Piezoelectric Materials

Vicente Lopes Jr. and Clayton Rodrigo Marqui

Abstract Piezoelectricity is a phenomenon in which certain crystalline substances develop an electric field when subjected to pressure force, or conversely, exhibit a mechanical deformation when subjected to an electric field. This reciprocal coupling between mechanical and electrical energy provides useful features for these materials. The dynamics of the piezoelectric sensor/actuator plays an increasing importance when higher performance from closed loop systems or damage monitoring is required for strategic applications. This chapter focuses on the development of the constitutive equations of smart structures. The incorporation of mass, stiffness, and electromechanical coupling of the piezoceramic patches has a significant influence on the dynamics properties of the system.

Keywords Electromechanical coupling • Piezoelectric material • Smart structure

1 Introduction

The dynamics of the piezoelectric sensor/actuator plays an increasing importance when higher performance from closed loop systems or damage monitoring is required for strategic applications. For a piezoceramic, the three direction (z-axis) is usually associated with the direction of poling and the material is approximately isotropic in the other two directions.

Materials that become electrically polarized when they are deformed present the direct piezoelectric effect, producing an electrical charge at the surface of the material. The converse piezoelectric effect results in a strain in the material when placed within an electric field. The direct and converse effects result an electromechanical coupling. While piezoelectric elements exhibit nonlinear hysteresis at high excitation levels, the response required in the current typical structural applications

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is approximately linear. The linear constitutive relations for piezoelectric materials are given by Leo (2007):

$$\mathbf{T} = \left[\mathbf{c}^{\mathrm{E}}\right]\{\mathbf{S}\} - [\mathbf{e}]\{\mathbf{E}\} \tag{1}$$

$$\mathbf{D} = [\mathbf{e}]^{\mathrm{T}}\{\mathbf{S}\} + \left[\varepsilon^{\mathrm{S}}\right]\{\mathbf{E}\} \tag{2}$$

where the superscript $(\ )^{\mathrm{S}}$ means that the values are measured at constant strain, the superscript $(\ )^{\mathrm{E}}$ means that the values are measured at constant electric field, **T** is the stress tensor [N/m^2], **D** is the electric displacement vector [C/m^2], {**S**} is the strain tensor [m/m], {**E**} is the electric field [V/m = N/C], [c^{E}] is the elasticity tensor at constant electric field [N/m^2], [**e**] is the dielectric permittivity tensor [N m/V m^2 = C/m^2], and is the dielectric tensor at constant mechanical strain (permittivity matrix) [N m/V^2 m]. The letters in brackets indicate the units of the variables (in the SI system of units) with N, m, V, and C denoting Newton, meter, Volts, and Coulomb, respectively.

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{23} & T_{13} & T_{12} \end{bmatrix}^{\mathrm{T}}$$
$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{22} & S_{33} & S_{23} & S_{13} & S_{12} \end{bmatrix}^{\mathrm{T}}$$
$$\mathbf{D} = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}^{\mathrm{T}}; \quad \mathbf{E} = \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^{\mathrm{T}}$$

$$\left[\boldsymbol{\varepsilon}^{\mathrm{S}}\right] = \begin{bmatrix} \varepsilon_1^{\mathrm{S}} & 0 & 0 \\ 0 & \varepsilon_2^{\mathrm{S}} & 0 \\ 0 & 0 & \varepsilon_3^{\mathrm{S}} \end{bmatrix}; \quad [\mathbf{e}] = \begin{bmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ 0 & e_{15} & 0 \end{bmatrix};$$

$$\left[\mathbf{c}^{E}\right] = \begin{bmatrix} c_{11}^{\mathrm{E}} & c_{12}^{\mathrm{E}} & c_{13}^{\mathrm{E}} & 0 & 0 & 0 \\ c_{12}^{\mathrm{E}} & c_{22}^{\mathrm{E}} & c_{23}^{\mathrm{E}} & 0 & 0 & 0 \\ c_{13}^{\mathrm{E}} & c_{23}^{\mathrm{E}} & c_{33}^{\mathrm{E}} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^{\mathrm{E}} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^{\mathrm{E}} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^{\mathrm{E}} \end{bmatrix}$$

If each element of the matrix of piezoelectric material constant, [**e**], is designed by e_{ij} where i corresponds to the row and j corresponds to the column of the matrix, then e_{ij} corresponds to the stress developed in the jth direction due to an electric field applied in the ith direction. The piezoelectric strain constants d_{ij}, relating the voltage applied in the *i*th direction to a strain developed in *j*th direction, are provided more often than the stress constants. However, the piezoelectric stress constants can be obtained from the strain constants since the constitutive equation can also be written as:

$$\mathbf{S} = \left[\mathbf{s}^{\mathrm{E}}\right]\{\mathbf{T}\} + [\mathbf{d}]\{\mathbf{E}\} \tag{3}$$

$$\mathbf{D} = [\mathbf{d}]^{\mathrm{T}}\{\mathbf{T}\} + \left[\varepsilon^{\mathrm{T}}\right]\{\mathbf{E}\} \tag{4}$$

where $\boldsymbol{\varepsilon}^{\mathrm{T}}$ is the dielectric tensor at constant stress. The relative dielectric constant, K^{T}, is the ratio of the permittivity of the material, ε^{T}, to the permittivity of the free space, ε_0, ($\varepsilon_0 = 8.9 \times 10^{-12}$ F/m or A s/V m). Then,

$$\left[\mathbf{c}^{\mathrm{E}}\right] = \left[\mathbf{s}^{\mathrm{E}}\right]^{-1}; \quad [\mathbf{e}] = \left[\mathbf{c}^{\mathrm{E}}\right][\mathbf{d}]$$

$$\left[\boldsymbol{\varepsilon}^{\mathrm{S}}\right] = \left[\boldsymbol{\varepsilon}^{\mathrm{T}}\right] - [\mathbf{d}]^{\mathrm{T}}\left[\mathbf{c}^{\mathrm{E}}\right][\mathbf{d}]; \quad K^{\mathrm{T}} = \frac{\varepsilon^{\mathrm{T}}}{\varepsilon_0}$$

with

$$[\mathbf{d}] = \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{31} \\ 0 & 0 & d_{33} \\ 0 & d_{15} & 0 \\ 0 & 0 & d_{15} \end{bmatrix}$$

2 Finite Element Formulation of Electromechanical Systems

Finite Element Method (FEM) is widely used in engineering problems allowing to obtain approximate solutions to differential equations that describe the dynamics of a system. Other methods for obtaining electromechanical models may be used as the assumed modes method. However, the biggest advantage of FEM is to model structures with complex geometry. The basic idea is to divide the region into a finite number of elements and assume that these elements are interconnected by nodes (Bathe and Wilson 1976).

The pioneers in the development of dynamic models for smart structures are the work Allik and Hughes (1970). They use the mechanical stress induced by the piezoelectric to contribute with the total mechanical stress of the host structure. However, the first research work that has developed a rigorous system for the design of electromechanical coupled structure was presented by Hagood et al. (1990), who applied the generalized Hamilton's principle, also known as variational principle applied to piezoelectric systems (Allik and Hughes 1970). The great contribution of Hagood et al. (1990) was formulated more clearly the electromechanical coupling.

The FEM is a method of transformation and approximation of an integral formulation, by an approximation linear algebraic formulation, where the coefficients are integral evaluations on the subarea of the area of resolution. The Rayleigh–Ritz formulation is used to derive the equations of motion of the electroelastic beam. The assumed displacement field shapes within the elastic body and electric potential field shapes will be combined through the piezoelectric properties to form a set of coupled electromechanical equations of motion. The generalized form of Hamilton's principle for a coupled electromechanical system is (Hagood et al. 1990)

$$\int_{t_1}^{t_2} [\delta(T - U + W_{\mathrm{e}} - W_{\mathrm{m}}) + \delta W] dt = 0 \tag{5}$$

where t_1 and t_2 are two arbitrary instants, T is the Kinetic energy, U is the potential energy, W_{e} is the work done by electrical energy, and W_{m} is the work done by magnetic energy, which is negligible for piezoceramic material.

$$T = T_{\mathrm{S}} + T_{\mathrm{P}} = \int_{V_{\mathrm{S}}} \frac{1}{2} \rho_{\mathrm{S}} \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}} \; dV + \int_{V_{\mathrm{P}}} \frac{1}{2} \rho_{\mathrm{P}} \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}} \; dV \tag{6}$$

$$U = U_{\mathrm{S}} + U_{\mathrm{P}} = \int_{V_{\mathrm{S}}} \frac{1}{2} \mathbf{S}^{\mathrm{T}} \mathbf{T} dV + \int_{V_{\mathrm{P}}} \frac{1}{2} \mathbf{S}^{\mathrm{T}} \mathbf{T} dV \tag{7}$$

$$W_{\mathrm{e}} = \int_{V_{\mathrm{p}}} \frac{1}{2} \mathbf{E}^{\mathrm{T}} \mathbf{D} dV \tag{8}$$

where ρ is the mass density and the subscript s and p refer to the structure and piezoelectric material, respectively. The virtual work, δW, done by external forces and the prescribed surface charge, Q, is,

$$\delta W = \int_{V_{\mathrm{S}}} \delta \mathbf{u}^{\mathrm{T}} P_{\mathrm{b}} dV + \int_{S_{\mathrm{S}}} \delta \mathbf{u}^{\mathrm{T}} P_{\mathrm{S}} ds_{\mathrm{S}} + \delta \mathbf{u}^{\mathrm{T}} P_{\mathrm{C}} - \int_{S_{\mathrm{P}}} \delta \Phi Q ds_{\mathrm{P}} \tag{9}$$

where P_{b} is the body force, P_{S} is the surface force, P_{C} is the concentrated load, and Q is the surface charge. To formulate the matrix of the electromechanical coupling using FEM, the displacement vector, u, and the electric potential, ϕ, must be expressed in terms of nodal value, i, via the interpolation function

$$\mathbf{u}(x) = [\mathbf{N}_u]\{\mathbf{u}_i\} \tag{10}$$

$$\Phi(x) = \left[\mathbf{N}_\phi\right]\{\phi_i\} \tag{11}$$

Substituting Eq. (10) into Eq. (6) yields

$$T = \iiint\limits_{V_s} \frac{1}{2}\rho_s \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}}\, dV_s + \iiint\limits_{V_p} \frac{1}{2}\rho_p \dot{\mathbf{u}}^{\mathrm{T}} \mathbf{u}\, dV_p \frac{1}{2} \tag{12}$$

The potential energy is the sum of the potential energy of the structure and of the piezoelectric material. The constitutive relation of the structure in matrix form is given by:

$$\mathbf{T}_s = \mathbf{G}_s\mathbf{S} \quad \text{and} \quad \mathbf{G}_s = \frac{E_s}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\upsilon}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\upsilon}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\upsilon}{2} \end{bmatrix} \tag{13}$$

$\mathbf{G}_s$ is the matrix containing the elastic coefficients of the material. E_s is the Young's modulus and υ is the Poisson ratio. The strain can be represented in matrix form by:

$$\mathbf{S} = \mathbf{L}_u\,\mathbf{u}; \quad \begin{Bmatrix} S_x \\ S_y \\ S_z \\ S_{xy} \\ S_{xz} \\ S_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}; \quad \mathbf{S} = \mathbf{L}_u\mathbf{N}_u\mathbf{u}_i \tag{14}$$

or

$$\mathbf{S} = \mathbf{B}_u \mathbf{u}_i \tag{15}$$

and

$$\mathbf{B}_u = \mathbf{L}_u \mathbf{N}_u \tag{16}$$

Substituting (15) in (13), one obtains the stress tensor in the host structure

$$\mathbf{T}_\mathrm{s} = \mathbf{G}_\mathrm{s} \mathbf{S} = \mathbf{G}_\mathrm{s} \mathbf{B}_u \mathbf{u}_i \tag{17}$$

Solving (7) in the structural domain, V_s, yields

$$\mathbf{U}_\mathrm{s} = \iiint\limits_{V_\mathrm{s}} \frac{1}{2} \mathbf{u}_i^\mathrm{T} \mathbf{B}_u^\mathrm{T} \mathbf{G}_\mathrm{s} \mathbf{B}_u \mathbf{u}_i dV_\mathrm{s} \tag{18}$$

Similarly from the mechanical strain, the electric field is described by

$$\mathbf{E} = \mathbf{L}_\phi \boldsymbol{\Phi} \tag{19}$$

or

$$\mathbf{E} = \mathbf{L}_\phi \mathbf{N}_\phi \phi_i = \mathbf{B}_\phi \phi_i \tag{20}$$

where

$$\mathbf{B}_\phi = \mathbf{L}_\phi \mathbf{N}_\phi \tag{21}$$

and $\mathbf{L}_\phi$ is the matrix containing the differential operators. Substituting (1) into (7) and using (20), the potential energy in the piezoelectric domain, V_p, yields

$$\mathbf{U}_\mathrm{p} = \iiint\limits_{V_\mathrm{p}} \frac{1}{2} \mathbf{u}_i^\mathrm{T} \mathbf{B}_u^\mathrm{T} \, \mathbf{c}^\mathrm{E} \, \mathbf{B}_u \, \mathbf{u}_i \, \mathrm{d}V_\mathrm{p} \; - \iiint\limits_{V_\mathrm{p}} \frac{1}{2} \mathrm{u}_i^\mathrm{T} \, \mathbf{B}_u^\mathrm{T} \, \mathbf{e} \, \mathbf{B}_\phi \, \phi_i \, \mathrm{d}V_\mathrm{p} \tag{22}$$

The potential energy of the piezostructure is obtained by adding (18) and (22)

$$\mathbf{U} = \iiint\limits_{V_\mathrm{s}} \frac{1}{2} \mathbf{u}_i^\mathrm{T} \mathbf{B}_u^\mathrm{T} \mathbf{G}_\mathrm{s} \mathbf{B}_u \mathrm{u}_i \mathrm{d}V_\mathrm{s} + \iiint\limits_{V_\mathrm{p}} \frac{1}{2} \mathbf{u}_i^\mathrm{T} \mathbf{B}_u^\mathrm{T} \mathbf{c}^\mathrm{E} \, \mathbf{B}_u \mathbf{u}_i \mathrm{d}V_\mathrm{p} - \iiint\limits_{V_\mathrm{p}} \frac{1}{2} \mathrm{u}_i^\mathrm{T} \mathbf{B}_u^\mathrm{T} \mathbf{e} \mathbf{B}_\phi \phi_i \mathrm{d}V_\mathrm{p} \tag{23}$$

The work done by electrical energy is

$$\mathbf{W}_{\mathrm{e}} = \iiint\limits_{V_{\mathrm{p}}} \frac{1}{2}\mathbf{E}^{\mathrm{T}}\,\mathbf{D}\,\mathrm{d}V_{\mathrm{P}} \tag{24}$$

Using the constitutive relations yields

$$\mathbf{W}_{\mathrm{e}} = \iiint\limits_{V_{\mathrm{P}}} \frac{1}{2}\phi_i^{\mathrm{T}}\mathbf{B}_{\phi}^{\mathrm{T}}\,\mathbf{e}^{\mathrm{T}}\,\mathbf{B}_u\mathbf{u}_i\mathrm{d}V_{\mathrm{P}} + \iiint\limits_{V_{\mathrm{P}}} \frac{1}{2}\phi_i^{\mathrm{T}}\mathbf{B}_{\phi}^{\mathrm{T}}\varepsilon^{\mathrm{S}}\mathbf{B}_{\phi}\phi_i\mathrm{d}V_{\mathrm{P}} \tag{25}$$

At this point, the coupled electromechanical system equation can be derived from the generalized form of Hamilton's principle. Allowing arbitrary variations of $\{\mathbf{u}_i\}$ and $\{\Phi_i\}$, two equilibrium matrix equations, in generalized coordinates, are obtained.

$$\left(\left[\mathbf{M}_{\mathrm{S}}^{\mathrm{e}}\right] + \left[\mathbf{M}_{\mathrm{P}}^{\mathrm{e}}\right]\right)\{u_i\} + \left(\left[\mathbf{K}_{\mathrm{S}}^{\mathrm{e}}\right] + \left[\mathbf{K}_{\mathrm{P}}^{\mathrm{e}}\right]\right)\{u_i\} - \left[K_{u\phi}^{\mathrm{e}}\right]\{\Phi_i\} = \{F^{\mathrm{e}}\} \tag{26}$$

$$\left[K_{\phi u}^{\mathrm{e}}\right]\{u_i\} - \left[K_{\phi\phi}^{\mathrm{e}}\right]\{\Phi_i\} = \{Q^{\mathrm{e}}\} \tag{27}$$

where $\mathbf{M}_S^e$ and $\mathbf{M}_P^e$ are the local matrix of mass for the host structure and the PZT, respectively:

$$\mathbf{M}_{\mathrm{s}}^{\mathrm{e}} = \iiint\limits_{V_{\mathrm{s}}} \rho_{\mathrm{s}}\mathbf{N}_{\mathrm{u}}^{\mathrm{T}}\mathbf{N}_{\mathrm{u}}dV_{\mathrm{s}} \tag{28}$$

$$\mathbf{M}_{\mathrm{p}}^{\mathrm{e}} = \iiint\limits_{V_{\mathrm{p}}} \rho_{\mathrm{p}}\mathbf{N}_{\mathrm{u}}^{\mathrm{T}}\mathbf{N}_{\mathrm{u}}dV_{\mathrm{p}} \tag{29}$$

and $\mathbf{K}_S^e$ and $\mathbf{K}_P^e$ are the local matrix of stiffness for the host structure and the PZT, respectively:

$$\mathbf{K}_{\mathrm{s}}^{\mathrm{e}} = \iiint\limits_{V_{\mathrm{s}}} \mathbf{B}_{\mathrm{u}}^{\mathrm{T}}\mathbf{G}_{\mathrm{s}}\mathbf{B}_{\mathrm{u}}dV_{\mathrm{s}} \tag{30}$$

$$\mathbf{K}_{\mathrm{p}}^{\mathrm{e}} = \iiint\limits_{V_{\mathrm{p}}} \mathbf{B}_u^{\mathrm{T}}\mathbf{c}^{\mathrm{E}}\mathbf{B}_u dV_{\mathrm{p}} \tag{31}$$

The electromechanical coupling matrix, $\mathbf{K}_{\mathrm{u}\phi}^{\mathrm{e}}$, and the piezoelectric capacitance matrix, $\mathbf{K}_{\phi\phi}^{\mathrm{e}}$, are

$$\mathbf{K}^{\mathrm{e}}_{\mathrm{u}\phi} = \iiint_{V_{\mathrm{p}}} \mathbf{B}_u^{\mathrm{T}}\, \mathbf{e}\, \mathbf{B}_{\phi} dV_{\mathrm{p}} \tag{32}$$

$$\mathbf{K}^{\mathrm{e}}_{\phi\phi} = \iiint_{V_{\mathrm{P}}} \mathbf{B}_{\phi}^{\mathrm{T}} \in^{\mathrm{S}} \mathbf{B}_{\phi} dV_{\mathrm{P}} \tag{33}$$

with $\left[K^{\mathrm{e}}_{\phi u}\right] = \left[K^{\mathrm{e}}_{u\phi}\right]^{\mathrm{T}}$. The force vectors are given by:

$$\{F^{\mathrm{e}}\} = \int_{V_{\mathrm{S}}} [N_u]^{\mathrm{T}}\{P_{\mathrm{B}}\} dV_{\mathrm{S}} + \int_{S_{\mathrm{S}}} [N_u]^{\mathrm{T}}\{P_{\mathrm{S}}\} dS_{\mathrm{s}} + [N_u]^{\mathrm{T}}\{P_{\mathrm{c}}\} \tag{34}$$

$$\{Q^{\mathrm{e}}\} = -\int_{S_{\mathrm{P}}} \left[N_{\phi}\right]^{T} Q\, dS_{\mathrm{P}} \tag{35}$$

For the entire structure, using the standard assembly technique for the FEM, we obtain the complete equation for a coupled electromechanical system as

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\Phi} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{\mathrm{uu}} & \mathbf{K}_{\mathrm{u}\phi} \\ \mathbf{K}_{\phi\mathrm{u}} & \mathbf{K}_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \Phi \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{Q} \end{Bmatrix} \tag{36}$$

where the global matrices are defined by

$$\mathbf{M} = \sum_{i=1}^{ne} \left(\mathbf{M}^{\mathrm{e}}_{\mathrm{s}}\right)_i + \sum_{j=1}^{np} \left(\mathbf{M}^{\mathrm{e}}_{\mathrm{p}}\right)_j \tag{37}$$

$$\mathbf{K}_{\mathrm{uu}} = \sum_{i=1}^{ne} \left(\mathbf{K}^{e}_{\mathrm{s}}\right)_i + \sum_{j=1}^{np} \left(\mathbf{K}^{e}_{\mathrm{p}}\right)_j \tag{38}$$

$$\mathbf{K}_{\mathrm{u}\phi} = \mathbf{K}^{\mathrm{T}}_{\phi\mathrm{u}} = -\sum_{j=1}^{np} \left(\mathbf{K}^{\mathrm{e}}_{\mathrm{u}\phi}\right)_j \tag{39}$$

$$\mathbf{K}_{\phi\phi} = -\sum_{j=1}^{np} \left(\mathbf{K}^{e}_{\phi\phi}\right)_j \tag{40}$$

where *ne* is the number of structural elements and *np* is the number of piezoelectric patches in the structure. The symbol summation, in the above equations, means finite element assembling matrices. At this point, it is important to note that the mass and stiffness matrices for a finite element and therefore for the complete structure are not positive definite.

The sensor equation is:

$$\mathbf{K}_{\phi \mathrm{u}}\,\mathbf{u} + \mathbf{K}_{\phi\phi}\,\Phi_{\mathrm{s}} = \mathbf{Q} \tag{41}$$

Making the electric charge Q to zero since there is no electric potential applied to the sensor, yields

$$\Phi_{\mathrm{s}} = -\mathbf{K}_{\phi\phi}^{-1}\,\mathbf{K}_{\phi \mathrm{u}}\mathbf{u} \tag{42}$$

To find the force generated in the actuator, one must consider the charge Q nonzero, then we can rewrite equation (41) as follows:

$$\mathbf{K}_{\phi \mathrm{u}}\,\mathbf{u} + \mathbf{K}_{\phi\phi}\,\Phi_{\mathrm{a}} = \mathbf{Q} \tag{43}$$

or

$$\Phi_{\mathrm{a}} = \mathbf{K}_{\phi\phi}^{-1}\left(\mathbf{Q} - \mathbf{K}_{\phi \mathrm{u}}\mathbf{u}\right) \tag{44}$$

Replacing the electric potential (44) in the global equation (36) yields

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} + \mathbf{F}_{\mathrm{el}} \tag{45}$$

where

$$\mathbf{K} = \mathbf{K}_{\mathrm{uu}} - \mathbf{K}_{\mathrm{u}\phi}\,\mathbf{K}_{\phi\phi}^{-1}\,\mathbf{K}_{\phi \mathrm{u}} \tag{46}$$

$$\mathbf{F}_{\mathrm{el}} = -\mathbf{K}_{\mathrm{u}\phi}\,\mathbf{K}_{\phi\phi}^{-1}\,\mathbf{Q} \tag{47}$$

where $\mathbf{F}_{\mathrm{el}}$ is the electric force generated in the actuator by applying an electrical charge.

The term $\mathbf{K}_{u\phi}\Phi$ can be divided in two parts dependent on the electric potential, one referring to the piezoelectric material used as sensor and the other for the piezoelectric material used as actuator.

$$\mathbf{K}_{\mathrm{u}\phi}\Phi = \mathbf{K}_{\mathrm{u}\phi}\,\Phi_{\mathrm{s}} + \mathbf{K}_{\mathrm{u}\phi}\,\Phi_{\mathrm{a}} \tag{48}$$

Substituting in the motion equation (36)

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}_{uu}\,\mathbf{u} + \mathbf{K}_{u\phi}\left(-\mathbf{K}_{\phi\phi}^{-1}\,\mathbf{K}_{\phi u}\,\mathbf{u}\right) = \mathbf{F} - \mathbf{K}_{u\phi}\Phi_{\mathrm{a}} \tag{49}$$

or

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} - \mathbf{K}_{u\phi}\Phi_{\mathrm{a}} \tag{50}$$

where

$$\mathbf{K} = \mathbf{K}_{uu} - \mathbf{K}_{u\phi}\,\mathbf{K}_{\phi\phi}^{-1}\,\mathbf{K}_{\phi u} \tag{51}$$

Every structure has some damping effect. Usually, this value is difficult to be defined precisely, but can be predicted. A practical approach is considering proportional damping, to the mass and stiffness.

$$\mathbf{D}_{\mathrm{a}} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{52}$$

The global equation of motion, considering damping matrix, is given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D_a}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} \;=\; \mathbf{F} - \mathbf{K}_{u\phi}\,\mathbf{K}_{\phi\phi}^{-1}\mathbf{Q} \tag{53}$$

where $\mathbf{M}$, $\mathbf{D}_{\mathrm{a}}$, and $\mathbf{K}$ are the global matrices of mass, damping, and stiffness, respectively.

3 Eigenvalue Problem for the Short Circuit Case

Natural frequencies and mode shapes can be obtained by reducing the assembled global matrices to a standard eigenvalue form. It can be done by suitable grounding the structure by specifying one or more nodal value of electrical potential. Then the new piezoelectric capacitance matrix, $K_{\phi\phi}^{*}$, is non-singular and the eigenvalue problem, for the undamped homogeneous system, can be written as (Lopes Jr. et al. 2000)

$$\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{\mathbf{u}\} = \{0\} \tag{54}$$

where

$$[\mathbf{M}] = [\mathbf{M}_{uu}] \tag{55}$$

$$[\mathbf{K}] = [\mathbf{K}_{uu}] - \left[\mathbf{K}_{u\phi}\right]\left[\mathbf{K}_{\phi\phi}^{*}\right]^{-1}\left[\mathbf{K}_{\phi u}\right] \tag{56}$$

and $\left[\,\circ\,\right]^{-1}$ indicates the inverse of the matrix.

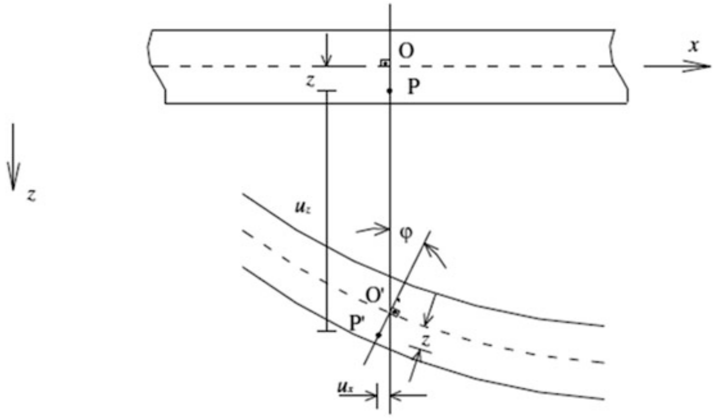

Fig. 1 Displacement of a point *P* at a distance *z* from the median line of the beam

4 Application: Clamped-Free Beam with Bonded PZT

The general equations from the previous section will be applied for the case of a clamped-free beam with a pair of bonded PZT (bimorph case). Different numbers and locations of PZTs can be considered.

The poling of the piezoelectric is in the *z*-direction. Figure 1 shows an Euler–Bernoulli beam, where the displacement of a point on a normal plane of the beam at a distance "*z*" from the median line in the direction "*x*" is

$$u_x = -ztg\varphi = -z\frac{\partial u_z}{\partial x} \tag{57}$$

The state of plane strain is given by

$$S_x = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 u_z}{\partial x^2} \tag{58}$$

Equation (14) can be rewritten as

$$\mathbf{S} = \mathbf{L}_\mathrm{u} u_z \tag{59}$$

where

$$\mathbf{L}_\mathrm{u} = \left[-z\frac{\partial^2}{\partial x^2}\right] \tag{60}$$

The stress is also rewritten as

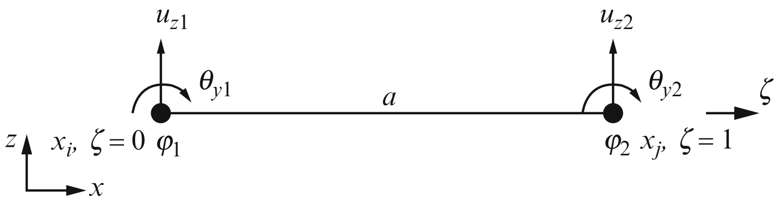

Fig. 2 Structural element with electromechanical coupling

$$\mathbf{T}_x = \mathbf{E}_s \mathbf{S}_x \tag{61}$$

Considering that the piezoelectric material is being modeled as Euler–Bernoulli beam element, their constitutive relations can be summarized as:

$$D_3 = e_{31} S_{11} + \varepsilon_{33}^{S} E_3 \quad \text{sensor equation} \tag{62}$$

$$T_{11} = c_{11}^{E} S_{11} - e_{31} E_3 \quad \text{actuator equation} \tag{63}$$

The goal is to obtain the interpolation function on the basis of generalized coordinates for the degrees of freedom of displacement and electrical potential. With these functions, one can determine the elementary matrices of electromechanical coupled system. Initially, it is considered the electromechanical coupling between the host structure and the piezo element, as shown in Fig. 2.

The element is composed by two nodes, with two structural degrees of freedom per node, translation denoted by "u_{zi}" in direction "z" and rotation in the plane "yz" denoted by "θ_{yi}," and one electric potential degree of freedom per node "ϕ_i." Considering x_i the point localized in the node i and ξ the generalized coordinate in function of x, as

$$\xi = \frac{x}{a} \tag{64}$$

One can rewrite the displacement vector of the ith element as

$$\mathbf{u}_i = \begin{bmatrix} u_{z1} & \theta_{y1} & u_{z2} & \theta_{y2} \end{bmatrix}^{\mathrm{T}} \tag{65}$$

and the electric potential is

$$\boldsymbol{\Phi}_i = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}^{\mathrm{T}} \tag{66}$$

or

$$\mathbf{u}(\xi) = \mathbf{N}_{u1}(\xi) u_{z1} + \mathbf{N}_{u2}(\xi) \theta_{y1} + \mathbf{N}_{u3}(\xi) u_{z2} + \mathbf{N}_{u4}(\xi) \theta_{y2} \tag{67}$$

and

$$\Phi(\xi) = \mathbf{N}_{\phi 1}(\xi)\phi_1 \; + \; \mathbf{N}_{\phi 2}(\xi)\phi_2 \tag{68}$$

Initially, one can find the interpolation functions of the mechanical displacements. For this, it is observed that the element is analyzed in only one dimension (ξ) and has four degrees of freedom. Therefore, one obtains the following interpolating function for displacement in the z-direction.

$$u_z(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3 \tag{69}$$

or

$$u = \mathbf{P}\alpha \tag{70}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \tag{71}$$

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}^{\mathrm{T}} \tag{72}$$

Considering small angles

$$\theta_\eta(\xi) = -\frac{\partial u_z(\xi)}{\partial \xi} = -\alpha_2 - 2\alpha_3\xi - 3\alpha_4\xi^2 \tag{73}$$

The values of the generalized coordinates for each element node can be obtained in matrix form as Eq. (74). The columns of the inverse matrix $\mathbf{P}n$ contain the interpolation functions. The values of the generalized coordinates for node 1 ($\xi = 0$) and node 2 ($\xi = 1$) yield

$$\begin{Bmatrix} u_{z1} \\ \theta_{\eta 1} \\ u_{z2} \\ \theta_{\eta 2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{74}$$

or

$$\delta = \mathbf{P}n \; \alpha \tag{75}$$

and,

$$\alpha = [\mathbf{P}n]^{-1} \; \delta \tag{76}$$

One can also write

$$\theta_y = -\frac{\partial u_z}{\partial x} = -\left(\frac{\partial u_z}{\partial \xi}\frac{\partial \xi}{\partial x}\right) = \frac{1}{a}\theta_\eta \rightarrow \theta_\eta = a\theta_y \tag{77}$$

where $\frac{\partial u_z}{\partial \xi} = -\theta_\eta$ and $\frac{\partial \xi}{\partial x} = \frac{1}{a}$, then

$$\begin{Bmatrix} u_{z1} \\ \theta_{\eta 1} \\ u_{z2} \\ \theta_{\eta 2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \tag{78}$$

or

$$\delta = \mathbf{Z}\mathbf{u}_i \tag{79}$$

Substituting equation (79) into (76) and after that into (70) yields

$$\mathbf{u} = \mathbf{P}[\mathbf{P}n]^{-1}\mathbf{Z}\mathbf{u}_i \tag{80}$$

One knows that $\mathbf{u} = \mathbf{N}_{\mathrm{u}}\mathbf{u}_i$, then

$$\mathbf{N}_u = \mathbf{P}[\mathbf{P}n]^{-1}\mathbf{Z} \tag{81}$$

and

$$\mathbf{N}_u = \begin{bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ -a\xi + 2a\xi^2 - a\xi^3 \\ 3\xi^2 - 2\xi^3 \\ a\xi^2 - a\xi^3 \end{bmatrix}^{\mathrm{T}} \tag{82}$$

In order to find the matrix $\mathbf{B}_{\mathrm{u}}$, one considers

$$\mathbf{L}_{\mathrm{u}} = \left[-\frac{z}{a^2}\frac{\partial^2}{\partial \xi^2}\right] \tag{83}$$

then

$$\mathbf{B}_u^{\mathrm{T}} = -\frac{z}{a^2}\frac{\partial^2 \mathbf{N}_u^{\mathrm{T}}}{\partial \xi^2} \tag{84}$$

and

$$\mathbf{B}_u = -\frac{z}{a^2}\begin{bmatrix} -6+12\xi \\ 4a-6\xi \\ 6-12\xi \\ 2a-6a\xi \end{bmatrix}^{\mathrm{T}} \tag{85}$$

Similarly, one can find the interpolation functions of the electric potential. The element has one dimension (ξ) and two electric degrees of freedom, thus one obtains the following polynomial basis to obtain the interpolation functions.

$$\mathbf{P} = [\,1 \quad \xi\,] \tag{86}$$

The values of the generalized coordinates for each element node, $\mathbf{P}n$, are given in equation (87). The columns of the inverse matrix $\mathbf{P}n$ contain the indices of the interpolation functions. The values of the generalized coordinate for the node 1 ($\xi = 0$) and node 2 ($\xi = 1$) yield

$$\mathbf{P}n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{87}$$

and

$$[\mathbf{P}n]^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tag{88}$$

The interpolation functions are given by multiplying equations (86) and (88)

$$\mathbf{N}_\phi = \mathbf{P}[\mathbf{P}n]^{-1} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}^{\mathrm{T}} \tag{89}$$

Whereas the electric field can be written directly proportional to the difference of the electric potential and inversely proportional to the distance of these potentials, then

$$\Phi = \mathbf{E}\delta \;\rightarrow\; \mathbf{E} = \frac{d\Phi}{d\delta} \tag{90}$$

where δ is the distance between the potentials, so $\phi = \phi(x) \rightarrow \delta = \delta(x)$, then

$$\mathbf{E} = \frac{\partial \Phi}{\partial x} \tag{91}$$

Rewritten in the matrix form

$$\mathbf{E}(x) = \left[\frac{\partial}{\partial x}\right]\Phi(x) \tag{92}$$

Comparing (92) with (19)

$$\mathbf{L}_\phi = \left[\frac{\partial}{\partial x}\right] \tag{93}$$

Considering the generalized coordinate $\xi = \frac{x}{a} \rightarrow \partial x = a\partial\xi$, one can rewrite (93) as

$$\mathbf{L}_\phi = \left[\frac{1}{a}\frac{\partial}{\partial\xi}\right] \tag{94}$$

then,

$$\mathbf{B}_\phi^{\mathrm{T}} = \left[\frac{1}{a}\frac{\partial\mathbf{N}_\phi^{\mathrm{T}}}{\partial\xi}\right] \tag{95}$$

$$\mathbf{B}_\phi = \frac{1}{a}\begin{bmatrix}-1\\1\end{bmatrix}^{\mathrm{T}} \tag{96}$$

The interpolation functions of the mechanical displacement and electric potential can now be used in equations from (28) to (33) in order to find the electromechanical coupled elementary matrices. The differential volume of the host structure element is

$$dV_{\mathrm{S}} = dz\,dx\,dy \tag{97}$$

Considering the generalized coordinates $\eta = \frac{y}{b}$, one can write the differential volume as

$$dV_{\mathrm{s}} = dz\,ab\;d\xi\,d\eta \tag{98}$$

Substituting (98), equations (28) and (29) are rewritten as:

$$\mathbf{M}_{\mathrm{s}}^{\mathrm{e}} = \int_0^1\int_0^1\int_{-t_s/2}^{t_s/2} dz\rho_{\mathrm{s}}\,ab\,\mathbf{N}_u^{\mathrm{T}}\mathbf{N}_u d\xi\,d\eta \tag{99}$$

where a is the length, b the width, and t_{s} the thickness of the element. Integrating in z- and η-directions yields

$$\mathbf{M}_{\mathrm{s}}^{\mathrm{e}} = \rho_{\mathrm{s}} t_{\mathrm{s}} ab \int_0^1 \mathbf{N}_u^{\mathrm{T}} \mathbf{N}_u d\xi \tag{100}$$

similarly,

$$\mathbf{M}_{\mathrm{p}}^{\mathrm{e}} = \rho_{\mathrm{p}} t_{\mathrm{p}} ab \int_0^1 \mathbf{N}_u^{\mathrm{T}} \mathbf{N}_u d\xi \tag{101}$$

The local matrix of stiffness for the host structure and the PZT, Eqs. (30) and (31), are obtained by substituting the differential volume

$$\mathbf{K}_{\mathrm{s}}^{\mathrm{e}} = \frac{E_{\mathrm{S}}\, t_{\mathrm{s}}^3\, b}{12a^3} \int_0^1 \mathbf{B}_u^{\mathrm{T}} \mathbf{B}_u d\xi \tag{102}$$

$$\mathbf{K}_{\mathrm{p}}^{\mathrm{e}} = \frac{c_{11}^{\mathrm{E}}\, t_{\mathrm{p}}^3\, b}{12a^3} \int_0^1 \mathbf{B}_u^{\mathrm{T}} \mathbf{B}_u d\xi \tag{103}$$

The electromechanical coupling matrix and the piezoelectric capacitance matrix, Eqs. (32) and (33), are

$$\mathbf{K}_{u\phi}^{\mathrm{e}} = \frac{e_{31}\, t_{\mathrm{p}}^2\, ab}{2} \int_0^1 \mathbf{B}_u^{\mathrm{T}} \mathbf{B}_\phi \, d\xi \tag{104}$$

and

$$\mathbf{K}_{\phi\phi}^{\mathrm{e}} = \varepsilon_{33}^{\mathrm{S}}\, t_{\mathrm{p}}\, ab \int_0^1 \mathbf{B}_\phi^{\mathrm{T}} \mathbf{B}_\phi \, d\xi \tag{105}$$

where $\left[\mathbf{K}_{u\phi}^{\mathrm{e}}\right]^{\mathrm{T}} = \mathbf{K}_{\phi u}^{\mathrm{e}}$

The general equations from the previous sections are applied for the case of an aluminum clamped-free beam, as shown in Fig. 3. The beam is modeled with 20 elements with 2 mechanical and 1 electrical DOF per node. Different numbers and locations of PZTs can be considered. The poling of the piezoelectric patches is in the z-direction. The geometrics and physics features of the beam are Young

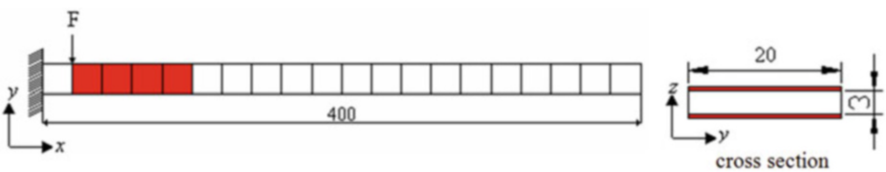

Fig. 3 Schematic drawing of the beam with PZT patches

Table 1 Six first natural frequencies for the aluminum clamped-free beam

Modes	Analytics f_n (Hz)	SMARTSYS f_n (Hz)	Difference (%)
1	15.47	15.39	0.52
2	96.98	96.47	0.52
3	271.56	270.13	0.53
4	532.14	529.36	0.52
5	879.67	875.17	0.51
6	1314.07	1307.62	0.49

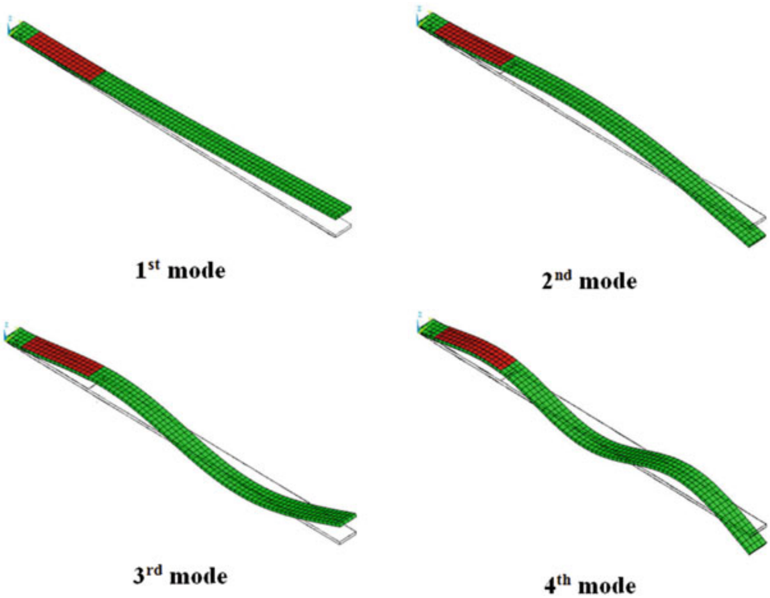

Fig. 4 Four first vibration modes for the electromechanical-coupled beam

modulus 70 GPa; Poisson coefficient 0.31; mass density 2710 kg/m^3; length 400 mm; width 20 mm; and thickness 3 mm.

A finite element code was developed using the previous equations, called SmartSys. Table 1 shows the six first natural frequencies obtained with the SmartSys code and analytically (INMAN 2013) for the case without PZT patches.

The incorporation of mass, stiffness, capacitance, and coupling matrix of the piezoelectric patch has a significant influence on the dynamic properties of the system. The disregarding of these terms may cause errors in many applications. In order to verify the influence of the electromechanical coupling, four pairs of PZT patches were bonded on both sides of the beam, as shown in Fig. 3. The beam is discretized with 20 beam elements, 21 nodes with two mechanical and 1 electrical DOF per node. The geometrics and physics features of the PZT patches are Young modulus 62 GPa; mass density 7500 kg/m^3; length of each PZT patch 20 mm; width 20 mm; thickness 3 mm; strain constant d_{31} 320e − 12; dielectric tensor at constant

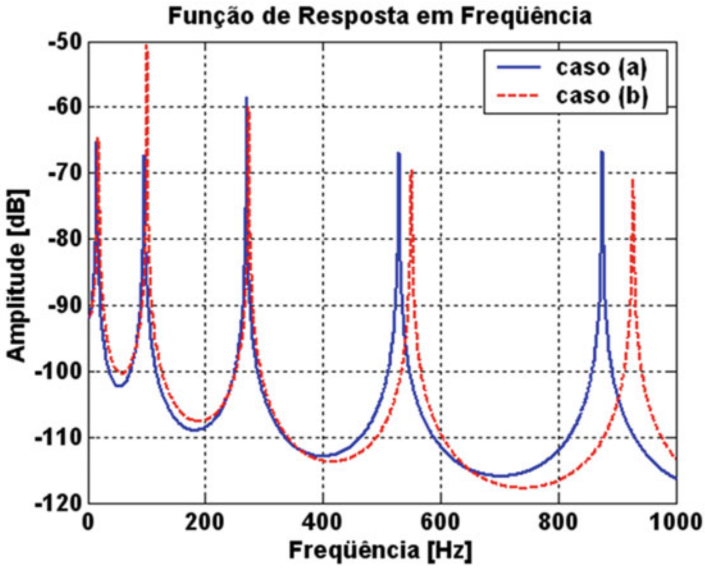

Fig. 5 FRF of the beam; (**a**) without PZTs and, (**b**) four pairs of PZT patches as shown in Fig. 3

Table 2 Five first natural frequencies for the beam without PZT patch and for the beam with four pairs of PZT patches

	Case (a)	Case (b)	
Modes	f_n (Hz)	f_n (Hz)	Difference (%)
1	15.39	18.45	19.88
2	96.47	101.02	4.72
3	270.13	274.18	1.50
4	529.36	550.53	4.00
5	875.17	927.39	5.97

mechanical strain ε_{33}^{S} 3.363e − 8 F/m; elasticity constant c_{11} 92.3e9 N/m^2; and dielectric permittivity e_{31} 16.27 C/m^2

Figure 4 shows the four first vibration modes for the electromechanical-coupled beam.

Figure 5 shows the Frequency Response Functions, FRF, for an impulsive excitation ($F = 1$ N) in node 2 and response in the free end of the beam, node 21. There were considered two cases: (a) beam without PZT patch, and (b) beam with four pairs of PZT patches as shown in Fig. 3.

The five first natural frequencies are shown in Table 2 for both cases, beam without PZT patch and beam with four pairs of PZT patches

The analytical model of a beam with piezoelectric material coupling the electrical and mechanical coordinates was derived using a generalized Hamilton's principle. It was found that the incorporation of mass, stiffness, capacitance, and coupling matrix of the piezoelectric patch has a significant influence on the dynamic properties of the system. This model of smart structure contains additional degrees of freedom at each node, the electrical potential, and it makes the global mass and stiffness matrices non-positive definite, which require special numerical preparation to solve the eigenvalue problem.

References

H. Allik, T.J.R. Hughes, Finite method for piezoelectric vibration. Int. J. Numer. Methods Eng. **2**, 151–157 (1970)

K.J. Bathe, E.L. Wilson, *Numerical Methods in Finite Element Analysis* (Prentice Hall, New Jersey, 1976)

N.W. Hagood, W.H. Chung, A. von Flotow, Modeling of piezoelectric actuator dynamics for active structural control. J. Intell. Mater. Syst. Struct. **1**, 4–25 (1990)

D.J. Inman, *Engineering Vibration*, 4th edn. (Prentice Hall, Upper Saddle River, 2013)

D. Leo, *Engineering Analysis of Smart Material Systems* (Wiley, New York, 2007)

V. Lopes Jr, J.A. Pereira, D.J. Inman, Structural FRF acquisition via electric impedance measurement applied to damage location, in *IMAC*, 2000