Chapter 6
Connections to Numeracy

Abstract Volume of rectangular prisms extended into an interesting scaling problem: One child asked if they could try to build a Soma cube as large as the teacher's demonstration Soma cube (made with very large individual cubes) using the smaller Soma figures. Initially they built four Soma cubes using the small sets of Soma figures and arranged them to look like a very large Soma figure #2. They eagerly attacked the problem of how many little individual cubes were in this figure each showing at least two ways to find the answer. The following week they built a huge cube using 27 smaller sets of Soma cubes and calculated 27×27 to find how many little individual cubes were in this model. Permutations within cake patterns: At the end of each year, the children were challenged to create a "cake" with the seven Soma figures. The cake had to have 24 cubes for a base and three "candles" on the second level. They created several 3×8 and 4×6 cakes and drew the top plan view coding patterns for these to submit to the baker. We share how children from three different cohorts discovered the permutations of each cake design from the three or four Soma figures (#1, #5, #6, and #7) that have the same 3-cube footprint.

6.1 Scaling up the Soma Cube

In Chap. 5, we described the connections the children made from the top plan view to represent rectangular prisms, enumeration of the total number of cubes in rectangular prisms, and the conceptualization of the volume formula for rectangular prisms. Here, we describe an interesting numeration problem that arose from the children's fascination with the Soma puzzle cube.

We frequently allowed free exploration time during which many of the children tried to build the Soma cube of 27 unit cubes made from all seven Soma figures. According to Wikipedia (https://en.wikipedia.org/wiki/Soma_cube) there are 240 distinct solutions, excluding rotations and reflections. On one particular day, one of the boys, David (pseudonym), posed an interesting problem: If we build a lot of Soma cubes each with the seven different Soma figures, how many would it take to

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_6

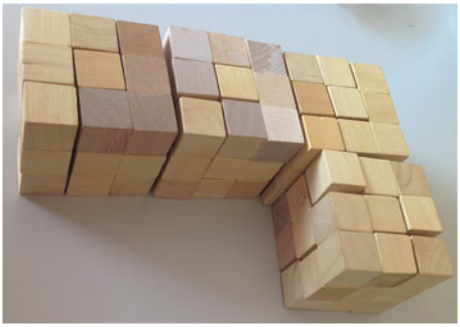


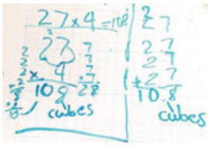
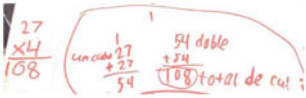
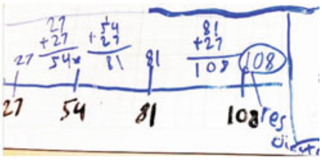
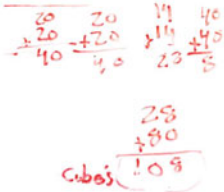
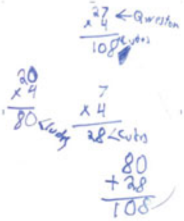
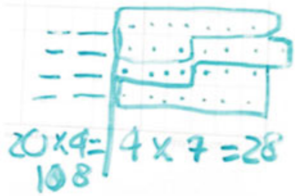
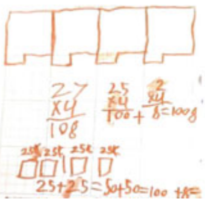
Fig. 6.1 The large Soma figure #2 made from four small sets of Soma figures

make a cube as big as the one made from the teacher's much larger Soma figure demonstration set? This became the problem for that particular session.

First, the children set about trying to build a cube from all seven Soma figures. We never emphasized this particular problem, since we always wanted that to be a self-determined problem choice that always remained interesting due to the large number of possible solutions. This took several minutes until Jason (pseudonym) was successful. He was charged with building more, using everyone else's Soma figure sets. David immediately set about carefully creating a very large Soma figure #2, using Jason's four completed Soma cubes (see Fig. 6.1).

Although David initially had wanted to build a very large cube with several of the small Soma cubes, Sack and Vazquez simultaneously asked, "How many unit cubes are in this large Soma #2 figure?" The children were excited about this problem, being a good extension from their developing multiplication skills. Those who struggled also had difficulty adding double-digit numbers fluently and often counted on their fingers, which is an indication that they may not have been ready for formal multiplication (Van Niekerk, personal communication, June, 2009). They all knew that this large Soma figure comprised 4 sets of 27 cubes. Doing a multiplication problem with 27 as a factor was totally new to most of them. In class, Vazquez had deliberately avoided the traditional algorithm, preferring to let the children "invent" their own algorithms. They shared their different methods to the whole class so that they all could benefit from different perspectives. She also emphasized the distributive property of multiplication by decomposing difficult numbers into two easier numbers using array models to show why this works. For example, 7×6 may be represented as an array of 7 rows with 6 squares in each. By drawing a thick line below the 5th row, they could see this was equivalent to 5 rows of 6 and then 2 more rows of 6, or $7 \times 6 = 5 \times 6 + 2 \times 6$. Decomposing numbers to facilitate multiplication through the distributive property is emphasized in the 3rd grade curriculum. During their daily mathematics instruction, teacher Vazquez, always asked for such explanations, in order for the presenting child to develop the mathematics process standards, to improve her or his communication skills and also for the other children to make sense of others' conceptualizations.

Table 6.1 Samples of children's calculations and researchers' comments

	Children's calculations of 27 × 4	Researcher's interpretations
A		Adept at double-digit, multiple number addition
B		Used "double the double": Added 27 + 27 and then doubled the sum: 54 + 54
C		Added 27 + 27 = 54. Then, 54 + 27 = 81. Then, 81 + 27 = 108 Showed each sum on a number line
D		Decomposed 27 into 7 + 20 Doubled 7 to get 14 (not shown) and doubled 14 to get 28 Doubled 20 to get 40 and doubled 40 to get 80. Then added 28 + 80
E		Wrote 27 × 4 as the "Question." Decomposed the problem into 20 × 4 + 7 × 4 Followed teacher's instruction to keep units (cubes) in the calculation
F		Represented 27 × 4 using the teacher's base-10 notation: 2 lines for two 10 s and 7 dots for seven 1 s. Set these up vertically as four sets of 2 lines and 7 dots Grouped the top 7 ones with 3 ones in the next row to make 10. Grouped the bottom 7 ones with 3 ones in the row above to make 10, for a total of 10 tens; and 8 ones grouped in the middle of the ones Also calculated using decomposition of 4 × 20 and 4 × 7
G		David used the grid on the board to draw four 27 s. He drew 25-unit squares on the grid andthen added 2 more squares to each to make 27 s. He decomposed 27 into 25 + 2 and added the products 25 × 4 and 2 × 4 When Irma asked how he knew how to multiply by 25, he said, "It's like money. A quarter is 25 cents and four quarters is $1, or 100 cents."

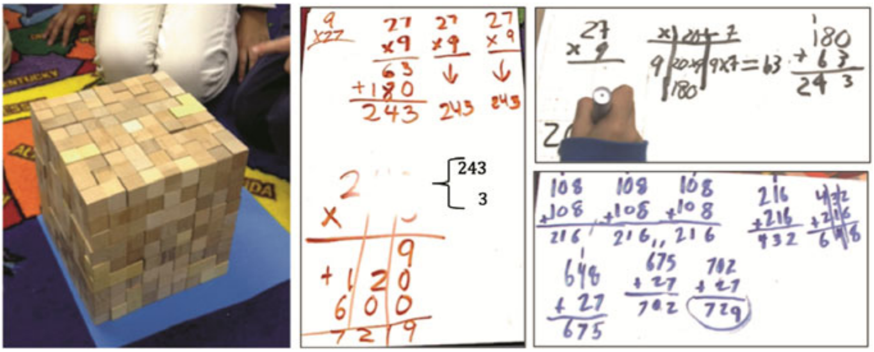


Fig. 6.2 The 27-by-27 cube and examples of children's calculations

The children set about solving this problem on small dry-erase boards. Vazquez told them to show the solution in at least two different ways. Afterwards, as during the instructional day, she asked each child to explain what he or she had written. Examples are shown in Table 6.1.

The following week, the children all set about building 27 small Soma cubes in order to construct a very large 3D array, using all of the 27 sets of small Soma figures. They immediately set about calculating how many unit cubes were in this 27-by-27 array. Examples are shown in Fig. 6.2.

In the left and top right examples, the large figure contained 3 slices, each with 9 small 27-cubes. In top right example, the child had calculated 9×27 by decomposing 27 into $20 + 7$. S/he was busy working the next step, $3 \times 9 \times 27$, or 3×243. In the left figure, the child was able to calculate $9 \times 27 = 243$; then, multiplied 243×3, an extension of 1-digit by 2-digit multiplication that is in the 3rd grade curriculum. In the lower right example, the child knew that each of six Soma figures contained four 27-cubes or 108 small unit cubes (from the previous week), plus Soma #1, with three 27-cubes.

This learner-generated problem provided the children an opportunity to apply their developing multiplication skills to a much more difficult problem than they were expected to compute during their regular mathematics classes. Each child used at least two different ways to solve the problem (not all illustrated in this chapter) to check their initial answers. We were surprised at the plethora of solution methods and particularly excited about David's enthusiasm with this problem. This supports Connell's (2001) view that children are capable of solving difficult problems based on their past experiences that have now become instantiated in their minds as objects on which to act. The different approaches to solving this problem show that each child had found ways to extend their prior knowledge using methods of personal choice. These episodes also confirm Presmeg's (1992) view that imagistic or spatial processing is essential for the development of abstraction and generalization in that the children abstracted routine skills to tackle their self-generated task of finding the total number of cubes (27 × 27) even though they are only expected to calculate 2-digit by 1-digit products in 3rd grade.

6.2 Permutations Within Cake Patterns

Since Year 1 we celebrated the last day of the program by inviting parents to see what their children have learned. To prepare, we asked the children to design a “cake” using all seven unique Soma figures. Since Soma figures #1–4 can lie flat, but Soma figures #5–7 have at least 1 cube on the second level, we specified that the cake have exactly 3 “candles”, from Soma figures #5, 6, and 7. Thus, the cake must be a 24-unit rectangle. We asked what dimensions the cake could have, which was a good application of their developing multiplication skills. Most said 3-by-8 or 4-by-6. We asked if they could make a 1-by-24 rectangle, but they said it was not possible since all of the Soma figures were at least 2-units wide, depending on how you laid them down. Every year, some children tried to build a 2-by-12 cake. Only by experiment did they eventually reason why this was not possible with the seven different Soma figures. We told them one pattern would be selected to be sent to the baker, to bake a puzzle cake for the party. They also had to represent their different cake designs on paper, using different colors for the “footprint” of each Soma figure. An example is shown in Fig. 6.3.

During Year 1, Sophia (pseudonym), noticed that Soma figures #5, 6, and 7 all had the same 3-cube footprint. Vazquez asked her to try to determine how many different cake patterns she could make just by moving these three Soma figures. Generally, Sophia struggled with reading and computation, both requiring adept decoding skills. She had demonstrated extremely strong visual skills throughout the year in this after-school program. This activity provided her an opportunity to build her numeracy using visual cues. She successfully found all six permutations of #5, 6, and 7, and listed them, using the F(ront), M(iddle) and B(ack) format that she created (see Fig. 6.4). She then transferred the knowledge to ordering the 3 digits. She had struggled with this concept in her regular class earlier that day.

Also in Year 1, but with the 4th grade group, Dawn made a 3 × 8 cake and showed how she could interchange Somas #1, 5, 6, and 7 (same footprint) to make additional cakes. We challenged her with “How many ways?” She, like Sophia, had some

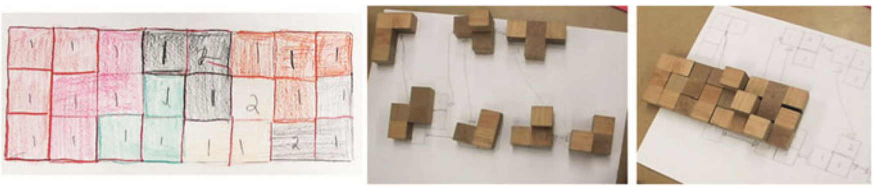


Fig. 6.3 Soma cake patterns

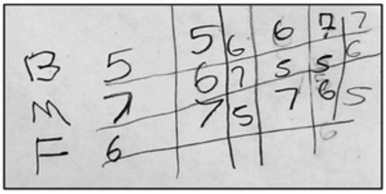


Fig. 6.4 Sophia's permutations of Soma figures #5, #6 and #7

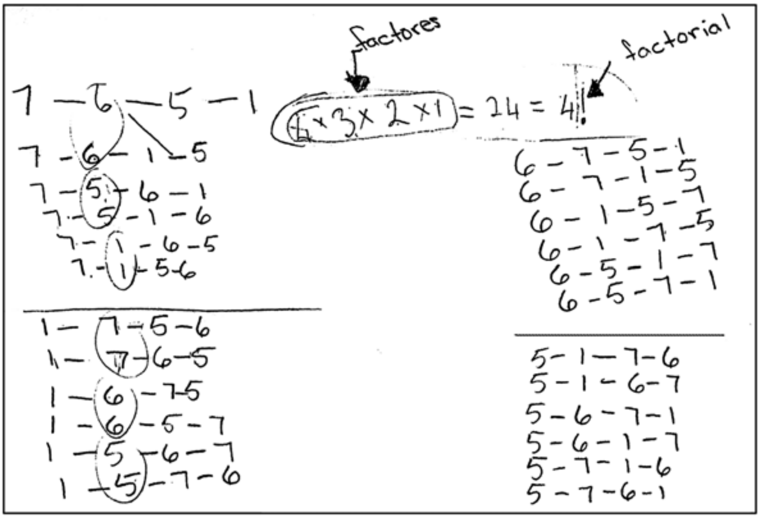


Fig. 6.5 Dawn's permutations of Soma figures #1, 5, 6 and 7

numeracy gaps. After pointing out the combination sequence on her cake pattern: 7–6–5–1, she started to write another random combination, starting with 1—… We suggested she keep the 7 and the 6 in place and think about another arrangement. She immediately wrote 7–6–5–1, and then 7–6–1–5 and continued until she had all 6 arrangements beginning with 7. She then proceeded to write down all six combinations beginning with 6—and had no trouble suggesting that there would be six more that began with 5 and six more that began with 1 for a total of 24 different combinations (See Fig. 6.5). Then, we asked Dawn to look at how we get 24 (four sets of 6; and three sets of 2 within each set of 6): $4 \times 3 \times 2 \times 1 = 24$. By this time, her friend, Emily, became involved. Emily remembered that the numbers 4, 3, 2, and 1 are all factors of 24. They learned a new word, *factorial*, and its symbol, 4! There was much excitement that this might be taught in another 3 years' time, in middle school.

Over the 8 years of the project, finding permutations occurred only if a child noticed the footprint pattern for these Soma figures. In Year 4, one remarkable child,

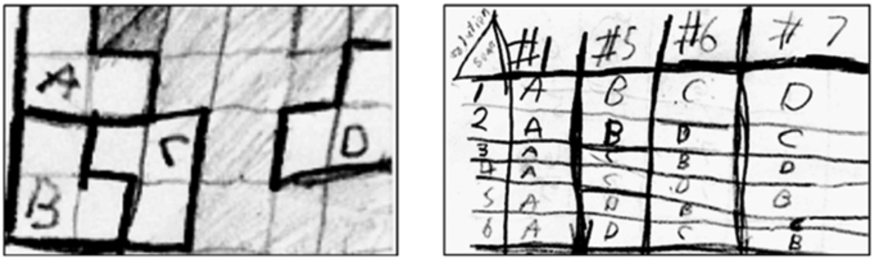


Fig. 6.6 Henry's permutations of Soma figures #1, 5, 6, and 7

Henry, organized his thinking using a table, partially shown in Fig. 6.6 alongside his cake pattern. He showed all 24 permutations in his table. The other children interpreted his pattern, explaining that A, B, C and D represented Soma figures #1, #5, #6, and #7 shown in the first row of his table. In row 2, the interchange of C and D meant that Soma figure #6 switched places with Soma figure #7. The children collectively built this pattern and showed how each solution evolved from the previous one in the table. While Henry had also multiplicatively reasoned why there are 24 permutations for each cake pattern, the others reasoned through his table patterns.

6.3 Conclusions

It is remarkable that the children were able to move from the abstract top plan numeric view and front, side and top views representations to the 3D models independently of the computer interface. The research team considered this to be evidence of the children's growing ability to visualize, abstract and generalize as they moved between abstract and 3D assembly models. Actions on objects (Connell 2001) occured concretely with the 3D models, virtually through the Geocadabra Construction Box interface and ultimately as mental imaging through the powerful problem-solving approaches developed by the research team. Through their different configurations the Soma figures provided a high degree of complexity and constraint to the instructional tasks and forced the children to engage in a variety of mathematical tasks including mental transformations in ways that would not be possible using loose cubes.

This research team, through this learning trajectory, attempts to fill the gap in the development of spatial thinking in the elementary mathematics curriculum as expressed by others. However, in a regular school classroom the day-to-day curriculum constraints include pressure from mandated high-stakes testing and rigidly scripted programs used within many school systems. Thus the research team faces the challenge of how to impart these visualization experiences to a wide audience of practitioners given the open-ended nature of the tasks in their learning trajectory.

References

Connell, M. L. (2001). Actions on objects: A metaphor for technology-enhanced mathematics instruction. *Computers in the Schools., 17*, 143–171.

Presmeg, N. C. (1992). Metaphors, metonymies and imaginative reality in high school mathematics. *Educational Studies in Mathematics, 23*, 595–610.