Chapter 2
Enforcing Linear Dynamics Through the Addition of Nonlinearity

G. Habib, C. Grappasonni, and G. Kerschen

Abstract The current trend of developing more slender structures is increasing the importance of nonlinearities in engineering design, which, in turn, gives rise to complicated dynamical phenomena. In this study, we evidence the somewhat paradoxical result that adding purposefully nonlinearity to an already nonlinear structure renders the behavior more linear.

Isochronicity, i.e., the invariance of natural frequencies with respect to oscillation amplitude, and the force-displacement proportionality are two key properties of linear systems that are lost for nonlinear systems. The objective of this research is to investigate how these properties can be enforced in a nonlinear system through the addition of nonlinearity. To this end, we exploit the nonlinear normal modes theory to derive simple rules, yet applicable to real structures, for the compensation of nonlinear effects. The developments are illustrated using numerical experiments on a cantilever beam possessing a geometrically nonlinear boundary condition.

Keywords Nonlinear normal modes • Linearization • Compensation of nonlinearity • Isochronicity • Perturbation

2.1 Introduction

Many engineering applications, as for instance tuned vibration absorbers [1], ultrasensitive mass and force sensing devices [2], time keeping devices [3], nanoscale imaging systems [4] and many others, rely on linear properties of mechanical systems, such as force-displacement proportionality and invariance of the resonant frequency. However, if high excitation amplitudes are considered, nonlinearities are activated, invalidating linear properties. This situation is particularly relevant for nano- and micro-electromechanical systems, where nonlinearities are activated already at moderate forcing amplitudes [5]. Furthermore, the current trend of developing more slender structures increases the importance of nonlinearities also in macro systems.

The loss of force-displacement proportionality, the dependence of the resonant frequency on the amplitude, the appearance of quasiperiodic or chaotic solutions, variations in stability properties, coexistence of different solutions, boundedness of basins of attraction, appearance of bifurcations are some of the effects typically generated by nonlinearities [6] that have no linear counterpart. Most of these phenomena have been studied in depth during the last decades, and it is now possible to predict the consequences of many different types of nonlinearities, although unexpected behaviors are always possible. Nevertheless, there are few studies that attempt to eliminate these usually unwanted phenomena. Most existing studies deal with the implementation of active controllers [7, 8], which is referred to as feedback linearization.

The objective of this paper is to enforce linear properties in a nonlinear system through the addition of passive nonlinear elements. The two target properties are the force-displacement proportionality and the invariance of the resonant frequency with respect to the amplitude, which are generally lost even at low level of excitation.

The developed procedure exploits the nonlinear normal modes (NNMs) [9] of undamped, unforced systems, because they give a good approximation of the system's backbone curves. Thanks to the energy balance criterion, the undamped, unforced dynamics can be related to the forced damped dynamics, thus giving complete (but approximate) information about the location of the resonant peaks in the force-frequency space. The resulting equations are solved through reduction to a single harmonics and a standard perturbation technique, which allows to derive equations that can be solved explicitly.

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This procedure prepares the ground for the definition of linear equations, whose solution is directly related to the target linear property. The analytical developments are validated using a two-degree-of-freedom (DoF) reduced-order model of a cantilever beam possessing a geometrically nonlinear boundary condition and a nonlinear attachment.

2.2 Model

We consider a general n-DoF mechanical system with concentrated polynomial nonlinearities of odd orders, subject to harmonic excitation. The system has the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \tilde{\mathbf{b}}(\mathbf{x}) = \sqrt{\varepsilon}\tilde{\mathbf{v}}f\cos(\omega t) \tag{2.1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, $\mathbf{x}$ is the position vector, $\tilde{\mathbf{b}}$ contains the nonlinear terms, $\tilde{\mathbf{v}}$ indicates which DoFs are excited, f is the forcing amplitude, ω is the excitation frequency and ε is a small parameter, while t is time. $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are assumed to be symmetric, real and positive-definite, while $\tilde{\mathbf{b}}$ has the generic form

$$\tilde{\mathbf{b}} = \begin{bmatrix} \vdots \\ \sum_{m=3,5,\dots} \sum_{h_1+\dots+h_n=m} \tilde{b}_{jh_1\dots h_n} \prod_{i=1}^{n} x_i^{h_i} \\ \vdots \end{bmatrix}, \tag{2.2}$$

where $j = \overline{1,n}$, such that, for example, for a 2-DoF system the cubic terms of the first raw of $\tilde{\mathbf{b}}$ are $\tilde{b}_{130}x_1^3 + \tilde{b}_{121}x_1^2x_2 + \tilde{b}_{112}x_1x_2^2 + \tilde{b}_{103}x_2^3$.

In order to decouple the linear part of the system, we apply classical modal analysis, i.e., denoting $\mathbf{U}$ the matrix containing the eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$, we apply the transformation $\mathbf{x} = \mathbf{U}\mathbf{y}$ to Eq. (2.1) and we pre-multiply it by $\mathbf{U}^{\mathrm{T}}$. The resulting system has the form

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}\ddot{\mathbf{y}} + \mathbf{U}^{\mathrm{T}}\mathbf{D}\mathbf{U}\dot{\mathbf{y}} + \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}\mathbf{y} + \mathbf{U}^{\mathrm{T}}\tilde{\mathbf{b}} = \sqrt{\varepsilon}\mathbf{U}^{\mathrm{T}}\tilde{\mathbf{v}}f\cos(\omega t), \tag{2.3}$$

where $\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}$ and $\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}$ are diagonal. Pre-multiplying then the system by the inverse of $\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}$ and applying the transformation $\mathbf{y} = \sqrt{\varepsilon}f\mathbf{q}$, we have

$$\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} + \mathbf{b} = \mathbf{v}\cos(\omega t), \tag{2.4}$$

where $\mathbf{C}$ is not symmetric (differently from $\mathbf{D}$), and $\mathbf{\Omega}$ is diagonal and contains the squares of the natural frequencies of the different modes of vibration. $\mathbf{\Omega}$, $\mathbf{C}$ and $\mathbf{b}$ have the general form

$$\mathbf{\Omega} = \begin{bmatrix} \ddots & & 0 \\ & \Omega_j^2 & \\ 0 & & \ddots \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \vdots \\ \sum_{m=3,5,\dots} \varepsilon^{\frac{m-1}{2}} f^{m-1} \sum_{h_1+\dots+h_n=m} b_{jh_1\dots h_n} \prod_{i=1}^{n} q_i^{h_i} \\ \vdots \end{bmatrix}. \tag{2.5}$$

The forcing amplitude f is contained uniquely in $\mathbf{b}$, i.e. the system depends on the forcing amplitude only through the coefficients of the nonlinear terms. This clearly illustrates that considering small nonlinearities or small forcing amplitudes is equivalent, in many cases. Furthermore, terms of order m are proportional to $\varepsilon^{\frac{m-1}{2}}$ (see $\mathbf{b}$), which prepares the ground for the perturbation procedure implemented in the following sections.

For the sake of simplicity, we consider only cubic nonlinearities in this analytical development. Neglecting terms higher than the third order allows us to develop the equations up to order ε^1. Nevertheless, the procedure can be extended to higher-order nonlinear terms.

2.3 Calculation of Unforced, Undamped Response Using Nonlinear Normal Modes

NNMs give a good approximation of the system backbone curves. In order to calculate them, we consider the undamped unforced equations of motion

$$\ddot{\mathbf{q}} + \boldsymbol{\Omega}\mathbf{q} + \mathbf{b} = \mathbf{0}. \tag{2.6}$$

Adopting a standard perturbation procedure, the solution of Eq. (2.6) can be approximated to a single harmonic by

$$\mathbf{q} = \left(\mathbf{q}_0 + \varepsilon\mathbf{q}_1 + O\left(\varepsilon^2\right)\right)\sin\left(\omega t\right), \tag{2.7}$$

where

$$\mathbf{q}_0 = \begin{bmatrix} \vdots \\ q_{j0} \\ \vdots \end{bmatrix}, \quad \mathbf{q}_1 = \begin{bmatrix} \vdots \\ q_{j1} \\ \vdots \end{bmatrix} \text{and } \omega = \omega_0 + \varepsilon\omega_1 + O\left(\varepsilon^2\right), \tag{2.8}$$

which is valid for small values of ε.

Substituting Eq. (2.7) into Eq. (2.6) and adopting the standard single harmonic approximation $\sin^3(\omega t) \approx 3/4\sin(\omega t)$, we obtain n equations of the form

$$-\left(\omega_0^2 + 2\varepsilon\omega_1\omega_0\right)\left(q_{j0} + \varepsilon q_{j1}\right) + \Omega_j^2\left(q_{j0} + \varepsilon q_{j1}\right) + \frac{3}{4}\varepsilon\left(\sum_{h_1+\ldots+h_n=3} b_{jh_1\ldots h_n}\prod_{i=1}^{n} q_i^{h_i}\right) + O\left(\varepsilon^2\right) = 0. \tag{2.9}$$

Considering the terms of order ε^0, related to the underlying linear system, we have

$$-\omega_0^2 q_{j0} + \Omega_j^2 q_{j0} = 0. \tag{2.10}$$

In order to obtain the NNM associated with the lth mode of vibration, we impose that the linear part of all other modes have zero amplitude, i.e.

$$\text{for } j \neq l \;\Rightarrow\; q_{j0} = 0, \tag{2.11}$$

while $q_{l0} \neq 0$, such that the we refer to the lth mode of vibration and $\omega_0 = \Omega_l$.

Considering now the terms of order ε^1 of Eq. (2.9), we obtain

$$j \neq l \qquad -\Omega_l^2 q_{j1} + \Omega_j^2 q_{j1} + \frac{3}{4} b_{j0\ldots3\ldots0} q_{l0}^3 = 0 \tag{2.12}$$

$$j = l \qquad -\Omega_l^2 q_{l1} - 2\omega_1\Omega_l q_{l0} + \Omega_l^2 q_{l1} + \frac{3}{4} b_{l0\ldots3\ldots0} q_{l0}^3 = 0. \tag{2.13}$$

($b_{j0\ldots3\ldots0}$, for example in a 4-DoF system where $l = 2$, would be b_{j0300}). Thus, from Eq. (2.12) we have

$$q_{j1} = \frac{3}{4}\frac{b_{j0\ldots3\ldots0} q_{l0}^3}{\Omega_l^2 - \Omega_j^2}, \quad j \neq l, \tag{2.14}$$

that indicates how the modes not directly excited by the force ($j \neq l$) are excited by the nonlinear coupling. While from Eq. (2.13) we have

$$\omega_1 = \frac{3}{4} \frac{b_{l0\ldots3\ldots0} q_{l0}^2}{2\Omega_l}, \tag{2.15}$$

where ω_1 represents the variation of the lth natural frequency with respect to the amplitude of oscillations, in first approximation.

2.4 Calculation of Forced, Damped Response Using Energy Balance

The energy balance criterion can be used to relate the undamped, unforced dynamics of the NNMs to the forced damped dynamics [10, 11], thus obtaining complete information of the resonant peaks. Given a general linearly-damped mechanical system, the balance between the dissipated and input energies is expressed by the equation

$$\int_0^T \dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{D} \dot{\mathbf{x}}(t) \mathrm{d}t = \int_0^T \dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{f}(t) \mathrm{d}t, \tag{2.16}$$

where T is the period of vibration and f is the external force. In the case of harmonic excitation $\mathbf{f} = \tilde{\mathbf{v}} f \cos(\omega t)$, approximating the solution to a single harmonics $\mathbf{x}(t) \approx \mathbf{x}_0 \sin(\omega t)$ in resonant conditions, Eq. (2.16) has the form

$$\omega^2 \mathbf{x}_0^{\mathrm{T}} \mathbf{D} \mathbf{x}_0 \int_0^T \cos(\omega t)^2 \,\mathrm{d}t \approx \omega \mathbf{x}_0^{\mathrm{T}} \tilde{\mathbf{v}} f \int_0^T \cos(\omega t)^2 \,\mathrm{d}t \quad \Rightarrow \quad \omega \mathbf{x}_0^{\mathrm{T}} \mathbf{D} \mathbf{x}_0 \approx \mathbf{x}_0^{\mathrm{T}} \tilde{\mathbf{v}} f. \tag{2.17}$$

Inserting in Eq. (2.17) the solution of the undamped unforced system, it is possible to estimate the ratio between the forcing amplitude and the amplitude of oscillation.

Applying the aforementioned procedure, considering the system in Eq. (2.4) and the tentative solution (2.7), we obtain the energy balance equation

$$(\Omega_l + \varepsilon\omega_1) \left(\begin{bmatrix} 0 \\ \vdots \\ q_{l0} \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon \begin{bmatrix} q_{11} \\ \vdots \\ q_{n1} \end{bmatrix}^{\mathrm{T}} \right) \mathbf{C} \left(\begin{bmatrix} 0 \\ \vdots \\ q_{l0} \\ \vdots \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} q_{11} \\ \vdots \\ q_{n1} \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 0 \\ \vdots \\ q_{l0} \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon \begin{bmatrix} q_{11} \\ \vdots \\ q_{n1} \end{bmatrix}^{\mathrm{T}} \right) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + O\left(\varepsilon^2\right). \tag{2.18}$$

Collecting terms of order ε^0 we obtain

$$\Omega_l q_{l0}^2 c_{ll} = q_{l0} v_l \quad \Rightarrow \quad q_{l0} = \frac{v_l}{\Omega_l c_{ll}}, \tag{2.19}$$

which yields the relation in the linear range between the forcing amplitude (here normalized) and the oscillation amplitude at resonance as a function of the modal damping.

Collecting terms of order ε^1 of Eq. (2.18) we have

$$\Omega_l q_{l0}\left(\sum_{\substack{j=1\\j\neq l}}^{n} c_{lj}q_{j1}+\sum_{\substack{j=1\\j\neq l}}^{n} c_{jl}q_{j1}\right)+2\Omega_l q_{l0}c_{ll}q_{l1}+\omega_1 q_{l0}^2 c_{ll}=\sum_{\substack{j=1\\j\neq l}}^{n} q_{j1}\upsilon_j+q_{l1}\upsilon_l, \tag{2.20}$$

thus

$$q_{l1}=\frac{\Omega_l q_{l0}\left(\sum_{\substack{j=1\\j\neq l}}^{n} c_{lj}q_{j1}+\sum_{\substack{j=1\\j\neq l}}^{n} c_{jl}q_{j1}\right)+\omega_1 q_{l0}^2 c_{ll}-\sum_{\substack{j=1\\j\neq l}}^{n} q_{j1}\upsilon_j}{\upsilon_l-2\Omega_l q_{l0}c_{ll}}, \tag{2.21}$$

which indicates the variation of the modal amplitude of oscillation due to nonlinearity.

2.5 Enforcement of Linear Properties

The set of Eqs. (2.11), (2.14), (2.15), (2.19) and (2.21) characterize the NNMs and forced resonant response of the lth structural mode. Equations (2.11) and (2.19) refer to the underlying linear system; Eq. (2.11) is due to the imposed resonant condition, while Eq. (2.19) gives the relationship between the forcing amplitude and the amplitude of oscillation in the linear case. Equations (2.14), (2.15) and (2.21) refer to the nonlinear properties of the system. q_{j1} ($j=\overline{1n}$, $j\neq l$) are directly proportional to $b_{j0\ldots3\ldots0}$, ω_1 is directly proportional to $b_{l0\ldots3\ldots0}$, while q_{l1} depends linearly on all the coefficients $b_{j0\ldots3\ldots0}$ ($j=\overline{1,n}$), which are the coefficients of the solely nonlinear terms relevant at order ε^1, when the lth resonance is excited. If the system includes higher-order nonlinear terms, other coefficients come into play.

The objective of this section is to show that, through an appropriate tuning of the coefficients $b_{j0\ldots3\ldots0}$, the dynamics of the nonlinear system can resemble that of a linear system.

2.5.1 *Enforcing Force-Displacement Proportionality*

We consider the general case for which the objective is to keep the force-displacement proportionality typical of linear systems for the kth DoF (x_k in the physical coordinate system), while the system vibrates at the lth resonant frequency. Recalling that $\mathbf{x}=\sqrt{\varepsilon f}\mathbf{Uq}$ and considering the lth resonance, it follows that

$$x_k=\sqrt{\varepsilon f}\left(u_{kl}q_{l0}+\varepsilon\left(\sum_{j=1}^{n} u_{kj}q_{j1}\right)+O\left(\varepsilon^2\right)\right). \tag{2.22}$$

where u_{ij} is an element of matrix $\mathbf{U}$. If we impose that

$$\sum_{j=1}^{n} u_{kj}q_{j1}=0 \tag{2.23}$$

x_k obeys, in first approximation, a force-displacement proportionality. Equation (2.23) depends on the n coefficients $b_{j0\ldots3\ldots0}$ ($j=\overline{1n}$), which, in turn, depend on the parameters characterizing the physical nonlinearities of the system.

2.5.2 *Enforcing Straight Line Frequency Backbone*

Another property, typical of linear systems and generally not satisfied in nonlinear systems, is the invariance of the resonance frequencies with respect to forcing amplitude, giving rise to straight backbone curves. Hardening (softening) nonlinearities shift resonance frequencies toward greater (lower) values for increasing amplitudes of oscillation. To enforce a straight backbone curve for the lth resonance, ω_1 should be set to 0, and, hence, $b_{l0\ldots3\ldots0} = 0$. An important feature of the proposed approach is that the final equations are fully explicit. Thus, in spite of their complexity, they can be rapidly implemented.

2.6 Beam Example

To validate the previous theoretical developments, a nonlinear cantilever beam with a nonlinear attachment, similar to that studied in [12, 13], is considered. The two nonlinearities of the attachment, k_{nl2} and k_{nl3}, are designed so as to enforce linear properties in the coupled system. A 2-DoF reduced-order model of this system is

$$\begin{bmatrix} 0.46 & 0 \\ 0 & 0.069 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0.52 & -0.15 \\ -0.15 & 0.25 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 14065 & -1709 \\ -1709 & 1709 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_{nl1}x_1^3 + k_{nl2}(x_1 - x_2)^3 \\ k_{nl2}(x_2 - x_1)^3 + k_{nl3}x_2^3 \end{bmatrix} = f \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.24}$$

The cantilever beam has a single concentrated nonlinearity, $k_{nl1} = 3.3 \times 10^9$ N/m^3.

Figure 2.1a illustrates the normalized frequency response of the first degree of freedom of the system without the additional nonlinearities, i.e., $k_{nl2} = k_{nl3} = 0$, for three forcing amplitudes. A substantial hardening effect is observed, while the amplitude of the first (second) resonance decreases (increases) when the forcing amplitude increases.

The displacement x_1 around the first resonance obeys force-displacement proportionality if

$$-0.0128 + 1.39 \times 10^{-10} k_{nl2} + 2.17 \times 10^{-10} k_{nl3} = 0. \tag{2.25}$$

which is verified if $k_{nl2} = 9.2 \times 10^7$ N/m^3 and $k_{nl3} = 0$. Figure 2.1b that depicts the corresponding normalized frequency response confirms that the amplitude of the first resonance is almost identical for the three forcing amplitudes. The linearization of the force-displacement relation can be further improved if a fifth-order spring is added between the two lumped masses. This is evidenced in Fig. 2.1c, which compares the envelopes of the resonant peaks in the different considered cases.

Aiming now to enforce isochronicity of the first resonance, i.e., $\omega_1 = 0$, we obtain

$$62.6 + 7.39 \times 10^{-7} k_{nl2} + 2.84 \times 10^{-6} k_{nl3} = 0. \tag{2.26}$$

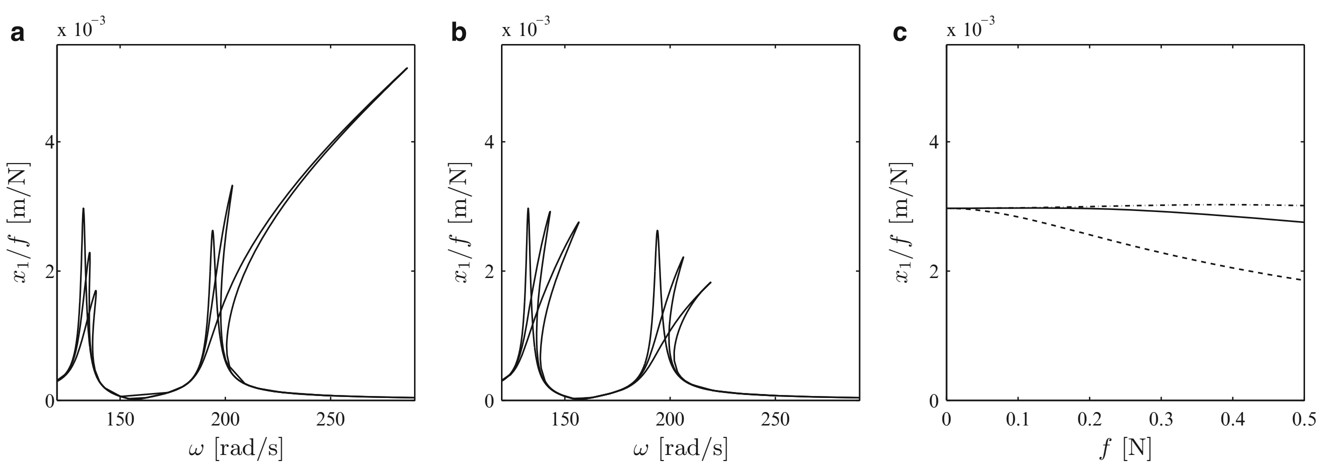

Fig. 2.1 (**a**), (**b**) frequency response of the system in Eq. (2.24) for forcing amplitudes $f \to 0$, $f = 0.3$ and $f = 0.5$ N, for $k_{nl2} = k_{nl3} = 0$ (**a**) and for $k_{nl2} = 9.2 \times 10^7$ N/m^3 and $k_{nl3} = 0$ (**b**); (**c**) envelope of the first resonant peak with respect to the forcing amplitude. *Dashed line*: $k_{nl2} = k_{nl3} = 0$; *solid line*: $k_{nl2} = 9.2 \times 10^7$ N/m^3 and $k_{nl3} = 0$; *dash-dotted line*: $k_{nl2} = 9.2 \times 10^7$ N/m^3, $k_{nl3} = 0$ and additional quintic spring $k_{nl4} = 3.9 \times 10^{12}$ N/m^5

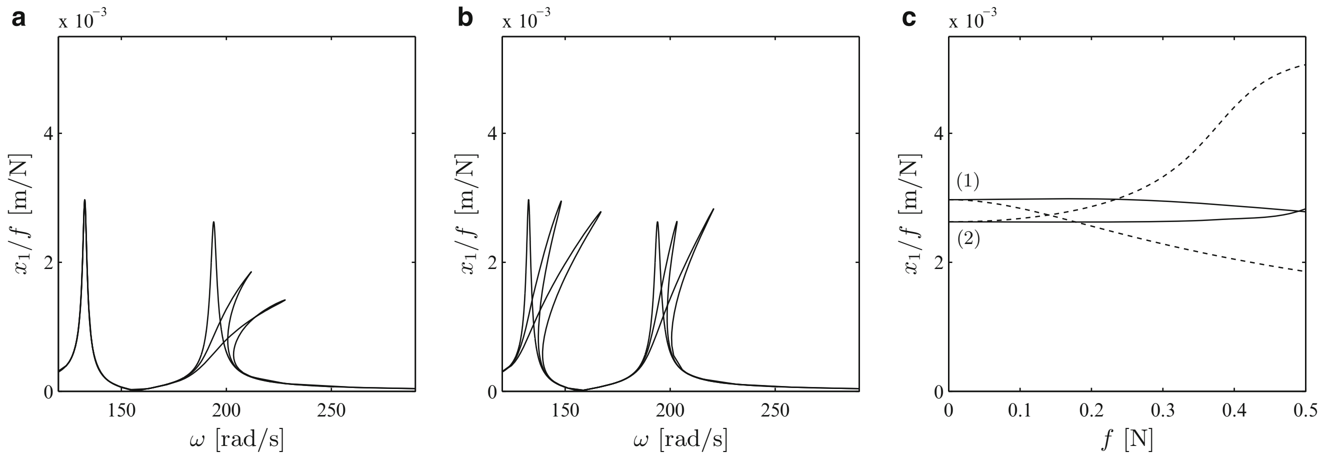

Fig. 2.2 Frequency response of the system in Eq. (2.24) for forcing amplitudes $f \to 0$, $f = 0.3$ and $f = 0.5$ N, for $k_{nl2} = 2.11 \times 10^7$ and $k_{nl3} = -7.7 \times 10^7$ (**a**) and for $k_{nl2} = 2.75 \times 10^7$ and $k_{nl3} = 4.14 \times 10^7$ (**b**); envelope of the two resonant peaks with respect to the forcing amplitude. *Dashed lines*: $k_{nl2} = k_{nl3} = 0$; *solid lines*: $k_{nl2} = 2.11 \times 10^8$ and $k_{nl3} = -7.7 \times 10^7$; the numbers in brackets indicate the first *(1)* and the second *(2)* resonant peak

Equations (2.25) and (2.26) can be simultaneously satisfied if and only if $k_{nl2} = 2.11 \times 10^8$ and $k_{nl3} = -7.7 \times 10^7$. Doing so, the first resonance peak can be made practically unchanged with respect to the linear resonance, as plotted in Fig. 2.2a.

The developed framework can go beyond operating on a single resonance. For instance, force-displacement proportionality of the second peak (with respect to x_1) can be imposed through

$$0.00654 - 1.53 \times 10^{-10} k_{nl2} - 5.63 \times 10^{-11} k_{nl3} = 0. \qquad (2.27)$$

For $k_{nl2} = 2.75 \times 10^7$ and $k_{nl3} = 4.14 \times 10^7$, Eqs. (2.25) and (2.27) are simultaneously verified, which means that both resonances obey approximately force-displacement proportionality. The corresponding normalized frequency response is illustrated in Fig. 2.2b. Figure 2.2c depicts the envelopes of the two peaks with respect to the forcing amplitude. The dashed lines, referring to the original system ($k_{nl2} = k_{nl3} = 0$), either decrease (first peak) or increase (second peak) whereas the solid lines are almost horizontal for a large range of forcing amplitudes.

2.7 Conclusions

This paper has demonstrated how it is possible to design nonlinearities of a mechanical system so that its dynamics resemble that of a linear system, which include force-displacement proportionality and isochronocity. The developments exploited a standard perturbation technique, combined with NNM theory and energy balance of periodic solutions at resonance, and were validated using a 2-DoF reduced-order model of a nonlinear cantilever beam.

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