SPRINGER BRIEFS IN EDUCATION

Jacqueline Sack
Irma Vazquez

A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children

Springer

SpringerBriefs in Education

More information about this series at http://www.springer.com/series/8914

Jacqueline Sack · Irma Vazquez

A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children

Springer

Jacqueline Sack
University of Houston-Downtown
Houston, TX
USA

Irma Vazquez
Houston Independent School District
Wharton Dual Language Academy
Houston, TX
USA

ISSN 2211-1921 ISSN 2211-193X (electronic)
SpringerBriefs in Education
ISBN 978-3-319-29798-9 ISBN 978-3-319-29799-6 (eBook)
DOI 10.1007/978-3-319-29799-6

Library of Congress Control Number: 2016932513

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Contents

Chapter 1
Introduction and Project Background

Abstract The project began as part of a PhD dissertation study in South Africa, by Retha Van Niekerk after she met with Pierre Van Hiele and with researchers within the Wiskobas project at the Freudenthal Institute in The Netherlands. Her work with Ton Lecluse, author of the Geocadabra Construction Box, developed for this project, and her professional association with authors, Sack and Vazquez, formed the basis for the work done in the US after she returned to South Africa. A brief overview of the book's organization will end this chapter.

1.1 The Start of a Task-Design Journey in the Context of Block Buildings

The journey started in 1993 when researcher Van Niekerk embarked on post-graduate mathematics education studies in South Africa. In 1994, she visited the Freudenthal Institute where she studied directly with Van Hiele (1986) and with researchers involved with the Wiskobas project (Treffers 1993). Van Niekerk was introduced to the teaching sequence, *building blocks* (Wijdeveld 1977), also called "4 Kubers" in the Dutch context. When she returned to South Africa she published "4 Kubers in Africa," which she presented at the 1995 Panama Najaars Conference in The Netherlands (Van Niekerk 1995, 1996a). She designed a framework, the Spatial Operational Capacity (SOC) model as a result of her PhD (Van Niekerk 1997). The SOC model assists in the design of activities or tasks that deal with the development of spatial knowledge and addresses the complex interrelated nature of the teaching and learning of geometry. This framework will be addressed in detail in Chap. 2.

The SOC model served as a framework to support the design of a preliminary curriculum (Van Niekerk 1996b) for geometry in South Africa in 1999 and ultimately served to guide the development of the *space and shape* curriculum strand of the South African national curriculum in the early 2000s (Wessels and Van Niekerk 2001). In 2001, Van Niekerk utilized the SOC model as a framework to develop an entire teaching and learning trajectory for young children utilizing

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_1

block buildings as a context. During this time, she became aware of the work of Ton Lecluse, designer of the *Geocadabra* dynamic geometry software (2005). In 2001, Van Niekerk visited Lecluse in The Netherlands and together they began to collaborate on the possibility of adapting Geocadabra for young learners in the context of block buildings. Later that year, van Niekerk relocated from South Africa to the US and joined forces with author, Sack, who, at that time, was a veteran middle school mathematics teacher, and introduced her to the SOC model as well as to Lecluse's Geocadabra software. Together, they co-taught two series of professional development workshops in geometry for elementary and for middle-grades teachers. Author, Vazquez, attended these and later became a co-instructor with Sack for the same series.

Van Niekerk returned to South Africa in 2003 and continued to develop a series of block building tasks mainly for primary grades. These tasks were incorporated into an Afrikaans-language booklet, *Geocadabra Meetmaatjie*, which served as the first attempt at a teaching trajectory incorporating Geocadabra as a tool for developing spatial knowledge. The Geocadabra interface was also translated from Dutch to Afrikaans for this intervention. This booklet, translated into English by Van Niekerk and Sack and re-named *Geobuddies*, was later used as a prototype for Sack's work in the US that began in 2007.

In 2004, Lecluse and Sack joined van Niekerk in South Africa to present their work at the Amesa Conference at North West University. At this stage it was decided that more work needed to be done. In 2007, Sack started to work with teacher Vazquez in her Grade 3 dual language (Spanish-English) classroom in Texas. They utilized the Geocadabra software, the English Geobuddies booklet and the pencil and paper activities that were initially designed in 2003 by van Niekerk in South Africa. Spanish was also integrated orally as needed. This intervention resulted in extended development and refinement by Sack and Vazquez of the tasks that constitute the teaching trajectory presented in this publication.

As a result of the collaboration among Lecluse, van Niekerk, Sack and Vazquez, based on the findings in the US classroom, the original SOC framework had now evolved. The *virtual models* strand was added to the original SOC design. This redesign of the SOC model, which was as a direct result of the impact of the task-sequences, had also culminated with the redesign of the Geocadabra building blocks interface, by Lecluse. The Geocadabra interface now included the updated simple interface, the *Construction Box,* and also a more advanced interface called the *Extended Construction Box*. It is important to mention here that the authors constantly considered the important role of the different media of execution and presentation and its effects on the entire learning thinking process (See Sack and Vazquez 2011).

Early progress of this work was published in NCTM's 71st Yearbook (Sack and Van Niekerk 2009) as a chapter, "Understanding the Spatial Operational Capacity of young children using wooden cubes and dynamic simulation software."

Chapter 2 deals with the theoretical frameworks undergirding the project, the design research methodology and the school context in which the study was conducted. In Chap. 3 we share introductory activities that move between loose cubes, the set of Soma figures and 2D conventional pictures of assemblies of two Soma figures. Then, in Chap. 4, we show how the children learn to use the Geocadabra Construction Box. In Chap. 5, abstract top-view plans and top, front, and side views are integrated with the 2D conventional and 3D model representations of various block figures. Finally, Chap. 6 shows how visualization activities integrated with numeracy development, in particular, with multiplication skills that are typically developed in 3rd grade.

References

Lecluse, A. (2005). Geocadabra (Computer software). Accessed August 26, 2015, from http://home.casema.nl/alecluse/setupeng.exe.

Sack, J. J., & Van Niekerk, R. (2009). Developing the spatial operational capacity of young children using wooden cubes and dynamic simulation software. In T. Craine, R. Rubenstein, & R. Reston (Eds.), *Understanding geometry for a changing world, 71st yearbook* (pp. 141–154). Reston: NCTM.

Sack, J., & Vazquez, I. (2011). The Geocadabra Construction Box: Open source geometry interface within a 3D visualization program for elementary children. In A. Méndez-Vilas (Ed.), *Education in a technological world: Communicating current and emerging research and technological efforts* (pp. 505–512). Badajoz: Formatex Research Center.

Treffers, A. (1993). Wiskobas and freudenthal: Realistic mathematics education. *Educational Studies in Mathematics, 25*, 1–2.

Van Hiele, P. M. (1986). *Structure and insight*. Orlando: Academic Press Inc.

Van Niekerk, R. (1995). From spatial orientation to spatial insight: A geometry curriculum for the primary school. *Pythagoras, 36*, 7–12.

Van Niekerk, R. (1996a). 4 Kubers in Africa. *Pythagoras, 40*, 28–33.

Van Niekerk, R. (1996b). What about the spatial development of the primary school child? *Tydskrif vir Primere Onderwys, 12*, 14–31.

Van Niekerk, (Retha) H. M. (1997). *A subject didactical analysis of the development of the spatial knowledge of young children through a problem-centered approach to mathematics teaching and learning*. Ph.D. Dissertation, Potchefstroom University for CHE, South Africa.

Wessels, Dirk C. J., & Van Niekerk, R. (2001). Building the spatial operational capacity (SOC) of the primary school child through rich learning experiences: A geometry curriculum for the Gauteng Province in South Africa. *Tydskrif vir Christelike Wetenskap, 37*, 89–113.

Wijdeveld, E. (1977). Vierkubers- Een onderwijs leerpakket voor de basisschool. *Wiskobas Bulletin.* 6. Instituut Ontwikkeling Wiskunde Ondewijs.

Chapter 2
Theoretical Frameworks and School Context

Abstract The rationale and Spatial Operational Capacity theoretical framework that support the project are described along with the design research methodology that evolved over the 7 years of research, the school context, and a pre-interview used to support the researchers' perspectives on each participant child's beginning spatial ability.

2.1 Why Are Visualization Skills Important?

Visual and spatial thinking and reasoning "generally refers to the ability to represent, transform, generalize, communicate, document, and reflect on visual information" (Hershkowitz et al. 1989, p. 75). Pittalis and Constantinou (2010) summarize spatial thinking as "a form of mental activity that enables individuals to create spatial images and to manipulate them in solving various practical and theoretical problems" (p. 191). This includes finding meaning in the shape, size, orientation, location, direction or trajectory, of objects, processes or phenomena, or the relative positions in space of multiple objects, processes or phenomena. The importance of visual thinking and reasoning has been expressed by researchers and standards organizations across mathematical and scientific disciplines. Researchers have shown that imagistic processing, in balance with algebraic thinking, is an essential component in developing proficiency with abstract axiomatic mathematics (e.g., Dreyfus 1991; Presmeg 1992; Tall et al. 2001). Outside of mathematics, e.g., in the physical and geosciences, a pertinent question of interest to researchers is, "How do people combine information gathered from multiple viewpoints into a single integrated mental model of the three-dimensional object or process, and how can that inherent human ability be harnessed to help students interpret 1D or 2D data sets in terms of 3D processes?" (Science Education Research Center 2009). Researchers have shown that spatial abilities can be learned through appropriate learning experiences (e.g., Clements and Battista 1992; Ganesh et al. 2009; Yakimanskaya 1991).

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_2

2.2 The Spatial Operational Capacity Framework

The Spatial Operation Capacity (SOC) theoretical framework (see Fig. 2.1), originally designed by Van Niekerk (1997), consists of four main categories of variables that can contribute to the complexity of a visual image as a stimulus in task design namely:

1. Perception: The stimulus with which visual information is presented to the learner is grouped among four different categories: (i) full-scale images, (ii) virtual real images, (iii) conventional graphic images, or (iv) iconic images. These categories are differentiated by the closeness of the representation to reality in both a visual and a tactile sense.
2. Dimensionality: The objects, which are presented via the visual information that the learner perceives, processes or acts on, can be (i) one-dimensional (points and lines), (ii) two-dimensional (e.g., triangles, quadrilaterals), or (iii) three-dimensional images (e.g., prisms, pyramids) and may be a part of or the entire presented stimulus.
3. Transformations: A critically important cognitive process that must be addressed during visual processing, while acting on the object/s represented by the image, is the ability to comprehend the nature of the changes that objects and situations can undergo during perception. In other words, this is the ability of the learner to keep track of what is fixed and what changes when manipulating objects and

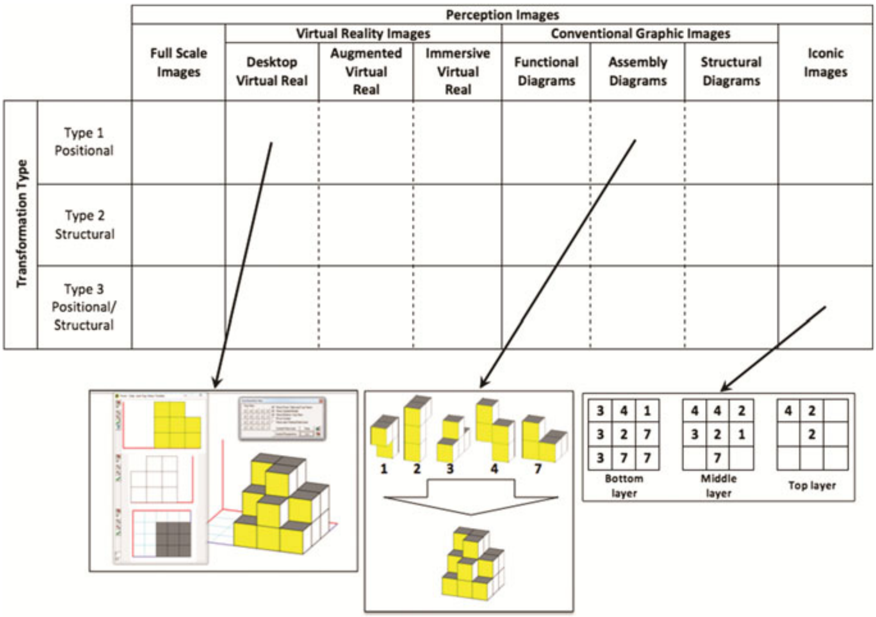


Fig. 2.1 The spatial operational capacity (SOC) model

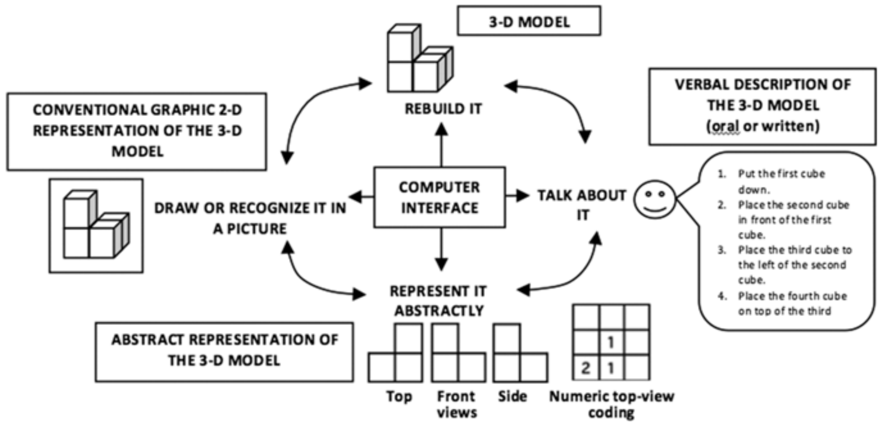


Fig. 2.2 Adapted SOC model

situations. The three different kinds of transformations that objects, which are represented via visual images, can undergo are (i) positional, (ii) structural, or (iii) combined positional-structural changes.

4. Mobility: The visual images contemporary learners encounter that can be represented with respect to mobility are determined by the nature of the visual image per se. This variable reflects the importance that the authors give to the role of the body in visual imaging (Hansen 2004). This mobility aspect can be represented as a continuum between a static medium (printed/typographic materials) and a potentially dynamic medium (digital and electronic materials) (Rückriem 2009, pp. 97–99). These different kinds of mobilities are represented in this model as (i) static (print) (ii) semi dynamic (e.g., PowerPoint slides, photo slides), or (iii) dynamic (video/film/television) images.

For this particular study, the SOC framework was adapted as shown in Fig. 2.2. Activities utilize actual 3D models, namely, loose cubes and puzzle figures, made from unit cubes glued together in different 3- or 4-cube arrangements (see Chap. 3); conventional 2D pictures of the 3D models that have hidden components since only up to 3 views can be shown in such pictures; abstract representations of the 3D models such as top-plan numeric views or front-side-top views that do not obviously resemble only one 3D figure; and the Geocadabra Construction Box dynamic computer interface that integrates these representations in real time (see Chap. 4).

2.3 Research Methodology

Design research principles guided the methodology for the entire project. While some design researchers focus their analyses on whole-group sense making (e.g., Cobb et al. 2001), others (e.g., Simon et al. 2010) focus on fine-grained individual participant's conceptualizations. Each lesson was part of a larger design experiment followed by a retrospective analysis in which the research team determined the actual outcomes and then planned the next lesson. This may have been an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that were deemed worthy of more attention. The overarching question that guided this research team's analysis was, how does the research team attend to individual children's sense making and to their collective understandings, in order to move to more complex and deeper challenges for the entire class? The data corpus included formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

2.4 School and Classroom Context

The study was conducted over a 7 year period beginning Fall 2007 in a dual-language urban elementary school serving approximately four hundred students within one of the largest public school districts in the mid-southwestern United States. Approximately 70 % of the students are Hispanic, 20 % are African American and the remaining 10 % are White or Asian. Three-quarters of the students are designated "At-Risk" and 55 % are English-language learners. The participants in the study represented a typical cross-section of the larger school community. Mathematics instruction is conducted in Spanish for students below fourth grade during the academic day but this project was conducted in English within an established after-school program at the school.

The research/instructional team initially consisted of university-based researcher, Sack, who had over 15 years of classroom experience, and two teacher-researchers, author Vazquez and another teacher, each with at least 8 years of classroom experience, who taught full-time at the third-grade and kindergarten levels in the school's dual-language program collaborating closely during the school day on matters related to their academic programs. During Year 2, the second teacher taught third grade but moved to a different school in Year 3. Sack and Vazquez continued to collaborate on the research aspects of the project over a total of 7 years. The project has become institutionalized within the school's after-school program under the direction of authors Vazquez and Sack.

For Year 1 of the study, at the beginning of the fall semester, all after-school third- and fourth-graders were invited to participate. This afforded the research team a classroom setting without the ongoing curricular and assessment pressures that have come to dominate the daily lives of school. Furthermore, the children who participated did so by choice. Fourteen fourth-graders and eleven third-graders started out in the study. However, various conflicting activities resulted in attrition and approximately eleven fourth-graders and eight third-graders attended the program consistently throughout the year. Teacher Vazquez had taught mathematics and science to all fourth-grade participants during their entire third-grade year. In Years 2–7, only third graders participated. However, during Year 2, some of the children who had participated as fourth-graders during Year 1 returned to get help with concepts that were being taught in very abstract ways.

Within this study, students worked independently or in small groups of two to four and all students were expected to ask each other for help or support before asking the teacher. Mutual respect was fostered in the classroom environment in which students felt comfortable expressing their understandings knowing they were safe to express their confusion or frustration in front of their peers. They were expected to explain and provide justification for their mathematical conclusions.

2.5 Pre-program Interview

A pre-program interview was designed to informally assess each participating child's ability to visualize the number of cubes in various conventional pictures shown in Fig. 2.3. The children were interviewed one-on-one with the researcher, who showed only 1 cube and asked how many would be needed to build the structures shown in each of the pictures. These were presented one at a time in the order shown. On average, for 10 participants, 9 out of 10 correctly determined the number of cubes

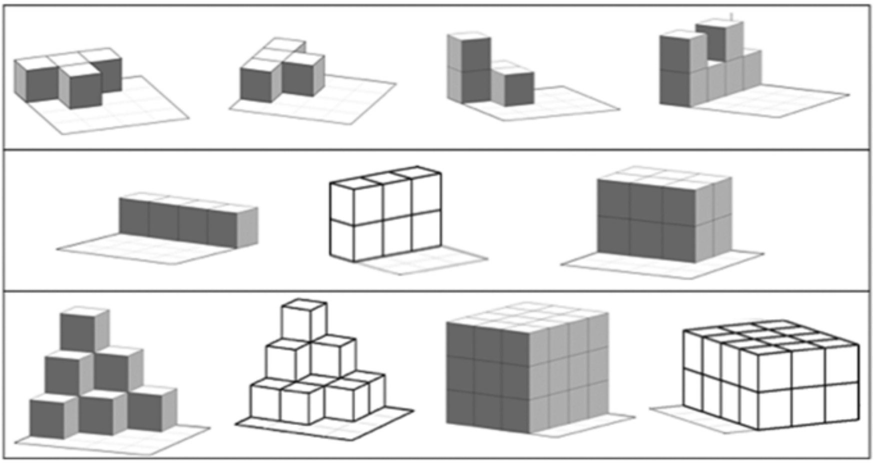


Fig. 2.3 Pre-interview figures

in the first row, where none of the cubes are invisible, and in the second row, where the structures are relatively simple. For the structures shown in the third row, only 1 or 2 out of 10 participants were able to correctly determine the number of cubes in each figure since several are hidden. Battista (1999) showed that for such structures children revert to counting visible faces or visible cubes since they are aware of but are unable to accurately count hidden cubes. The interview results aligned perfectly with Battista's findings and provided the research team a reference marker for each child before the program began.

In Chap. 3 we share introductory activities that move between loose cubes, the set of Soma figures and 2D conventional pictures of assemblies of two Soma figures. Then, in Chap. 4, we show how the children learn to use the Geocadabra Construction Box. In Chap. 5, abstract top-view plans and top, front, and side views are integrated with the 2D conventional and 3D model representations of various block figures. Chapter 6 shows how visualization activities integrated with numeracy development, in particular, with multiplication skills that are typically developed in 3rd grade.

References

Battista, M. (1999). Fifth graders enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education, 30*, 417–448.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*, 113–163.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In F. Furinghetti (Ed.), *Proceedings of the 15th P.M.E. Conference* (pp. 33–48). Genova, Italy: Università degli Studi di Genova.

Ganesh, B., Wilhelm, J., & Sherrod, S. (2009). Development of a geometric spatial visualization tool. *School Science and Mathematics, 109*, 461–472.

Hansen, M. B. N. (2004). *New philosophy for new media*. Cambridge: MIT Press.

Hershkowtiz, R., Ben-Chaim, D., Hoyles, C., Lappan, G., Michelmore, M., & Vinner, S. (1989). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition (ICME study series)* (pp. 70–95). Cambridge: Cambridge University Press.

Pittalis, M., & Constantinou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics, 75*, 191–212.

Presmeg, N. C. (1992). Metaphors, metonymies and imaginative reality in high school mathematics. *Educational Studies in Mathematics, 23*, 595–610.

Rückriem, G. (2009). Digital technology and mediation: A challenge to activity theory. In A. Sannino, D. H. Daniels, & K. D. Gutiérrez (Eds.), *Learning and Expanding with Activity Theory* (pp. 88–111). Cambridge University Press.

Science Education Research Center. (2009). Spatial thinking in geosciences. Accessed August 26 2015, from http://serc.carleton.edu/research_on_learning/synthesis/spatial.html.

Simon, M., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Ozgur Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity, cognition and instruction. *Cognition and Instruction, 28*, 70–112.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., et al. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics and Technology Education* 1: 81–84.

Van Niekerk, (Retha) H. M. (1997). *A subject didactical analysis of the development of the spatial knowledge of young children through a problem-centered approach to mathematics teaching and learning*. Ph.D. Dissertation, Potchefstroom University for CHE, South Africa.

Yakimanskaya, I. S. (1991). The development of spatial thinking in school children. In P. S. Wilson & E. J. Davis (trans. R. H. Silverman) (Eds.), *Soviet Studies in Mathematics Education*. Reston, VA: NCTM.

Chapter 3
Introductory Activities

Abstract The trajectory's introductory activities developed children's abilities to coordinate 3D models with 2D conventional pictures as shown in the adapted SOC model (Fig. 2.2). These included building four block "houses" using appropriate positional language; the set of Soma figures (made from 3 or 4 cubes) and their identification and construction; and, coordinating assemblies of two Soma figures in various combinations with their 2D conventional pictures. The researchers' attention to differentiating instructional activities for different ability levels is addressed.

3.1 Four Block Houses

The project ran for seven consecutive years from 2007 to 2014. During Year 1 (2007–2008), we started with activities to familiarize the children with the Soma figures that would later be used extensively among the SOC representations as shown in the adapted SOC model in Fig. 2.2. We designed all activities around these representations and will refer to these to ensure that readers are able to make sense of our program design choices.

Using a contextual story about building a new community in outer space whose houses would consist of rooms made by connecting four cubes face-to-face without overlaps or gaps, in any orientation that can stand freely, the teacher challenged the children to see how many different houses could be built using loose cubes (SOC 3D and verbal stimuli to 3D product). See Fig. 3.2 for an example of a student's structures.

The solutions to this activity include six of the seven Soma figures (#2 through #7 in Fig. 3.1). The children engaged in rich discussion about whether Soma #2, when placed flat as a 1-story house, can be reflected to make two different houses or not and whether figures like Soma #6 and #7 are different or not. In the context of houses, the two flat orientations of Soma #2 are different, since if one imagines entering the house through a door and walking straight through the three in-line rooms, one requires a left turn and the other a right turn to enter the fourth room as shown in Fig. 3.3a. In addition, language conventions, such as "vertical" and

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_3

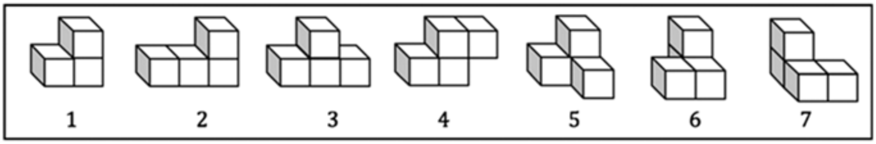


Fig. 3.1 The seven Soma figures made from unit cubes

Fig. 3.2 Four-block houses

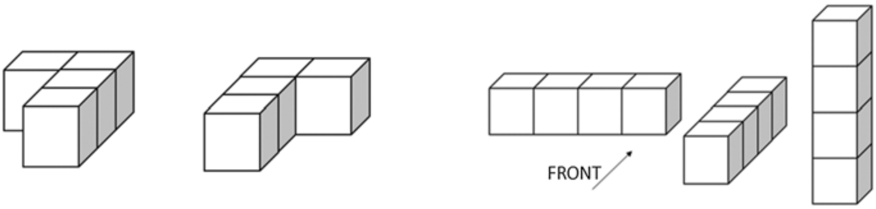


Fig. 3.3 **a** Soma #2 house turns *left* or *right*. **b** *Horizontal* and *vertical* houses

"horizontal" when used in 3D space or when used in 2D space, are discussed. Some interesting findings about the use of conventional directional language have emerged from our work. One child stated that a single row of four cubes lying flat but from front-to-back was "vertical down," while a 4-cube stack was "vertical up" as shown in Fig. 3.3b (Sack and Vazquez 2008).

The children engaged in justifying how their 4-cube houses were alike and different, and how transformations produced figures that were alike or different from their pre-images. Each child who shared his or her understandings impacted the class' collective knowledge about what constituted unique figures in the context of the 4-cube houses problem.

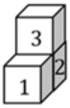
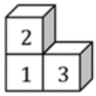
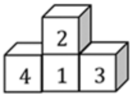
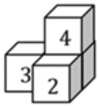
Example item	I
1. Put the first block down. 2. Place the second block **behind** the first block. 3. Place the third block **on top of** the second block.  	1. Put the first block down. 2. Place the second block on top of the first block. 3. Place the third block **to the right** of the first block. 
II	**III**
1. Put the first block down. 2. Place the second block on top of the first block. 3. Place the third block **to the right** of the first block. 4. Place the fourth block **to the left** of the first block.  	1. Put the first block down. 2. Place the second block **in front** of the first block. 3. Place the third block **to the left** of the first block. 4. Place the fourth block on top of the first block. 

Fig. 3.4 Sample oral direction task cards

Following this activity, in Year 1, we decided to see whether children could follow written or oral verbal directions for building 4-block houses (SOC verbal stimulus to 3D product). We provided task cards with written directions (see examples in Fig. 3.4).

One child read the directions while a peer tried to build the figure. The 2D picture solution was not shown to the peer builder until she or he had finished building. If the 3D figure matched the 2D picture then they switched roles and the reader selected a new card. We found that almost all of the readers did not pay close attention to the orientations of the peers' 3D figures and so this activity was not repeated for future cohorts. However, as we moved about and helped to facilitate we noticed significant numbers of children placing blocks in the wrong position when told to place them "in front" or "behind" a given block as in Fig. 3.4's Example and III cards (see Sack and Vazquez 2008 for detailed analysis). Van Niekerk (1997) noted similar terminology issues in her earlier work. She refers to children's "use of different deictic terms for the same position in space." The child who sees himself in the referent position will refer to the face corresponding to his own front as the front (as in a translation of position). The child who uses common convention places himself facing the figure and claims its nearest face to be the front (as in a reflection across a plane between the child and the figure).

To develop conventional positional language we now explicitly work with the whole group rather than individuals or small groups. We have also tried similar activities with adult learners (in-service and pre-service school teachers at various grade levels and content areas) using oral directions (as in the Example and Item III above) and found several instances of non-conventional use of these terms. The following whole class activities have replaced the small group/pair activities to ensure appropriate directional language understanding. By modeling a line of three children standing one in front of each other, first facing away from the whole group and then turning to face the group, the class understood the need for a convention when the child in the front then became the child at the back. By replacing the line with inanimate blocks, indicated only by different colored discs on their top faces, the class used the new convention to name the closest and most visible block to be the front and the one in the back as partially hidden from view. Another analogy we have used is that of facing a house or the school from the street. Can we see the front of the house/school? (Yes) Can we see the back of the house/school? (No) Where is the school playground (At the back)? Can we see the school playground from the street (No)?

3.2 The Soma Puzzle Pieces

Following the four-block house activity, typically by the second session, we provided each child a bag containing the Soma figures and a laminated figure strip, as in Fig. 3.1. They matched each 3D Soma figure with its 2D picture by laying the 3D figures out alongside the strip (2D conventional stimulus to 3D model). This became a required first task for many subsequent activities that used the 3D Soma figures, to ensure that each child had the correct collection of Soma figures. They quickly began to associate each figure with its number name. Over the 7 year duration of the project, we became quite efficient about developing naming conventions within the project. We began by asking them how many individual cubes were associated with the entire set. Since their multiplication skills are developing at this grade level, this was a good initial activity to integrate analytical, symbolic skills with visual skills. They quickly noticed that the six Soma figures #2 through #7 all have four cubes (for a total of 24 unit cubes), like many of the four-cube houses they had built in the earlier activity. Only Soma figure #1 has 3 cubes, in an L-shape (for a total of 27 cubes). Holding Soma figure #1 in one hand and a loose cube in the other hand, they went through the process of creating Somas figures #2 through #7 by moving the loose cube to different positions on Soma #1. They also had to find a way to distinguish Soma figure #7 from Soma figure #6, being mirror images of each other. Usually, someone in the class would hold up her/his left hand with thumb pointing up and fingers curled. This looks like Soma figure #6. Soma figure #7 corresponds with the right hand in this position.

Once the children had correctly matched the individual Soma figures to their 2D counterparts on the identification strip, we provided them with task cards representing assembly figures of two of the Soma figures. The goal was to identify which two Soma figures formed the assembly figure shown in each card (SOC 2D conventional picture stimulus to 3D model product). This activity is described in the next section.

3.3 The Coordinate Grid

The research team had created a set of over 400 two-Soma assembly task cards. These task cards also provided the children a wide range of choices in levels of challenge, abstraction and preference. Cards with one Soma figure shaded and the other unshaded helped some children discriminate among the possible Soma figures. Examples are shown in Fig. 3.5. Many of the cards with or without shading could often be solved with more than one combination of two Soma figures. Some of these cards were also color-coded. For example, purple cards had assembly figures that all included Soma figure #2; yellow cards had assembly figures that all included Soma figure #5. Initially, the children in small groups also used a laminated 7×7 coordinate grid with the seven Soma figures along the top and down the left side. They placed their task cards in the appropriate cells relative to the two Soma figures identified from each card. The research team challenged the children to discover the "rule" behind the color-coding. The purple cards all lay along the row or column aligned with Soma figure #2; yellow cards along the row or column aligned with Soma figure #5. The color code discovery helped less visually inclined learners to solve more cards of that color knowing which Soma figure had to be used. This coordinate grid provided an early experience in using two-dimensional coordinate system, a concept typically introduced in fifth grade.

Additional sets without color-coding were also provided, one with all figures that included the shading, and one with no color-coding and no shading. These sets of task cards were more challenging as there was no color key to determine at least one of the Soma figures in the assembly picture. Some children needed to build the figures represented on the cards using loose cubes and then identified the two Soma figures that could be assembled together to form the figure while others were able to identify the two Soma figures directly from the pictures. Within two sessions, most of the children were able to identify the two Soma figures directly from the pictures without needing to assemble them except to provide evidence to the research team of their thinking. Occasionally, we would have a child who had very weak visual skills who needed scaffolding help. We asked the child to select one of the Soma figures and combine it with Soma figure #1 and then to situate a loose cube against Soma figure #1 to complete the assembly. By separating the other Soma figure the child was able to identify which of Soma figures #2–7 could be used.

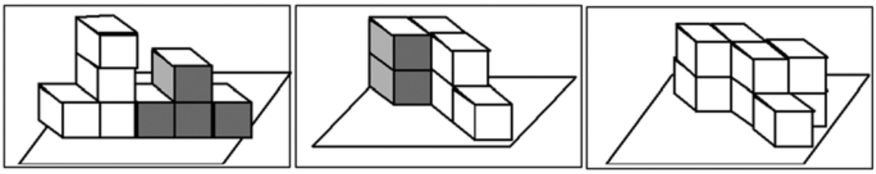


Fig. 3.5 Examples of two-Soma assembly task cards

Levels of complexity of these assembly activities varied according to children's interest and readiness by using these different types of cards, or by allowing children to use loose cubes, Soma figures or mental imaging as needed to solve these problems. This activity moved among single 2D Soma pictures, assembly 2D Soma pictures and 3D models. We typically stayed with this activity for 2–3 sessions prior to introducing them to the Construction Box dynamic digital interface. In Chap. 4, we describe the activities associated with the Geocadabra Construction Box dynamic interface. As soon as the children were proficient users of the interface, they had opportunities to create their own assembly task cards to challenge each other.

References

Sack, J., & Vazquez, I. (2008). Three-dimensional visualization: Children's non-conventional verbal representations. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the joint meeting of PME 32 and PME-NA XXX* (pp. 217–224). Mexico: Cinvestav-UMSNH.

Van Niekerk, (Retha) H. M. (1997). *A subject didactical analysis of the development of the spatial knowledge of young children through a problem-centered approach to mathematics teaching and learning*. Ph.D. Dissertation, Potchefstroom University for CHE, South Africa.

Chapter 4
The Geocadabra Construction Box Dynamic Geometry Interface

Abstract The children learned to use the Geocadabra Construction Box by constructing virtual cube structures to match the conventional 2D figures in the printed *Build and Explore with Geocadabra* manual. The process builds proficiency with top-plan numeric views. As they progressed, they were able to digitally reproduce cube structures like those they constructed with loose cubes on the desktop next to their computers. Activities integrating top-plan numeric views and top, front and side views are shared.

4.1 Learning to Use the Construction Box

The Geocadabra Construction Box (Lecluse 2005) module allows learners to construct, view and manipulate complex, multi-cube structures as 2D conventional representations, or as top, side and front views, or as top-plan numeric view grid codings, as shown in Fig. 4.1. By clicking successively on a grid position on the key pad shown in Fig. 4.1, a corresponding stack of cubes appears. Two clicks results in a 2-cube high tower. By right-clicking, the stack may be reduced in height or removed. The number grid is essentially a top-plan numeric view but includes 0 s in the spaces with no cubes. As the figure is constructed, the front, side and top views dynamically change. The *Show* (*Hide*) *Front, Side and Top Views*, numeric key pad or 2D figure options can be pre-selected according to instructional goals. The *Control View Line* option allows the user to move the figure dynamically using the mouse or by clicking on the arrows at the ends of the space's triaxial system that appears on the Construction Box control window. Currently, the size of the top-view rectangular grid can be adjusted from 2 to 8 units in width and depth according to user preference per authors' request to program developer Lecluse.

The *Build and Explore with Geocadabra* manual (formerly named *Geobuddies Manual*) has been adapted over the course of the project to align with the researchers' perceptions of learners needs. Some sections have been removed; others revised. We share activities from the latest version of the manual. For the first set of activities, children dynamically reproduce the 2D conventional figures printed in the manual.

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_4

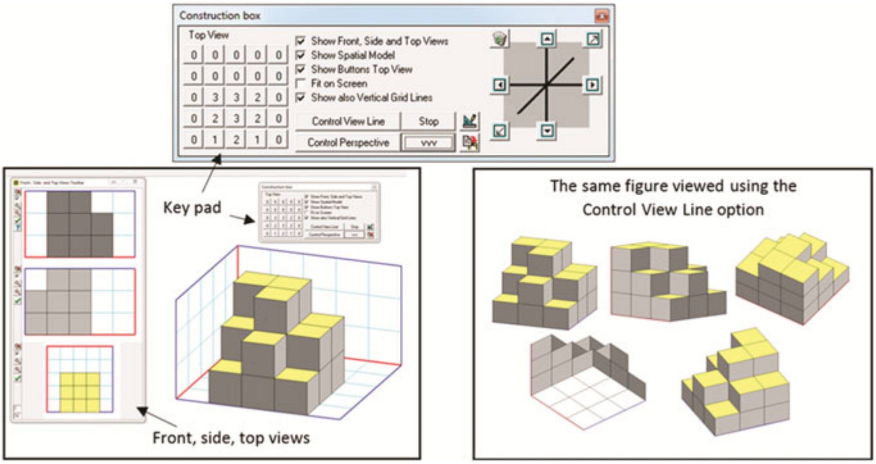


Fig. 4.1 The Geocadabra construction box

They first predict and record the total number of cubes in each printed figure. A blank grid is provided for each of the first 3 figures and users copy the numbers from the Construction Box grid to develop proficiency with top-plan numeric representations. They also compare the total number of cubes predicted and then evident in each structure with an explanation of how they determined that number. They quickly discover that the sum of the numbers in the grid represents the total number of cubes in the figure. As with the pre-interview, the initial figures are quite simple, with no hidden cubes. They gradually increase in complexity, including figures with several hidden cubes. There are 18 figures in all, sequenced alphabetically *a* through *s*. Figures *a*, *e, f, m* and *r* are represented in Fig. 4.2. After figure *c*, learners must reproduce the top-plan numeric view freehand in the spaces provided below each figure. As they develop proficiency, they begin to predict and then verify the top-plan numeric view. Then, from the seventh figure, learners switch on the *Front, Side and Top Views* button and copy these views next to each printed figure. They are challenged to draw their predictions of these views as they develop proficiency.

Learners generally worked independently, with a partner, or in a small group of up to 4 children, depending on the activity. We found that they needed to work independently on anything new, in order to develop some sense of the concepts at hand. Then, we called them together for whole group debriefing, particularly to encourage them to share their thinking about difficult tasks. Their independent struggles helped all of them make sense of others' ideas and of our guiding questions at that time.

During Year 1, after the children had spent at least one session learning to use Geocadabra, we called them together and had one student, Ethan (pseudonym), build one of the four block houses resembling Soma Figure 3 as shown in Fig. 4.3 using

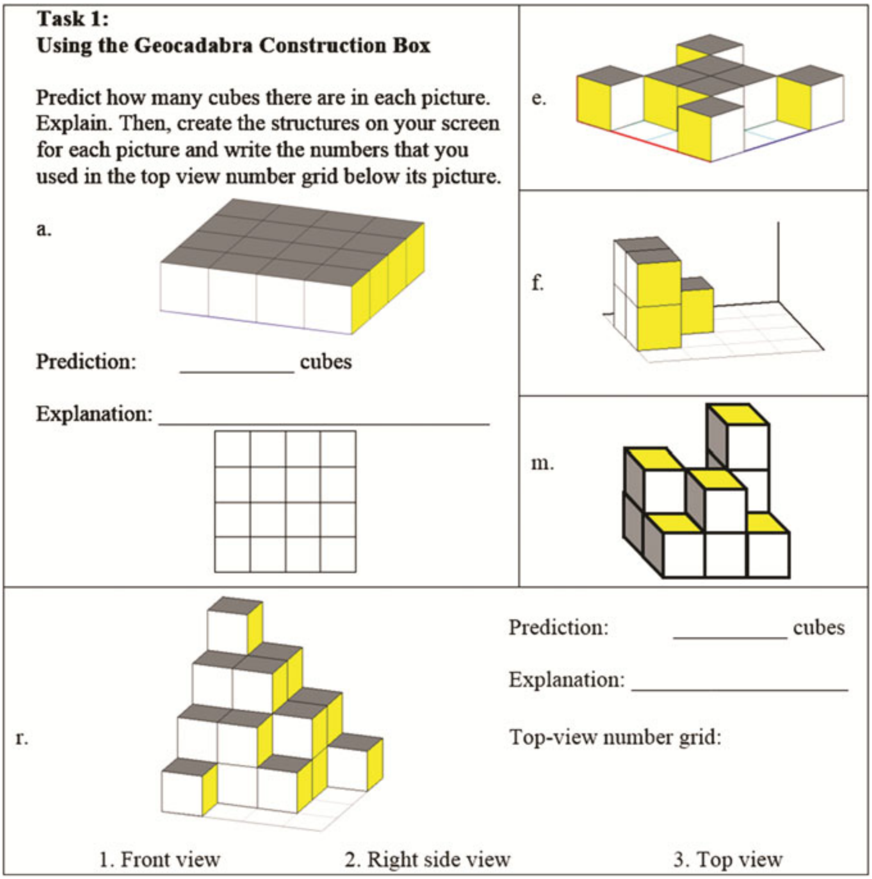
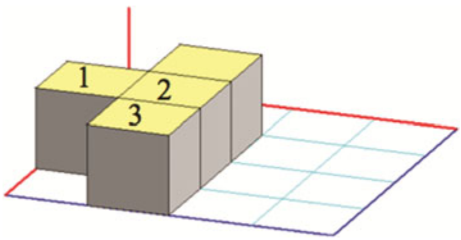


Fig. 4.2 Sample tasks from the *Build and Explore with Geocadabra* manual

Fig. 4.3 Soma figure #3



the projection camera so that all could see his moves. The numbers 1, 2 and 3 have been added to the figure to clarify the order in which Ethan constructed the figure.

He described each move, using the number grid and also the conventional figure as he added new cubes. First, he placed one cube in the middle on the left side; then, he placed a second cube to its immediate right. As he placed the third cube to the front of the second cube (forming an L-shape), he said, "and on the side of… under that one there was another one" (meaning the third cube)…. "on top of that one (the second cube) it had another one." At this point, teacher Vazquez asked him, "When you say, 'on top' are you referring to the [number] grid or your building [the conventional picture]? Do you mean on top right here [pointing to the Construction Box number grid on the vertical projected image]; or this block is on top of the other [pointing to the conventional structure, in particular, to the cube behind the second cube]?" This interaction, made us aware of positional language and positional coordination that had become confusing for all of the children. The vertical computer screen (or projection) shows a 3D figure made from cubes, while the cubes' positions are shown vertically on the screen on the Construction Box number grid, which actually corresponds with the horizontal base plane of the emerging 3D figure. The top row of the number grid corresponds to the back row of the conventional figure. Ethan also used the term, "under" for a position on the vertically oriented number grid that was frontward on the conventional figure. Having a group discussion on conventional vocabulary and corresponding grid positions after only one or two sessions on Geocadabra became a consistent and vitally important part of the learning trajectory. This experience also alerted us to the general problem of coordinating a vertical plane with a horizontal plane, regardless of the medium of instruction, for example, by projecting a printed image while learners look at the same image lying horizontally on their desks. When asked to point to "north," many people point up to the ceiling or sky, as this is where north usually is on wall maps. In the next section we describe a more extreme problem that one child had tracking his eyes from the Geocadabra manual lying horizontally next to his computer screen on the table to the Construction Box images on the vertical computer screen.

4.2 Supporting Learners with Relatively Weak Visual Skills

Within the first 2 years of the project we became acutely aware of the different visual abilities among each group of children. This prompted us to develop the pre-program interview in later years so that we could quickly identify those who would probably need more support. As the children worked through the Geocadabra manual, we realized that the strugglers kept pace with their peers on the more difficult items by simply writing and drawing whatever came to mind, without using self-checking mechanisms. They were aware of hidden cubes in the back but were unable to recognize how many cubes supported the upper visible cubes. While the children worked through the manual we moved about and stopped to ask individuals probing questions especially if we noted inconsistencies between printed conventional figures and recorded numeric grids. Task 1*r* shown in Fig. 4.2 became our key focal item for these inconsistencies. The total number of unit cubes in the figure is 24. One particular child had recorded 20 cubes. She explained that at the back there were 1, 2, 3, and 4 for a total of 10. When she tried to count the other cubes, she was unable to make sense

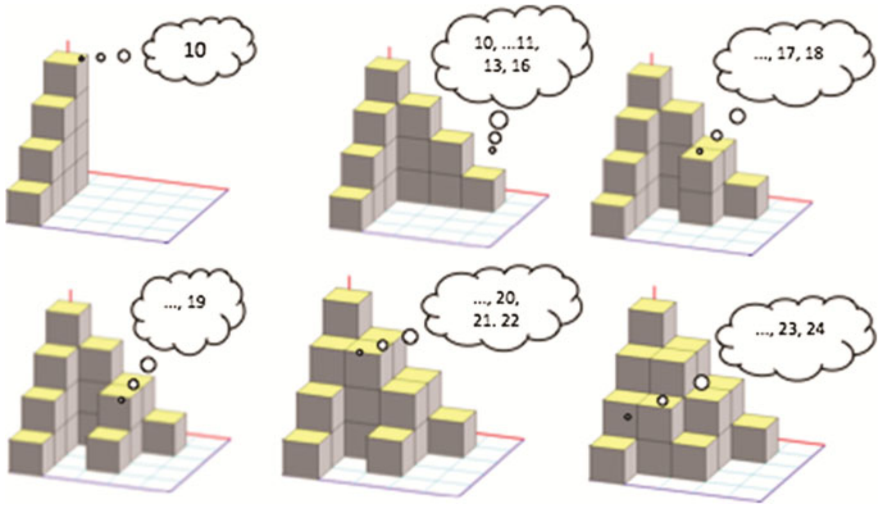


Fig. 4.4 Constructing and enumerating Task 1r

of the heights of each visible cube and counted only the visible cubes for a new total of 22. In order to establish whether she had difficulty constructing the figure on the Construction Box we asked her to rebuild it. Most children will first build the cubes that they can see, but she opted to start from the back and build the left side of the figure, then the back, and finally the cubes toward the front. Figure 4.4 shows the sequence that she presented.

During Year 2, we worked closely with Gary (pseudonym), who had unusual visual issues that had not been adequately addressed. During the regular morning schedule, he had difficulty copying items from the board. His ability to read and decode numbers was limited and this had impacted his ability to make appropriate connections in his numeracy reasoning. The following sequence was summarized from a video recording that demonstrated how we intervened to support him. We worked with him on Task 1r. First, we asked him to build the figure using the Construction Box grid. The sequence shown in Fig. 4.5, from left to right and top to bottom summarizes his initial attempts to reproduce the figure.

The printed manual lay on the table next to the computer keyboard. He appeared to number the squares toward the right and front of the figure correctly, but the 3-cube and 4-cube structures to the left and back of the figure confused him. He made several corrections to the left side of the figure and did not attempt to complete the back row. Then, teacher Vazquez lifted the printed manual and held it vertically, next to the computer screen. He immediately corrected the three numbers on left side of the grid to 1, 2, and 3 and replaced the 0 (shown in Fig. 4.5) with 2. Educators need to be aware that learners with visual tracking issues have difficulty copying from the vertical board or projection screen to their notebooks on their desks or

4	3	2	1
3	3	2	
2	2	1	
1			

Task 1*r*
Correct numbering

		2	
2	0	2	

		2	1
2	0	2	
3	3	1	
1			

		2	1
2	0	2	
2	3	1	
1			

		2	1
2	0	2	
1	3	1	
1			

		2	1
3	0	2	
1	3	1	
1			

		2	1
4	0	2	
1	3	1	
1			

		2	1
3	0	2	
1	3	1	
1			

		2	1
2	0	2	
2	2	1	
1			

Fig. 4.5 Gary's attempts to construct Task 1*r*

tables. If they notice errors in copying, they may need to place a copy of the notes on that learner's desk to compensate for these tracking problems.

Gary then tried to correct the left side of his figure. He pointed at the cube three rows back on the computer screen and stated, "I need to put a cube here," but was unsure how to change this on his number grid. Vazquez then pointed to the left side and suggested that he think of a frog jumping from one cube to the next. We also rotated the digital figure so that he could see the structure of the staircase on the left side. He said, "I can jump 1, 2, 3," and then changed the number on the grid accordingly. By thinking about the step-jumping frog, he was able to number the squares along the back, 1, 2, 3, 3, from right to left. He stated that the cubes on the 3rd level resembled a square, and then changed the left back number to a 4, based on the jumping steps.

In order to affirm Gary's developing position coordination, we provided him with a blank four-by-four grid with his completed 3D computer figure of Task 1*r*. We asked him to write in the numbers as he had entered them on the Construction Box grid (*Show Buttons Top View* was deselected to hide the number grid on the screen). He correctly placed the 1 s and 2 s on the left column and back row but became confused with the numbers in the middle of the grid. He needed to be reminded about the jumping frog and then wrote out the correct number grid. To reinforce his developing visualization skills, teacher Vazquez asked him to build the figure with loose cubes on a blank four-by-four grid. Beginning with the frog's jumping steps, he built the left and back of the figure correctly. The middle of the figure still was challenging, so we pointed at cubes in the digital 3D figure and he built these towers to represent the figure correctly.

During our one-to-one interviews with children working on Task 1*r*, we also noted that some had difficulty with the total number of cubes if they had focused on hidden cubes "behind" the visible cubes in the picture. By helping them to focus instead on how many hidden cubes are "below" each visible cube, they usually made better sense of the number of hidden cubes altogether. We asked them to record the number of cubes in each "tower" on the visible cube at the top of the tower.

4.3 Front-Side-Top Views

Top, side and front view activities were added to the Geocadabra manual in Year 3 as soon as children showed proficiency in enumerating cubes and drawing correct top plan views for complex figures such as Task 1*r*. In this section of the manual

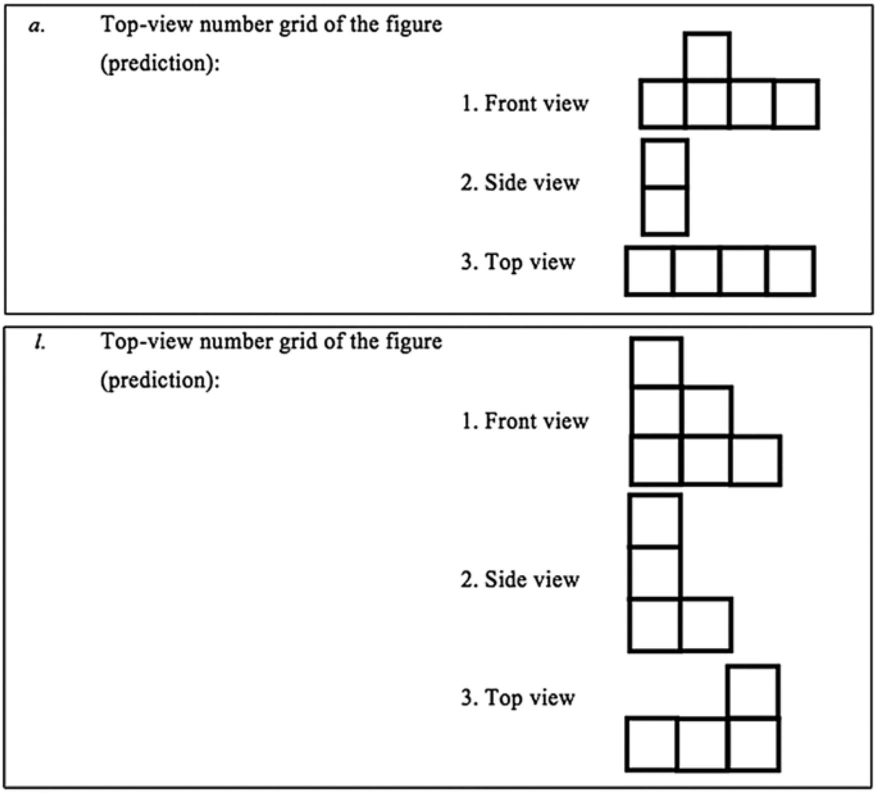


Fig. 4.6 Top, side and front view activities from the Geocadabra manual

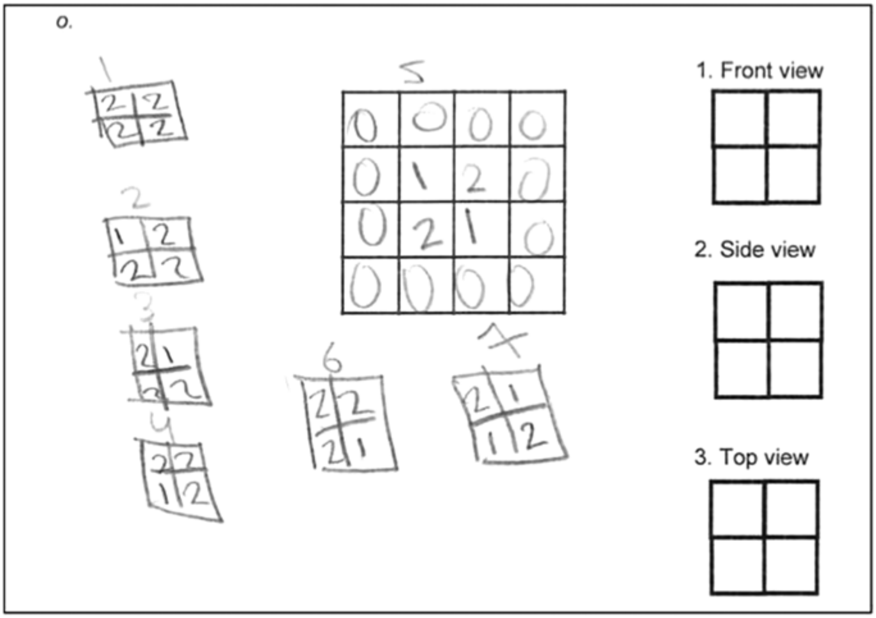


Fig. 4.7 Multiple structures for one set of front, side and top views

learners must predict what the top-plan numeric view of the figure would be given its front, side and top views. We introduced this concept by projecting a fairly simple figure on the board and then showed them the three view figures (selecting this option on the Construction Box menu). By rotating the figure using the *Control view line* arrows, the different views could be discerned. They drew the numeric view and then checked their predictions using Geocadabra by entering the numbers into the top-plan grid while selecting to show the front side and top views. The children then worked through the different views items (see examples in Fig. 4.6). We encouraged them to use the *Hide/Show Top Side Front Views* switch, first to turn off the views, then to build the predicted figure using the numeric grid, and finally to turn the views option on again to check. About half of the children were able to do this, while the others needed to keep the views on as they experimented with different structures using the numeric grid.

We noted that the children tended to defer to the Construction Box views too soon and this prevented them from having to really think hard about the different structures associated with given sets of views. Over the next few years, we introduced these views earlier, integrating them with top plan view activities. We were stunned when one child with unusually strong visualization skills produced 7 different top plan view structures for one set of front, side and top views, shown in Fig. 4.7.

In the next chapter we share activities developed to specifically help children integrate their developing multiplication skills with the concepts undergirding rectangular prism volume.

Reference

Lecluse, A. (2005). Geocadabra (Computer software). Accessed August 26, 2015, from http://home.casema.nl/alecluse/setupeng.exe.

Chapter 5
3D to 2D via Top-View Plans

Abstract Having become proficient at correlating 2D pictures with top plan numeric view representations learners used the Construction Box to create their own task card puzzles for peers to solve without the aid of the digital interface. These included 2D pictures of Soma assemblies and later, top plan view diagrams that they drew by hand. These tasks required adept mental transformation skills moving among the SOC representations. The Extended Construction Box module was created to allow users to construct digital cube figures with holes or overhangs within the first octant of a 3D coordinate space. Learners then developed a top plan view coding system that allowed for holes or overhangs. They also used their knowledge of top plan views to represent rectangular prisms made up of unit cubes.

5.1 Self-created Task Card Puzzles

Even though we had created hundreds of Soma figure assembly task cards over time, the children were motivated to work with task puzzles that their peers created. First each child selected two Soma figures to form an assembly figure; then, using the Construction Box number grid, the virtual conventional assembly figure was created and rotated to the user's satisfaction. At this point they learned to copy the virtual figure to the computer's clipboard and paste into a word processing document. They saved each day's work using their secret code names. These were designed for everyone to be able identify individuals but also to protect each child's identity in publication. Researcher Sack then adjusted the size of each child's figures to approximately 3 cm by 5 cm, and printed and laminated them for use as peer puzzles the following week. Examples are shown in Fig. 5.1. Using these puzzle cards, the children were challenged to find more than one pair of Soma figures that replicated the figure on the printed card even though the card creator most likely used only one pair of Soma figures.

In Fig. 5.2, from left to right: The first figure can be assembled using Soma figures 7 and 3, or Soma figures 5 and 4. The middle figure can be assembled using Soma figures 3 and 2, or Soma figures 5 and 7. The third figure can be assembled

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_5

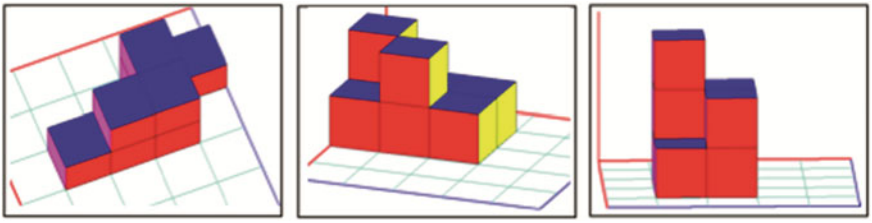


Fig. 5.1 Learner-created assembly task puzzles

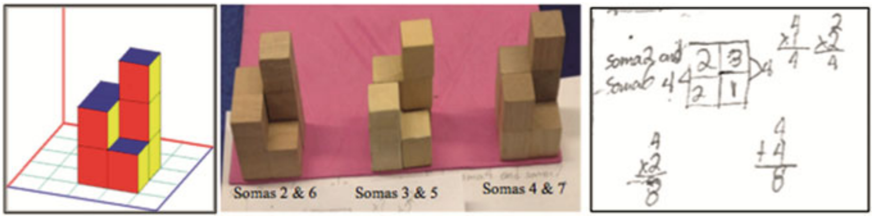


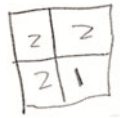
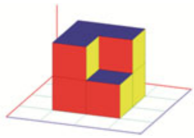
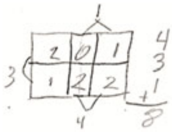
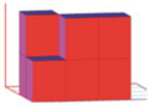
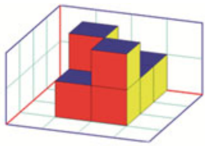
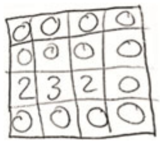
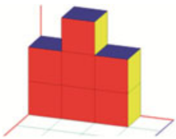
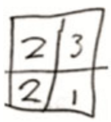
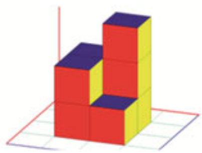
Fig. 5.2 Rosa's multiple solutions for one assembly puzzle using Soma figures 2, 3, 4, 5, 6 and 7

using Soma figures 2 and 7, or Soma figures 3 and 5. Other combinations may be possible.

During Year 7, one particular child, Rosa (pseudonym) had been retained to repeat 3rd grade due to low reading and numeration ability. She was encouraged to join the Geocadabra class to develop her visualization skills that would in turn support her numeration skills. She was a relatively reserved child who liked to engage in the tasks quietly. She progressed very well demonstrating good visualization skills but then stunned us by solving one particular task puzzle using all of Soma figures 2 through 7 (those with 4 cubes) in pairs. The puzzle card and her solutions are shown in Fig. 5.2. Rosa's superior visualization skills were never put to the test during the school day. Individuals who serve on committees to define formal curriculum standards should balance analytical skills that depend on symbolic decoding (numbers or variables) with visualization skills. By attaching contextual situations to analytical skill development one may integrate visualization skills since learners can mentally make sense of a given situation and imagine the objects of the problem.

In a follow-up activity the children drew the top-plan views of their own task card figures. The top-plan views became the new puzzles for others to solve. This required mental imaging of the actual figure while manipulating the seven Soma figures to create the assembly figure without the aid of the Construction Box interface. Table 5.1 shows some examples with peer solutions. Only the top plan codes were provided. Note that some of the children drew 4-by-4 grids like those on the Construction Box' default grid even though their figures did not need all of the rows or columns. However, they did not all coordinate their zeros in their drawings with

Table 5.1 Examples of top plan view puzzles of two-Soma assemblies and peer solutions

Puzzle	Top plan coding and task card	Peer solutions
A	 	Soma #1 and Soma #5 OR Soma #1 and Soma #6
B	 	Soma #6 and Soma #7
C	 	Soma #5 and Soma #6 OR Soma #2 and Soma #3
D	 	Soma #4 and Soma #1 OR Soma #2 and Soma #1
E	 	Soma #7 and Soma #4 Soma #3 and Soma #5 OR Soma #6 and Soma #2

the zero positions on their printed task cards. Two different children solved puzzle E using three different Soma figure combination.

During Year 1, we developed activities based on our perceptions of natural extensions to activities just experienced. We debriefed together after each session, taking detailed notes of our observations. These were supported by video-recordings and subsequent transcriptions to help us triangulate observation data. Sometimes the children posed interesting questions and we allowed them to explore these to see what would evolve. We did not have a set trajectory in mind other than having the children develop proficiency moving among 3D models, 2-D conventional pictures (usually assemblies of Soma figures) and top plan views using the Construction Box. When they began to construct assembly task puzzles, some became frustrated that they had used the Soma figures to assemble interesting figures with holes or overhangs that could not be constructed using the bottom-up process on the Construction Box. We contacted Lecluse, the developer of the Geocadabra platform to ask about the possibility of constructing such figures. Within three days he sent us a new module, the Extended Construction box.

5.2 Extended Construction Box

The new module allowed the user to build block figures within a 3D octant space. When *New Block* is pressed, a new block appears in the (1, 1, 1) position as shown in Fig. 5.3. By clicking on the arrows (up, down; left, right; forward, backward) the block moves one space for each click. If the block lies against one of the walls or on the floor of the space it will not move through that wall. By selecting *Shadowing* on the bottom, back wall and side wall, the exact position of each cube can be seen. If several blocks are connected, then by selecting the button that looks like Soma figure #2, one can translate or rotate the entire figure using the arrows. Using the *Control View Line* option one can turn the figure using the mouse to show sides that are hidden.

The children were now able to construct 2D assembly task card puzzles with holes or overhangs as shown in Fig. 5.4. They set their own levels of challenge according to their confidence with the new interface and with their ability to move from the 3D assembly figures to the virtual 2D image on the interface.

As soon as they had begun to construct these task cards, they immediately started to make coding puzzles as they had for the bottom-up task figures they had previously created on the regular Construction Box interface. This posed a new problem for the research team since they were not aware of a coding system for figures with holes or overhangs that would uniquely determine each 3D figure. They knew that a set of face views, shown as shadows on the bottom and two walls could be produced from a variety of 3D figures. The children enthusiastically set out to invent a new coding system. They created several interesting but very complicated ways of showing holes or overhangs. However, for each coding puzzle, the coder would write a key for peers to read and use to make sense of the puzzle. Most of these were very complicated and the decoder would have to interrupt the coder to clarify. We led them to the challenge of selecting the "best" coding system that all would use.

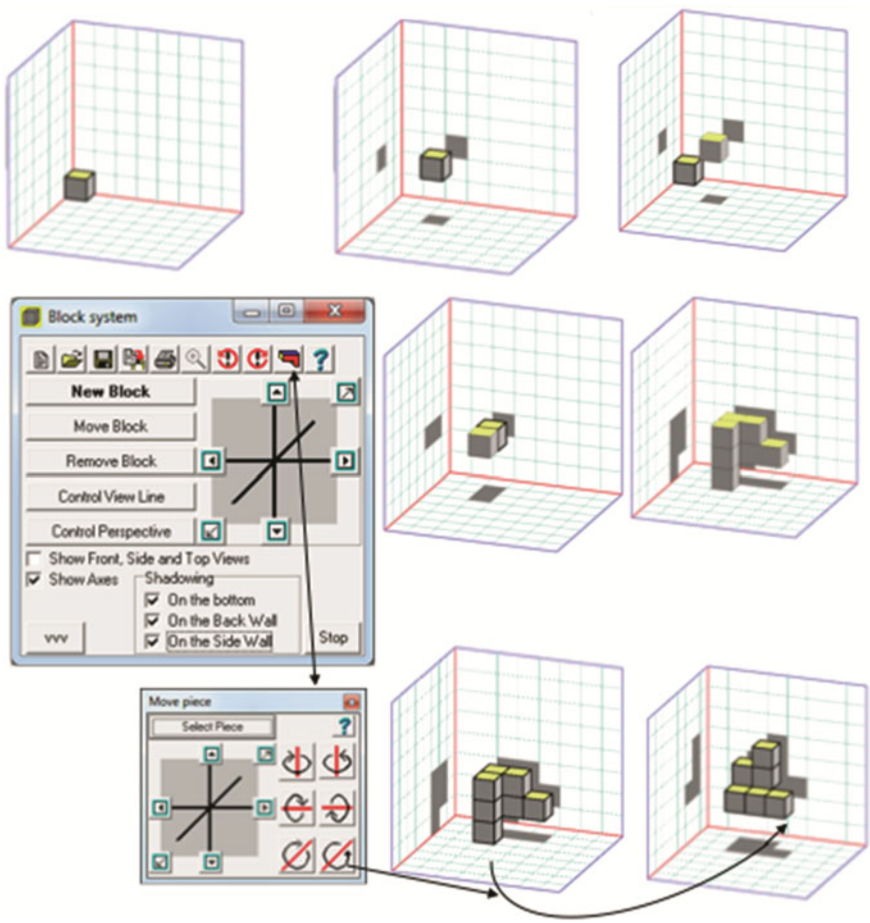


Fig. 5.3 The extended construction box

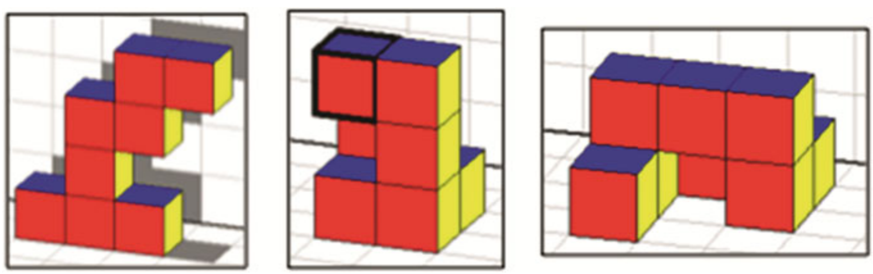


Fig. 5.4 Two-Soma assembly task card puzzles with holes or overhangs

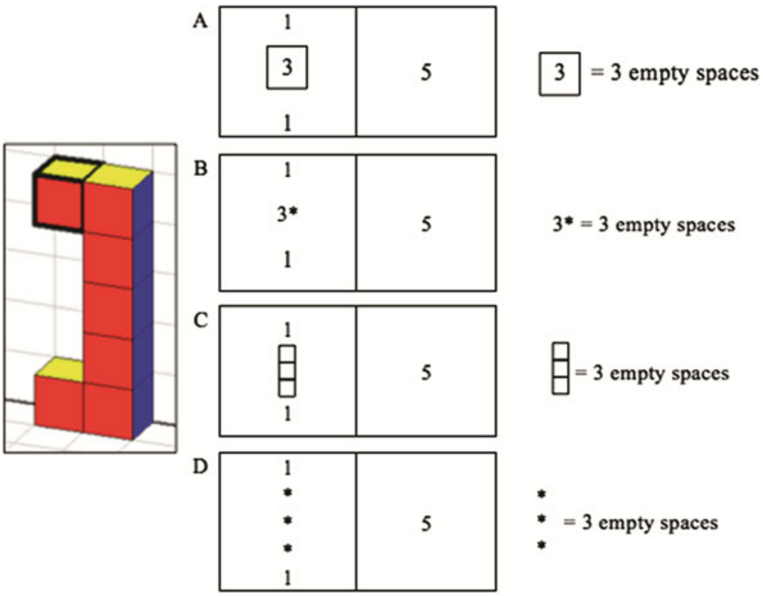


Fig. 5.5 Third graders' invented coding for figures with holes or overhangs

For our 3rd grade group, we selected the task card shown in Fig. 5.5, constructed with Somas #1 and #2, since it was quite simple and would not be challenging for those who may still be developing proficiency with standard top plan view coding. We asked the children to code this figure so that we could compare and contrast the different codings and, as a group, select one that everyone would then easily use. Some children drew the picture as a front view with 1s in each square. We showed them how exchanging Soma #1 with Soma #5 would be very problematic with that coding. We also suggested that they find ways to adapt the conventional top plan view format from the basic Construction Box to depict empty spaces within a structure. After some time, four different coding systems emerged also shown in Fig. 5.5.

During whole class discussion, where the children who had coded the figure as shown in Fig. 5.5A–D explained their work, the class agreed that codings A and B were more efficient than codings C or D, especially when you have to consider say, 10 empty spaces. They selected coding A and also decided that instead of placing the number for empty spaces in a square, they would circle the number, to represent "zero" cubes.

We provided a slightly more challenging figure, shown in Fig. 5.6, with a gap and an overhang for the 4th grade class. Again, the children produced very complicated coding systems that the class felt would be difficult to remember. One child produced a coding system similar to some of the 3rd graders' (Fig. 5.6E). We shared two of the 3rd grade codes, Fig. 5.5A, B, and the 4th grade class also agreed to use a circle

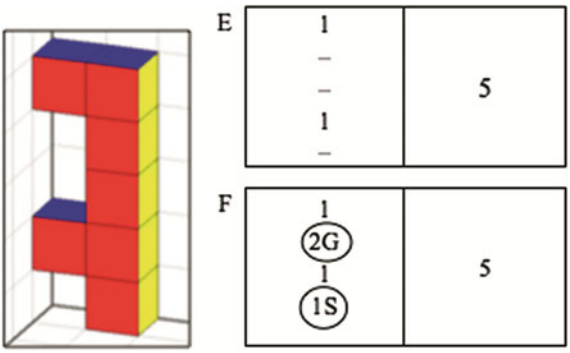


Fig. 5.6 Fourth graders' invented coding for figures with holes or overhangs

to show the number of empty spaces, with the letters S (for spaces) or G (for gaps) shown in Fig. 5.6F.

5.3 Rectangular Prisms and Their Volumes

During Year 2, five children who had attended during Year 1 returned to seek help with making sense of rectangular prisms. During their regular mathematics class they were expected to understand and use the Volume = length × width × height formula. They struggled to make sense of these measurements and the notion of volume being the number of cubic units enclosed by the figure. We provided them with a bag of loose cubes and asked them to build prisms with 24 cubes, since the large number of factors could produce a variety of different rectangular prisms. As soon as we had a nice collection of prisms, standing on different size bases, we asked them to record the length, width and height dimensions. This appeared to be confusing to all of them and so we asked how they could represent their buildings. We were stunned when every single child drew a rectangle, gridded in according to the number of rows and columns for the base of each figure, and then represented the prism's height as a number in each square. They had recalled the top plan view representation from the previous year and used this with ease. They now had no difficulty understanding that the number of squares in each row and the number of rows in their grids represented the length and the width of the figure; and that the number in each square, which was the same for every square on the base represented the height, or the number of layers in each figure. We were very careful to show them how the linear measurements translated into rows and columns on the base.

As a result, we added and retained this activity in our 3rd grade program to ensure success for every participant as they made sense of geometric measurement formulas. We provided each group a large bag of loose cubes and asked them to build as many 24-cube rectangular prisms as they could. Then, we asked them to sketch all of the top plan view representations for these prisms. When everyone had

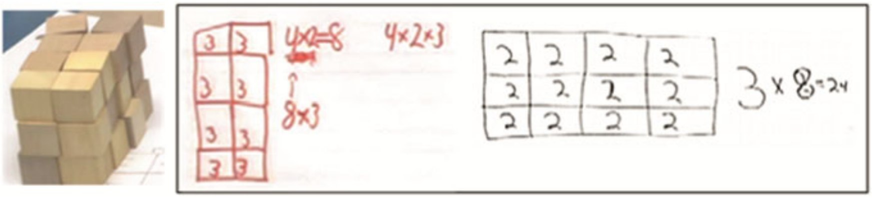


Fig. 5.7 Representing rectangular prism volume

completed at least one of these, we called them all together to share their drawings. From these we asked what mathematical expression they could give to a friend as a puzzle to see if he or she could reproduce the top plan view grid for that particular prism. Examples of grids and multiplication expressions for total number of cubes that children drew on the board are shown in Fig. 5.7.

All of the children in the project were able to make strong conceptual connections between the top plan view and the volume of a rectangular prism. They were able to mentally picture the figure directly from the top plan view. Furthermore, they now had the means to represent the 3D figure, albeit abstract, compared to representing a conventional 2D rectangular prism figure. This was a significant result of the project, as other researchers have commented about the difficulties in enumerating hidden cubes (Battista 1999) and also in representing 3D prisms on paper (Outhred et al. 2003). This activity integrates visualization with numeration, the topic of Chap. 6, in which we describe additional numeration activities linked to our visualization program.

References

Battista, M. (1999). Fifth graders enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education, 30*, 417–448.

Outhred, L., Mitchelmore, M., McPhail, D., & Gould, P. (2003). Count me into measurement: A program for the early elementary school. In Douglas Clements & George Bright (Eds.), *Learning and teaching measurement: 2003 NCTM yearbook* (pp. 81–99). Reston: National Council of Teachers of Mathematics.

Chapter 6
Connections to Numeracy

Abstract Volume of rectangular prisms extended into an interesting scaling problem: One child asked if they could try to build a Soma cube as large as the teacher's demonstration Soma cube (made with very large individual cubes) using the smaller Soma figures. Initially they built four Soma cubes using the small sets of Soma figures and arranged them to look like a very large Soma figure #2. They eagerly attacked the problem of how many little individual cubes were in this figure each showing at least two ways to find the answer. The following week they built a huge cube using 27 smaller sets of Soma cubes and calculated 27×27 to find how many little individual cubes were in this model. Permutations within cake patterns: At the end of each year, the children were challenged to create a "cake" with the seven Soma figures. The cake had to have 24 cubes for a base and three "candles" on the second level. They created several 3×8 and 4×6 cakes and drew the top plan view coding patterns for these to submit to the baker. We share how children from three different cohorts discovered the permutations of each cake design from the three or four Soma figures (#1, #5, #6, and #7) that have the same 3-cube footprint.

6.1 Scaling up the Soma Cube

In Chap. 5, we described the connections the children made from the top plan view to represent rectangular prisms, enumeration of the total number of cubes in rectangular prisms, and the conceptualization of the volume formula for rectangular prisms. Here, we describe an interesting numeration problem that arose from the children's fascination with the Soma puzzle cube.

We frequently allowed free exploration time during which many of the children tried to build the Soma cube of 27 unit cubes made from all seven Soma figures. According to Wikipedia (https://en.wikipedia.org/wiki/Soma_cube) there are 240 distinct solutions, excluding rotations and reflections. On one particular day, one of the boys, David (pseudonym), posed an interesting problem: If we build a lot of Soma cubes each with the seven different Soma figures, how many would it take to

J. Sack and I. Vazquez, *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children*, SpringerBriefs in Education,
DOI 10.1007/978-3-319-29799-6_6

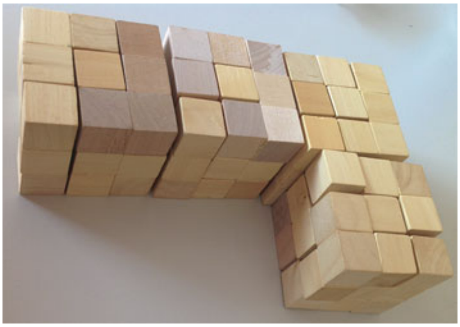


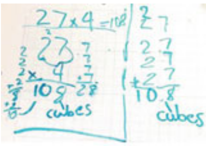
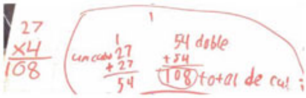
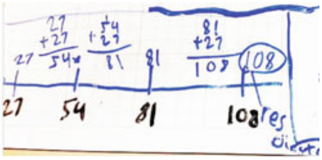
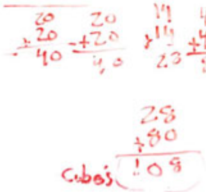
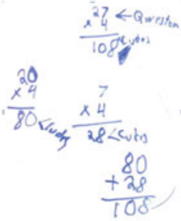
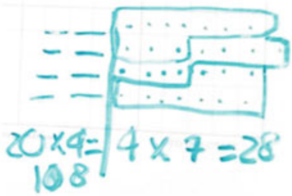
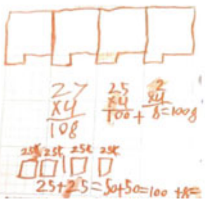
Fig. 6.1 The large Soma figure #2 made from four small sets of Soma figures

make a cube as big as the one made from the teacher's much larger Soma figure demonstration set? This became the problem for that particular session.

First, the children set about trying to build a cube from all seven Soma figures. We never emphasized this particular problem, since we always wanted that to be a self-determined problem choice that always remained interesting due to the large number of possible solutions. This took several minutes until Jason (pseudonym) was successful. He was charged with building more, using everyone else's Soma figure sets. David immediately set about carefully creating a very large Soma figure #2, using Jason's four completed Soma cubes (see Fig. 6.1).

Although David initially had wanted to build a very large cube with several of the small Soma cubes, Sack and Vazquez simultaneously asked, "How many unit cubes are in this large Soma #2 figure?" The children were excited about this problem, being a good extension from their developing multiplication skills. Those who struggled also had difficulty adding double-digit numbers fluently and often counted on their fingers, which is an indication that they may not have been ready for formal multiplication (Van Niekerk, personal communication, June, 2009). They all knew that this large Soma figure comprised 4 sets of 27 cubes. Doing a multiplication problem with 27 as a factor was totally new to most of them. In class, Vazquez had deliberately avoided the traditional algorithm, preferring to let the children "invent" their own algorithms. They shared their different methods to the whole class so that they all could benefit from different perspectives. She also emphasized the distributive property of multiplication by decomposing difficult numbers into two easier numbers using array models to show why this works. For example, 7×6 may be represented as an array of 7 rows with 6 squares in each. By drawing a thick line below the 5th row, they could see this was equivalent to 5 rows of 6 and then 2 more rows of 6, or $7 \times 6 = 5 \times 6 + 2 \times 6$. Decomposing numbers to facilitate multiplication through the distributive property is emphasized in the 3rd grade curriculum. During their daily mathematics instruction, teacher Vazquez, always asked for such explanations, in order for the presenting child to develop the mathematics process standards, to improve her or his communication skills and also for the other children to make sense of others' conceptualizations.

Table 6.1 Samples of children's calculations and researchers' comments

Children's calculations of 27 × 4	Researcher's interpretations
A 	Adept at double-digit, multiple number addition
B 	Used "double the double": Added 27 + 27 and then doubled the sum: 54 + 54
C 	Added 27 + 27 = 54. Then, 54 + 27 = 81. Then, 81 + 27 = 108 Showed each sum on a number line
D 	Decomposed 27 into 7 + 20 Doubled 7 to get 14 (not shown) and doubled 14 to get 28 Doubled 20 to get 40 and doubled 40 to get 80. Then added 28 + 80
E 	Wrote 27 × 4 as the "Question." Decomposed the problem into 20 × 4 + 7 × 4 Followed teacher's instruction to keep units (cubes) in the calculation
F 	Represented 27 × 4 using the teacher's base-10 notation: 2 lines for two 10 s and 7 dots for seven 1 s. Set these up vertically as four sets of 2 lines and 7 dots Grouped the top 7 ones with 3 ones in the next row to make 10. Grouped the bottom 7 ones with 3 ones in the row above to make 10, for a total of 10 tens; and 8 ones grouped in the middle of the ones Also calculated using decomposition of 4 × 20 and 4 × 7
G 	David used the grid on the board to draw four 27 s. He drew 25-unit squares on the grid andthen added 2 more squares to each to make 27 s. He decomposed 27 into 25 + 2 and added the products 25 × 4 and 2 × 4 When Irma asked how he knew how to multiply by 25, he said, "It's like money. A quarter is 25 cents and four quarters is $1, or 100 cents."

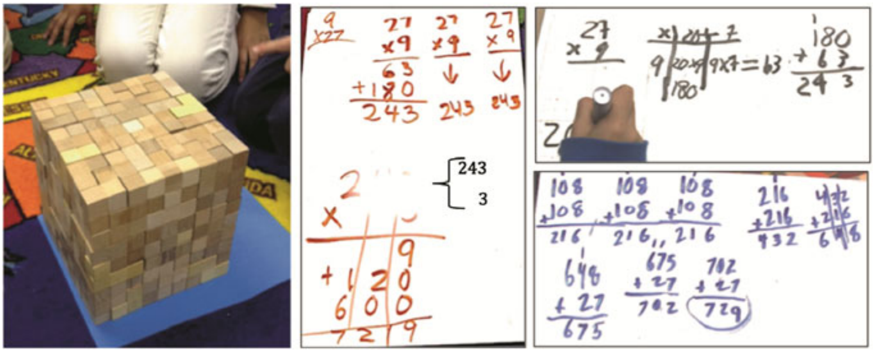


Fig. 6.2 The 27-by-27 cube and examples of children's calculations

The children set about solving this problem on small dry-erase boards. Vazquez told them to show the solution in at least two different ways. Afterwards, as during the instructional day, she asked each child to explain what he or she had written. Examples are shown in Table 6.1.

The following week, the children all set about building 27 small Soma cubes in order to construct a very large 3D array, using all of the 27 sets of small Soma figures. They immediately set about calculating how many unit cubes were in this 27-by-27 array. Examples are shown in Fig. 6.2.

In the left and top right examples, the large figure contained 3 slices, each with 9 small 27-cubes. In top right example, the child had calculated 9×27 by decomposing 27 into $20 + 7$. S/he was busy working the next step, $3 \times 9 \times 27$, or 3×243. In the left figure, the child was able to calculate $9 \times 27 = 243$; then, multiplied 243×3, an extension of 1-digit by 2-digit multiplication that is in the 3rd grade curriculum. In the lower right example, the child knew that each of six Soma figures contained four 27-cubes or 108 small unit cubes (from the previous week), plus Soma #1, with three 27-cubes.

This learner-generated problem provided the children an opportunity to apply their developing multiplication skills to a much more difficult problem than they were expected to compute during their regular mathematics classes. Each child used at least two different ways to solve the problem (not all illustrated in this chapter) to check their initial answers. We were surprised at the plethora of solution methods and particularly excited about David's enthusiasm with this problem. This supports Connell's (2001) view that children are capable of solving difficult problems based on their past experiences that have now become instantiated in their minds as objects on which to act. The different approaches to solving this problem show that each child had found ways to extend their prior knowledge using methods of personal choice. These episodes also confirm Presmeg's (1992) view that imagistic or spatial processing is essential for the development of abstraction and generalization in that the children abstracted routine skills to tackle their self-generated task of finding the total number of cubes (27 × 27) even though they are only expected to calculate 2-digit by 1-digit products in 3rd grade.

6.2 Permutations Within Cake Patterns

Since Year 1 we celebrated the last day of the program by inviting parents to see what their children have learned. To prepare, we asked the children to design a "cake" using all seven unique Soma figures. Since Soma figures #1–4 can lie flat, but Soma figures #5–7 have at least 1 cube on the second level, we specified that the cake have exactly 3 "candles", from Soma figures #5, 6, and 7. Thus, the cake must be a 24-unit rectangle. We asked what dimensions the cake could have, which was a good application of their developing multiplication skills. Most said 3-by-8 or 4-by-6. We asked if they could make a 1-by-24 rectangle, but they said it was not possible since all of the Soma figures were at least 2-units wide, depending on how you laid them down. Every year, some children tried to build a 2-by-12 cake. Only by experiment did they eventually reason why this was not possible with the seven different Soma figures. We told them one pattern would be selected to be sent to the baker, to bake a puzzle cake for the party. They also had to represent their different cake designs on paper, using different colors for the "footprint" of each Soma figure. An example is shown in Fig. 6.3.

During Year 1, Sophia (pseudonym), noticed that Soma figures #5, 6, and 7 all had the same 3-cube footprint. Vazquez asked her to try to determine how many different cake patterns she could make just by moving these three Soma figures. Generally, Sophia struggled with reading and computation, both requiring adept decoding skills. She had demonstrated extremely strong visual skills throughout the year in this after-school program. This activity provided her an opportunity to build her numeracy using visual cues. She successfully found all six permutations of #5, 6, and 7, and listed them, using the F(ront), M(iddle) and B(ack) format that she created (see Fig. 6.4). She then transferred the knowledge to ordering the 3 digits. She had struggled with this concept in her regular class earlier that day.

Also in Year 1, but with the 4th grade group, Dawn made a 3 × 8 cake and showed how she could interchange Somas #1, 5, 6, and 7 (same footprint) to make additional cakes. We challenged her with "How many ways?" She, like Sophia, had some

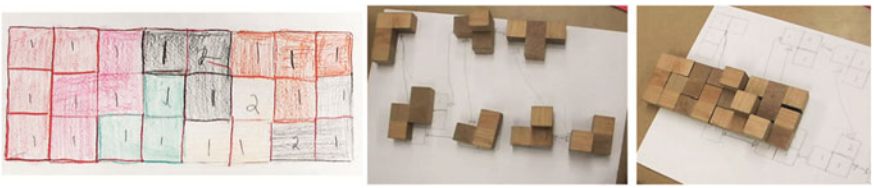


Fig. 6.3 Soma cake patterns

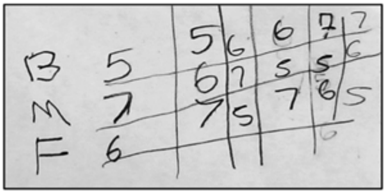


Fig. 6.4 Sophia's permutations of Soma figures #5, #6 and #7

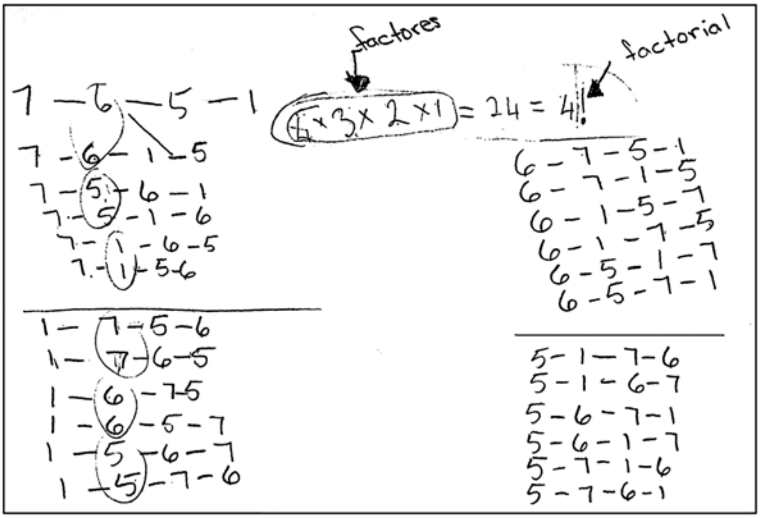


Fig. 6.5 Dawn's permutations of Soma figures #1, 5, 6 and 7

numeracy gaps. After pointing out the combination sequence on her cake pattern: 7–6–5–1, she started to write another random combination, starting with 1—… We suggested she keep the 7 and the 6 in place and think about another arrangement. She immediately wrote 7–6–5–1, and then 7–6–1–5 and continued until she had all 6 arrangements beginning with 7. She then proceeded to write down all six combinations beginning with 6—and had no trouble suggesting that there would be six more that began with 5 and six more that began with 1 for a total of 24 different combinations (See Fig. 6.5). Then, we asked Dawn to look at how we get 24 (four sets of 6; and three sets of 2 within each set of 6): $4 \times 3 \times 2 \times 1 = 24$. By this time, her friend, Emily, became involved. Emily remembered that the numbers 4, 3, 2, and 1 are all factors of 24. They learned a new word, *factorial*, and its symbol, 4! There was much excitement that this might be taught in another 3 years' time, in middle school.

Over the 8 years of the project, finding permutations occurred only if a child noticed the footprint pattern for these Soma figures. In Year 4, one remarkable child,

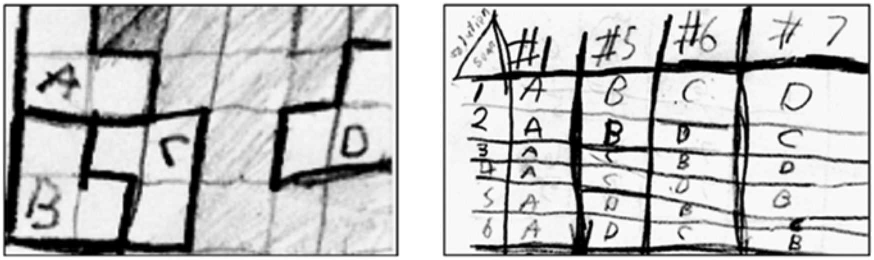


Fig. 6.6 Henry's permutations of Soma figures #1, 5, 6, and 7

Henry, organized his thinking using a table, partially shown in Fig. 6.6 alongside his cake pattern. He showed all 24 permutations in his table. The other children interpreted his pattern, explaining that A, B, C and D represented Soma figures #1, #5, #6, and #7 shown in the first row of his table. In row 2, the interchange of C and D meant that Soma figure #6 switched places with Soma figure #7. The children collectively built this pattern and showed how each solution evolved from the previous one in the table. While Henry had also multiplicatively reasoned why there are 24 permutations for each cake pattern, the others reasoned through his table patterns.

6.3 Conclusions

It is remarkable that the children were able to move from the abstract top plan numeric view and front, side and top views representations to the 3D models independently of the computer interface. The research team considered this to be evidence of the children's growing ability to visualize, abstract and generalize as they moved between abstract and 3D assembly models. Actions on objects (Connell 2001) occured concretely with the 3D models, virtually through the Geocadabra Construction Box interface and ultimately as mental imaging through the powerful problem-solving approaches developed by the research team. Through their different configurations the Soma figures provided a high degree of complexity and constraint to the instructional tasks and forced the children to engage in a variety of mathematical tasks including mental transformations in ways that would not be possible using loose cubes.

This research team, through this learning trajectory, attempts to fill the gap in the development of spatial thinking in the elementary mathematics curriculum as expressed by others. However, in a regular school classroom the day-to-day curriculum constraints include pressure from mandated high-stakes testing and rigidly scripted programs used within many school systems. Thus the research team faces the challenge of how to impart these visualization experiences to a wide audience of practitioners given the open-ended nature of the tasks in their learning trajectory.

References

Connell, M. L. (2001). Actions on objects: A metaphor for technology-enhanced mathematics instruction. *Computers in the Schools., 17*, 143–171.

Presmeg, N. C. (1992). Metaphors, metonymies and imaginative reality in high school mathematics. *Educational Studies in Mathematics, 23*, 595–610.