infty$ (see corollary 6.8). Next we will see that, in general $\|\cdot\|_{(p,q)}$ is not a norm but equivalent to one if $1 < p < q \leq \infty$.

Theorem 6.5. *Let $(X, \mathscr{A}, \mu)$ be a nonatomic measure space. Then*

$$\|\cdot\|_{(p,q)} : \mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu) \to \mathbb{R}^+$$

is not a norm for:

(a) $1 \leq p < q \leq \infty$.
(b) $0 < p < 1$, $0 < q \leq \infty$.
(c) $0 < p < 1 \leq q < \infty$.

Proof. (a) We start with the case when $1 \leq p < q < \infty$. Take

$$f(x) = (1+\varepsilon)\chi_{[0,a+h]}(x) + \chi_{[a+h,a+2h]}(x),$$

and

$$g(x) = \chi_{[0,h]}(x) + (1+\varepsilon)\chi_{[a+h,a+2h]}(x),$$

where a, h, $\varepsilon > 0$. It is easy to see that $f^*(t) = g^*(t) = f(x)$ and since

$$(f+g)^*(t) = (2+2\varepsilon)\chi_{[0,a]}(t) + (2+\varepsilon)\chi_{[a,a+2h]}(t),$$

it follows that we can evaluate the norm of f, g and $f+g$ by

$$\|f\|_{(p,q)}^q = \|g\|_{(p,q)}^q = \frac{p}{q}\left[(1+\varepsilon)^q(a+h)^{q/p} + (2a+h)^{q/p} - (a+h)^{q/p}\right],$$

and

$$\|f+g\|_{(p,q)}^q = \frac{p}{q}\left[(2+2\varepsilon)^q a^{q/p} + (2+\varepsilon)^q\left((a+2h)^{q/p} - a^{q/p}\right)\right],$$

respectively. Now, let us assume that the triangle inequality holds, that is

$$\|f+g\|_{(p,q)} \leq \|f\|_{(p,q)} + \|g\|_{(p,q)}.$$

Then the inequality

$$(2+2\varepsilon)^q a^{q/p} + (2+\varepsilon)^q \Big((a+2h)^{q/p} - a^{q/p} \Big) \leq 2^q \Bigg((1+\varepsilon)^q (a+h)^{q/p} + (a+2h)^{q/p} - (a+h)^{q/p} \Bigg),$$

holds and it can be written as

$$(a+2h)^{q/p} - (a+h)^{q/p} \leq \frac{(1+\varepsilon)^q - (1+\varepsilon/2)^q}{(1+\varepsilon/2)^q - 1} \Big((a+h)^{q/p} - a^{q/p} \Big).$$

Taking $\varepsilon \to 0$, we obtain that

$$(a+2h)^{q/p} + a^{q/p} \leq 2(a+h)^{q/p}. \tag{6.4}$$

If we define a function f as

$$f(x) = \int_0^x t^{\frac{q}{p}-1}\, \mathrm{d}t,$$

then we can rewrite inequality (6.4) as

$$f(a+2h) + f(a) \leq 2f(a+h),$$

which implies, together with the fact that f is continuous, that it is a concave function. By the concavity of f the derivative $f'(x) = x^{\frac{q}{p}-1}$must be a decreasing function, that is, it must be that $q \leq p$. This contradicts $q > p$, and we concluded that the triangle inequality does not hold. For $q = \infty$, take measurable sets $A \subset B \subset X$ such that

$$a = \left(\frac{\mu(B)}{\mu(A)} \right)^{1/p} > 1,$$

and $\mu(B \backslash A) \leq \mu(A)$. If we let

$$f(x) = a\chi_A(x) + \chi_{B\backslash A}(x),$$

and

$$g(x) = \chi_A(x) + a\chi_{B\backslash A}(x),$$

then

$$f^*(t) = a\chi_{[0,\mu(A)]}(t) + \chi_{[\mu(A),\mu(B)]}(t),$$

and

$$g^*(t) = a\chi_{[0,\mu(B\backslash A)]}(t) + \chi_{[\mu(B\backslash A),\mu(B)]}(t).$$

Thus

$$\|f\|_{(p,\infty)} = \max\left(a\Big(\mu(A)\Big)^{1/p}, \Big(\mu(B)\Big)^{1/p}\right) = \Big(\mu(B)\Big)^{1/p}$$

$$\|g\|_{(p,\infty)} = \max\left(a\Big(\mu(B\backslash A)\Big)^{1/p}, \Big(\mu(B)\Big)^{1/p}\right) = \Big(\mu(B)\Big)^{1/p},$$

and since $f(x)+g(x) = (a+1)\chi_B$, we have $\|f+g\|_{(p,\infty)} = (a+1)\chi_B$. Then

$$\begin{aligned}\|f+g\|_{(p,\infty)} &= (a+1)\Big(\mu(B)\Big)^{1/p} \\ &> 2\Big(\mu(B)\Big)^{1/p} \\ &= \|f\|_{(p,\infty)} + \|g\|_{(p,\infty)},\end{aligned}$$

which shows that the triangle inequality does not hold. Hence the proof of the first case is complete.

(b) Let $0 < p, q < \infty$ and A, B be two measurable sets such that $A \cap B \neq \emptyset$. Then

$$\|\chi_A + \chi_B\|_{(p,q)} = \left(\frac{p}{q}\right)^{1/q}\Big(\mu(A)+\mu(B)\Big)^{1/p},$$

and

$$\|\chi_A\|_{(p,q)} + \|\chi_B\|_{(p,q)} = \left(\frac{p}{q}\right)^{1/q}\left(\Big[\mu(A)\Big]^{1/p} + \Big[\mu(B)\Big]^{1/p}\right).$$

The triangle inequality gives

$$\Big(\mu(A)+\mu(B)\Big)^{1/p} \le \Big[\mu(A)\Big]^{1/p} + \Big[\mu(B)\Big]^{1/p},$$

and it fails for any $0 < q < \infty$ if $0 < p < 1$.
If $q = \infty$, we get the same norms.

(c) Let $0 < p < \infty$ and $0 < q < 1$. Define

$$f(x) = \begin{cases} 2 \text{ if } 0 < x < 2^{-p} \\ 0 \text{ otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 4 \text{ if } 0 < x < 2^{-2p} \\ 0 \text{ otherwise.} \end{cases}$$

Then

$$\|f\|_{(p,q)} = \|g\|_{(p,q)} = \left(\frac{p}{q}\right)^{1/q},$$

and since

$$f(x)+g(x) = \begin{cases} 6 \text{ if } 0 < x < 2^{-2p} \\ 2 \text{ if } 2^{-2p} < x < 2^{-p}, \end{cases}$$

the decreasing rearrangement of $f+g$ is equal to $f+g$, i.e., $(f+g)^* = f+g$. Thus

$$\|f+g\|_{(p,q)} = \left(\frac{p}{q}\right)^{1/q} \left(2^{2-2q} + 2^{1-q}\right)^{1/q}.$$

Assume that the triangle inequality holds. Then

$$2^{2-2q} + 2^{1-q} \le 2^q,$$

which can be written as

$$4 \le 2^q(2^{2q} - 2) < 2(2^2 - 2) = 4.$$

Hence, we have a contradiction and the assumption that the triangle inequality holds is wrong. This proves the third case. □

The following result gives us a characterization of $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$ in terms of the distribution function.

Theorem 6.6. *Let* $(X,\mathscr{A},\mu)$ *be a* σ*-finite measure space. For* $0 < p < \infty$ *and* $0 < q \le \infty$ *we have the identity*

$$\|f\|_{(p,q)} = p^{1/q} \left(\int_0^\infty \left[\lambda \left(D_f(\lambda) \right)^{1/p} \right]^q \frac{d\lambda}{\lambda} \right)^{1/q}.$$

Proof. Case $q = \infty$. For this case let us define

$$C = \sup_{\lambda > 0} \left\{ \lambda^p D_f(\lambda) \right\}^{1/p},$$

then

$$D_f(\lambda) \le \frac{C^p}{\lambda^p}.$$

Choosing $t = \frac{C^p}{\lambda^p}$ we have $\lambda = \frac{C}{\lambda^{1/p}}$, and thus it is clear that

$$f^*(t) = \inf\left\{ \lambda > 0 : D_f(\lambda) \le t \right\} \le \frac{C}{t^{1/p}}.$$

Hence $t^{1/p} f^*(t) \le C$, for all $t > 0$, then

$$\sup_{t>0} t^{1/p} f^*(t) \le C. \tag{6.5}$$

On the other hand, given $\lambda > 0$ choose ε satisfying $0 < \varepsilon < \lambda$, Theorem 4.5(b) yields $f^*(D_f(\lambda) - \varepsilon) > \lambda$ which implies that

$$\begin{aligned}\sup_{t>0} t^{\frac{1}{p}} f^*(t) &\ge (D_f(\lambda) - \varepsilon)^{\frac{1}{p}} f^*\left(D_f(\lambda) - \varepsilon\right)\\ &> (D_f(\lambda) - \varepsilon)^{\frac{1}{p}} \lambda.\end{aligned}$$

We first let $\varepsilon \to 0$ and take the supremum over all $\lambda > 0$ to obtain

$$\begin{aligned}\sup_{t>0} t^{\frac{1}{p}} f^*(t) &\ge \sup_{\lambda>0} \lambda \left(D_f(\lambda)\right)^{\frac{1}{p}}\\ &= \sup_{\lambda>0} \left\{\lambda^p D_f(\lambda)\right\}^{\frac{1}{p}}.\end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6) we obtain

$$\|f\|_{(p,\infty)} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \left\{\lambda^p D_f(\lambda)\right\}^{1/p}.$$

Case $0 < q < \infty$. In this case we use Theorem 4.5(b) and the Fubini theorem, indeed

$$\begin{aligned}\|f\|_{(p,q)}^q &= \int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \left(f^*(t)\right)^q t^{\frac{q}{p}-1}\,\mathrm{d}t\\ &= \int_0^\infty \left(\int_0^{f^*(t)} q\lambda^{q-1}\,\mathrm{d}\lambda\right) t^{\frac{q}{p}-1}\,\mathrm{d}t\\ &= \int_0^\infty \left(\int_0^\infty q\lambda^{q-1} \chi_{\{\lambda>0: f^*(t)>\lambda\}}(\lambda)\,\mathrm{d}\lambda\right) t^{\frac{q}{p}-1}\,\mathrm{d}t\\ &= \int_0^\infty \left(\int_0^\infty q\lambda^{q-1} t^{\frac{q}{p}-1} \chi_{\{t\ge 0: D_f(\lambda)>t\}}(t)\,\mathrm{d}t\right) \mathrm{d}\lambda\\ &= \int_0^\infty q\lambda^{q-1} \left(\int_0^\infty t^{\frac{q}{p}-1} \chi_{\left(0, D_f(\lambda)\right)}(t)\,\mathrm{d}t\right) \mathrm{d}\lambda\end{aligned}$$

$$= \int_0^\infty q\lambda^{q-1}\left(\int_0^{D_f(\lambda)} t^{\frac{q}{p}-1}\,\mathrm{d}t\right)\mathrm{d}\lambda$$

$$= p\int_0^\infty \lambda^{q-1}\left(D_f(\lambda)\right)^{\frac{q}{p}}\mathrm{d}\lambda$$

$$= p\int_0^\infty \left[\lambda\left(D_f(\lambda)\right)^{\frac{1}{p}}\right]^q \frac{\mathrm{d}\lambda}{\lambda}.$$

Finally

$$\|f\|_{(p,q)} = p^{1/q}\left(\int_0^\infty \left[\lambda\left(D_f(\lambda)\right)^{\frac{1}{p}}\right]^q \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}$$

□

The following result together with Theorem 6.12 (Hardy) will help us to prove the triangle inequality for $\|\cdot\|_{(p,q)}$.

Theorem 6.7. *Suppose that $(X,\mathscr{A},\mu)$ is a nonatomic $\sigma-$finite measure space, that p and q are two numbers satisfying $1 \le q \le p < \infty$. In addition, let q' be the conjugate exponent to q and let $\mathfrak{F}_0$ be the set of nonnegative-value nonincreasing function on $[0,\infty)$. If h is any function in $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$, then*

$$\|h\|_{(p,q)} = \sup\left\{\int_0^\infty h^*(t)t^{\frac{1}{p}-\frac{1}{q}}k(t)\,\mathrm{d}t : k \in \mathfrak{F}_0 \quad \textit{and} \quad \|k\|_{L_{q'(0,\infty)}} = 1\right\}.$$

Proof. By Hölder inequality we have

$$\int_0^\infty h^*(t)t^{\frac{1}{p}-\frac{1}{q}}k(t)\,\mathrm{d}t \le \left(\int_0^\infty \left(t^{1/p}h^*(t)\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q}\left(\int_0^\infty \left(k(t)\right)^{q'}\mathrm{d}t\right)^{1/q'},$$

then

$$\sup\left\{\int_0^\infty h^*(t)t^{\frac{1}{p}-\frac{1}{q}}k(t)\,\mathrm{d}t : k \in \mathfrak{F}_0 \quad \text{and} \quad \|k\|_{L_{q'(0,\infty)}} = 1\right\} \le \|h\|_{(p,q)}. \tag{6.7}$$

Now, let $k(t) = c\left[t^{\frac{1}{p}-\frac{1}{q}} h^*(t)\right]^{q-1}$, clearly k is a nonnegative valued non-decreasing function on $[0,\infty)$ and

$$\left(\int_0^\infty \left(k(t)\right)^{q'} \mathrm{d}t\right)^{1/q'} = c\|h\|_{(p,q)}^{q/q'},$$

taking $c = \|h\|_{(p,q)}^{1-q}$, we obtain $\|k\|_{L_{q'(0,\infty)}} = 1$.
On the other hand, since $k(t) = c\left[t^{\frac{1}{p}-\frac{1}{q}} h^*(t)\right]^{q-1}$ we have

$$t^{\frac{1}{p}-\frac{1}{q}} h^*(t)k(t) = c\left[t^{\frac{1}{p}-\frac{1}{q}} h^*(t)\right]^{q-1},$$

then

$$\int_0^\infty h^*(t)t^{\frac{1}{p}-\frac{1}{q}} k(t)\,\mathrm{d}t = \|h\|_{(p,q)},$$

therefore

$$\|h\|_{(p,q)} = \sup\left\{\int_0^\infty h^*(t)t^{\frac{1}{p}-\frac{1}{q}} k(t)\,\mathrm{d}t : k \in \mathfrak{F}_0 \quad \text{and} \quad \|k\|_{L_{q'(0,\infty)}} = 1\right\} \tag{6.8}$$

Finally by (6.6) and (6.7) the result follows. □

Corollary 6.8 *Let $(X, \mathscr{A}, \mu)$ be a $\sigma-$finite measure space. Suppose $1 \le q \le p < \infty$ with $q' = \frac{q}{q-1}$ and that f and g are two functions in $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$. Then*

$$\|f+g\|_{(p,q)} \le \|f\|_{(p,q)} + \|g\|_{(p,q)}. \tag{6.9}$$

Proof. The hypothesis $q \le p$ implies that $t^{\frac{1}{p}-\frac{1}{q}}$ is decreasing, hence $t^{\frac{1}{p}-\frac{1}{q}} k(t)$ is decreasing. We may apply Theorem 4.19 and Hölder's inequality to obtain

$$\begin{aligned}\int_0^\infty t^{\frac{1}{p}-\frac{1}{q}} (f+g)^*(t)k(t)\,\mathrm{d}t &\le \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}} (f)^*(t)k(t)\,\mathrm{d}t + \int_0^\infty t^{\frac{1}{p}-\frac{1}{q}} (g)^*(t)k(t)\,\mathrm{d}t \\ &\le \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \|k\|_{L_{q'(0,\infty)}} \\ &\quad + \left(\int_0^\infty \left(t^{1/p} g^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \|k\|_{L_{q'(0,\infty)}}\end{aligned}$$

$$= \|f\|_{(p,q)} + \|g\|_{(p,q)}.$$

Since $\|k\|_{L_{q'(0,\infty)}} = 1$, this together with Theorem 6.6 establishes (6.9). □

The following result is a Hölder's type inequality.

Theorem 6.9. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let p and q be two extended real numbers in $[1, \infty]$ and let p' and q' be their conjugate exponents. If $f \in \mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$ and $g \in \mathbf{L}_{(p',q')}(X, \mathscr{A}, \mu)$, then*

$$\|fg\|_1 \le \|f\|_{(p,q)} \|g\|_{(p',q')}.$$

Proof. By Theorem 4.15 and Hölder's inequality we have

$$\begin{aligned}
\int_X |fg| \, d\mu &\le \int_0^\infty f^*(t) g^*(t) \, dt \\
&= \int_0^\infty \left(t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right) \left(t^{\frac{1}{p'} - \frac{1}{q'}} g^*(t) \right) dt \\
&\le \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} \left(\int_0^\infty \left(t^{1/p'} g^*(t) \right)^{q'} \frac{dt}{t} \right)^{1/q'} \\
&= \|f\|_{(p,q)} \|g\|_{(p',q')}.
\end{aligned}$$

□

6.2 Normability

One can associate the so-called Lorentz spaces with the pre-Lorentz spaces. In order to do that let us define a relation $\sim$ on $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$ as follows:

$$f \sim g \quad \text{if and only if} \quad f = g \quad \mu - a.e.$$

It is not hard to prove that $\sim$ is an equivalence relation. Let us write

$$[f] = \left\{ g \in \mathfrak{F}(X, \mathscr{A}) : f = g \quad \mu - a.e \right\}$$

for each function $f \in \mathfrak{F}(X, \mathscr{A})$ and

$$L_{(p,q)}(X, \mathscr{A}, \mu) = \left\{ [f] : f \in \mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu) \right\}.$$

It was stated in Theorem 6.4 (c) and Corollary 6.8 that $\|\cdot\|_{(p,q)}$ is norm on $\mathbf{L}_{(p,q)}(X,\mathscr{A},\mu)$ provided that $1 \le q \le p \le \infty$ but, in general it is not a norm for the remaining case. The reason for this is that the nonincreasing rearrangement operator is not sub-additive in the sense that, in general, the inequality $(f+g)^* \le f^* + g^*$ does not hold for any two measurable functions f and g. This means that one should not expect to be able to define a norm in terms of the nonincreasing rearrangement operator since the triangle inequality is not likely to be satisfied. The aim of the present section is to define an operator that is related to the nonincreasing rearrangement operator and that is sub-additive and which for $p > 1$, defines a norm on $L_{(p,q)}(X,\mathscr{A},\mu)$ equivalent to $\|\cdot\|_{(p,q)}$.

One can use the maximal function (see Definition 4.24) of f in place of f^* to define another two parameter family of functions on $\mathfrak{F}(X,\mathscr{A})$.

Definition 6.10. For $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz spaces $L_{(p,q)}(X,\mathscr{A},\mu)$ is defined as

$$L_{(p,q)}(X,\mathscr{A},\mu) = \left\{ f \in \mathfrak{F}(X,\mathscr{A}) : \|f\|_{pq} < \infty \right\}$$

where $\|\cdot\|_{pq}$ is defined by

$$\left\| f \mid L_{pq} \right\| = \|f\|_{pq} = \begin{cases} \left(\int\limits_0^\infty \left(t^{1/p} f^{**}(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & 1 \le p < \infty,\ 1 \le q < \infty \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 \le p < \infty,\ q = \infty. \end{cases} \tag{6.10}$$

⊘

Remark 6.11. If $(X,\mathscr{A},\mu)$ is a nonatomic measure space, it follows from Remark 4.18 that

$$\int_0^t (f+g)^*(s)\,\mathrm{d}s \le \int_0^t f^*(s)\,\mathrm{d}s + \int_0^t g^*(s)\,\mathrm{d}s \tag{6.11}$$

where $t = \mu(E)$ for $E \in \mathscr{A}$. Now, using Definition 4.24 and (6.11) we have

$$\begin{aligned} (f+g)^{**}(t) &= \frac{1}{t} \int_0^t (f+g)^*(s)\,\mathrm{d}s \\ &\le \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s + \frac{1}{t} \int_0^t g^*(s)\,\mathrm{d}s \\ &= f^{**}(t) + g^{**}(t), \end{aligned}$$

that is

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t) \tag{6.12}$$

for any two function $f, g \in \mathfrak{F}(X, \mathscr{A})$.

The sub-additivity of the maximal operator (6.12) means that if the set

$$\left\{ f \in \mathfrak{F}(X, \mathscr{A}) : \|f\|_{pq} < \infty \right\}$$

is a vector space, then $\|\cdot\|_{pq}$ is a norm on this spaces.

Now this spaces turns out to be identical to $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$ provided that $p > 1$ (see Theorem 6.6) and these spaces are therefore normed space. On the other hand, just as f^{**} tends to be a more complicated function than f^*, the quantity $\|f\|_{pq}$ tends to be more difficult to work with than $\|f\|_{(p,q)}$. For example, if A is a set in $\mathscr{A}$ for which $0 < \mu(A) < \infty$, then

$$\|\chi_A\|_{pq} = \begin{cases} \Big[\mu(A)\Big]^{1/p} \left(\frac{p^2}{q(p-1)}\right)^{1/q}, & 1 \le q < \infty,\ 1 \le p < \infty \\ \infty, & p = 1,\ q < \infty. \\ \infty, & p = \infty,\ q < \infty \\ \Big(\mu(A)\Big)^{1/p}, & 1 \le p < \infty,\ q = \infty \\ 1, & p = q = \infty. \end{cases} \tag{6.13}$$

For fixed p and q in $[1, \infty]$ there are several inequalities relating the functions $\|\cdot\|_{(p,q)}$ and $\|.\|_{pq}$ and the key to deducing them is the following integral inequality due to Hardy.

The following integral inequality comes in different shapes; we will use the one given below since it is the key to deduce inequalities related to the functionals $\|\cdot\|_{(p,q)}$ and $\|.\|_{pq}$ for fixed p and q in $[1, \infty]$.

Theorem 6.12 (G. H. Hardy). *If f is a nonnegative "valued-measurable function on $[0, \infty)$ and if q and r are two numbers satisfying $1 \le q < \infty$ and $0 < r < \infty$, then*

(a)

$$\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s \right)^q t^{-r-1}\,\mathrm{d}t \le \left(\frac{q}{r}\right)^q \int_0^\infty \Big(sf(s)\Big)^q s^{-r-1}\,\mathrm{d}s.$$

(b)

$$\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s \right)^q t^{r-1}\,\mathrm{d}t \le \left(\frac{q}{r}\right)^q \int_0^\infty \Big(sf(s)\Big)^q s^{r-1}\,\mathrm{d}s.$$

Proof. (a) If $q = 1$, then by Fubini's theorem we have

$$\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s \right) t^{-r-1}\,\mathrm{d}t = \int_0^\infty \int_s^\infty t^{-r-1} f(s)\,\mathrm{d}t\,\mathrm{d}s$$

$$= \frac{1}{r}\int_0^\infty \left[sf(s)\right] s^{-r-1}\,\mathrm{d}s$$

$$= \left(\frac{q}{r}\right)^q \int_0^\infty (sf(s))^q s^{-r-1}\,\mathrm{d}s.$$

Now, suppose that $q > 1$ and let p be the conjugate exponent of q. Then by Hölder's inequality with respect to the measure $s^{\frac{r}{q}-1}\,\mathrm{d}s$ we have

$$\left(\int_0^t f(s)\,\mathrm{d}s\right)^q = \left(\int_0^t f(s)s^{1-\frac{r}{q}}s^{\frac{r}{q}-1}\,\mathrm{d}s\right)^q$$

$$\leq \left(\int_0^t \left[f(s)\right]^q s^{q-r}s^{\frac{r}{q}-1}\,\mathrm{d}s\right)\left(\int_0^t s^{\frac{r}{q}-1}\,\mathrm{d}s\right)^{q/p}$$

$$= \left(\frac{q}{r}t^{\frac{r}{q}}\right)^{q/p}\int_0^t \left[f(s)\right]^q s^{q-r}s^{\frac{r}{q}-1}\,\mathrm{d}s$$

$$= \left(\frac{q}{r}\right)^{q/p} t^{\frac{r}{p}}\int_0^t \left[f(s)\right]^q s^{q-r}s^{\frac{r}{q}-1}\,\mathrm{d}s.$$

By integrating both sides from zero to infinity and using Fubini's theorem we have

$$\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s\right)^q t^{-r-1}\,\mathrm{d}t \leq \left(\frac{q}{r}\right)^{q/p}\int_0^\infty t^{-r-1+r(1-\frac{1}{q})}\int_0^t \left[f(s)\right]^q s^{q-r}s^{\frac{r}{q}-1}\,\mathrm{d}s\,\mathrm{d}t$$

$$= \left(\frac{q}{r}\right)^{q/p}\int_0^\infty \left[f(s)\right]^q s^{q-r}s^{\frac{r}{q}-1}\left(\int_s^\infty t^{-1-\frac{r}{q}}\,\mathrm{d}t\right)\mathrm{d}s$$

$$= \left(\frac{q}{r}\right)^{\frac{q}{p}+1}\int_0^\infty \left[sf(s)\right]^q s^{-r+\frac{r}{q}-1}s^{-r/q}\,\mathrm{d}s$$

$$= \left(\frac{q}{r}\right)^q \int_0^\infty \left[sf(s)\right]^q s^{-r-1}\,\mathrm{d}s.$$

Hence

$$\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s\right)^q t^{-r-1}\,\mathrm{d}t \le \left(\frac{q}{r}\right)^q \int_0^\infty \Big(sf(s)\Big)^q s^{-r-1}\,\mathrm{d}s.$$

(b) If $q=1$ one more time by Fubini's theorem we have

$$\begin{aligned}\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q t^{r-1}\,\mathrm{d}t &= \int_0^\infty \int_0^s t^{r-1} f(s)\,\mathrm{d}t\,\mathrm{d}s\\ &= \frac{1}{r}\int_0^\infty \Big(sf(s)\Big) s^{r-1}\,\mathrm{d}s\\ &= \left(\frac{q}{r}\right)^q \int_0^\infty \Big(sf(s)\Big)^q s^{r-1}\,\mathrm{d}s.\end{aligned}$$

Next, let $q>1$ and p be its conjugate exponent. Then from Hölder's inequality with respect to the measure $s^{-\frac{r}{q}-1}\,\mathrm{d}s$ we have

$$\begin{aligned}\left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q &= \left(\int_t^\infty f(s)s^{\frac{r}{q}+1}s^{-\frac{r}{q}-1}\,\mathrm{d}s\right)\\ &\le \left(\int_t^\infty \Big[f(s)\Big]^q s^{r+q}s^{-\frac{r}{q}-1}\,\mathrm{d}s\right)\left(\int_t^\infty s^{-\frac{r}{q}-1}\,\mathrm{d}s\right)^{q/p}\\ &= \left(\frac{q}{r}\right)^{\frac{q}{p}} t^{-\frac{r}{p}}\left(\int_t^\infty \Big[sf(s)\Big]^q s^{r-\frac{r}{q}-1}\,\mathrm{d}s\right).\end{aligned}$$

By integrating both sides from zero to infinity and using Fubini's theorem we obtain

$$\begin{aligned}\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q t^{r-1}\,\mathrm{d}t &\le \left(\frac{q}{r}\right)^{q/p}\int_0^\infty t^{-\frac{r}{p}+r-1}\left(\int_t^\infty \Big[sf(s)\Big]^q s^{r-\frac{r}{q}-1}\,\mathrm{d}s\right)\mathrm{d}t\\ &= \left(\frac{q}{r}\right)^{q/p}\int_0^\infty t^{\frac{r}{q}-1}\left(\int_t^\infty \Big[sf(s)\Big]^q s^{r-\frac{r}{q}-1}\,\mathrm{d}s\right)\mathrm{d}t\\ &= \left(\frac{q}{r}\right)^{q/p}\int_0^\infty \Big[sf(s)\Big]^q s^{r-\frac{r}{q}-1}\left(\int_0^s t^{\frac{r}{q}-1}\,\mathrm{d}t\right)\mathrm{d}s\end{aligned}$$

$$= \left(\frac{q}{r}\right)^{\frac{q}{p}+1} \int_0^\infty \left[sf(s)\right]^q s^{r-\frac{r}{q}-1} s^{r/q}\,\mathrm{d}s$$

$$= \left(\frac{q}{r}\right)^q \int_0^\infty \left[sf(s)\right]^q s^{r-1}\,\mathrm{d}s.$$

Hence

$$\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q t^{r-1}\,\mathrm{d}t \le \left(\frac{q}{r}\right)^q \int_0^\infty \left(sf(s)\right)^q s^{r-1}\,\mathrm{d}s.$$

□

We now prove that the norm $\|\cdot\|_{pp}$ is equivalent with the Lebesgue norm $\|\cdot\|_p$, i.e., the diagonal case of Lorentz spaces coincides with the Lebesgue space.

Theorem 6.13 *If* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\|f\|_p \le \|f\|_{pp} \le \frac{p}{p-1}\|f\|_p.$$

Proof. Since $f^* \le f^{**}$ by Remark 4.14 (ii) we have

$$\|f\|_p^p = \int_0^\infty \left[f^*(t)\right]^p \mathrm{d}t \tag{6.14}$$

$$= \int_0^\infty \left[t^{1/p} f^*(t)\right] \frac{\mathrm{d}t}{t}$$

$$\le \int_0^\infty \left[t^{1/p} f^{**}(t)\right]^p \mathrm{d}t$$

$$= \|f\|_{pp}. \tag{6.15}$$

On the other hand, the second inequality follows immediately from the definition of f^{**} and the Hardy's inequality with $r = p-1$, that is,

$$\left(\int_0^\infty \left(t^{\frac{1}{p}-1} \int_0^t f(s)\,\mathrm{d}s\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^\infty \left(f(s)\right)^p \mathrm{d}s\right)^{1/p}$$

thus

$$\|f\|_{pq} \le \frac{p}{p-1}\|f\|_p. \tag{6.16}$$

By (6.16) and (6.15) we obtain

$$\|f\|_p \le \|f\|_{pp} \le \frac{p}{p-1}\|f\|_p,$$

therefore the two norms are equivalent. □

We also have that the norms $\|\cdot\|_{(p,q)}$ and $\|\cdot\|_{pq}$ are equivalent.

Theorem 6.14 *If $(X,\mathscr{A},\mu)$ is a measure space and if p and q are two extended real numbers satisfying $1<p\le\infty$ and $1\le q\le\infty$, then*

$$\|f\|_{(p,q)} \le \|f\|_{pq} \le \frac{p}{p-1}\|f\|_{(p,q)}$$

for any $f\in\mathfrak{F}(X,\mathscr{A})$, where $\frac{p}{p-1}$ is to be interpreted as 1 if $p=\infty$.

Proof. Since $f^*\le f^{**}$, then

$$\begin{aligned}\|f\|_{(p,q)}^q &= \int_0^\infty \left(t^{1/p}f^*(t)\right)^q \frac{\mathrm{d}t}{t}\\ &\le \int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}\\ &= \|f\|_{pq}^q.\end{aligned}$$

Next, if $q<\infty$ and $p=\infty$, then, as was pointed out following Definition 6.2, either $f=0$ a.e. or else $\|f\|_{(p,q)}=\infty$, and the second inequality is obvious in either case. If both q and p are finite, by Hardy's inequality Theorem (6.12) we have

$$\begin{aligned}\|f\|_{pq}^q &= \int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \left(t^{\frac{1}{p}-1}\int_0^t f^*(s)\,\mathrm{d}s\right)^q \frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^q t^{\frac{q}{p}-q-1}\,\mathrm{d}t\\ &\le \left(\frac{p}{p-1}\right)^q \|f\|_{(p,q)}^q.\end{aligned}$$

And, finally, if $q=\infty$, then

$$t^{1/p}f^{**}(t) = t^{\frac{1}{p}-1}\int_0^t f^*(s)\,\mathrm{d}s$$

$$= t^{\frac{1}{p}-1} \int_0^t s^{1/p} f^*(s) s^{-1/p} \, \mathrm{d}s$$

$$\leq t^{\frac{1}{p}-1} \|f\|_{(p,q)} \int_0^t s^{-1/p} \, \mathrm{d}s$$

$$= \frac{p}{p-1} \|f\|_{(p,q)},$$

for all $t > 0$ regardless of whether p is finite or infinite. □

The significance of this theorem is, of course, that for $1 < p \leq \infty$ and $1 \leq q \leq \infty$ the space $\mathbf{L}_{(p,q)}(X, \mathscr{A}, \mu)$ could equally well be defined to consist of those functions $f \in \mathfrak{F}(X, \mathscr{A})$ for which $\|f\|_{(p,q)} < \infty$ or for which $\|f\|_{pq} < \infty$. Note that for such p and q, the Theorem 6.4(c) and (6.12) imply that the function $\|\cdot\|_{pq}$ determines a norm on $L_{(p,q)}(X, \mathscr{A}, \mu)$.

The following two lemmas give embedding information regarding Lorentz spaces, namely they provide some comparison between the spaces $L_{(p,q)}(X, \mathscr{A}, \mu)$ and $L_{(p,r)}(X, \mathscr{A}, \mu)$.

Lemma 6.15. *Let $f \in L_{(p,q)}$. Then*

$$f^{**}(t) \leq \left(\frac{q}{p}\right)^{1/q} \frac{\|f\|_{pq}}{t^{1/p}}. \tag{6.17}$$

Proof. Note that

$$\|f\|_{pq}^q = \int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}$$

$$\geq \int_0^t \left(f^{**}(s)\right)^q s^{\frac{q}{p}-1} \, \mathrm{d}s$$

$$\geq \left[f^{**}(t)\right]^q \int_0^t s^{\frac{q}{p}-1} \, \mathrm{d}s$$

$$= \frac{p}{q} \left[f^{**}(t)\right]^q t^{q/p},$$

from which (6.17) follows. □

Corollary 6.16 *Let $f \in L_{(p,q)}$. Then $\|f\|_{(p,\infty)} \leq \|f\|_{p\infty} \leq \left(\frac{q}{p}\right)^{1/q} \|f\|_{pq}$.*

We now show an embedding result in the framework of Lorentz spaces, namely $L_{(p,q)} \hookrightarrow L_{(p,r)}$ whenever $q < r$.

Lemma 6.17 (Calderón). *If* $1 < p < \infty$ *and* $1 \leq q < r \leq \infty$, *then*

$$\|f\|_{pr} \leq \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{r}} \|f\|_{pq}.$$

Proof. Using Lemma 6.15 we have

$$\begin{aligned}
\|f\|_{pr}^r &= \int_0^\infty \left(f^{**}(t)\right)^r t^{\frac{r}{p}-1}\,\mathrm{d}t \\
&= \int_0^\infty \left(f^{**}(t)\right)^q \left(f^{**}(t)\right)^{r-q} t^{\frac{r}{p}-1}\,\mathrm{d}t \\
&\leq \int_0^\infty \left(f^{**}(t)\right)^q \left[\left(\frac{q}{p}\right)^{1/q} \frac{\|f\|_{pq}}{t^{1/p}}\right]^{r-q} t^{\frac{r}{p}-1}\,\mathrm{d}t \\
&= \left(\frac{q}{p}\right)^{\frac{r}{q}-1} \left(\|f\|_{pq}\right)^{r-q} (\|f\|_{pq})^q \\
&= \left(\frac{q}{p}\right)^{\frac{1}{q}-1} \left(\|f\|_{pq}\right)^r,
\end{aligned}$$

which ends the proof. □

6.3 Completeness

We are now ready to prove completeness which follows, as in the ordinary L_p case, from the Riesz theorem.

Theorem 6.18 (Completeness). *The normed space* $(L_{(p,q)}(X,\mathscr{A},\mu), \|\cdot\|_{pq})$ *is complete (Banach space) for all* $0 < p < \infty$ *and* $0 < q \leq \infty$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $L_{(p,q)}(X,\mathscr{A},\mu)$. Then

$$\|f_m - f_n\|_{pq} \to 0 \quad \text{as} \quad n,m \to \infty,$$

and by Corollary 6.16 we have

$$\|f_m - f_n\|_{(p,\infty)} \leq \left(\frac{q}{p}\right)^{1/q} \|f_m - f_n\|_{pq} \to 0$$

as $n,m \to \infty$.

Thus

$$\sup_{t>0} t^{1/p}(f_m - f_n)^*(t) = \|f_m - f_n\|_{(p,\infty)} \to 0$$

as $n,m \to \infty$.

By Theorem 6.6 (case $q = \infty$) we have

$$\sup_{\lambda>0}\left\{\lambda^p D_{f_m-f_n}(\lambda)\right\}^{1/p} = \sup_{t>0} t^{1/p}\Big(f_m - f_n\Big)^*(t) \to 0$$

as $m,n \to \infty$, then

$$\sup_{\lambda>0}\left\{\lambda^p \mu\Big(\{x \in X : |f_m(x) - f_n(x)| > \lambda\}\Big)\right\}^{1/p} = \sup_{\lambda>0}\left\{\lambda^p D_{f_m-f_n}(\lambda)\right\}^{1/p} \to 0$$

as $m,n \to \infty$, this implies that

$$\mu\Bigg(\{x \in X : |f_m(x) - f_n(x)| > \lambda\}\Bigg) \to 0$$

as $m,n \to \infty$ for any $\lambda > 0$.

We showed that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the measure μ. We can therefore apply F. Riesz's theorem and conclude that there exists an $\mathscr{A}$-measurable function f such that f_n converges to f in the measure μ. This implies again by a theorem of F. Riesz that there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ which converges to f μ-a.e. on X.

Let $\varepsilon > 0$ be arbitrary. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy there exists an $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_{n_0}\|_{pq} < \varepsilon \qquad (n > n_0)$$

and $f_{n_k} - f_{n_0}$ converge to $f - f_{n_0}$ μ-a.e. on X.

It follows now by Theorem 4.5 (g) that

$$(f - f_{n_0})^*(t) \le \liminf_{k\to\infty}(f_{n_k} - f_{n_0})^*(t)$$

for all $t > 0$. Using the Fatou Lemma we have

$$(f - f_{n_0})^{**}(t) \le \liminf(f_{n_k} - f_{n_0})^{**}(t)$$

for all $t > 0$. One more time by Fatou's Lemma we have

$$\|f - f_{n_0}\|_{pq} = \left(\int_0^\infty \Big(t^{1/p}(f - f_{n_0})^{**}(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

$$\leq \left(\int_0^\infty \left(t^{1/p} \liminf_{k\to\infty} (f_{n_k} - f_{n_0})^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\leq \liminf_{k\to\infty} \left(\int_0^\infty \left(t^{1/p} (f_{n_k} - f_{n_0})^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q}$$

$$= \liminf_{k\to\infty} \|f_{n_k} - f_{n_0}\|_{pq} < \varepsilon$$

whenever $n_k > n_0$. Since $f = (f - f_{n_0}) + f_{n_0} \in L_{(p,q)}(X, \mathscr{A}, \mu)$ and this proves that $L_{(p,q)}(X, \mathscr{A}, \mu)$ is complete for all $0 < p < \infty$ and $0 < q \leq \infty$. □

6.4 Separability

To show that $L_{(p,q)}(X, \mathscr{A}, \mu)$ is separable we need to show that the set of all simple functions is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$. This desirable property means that any function in $L_{(p,q)}(X, \mathscr{A}, \mu)$ can be approximated by a simple function in the norm of $L_{(p,q)}(X, \mathscr{A}, \mu)$.

Theorem 6.19. *The set of all simple functions S is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$ for $0 < p < \infty$ and $0 < q < \infty$.*

Proof. Let $q < \infty$ and $f \in L_{(p,q)}(X, \mathscr{A}, \mu)$ be arbitrary. We can without loss of generality assume that f is positive, and then there exists a sequence of simple integrable functions such that $0 \leq s_n \leq f$ for all $n \in \mathbb{N}$ and $s_n \to f$ as $n \to \infty$. Hence, by Theorem 4.11 we have

$$\begin{aligned}(f - s_n)^*(t) &\leq f^*(t/2) + s_n^*(t/2)\\ &\leq 2f^*(t/2),\end{aligned}$$

since $s_n^*(t/2) \leq f^*(t/2)$ for all $n \in \mathbb{N}$, and thus, if we apply Lebesgue's dominated convergence theorem, and Theorem 6.14 we have

$$\lim_{n\to\infty} \|f - s_n\|_{pq} \leq \frac{p}{p-1} \lim_{n\to\infty} \|f - s_n\|_{(p,q)} = 0.$$

Since f was arbitrary this shows that $\overline{S} = L_{(p,q)}(X, \mathscr{A}, \mu)$ that is, S is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$. □

The separability of $L_{(p,q)}(X, \mathscr{A}, \mu)$ will now follow by showing that the set S from the previous theorem is countable and this is the case if and only if the measure is separable. Before we give the proof of this fact, we state the following definition.

Definition 6.20. A measure μ is separable if there exists a countable family $\mathfrak{H}$ of sets from $\mathscr{A}$ of finite measure such that for any $\varepsilon > 0$ and any set $A \in \mathscr{A}$ of finite measure we can find a set $B \in \mathfrak{H}$ with $\mu(A \triangle B) < \varepsilon$. ⊘

Theorem 6.21 (Separability). *The Lorentz $L_{(p,q)}(X, \mathscr{A}, \mu)$ space is separable for $0 < p < \infty$ and $0 < q < \infty$ if and only if the measure μ is separable.*

Proof. Assume first that μ is a separable measure and let A be any measurable set with finite measure and $\varepsilon > 0$ be arbitrary. Then there exists a countable family $\mathfrak{H}$ of subsets of X of finite measure and a set $B \in \mathfrak{H}$ such that

$$\mu\Big(A \Delta B\Big) = \mu\Big[(A \backslash B) \cup (B \backslash A)\Big] < \varepsilon.$$

It follows that

$$\begin{aligned}
\|\chi_A - \chi_B\|_{pq} &= \left(\int_0^\infty \Big(t^{1/p} (\chi_A - \chi_B)^{**}(t) \Big)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \left(\int_0^\infty t^{\frac{q}{p}-1} \chi^{**}_{A\Delta B}(t)\, \mathrm{d}t \right)^{1/q} \\
&= \left(\frac{p^2}{q(p-1)} \right)^{1/q} \Big(\mu\Big(A \Delta B\Big) \Big)^{1/p} \\
&< \left(\frac{p^2}{q(p-1)} \right)^{1/q} \varepsilon^{1/p}.
\end{aligned}$$

Hence, for any characteristic function of any $\mathscr{A}$-measurable set A of finite measure we can always find another characteristic function of a set $B \in \mathfrak{H}$ such that the norm of the difference between the two functions is as small as we wish.

Let s be a simple function of the following form

$$s = \sum_{j=1}^{m} \alpha_j \chi_{A_j}$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_m > 0$ and

$$A_j = \{x \in X : s(x) = \alpha_j\}$$

for all $j = 1, 2, \ldots, m$.
We can then define a new simple function

$$\widetilde{s} = \sum_{j=1}^{m} \alpha_j \chi_{B_j}$$

where $B_j \in \mathfrak{H}$ is chosen such that

$$\mu(A_j \Delta B_j) < \left(\frac{p^2}{q(p-1)} \right)^{1/q} \varepsilon^{1/p}$$

for all $j = 1, 2, \ldots, m$. It follows that

$$\begin{aligned} \|s - \widetilde{s}\|_{pq} &= \Big\| \sum_{j=1}^{m} \alpha_j \chi_{A_j \Delta B_j} \Big\|_{pq} \\ &< m \left(\frac{p^2}{q(p-1)} \right)^{1/q} \varepsilon^{1/p}. \end{aligned}$$

That is, for any simple function s defined on $\mathscr{A}$-measurable sets $A_1, A_2, \ldots, A_m$, we can always find another simple functions $\widetilde{s}$, defined on sets $B_1, B_2, \ldots, B_m$ where $B_j \in \mathfrak{H}$; $j = 1, 2, \ldots, m$. Since the set of all simple functions is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$ it follows that the set

$$\widetilde{S} = \left\{ s = \sum_{j=1}^{m} \alpha_j \chi_{B_j} : B_j \in \mathfrak{H},\ \alpha_j \in \mathbb{R},\ m = 1, 2, 3, \ldots \right\}$$

is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$.

Moreover, since the countable set

$$\widetilde{S}_{\mathbb{Q}} = \left\{ \widetilde{s} = \sum_{j=1}^{m} \alpha_j \chi_{B_j} : B_j \in \mathfrak{H},\ \alpha_j \in \mathbb{Q},\ m = 1, 2, 3, \ldots \right\}$$

is dense in $\widetilde{S}$, it follows that $\widetilde{S}_{\mathbb{Q}}$ is dense in $L_{(p,q)}(X, \mathscr{A}, \mu)$. Hence $L_{(p,q)}(X, \mathscr{A}, \mu)$ is separable.

Now, assume that μ is not separable. Then there exists an $\varepsilon > 0$ and uncountable family of sets $\mathfrak{H}$ such that for A, B in $\mathfrak{H}$

$$\mu\Big(A \Delta B\Big) \geq \varepsilon.$$

Thus the set

$$H = \{ f = \chi_A : A \in \mathfrak{H} \}$$

is uncountable and for $f, g \in H$ we have

$$
\begin{aligned}
\|f-g\|_{pq} &= \|\chi_A - \chi_B\|_{pq} \\
&= \|\chi_{A\Delta B}\|_{pq} \\
&= \left(\frac{p^2}{q(p-1)}\right)^{1/q} \left(\mu\left(A\Delta B\right)\right)^{1/p} \\
&\geq \left(\frac{p^2}{q(p-1)}\right)^{1/q} \varepsilon^{1/p}.
\end{aligned}
$$

where $A \neq B$. Hence, we have an uncountable set $H \subset L_{(p,q)}(X, \mathscr{A}, \mu)$ such that for two functions $f, g \in H$, $\|f-g\|_{pq}$ is not as small as we wish, that is $L_{(p,q)}(X, \mathscr{A}, \mu)$ is not separable. □

6.5 Duality

We will now take a closer look on the space of all bounded linear functionals on $L_{(p,q)}(X, \mathscr{A}, \mu)$ which we will denote by

$$\left[L_{(p,q)}(X, \mathscr{A}, \mu)\right]^*.$$

In the next theorem we collect the duality of Lorentz spaces depending on the different parameters.

Theorem 6.22. *Suppose that $(X, \mathscr{A}, \mu)$ is a nonatomic σ-finite measure space. Then:*

(a) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* = \{0\}$ *when* $0 < p < 1$, $0 < q \leq \infty$,

(b) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* = L_\infty(X, \mathscr{A}, \mu)$ *when* $p = 1$, $0 < q < 1$,

(c) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* = \{0\}$ *when* $p = 1$, $1 < q < \infty$,

(d) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* \neq \{0\}$ *when* $p = 1$, $q = \infty$,

(e) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* = L_{(p',\infty)}(X, \mathscr{A}, \mu)$ *when* $1 < p < \infty$, $0 < q \leq 1$,

(f) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* = L_{(p',q')}(X, \mathscr{A}, \mu)$ *when* $1 < p < \infty$, $1 < q < \infty$,

(g) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* \neq \{0\}$ *when* $1 < p < \infty$, $q = \infty$,

(h) $\left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^* \neq \{0\}$ *when* $p = q = \infty$.

Proof. Since X is σ-finite, we have that $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is an increasing sequence of sets with $\mu\left(X_n\right) < \infty$ for all $n \in \mathbb{N}$. Given $T \in \left(L_{(p,q)}(X, \mathscr{A}, \mu)\right)^*$ where $0 < p < \infty$ and $0 < q < \infty$. Let us define $\sigma(E) = T\left(\chi_E\right)$ for all $E \in \mathscr{A}$.

Next, we like to show that:

(i) σ defines a signed measure on X, and

(ii)
$$|\sigma(E)| \leq \left(\frac{p}{q}\right)^{1/q} \left(\mu(E)\right)^{1/p} \|T\| \quad \text{when} \quad q < \infty,$$
and
$$|\sigma(E)| \leq \|T\| \left(\mu(E)\right)^{1/p} \quad \text{for} \quad q = \infty.$$

(i) Note that
$$\sigma(\emptyset) = T\left(\chi_\emptyset\right) = T(0) = 0$$
Since T is a linear functional. On the other hand, let $\{E_n\}_{n\in\mathbb{N}} \in \mathscr{A}$ such that $E_n \cap E_m = \emptyset$ if $n \neq m$. Then
$$\sigma\left(\bigcup_{n=1}^{\infty} E_n\right) = T\left(\chi_{\bigcup_{n=1}^{\infty} E_n}\right) = T\left(\sum_{n=1}^{\infty} \chi_{E_n}\right)$$
$$\sum_{n=1}^{\infty} T\left(\chi_{E_n}\right) = \sum_{n=1}^{\infty} \sigma\left(E_n\right).$$

(ii) Observe that
$$\left|\sigma(E)\right| = \left|T\left(\chi_E\right)\right| \leq \|\chi_E\|_{(p,q)} \|T\| \tag{6.18}$$
for $q < \infty$, and
$$\begin{aligned}
\|\chi_E\|_{(p,q)} &= \left(\int_0^{\infty} \left(t^{1/p} \chi_E^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= \left(\int_0^{\infty} \left(t^{1/p} \chi_{\left(0,\mu(E)\right)}(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= \left(\int_0^{\mu(E)} t^{\frac{q}{p}-1}\, \mathrm{d}t\right)^{1/q}
\end{aligned}$$

$$= \left(\frac{p}{q}\right)^{1/q} \left(\mu(E)\right)^{1/p}. \tag{6.19}$$

Putting (6.19) into (6.18) we have

$$\left|\sigma(E)\right| \le \left(\frac{p}{q}\right)^{1/q} \left(\mu(E)\right)^{1/p} \|T\|,$$

for $q < \infty$.
If $q = \infty$, then

$$\|\chi_E\|_{(p,\infty)} = \sup_{t>0} t^{1/p} \chi_E^*(t) = \sup_{t>0} t^{1/p} \chi_{\left(0,\mu(E)\right)}(t) = \left[\mu\left(E\right)\right]^{1/p},$$

thus

$$\left|\sigma(E)\right| \le \|T\| \left(\mu(E)\right)^{1/p},$$

for $q = \infty$.

Once we proved (i) and (ii) we easily see that σ is absolutely continuous with respect to the measure μ. By Radon-Nikodym theorem, there exists a complex-valued measurable function g (which satisfies $\int\limits_{X_n} |g|\,\mathrm{d}\mu < \infty$ for all n) such that

$$\sigma(E) = T(\chi_E) = \int\limits_X |g| \chi_E \,\mathrm{d}\mu, \tag{6.20}$$

Linearity implies that (6.20) holds for any simple function on X. The continuity of T and the density of the simple functions on $L_{(p,q)}(X, \mathscr{A}, \mu)$ (when $q < \infty$) give

$$T(f) = \int\limits_X |gf| \,\mathrm{d}\mu, \tag{6.21}$$

for every $f \in L_{(p,q)}(X, \mathscr{A}, \mu)$. We now examine each case (a), (e), (f) separately, for the remaining cases see Grafakos [22] and the reference therein.

(a) We first consider the case $0 < p < 1$. Let $f = \sum_n a_n \chi_{E_n}$ be a simple function on X (take f to be countably simple when $q = \infty$). If X is nonatomic, we can split each E_n as $E_n = \bigcup_{j=1}^N E_{jn}$, where E_{jn} are disjoint sets and $\mu(E_{jn}) = \frac{\mu(E_n)}{N}$.
Let $f_j = \sum_n a_n \chi_{E_{jn}}$, then

$$\|f\|_{(p,q)} = \left(\int\limits_0^\infty \left(t^{1/p} \left(\sum_n a_n \chi_{E_n} \right)^* (t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

$$
\begin{aligned}
&= \left(\sum_n a_n \int_0^\infty \left(t^{1/p} (\chi_{E_n})^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \left(\sum_n a_n \int_0^\infty \left(t^{1/p} \chi_{\left(0,\mu(E_n)\right)}(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \left(\sum_n a_n \int_0^{\mu(E_n)} t^{\frac{q}{p}-1}\,\mathrm{d}t \right)^{1/q} \\
&= \sum_n a_n \left(\frac{p}{q}\right)^{1/q} \left(\mu(E_n) \right)^{1/p} \\
&= \sum_n a_n \left(\frac{p}{q}\right)^{1/q} \left(N\mu(E_{jn}) \right)^{1/p} \\
&= N^{1/p} \sum_n a_n \left(\frac{p}{q}\right)^{1/q} \left(\mu(E_{jn}) \right)^{1/p} \\
&= N^{1/p} \left(\int_0^\infty \left(t^{1/p} \left(\sum_n a_n \chi_{E_{jn}} \right)^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= N^{1/p} \|f_j\|_{(p,q)}.
\end{aligned}
$$

Thus $\|f_j\|_{(p,q)} = N^{-1/p}\|f\|_{(p,q)}$. Now, if $T \in \left(L_{(p,q)}(X,\mathscr{A},\mu)\right)^*$, it follows that

$$
\begin{aligned}
\left|T(t)\right| &= \left| T\left(\sum_n a_n \chi_{E_n} \right) \right| \\
&= \left| T\left(\sum_n a_n \chi_{\bigcup_{j=1}^N E_{jn}} \right) \right| \\
&= \left| T\left(\sum_n a_n \sum_{j=1}^N \chi_{E_{jn}} \right) \right| \\
&= \left| T\left(\sum_n \sum_{j=1}^N a_n \chi_{E_{jn}} \right) \right| \\
&= \left| T\left(\sum_{j=1}^N \sum_n a_n \chi_{E_{jn}} \right) \right| \\
&= \left| T\left(\sum_{j=1}^N f_j \right) \right|
\end{aligned}
$$

$$\le \sum_{j=1}^{N} \left|T(f_j)\right|$$
$$\le \sum_{j=1}^{N} \|T\|\|f_j\|_{(p,q)}$$
$$= \|T\| \sum_{j=1}^{N} \|f_j\|_{(p,q)}$$
$$= N^{1-\frac{1}{p}} \|T\|\|f_j\|_{(p,q)}.$$

Let $N \to \infty$ and use that $p < 1$ to obtain that $T = 0$.

(e) We now take up case $p > 1$ and $0 < q \le 1$. By Theorem 4.15 and Theorem 6.3 we see that if $g \in L_{(p',\infty)}(X, \mathscr{A}, \mu)$, then

$$\left|\int_X fg\,\mathrm{d}\mu\right| \le \int_0^\infty t^{1/p} f^*(t) t^{1/p'} g^*(t) \frac{\mathrm{d}t}{t}$$
$$\le \|f\|_{(p,1)} \|g\|_{(p',\infty)}$$
$$\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1} \|f\|_{(p,q)} \|g\|_{(p',\infty)},$$

from which we have

$$\|T\| \le \left(\frac{q}{p}\right)^{\frac{1}{q}-1} \|g\|_{(p',q)}. \tag{6.22}$$

Conversely, suppose that $T \in \Big(L_{(p,q)}(X, \mathscr{A}, \mu)\Big)^*$ when $1 < p < \infty$ and $0 < q \le 1$. Let g satisfy (6.21). Taking $f = \overline{g}|g|^{-1}\chi_{\{|g|>\lambda\}}$ then

$$\int_X fg\,\mathrm{d}\mu = \int_{\{|g|>\lambda\}} g\overline{g}|g|^{-1}\,\mathrm{d}\mu$$
$$= \int_{\{|g|>\lambda\}} |g|^2|g|^{-1}\,\mathrm{d}\mu$$
$$= \int_{\{|g|>\lambda\}} |g|\,\mathrm{d}\mu,$$

and

$$\lambda\mu\Big(\{|g| > \lambda\}\Big) \le \left|\int_X fg\,\mathrm{d}\mu\right| \le \|T\|\|f\|_{(p,q)}. \tag{6.23}$$

Since

$$|f| = |\overline{g}||g|^{-1}\chi_{\{|g|>\lambda\}} = \chi_{\{|g|>\lambda\}},$$

hence

$$f^*(t) = |f|^*(t) = \chi_{\left(0,\mu(\{|g|>\lambda\})\right)}$$

thus

$$\begin{aligned}
\|f\|_{(p,q)} &= \left(\int_0^\infty \left(t^{1/p}f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= \left(\int_0^\infty \left(t^{1/p}\chi_{\left(0,\mu(\{|g|>\lambda\})\right)}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= \int_0^{\mu\left(\{|g|>\lambda\}\right)} t^{\frac{q}{p}-1}\,\mathrm{d}t \\
&= \left(\frac{p}{q}\right)^{1/q}\left(\mu\left(\{|g|>\lambda\}\right)\right)^{1/p}.
\end{aligned}$$

Now, back to (6.22) we have

$$\begin{aligned}
\lambda\mu\left(\{|g|>\lambda\}\right) &\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1}\|T\|\left(\mu\left(\{|g|>\lambda\}\right)\right)^{1/p} \\
\lambda\left[\mu\left(\{|g|>\lambda\}\right)\right]^{1/p'} &\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1}\|T\| \\
\lambda\left(D_g(\lambda)\right)^{1/p'} &\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1}\|T\| \\
\sup_{t>0} t^{1/p'}g^*(t) &\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1}\|T\| \\
\|g\|_{(p',\infty)} &\le \left(\frac{q}{p}\right)^{\frac{1}{q}-1}\|T\|.
\end{aligned} \tag{6.24}$$

Finally by (6.22) and (6.24) we have

$$\|g\|_{(p',\infty)} \approx \|T\|.$$

(6) Using Theorem 4.15 and Hölder's inequality, we obtain

$$\begin{aligned}\left|T(f)\right| &= \left|\int_X fg\,\mathrm{d}\mu\right| \\ &\le \int_0^\infty t^{1/p} f^*(t) t^{1/p'} g^*(t)\frac{\mathrm{d}t}{t} \\ &\le \|f\|_{(p,q)}\|g\|_{(p',q')}.\end{aligned}$$

Hence T is bounded and if we take the supremum on both sides over all functions f with norm 1 we have

$$\|T\| \le \|g\|_{(p',q')}. \tag{6.25}$$

Thus, for every g in $L_{(p',q')}$ we can find a linear bounded functional on the space $L_{(p,q)}(X,\mathscr{A},\mu)$.

Conversely, let T be in $\left[L_{(p,q)}(X,\mathscr{A},\mu)\right]^*$. Note that T is given by integration against a locally integrable function g. It remains to prove that $g \in L_{(p',q')}(X,\mathscr{A},\mu)$. Using Theorem 4.26 for all f in $L_{(p,q)}(X,\mathscr{A},\mu)$ we have

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t = \sup\left|\int_X f\widetilde{g}\mathrm{d}\mu\right| \le \|T\|\|f\|_{(p,q)},$$

where the supremum is taken over all $\mathscr{A}$-measurable functions $\widetilde{g}$ equimeasurables with g. Next, by Theorem 4.29 there exists a measurable function on X such that

$$f^*(t) = \int_{t/2}^\infty h(s)\frac{\mathrm{d}s}{s}.$$

where $h(s) = s^{\frac{q'}{p'}-1}\left(g^*(s)\right)^{q'-1}$. Then by Theorem 6.12 (b) with $r = p$

$$\begin{aligned}\|f\|_{(p,q)}^q &= \int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \left(\int_{t/2}^\infty h(s)\frac{\mathrm{d}s}{s}\right)^q t^{\frac{q}{p}-1}\,\mathrm{d}t \\ &= 2^{q/p}\int_0^\infty \left(\int_u^\infty h(s)\frac{\mathrm{d}s}{s}\right)^q u^{\frac{q}{p}-1}\,\mathrm{d}u\end{aligned}$$

$$\le \left(\frac{2^{1/p}q}{p}\right)^q \int_0^\infty u^{\frac{q}{p}-1}\Big(h(u)\Big)^q \,\mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p}\right)^q \int_0^\infty u^{\frac{q}{p}-1+\frac{qq'}{p'}-q}\Big(g^*(u)\Big)^{q(q'-1)} \,\mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p}\right)^q \int_0^\infty u^{\frac{q'}{p'}-1}\Big(g^*(u)\Big)^{q'} \,\mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p}\right)^q \int_0^\infty \Big(u^{\frac{1}{p'}-1}g^*(u)\Big)^{q'} \frac{\mathrm{d}u}{u}$$

$$= \left(\frac{2^{1/p}q}{p}\right)^q \|g\|_{(p',q')}^{q'},$$

thus

$$\|f\|_{(p,q)} \le \left(\frac{2^{1/p}q}{p}\right)\|g\|_{(p',q')}^{q'/q}. \tag{6.26}$$

On the other hand, we have

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \ge \int_0^\infty \int_{t/2}^t s^{\frac{q'}{p'}-1}\Big(g^*(s)\Big)^{q'-1}\frac{\mathrm{d}s}{s}g^*(t)\,\mathrm{d}t$$

$$\ge \int_0^\infty \Big(g^*(t)\Big)^{q'}\int_{t/2}^t s^{\frac{q'}{p'}-1}\frac{\mathrm{d}s}{s}\,\mathrm{d}t = \int_0^\infty \Big(g^*(t)\Big)^{q'}\int_{t/2}^t s^{\frac{q'}{p'}-2}\,\mathrm{d}s\,\mathrm{d}t$$

$$= \frac{p'}{q'-p'}\Big(1-2^{1-\frac{q'}{p'}}\Big)\int_0^\infty t^{\frac{q'}{p'}-1}\Big(g^*(t)\Big)^{q'}\,\mathrm{d}t$$

$$= \frac{p'}{q'-p'}\Big(1-2^{1-\frac{q'}{p'}}\Big)\|g\|_{(p',q')}^{q'}. \tag{6.27}$$

Combining (6.26) and (6.27) we obtain

$$\|g\|_{(p',q')} \le \left[\frac{q'-p'}{p'\Big(1-2^{1-\frac{q'}{p'}}\Big)}\right]^{1/q'}\|T\|. \tag{6.28}$$

Finally by (6.25) and (6.28) we have the required conclusion. □

Theorem 6.23 (Duality). *Let* $1<p<\infty$ *and* $1\le q<\infty$ *or* $p=q=1$. *Then the space of all bounded linear functionals on* $L_{(p,q)}(X,\mathscr{A},\mu)$, *denoted by* $\Big[L_{(p,q)}(X,\mathscr{A},\mu)\Big]^*$ *is isomorphic to* $L_{(p',q')}(X,\mathscr{A},\mu)$ *where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$.

Proof. If $p=q$ then we have that $L_{(p,p)}(X,\mathscr{A},\mu)=L_p(X,\mathscr{A},\mu)$ which is isomorphic to $L_{p'}(X,\mathscr{A},\mu)$ for all $1\le p<\infty$.
Thus, we need to only consider the case when $1<p<\infty$, $1<q<\infty$ and $p\ne q$. To prove that $\left[L_{(p,q)}(X,\mathscr{A},\mu)\right]^*$ is isomorphic to $L_{(p',q')}(X,\mathscr{A},\mu)$ we must show that for each element in $L_{(p',q')}(X,\mathscr{A},\mu)$ there exists a unique corresponding element in $\left[L_{(p,q)}(X,\mathscr{A},\mu)\right]^*$ and vice versa.

We start with the case when $1<p,q<\infty$, $p\ne q$. Let $g\in L_{p',q'}(X,\mathscr{A},\mu)$ be arbitrary and define the functional T as

$$T(f)=\int_X fg\,\mathrm{d}\mu,$$

for all $f\in L_{(p,q)}(X,\mathscr{A},\mu)$. By Theorem 4.15 and Hölder's inequality we get that

$$\begin{aligned}
|T(f)| &= \left|\int_X fg\,\mathrm{d}\mu\right| \\
&\le \int_X |fg|\,\mathrm{d}\mu \\
&\le \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \\
&\le \int_0^\infty f^{**}(t)g^{**}(t)\,\mathrm{d}t \\
&\le \int_0^\infty t^{1/p}f^{**}(t)t^{1/p'}g^{**}(t)\frac{\mathrm{d}t}{t} \\
&\le \left(\int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q}\left(\int_0^\infty \left(t^{1/p'}g^{**}(t)\right)^{q'}\frac{\mathrm{d}t}{t}\right)^{1/q'} \\
&= \|f\|_{pq}\|g\|_{p'q'}.
\end{aligned}$$

Hence, T is bounded and if we take the supremum on both sides over all functions f with norm 1 we have that

$$\|T\|\le\|g\|_{p'q'}.$$

If $1<p<\infty$ and $q=1$ we can use that

$$\int_0^\infty t^{1/p}f^{**}(t)t^{1/p'}g^{**}(t)\frac{\mathrm{d}t}{t}\le\int_0^\infty t^{1/p}f^{**}(t)\left(\sup_{s>0}s^{1/p}g^{**}(s)\right)\frac{\mathrm{d}t}{t}$$

to obtain that

$$\|T\| \leq \|g\|_{p'\infty}.$$

Hence, for all functions in $L_{(p',q')}(X,\mathscr{A},\mu)$ we can find a linear bounded functional on $L_{(p,q)}(X,\mathscr{A},\mu)$, that is an element in $\left[L_{(p,q)}(X,\mathscr{A},\mu)\right]^*$. In Theorem 6.22 we showed that

$$\sigma(E) = T\left(\chi_E\right)$$

is absolutely continuous with respect to μ and by Radon-Nikodym theorem there exists a unique function $g \in L_1(X,\mathscr{A},\mu) = L_{(1,1)}(X,\mathscr{A},\mu)$ such that

$$\sigma(E) = T\left(\chi_E\right) = \int_X g\chi_E \,\mathrm{d}\mu.$$

By the linearity of the integral and density of simple functions it follows that

$$T(f) = \int_X gf \,\mathrm{d}\mu$$

for all $f \in L_{(p,q)}(X,\mathscr{A},\mu)$.

Once again, by Theorem 4.29 there exists a measurable function f on X such that

$$f^*(t) = \int_{t/2}^{\infty} h(s)\frac{\mathrm{d}s}{s},$$

where $h(s) = s^{\frac{q'}{p'}-1}\left(g^*(s)\right)^{q'-1}$.

Then by Theorem and Theorem 6.12 with $r = p$ we have

$$\begin{aligned}
\|f\|_{pq}^q &\leq \left(\frac{p}{p-1}\|f\|_{(p,q)}\right)^q \\
&= \left(\frac{p}{p-1}\right)^q \int_0^{\infty} \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \\
&= \left(\frac{p}{p-1}\right)^q \int_0^{\infty} \left(\int_{t/2}^{\infty} h(s)\frac{\mathrm{d}s}{s}\right)^q t^{\frac{q}{p}-1}\,\mathrm{d}t \\
&= 2^{q/p}\left(\frac{p}{p-1}\right)^q \int_0^{\infty} \left(\int_u^{\infty} h(s)\frac{\mathrm{d}s}{s}\right)^q u^{\frac{q}{p}-1}\,\mathrm{d}u
\end{aligned}$$

$$\leq \left(\frac{2^{1/p}q}{p-1}\right)^q \int_0^\infty u^{\frac{q}{p}-1}\Big(h(u)\Big)^q \, \mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p-1}\right)^q \int_0^\infty u^{\frac{q}{p}-1+\frac{qq'}{p'}-q}\Big(g^*(u)\Big)^{q(q'-1)} \, \mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p-1}\right)^q \int_0^\infty u^{\frac{q'}{p'}-1}\Big(g^*(u)\Big)^{q'} \, \mathrm{d}u$$

$$= \left(\frac{2^{1/p}q}{p-1}\right)^q \int_0^\infty \Big(u^{\frac{1}{p'}} g^*(u)\Big)^{q'} \frac{\mathrm{d}u}{u}$$

$$= \left(\frac{2^{1/p}q}{p-1}\right)^q \|g\|^{q'}_{(p',q')},$$

thus

$$\|f\|_{pq} \leq \left(\frac{2^{1/p}q}{p-1}\right)^q \|g\|^{q'/q}_{(p'q')}.$$

On the other hand, using the definition of the norm of T and Theorem 4.23 we have

$$\|T\| = \sup_{f\in L_{(p,q)}} \frac{\big|T(f)\big|}{\|f\|_{pq}} = \sup_{f\in L_{(p,q)}} \frac{\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t}{\|f\|_{pq}},$$

thus

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \leq \|T\|\|f\|_{pq}.$$

Now, observe that

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \geq \int_0^\infty \left(\int_{t/2}^\infty s^{\frac{q'}{p'}-1}\Big(g^*(s)\Big)^{q'-1}\frac{\mathrm{d}s}{s}\right) g^*(t)\,\mathrm{d}t$$

$$\geq \int_0^\infty \Big(g^*(t)\Big)^{q'} \int_{t/2}^t s^{\frac{q'}{p'}-1}\frac{\mathrm{d}s}{s}\,\mathrm{d}t$$

$$= \int_0^\infty \Big(g^*(t)\Big)^{q'} \int_{t/2}^t s^{\frac{q'}{p'}-2}\,\mathrm{d}s\,\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{p'}{q'-p'}\left(1-2^{1-\frac{q'}{p'}}\right)\int_0^\infty t^{\frac{q'}{p'}-1}\left(g^*(t)\right)^{q'}\,\mathrm{d}t \\
&= \frac{p'}{q'-p'}\left(1-2^{1-\frac{q'}{p'}}\right)\|g\|_{(p',q')}^{q'} \\
&\geq \frac{p'}{q'-p'}\left(1-2^{1-\frac{q'}{p'}}\right)\left(\frac{1}{p}\|g\|_{p'q'}\right)^{q'}.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
\frac{\left(\frac{1}{p}\|g\|_{p'q'}\right)^{q'}}{\|f\|_{pq}}\frac{p'}{q'-p'}\left(1-2^{1-\frac{q'}{p'}}\right) &\leq \|T\| \\
C\|g\|_{p'q'}^{q'-\frac{q'}{p}} &\leq \|T\| \\
C\|g\|_{p'q'} &\leq \|T\|,
\end{aligned}
$$

where

$$
C = \left(\frac{1}{p}\right)^q \frac{p-1}{2^{1/p}q}\left(\frac{p'}{q'-p'}\right)\left(1-2^{1-\frac{q'}{p'}}\right).
$$

This shows that $\|g\|_{p'q'} < \infty$ and thus $g \in L_{(p',q')}(X,\mathscr{A},\mu)$. Also we have that

$$
C\|g\|_{p'q'} \leq \|T\| \leq \|g\|_{p'q'},
$$

hence $L^*_{(p,q)}$ and $L_{(p',q')}$ are isomorphic for $1 < q < p < \infty$. □

6.6 $L_1 + L_\infty$ Space

We now introduce a space based upon the concept of sum space.

Definition 6.24. Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space. The space L_1+L_∞ consists of all functions $f \in \mathfrak{F}(X,\mathscr{A})$ that are representable as a sum $f = g+h$ of functions $g \in L_1$ and $h \in L_\infty$. For each $f \in L_1+L_\infty$, let

$$
\|f\|_{L_1+L_\infty} := \inf_{f=g+h}\left\{\|g\|_{L_1}+\|h\|_{L_\infty}\right\}. \tag{6.29}
$$

where the infimum is taken over all representations $f = g+h$, where $g \in L_1$ and $h \in L_\infty$. ⊘

The next result provides an analogous description of the norm in L_1+L_∞.

Theorem 6.25. *Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space and suppose f belongs to $\mathfrak{F}(X,\mathscr{A})$. Then*

$$\inf_{f=g+h} \left\{ \|g\|_{L_1} + \|h\|_{L_\infty} \right\} = \int_0^t f^*(s)\,\mathrm{d}s = t f^{**}(t), \tag{6.30}$$

for all $t > 0$.

Proof. For $t > 0$ denote

$$\alpha_t = \inf_{f=g+h} \left\{ \|g\|_{L_1} + \|h\|_{L_\infty} \right\}.$$

Next, we like to show that

$$\int_0^t f^*(s)\,\mathrm{d}s \le \alpha_t. \tag{6.31}$$

We may assume that f belongs to $L_1 + L_\infty$. Since, otherwise the infimum α_t is infinite and there is nothing to prove. In this case f may be expressed as a sum $f = g + h$ with $g \in L_1$ and $h \in L_\infty$. The sub-additivity of f^{**} (see (6.12)) gives

$$\int_0^t f^*(s)ds = \int_0^t (g+h)^*(s)ds \le \int_0^t g^*(s)ds + \int_0^t h^*(s)\,\mathrm{d}s.$$

Since $h^*(s) \le h^*(0)$ for $s > 0$, we have

$$\int_0^t f^*(s)\,\mathrm{d}s \le \int_0^\infty g^*(s)\,\mathrm{d}s + \int_0^t h^*(0)\,\mathrm{d}s = \int_X |g|\,\mathrm{d}\mu + t\|h\|_\infty = \|g\|_{L_1} + t\|h\|_\infty.$$

Taking the infimum over all possible representations $f = g + h$, we obtain

$$\int_0^t f^*(s)\,\mathrm{d}s \le \alpha_t. \tag{6.32}$$

For the reverse inequality of (6.32) it suffices to construct functions $g \in L_1$ and $h \in L_\infty$ such that $f = g + h$ and assume that $\int_0^t f^*(s)\,\mathrm{d}s.\infty$.

Let $E = \left\{ x \in X : |f(x)| > f^*(t) \right\}$ and $\mu(E) = t_0$. Since $D_f(f^*(t)) \le t$, then $t_0 = \mu(E) = D_f(f^*(t)) \le t$, thus $t_0 \le t$, then by the Hardy-Littlewood inequality we have

$$\begin{aligned} \int_E |f|\,\mathrm{d}\mu &= \int_X |f|\chi_E\,\mathrm{d}\mu \\ &\le \int_0^\infty f^*(s)\chi_{(0,\mu(E))}(s)\,\mathrm{d}s \end{aligned}$$

$$= \int_0^{t_0} f^*(s)\,\mathrm{d}s$$
$$\leq \int_0^{t} f^*(s)\,\mathrm{d}s < \infty,$$

thus f is integrable over E. Now, let us define

$$g(x) = \max\left\{|f(x)| - f^*(t), 0\right\} \operatorname{sgn} f(x)$$

and

$$h(x) = \max\left\{|f(x)|, f^*(t)\right\} \operatorname{sgn} f(x).$$

The L_1 norm of g can be calculated as

$$\|g\|_{L_1} = \int_X |g|\,\mathrm{d}\mu = \int_E |g|\,\mathrm{d}\mu + \int_{E^\complement} |g|\,\mathrm{d}\mu.$$

Note that the second integral is null on $E^\complement$ and so

$$\begin{aligned}
\|g\|_{L_1} &= \int_E |g|\,\mathrm{d}\mu \\
&= \int_E |f(x)| d\mu - f^*(t) \int_E \mathrm{d}\mu \\
&= \int_E |f(x)| d|m - t_0 f^*(t) \\
&\leq \int_E |f(x)| d\mu < \infty,
\end{aligned}$$

thus g belongs to L_1. Next, observe that

$$\begin{aligned}
&\mu\left(\left\{x \in X : |h(x)| > f^*(t)\right\}\right) \\
&= \mu\left(\left\{x \in E : |h(x)| > f^*(t)\right\}\right) + \mu\left(\left\{x \in E^\complement : |h(x)| > f^*(t)\right\}\right) \\
&= \mu\left(\left\{x \in E : f^*(t) > f^*(t)\right\}\right) + \mu\left(\left\{x \in E^\complement : |f(x)| > f^*(t)\right\}\right) \\
&= 0
\end{aligned}$$

since the sets in the penultimate equality are the empty set, therefore

$$\|h\|_\infty = \inf\left\{M > 0 : \mu\left(\left\{x \in X : |h(x)| > M\right\}\right) = 0\right\} = f^*(t),$$

hence h belongs to L_∞.

On the other hand, observe that

$$\begin{aligned}
&g(x) + h(x) \\
&\quad = \max\left\{|f(x)| - f^*(t), 0\right\} \operatorname{sgn} f(x) + \min\left\{|f(x)|, f^*(t)\right\} \operatorname{sgn} f(x) \\
&\quad = \begin{cases} |f(x)| - f^*(t) + f^*(t) & if\ |f(x)| > f^*(t) \\ 0 + |f(x)| & if\ |f(x)| \leq f^*(t) \end{cases} \\
&\quad = |f(x)|.
\end{aligned}$$

Since

$$\|g\|_{L_1} = \int_E |g| d\mu = \int_E |f(x)| d\mu - t_0 f^*(t) \leq \int_0^{t_0} f^*(s)\,\mathrm{d}s - t_0 f^*(t)$$

and thus

$$\begin{aligned}
\|g\|_{L_1} + t\|h\|_\infty &\leq \int_0^{t_0} f^*(s)\,\mathrm{d}s + t\|h\|_\infty - t_0 f^*(t) \\
&\leq \int_0^{t_0} f^*(s)\,\mathrm{d}s + (t - t_0) f^*(t) \\
&= \int_0^{t_0} f^*(s)\,\mathrm{d}s + \int_{t_0}^{t} f^*(t)\,\mathrm{d}s \\
&\leq \int_0^{t_0} f^*(s)\,\mathrm{d}s + \int_{t_0}^{t} f^*(s)\,\mathrm{d}s \\
&= \int_0^{t} f^*(s)\,\mathrm{d}s
\end{aligned}$$

which entails

$$\alpha_t \leq \int_0^t f^*(s)\,\mathrm{d}s. \tag{6.33}$$

Combining (6.31) and (6.33) we have

$$\int_0^t f^*(s)\,\mathrm{d}s = \inf_{f=g+h}\left\{\|g\|_{L_1} + t\|h\|_\infty\right\}$$

and

$$tf^{**}(t) = \int_0^t f^*(s)ds = \inf_{f=g+h}\left\{\|g\|_{L_1} + t\|h\|_\infty\right\}.$$

□

Note that we can calculate the $L_1 + L_\infty$ norm of a function via:

$$f^{**}(1) = \inf_{f=g+h}\left\{\|g\|_{L_1} + \|h\|_\infty\right\} = \|f\|_{L_1+L_\infty}$$

and

$$\int_0^1 f^*(s)\,\mathrm{d}s = \|f\|_{L_1+L_\infty}.$$

We now introduce the notion of maximal function $f \mapsto Mf$, which will be studied in more detail in Chapter 9.

Definition 6.26. Let $f \in L_{1,loc}(\mathbb{R}^n)$. The *Hardy-Littlewood maximal function* is defined as

$$Mf(x) = \sup_{0<r<\infty} \frac{1}{m\Big(B(x,r)\Big)} \int_{B(x,r)} |f(y)|\,\mathrm{d}y. \tag{6.34}$$

where $B(x,r) = \Big\{y \in \mathbb{R}^n : |y-x| < r\Big\}$ is an open ball in $\mathbb{R}^n$. ⊘

With the notion of maximal function at hand, we now prove that the maximal function given in (4.17) and the new one given in (6.34) are related in the following sense.

Theorem 6.27. *There exists a constant c depending only on n, such that*

$$(Mf)^*(f) \le cf^{**}(t)$$

for every locally integrable function f on $\mathbb{R}^n$.

Proof. Fix $t > 0$. For the left-hand side inequality we may suppose $f^{**}(t) < \infty$, otherwise there is nothing to prove. In this case by Theorem 6.25, given $\varepsilon > 0$ there are functions $g_t \in L_1$ and $h_t \in L_\infty$ such that $f = g_t + h_t$ and

$$\|g_T\|_{L_1} + t\|h_T\|_\infty \le tf^{**}(t) + \varepsilon, \tag{6.35}$$

then by Theorem 7.29 and Theorem 3.38, for any $s > 0$

$$(Mf)^*(s) \le (Mg_t)^*\left(\frac{s}{2}\right) + (Mh_t)^*\left(\frac{s}{2}\right) \\ \le \frac{C}{s}\|g_t\|_{L_1} + \|h_t\|_\infty = \frac{C}{s}\left(\|g_t\|_{L_1} + s\|h_t\|_\infty\right).$$

□

Theorem 6.28 (Hardy-Littlewood inequality). *Let* $1 < p \le \infty$ *and suppose that* $f \in L_p(\mathbb{R}^n)$*. Then* $Mf \in L_p(\mathbb{R}^n)$ *and*

$$\|Mf\|_p \le c\|f\|_p$$

where c *is a constant depending only on* p *and* n

We wish to point out that using the rearrangement (6.30) a proof of Theorem 6.28 can be obtained directly and without using any covering technique.

Proof (Proof of Theorem 6.28). If $f \in L_\infty(\mathbb{R}^n)$, by Theorem

$$\|Mf\|_{L_\infty} = \sup_{t>0} (Mf)^*(t) \le c \sup_{t>0} f^{**}(t) \le c\|f\|_{L_\infty}.$$

Now, if $f \in L_p(\mathbb{R}^n)$ with $1 < p < \infty$ then one more time by Theorem 6.27 we have

$$\|Mf\|_p = \left(\int_0^\infty \left((Mf)^*(t)\right)^p \mathrm{d}t\right)^{\frac{1}{p}} \le c\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{\frac{1}{p}} \\ \le \frac{cp}{p-1}\left(\int_0^\infty (f^*(t))^p\,\mathrm{d}t\right)^{\frac{1}{p}} = \frac{cp}{p-1}\|f\|_p.$$

□

6.7 $L\exp$ and $L\log L$ Spaces

We now introduce another function spaces.

Definition 6.29. The Zygmund space $L\exp$ consists of all $f \in \mathfrak{F}(X, \mathscr{A})$ for which there is a constant $\alpha = \alpha(f)$ such that

$$\int_E \exp\left(\alpha|f(x)|\right) \mathrm{d}\mu(x) < \infty \tag{6.36}$$

for all $E \in \mathscr{A}$. The Zygmund space $L\log L$ consists of all $f \in \mathfrak{F}(X, \mathscr{A})$ for which

$$\int_X |f(x)| \log^+ |f(x)| \, \mathrm{d}\mu(x) < \infty \tag{6.37}$$

where $\log^+ x = \max\{\log x, 0\}$. $\oslash$

The quantities introduced in (6.36) and (6.37) are evidently far from satisfying the properties of norm.

The expression introduced in the next theorem, defined in terms of the decreasing rearrangement, will prove more manageable.

Theorem 6.30. *Let* $f \in \mathfrak{F}(X, \mathscr{A})$ *and* $A \in \mathscr{A}$ *such that* $0 < \mu(E) < \infty$. *Then*

(a) $\int_E |f(x)| \log^+ |f(x)| \, \mathrm{d}\mu(x) < \infty$ *if and only if* $\int_0^{\mu(E)} f^*(t) \log\left(\frac{\mu(E)}{t}\right) \mathrm{d}t < \infty$.

(b) $\int_E \exp\left(\alpha|f(x)|\right) \mathrm{d}\mu(x) < \infty$ *for some constant* $\alpha = \alpha(f)$ *if and only if there is a constant* $c = c(f)$ *such that*

$$f^*(t) \leq c\left(1 + \log\left(\frac{\mu(E)}{t}\right)\right)$$

for $0 < t < \mu(E)$.

Proof. To this end, we first apply Theorem 4.13 to obtain

$$\int_E |f(x)| \log^+ |f(x)| \, \mathrm{d}\mu = \int_0^\infty f^*(t) \log^+ f^*(t) \, \mathrm{d}t,$$

thus $\int_0^{\mu(E)} f^*(t) \log^+ f^*(t) \mathrm{d}t < \infty$. On the other hand, note that

$$f^*(t) \leq f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s \leq \frac{1}{t} \int_0^{\mu(E)} f^*(s) \, \mathrm{d}s = \frac{1}{t} \int_E |f| d\mu = \frac{\|f\|_{L_1}}{t},$$

since $\log^+ x$ is an increasing function and $\log \frac{1}{x}$ is a decreasing function. On the one hand, if

$$\int_0^{\mu(E)} f^*(t) \log\left(\frac{\mu(E)}{t}\right) \mathrm{d}t < \infty,$$

then

$$\int_0^{\mu(E)} f^*(t)\log^+ f^*(t)\,\mathrm{d}t \le \int_0^{\mu(E)} f^*(t)\log^+ \frac{\|f\|_{L_1}}{t}\,\mathrm{d}t$$
$$= \int_0^{\min\left\{\|f\|_{L_1},\mu(E)\right\}} f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t,$$

since $\log^+ \frac{\|f\|_{L_1}}{t} = \log\left(\frac{\|f\|_{L_1}}{t}\right)$ if $0 < t < \|f\|_{L_1}$ and 0 otherwise.

Next,

$$\int_0^{\min\{\|f\|_{L_1},\mu(E)\}} f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t \le \int_0^{\|f\|_{L_1}} f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t$$
$$= \left(\int_0^{\mu(E)} + \int_{\mu(E)}^{\|f\|_{L_1}}\right) f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t$$
$$\le \int_0^{\mu(E)} f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t + \|f\|_{L_1}\int_{\mu(E)}^{\|f\|_{L_1}} \frac{1}{t}\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t$$
$$= \int_0^{\mu(E)} f^*(t)\log\left(\frac{\|f\|_{L_1}}{t}\right)\mathrm{d}t + \frac{\|f\|_{L_1}}{2}\left(\log\frac{\|f\|_{L_1}}{\mu(E)}\right)^2.$$

Thus $\int_E |f(x)|\log^+|f(x)|\,\mathrm{d}\mu < \infty$. On the other hand, if $\int_E |f(x)|\log^+|f(x)|\,\mathrm{d}\mu < \infty$ we consider the following set

$$A = \left\{t \in [0,\mu(E)] : f^*(t) > \left(\frac{\mu(E)}{t}\right)^{\frac{1}{2}}\right\} \quad \text{and} \quad B = [0,\mu(E)]\setminus A,$$

either of which may be empty. Next, we can write

$$\int_0^{\mu(E)} f^*(t)\log\left(\frac{\mu(E)}{t}\right)\mathrm{d}t = \left(\int_A + \int_B\right) f^*(t)\log\left(\frac{\mu(E)}{t}\right)\mathrm{d}t$$
$$\le \int_0^{\mu(E)} f^*(t)\log(f^*(t))^2\,\mathrm{d}t + \big(\mu(E)\big)^{\frac{1}{2}}\int_0^{\mu(E)} t^{-\frac{1}{2}}\log\left(\frac{\mu(E)}{t}\right)\mathrm{d}t$$
$$= 2\int_0^{\mu(E)} f^*(t)\log(f^*(t))\,\mathrm{d}t + \frac{(\mu(E))^{\frac{3}{2}}}{2},$$

hence $\int_0^{\mu(E)} f^*(t)\log\left(\frac{\mu(E)}{t}\right)\mathrm{d}t < \infty$. Turning now to the equivalence in (b) we suppose first that $f^*(t) \le C\left(1+\log\left(\frac{\mu(E)}{t}\right)\right)$ for some constant $C > 0$ with $0 < t < \mu(E)$. Then

$$\int_0^{\mu(E)} \exp\big(\alpha f^*(t)\big)\,\mathrm{d}t \le \int_0^{\mu(E)} \exp\left[\alpha C\left(1+\log\left(\frac{\mu(E)}{t}\right)\right)\right]\mathrm{d}t$$
$$= e^{C\alpha}\left[\mu(E)\right]^{C\alpha}\int_0^{\mu(E)} t^{-C\alpha}\,\mathrm{d}t.$$

from which $\int_0^{\mu(E)} \exp\big(\alpha f^*(t)\big)\,\mathrm{d}t < \infty$ for any constant $\alpha < 1/C$.

Conversely, suppose that

$$M = \int_0^{\mu(E)} \exp\big(\alpha f^*(t)\big)\,\mathrm{d}t < \infty.$$

Clearly $M \ge \mu(E)$. Since f^* is decreasing we have

$$f^*(t) = f^*(t)\frac{1}{t}\int_0^t \mathrm{d}s \le \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s.$$

Then by Jensen's inequality we have

$$\exp\big(\alpha f^*(t)\big) \le \exp\left(\frac{1}{t}\int_0^t \alpha f^*(s)\,\mathrm{d}s\right) \le \frac{1}{t}\int_0^t \exp\big(\alpha f^*(s)\big)\,\mathrm{d}s,$$

which entails

$$f^*(t) \le \frac{1}{\alpha}\log\left(\frac{M}{t}\right) \le \frac{1}{\alpha}\left(1+\log\left(\frac{M}{\mu(E)}\right)\right)\left(1+\log\left(\frac{\mu(E)}{t}\right)\right).$$

The proof is now complete. □

An integration by parts shows that

$$\int_0^{\mu(E)} f^*(t)\log\left(\frac{\mu(E)}{t}\right)\mathrm{d}t = \int_0^{\mu(E)} f^{**}(t)\,\mathrm{d}t. \tag{6.38}$$

The latter quantity involving the sub-additive function $f \to f^{**}$ satisfies the triangle inequality and so may be used to directly define a norm in $L\log L$.

On the other hand the expression

$$f^*(t) \le C\left(1+\log\left(\frac{\mu(E)}{t}\right)\right) \quad 0<t<\mu(E) \tag{6.39}$$

for the space $L\exp$ involve f^* rather than the sub-additive f^{**}. This present no problem, however

$$\begin{aligned} f^{**}(t) &= \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s \\ &\le \frac{C}{t}\int_0^t \left(1+\log\left(\frac{\mu(E)}{s}\right)\right)\mathrm{d}s \\ &\le 2C\left(1+\log\left(\frac{\mu(E)}{t}\right)\right). \end{aligned} \tag{6.40}$$

Hence (6.39) (with constant C) implies (6.40) (with constant $2C$) and so (6.39) and (6.40) are equivalent. We use the relation (6.40) to define a norm on $L\exp$ as follows.

Definition 6.31. Let $f \in \mathfrak{F}(X,\mathscr{A})$. Set

$$\|f\|_{L\log L} = \int_0^{\mu(E)} f^*(t)\log\left(\frac{\mu(E)}{t}\right)dt = \int_0^{\mu(E)} f^{**}(t)\,\mathrm{d}t. \tag{6.41}$$

and

$$\|f\|_{L\exp}\|f\|_{L\exp} = \sup_{0<t<\mu(E)} \frac{f^{**}(t)}{\left(1+\log\left(\frac{\mu(E)}{t}\right)\right)}. \tag{6.42}$$

⊘

It follows from Theorem 6.30 and the observation made above that $L\log L$ and $L\exp$ consist of all function $f \in \mathfrak{F}(X,\mathscr{A})$ for which the representative quantities (6.41) and (6.42) are finite. Since $f \to f^{**}$ is sub-additive, it is easy to prove directly that this qualities define norms under which $L\log L$ and $L\exp$ are rearrangement-invariant Banach spaces. The following result gives a Hölder type inequality.

Theorem 6.32. *Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space and $f \in L\log L$ and $g \in L\exp$. Then*

$$\int_E |fg|\,\mathrm{d}\mu \le 2\|f\|_{L\log L}\|g\|_{L\exp}$$

for any $E \in \mathscr{A}$.

Proof. Let $E \in \mathscr{A}$ and $f \in L\log L$, $g \in L\exp$. Then from Theorem 4.13 and the fact that $f^* \le f^{**}$ we have

$$\begin{aligned}
\int_E |fg|d\mu &= \int_X |fg\chi_E|\,\mathrm{d}\mu \le \int_0^\infty f^*(t)g^*(t)\chi_{(0,\mu(E))}(t)\,\mathrm{d}t = \int_0^{\mu(E)} f^*(t)g^*(t)\,\mathrm{d}t \\
&\le \int_0^{\mu(E)} f^*(t)\left(1+\log\frac{\mu(E)}{t}\right)\frac{g^{**}(t)}{1+\log\frac{\mu(E)}{t}}\,\mathrm{d}t \\
&\le \left[\int_0^{\mu(E)} f^*(t)\left(1+\log\frac{\mu(E)}{t}\right)\mathrm{d}t\right]\|g\|_{L\exp} \\
&\le 2\left(\int_0^{\mu(E)} f^{**}(t)\,\mathrm{d}t\right)\|g\|_{L\exp} \\
&= 2\|f\|_{L\log L}\|g\|_{L\exp},
\end{aligned}$$

and the assertion of the theorem holds. □

6.8 Lorentz Sequence Spaces

In this section we will investigate the Lorentz sequence spaces.

For $X = \mathbb{N}$ with $A = 2^{\mathbb{N}}$, the power set of X and μ = counting measure, the distribution function of any complex-valued function $a = \{a(n)\}_{n\ge1}$ can be written as

$$D_a(\lambda) = \mu(\{n \in \mathbb{N} : |a(n)| > \lambda\}) \quad (\lambda \ge 0).$$

The decreasing rearrangement a^* of a is given as

$$a^*(t) = \inf\{\lambda > 0 : D_a(\lambda) \le t\} \quad (t \ge 0).$$

We can interpret the decreasing rearrangement of a with $D_a(\lambda) < \infty$, $\lambda > 0$ as a sequence $\{a^*(n)\}$ if we define for $n-1 \le t \le n$

$$a^*(n) = a^*(t) = \inf\{\lambda > 0 : D_a(\lambda) \le n-1\} \tag{6.43}$$

Then the sequence $a^* = \{a^*(n)\}$ is obtained by permuting $\{|a(n)|\}_{n\in S}$ where $S = \{n : a(n) \ne 0\}$, in the decreasing order with $a^*(n) = 0$ for $n > \mu(S)$ if $\mu(S) < \infty$.

Definition 6.33. The Lorentz sequence space $\ell_{(p,q)}$, $1 < p \le \infty$, $1 \le q \le \infty$, is the set of all complex sequences $a = \{a(n)\}$ such that $\|a\|^s_{(p,q)} < \infty$ where

$$\|a\|^s_{(p,q)} = \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{1/p} a^*(n)\right)^q \frac{1}{n}\right)^{1/q}, & 1 < p < \infty, 1 \le q < \infty \\ \sup_{n\ge 1} n^{1/p} a^*(n), & 1 < p \le \infty, q = \infty. \end{cases}$$

where a^* is given in (6.43). ⊘

The Lorentz sequence space $\ell_{(p,q)}$, $1 < p \le \infty$, $1 < p \le \infty$, $q = \infty$, is a linear space and $\|\cdot\|^s_{(p,q)}$ is a quasi-norm. Moreover, $\ell_{(p,q)}$ $1 < p \le \infty$, $1 \le q \le \infty$, is complete with respect to the quasi-norm $\|\cdot\|^s_{(p,q)}$.

The Lorentz sequence space $\ell_{(p,q)}$ and $L_{(p,q)}$ when $X = \mathbb{N}$, $\mathscr{A} = 2^{\mathbb{N}}$ and $\mu(\{n\}) = 1$ are equivalent for $0 < p < \infty$, $0 < q < \infty$. In fact, if we let $a^*(n) = f^*(t)$ for $n-1 \le t < n$, we have

$$\begin{aligned} \|f\|_{(p,q)} &= \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\ &= \left(\sum_{n=1}^{\infty} \left[a^*(n)\right]^q \int_{n-1}^{n} t^{q/p-1}\,\mathrm{d}t\right)^{1/q} \end{aligned}$$

and since

$$\left(\frac{1}{2}\right)^{q/p} n^{q/p-1} \le \int_{n-1}^{n} t^{q/p-1}\,\mathrm{d}t \le 2n^{q/p-1}$$

we obtain

$$\left(\frac{1}{2}\right)^{q/p} \sum_{n=1}^{\infty} (a^*(n))^q n^{q/p-1} \le \sum_{n=1}^{\infty} (a^*(n))^q \int_{n-1}^{n} t^{q/p-1}\,\mathrm{d}t \le 2\sum_{n=1}^{\infty} (a^*(n))^q n^{q/p-1},$$

from which we get

$$\left(\frac{1}{2}\right)^{1/p}\left(\sum_{n=1}^{\infty}(n^{1/p}a^*(n))^q\frac{1}{n}\right)^{1/q} \leq \|f\|_{(p,q)} \leq 2^{1/q}\left(\sum_{n=1}^{\infty}(n^{1/p}a^*(n))^q\frac{1}{n}\right)^{1/q}$$

thus

$$\left(\frac{1}{2}\right)^{1/p}\|a\|^s_{(p,q)} \leq \|f\|_{(p,q)} \leq 2^{1/q}\|a\|^s_{(p,q)}.$$

Observe that the space $\ell_{(p,q)}$ is not empty when $p=\infty$. For example, all sequences which only have a finite number of nonzero elements are in $\ell_{(p,q)}$ for all $0<q\leq\infty$. This show that there is a fundamental difference between $L_{(\infty,q)}$ and $\ell_{(\infty,q)}$.

The following result will be of great utility in our study, and we include a short proof for the benefit of the reader.

Theorem 6.34. *If $a=\{a(n)\}_{n\in\mathbb{N}}$ and $b=\{b(n)\}_{n\in\mathbb{N}}$ are complex sequences, $b\in\ell_{(p,q)}$ with $1<p\leq\infty$, $1\leq q\leq\infty$ and $|a(n)|\leq|b(n)|$ for all $n\in\mathbb{N}$, then $a\in\ell_{(p,q)}$ and $\|a\|^s_{(p,q)}\leq\|b\|^s_{(p,q)}$.*

Proof. If $|a(n)|\leq|b(n)|$ for all $n\in\mathbb{N}$, then

$$\left\{n\in\mathbb{N}:|a(n)|>\lambda\right\}\subset\left\{n\in\mathbb{N}:|b(n)|>\lambda\right\},$$

by the monotonicity of the measure, we have $D_a(\lambda)\leq D_b(\lambda)$ for all $\lambda>0$. Thus, for any $m\in\mathbb{N}$, we obtain

$$\{\lambda>0:D_b(\lambda)\leq m-1\}\subset\{\lambda>0:D_a(\lambda)\leq m-1\}$$

and hence $a^*(m)\leq b^*(m)$. From this last fact, the result follows easily. □

The aim of this section is to present basic results about Lorentz sequence spaces. The Lorentz sequence space $\ell_{(p,q)}$ is a normed linear space if and only if $1\leq q\leq p<\infty$. Moreover, $\ell_{(p,q)}$ is normable when $1<p<q\leq\infty$, that is there exists a norm equivalent to $\|\cdot\|^s_{(p,q)}$. For the remaining cases $\ell_{(p,q)}$ cannot be equipped with an equivalent norm.

The normable case for $p<q$ comes up in the following way

$$\|a\|^*_{(p,q)}=\begin{cases}\left(\sum_{n=1}^{\infty}\left(a^{**}(n)\right)^q n^{q/p-1}\right)^{1/q}, & q<\infty\\ \sup_{n\geq1}\left\{n^{1/p}a^{**}(n)\right\}, & q=\infty\end{cases}$$

where $a^{**} = \left\{a^{**}(n)\right\}_n$ is called the maximal sequence of $a^* = \left\{a^*(n)\right\}_n$ and it is defined as

$$a^{**}(n) = \frac{1}{n}\sum_{k=1}^{n} a^*(k).$$

We now prove that the functionals $\|\cdot\|^*$ and $\|\cdot\|^s$ are equivalent.

Theorem 6.35. *Let $a = \{a(n)\}_{n\geq 1}$ be a complex sequence, $1 < p \leq q < \infty$ then*

$$\|a\|^s_{(p,q)} \leq \|a\|^*_{(p,q)} \leq \left(\frac{p}{p-1}\right)^q \|a\|^s_{(p,q)}.$$

Proof. The inequality

$$\|a\|^s_{(p,q)} \leq \|a\|^*_{(p,q)}$$

is an easy consequence of the fact that

$$a^*(n) \leq a^{**}(n)$$

for all $n \in \mathbb{N}$. Hence, we just need to show that

$$\|a\|^*_{(p,q)} \leq \left(\frac{p}{p-1}\right)^q \|a\|^s_{(p,q)}.$$

In fact, let $r = q - \frac{q}{p}$. Using the fact that the function $g(t) = t^{r/q-1}$ is decreasing we apply the Hölder inequality to obtain

$$\begin{aligned}
\left(\sum_{k=1}^{n} a^*(k)\right)^q &= \left(\sum_{k=1}^{n} a^*(k)k^{1-\frac{r}{q}}k^{\frac{r}{q}-1}\right)^q \\
&\leq \left(\sum_{k=1}^{n}(a^*(k))^q k^{q-r}k^{\frac{r}{q}-1}\right)\left(\sum_{k=1}^{n} k^{\frac{r}{q}-1}\right)^{q/q'} \\
&\leq \left(\sum_{k=1}^{n}(a^*(k))^q k^{q-r}k^{\frac{r}{q}-1}\right)\left(\int_0^n t^{\frac{r}{q}-1}\,\mathrm{d}t\right)^{q/q'} \\
&= \left(\frac{q}{r}\right)^{q/q'} n^{r/q'} \sum_{k=1}^{n}\left(a^*(k)\right)^q k^{q-r}k^{\frac{r}{q}-1}.
\end{aligned}$$

Moreover since $f(t) = t^{-1-\frac{r}{q}}$ is decreasing and $\int_k^\infty f(t)\,\mathrm{d}t < \infty$ we have

$$\sum_{n=k-1}^{k} f(n) \leq \int_{k-2}^{k} f(t)\,\mathrm{d}t,$$

from which we get

$$\sum_{k=m+1}^{\infty}\left(\sum_{n=k-1}^{k} f(n)\right) \le \sum_{k=m+1}^{\infty}\left(\int_{k-2}^{k} f(t)\,\mathrm{d}t\right),$$

and

$$\sum_{n=m}^{\infty} f(n) \le \int_{m-1}^{\infty} f(t)\,\mathrm{d}t.$$

Next using the above inequality and Fubini's Theorem we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a^*(k)\right)^q n^{-r-1}\\
&\le \left(\frac{q}{r}\right)^{q/q'} \sum_{n=1}^{\infty} n^{-r-1+r\left(1-\frac{1}{q}\right)} \left(\sum_{k=1}^{n} (a^*(k))^q k^{q-r} k^{\frac{r}{q}-1}\right)\\
&= \left(\frac{q}{r}\right)^{q/q'} \sum_{k=1}^{\infty} (a^*(k))^q k^{q-r} k^{\frac{r}{q}-1} \left(\sum_{n=k}^{\infty} n^{-1-\frac{r}{q}}\right)\\
&= \left(\frac{q}{r}\right)^{\frac{q}{q'}+1} \sum_{k=1}^{\infty} (a^*(k))^q k^{q-r-1}\\
&= \left(\frac{p}{p-1}\right)^{q} \sum_{k=1}^{\infty} (a^*(k))^q k^{\frac{q}{p}-1}.
\end{aligned}$$

That is

$$\|a\|^*_{(p,q)} \le \left(\frac{p}{p-1}\right)^q \|a\|^s_{(p,q)}.$$

□

Note that if $\{a_k\}_{k=1,2,\dots,N} \in \ell_{(p,q)}$, then

$$\left\|\sum_{k=1}^{N} a_k\right\|^s_{(p,q)} \le \left\|\sum_{k=1}^{N} a_k\right\|^*_{(p,q)} \tag{6.44}$$

$$\le \left(\frac{p}{p-1}\right)^q \sum_{k=1}^{N} \|a_k\|^s_{(p,q)} \tag{6.45}$$

That is, $\|\cdot\|^s_{(p,q)}$ is a quasi-norm. On the other hand, if (6.44) holds, then $\|\cdot\|^s_{(p,q)}$ is equivalent to a norm, this norm is called decomposition norm and it is defined as

$$\|a\|_{pq} = \inf\left\{\sum_{k=1}^{N} \|a_k\|^s_{(p,q)} : a = \sum_{k=1}^{N} a_k\right\}.$$

The functional $\|\cdot\|_{pq}$ is an equivalent norm to $\|\cdot\|^s_{(p,q)}$ when $1 \le p,q \le \infty$. Moreover

$$\|\cdot\|_{pq} = \|\cdot\|^s_{(p,q)} \text{ if } 1 \le q \le p.$$

The following result is due to Hardy and Littlewood.

Theorem 6.36. *If $a = \{a(k)\}_{k\in\mathbb{N}}$ and $b = \{b(k)\}_{k\in\mathbb{N}}$ are complex sequences, then*

$$\sum_{k=1}^{\infty} \left|a(k)b(k)\right| \le \sum_{k=1}^{\infty} a^*(k)b^*(k). \tag{6.46}$$

Proof. It will be enough to show (6.46) for nonnegative sequences. For $n \in \mathbb{N}$ fixed, we set $E = \{1,2,\cdots,n\}$. Let us consider $c(1), c(2), \cdots, c(m)$ the different elements of the set $\{a(k) : k \in E\}$. Then it is clear that $m \le n = \mu(E)$. Thus, for $j \in \{1,2,\cdots,m\}$ we can define the sets

$$F_j = \{k \in E : a(k) = c(j)\}$$

Note that the sets F_j are pairwise disjoint and $\cup_{j=1}^m F_j = E$. Observe that

$$\sum_{k\in E} a(k) = \sum_{j=1}^{m} c(j)\mu\left(F_j\right). \tag{6.47}$$

Furthermore, for any $k \in E$ there exists an unique $j_k \in \{1,2,\cdots,m\}$ such that $k \in F_{j_k}$ and therefore

$$a(k) = \sum_{j=1}^{m} c(j)\chi_{F_j}(k),$$

then

$$a^*(k) = \sum_{j=1}^{m} c(j)\chi_{[1,\mu(F_j)]}(k).$$

Therefore from (6.47), we have

$$\begin{aligned}\sum_{k\in E} a(k) &= \sum_{j=1}^{m} c(j)\mu(F_j)\\ &\le \sum_{j=1}^{m} c(j) \sum_{k=1}^{\mu(E)} \chi_{[1,\mu(F_j)]}(k)\\ &= \sum_{k=1}^{\mu(E)} \sum_{j=1}^{m} c(j)\chi_{[1,\mu(F_j)]}(k)\\ &= \sum_{k=1}^{\mu(E)} a^*(k).\end{aligned}$$

That is

$$\sum_{k\in E} a(k) \le \sum_{k=1}^{\infty} a^*(k), \tag{6.48}$$

and since $n \in \mathbb{N}$ was arbitrary, we conclude

$$\sum_{k=1}^{\infty} a(k) \le \sum_{k=1}^{\infty} a^*(k).$$

On the other hand, employing inequality (6.48) we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{N}} a(k)b(k) &= \sum_{k\in\mathbb{N}} \left(\sum_{j=1}^{m} c(j)\chi_{F_j}(k) \right) b(k) \\
&= \sum_{j=1}^{m} c(j) \sum_{k\in\mathbb{N}} b(k)\chi_{F_j}(k) \\
&= \sum_{j=1}^{m} c(j) \sum_{k\in F_j} b(k) \\
&\le \sum_{j=1}^{m} c(j) \sum_{k=1}^{\mu(F_j)} b^*(k) \\
&\le \sum_{k=1}^{\infty} \sum_{j=1}^{m} c(j)\chi_{[1,\mu(F_j)]}(k) b^*(k) \\
&= \sum_{k=1}^{\infty} a^*(k)b^*(k).
\end{aligned}$$

Hence

$$\sum_{k\in\mathbb{N}} a(k)b(k) \le \sum_{k=1}^{\infty} a^*(k)b^*(k),$$

which ends the proof. □

We now show a Hölder type inequality for Lorentz sequence spaces.

Theorem 6.37. *Let* $a = \{a(k)\} \in \ell_{(p,t)}$ *and* $b = \{b(k)\} \in \ell_{(q,r)}$ *where* $\frac{1}{t} + \frac{1}{r} = 1$, *then*

$$\sum_{k=1}^{\infty} \left| a(k)b(k) \right| \le \|a\|_{(p,t)}^{s} \|b\|_{(q,r)}^{s}. \tag{6.49}$$

Proof. By virtue of Theorem 6.36 and by Hölder's inequality we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}} \left| a(k)b(k) \right| &\le \sum_{k=1}^{\infty} a^*(k)b^*(k) \\
&= \sum_{k=1}^{\infty} k^{\frac{1}{p}-\frac{1}{t}} a^*(k) k^{\frac{1}{q}-\frac{1}{r}} b^*(k)
\end{aligned}$$

$$\leq \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p}-\frac{1}{t}} a^*(k) \right)^t \right)^{1/t} \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{q}-\frac{1}{r}} b^*(k) \right)^r \right)^{1/r}$$
$$= \left(\sum_{k=1}^{\infty} (a^*(k))^t k^{t/p-1} \right)^{1/t} \left(\sum_{k=1}^{\infty} (b^*(k))^r k^{r/q-1} \right)^{1/r},$$

from which (6.49) follows. □

6.9 Problems

6.38. Let μ denote the Lebesgue measure on the σ-algebra $\mathscr{B}$ of Borel set of $[0,1]$ and put $f(x) = x$ and $g(x) = 1-x$ for $x \in [0,1]$. Express $\|f\|_{(p,q)}$ and $\|g\|_{(p,q)}$ for $p \in [0,1]$, $q \in [1,\infty)$ in terms of the gamma function.

Hint. First express these numbers in terms of the Beta function.

6.39. Suppose that μ denotes the Lebesgue measure on the σ-algebra $\mathscr{B}$ of Borel subset of $\mathbb{R}$ and for each $a > 0$ put $f_a(x) = \mathrm{e}^{-a|x|}$ and $g_a(x) = \mathrm{e}^{-ax^2}$ for $x \in \mathbb{R}$.

(a) Calculate $\|f_a\|_{(p,q)}$ for $a > 0$ and $p,q \in [1,\infty]$.
(b) Calculate $\|g_a\|_{(p,q)}$ for $a > 0$ and $p,q \in [1,\infty]$.

6.40. On $\mathbb{R}^n$, let $\delta^\varepsilon(f)(x) = f(\varepsilon x)$, $\varepsilon > 0$, be the dilation operator. Show that $\left\|\delta^\varepsilon(f)\right\|_{(p,q)} = \varepsilon^{-n/p}\|f\|_{(p,q)}$.

6.41. Show that

$$\sup_{t>0} t^{1/p} f^*(t) \leq \|f^*\|_p.$$

6.42. Let $(X,\mathscr{A},\mu)$ be a measure space, let p and q be two extended real numbers in $[1,\infty]$, and let p' and q' their conjugate exponents. If $f \in L_{(p,q)}(X,\mathscr{A},\mu)$ and $g \in L_{(p',q')}(X,\mathscr{A},\mu)$, prove that

$$\int_0^\infty f^{**}(t) g^{**}(t)\,\mathrm{d}t \leq \|f\|_{pq}\|g\|_{p'q'}.$$

6.43. Let f and g be nonnegative μ-measurable functions on $\mathbb{R}^+$. Prove that

$$\int_{\mathbb{R}} fg\,\mathrm{d}\mu \leq \frac{1}{2}\int_0^\infty f^{**}(t) g^{**}(t)\,\mathrm{d}t$$

the constant $1/2$ is optimal.

6.44. Let $1 < p_0 < \infty$ and assume that $f \in L(p,\infty)(\mathbb{R}^+)$ for every $1 < p \le p_0 < \infty$. Prove that

$$\lim_{p\to 1} \|f\|_{p\infty} = \|f\|_1.$$

6.45. Let $1 < p_0 < \infty$ and assume that $\varphi \in L_{(p,1)}(\mathbb{R}^+)$ for every $1 < p_0 \le p < \infty$. Prove that

$$\lim_{p\to\infty} \frac{1}{p}\|\varphi\|_{(p,1)} = \|\varphi\|_\infty.$$

6.46. Let $f \in L_p(X, \mathscr{A}, \mu)$. Prove that

$$\|Mf\|_{L_{(p,\infty)}} \le C\|f\|_p$$

where C is a positive constant and M stands for the Hardy-Littlewood maximal operator (9.2).

6.47. Let $f \in L_1(X, \mathscr{A}, \mu)$. Prove that

$$\|I_\alpha f\|_{L_{\left(\frac{n}{n-\alpha},\infty\right)}} \le C\|f\|_1$$

where C is a positive constant and I^α stands for the Riesz potential operator (11.9).

6.48. We say that $h \in \frac{L^q}{\log^\alpha L}(\Omega)$, $\alpha > 0$, if

$$\int_\Omega \frac{|h(x)|^q}{\log^\alpha(\mathrm{e} + |h(x)|)}\,\mathrm{d}x < \infty.$$

Show that $h \in \frac{L}{\log^\alpha L}[(0, 1/\mathrm{e})]$ if and only if $\alpha + \beta > 1$, where

$$h(x) = \frac{1}{x|\log x|^\beta}, \quad \beta \in \mathbb{R}.$$

6.10 Notes and Bibliographic References

The Lorentz spaces were introduced in Lorentz [44, 45]. It seems that the first expository paper on the topic is Hunt [34].

The duality problem regarding Lorentz spaces was investigated in Cwikel [10], Cwikel and Fefferman [11, 12]

The space $L_1 + L_\infty$ was studied in Gould [21] and Luxemburg and Zaanen [47].

The spaces $L\exp$ and $L\log L$ were introduced independently by Zygmund [85] and Titchmarsh [78, 79].