Chapter 2
Lebesgue Sequence Spaces

Abstract In this chapter, we will introduce the so-called Lebesgue sequence spaces, in the finite and also in the infinite dimensional case. We study some properties of the spaces, e.g., completeness, separability, duality, and embedding. We also examine the validity of Hölder, Minkowski, Hardy, and Hilbert inequality which are related to the aforementioned spaces. Although Lebesgue sequence spaces can be obtained from Lebesgue spaces using a discrete measure, we will not follow that approach and will prove the results in a direct manner. This will highlight some techniques that will be used in the subsequent chapters.

2.1 Hölder and Minkowski Inequalities

In this section we study the Hölder and Minkowski inequality for sums. Due to their importance in all its forms, they are sometimes called the *workhorses of analysis*.

Definition 2.1. The space ℓ_p^n, with $1 \leq p < \infty$, denotes the n-dimensional vector space $\mathbb{R}^n$ for which the functional

$$\|\mathbf{x}\|_{\ell_p^n} = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \tag{2.1}$$

is finite, where $\mathbf{x} = (x_1, \ldots, x_n)$. In the case of $p = \infty$, we define ℓ_∞^n as

$$\|\mathbf{x}\|_{\ell_\infty^n} = \sup_{i \in \{1,\ldots,n\}} |x_i|.$$

$\oslash$

From Lemma 2.4 we obtain in fact that $\|\cdot\|_{\ell_p^n}$ defines a norm in $\mathbb{R}^n$.

R.E. Castillo, H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, CMS Books in Mathematics, DOI 10.1007/978-3-319-30034-4_2

Example 2.2. Let us draw the unit ball for particular values of p for $n = 2$, as in Figs. 2.1, 2.2, and 2.3.

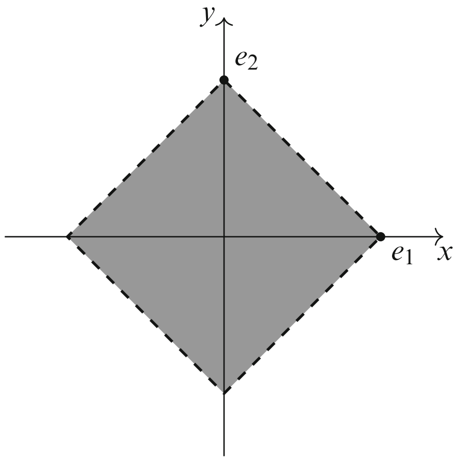

Fig. 2.1 Unit ball for ℓ_1^2

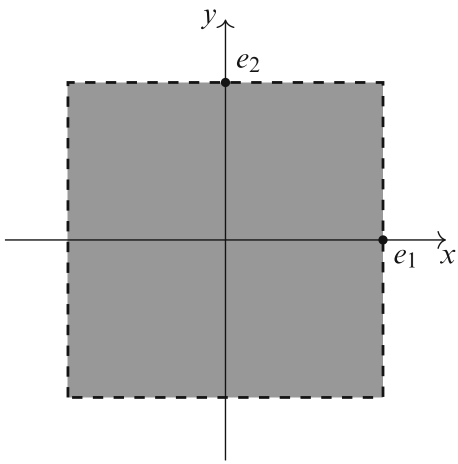

Fig. 2.2 Unit ball for ℓ_∞^2

Lemma 2.3 (Hölder's inequality). *Let p and q be real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q} . \tag{2.2}$$

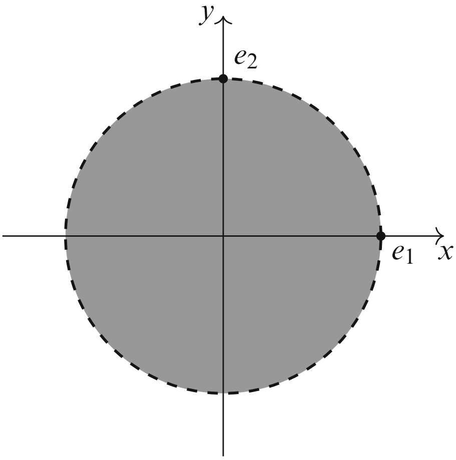

Fig. 2.3 Unit ball for ℓ_2^2

for $x_k,\, y_k \in \mathbb{R}$.

Proof. Let us take

$$\alpha = \frac{|x_k|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}}, \quad \beta = \frac{|y_k|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}.$$

By Young's inequality (1.15) we get

$$\frac{|x_k||y_k|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \le \frac{1}{p}\frac{|x_k|^p}{\sum_{k=1}^n |x_k|^p} + \frac{1}{q}\frac{|y_k|^q}{\sum_{k=1}^n |y_k|^q}.$$

Termwise summation gives

$$\frac{\sum_{k=1}^n |x_k||y_k|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \le \frac{1}{p} + \frac{1}{q}$$

and from this we get

$$\sum_{k=1}^n |x_k y_k| \le \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}.$$

□

We can interpret the inequality (2.2) in the following way: If $\mathbf{x} \in \ell_p^n$ and $\mathbf{y} \in \ell_q^n$ then $\mathbf{x} \odot \mathbf{y} \in \ell_1^n$ where $\odot$ stands for component-wise multiplication and moreover

$$\|\mathbf{x} \odot \mathbf{y}\|_{\ell_1^n} \le \|\mathbf{x}\|_{\ell_p^n} \|\mathbf{y}\|_{\ell_q^n}.$$

Lemma 2.4 (Minkowski's inequality). *Let* $p \geq 1$*, then*

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \tag{2.3}$$

for $x_k, y_k \in \mathbb{R}$.

Proof. We have

$$\begin{aligned}\sum_{k=1}^{n} |x_k + y_k|^p &= \sum_{k=1}^{n} |x_k + y_k|^{p-1}|x_k + y_k| \\ &\leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}\end{aligned}$$

By Lemma 2.3 we get

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \left[\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}\right]\left(\sum_{k=1}^{n} |x_k + y_k|^{(p-1)q}\right)^{1/q}.$$

Since $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $p = (p-1)q$, from which

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \left[\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}\right]\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q},$$

then

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p},$$

which entails (2.3). □

2.2 Lebesgue Sequence Spaces

We now want to extend the n-dimensional ℓ_p^n space into an infinite dimensional sequence space in a natural way.

Definition 2.5. The *Lebesgue sequence space* (also known as *discrete Lebesgue space*) with $1 \leq p < \infty$, denoted by ℓ_p or sometimes also by $\ell_p(\mathbb{N})$, stands for the set of all sequences of real numbers $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$. We endow the Lebesgue sequence space with the norm,

$$\|\mathbf{x}\|_{\ell_p} = \|\{x_n\}_{n\in\mathbb{N}}\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, \tag{2.4}$$

where $\mathbf{x} \in \ell_p$. ⊘

We leave as Problem 2.24 to show that this is indeed a norm in ℓ_p, therefore $(\ell_p, \|\cdot\|_{\ell_p})$ is a normed space.

We will denote by $\mathbb{R}^\infty$ the set of all sequences of real numbers $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$.

Example 2.6. The *Hilbert cube* $\mathfrak{H}$ is defined as the set of all real sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $0 \le x_n \le 1/n$, i.e.

$$\mathfrak{H} := \{\mathbf{x} \in \mathbb{R}^\infty : 0 \le x_n \le 1/n\}.$$

By the hyper-harmonic series we have that the Hilbert cube is not contained in ℓ_1 but is contained in all ℓ_p with $p > 1$. ⊘

Let us show that ℓ_p is a subspace of the space $\mathbb{R}^\infty$. Let $\mathbf{x}$ and $\mathbf{y}$ be elements of ℓ_p and α, β be real numbers. By Lemma 2.4 we have that

$$\left(\sum_{k=1}^{n} |\alpha x_k + \beta y_k|^p\right)^{1/p} \le |\alpha| \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + |\beta| \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}. \tag{2.5}$$

Taking limits in (2.5), first to the right-hand side and after to the left-hand side, we arrive at

$$\left(\sum_{k=1}^{\infty} |\alpha x_k + \beta y_k|^p\right)^{1/p} \le |\alpha| \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + |\beta| \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}, \tag{2.6}$$

and this shows that $\alpha\mathbf{x} + \beta\mathbf{y}$ is an element of ℓ_p and therefore ℓ_p is a subspace of $\mathbb{R}^\infty$.

The Lebesgue sequence space ℓ_p is a complete normed space for all $1 \le p \le \infty$. We first prove for the case of finite exponent and for the case of $p = \infty$ it will be shown in Theorem 2.11.

Theorem 2.7. *The space $\ell_p(\mathbb{N})$ is a Banach space when $1 \le p < \infty$.*

Proof. Let $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell_p(\mathbb{N})$, where we take the sequence $\mathbf{x}_n$ as $\mathbf{x}_n = (x_1^{(n)}, x_2^{(n)}, \ldots)$. Then for any $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that if $n, m \ge n_0$, then $\|\mathbf{x}_n - \mathbf{x}_m\|_{\ell_p} < \varepsilon$, i.e.

$$\left(\sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p\right)^{1/p} < \varepsilon, \tag{2.7}$$

whenever $n, m \ge n_0$. From (2.7) it is immediate that for all $j = 1, 2, 3, \ldots$

$$|x_j^{(n)} - x_j^{(m)}| < \varepsilon, \tag{2.8}$$

whenever $n, m \ge n_0$. Taking a fixed j from (2.8) we see that $(x_j^{(1)}, x_j^{(2)}, \ldots)$ is a Cauchy sequence in $\mathbb{R}$, therefore there exists $x_j \in \mathbb{R}$ such that $\lim_{m\to\infty} x_j^{(m)} = x_j$.

Let us define $\mathbf{x} = (x_1, x_2, \ldots)$ and show that $\mathbf{x}$ is in ℓ_p and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. From (2.7) we have that for all n, $m \geq n_0$

$$\sum_{j=1}^{k} |x_j^{(m)} - x_j^{(n)}|^p < \varepsilon^p, \quad k = 1, 2, 3, \ldots$$

from which

$$\sum_{j=1}^{k} |x_j - x_j^{(n)}|^p = \sum_{j=1}^{k} |\lim_{m\to\infty} x_j^{(m)} - x_j^{(n)}|^p \leq \varepsilon^p,$$

whenever $n \geq n_0$, This shows that $\mathbf{x} - \mathbf{x}_n \in \ell_p$ and we also deduce that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. Finally in virtue of the Minkowski inequality we have

$$\begin{aligned} \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} &= \left(\sum_{j=1}^{\infty} |x_j^{(n)} + x_j - x_j^{(n)}|^p \right)^{1/p} \\ &\leq \left(\sum_{j=1}^{\infty} |x_j^{(n)}|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |x_j - x_j^{(n)}|^p \right)^{1/p}, \end{aligned}$$

which shows that $\mathbf{x}$ is in $\ell_p(\mathbb{N})$ and this completes the proof. □

The next result shows that the Lebesgue sequence spaces are separable when the exponent p is finite, i.e., the space ℓ_p admits an enumerable dense subset.

Theorem 2.8. *The space $\ell_p(\mathbb{N})$ is separable whenever $1 \leq p < \infty$.*

Proof. Let M be the set of all sequences of the form $\mathbf{q} = (q_1, q_2, \ldots, q_n, 0, 0, \ldots)$ where $n \in \mathbb{N}$ and $q_k \in \mathbb{Q}$. We will show that M is dense in ℓ_p. Let $\mathbf{x} = \{x_k\}_{k\in\mathbb{N}}$ be an arbitrary element of ℓ_p, then for $\varepsilon > 0$ there exists n which depends on ε such that

$$\sum_{k=n+1}^{\infty} |x_k|^p < \varepsilon^p/2.$$

Now, since $\overline{\mathbb{Q}} = \mathbb{R}$, we have that for each x_k there exists a rational q_k such that

$$|x_k - q_k| < \frac{\varepsilon}{\sqrt[p]{2^n}},$$

then

$$\sum_{k=1}^{n} |x_k - q_k|^p < \varepsilon^p/2,$$

which entails

$$\|\mathbf{x} - \mathbf{q}\|_{\ell_p}^p = \sum_{k=1}^{n} |x_k - q_k|^p + \sum_{k=n+1}^{\infty} |x_k|^p < \varepsilon^p,$$

and we arrive at $\|\mathbf{x} - \mathbf{q}\|_{\ell_p} < \varepsilon$. This shows that M is dense in ℓ_p, implying that ℓ_p is separable since M is enumerable. □

With the notion of Schauder basis (recall the definition of Schauder basis in Definition B.3), we now study the problem of duality for the Lebesgue sequence space.

Theorem 2.9. *Let* $1 < p < \infty$*. The dual space of* $\ell_p(\mathbb{N})$ *is* $\ell_q(\mathbb{N})$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. A Schauder basis of ℓ_p is $e_k = \{\delta_{kj}\}_{j\in\mathbb{N}}$ where $k \in \mathbb{N}$ and δ_{kj} stands for the Kronecker delta, i.e., $\delta_{kj} = 1$ if $k = j$ and 0 otherwise. If $f \in (\ell_p)^*$, then $f(\mathbf{x}) = \sum_{k\in\mathbb{N}} \alpha_k f(e_k)$, $\mathbf{x} = \{\alpha_k\}_{k\in\mathbb{N}}$. We define $T(f) = \{f(e_k)\}_{k\in\mathbb{N}}$. We want to show that the image of T is in ℓ_q, for that we define for each n, the sequence $\mathbf{x}^n = (\xi_k^{(n)})_{k=1}^{\infty}$ with

$$\xi_k^{(n)} = \begin{cases} \frac{|f(e_k)|^q}{f(e_k)} & \text{if} \quad k \leq n \text{ and } f(e_k) \neq 0, \\ 0 & \text{if} \quad k > n \text{ or } f(e_k) = 0. \end{cases}$$

Then

$$f(\mathbf{x}^n) = \sum_{k\in\mathbb{N}} \xi_k^{(n)} f(e_k) = \sum_{k=1}^{n} |f(e_k)|^q.$$

Moreover

$$\begin{aligned} f(\mathbf{x}^n) &\leq \|f\| \|\mathbf{x}^n\|_p \\ &= \|f\| \left(\sum_{k=1}^{n} |\xi_k^{(n)}|^p \right)^{\frac{1}{p}} \\ &= \|f\| \left(\sum_{k=1}^{n} |f(e_k)|^{qp-p} \right)^{\frac{1}{p}} \\ &= \|f\| \left(\sum_{k=1}^{n} |f(e_k)|^q \right)^{\frac{1}{p}}, \end{aligned}$$

from which

$$\begin{aligned} \left(\sum_{k=1}^{n} |f(e_k)|^q \right)^{1-\frac{1}{p}} &= \left(\sum_{k=1}^{n} |f(e_k)|^q \right)^{\frac{1}{q}} \\ &\leq \|f\|. \end{aligned}$$

Taking $n \to \infty$, we obtain

$$\left(\sum_{k=1}^{\infty} |f(e_k)|^q \right)^{\frac{1}{q}} \leq \|f\|$$

where $\{f(e_k)\}_{k\in\mathbb{N}} \in \ell_q$.

Now, we affirm that:

(i) T is onto. In effect given $b = (\beta_k)_{k\in\mathbb{N}} \in \ell_q$, we can associate a bounded linear functional $g \in (\ell_p)^*$, given by $g(\mathbf{x}) = \sum_{k=1}^{\infty} \alpha_k \beta_k$ with $\mathbf{x} = (\alpha_k)_{k\in\mathbb{N}} \in \ell_p$ (the boundedness is deduced by Hölder's inequality). Then $g \in (\ell_p)^*$.

(ii) T is 1-1. This is almost straightforward to check.

(iii) T is an isometry. We see that the norm of f is the ℓ_q norm of Tf

$$\begin{aligned} |f(\mathbf{x})| &= \left|\sum_{k\in\mathbb{N}} \alpha_k f(e_k)\right| \\ &\le \left(\sum_{k\in\mathbb{N}} |\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} |f(e_k)|^q\right)^{\frac{1}{q}} \\ &= \|x\| \left(\sum_{k\in\mathbb{N}} |f(e_k)|^q\right)^{\frac{1}{q}}. \end{aligned}$$

Taking the supremum over all x of norm 1, we have that

$$\|f\| \le \left(\sum_{k\in\mathbb{N}} |f(e_k)|^q\right)^{\frac{1}{q}}.$$

Since the other inequality is also true, we can deduce the equality

$$\|f\| = \left(\sum_{k\in\mathbb{N}} |f(e_k)|^q\right)^{\frac{1}{q}},$$

with which we establish the desired isomorphism $f \to \{f(e_k)\}_{k\in\mathbb{N}}$.

□

The ℓ_p spaces satisfy an embedding property, forming a nested sequence of Lebesgue sequences spaces.

Theorem 2.10. *If* $0 < p < q < \infty$, *then* $\ell_p(\mathbb{N}) \subsetneq \ell_q(\mathbb{N})$.

Proof. Let $\mathbf{x} \in \ell_p$, then $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, then $|x_n| < 1$. Now, since $0 < p < q$, then $0 < q - p$ and $|x_n|^{q-p} < 1$ if $n > n_0$, by which $|x_n|^q < |x_n|^p$ if $n > n_0$. Let $M = \max\{|x_1|^{q-p}, |x_2|^{q-p}, \ldots, |x_{n_0}|^{q-p}, 1\}$, then

$$\sum_{n=1}^{\infty} |x_n|^q = \sum_{n=1}^{\infty} |x_n|^p |x_n|^{q-p} < M \sum_{n=1}^{\infty} |x_n|^p < +\infty,$$

implying that $\mathbf{x} \in \ell_q$.

To show that $\ell_p(\mathbb{N}) \neq \ell_q(\mathbb{N})$, we take the following sequence $x_n = n^{-1/p}$ for all $n \in \mathbb{N}$ with $1 \leq p < q \leq \infty$, and since $p < q$, then $\dfrac{q}{p} > 1$. Now we have

$$\sum_{n=1}^{\infty} |x_n|^q = \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} < \infty.$$

The last series is convergent since it is a hyper-harmonic series with exponent bigger than 1, therefore $\mathbf{x} \in \ell_q(\mathbb{N})$. On the other hand

$$\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} \frac{1}{n}$$

and we get the harmonic series, which entails that $\mathbf{x} \notin \ell_p(\mathbb{N})$. □

2.3 Space of Bounded Sequences

The *space of bounded sequences*, denoted by ℓ_∞ or sometimes $\ell_\infty(\mathbb{N})$, is the set of all real bounded sequences $\{x_n\}_{n\in\mathbb{N}}$ (it is clear that ℓ_∞ is a vector space). We will take the norm in this space as

$$\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_{\ell_\infty} = \sup_{n\in\mathbb{N}} |x_n|, \tag{2.9}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$. The verification that (2.9) is indeed a norm is left to the reader.

An almost immediate property of the ℓ_∞-space is its completeness, inheriting this property from the completeness of the real line.

Theorem 2.11. *The space ℓ_∞ is a Banach space.*

Proof. Let $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ_∞, where $\mathbf{x}_n = (x_1^{(n)}, x_2^{(n)}, \ldots)$. Then for any $\varepsilon > 0$ there exists $n_0 > 0$ such that if $m, n \geq n_0$ then

$$\|\mathbf{x}_m - \mathbf{x}_n\|_\infty < \varepsilon.$$

Therefore for fixed j we have that if $m, n \geq n_0$, then

$$|x_j^{(m)} - x_j^{(n)}| < \varepsilon \tag{2.10}$$

resulting that for all fixed j the sequence $(x_j^{(1)}, x_j^{(2)}, \ldots)$ is a Cauchy sequence in $\mathbb{R}$, and this implies that there exists $x_j \in \mathbb{R}$ such that $\lim_{m\to\infty} x_j^{(m)} = x_j$.

Let us define $\mathbf{x} = (x_1, x_2, \ldots)$. Now we want to show that $\mathbf{x} \in \ell_\infty$ and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$.

From (2.10) we have that for $n \geq n_0$, then

$$\left|x_j - x_j^{(n)}\right| = \left|\lim_{n\to\infty} x_j^{(m)} - x_j^{(n)}\right| \leq \varepsilon, \tag{2.11}$$

since $\mathbf{x}_n = \{x_j^{(n)}\}_{j\in\mathbb{N}} \in \ell_\infty$, there exists a real number M_n such that $\left|x_j^{(n)}\right| \leq M_n$ for all j.

By the triangle inequality, we have

$$\left|x_j\right| \leq \left|x_j - x_j^{(n)}\right| + \left|x_j^{(n)}\right| \leq \varepsilon + M_n$$

whenever $n \geq n_0$, this inequality being true for any j. Moreover, since the right-hand side does not depend on j, therefore $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of bounded real numbers, this implies that $\mathbf{x} = \{x_j\}_{j\in\mathbb{N}} \in \ell_\infty$.

From (2.11) we also obtain

$$\|\mathbf{x}_n - \mathbf{x}\|_{\ell_\infty} = \sup_{j\in\mathbb{N}} \left|x_j^{(n)} - x_j\right| < \varepsilon.$$

whenever $n \geq n_0$. From this we conclude that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ and therefore ℓ_∞ is complete. □

The following result shows a "natural" way to introduce the norm in the ℓ_∞ space via a limiting process.

Theorem 2.12. *Taking the norm of Lebesgue sequence space as in* (2.4) *we have that* $\lim_{p\to\infty} \|\mathbf{x}\|_{\ell_p} = \|\mathbf{x}\|_{\ell_\infty}$.

Proof. Observe that $|x_k| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}}$, therefore $|x_k| \leq \|\mathbf{x}\|_{\ell_p}$ for $k = 1,2,3,\ldots,n$, from which

$$\sup_{1\leq k\leq n} |x_k| \leq \|\mathbf{x}\|_{\ell_p},$$

whence

$$\|\mathbf{x}\|_{\ell_\infty} \leq \liminf_{p\to\infty} \|\mathbf{x}\|_{\ell_p}. \tag{2.12}$$

On the other hand, note that

$$\left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n \left(\sup_{1\leq k\leq n} |x_k|\right)^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \|\mathbf{x}\|_{\ell_\infty},$$

then for all $\varepsilon > 0$, there exists N such that

$$\|\mathbf{x}\|_{\ell_p} \leq \left(\sum_{k=1}^N |x_k|^p + \varepsilon\right)^{\frac{1}{p}} \leq \left(\|\mathbf{x}\|_{\ell_\infty}^p N + \varepsilon\right)^{\frac{1}{p}} \leq \|\mathbf{x}\|_{\ell_\infty} \left(N + \frac{\varepsilon}{\|\mathbf{x}\|_{\ell_\infty}^p}\right)^{\frac{1}{p}},$$

therefore

$$\limsup_{p\to\infty} \|\mathbf{x}\|_{\ell_p} \leq \|\mathbf{x}\|_{\ell_\infty}. \tag{2.13}$$

Combining (2.12) and (2.13) results

$$\|\mathbf{x}\|_{\ell_\infty} \leq \liminf_{p\to\infty} \|\mathbf{x}\|_{\ell_p} \leq \limsup_{p\to\infty} \|\mathbf{x}\|_{\ell_p} \leq \|\mathbf{x}\|_{\ell_\infty},$$

and from this we conclude that $\lim_{p\to\infty} \|\mathbf{x}\|_{\ell_p} = \|\mathbf{x}\|_{\ell_\infty}$. □

Now we study the dual space of ℓ_1 which is ℓ_∞.

Theorem 2.13. *The dual space of ℓ_1 is ℓ_∞.*

Proof. For all $\mathbf{x} \in \ell_1$, we can write $\mathbf{x} = \sum_{k=1}^{\infty} \alpha_k e_k$, where $e_k = (\delta_{kj})_{j=1}^{\infty}$ forms a Schauder basis in ℓ_1, since

$$\mathbf{x} - \sum_{k=1}^{n} \alpha_k e_k = (\underbrace{0,\dots,0}_{n}, \alpha_{n+1}, \dots)$$

and

$$\left\| \mathbf{x} - \sum_{k=1}^{n} \alpha_k e_k \right\|_{\ell_1} = \left\| \sum_{k=n+1}^{\infty} \alpha_k e_k \right\|_{\ell_1} \to 0$$

since the series $\sum_{k=1}^{\infty} \alpha_k e_k$ is convergent.

Let us define $T(f) = \{f(e_k)\}_{k\in\mathbb{N}}$, for all $f \in (\ell_1)^*$. Since $f(\mathbf{x}) = \sum_{k\in\mathbb{N}} \alpha_k f(e_k)$, then $|f(e_k) \leq \|f\|$, since $\|e_k\|_{\ell_1} = 1$. In consequence, $\sup_{k\in\mathbb{N}} |f(e_k)| \leq \|f\|$, therefore $\{f(e_k)\}_{k\in\mathbb{N}} \in \ell_\infty$.

We affirm that:

(i) T is onto. In fact, for all $b = \{\beta_k\}_{k\in\mathbb{N}} \in \ell_\infty$, let us define $q : \ell_1 \to \mathbb{R}$ as $g(\mathbf{x}) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k$ if $\mathbf{x} = \{\alpha_k\}_{k\in\mathbb{N}} \in \ell_\infty$. The functional g is bounded and linear since

$$|g(\mathbf{x})| \leq \sum_{k\in\mathbb{N}} |\alpha_k \beta_k| \leq \sup_{k\in\mathbb{N}} |\beta_k| \sum_{k\in\mathbb{N}} |\alpha_k| = \|\mathbf{x}\|_{\ell_1} \cdot \sup_{k\in\mathbb{N}} |\beta_k|,$$

then $g \in (\ell_1)^*$. Moreover, since $g(e_k) = \sum_{j\in\mathbb{N}} \delta_{kj} \beta_j$,

$$T(g) = \{g(e_k)\}_{k\in\mathbb{N}} = \{\beta_k\}_{k\in\mathbb{N}} = b.$$

(ii) T is 1-1. If $Tf_1 = Tf_2$, then $f_1(e_k) = f_2(e_k)$, for all k. Since we have $f_1(\mathbf{x}) = \sum_{k\in\mathbb{N}} \alpha_k f_1(e_k)$ and $f_2(\mathbf{x}) = \sum_{k\in\mathbb{N}} \alpha_k f_2(e_k)$, then $f_1 = f_2$.

(iii) T is an isometry. In fact,

$$\|Tf\|_\infty = \sup_{k\in\mathbb{N}} |f(e_k)| \leq \|f\| \tag{2.14}$$

and

$$|f(\mathbf{x})| = \left|\sum_{k\in\mathbb{N}} \alpha_k f(e_k)\right| \leq \sup_{j\in\mathbb{N}} |f(e_k)| \sum_{k\in\mathbb{N}} |\alpha_k| = \|\mathbf{x}\|_{\ell_1} \sup_{k\in\mathbb{N}} |f(e_k)|.$$

Then

$$\|f\| \leq \sup_{k\in\mathbb{N}} |f(e_k)| = \|Tf\|_\infty. \tag{2.15}$$

Combining (2.14) and (2.15) we get that $\|Tf\|_\infty = \|f\|$. We thus showed that the spaces $(\ell_1)^*$ and ℓ_∞ are isometric.

□

One of the main difference between ℓ_p and ℓ_∞ spaces is the separability issue. The space of bounded sequence ℓ_∞ is not separable, contrasting with the separability of the ℓ_p spaces whenever $1 \leq p < \infty$, see Theorem 2.8.

Theorem 2.14. *The space ℓ_∞ is not separable.*

Proof. Let us take any enumerable sequence of elements of ℓ_∞, namely $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, where we take the sequences in the form

$$\begin{aligned}
\mathbf{x}_1 &= \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_k^{(1)}, \ldots\right)\\
\mathbf{x}_2 &= \left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_k^{(2)}, \ldots\right)\\
\mathbf{x}_3 &= \left(x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \ldots, x_k^{(3)}, \ldots\right)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\mathbf{x}_k &= \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_k^{(k)}, \ldots\right)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

We now show that there exists an element in ℓ_∞ which is at a distance bigger than 1 for all elements of $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, showing the non-separability nature of the ℓ_∞ space.

Let us take $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ as

$$x_n = \begin{cases} 0, & \text{if } |x_n^{(n)}| \geq 1; \\ x_n = x_n^{(n)} + 1, & \text{if } |x_n^{(n)}| < 1. \end{cases}$$

It is clear that $\mathbf{x} \in \ell_\infty$ and $\|\mathbf{x} - \mathbf{x}_n\|_{\ell_\infty} > 1$ for all $n \in \mathbb{N}$, which entails that ℓ_∞ is not separable. □

We now define some subspaces of ℓ_∞, which are widely used in functional analysis, for example, to construct counter-examples.

Definition 2.15. Let $\mathbf{x} = (x_1, x_1, \ldots)$.

By c we denote the subspace of ℓ_∞ such that $\lim_{n\to\infty} x_n$ exists and is finite.

By c_0 we denote the subspace of ℓ_∞ such that $\lim_{n\to\infty} x_n = 0$.

By c_{00} we denote the subspace of ℓ_∞ such that $\operatorname{supp}(\mathbf{x})$ is finite. ⊘

These newly introduced spaces enjoy some interesting properties, e.g., c_0 is the closure of c_{00} in ℓ_∞. For more properties, see Problem 2.20.

2.4 Hardy and Hilbert Inequalities

We now deal with the discrete version of the well-known Hardy inequality.

Theorem 2.16 (Hardy's inequality). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real positive numbers such that $\sum_{n=1}^{\infty} a_n^p < \infty$. Then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

Proof. Let $\alpha_n = \frac{A_n}{n}$ where $A_n = a_1 + a_2 + \cdots + a_n$, i.e., $A_n = n\alpha_n$, then

$$a_1 + a_2 + \cdots + a_n = n\alpha_n, \tag{2.16}$$

from which we get that $a_n = n\alpha_n - (n-1)\alpha_{n-1}$. Let us consider now

$$\begin{aligned}\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n &= \alpha_n^p - \frac{p}{p-1}\left[n\alpha_n - (n-1)\alpha_{n-1}\right]\alpha_n^{p-1}\\ &= \alpha_n^p - \frac{pn}{p-1}\alpha_n\alpha_n^{p-1} + \frac{p(n-1)}{p-1}\alpha_{n-1}\alpha_n^{p-1}.\end{aligned}$$

In virtue of Corollary 1.10 we have

$$\begin{aligned}\frac{p(n-1)}{p-1}\alpha_{n-1}\alpha_n^{p-1} &\leq \frac{p(n-1)}{p-1}\frac{\alpha_{n-1}^p}{p} + \frac{p(n-1)}{p-1}\frac{\alpha_n^{q(p-1)}}{q}\\ &= \frac{n-1}{p-1}\alpha_{n-1}^p + \frac{p(n-1)}{p-1}\left(1-\frac{1}{p}\right)\alpha_n^p\\ &= \frac{n-1}{p-1}\alpha_{n-1}^p + (n-1)\alpha_n^p,\end{aligned}$$

therefore

$$\begin{aligned}\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n &\leq \alpha_n^p - \frac{pn}{p-1}\alpha_n^p + \frac{n-1}{p-1}\alpha_{n-1}^p + (n-1)\alpha_n^p\\ &= \frac{p\alpha_n^p - \alpha_n^p - pn\alpha_n^p}{p-1} + \frac{(n-1)\alpha_{n-1}^p + (p-1)(n-1)\alpha_n^p}{p-1}\\ &= \frac{p\alpha_n^p - \alpha_n^p - pn\alpha_n^p + (n-1)\alpha_{n-1}^p + (pn-p-n+1)\alpha_n^p}{p-1}\\ &= \frac{1}{p-1}\left[(n-1)\alpha_{n-1}^p - n\alpha_n^p\right],\end{aligned}$$

from which

$$\begin{aligned}\sum_{n=1}^{N}\alpha_n^p-\frac{p}{p-1}\sum_{n=1}^{N}\alpha_n^{p-1}a_n &\leq \frac{1}{p-1}\sum_{n=1}^{N}\left[(n-1)\alpha_{n-1}^p-n\alpha_n^p\right]\\ &=\frac{1}{p-1}\left[-\alpha_1^p+\alpha_1^p-2\alpha_2^p+\cdots-N\alpha_N^p\right]\\ &=-\frac{N\alpha_N^p}{p-1}\leq 0.\end{aligned}$$

Then

$$\sum_{n=1}^{N}\alpha_n^p\leq\frac{p}{p-1}\sum_{n=1}^{N}\alpha_n^{p-1}a_n.$$

By Hölder's inequality we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha_n^p &\leq \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\alpha_n^{q(p-1)}\right)^{\frac{1}{q}}\\ &=\frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\alpha_n^p\right)^{\frac{1}{q}},\end{aligned}$$

then

$$\left(\sum_{n=1}^{\infty}\alpha_n^p\right)^{1-\frac{1}{q}}\leq\frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}$$

and this implies

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{\infty}a_k\right)^p\leq\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p.$$

□

We now want to study the so-called Hilbert inequality. We need to remember some basic facts about complex analysis, namely

$$\frac{\pi}{\sin(\pi z)}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z+n}+\frac{1}{z-n}\right). \tag{2.17}$$

Let us consider the function

$$f(z)=\frac{1}{\sqrt[p]{z}(z+1)}\qquad(p>1)$$

defined in the region $D_1=\{z\in\mathbb{C}:0<|z|<1\}$. We want to obtain the Laurent expansion. In fact, if $|z|<1$, then

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^n z^n,$$

therefore

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{n-\frac{1}{p}}. \tag{2.18}$$

By the same reasoning, let us consider

$$g(z) = \frac{1}{z^{1+\frac{1}{p}}\left(1+\frac{1}{z}\right)}$$

defined in the region $D_2 = \{z \in \mathbb{C} : |z| > 1\}$. Since $\left|\frac{1}{z}\right| < 1$, then

$$\frac{1}{1+\frac{1}{z}} = \frac{1}{1-(-\frac{1}{z})} = \sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}(-1)^n z^{-n}.$$

Therefore

$$g(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1-\frac{1}{p}}. \tag{2.19}$$

We now obtain some auxiliary inequality before showing the validity of the Hilbert inequality (2.20).

Theorem 2.17 *For each positive number m and for all real $p > 1$ we have*

$$\sum_{n=1}^{\infty}\frac{m^{\frac{1}{p}}}{n^{\frac{1}{p}}(m+n)} \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}.$$

Proof. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{m^{\frac{1}{p}}}{n^{\frac{1}{p}}(m+n)} &\leq \int_0^{\infty}\frac{m^{\frac{1}{p}}}{x^{\frac{1}{p}}(m+x)}\,\mathrm{d}x \\
&= \int_0^{\infty}\frac{\mathrm{d}z}{z^{\frac{1}{p}}(1+z)} \\
&= \int_0^{1}\frac{\mathrm{d}z}{z^{\frac{1}{p}}(1+z)} + \int_1^{\infty}\frac{\mathrm{d}z}{z^{1+\frac{1}{p}}\left(1+\frac{1}{z}\right)}.
\end{aligned}$$

By (2.18) and (2.19) we deduce that

$$\sum_{n=1}^{\infty}\frac{m^{\frac{1}{p}}}{n^{\frac{1}{p}}(m+n)} \leq \int_0^{1}\left(\sum_{n=0}^{\infty}(-1)^n z^{n-\frac{1}{p}}\right)\mathrm{d}z + \int_1^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n z^{-n-1-\frac{1}{p}}\right)\mathrm{d}z$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(-1)^n \int_0^1 z^{n-\frac{1}{p}}\,\mathrm{d}z + \sum_{n=0}^{\infty}(-1)^n \int_1^{\infty} z^{-n-1-\frac{1}{p}}\,\mathrm{d}z \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n-\frac{1}{p}+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{1}{p}+n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{p}-n} + p + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{p}+n} \\
&= p + \sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{\frac{1}{p}-n} + \frac{1}{\frac{1}{p}+n} \right) \\
&= \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}.
\end{aligned}
$$

This last one is obtained by (2.17) with $z = \frac{1}{p}$. □

Remark 2.18. In fact the proof of Theorem 2.17 is a two line proof if we remember that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,\mathrm{d}x = B(\alpha,\beta)$$

and the fact that $B(1-\alpha,\alpha) = \frac{\pi}{\sin\pi\alpha}$, $0 < \alpha < 1$, see Appendix C. ⊘

Before stating and proving the Hilbert inequality we need to digress into the concept of double series. Let $\{x_{k,j}\}_{j,k\in\mathbb{N}}$ be a double sequence, viz. a real-valued function $x : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$. We say that a number L is the limit of the double sequence, denoted by

$$\lim_{k,j\to\infty} x_{k,j} = L,$$

if, for all $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that

$$|x_{k,j} - L| < \varepsilon$$

whenever $k > n$ and $j > n$. We can now introduce the notion of double series using the known construction for the series, namely

$$\sum_{k,j=1}^{\infty} x_{k,j} = \Sigma$$

if there exists the double limit

$$\lim_{k,j\to\infty} \Sigma_{k,j} = \Sigma$$

where $\Sigma_{k,j}$ is the rectangular partial sum given by

$$\Sigma_{k,j} = \sum_{m=1}^{k} \sum_{n=1}^{j} x_{m,n}.$$

A notion related to the double series is the notion of iterated series, given by

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{k,j} \right) \quad \text{and} \quad \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} x_{k,j} \right).$$

We can visualize the iterated series in the following way. We first represent the double sequence as numbers in an infinite rectangular array and then sum by lines and by columns in the following way:

$$\begin{array}{cccl}
x_{1,1} & x_{1,2} & x_{1,3} & \cdots \to \sum_{j=1}^{\infty} x_{1,j} =: L_1 \\
x_{2,1} & x_{2,2} & x_{2,3} & \cdots \to \sum_{j=1}^{\infty} x_{2,j} =: L_2 \\
x_{3,1} & x_{3,2} & x_{3,3} & \cdots \to \sum_{j=1}^{\infty} x_{2,j} =: L_3 \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
C_1 := \sum_{k=1}^{\infty} x_{k,1} & C_2 := \sum_{k=1}^{\infty} x_{k,2} & C_3 := \sum_{k=1}^{\infty} x_{k,3} &
\end{array}$$

and now the iterated series are given by $\sum_{j=1}^{\infty} C_j$ and $\sum_{k=1}^{\infty} L_k$.

It is necessary some caution when dealing with iterated series since the equality $\sum_{j=1}^{\infty} C_j = \sum_{k=1}^{\infty} L_k$ is in general not true even if the series converges, as the following example shows

$$\begin{array}{cccccl}
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \cdots \to 0 \\
0 & \frac{3}{4} & -\frac{3}{4} & 0 & 0 & \cdots \to 0 \\
0 & 0 & \frac{7}{8} & -\frac{7}{8} & 0 & \cdots \to 0 \\
0 & 0 & 0 & \frac{15}{16} & -\frac{15}{16} & \cdots \to 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} &
\end{array}$$

and clearly the obtained series are different. Fortunately we have a Fubini type theorem for series which states that when a double series is absolutely convergent then the double series and the iterated series are the same, i.e.

$$\sum_{k,j=1}^{\infty} x_{k,j} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{k,j} \right) = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} x_{k,j} \right).$$

Not only that, it is also possible to show a stronger result, that if the terms of an absolutely convergent double series are permuted in any order as a simple series, their sum tends to the same limit.

Theorem 2.19 (Hilbert's inequality). *Let $p,q>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$ and $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be sequences of nonnegative numbers such that $\sum_{m=1}^{\infty} a_m^p$ and $\sum_{n=1}^{\infty} b_n^q$ are convergent. Then*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{2.20}$$

Proof. Using Hölder's inequality and Proposition 2.17 we get

$$\begin{aligned}
&\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \\
&= \sum_{m,n=1}^{\infty} \frac{m^{\frac{1}{pq}}}{n^{\frac{1}{pq}}} \frac{a_m}{(m+n)^{\frac{1}{p}}} \frac{n^{\frac{1}{pq}}}{m^{\frac{1}{pq}}} \frac{b_n}{(m+n)^{\frac{1}{q}}} \\
&\le \left(\sum_{m,n=1}^{\infty} \left(\frac{m^{\frac{1}{q}}}{n^{\frac{1}{q}}(m+n)}\right) a_m^p\right)^{\frac{1}{p}} \left(\sum_{m,n=1}^{\infty} \left(\frac{n^{\frac{1}{p}}}{m^{\frac{1}{p}}(m+n)}\right) b_n^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{m^{\frac{1}{q}}}{n^{\frac{1}{q}}(m+n)}\right) a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n^{\frac{1}{p}}}{m^{\frac{1}{p}}(m+n)}\right) b_n^q\right)^{\frac{1}{q}} \\
&\le \left(\sum_{m=1}^{\infty} \frac{\pi}{\sin\frac{\pi}{q}} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{\pi}{\sin\frac{\pi}{p}} b_n^q\right)^{\frac{1}{q}} \\
&\le \left(\sum_{m=1}^{\infty} \frac{\pi}{\sin\frac{\pi}{p}} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{\pi}{\sin\frac{\pi}{p}} b_n^q\right)^{\frac{1}{q}} \\
&= \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{\frac{1}{p}} \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{\frac{1}{q}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}} \\
&= \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}},
\end{aligned}$$

which shows the result. □

2.5 Problems

2.20. Prove the following properties of the subspaces of ℓ_∞ introduced in Definition 2.15

(a) The space c_0 is the closure of c_{00} in ℓ_∞.
(b) The space c and c_0 are Banach spaces.
(c) The space c_{00} is not complete.

2.21. Show that (s,ρ) is a complete metric space, where s is the set of all sequences $\mathbf{x}=(x_1,x_2,\dots)$ and ρ is given by

$$\rho(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|x_k-y_k|}{1+|x_k-yk|}.$$

2.22. Let $\ell_p(\mathbf{w})$, $p\geq 1$ be the set of all real sequences $\mathbf{x}=(x_1,x_2,\dots)$ such that

$$\sum_{k=1}^{\infty}|x_k|^p w_k<\infty$$

where $\mathbf{w}=(w_1,w_2,\dots)$ and $w_k>0$. Does $\mathscr{N}:\ell_p(\mathbf{w})\longrightarrow\mathbb{R}$ given by

$$\mathscr{N}(\mathbf{x}):=\left(\sum_{k=1}^{\infty}|x_k|^p w_k\right)^{\frac{1}{p}}$$

defines a norm in $\ell_p(\mathbf{w})$?

2.23. As in the case of Example 2.2, draw the unit ball for ℓ_1^3, ℓ_∞^3, and ℓ_2^3.

2.24. Prove that (2.4) defines a norm in the space $\ell_p(\mathbb{N})$.

2.25. Prove the *Cauchy-Bunyakovsky-Schwarz inequality*

$$\left(\sum_{i=1}^{n}x_iy_i\right)^2\leqslant\left(\sum_{i=1}^{n}x_i^2\right)\left(\sum_{i=1}^{n}y_i^2\right)$$

without using Jensen's inequality. This inequality is sometimes called *Cauchy*, *Cauchy-Schwarz* or *Cauchy-Bunyakovsky*.
Hint: Analyze the quadratic form $\sum_{i=1}^{n}(x_iu+y_iv)^2=u^2\sum_{i=1}^{n}x_i^2+2uv\sum_{i=1}^{n}x_iy_i+v^2\sum_{i=1}^{n}y_i^2$.

2.26. Let $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ be sequences of real numbers such that

$$k=\sum_{n=-\infty}^{\infty}|a_n|<\infty \quad\text{and}\quad \sum_{m=-\infty}^{\infty}|b_m|^p<\infty$$

where $p>1$. Let $C_n=\sum_{m=-\infty}^{\infty}a_{n-m}b_m$. Prove that

(a) $|C_n|\leq k^{1/q}\left(\sum_{m=-\infty}^{\infty}|a_{n-m}||b_m|^p\right)^{1/p}$ where $\frac{1}{p}+\frac{1}{q}=1$.
(b) $\left(\sum_{n=-\infty}^{\infty}|C_n|^p\right)^{1/p}\leq k\left(\sum_{n=-\infty}^{\infty}|b_n|^p\right)^{1/p}$.

2.27. If $a_n>0$ for $n=1,2,3,\dots$ show that

$$\sum_{n=1}^{\infty}\sqrt[n]{a_1a_2\cdots a_n}\leq \mathrm{e}\sum_{n=1}^{\infty}a_n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_k \geq \cdots \geq a_n \geq 0$ and $\alpha \geq \beta > 0$. Demonstrate that

$$\left(\sum_{k=1}^{n} a_k^{\alpha}\right)^{1/\alpha} \leq \left(\sum_{k=1}^{n} a_k^{\beta}\right)^{1/\beta}.$$

2.28. Use Theorem 10.5 to show the Theorem 2.16.
Hint: Choose a sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$. Consider $A_N = \sum_{n=1}^{N} a_n$ and define $f = \sum_{n=1}^{\infty} a_n \chi_{(n-1,n)}$.

2.29. Demonstrate that ℓ_1 is not the dual space of ℓ_∞.

2.30. Show that

$$\|\mathbf{x}\|_{\ell_q} \leq \|\mathbf{x}\|_{\ell_p} \tag{2.21}$$

whenever $1 \leq p < q < \infty$.
Hint: First, show the inequality (2.21) when $\|\mathbf{x}\|_{\ell_p} \leq 1$. Use that result and the homogeneity of the norm to get the general case.

2.6 Notes and Bibliographic References

The history of Hölder's inequality can be traced back to Hölder [32] but the paper of Rogers [61] preceded the one from Hölder just by one year, for the complete history see Maligranda [48].

The Minkowski inequality is due to Minkowski [51] but it seems that the classical approach to the Minkowski inequality via Hölder's inequality is due to Riesz [58].

The Hardy inequality (Theorem 2.16) appeared in Hardy [26] as a generalization of a tool to prove a certain theorem of Hilbert.

According to Hardy, Littlewood, and Pólya [30], the Hilbert inequality (Theorem 2.19) was included by Hilbert for $p = 2$ in his lectures, and it was published by Weyl [82], the general case $p > 1$ appeared in Hardy [27].

The Cauchy-Bunyakovsky-Schwarz inequality, which appears in Problem 2.25, was first proved by Cauchy [6].