Chapter 8
Heat Transfer in a Complex Medium

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Abstract The heat equation is considered in the complex medium consisting of many small bodies (particles) embedded in a given material. On the surfaces of the small bodies an impedance boundary condition is imposed. An equation for the limiting field is derived when the characteristic size a of the small bodies tends to zero, their total number $\mathcal{N}(a)$ tends to infinity at a suitable rate, and the distance $d = d(a)$ between neighboring small bodies tends to zero: $a << d$, $\lim_{a\to 0} \frac{a}{d(a)} = 0$. No periodicity is assumed about the distribution of the small bodies. These results are basic for a method of creating a medium in which heat signals are transmitted along a given line. The technical part for this method is based on an inverse problem of finding potential with prescribed eigenvalues.

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8.1 Introduction and Results

In this paper the problem of heat transfer in a complex medium consisting of many small impedance particles of an arbitrary shape is solved. Equation for the effective limiting temperature is derived when the characteristic size a of the particles tends to zero while their number tends to infinity at a suitable rate while the distance d between closest neighboring particles is much larger than a, $d >> a$.

These results are used for developing a method for creating materials in which heat is transmitted along a line. Thus, the information can be transmitted by a heat signals.

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The contents of this paper is based on the earlier papers of the author cited in the bibliography, especially [13, 16, 17].

Let many small bodies (particles) D_m, $1 \le m \le M$, of an arbitrary shape be distributed in a bounded domain $D \subset \mathbb{R}^3$, diam$D_m = 2a$, and the boundary of D_m is denoted by $\mathcal{S}_m$ and is assumed twice continuously differentiable. The small bodies are distributed according to the law

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0. \tag{8.1}$$

Here $\Delta \subset D$ is an arbitrary open subdomain of D, $\kappa \in [0, 1)$ is a constant, $N(x) \ge 0$ is a continuous function, and $\mathcal{N}(\Delta)$ is the number of the small bodies D_m in Δ. The heat equation can be stated as follows:

$$u_t = \nabla^2 u + f(x) \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, := \Omega, \quad u|_{t=0} = 0, \tag{8.2}$$

$$u_N = \zeta_m u \text{ on } \mathcal{S}_m, \quad 1 \le m \le M, \qquad Re\zeta_m \ge 0. \tag{8.3}$$

Here N is the outer unit normal to $\mathcal{S}$,

$$\mathcal{S} := \bigcup_{m=1}^M \mathcal{S}_m, \quad \zeta_m = \frac{h(x_m)}{a^\kappa}, \quad x_m \in D_m, \quad 1 \le m \le M,$$

and $h(x)$ is a continuous function in D, Re$h \ge 0$.

Denote

$$\mathcal{U} := \mathcal{U}(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t)dt.$$

Then, taking the Laplace transform of Eqs. (8.2)–(8.3) one gets:

$$-\nabla^2 \mathcal{U} + \lambda \mathcal{U} = \lambda^{-1} f(x) \text{ in } \Omega, \tag{8.4}$$

$$\mathcal{U}_N = \zeta_m \mathcal{U} \text{ on } \mathcal{S}_m, 1 \le m \le M. \tag{8.5}$$

Let

$$g(x, y) := g(x, y, \lambda) := \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi |x - y|}, \tag{8.6}$$

$$F(x, \lambda) := \frac{1}{\lambda} \int_{\mathbb{R}^3} g(x, y) f(y) dy. \tag{8.7}$$

Look for the solution to (8.4)–(8.5) of the form

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} \int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds, \tag{8.8}$$

where

$$\mathcal{U}(x,\lambda) := \mathcal{U}(x) := \mathcal{U}, \tag{8.9}$$

and $\mathcal{U}(x)$ depends on λ.

The functions σ_m are unknown and should be found from the boundary conditions (8.5). Equation (8.4) is satisfied by $\mathcal{U}$ of the form (8.8) with arbitrary continuous σ_m. To satisfy the boundary condition (8.5) one has to solve the following equation obtained from the boundary condition (8.5):

$$\frac{\partial \mathcal{U}_e(x)}{\partial N} + \frac{A_m\sigma_m - \sigma_m}{2} - \zeta_m\mathcal{U}_e - \zeta_m T_m\sigma_m = 0 \text{ on } \mathcal{S}_m, \quad 1 \le m \le M, \tag{8.10}$$

where the effective field $\mathcal{U}_e(x)$ is defined by the formula:

$$\mathcal{U}_e(x) := \mathcal{U}_{e,m}(x) := \mathcal{U}(x) - \int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds, \tag{8.11}$$

the operator T_m is defined by the formula:

$$T_m\sigma_m = \int_{\mathcal{S}_m} g(s,s')\sigma_m(s')ds', \tag{8.12}$$

and A_m is:

$$A_m\sigma_m = 2\int_{\mathcal{S}_m} \frac{\partial g(s,s')}{\partial N_s}\sigma_m(s')ds'. \tag{8.13}$$

In deriving Eq. (8.10) we have used the known formula for the outer limiting value on $\mathcal{S}_m$ of the normal derivative of a simple layer potential.

We now apply the ideas and methods for solving many-body scattering problems developed in [12–15].

Let us call $\mathcal{U}_{e,m}$ the effective (self-consistent) value of $\mathcal{U}$, acting on the m-th body. As $a \to 0$, the dependence on m disappears, since

$$\int_{\mathcal{S}_m} g(x,s)\sigma_m(s)ds \to 0 \text{ as } a \to 0.$$

One has

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m + \mathcal{J}_2, \quad x_m \in D_m, \tag{8.14}$$

where

$$Q_m := \int_{\mathcal{S}_m} \sigma_m(s)ds,$$

$$\mathcal{J}_2 := \sum_{m=1}^{M} \int_{\mathcal{S}_m} [g(x,s') - g(x,x_m)]\sigma_m(s')ds'. \tag{8.15}$$

Define

$$\mathcal{J}_1 := \sum_{m=1}^{M} g(x,x_m)Q_m. \tag{8.16}$$

We prove in Lemma 3, Sect. 8.4 (see also [13, 16]) that

$$|\mathcal{J}_2| << |\mathcal{J}_1| \text{ as } a \to 0 \tag{8.17}$$

provided that

$$\lim_{a\to 0} \frac{a}{d(a)} = 0, \tag{8.18}$$

where $d(a) = d$ is the minimal distance between neighboring particles.

If (8.17) holds, then problem (8.4)–(8.5) is solved asymptotically by the formula

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m, \quad a \to 0, \tag{8.19}$$

provided that asymptotic formulas for Q_m, as $a \to 0$, are found.

To find formulas for Q_m, let us integrate (8.10) over $\mathcal{S}_m$, estimate the order of the terms in the resulting equation as $a \to 0$, and keep the main terms, that is, neglect the terms of higher order of smallness as $a \to 0$.

We get

$$\int_{\mathcal{S}_m} \frac{\partial \mathcal{U}_e}{\partial N} ds = \int_{D_m} \nabla^2 \mathcal{U}_e dx = O(a^3). \tag{8.20}$$

Here we assumed that $|\nabla^2 \mathcal{U}_e| = O(1), a \to 0$. This assumption is valid since $\mathcal{U} = \lim_{a\to 0} \mathcal{U}_e$ is smooth as a solution to an elliptic equation. One has

$$\int_{S_m} \frac{A_m\sigma_m - \sigma_m}{2} ds = -Q_m[1 + o(1)],\ a \to 0. \tag{8.21}$$

This relation is proved in Lemma 2, Sect. 8.4, see also [13]. Furthermore,

$$-\zeta_m \int_{S_m} \mathcal{U}_e ds = -\zeta_m |S_m| \mathcal{U}_e(x_m) = O(a^{2-\kappa}), \quad a \to 0, \tag{8.22}$$

where $|S_m| = O(a^2)$ is the surface area of S_m. Finally,

$$\begin{aligned} -\zeta_m \int_{S_m} ds \int_{S_m} g(s,s')\sigma_m(s')ds' &= -\zeta_m \int_{S_m} ds'\sigma_m(s') \int_{S_m} dsg(s,s') \\ &= Q_m O(a^{1-\kappa}), \qquad a \to 0. \end{aligned} \tag{8.23}$$

Thus, the main term of the asymptotic of Q_m, as $a \to 0$, is

$$Q_m = -\zeta_m |S_m| \mathcal{U}_e(x_m). \tag{8.24}$$

Formulas (8.24) and (8.19) yield

$$\mathcal{U}(x,\lambda) = F(x,\lambda) - \sum_{m=1}^{M} g(x,x_m)\zeta_m |S_m| \mathcal{U}_e(x_m,\lambda), \tag{8.25}$$

and

$$\mathcal{U}_e(x_m,\lambda) = F(x_m,\lambda) - \sum_{m'\neq m, m'=1}^{M} g(x_m,x_{m'})\zeta_{m'}|S_{m'}|\mathcal{U}_e(x_{m'},\lambda). \tag{8.26}$$

Denote

$$\mathcal{U}_e(x_m,\lambda) := \mathcal{U}_m, \quad F(x_m,\lambda) := F_m, \quad g(x_m,x_{m'}) := g_{mm'},$$

and write (8.26) as a linear algebraic system for $\mathcal{U}_m$:

$$\mathcal{U}_m = F_m - a^{2-\kappa} \sum_{m'\neq m} g_{mm'} h_{m'} c_{m'} \mathcal{U}_{m'}, \quad 1 \le m \le M, \tag{8.27}$$

where $h_{m'} = h(x_{m'})$, $\zeta_{m'} = \frac{h_{m'}}{a^\kappa}$, $c_{m'} := |S_{m'}|a^{-2}$.

Consider a partition of the bounded domain D, in which the small bodies are distributed, into a union of $P << M$ small nonintersecting cubes Δ_p, $1 \le p \le P$, of side b,

$$b >> d, \quad b = b(a) \to 0 \quad \text{as } a \to 0 \quad \lim_{a\to 0} \frac{d(a)}{b(a)} = 0.$$

Let $x_p \in \Delta_p$, $|\Delta_p| =$ volume of Δ_p. One has

$$a^{2-\kappa} \sum_{m'=1, m'\neq m}^{M} g_{mm'} h_{m'} c_{m'} \mathcal{U}_{m'} = a^{2-\kappa} \sum_{p'=1, p'\neq p}^{P} g_{pp'} h_{p'} c_{p'} \mathcal{U}_{p'} \sum_{x_{m'}\in\Delta_{p'}} 1 =$$
$$= \sum_{p'\neq p} g_{pp'} h_{p'} c_{p'} \mathcal{U}_{p'} N(x_{p'}) |\Delta_{p'}| [1 + o(1)], \quad a \to 0. \tag{8.28}$$

Thus, (8.27) yields a linear algebraic system (LAS) of order $P << M$ for the unknowns $\mathcal{U}_p$:

$$\mathcal{U}_p = F_p - \sum_{p'\neq p, p'=1}^{P} g_{pp'} h_{p'} c_{p'} N_{p'} \mathcal{U}_{p'} |\Delta_{p'}|, \quad 1 \le p \le P. \tag{8.29}$$

Since $P << M$, the order of the original LAS (8.27) is drastically reduced. This is crucial when the number of particles tends to infinity and their size a tends to zero. We have assumed that

$$h_{m'} = h_{p'}[1 + o(1)], \quad c_{m'} = c_{p'}[1 + o(1)], \quad \mathcal{U}_{m'} = \mathcal{U}_{p'}[1 + o(1)], \ a \to 0, \tag{8.30}$$

for $x_{m'} \in \Delta_{p'}$. This assumption is justified, for example, if the functions $h(x)$, $\mathcal{U}(x, \lambda)$,

$$c(x) = \lim_{x_{m'}\in\Delta_x, a\to 0} \frac{|S_{m'}|}{a^2},$$

and $N(x)$ are continuous, but these assumptions can be relaxed.

The continuity of the $\mathcal{U}(x, \lambda)$ is a consequence of the fact that this function satisfies elliptic equation, and the continuity of $c(x)$ is assumed. If all the small bodies are identical, then $c(x) = c = const$, so in this case the function $c(x)$ is certainly continuous.

The sum in the right-hand side of (8.29) is the Riemannian sum for the integral

$$lim_{a\to 0} \sum_{p'=1, p'\neq p}^{P} g_{pp'} h_{p'} c_{p'} N(x_{p'}) \mathcal{U}_{p'} |\Delta_p'| =$$
$$\int_D g(x, y) h(y) c(y) N(y) \mathcal{U}(y, \lambda) dy$$

Therefore, linear algebraic system (8.29) is a collocation method for solving integral equation

$$\mathcal{U}(x,\lambda) = F(x,\lambda) - \int_D g(x,y)c(y)h(y)N(y)\mathcal{U}(y,\lambda)dy. \tag{8.31}$$

Convergence of this method for solving equations with weakly singular kernels is proved in [10], see also [11, 20].

Applying the operator $-\nabla^2 + \lambda$ to Eq. (8.31) one gets an elliptic differential equation:

$$(-\Delta + \lambda)\mathcal{U}(x,\lambda) = \frac{f(x)}{\lambda} - c(x)h(x)N(x)\mathcal{U}(x,\lambda). \tag{8.32}$$

Taking the inverse Laplace transform of this equation yields

$$u_t = \Delta u + f(x) - q(x)u, \quad q(x) := c(x)h(x)N(x). \tag{8.33}$$

Therefore, the limiting equation for the temperature contains the term $q(x)u$. Thus, the embedding of many small particles creates a distribution of source and sink terms in the medium, the distribution of which is described by the term $q(x)u$.

If one solves Eq. (8.31) for $\mathcal{U}(x,\lambda)$, or linear algebraic system (8.29) for $\mathcal{U}_p(\lambda)$, then one can Laplace-invert $\mathcal{U}(x,\lambda)$ for $\mathcal{U}(x,t)$. Numerical methods for Laplace inversion from the real axis are discussed in [4, 19].

If one is interested only in the average temperature, one can use the relation

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u(x,t)dt = \lim_{\lambda\to 0} \lambda\mathcal{U}(x,\lambda). \tag{8.34}$$

Relation (8.34) is proved in Lemma 1, Sect. 8.4. It holds if the limit on one of its sides exists. The limit on the right-hand side of (8.34) let us denote by $\psi(x)$. From Eqs. (8.7) and (8.31) it follows that ψ satisfies the equation

$$\psi = \varphi - B\varphi,$$

where

$$\varphi := \int_\Omega g_0(x,y)f(y)dy,$$

$$g_0(x,y) := \frac{1}{4\pi|x-y|},$$

$$B\psi := \int_\Omega g_0(x,y)q(y)\psi(y)dy,$$

and

$$q(x) := c(x)h(x)N(x).$$

The function ψ can be calculated by the formula

$$\psi(x) = (I + B)^{-1}\varphi. \tag{8.35}$$

From the physical point of view the function $h(x)$ is non-negative because the flux $-\nabla u$ of the heat flow is proportional to the temperature u and is directed along the outer normal N: $-u_N = h_1 u$, where $h_1 = -h < 0$. Thus, $q \geq 0$.

It is proved in [5, 6] that zero is not an eigenvalue of the operator $-\nabla^2 + q(x)$ provided that $q(x) \geq 0$ and

$$q = O\Big(\frac{1}{|x|^{2+\epsilon}}\Big), \quad |x| \to \infty,$$

and $\epsilon > 0$.

In our case, $q(x) = 0$ outside of the bounded region D, so the operator $(I + B)^{-1}$ exists and is bounded in $C(D)$.

Let us formulate our basic result.

Theorem 1 *Assume (8.1), (8.18), and* $h \geq 0$. *Then, there exists the limit* $\mathcal{U}(x, \lambda)$ *of* $\mathcal{U}_e(x, \lambda)$ *as* $a \to 0$, $\mathcal{U}(x, \lambda)$ *solves Eq. (8.31), and there exists the limit (8.34), where* $\psi(x)$ *is given by formula (8.35).*

Methods of our proof of Theorem 1 are quite different from the proof of homogenization theory results in [1, 3].

The author's plenary talk at Chaos-2015 Conference was published in [18].

8.2 Creating Materials Which Allows One to Transmit Heat Signals Along a Line

In applications it is of interest to have materials in which heat propagates along a line and decays fast in all the directions orthogonal to this line.

In this section a construction of such material is given. We follow [17] with some simplifications.

The idea is to create first the medium in which the heat transfer is governed by the equation

$$u_t = \Delta u - q(x)u \quad \text{in } D, \quad u|_S = 0, \quad u|_{t=0} = f(x), \tag{8.36}$$

where D is a bounded domain with a piece-wise smooth boundary S, $D = D_0 \times [0, L]$, $D_0 \subset \mathbb{R}^2$ is a smooth domain orthogonal to the axis x_1, $x = (x_1, x_2, x_3)$, $x_2, x_3 \in D_0$, $0 \leq x_1 \leq L$.

Such a medium is created by embedding many small impedance particles into a given domain D filled with a homogeneous material. A detailed argument, given in Sect. 8.1 (see also [13, 16]), yields the following result.

Assume that in every open subset Δ of D the number of small particles is defined by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x)dx[1 + o(1)], \quad a \to 0, \tag{8.37}$$

where $a > 0$ is the characteristic size of a small particle, $\kappa \in [0, 1)$ is a given number and $N(x) \geq 0$ is a continuous in D function.

Assume also that on the surface S_m of the m-th particle D_m the impedance boundary condition holds. Here

$$1 \leq m \leq M = \mathcal{N}(D) = O\left(\frac{1}{a^{2-\kappa}}\right), \quad a \to 0,$$

and the impedance boundary conditions are:

$$u_N = \zeta_m u \quad \text{on } S_m, \quad \text{Re}\zeta_m \geq 0, \tag{8.38}$$

where

$$\zeta_m := \frac{h(x_m)}{a^{\kappa}}$$

is the boundary impedance, $x_m \in D_m$ is an arbitrary point (since D_m is small the position of x_m in D_m is not important), κ is the same parameter as in (8.37) and $h(x)$ is a continuous in D function, Re$h \geq 0$, N is the unit normal to S_m pointing out of D_m. The functions $h(x)$, $N(x)$ and the number κ can be chosen as the experimenter wishes.

It is proved in Sect. 8.1 (see also [13, 16]) that, as $a \to 0$, the solution of the problem

$$u_t = \Delta u \quad \text{in} D \setminus \bigcup_{m=1}^{M} D_m, \; u_N = \zeta_m u \quad \text{on } S_m, \; 1 \leq m \leq M, \tag{8.39}$$

$$u|_S = 0, \tag{8.40}$$

and

$$u|_{t=0} = f(x), \tag{8.41}$$

has a limit $u(x, t)$. This limit solves problem (8.36) with

$$q(x) = c_S N(x)h(x), \tag{8.42}$$

where

$$c_S := \frac{|S_m|}{a^2} = const, \tag{8.43}$$

and $|S_m|$ is the surface area of S_m. By assuming that c_S is a constant, we assume, for simplicity only, that the small particles are identical in shape, see [13].

Since $N(x) \geq 0$ is an arbitrary continuous function and $h(x)$, $Reh \geq 0$, is an arbitrary continuous function, and both functions can be chosen by experimenter as he/she wishes, it is clear that an arbitrary real-valued potential q can be obtained by formula (8.42).

Suppose that

$$(-\Delta + q(x))\phi(x) = \lambda_n \phi_n, \quad \phi_n|_S = 0, \quad ||\phi_n||_{L^2(D)} = ||\phi_n|| = 1, \tag{8.44}$$

where $\{\phi_n\}$ is an orthonormal basis of $L^2(D) := H$. Then the unique solution to (8.36) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}(f, \phi_n)\phi_n(x). \tag{8.45}$$

If $q(x)$ is such that $\lambda_1 = 0$, $\lambda_2 \gg 1$, and $\lambda_2 \leq \lambda_3 \leq \dots$, then, as $t \to \infty$, the series (8.45) is well approximated by its first term

$$u(x,t) = (f, \phi_1)\phi_1 + O(e^{-10t}), \quad t \to \infty. \tag{8.46}$$

If $\lambda_1 > 0$ is very small, then the main term of the solution is

$$u(x,t) = (f, \phi_1)\phi_1 e^{-\lambda_1 t} + O(e^{-10t})$$

as $t \to \infty$. The term $e^{-\lambda_1 t} \sim 1$ if $t << \frac{1}{\lambda_1}$.

Thus, our problem is solved if $q(x)$ has the following property:

$$|\phi_1(x)| \text{decays as } \rho \text{ grows}, \quad \rho = (x_2^2 + x_3^2)^{1/2}. \tag{8.47}$$

Since the eigenfunction is normalized, $||\phi_1|| = 1$, this function will not tend to zero in a neighborhood of the line $\rho = 0$, so information can be transformed by the heat signals along the line $\rho = 0$, that is, along $s-$axis. Here we use the cylindrical coordinates:

$$x = (x_1, x_2, x_3) = (s, \rho, \theta), \quad s = x_1, \quad \rho = (x_2^2 + x_3^2)^{1/2}.$$

In Sect. 8.3 the domain D_0 is a disc and the potential $q(x)$ does not depend on θ.

The technical part of solving our problem consists of the construction of $q(x) = c_S N(x)h(x)$ such that

$$\lambda_1 = 0, \quad \lambda_2 \gg 1; \quad |\phi_1(x)| \text{ decays as } \rho \text{ grows.} \tag{8.48}$$

Since the function $N(x) \geq 0$ and $h(x), \mathrm{Re}h \geq 0$, are at our disposal, any desirable $q, \mathrm{Re}\, q \geq 0$, can be obtained by embedding many small impedance particles in a given domain D. In Sect. 8.3, a potential q with the desired properties is constructed. This construction allows one to transform information along a straight line using heat signals.

8.3 Construction of $q(x)$

Let

$$q(x) = p(\rho) + Q(s),$$

where $s := x_1$, $\rho := (x_2^2 + x_3^2)^{1/2}$. Then the solution to problem (8.44) is $u = v(\rho)w(s)$, where

$$-v_m'' - \rho^{-1}v_m' + p(\rho)v_m = \mu_m v_m, \quad 0 \leq \rho \leq R,$$
$$|v_m(0)| < \infty, \quad v_m(R) = 0, \tag{8.49}$$

and

$$-w_l'' + Q(s)w_l = \nu_l w_l, \quad 0 \leq s \leq L,$$
$$w_l(0) = 0, \quad w_l(L) = 0. \tag{8.50}$$

One has

$$\lambda_n = \mu_m + \nu_l, \quad n = n(m, l). \tag{8.51}$$

Our task is to find a potential $Q(s)$ such that $\nu_1 = 0$, $\nu_2 \gg 1$ and a potential $p(\rho)$ such that $\mu_1 = 0, \mu_2 \gg 1$ and $|v_m(\rho)|$ decays as ρ grows.

It is known how to construct $q(s)$ with the desired properties: the Gel'fand-Levitan method allows one to do this, see [7]. Let us recall this construction. One has $\nu_{l0} = l^2$, where we set $L = \pi$ and denote by ν_{l0} the eigenvalues of the problem (8.50) with $Q(s) = 0$. Let the eigenvalues of the operator (8.50) with $Q \neq 0$ be $\nu_1 = 0, \nu_2 = 11, \nu_3 = 14, \nu_l = \nu_{l0}$ for $l \geq 4$.

The kernel $L(x, y)$ in the Gel'fand-Levitan theory is defined as follows:

$$L(x, y) = \int_{-\infty}^{\infty} \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \frac{\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}} d(\varrho(\lambda) - \varrho_0(\lambda)),$$

where $\varrho(\lambda)$ is the spectral function of the operator (8.50) with the potential $Q = Q(s)$, and $\varrho_0(\lambda)$ is the spectral function of the operator (8.50) with the potential $Q = 0$ and the same boundary conditions as for the operator with $Q \neq 0$.

Due to our choice of ν_l and the normalizing constants α_j, namely: $\alpha_j = \frac{\pi}{2}$ for $j \geq 2$ and $\alpha_1 = \frac{\pi^3}{3}$, the kernel $L(x, y)$ is given explicitly by the formula:

$$L(x, y) = \frac{3xy}{\pi^3} + \frac{2}{\pi}\Big(\frac{\sin(\sqrt{\nu_2}x)}{\sqrt{\nu_2}} \frac{\sin(\sqrt{\nu_2}y)}{\sqrt{\nu_2}} + \frac{\sin(\sqrt{\nu_3}x)}{\sqrt{\nu_3}} \frac{\sin(\sqrt{\nu_3}y)}{\sqrt{\nu_3}}\Big) - \\ - \frac{2}{\pi}\Big(\sin x \sin y + \sin(2x)\sin(2y) + \sin(3x)\sin(3y)\Big), \quad (8.52)$$

where $\nu_1 = 0$, $\nu_2 = 11$ and $\nu_3 = 14$. This is a finite rank kernel. The term xy is the value of the function $\frac{\sin \nu x}{\nu}\frac{\sin \nu y}{\nu}$ at $\nu = 0$, and the corresponding normalizing constant is $\frac{\pi^3}{3} = ||x||^2 = \int_0^\pi x^2 dx$.

Solve the Gel'fand-Levitan equation:

$$K(s, \tau) + \int_0^s K(s, s')L(s', \tau)ds' = -L(s, \tau), \quad 0 \leq \tau \leq s, \quad (8.53)$$

which is uniquely solvable (see [7]). Since Eq. (8.53) has finite-rank kernel it can be solved analytically being equivalent to a linear algebraic system.

If the function $K(s, \tau)$ is found, then the potential $Q(s)$ is computed by the formula [2, 7]:

$$Q(s) = 2\frac{dK(s, s)}{ds}, \quad (8.54)$$

and this $Q(s)$ has the required properties: $\nu_1 = 0, \nu_2 \gg 1, \nu_l \leq \nu_{l+1}$.

Consider now the operator (8.49) for $v(\rho)$. Our problem is to calculate $p(\rho)$ which has the required properties:

$$\mu_1 = 0, \quad \mu_2 \gg 1, \quad \mu_m \leq \mu_{m+1},$$

and $|\phi_m(\rho)|$ decays as ρ grows.

We reduce this problem to the previous one that was solved above. To do this, set $v = \frac{\psi}{\sqrt{\rho}}$. Then equation

$$-v'' - \frac{1}{\rho}v' + p(\rho)v = \mu v,$$

is transformed to the equation

$$-\psi'' - \frac{1}{4\rho^2}\psi + p(\rho)\psi = \mu\psi. \tag{8.55}$$

Let

$$p(\rho) = \frac{1}{4\rho^2} + Q(\rho), \tag{8.56}$$

where $Q(\rho)$ is constructed above. Then Eq. (8.55) becomes

$$-\psi'' + Q(\rho)\psi = \mu\psi, \tag{8.57}$$

and the boundary conditions are:

$$\psi(R) = 0, \quad \psi(0) = 0. \tag{8.58}$$

The problem (8.57)–(8.58) has the desired eigenvalues $\mu_1 = 0, \mu_2 \gg 1, \mu_m \leq \mu_{m+1}$.

The eigenfunction

$$\phi_1(x) = v_1(\rho)w_1(s),$$

where $v_1(\rho) = \frac{\psi_1(\rho)}{\sqrt{\rho}}$, decays as ρ grows, and the eigenvalues λ_n can be calculated by the formula:

$$\lambda_n = \mu_m + \nu_l, \quad m, l \geq 1, \quad n = n(m, l).$$

Since $\mu_1 = \nu_1 = 0$ one has $\lambda_1 = 0$. Since $\nu_2 = 11$ and $\mu_2 = 11$, one has $\lambda_2 = 11 \gg 1$.

Thus, the desired potential is constructed:

$$q(x) = Q(s) + (\frac{1}{4\rho^2} + Q(\rho)),$$

where $Q(s)$ is given by formula (8.54).

This concludes the description of our procedure for the construction of q.

Remark 1 It is known (see, for example, [2]) that the normalizing constants

$$\alpha_j := \int_0^\pi \varphi_j^2(s)ds$$

and the eigenvalues λ_j, defined by the differential equation

$$-\frac{d^2\varphi_j}{ds^2} + Q(s)\varphi_j = \lambda_j\varphi_j,$$

the boundary conditions

$$\varphi_j'(0) = 0, \quad \varphi_j'(\pi) = 0,$$

and the normalizing condition $\varphi_j(0) = 1$, have the following asymptotic:

$$\alpha_j = \frac{\pi}{2} + O(\frac{1}{j^2}) \quad \text{as}\, j \to \infty,$$

and

$$\sqrt{\lambda_j} = j + O(\frac{1}{j}) \quad \text{as}\, j \to \infty.$$

The differential equation

$$-\Psi_j'' + Q(s)\Psi_j = \nu_j\Psi_j,$$

the boundary condition

$$\Psi_j(0) = 0, \quad \Psi_j(\pi) = 0,$$

and the normalizing condition $\Psi_j'(0) = 1$ imply

$$\sqrt{\lambda_j} = j + O(\frac{1}{j}) \quad \text{as} \quad j \to \infty,$$

$$\Psi_j(s) \sim \frac{\sin(js)}{j} \quad \text{as} \quad j \to \infty.$$

The main term of the normalized eigenfunction is:

$$\frac{\Psi_j}{||\Psi_j||} \sim \sqrt{2/\pi}\sin(js) \quad \text{as} \quad j \to \infty,$$

and the main term of the normalizing constant is:

$$\alpha_j \sim \frac{\pi}{2j^2} \quad \text{as} \quad j \to \infty.$$

8.4 Auxiliary Results

Lemma 1 *If one of the limits* $\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds$ *or* $\lim_{\lambda\to 0}\lambda\mathcal{U}(\lambda)$ *exists, then the other also exists and they are equal to each other:*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds = \lim_{\lambda\to 0}\lambda\mathcal{U}(\lambda),$$

where

$$\mathcal{U}(\lambda) := \int_0^\infty e^{-\lambda t}u(t)dt := \bar{u}(\lambda).$$

Proof Denote

$$\frac{1}{t}\int_0^t u(t)dt := v(t), \quad \bar{u}(\sigma) := \int_0^\infty e^{-\sigma t}u(t)dt.$$

Then

$$\bar{v}(\lambda) = \int_\lambda^\infty \frac{\bar{u}(\sigma)}{\sigma}d\sigma$$

by the properties of the Laplace transform.

Assume that the limit $v(\infty) := v_\infty$ exists:

$$\lim_{t\to\infty} v(t) = v_\infty. \tag{8.59}$$

Then,

$$v_\infty = \lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}v(t)dt = \lim_{\lambda\to 0}\lambda\bar{v}(\lambda).$$

Indeed $\lambda\int_0^\infty e^{-\lambda t}dt = 1$, so

$$\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}(v(t)-v_\infty)dt = 0,$$

and (8.59) is verified.

One has

$$\lim_{\lambda\to 0}\lambda\bar{v}(\lambda) = \lim_{\lambda\to 0}\int_\lambda^\infty \frac{\lambda}{\sigma}\bar{u}(\sigma)d\sigma = \lim_{\lambda\to 0}\lambda\bar{u}(\lambda), \tag{8.60}$$

as follows from a simple calculation:

$$\lim_{\lambda\to 0}\int_\lambda^\infty \frac{\lambda}{\sigma}\bar{u}(\sigma)d\sigma = \lim_{\lambda\to 0}\int_\lambda^\infty \frac{\lambda}{\sigma^2}\sigma\bar{u}(\sigma)d\sigma = \lim_{\sigma\to 0}\sigma\bar{u}(\sigma), \tag{8.61}$$

where we have used the relation $\int_\lambda^\infty \frac{\lambda}{\sigma^2}d\sigma = 1$.

Alternatively, let $\sigma^{-1} = \gamma$. Then,

$$\int_\lambda^\infty \frac{\lambda}{\sigma^2}\sigma\bar{u}(\sigma)d\sigma = \frac{1}{1/\lambda}\int_0^{1/\lambda}\frac{1}{\gamma}\bar{u}(\frac{1}{\gamma})d\gamma = \frac{1}{\omega}\int_0^\omega \frac{1}{\gamma}\bar{u}(\frac{1}{\gamma})d\gamma. \tag{8.62}$$

If $\lambda\to 0$, then $\omega = \lambda^{-1}\to\infty$, and if

$$\psi := \gamma^{-1}\bar{u}(\gamma^{-1}),$$

then

$$\lim_{\omega\to\infty}\frac{1}{\omega}\int_0^\omega \psi d\gamma = \psi(\infty) = \lim_{\gamma\to 0}\gamma^{-1}\bar{u}(\gamma^{-1}) = \lim_{\sigma\to\infty}\sigma\bar{u}(\sigma). \tag{8.63}$$

Lemma 1 is proved. □

Lemma 2 *Equation (8.21) holds.*

Proof As $a\to 0$, one has

$$\frac{\partial}{\partial N_s}\frac{e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|} = \frac{\partial}{\partial N_s}\frac{1}{4\pi|s-s'|} + \frac{\partial}{\partial N_s}\frac{e^{-\sqrt{\lambda}|s-s'|}-1}{4\pi|s-s'|}. \tag{8.64}$$

It is known (see [8]) that

$$\int_{S_m}ds\int_{S_m}\frac{\partial}{\partial N_s}\frac{1}{4\pi|s-s'|}\sigma_m(s')ds' = -\frac{1}{2}\int_{S_m}\sigma_m(s')ds' = -\frac{1}{2}Q_m. \tag{8.65}$$

On the other hand, as $a\to 0$, one has

$$\left|\int_{S_m}ds\int_{S_m}\frac{e^{-\sqrt{\lambda}|s-s'|}-1}{4\pi|s-s'|}\sigma_m(s')ds'\right| \le |Q_m|\int_{S_m}ds\frac{1-e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|} = o(Q_m). \tag{8.66}$$

The relations (8.65) and (8.66) justify (8.21).

Lemma 2 is proved. □

Lemma 3 *If assumption (8.18) holds, then inequality (8.17) holds.*

Proof One has

$$\mathcal{J}_{1,m} := |g(x, x_m)Q| = \frac{|Q_m|e^{-\sqrt{\lambda}|x-x_m|}}{4\pi|x-x_m|}, \tag{8.67}$$

and

$$\mathcal{J}_{2,m} \le \frac{e^{-\sqrt{\lambda}|x-x_m|}}{4\pi|x-x_m|} \max\left(\sqrt{\lambda}a, \frac{a}{|x-x_m|}\right) \int_{\mathcal{S}_m} |\sigma_m(s')|ds' \tag{8.68}$$

where $|x - x_m| \ge d$, and $d > 0$ is the smallest distance between two neighboring particles. One may consider only those values of λ for which $\lambda^{1/4}a < \frac{a}{d}$, because for the large values of λ, such that $\lambda^{1/4} \ge \frac{1}{d}$ the value of $e^{-\sqrt{\lambda}|x-x_m|}$ is negligibly small. The average temperature depends on the behavior of $\mathcal{U}$ for small λ, see Lemma 1.

One has $|Q_m| = \int_{\mathcal{S}_m} |\sigma_m(s')|ds' > 0$ because σ_m keeps sign on $\mathcal{S}_m$, as follows from Eq. (8.24) as $a \to 0$.

It follows from (8.67)–(8.68) that

$$\left|\frac{\mathcal{J}_{2,m}}{\mathcal{J}_{1,m}}\right| \le O\left(\left|\frac{a}{x-x_m}\right|\right) \le O\left(\frac{a}{d}\right) << 1. \tag{8.69}$$

From (8.69) by the arguments similar to the given in [9] one obtains (8.17).

Lemma 3 is proved. □

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