

Chapter 29

Adaptive Harmonic Balance Methods, A Comparison

Onur Sert and Ender Cigeroğlu

Abstract Harmonic balance method (HBM) is one of the most popular and powerful methods, which is used to obtain response of nonlinear vibratory systems in frequency domain. The main idea of the method is to express the response of the system in Fourier series and converting the nonlinear differential equations of motion into a set of nonlinear algebraic equations. System response can be obtained by solving this nonlinear equation set in terms of the unknown Fourier coefficients. The accuracy of the solution is greatly affected by the number of harmonics included in the solution; hence, increasing the number of harmonics increases the accuracy of the solution at the expense of computational effort. Therefore, it is desirable to use an adaptive algorithm where the number of harmonics can be optimized in terms of both accuracy and computational effort. Until now, various adaptive harmonic balance methods have been formulated to perform this task. This paper presents an overview and a comparison of these adaptive harmonic balance methods in terms of their effectiveness.

Keywords Harmonic balance method • Adaptive harmonic balance method • Harmonic selection • Nonlinear vibrations • Nonlinear dynamics

29.1 Introduction

Mathematical methods of solving nonlinear equation systems are an essential part of nonlinear vibrations research. Most common methods for solving nonlinear differential equations are dependent on performing numerical integration in time domain. However, these time marching methods are known to require a great deal of computational effort [1, 2]. For vibration problems, most of the time the amplitude of steady-state response over a certain frequency spectrum is sought. In this case, time marching methods require even more computational time to reach and identify the steady-state response. Hence, when one is after the steady-state response, methods that can operate directly in the frequency domain are computationally favorable.

Harmonic Balance Method (HBM) is a frequency domain method which is used to determine the steady-state response. The mathematical basis of the method is to represent time periodic phenomena as truncated Fourier series with a finite number of harmonics. It reduces a set of nonlinear ordinary differential equations (ODEs) into a set of nonlinear algebraic equations, leaving the Fourier coefficients as the only unknowns to be determined. Therefore, HBM is computationally more efficient than time integration methods and it is capable of representing very strong nonlinearities with sufficient accuracy [3]. Therefore, HBM is extensively used in the literature and many different variants of HBM have been developed over the time [3–8].

Despite its advantages, for large scale applications, multi harmonic HBM can produce an excessive number of equations to be solved simultaneously and extensive computational time spent for unnecessary precision. In order to eliminate the use of unnecessary harmonics and increase the computational efficiency of HBM even more, Adaptive Harmonic Balance Methods (AHBMs), come into the picture. Together with a condensation method [2, 9], AHBMs have the potential to drastically decrease the computational time and the storage space required to solve a nonlinear system. For this purpose numerous AHBMs are introduced in the literature by different researchers in the recent years [9–14].

O. Sert • E. Cigeroğlu (✉)
Middle East Technical University, 06800 Ankara, Turkey
e-mail: ender@metu.edu.tr

In this paper, it is aimed to compare the AHBM which are developed for structural dynamics. In the following sections, firstly, the basics of HBM is explained in detail. Secondly, the methods taken from the literature and their selection algorithms are summarized. Finally, a case study is conducted in order to compare these methods. The methodology of the method is explained and using that methodology, the same nonlinear system is analyzed separately by all of the studied AHBM and the obtained results are compared in terms of computational time and accuracy.

29.2 Theory

29.2.1 Multi Harmonic HBM

The most general form of the equation of motion for an n degree of freedom (DOF) nonlinear vibratory system can be written as

$$[M] \{\ddot{q}(t)\} + [D] \{\dot{q}(t)\} + [K] \{q(t)\} + \{f_N(t, q, \dot{q})\} = \{F_{ext}(t)\}, \quad (29.1)$$

where $\{q(t)\}$ is the displacement vector, $[M]$ is the mass matrix, $[D]$ is the damping matrix (including damping and gyroscopic effects), $[K]$ is the stiffness matrix which may include structural damping, $\{f_N(t, q, \dot{q})\}$ is the nonlinear internal forcing vector and $\{F_{ext}(t)\}$ is the external forcing vector. If the external forcing is assumed to be periodic, then it can be represented as an infinite Fourier series. However, since using infinite number of terms is not practical, an m -harmonic representation can be used. In this case, the external forcing can be written as

$$\{F_{ext}(t)\} = \{F_0\} + \sum_{k=1}^m \{F_{sk}\} \sin(k\theta) + \sum_{k=1}^m \{F_{ck}\} \cos(k\theta), \quad (29.2)$$

where $\theta = \omega \cdot t$, $\{F_0\}$, $\{F_{sk}\}$ and $\{F_{ck}\}$ are $(n \times 1)$ vectors containing the known time independent bias terms and k^{th} harmonic coefficients of the sine and cosine components of the external forcing, respectively. For periodic excitation, the response of the system is as well expected to be periodic. Therefore, periodic response of the nonlinear system can be represented as:

$$\{q(t)\} = \{q_0\} + \sum_{k=1}^m \{q_{sk}\} \sin(k\theta) + \sum_{k=1}^m \{q_{ck}\} \cos(k\theta), \quad (29.3)$$

where $\{q_0\}$, $\{q_{sk}\}$ and $\{q_{ck}\}$ are $(n \times 1)$ unknown Fourier coefficient vectors of the bias component, sine and cosine components of the k^{th} harmonic, respectively and $\{q_0\}$ corresponds to the bias term. Similarly, the nonlinear forcing vector can be expressed as follows

$$\{f_N(t, q, \dot{q})\} = \{f_{N0}\} + \sum_{k=1}^m \{f_{Nsk}\} \sin(k\theta) + \sum_{k=1}^m \{f_{Nck}\} \cos(k\theta), \quad (29.4)$$

where, $\{f_{Nsk}\}$ and $\{f_{Nck}\}$ are $(n \times 1)$ vectors containing sine and cosine component of the k^{th} harmonic of the nonlinear internal forcing vector, respectively and $\{f_{N0}\}$ corresponds to the bias term. However, the nonlinear internal forces are not only dependent on time but also the response and its derivatives. Therefore, $\{f_{N0}\}$, $\{f_{Nsk}\}$ and $\{f_{Nck}\}$ are functions of $\{q_0\}$, $\{q_{sk}\}$ and $\{q_{ck}\}$. The Fourier coefficients of the internal nonlinear forces can be calculated as follows [2]:

$$\begin{aligned} \{f_{N0}\} &= \frac{1}{2\pi} \int_0^{2\pi} f_N(t, q, \dot{q}) d\theta \\ \{f_{Nsk}\} &= \frac{1}{\pi} \int_0^{2\pi} f_N(t, q, \dot{q}) \sin(k\theta) d\theta \\ \{f_{Nck}\} &= \frac{1}{\pi} \int_0^{2\pi} f_N(t, q, \dot{q}) \cos(k\theta) d\theta. \end{aligned} \quad (29.5)$$

For the cases where the internal nonlinear forces at steady-state cannot be represented analytically, Cameron and Griffin introduced an alternate frequency time (AFT) method in order to calculate nonlinear forcing coefficients [2, 15].

Substituting Eqs. (29.2), (29.3), and (29.4) into Eq. (29.1), equating the coefficients of like terms on both sides to each other, one obtains $n(2m + 1)$ nonlinear algebraic equations with $n(2m + 1)$ unknowns in terms of frequency, ω , which can be put in matrix form as follows:

$$\begin{bmatrix} [K] & [0] & [0] & \dots & [0] \\ [0] & [\Lambda(\omega)] & [0] & \dots & [0] \\ [0] & [0] & [\Lambda(2\omega)] & \dots & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & [0] & \dots & [\Lambda(m\omega)] \end{bmatrix} \begin{Bmatrix} \{q_0\} \\ \{q_{s1}\} \\ \{q_{c1}\} \\ \{q_{s2}\} \\ \{q_{c2}\} \\ \vdots \\ \{q_{sm}\} \\ \{q_{cm}\} \end{Bmatrix} + \begin{Bmatrix} \{f_{N0}\} \\ \{f_{Ns1}\} \\ \{f_{Nc1}\} \\ \{f_{Ns2}\} \\ \{f_{Nc2}\} \\ \vdots \\ \{f_{Nsm}\} \\ \{f_{Ncm}\} \end{Bmatrix} = \begin{Bmatrix} \{F_0\} \\ \{F_{s1}\} \\ \{F_{c1}\} \\ \{F_{s2}\} \\ \{F_{c2}\} \\ \vdots \\ \{F_{sm}\} \\ \{F_{cm}\} \end{Bmatrix}, \quad (29.6)$$

where

$$[\Lambda(\omega)] = \begin{bmatrix} [K] - \omega^2 [M] & -\omega [D] \\ \omega [D] & [K] - \omega^2 [M] \end{bmatrix}. \quad (29.7)$$

Equation (29.6) can be solved by using a nonlinear equation solver. In order to obtain the frequency response of a dynamic system over a certain frequency range $[\omega_1, \omega_2]$, one needs to solve Eq. (29.6) for different values of $\omega \in [\omega_1, \omega_2]$. In order to select these frequency values and generate initial guesses in a systematic way, and to obtain the frequency response curve in full, a path following method can be combined with a suitable predictor [2]. Newton's Method and arc length continuation are utilized for the solution of the resulting set of nonlinear algebraic equations in this paper.

29.2.2 Adaptive Harmonic Balance Methods

In the previous section, it is indicated that application of the classical multi harmonic HBM method yields $n(2m + 1)$ algebraic equations. For precision, keeping m high is desirable; however, it is computationally expensive. Therefore, using an algorithm that can predict and neglect the harmonics which do not contribute significantly to the total solution is favorable. The basis of adaptive harmonic balance methods (AHBMs) depends on this idea. In the following sections, the AHBMs available in the field of structural dynamics are discussed.

29.2.2.1 Jamouillé, Sinou and Petitjean's Method

In 2010, Jamouillé, Sinou and Petitjean [9] introduced an AHBM (Jamouillé's method) which is originally intended to be used in analyzing nonlinear bolted joint models. In order to identify the harmonics that are to be used, the method uses the concept of approximate strain energy. It is basically equal to the mean potential energy stored in the linear springs during one period of vibratory motion. At every frequency point, a k -harmonic response is calculated. Then, by using this response, the corresponding approximate strain energy is obtained. For the same frequency point, the number of harmonics is increased by 1 to $k + 1$, the response and the approximate strain energy is computed once more. This process continues until the relative approximate strain energy ratio given below falls down a certain threshold value:

$$\varepsilon = \frac{U_m^{k+1} - U_m^k}{U_m^k} \quad (29.8)$$

where U_m^k is the approximate strain energy calculated from the response containing k harmonics and U_m^{k+1} is the approximate strain energy calculated with $k + 1$ harmonics and m stands for mean. When ε becomes less than the threshold value, it means that the contribution from the last added harmonic is small enough. In such a case, no more harmonics are added, the solution is stored and the starting value of k is set to 1 for the next frequency step. The whole algorithm is summarized in Fig. 29.1.

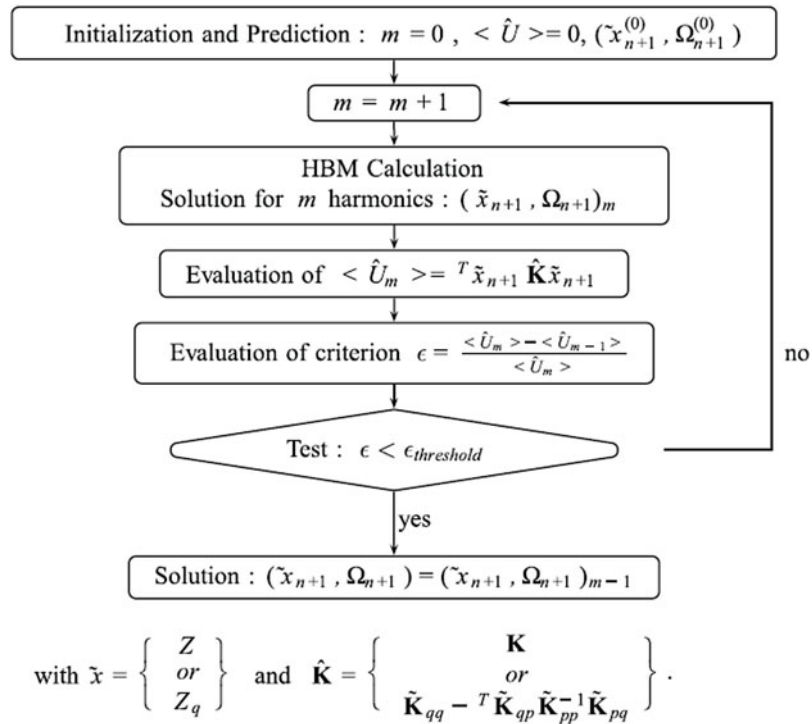


Fig. 29.1 Algorithm for the method presented by Jamouillé et al. [9]

Since this method assigns the same number of harmonics for all DOFs and increases the number of harmonics one by one, it is referred as a global method and an incremental method, respectively [10].

29.2.2.2 Yümer’s Method

In 2010, Yümer [11] presented an AHBM (Yümer’s method), which is initially intended to be used in vibration analysis of bladed disks. Like the method presented by Jaumoillé et al., it is a global and an incremental method. But different from the previous method, the harmonic selection criterion does not require repeated computation of system response for the same frequency point. Instead, at each frequency point, it tests the nonlinear forcing vector according to the following criterion:

$$|f_0| + |f_{s1}| + |f_{c1}| + \dots + |f_{sk}| + |f_{ck}| \geq a (|f_{s(k+1)}| + |f_{c(k+1)}| + \dots + |f_{sN_h^m}| + |f_{cN_h^m}|), \tag{29.9}$$

where the subscripts 0, s and c indicate the bias term, sine and cosine components of the nonlinear forcing, respectively. Subscript k stands for the harmonic number, N_h^m is the maximum number of harmonics allowed in the solution, a is an accuracy parameter set by the user. Initially, k is taken as 1. Then, k is increased one by one until the criterion given in Eq. (29.9) is satisfied. It can be seen from the equation that, increasing a forces the algorithm to increase k closer to N_h^m , whereas doing the opposite keeps k closer to 1, leading to a more coarse solution. Another implication of Eq. (29.9) is that, one needs to compute a N_h^m nonlinear forcing vector for every solution point, even when the response representation has less harmonics. This can be achieved by taking the unused harmonics in the response as zero and evaluating the integrals given in Eq. (29.5) for $k = 1, 2, \dots, N_h^m$.

29.2.2.3 Grolet and Thouverez’s Method

In 2012, Grolet and Thouverez proposed a new AHBM (Grolet and Thouverez’s method) [10]. Unlike the method described above; this method is local, that is, the number of harmonics is arranged separately for each DOF. In addition, the method is not incremental.

The harmonic selection process of the method is based on the tangent predictor. Assume that the Fourier coefficients are solved for the solution point i . At that point, due to the nature of AHBM, some harmonics are neglected and the resulting Fourier coefficient vector $\{q\}_r^i$ has less than the maximum number of harmonics, N_h^m , which is defined by the user. It is assumed that the neglected harmonics at the current solution step are so small that, they can be assumed zero. An $n(2N_h^m + 1) \times 1$ vector, $\{q\}_r^i$, is formed by adding zeros into $\{q\}_r^i$. From this new vector, by using the tangent predictor, an initial guess, $\{q\}_p$, having N_h^m harmonics is computed for the next solution point. After that, $\{q\}_p$ is divided into n sub-vectors having $2N_h^m + 1$ elements, each of which associated to a different DOF. The harmonic selection is performed through these vectors.

Considering the sub-vector of DOF d , $\{u\}_p^d$. The fraction of spectral energy for the k^{th} harmonic of DOF d is defined as:

$$\varepsilon_k^d = \frac{(u_{p_{sk}}^d)^2 + (u_{p_{ck}}^d)^2}{\|\{u\}_p^d\|}, \quad (29.10)$$

where $\|\cdot\|$ is the Euclidean norm, $u_{p_{sk}}^d$ and $u_{p_{ck}}^d$ are the sine and cosine coefficients of the k^{th} harmonic, respectively. Then, the total spectral energy ratio of the harmonics that are retained at the i^{th} solution point can be calculated as:

$$E_r^d = \sum_{k \in H_r^d} \varepsilon_k^d, \quad (29.11)$$

where $[H]_r^d$ is the row matrix, in which the retained harmonic indices of DOF d , at the solution point i are stored. If $[H]_r^d$ is equal to $\begin{bmatrix} 0 & 1 & 2 & \dots & N_h^m \end{bmatrix}$, E_r^d becomes 1. This leads to the definition of the fraction of residual energy:

$$\rho_d = 1 - E_r^d. \quad (29.12)$$

During the harmonic selection procedure, ρ_d is compared to two thresholds ρ_f and ρ_b , which are defined by the user such that, they satisfy the inequality $0 \leq \rho_b \leq \rho_f \leq 1$. If ρ_d is greater than ρ_f , it means that the amount of energy contained in the neglected harmonics is very large. Therefore the neglected harmonics with largest spectral energy values are added one by one into $[H]_r^d$ until ρ_d becomes less than ρ_f . If ρ_d turns out to be less than ρ_b this means that the error obtained is much smaller than the allowable error. Therefore the harmonics with minimum spectral energy values are deleted one by one from $[H]_r^d$ until ρ_d becomes larger than ρ_b . If ρ_d lies between two threshold values, the solution scheme continues with the current harmonics. This process is repeated for every DOF in the system. The algorithm for harmonic selection process is summarized in Fig. 29.2.

29.3 Comparison

29.3.1 Methodology

The aim of this study is to compare the AHBM in terms of solution accuracy and computational time. Naturally, the methods which obtain the same accuracy using the shortest time are regarded as the most effective ones.

In structural dynamics, one of the most important concerns is the magnitude of oscillations throughout a certain frequency spectrum. Therefore, in the case of a nonlinear frequency response obtained by HBM, it is natural for one to be interested in the total response rather than the harmonics themselves. Therefore, for this study, the accuracy measurements are all based on the total response.

In the case study, the discrete nonlinear dynamic system to be analyzed is firstly solved by a 10-harmonic classical HBM script. After the amplitudes of each harmonic are examined, it is decided that a 10 harmonic solution is sufficient to express the system response with an acceptable degree. The total response curve obtained from this solution, is assumed to be correct and is taken as a reference in the error calculations.

Afterwards, the same dynamic system is analyzed by using the AHBM. In order to form a data repository for each method, analyses are done for a wide range of harmonic selection parameters. Accuracies of the response curves obtained from AHBM are evaluated by how much they deviate from the 10-harmonic solution curve at the chosen frequencies. Since it is very difficult to measure the distance between points inside the unstable region and also it is difficult to observe the unstable region in a real-life experiment, the frequency points inside that part are excluded.

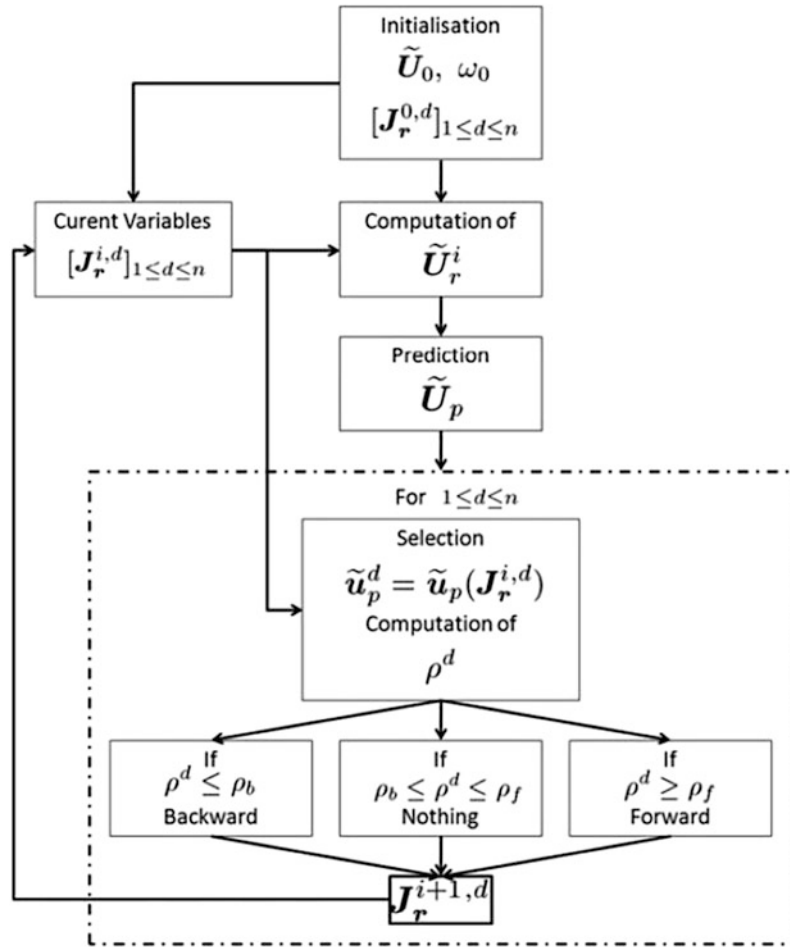


Fig. 29.2 The algorithm for Grolet and Thouverez’s method [10]

When comparing two different total response values, one taken from the classical solution curve and the other from an AHBM solution curve, it has to be guaranteed that the two points are located at the same frequency. In order to make the frequencies correspond, the step sizes are taken as constant in the path following schemes. If some of the chosen frequencies do not overlap, linear interpolation is performed. For the purpose of preventing the interpolation from being inaccurate, the step size is chosen as a small value. After the error values are obtained for all of the chosen frequency points, integral error values are calculated. For computing the integral error, the trapz function of MATLAB® is used. Since the step size is chosen as a small value, it is considered that the number of partitions to be used in the trapezoidal rule and the accuracy obtained are sufficient to give a good idea about the order of integral error.

The time spent during the solutions are measured by the tic-toc function of MATLAB®. All of the analyses are run on the same Dell XPS 15-L502X laptop computer having a 2.0 Intel Core i7-2630 (quad core) processor and 4 GB of 667 MHz DDR3 RAM.

29.3.2 The Analyzed System

The studied system is illustrated in Fig. 29.3. Parameters of the system and parameters of the gap nonlinearities are given in Tables 29.1 and 29.2, respectively.

When the linear system is studied, it can be seen that the natural frequencies are located at 48.21 and 119.76 rad/s. The frequency range in consideration is [35, 65] rad/s, which covers the first resonance frequency. Since the external forcing contains three harmonics, two super-harmonic resonances, located at 39.92 and 59.88 rad/s, the points where 2ω and 3ω are equal to the second natural frequency, are expected. Also, one more resonance must appear at the first natural frequency.

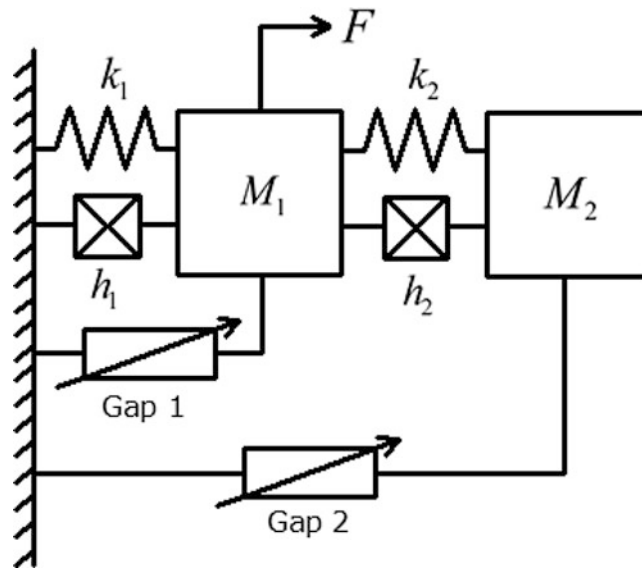


Fig. 29.3 A 2-DOF system with gap nonlinearity

Table 29.1 Parameters for the 2-DOF System

M_1 (kg)	M_2 (kg)	k_1, k_2 (N/m)	h_1, h_2 (N/m)	F (N)
1	0.75	5000	50	$30 \sin(\theta) + 30 \sin(2\theta) + 30 \sin(3\theta)$

Table 29.2 Parameters of Gap Elements

Nonlinear element	δ (m)	k_g (N/m)
Gap 1	0.05	500
Gap 2	0.1	500

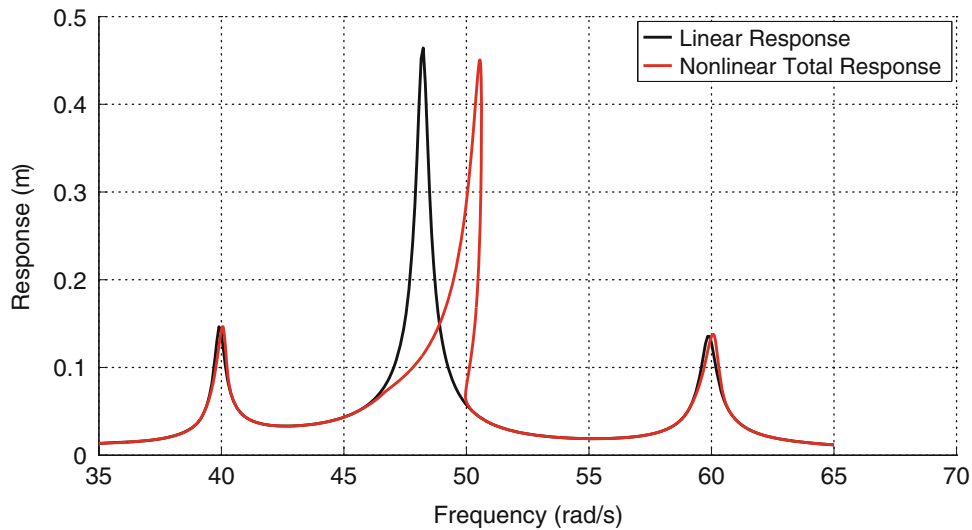


Fig. 29.4 Total Response of the first DOF

Gap nonlinearity is chosen for this study, since it excites higher harmonics when the system makes large oscillations; whereas for small oscillations, the system acts as a linear structure. Therefore adaptive algorithms analyzing systems with gap nonlinearity are forced to change the harmonic content often. The total response and contribution of each harmonic obtained by using a 10-harmonic solution, i.e. baseline solution for error calculations, are given in Figs. 29.4 and 29.5 for the first DOF, respectively.

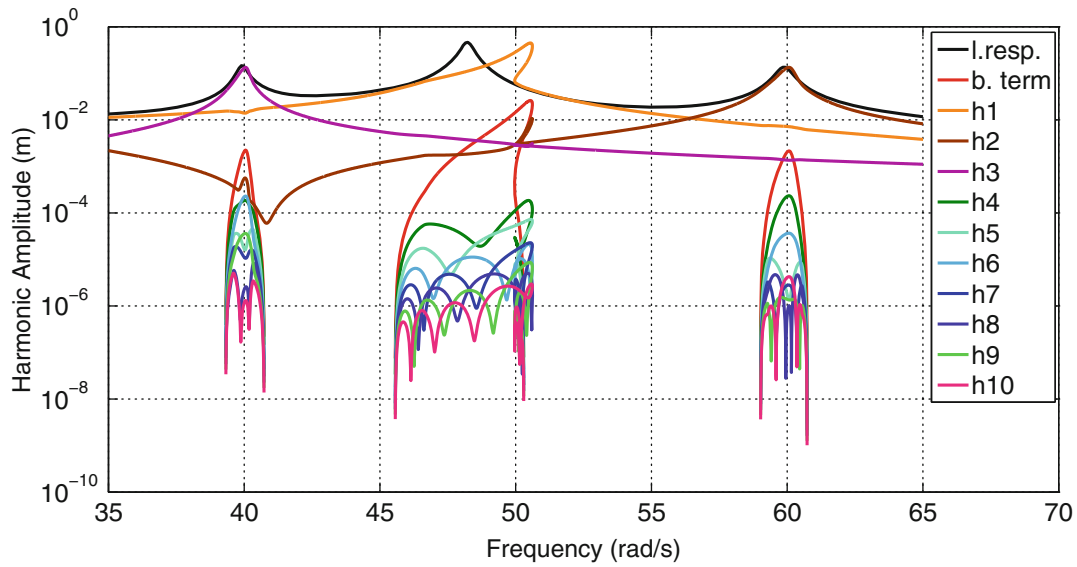


Fig. 29.5 Amplitudes of harmonics for the first DOF

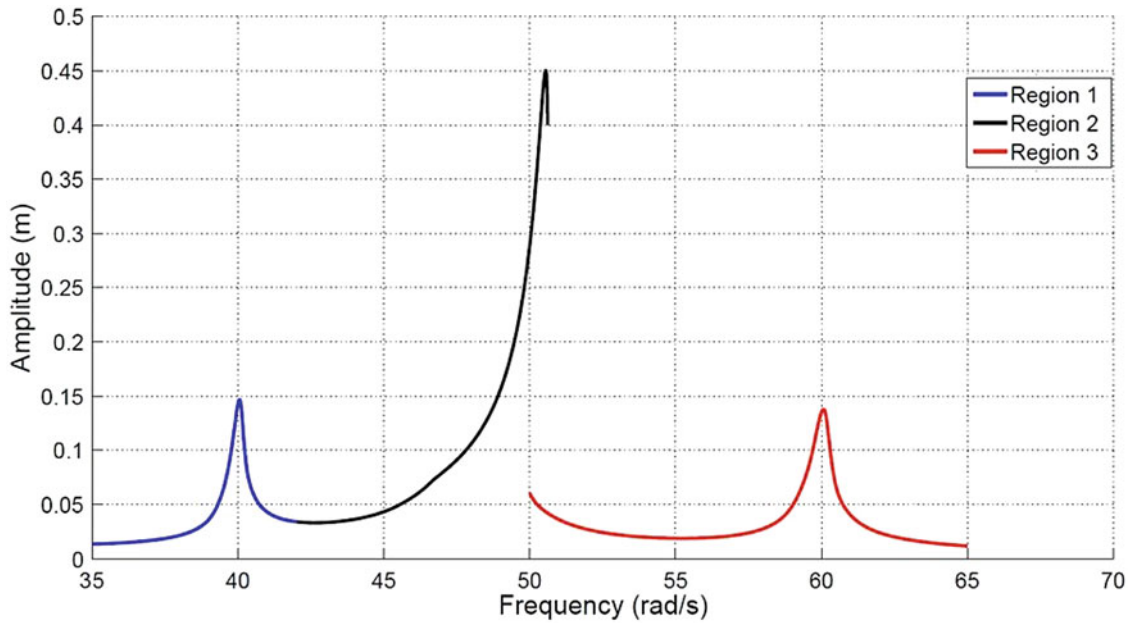


Fig. 29.6 Regions defined for error analysis

For the error analysis, the total response curve is divided into three parts. As explained in the previous section, the unstable region is excluded and the resulting regions are shown in Fig. 29.6.

29.3.3 Results

In this section, two criteria used for evaluation are presented in graphical form. In these plots, the horizontal axis is the computational time, and the vertical axis is the error. Every point on the given curve corresponds to an analysis performed by one of the AHBMs, with a different harmonic selection parameter set. The methods for which the points lie closest to the lower left corner are more favorable methods, since they manage to achieve greater accuracy in a shorter time. Results are presented in Figs. 29.7 and 29.8.

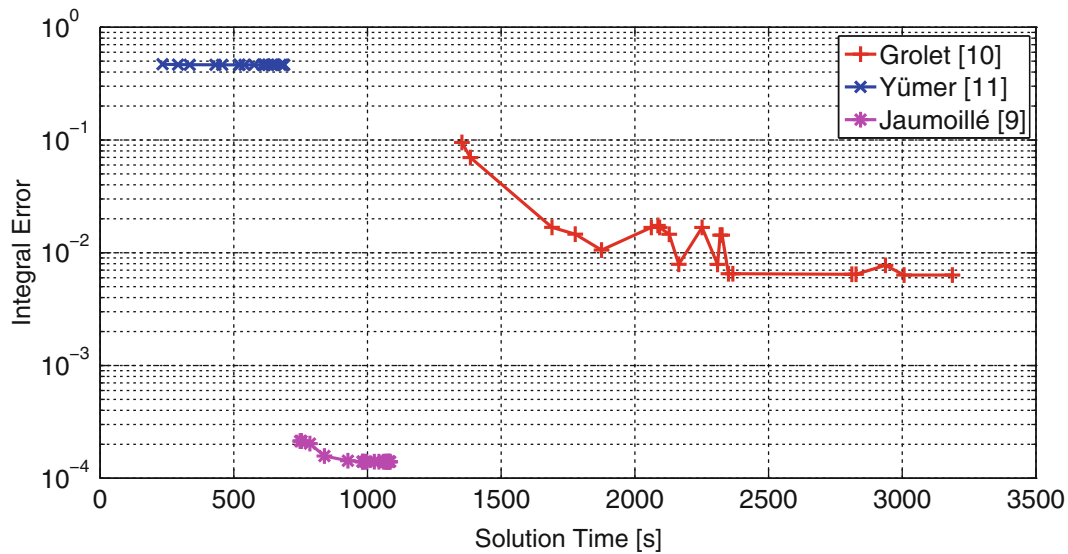


Fig. 29.7 Computational time vs integral error

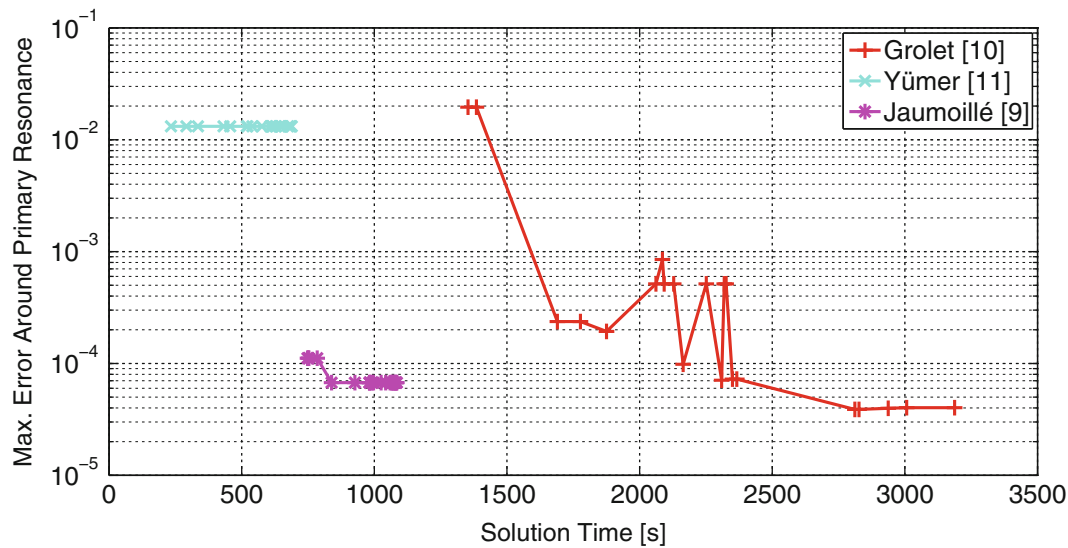


Fig. 29.8 Computational time vs maximum error around the first natural frequency

The results indicate that the computational times are relatively low, as desired for Yümer's method. However the error values are significantly high and they tend to stay constant, although the control parameter changes. It is observed that the method failed to increase the number of harmonics around the super-harmonic resonances. Therefore, the error values around those regions are high even though they are quite low near the first natural frequency. However, since Region 2 starts from 42 rad/s, the error values corresponding to region 2 turn out to be high as well. Around the secondary resonances, a single harmonic representation is used regardless of the value of a and the same amount of error is introduced, in almost every analysis. This phenomenon can be clearly seen in the overlapping absolute error plots given in Fig. 29.9.

The results also show that, even though the harmonic selection parameters change, the error obtained by Jaumoillé et al.'s method does not vary greatly. In Fig. 29.10, the number of harmonics for two different threshold parameters are shown. For $\varepsilon = 10^{-20}$, the algorithm is expected to retain much more harmonics than the case where $\varepsilon = 10^{-2}$. It can be seen from the figure that, this result occurs only in the neighborhood of the resonances. In other regions, the harmonic contents are almost the same. In addition to this in Figs. 29.10 and 29.11 one can see that, around 40 rad/s one can see that the algorithm increases the number of harmonics rapidly at first. Then a sudden drop occurs until the number of harmonics starts to rise again. The same behavior happens at other resonances as well. When the drop in the retained harmonics occurs, as expected, the error increases rapidly and reaches to a maximum. The algorithm shows this behavior for every value of ε considered in

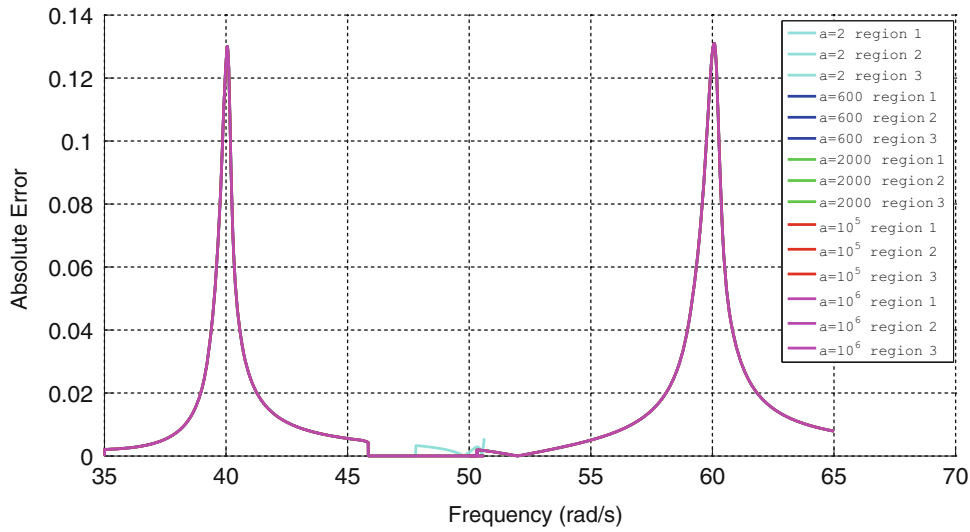


Fig. 29.9 Error plots for Yümer's method

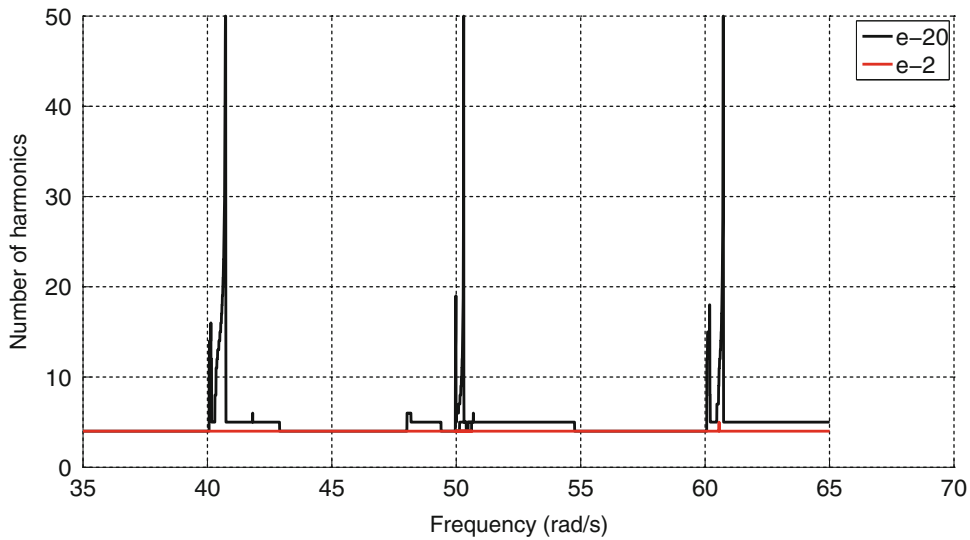


Fig. 29.10 Number of retained Harmonics for Jaumoillé's method

this study, therefore the maximum error values do not change significantly. For the case where $\varepsilon = 10^{-20}$, the number of harmonics reach up to 50 at 3 points however, since this number is extremely high for the current nonlinear system under study, its contribution to the accuracy is very low.

The method developed by Grolet and Thouverez obtains higher integral error values than Jaumoillé et al.'s method; however, it is more responsive to the changes in harmonic selection parameters. The method is actually accurate around the primary resonance; however, the higher integral error shows that the method is somewhat less accurate around the superharmonic resonances.

29.4 Discussion and Conclusion

In this paper, a short summary on AHBM's currently available in the field of structural dynamics are presented. Using a two-DOF lumped parameter model with gap nonlinearities, AHBM's are compared with each other in terms of computational time and accuracy. For comparison purposes, 10-harmonic solution is used as the baseline. It is observed, that the AHBM's

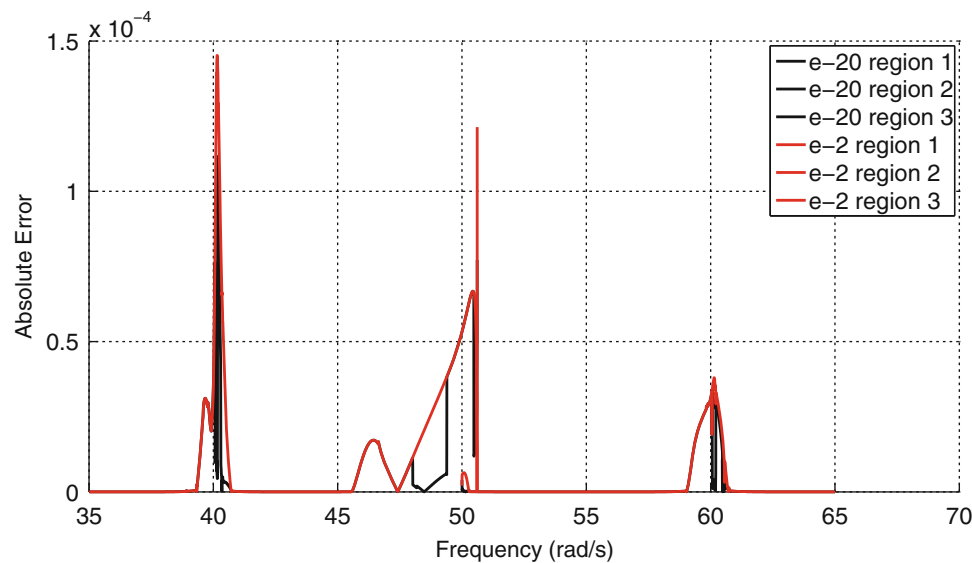


Fig. 29.11 Error plots for Jaumoillé's method

considered are perform better around the primary resonance whereas their performance around the superharmonic resonances are limited. Yümer's method is the fastest; however, its accuracy very limited. In terms of computational time and accuracy, Jaumoillé's method is better than the other two methods.

References

1. LaBryer, A., Attar, P.J.: A harmonic balance approach for large-scale problems in nonlinear structural dynamics. *Comput. Struct.* **88**(17–18), 1002–1014 (2010)
2. Sarrouy, E., Sinou, J.: Non-linear periodic and quasi-periodic vibrations in mechanical systems-on the use of the harmonic balance methods. In: Ebrahimi, D.F. (ed.) *Advances in Vibration Analysis Research InTech*, Rijeka, Croatia, pp. 419–434. (2011)
3. Kim, Y., Choi, S.: A multiple harmonic balance method for the internal resonant vibration of a non-linear Jeffcott rotor. *J. Sound Vib.* **208**, 745–761 (1997)
4. von Groll, G., Ewins, D.J.: The harmonic balance method with arc-length continuation in rotor/stator contact problems. *J. Sound Vib.* **241**(2), 223–233 (2001)
5. Lau, S.L., Cheung, Y.K.: Amplitude incremental variational principle for nonlinear vibration of elastic systems. *J. Appl. Mech.* **48**(4), 959 (1981)
6. Yuste, B.S.: Comments on the method of harmonic balance in which Jacobi elliptic functions are used. *J. Sound Vib.* **145**(3), 381–390 (1991)
7. Kim, Y., Noah, S.: Quasi-periodic response and stability analysis for a non-linear Jeffcott rotor. *J. Sound Vib.* **190**, 239–253 (1996)
8. Kim, T.C., Rook, T.E., Singh, R.: Super- and sub-harmonic response calculations for a torsional system with clearance nonlinearity using the harmonic balance method. *J. Sound Vib.* **281**(3–5), 965–993 (2005)
9. Jaumoillé, V., Sinou, J.-J., Petitjean, B.: An adaptive harmonic balance method for predicting the nonlinear dynamic responses of mechanical systems—application to bolted structures. *J. Sound Vib.* **329**(19), 4048–4067 (2010)
10. Grolet, A., Thouverez, F.: On a new harmonic selection technique for harmonic balance method. *Mech. Syst. Sig. Process.* **30**, 43–60 (2012)
11. Yümer, M.E.: *On the Non-Linear Vibration and Mistuning Identification of Bladed Disks*. Middle East Technical University, Ankara, Turkey (2010)
12. Zhu, L., Christoffersen, C.E.: Adaptive harmonic balance analysis of oscillators using multiple time scales. In *IEEE-NEWCAS Conference, 2005. The 3rd International*, Quebec City, 187–190 (2005), doi: [10.1109/NEWCAS.2005.1496738](https://doi.org/10.1109/NEWCAS.2005.1496738)
13. Gourary, M.M., Rusakov, S.G., Ulyanov, S.L., Zharov, M.M., Gullapalli, K.K., Mulvaney, B.J.: A New Computational Approach to Simulate Highly Nonlinear Systems by Harmonic Balance Method. In *16th IMACS WORLD CONGRESS 2000 on Scientific Computation, Applied Mathematics and Simulation*. Lausanne, 366 (2000)
14. Maple, R.C., King, P.I., Orkwis, P.D., Mitch Wolff, J.: Adaptive harmonic balance method for nonlinear time-periodic flows. *J. Comput. Phys.* **193**(2), 620–641 (2004)
15. Cameron, T.M., Griffin, J.H.: An alternating frequency/time domain method for calculating the steady-state response of nonlinear dynamic systems. *J. Appl. Mech.* **56**(1), 149 (1989)