Chapter 2
Inductive Properties of Electric Circuits

Characterizing the inductive properties of the power and ground interconnect is essential in determining the impedance characteristics of a power distribution system. Several of the following chapters are dedicated to the inductive properties of on-chip power distribution networks. The objective of this chapter is to introduce the concepts used in these chapters to describe the inductive characteristics of complex interconnect structures.

The magnetic properties of circuits are typically described using circuits with inductive coils. The inductive characteristics of such circuits are dominated by the self- and mutual inductances of these coils. The inductance of a coil is well described by the classical definition of inductance based on the magnetic flux through a current loop. The situation is more complex in circuits with no coils where no part of the circuit is inductively dominant and the circuit elements are strongly inductively coupled. The magnetic properties in this case are determined by the physical structure of the entire circuit, resulting in complex inductive behavior. The loop inductance formulation is inconvenient to represent the inductive characteristics of these circuits. The objective of this chapter is to describe various ways to represent a circuit inductance, highlighting specific assumptions. Intuitive interpretations are offered to develop a deeper understanding of the limitations and interrelations of these approaches. The variation of inductance with frequency and the relationship between the absolute inductance and the inductive behavior are also discussed in this chapter.

These topics are discussed in the following order. Several formulations of the circuit inductive characteristics as well as advantages and limitations of these formulations are described in Sect. 2.1. Mechanisms underlying the variation of inductance with frequency are examined in Sect. 2.2. The relationship between the absolute inductance and the inductive behavior of circuits is discussed in Sect. 2.3. The inductive properties of on-chip interconnect structures are analyzed in Sect. 2.4. The chapter is summarized in Sect. 2.5.

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2.1 Definitions of Inductance

There are several ways to represent the magnetic characteristics of a circuit. Understanding the advantages and limitations of these approaches presents the opportunity to choose the approach most suitable for a particular task. Several representations of the inductive properties of a circuit are presented in this section. The field energy formulation of inductive characteristics is described in Sect. 2.1.1. The loop flux definition of inductance is discussed in Sect. 2.1.2. The concept of a partial inductance is described in Sect. 2.1.3. The net inductance formulation is described in Sect. 2.1.4.

2.1.1 Field Energy Definition

Inductance represents the capability of a circuit to store energy in the form of a magnetic field. Specifically, the inductance relates the electrical current to the magnetic flux and magnetic field energy. The magnetic field is interrelated with the electric field and current, as determined by Maxwell's equations and constitutive relations,[1]

$$\nabla \boldsymbol{D} = \rho, \tag{2.1}$$

$$\nabla \boldsymbol{B} = 0, \tag{2.2}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{2.3}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.4}$$

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \tag{2.5}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2.6}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \tag{2.7}$$

assuming a linear media. The domain of circuit analysis is typically confined to those operational conditions where the electromagnetic radiation phenomena are negligible. The direct effect of the displacement current $\frac{\partial \boldsymbol{D}}{\partial t}$ on the magnetic field, as expressed by (2.3), can be neglected under these conditions (although the displacement current can be essential to determine the current density $\boldsymbol{J}$). The magnetic field is therefore determined only by the circuit currents. The local current density determines the local behavior of the magnetic field, as expressed by Ampere's law in the differential form,

[1] Vector quantities are denoted with bold italics, such as $\boldsymbol{H}$.

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}. \tag{2.8}$$

Equivalently, the elemental contribution to the magnetic field $d\boldsymbol{H}$ is expressed in terms of an elemental current $d\boldsymbol{J}$, according to the Biot-Savart law,

$$d\boldsymbol{H} = \frac{d\boldsymbol{J} \times \boldsymbol{r}}{4\pi r^3}, \tag{2.9}$$

where $\boldsymbol{r}$ is the distance vector from the point of interest to the current element $d\boldsymbol{J}$ and $r = |\boldsymbol{r}|$.

It can be demonstrated that the magnetic energy in a linear media can be expressed as [43]

$$W_{\mathrm{m}} = \frac{1}{2} \int \boldsymbol{J} \cdot \boldsymbol{A} \, d\boldsymbol{r}, \tag{2.10}$$

where $\boldsymbol{A}$ is the magnetic vector potential of the system, determined as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \, d\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{2.11}$$

Substituting (2.11) into (2.10) yields the expression of the magnetic energy in terms of the current distribution in a system,

$$W_{\mathrm{m}} = \frac{\mu}{8\pi} \iint \frac{\boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d\boldsymbol{r} \, d\boldsymbol{r}'. \tag{2.12}$$

If the system is divided into several parts, each contained in a volume V_i, the magnetic energy expression (2.12) can be rewritten as

$$W_{\mathrm{m}} = \frac{\mu}{8\pi} \sum_i \sum_j \int_{V_i} \int_{V_j} \frac{\boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d\boldsymbol{r} \, d\boldsymbol{r}'. \tag{2.13}$$

Assuming that the relative distribution of the current in each volume V_i is independent of the current magnitude, the current density distribution $\boldsymbol{J}$ can be expressed in terms of the overall current magnitude I and current distribution function $\boldsymbol{u}(\boldsymbol{r})$, so that $\boldsymbol{J}(\boldsymbol{r}) = I\boldsymbol{u}(\boldsymbol{r})$. The magnetic field energy can be expressed in terms of the overall current magnitudes I_i,

$$W_{\mathrm{m}} = \frac{1}{2} \sum_i \sum_j L_{ij} I_i I_j, \tag{2.14}$$

where

$$L_{ij} \equiv \frac{\mu}{4\pi} \int_{V_i} \int_{V_j} \frac{\boldsymbol{u}(\boldsymbol{r}) \cdot \boldsymbol{u}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d\boldsymbol{r} \, d\boldsymbol{r}' \tag{2.15}$$

is a mutual inductance between the system parts i and j. In a matrix formulation, the magnetic energy of a system consisting of N parts can be expressed as a positively defined binary form[2] $\mathbf{L}$ of a current vector $\boldsymbol{I} = \{I_1, \ldots, I_N\}$,

$$W_{\mathrm{m}} = \frac{1}{2}\boldsymbol{I}^T \mathbf{L} \boldsymbol{I} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij} I_i I_j. \tag{2.16}$$

Each diagonal element L_{ii} of the binary form $\mathbf{L}$ is a self-inductance of the corresponding current I_i and each non-diagonal element L_{ij} is a mutual inductance between currents I_i and I_j. Note that according to the definition of (2.15), the inductance matrix is symmetric, i.e., $L_{ij} = L_{ji}$.

While the field approach is general and has no limitations, determining the circuit inductance through this approach is a laborious process, requiring numerical field analysis except for the simplest structures. The goal of circuit analysis is to provide an efficient yet accurate description of the system in those cases where the detail and accuracy of a full field analysis are unnecessary. Resorting to a field analysis to determine specific circuit characteristics greatly diminishes the efficiency of the circuit analysis formulation.

2.1.2 Magnetic Flux Definition

The concept of inductance is commonly described as a constant L relating a magnetic flux Φ through a circuit loop to a current I' in another loop,

$$\Phi = LI'. \tag{2.17}$$

In the special case where the two circuit loops are the same, the coefficient is referred to as a loop self-inductance; otherwise, the coefficient is referred to as a mutual loop inductance.

For example, consider two isolated complete current loops ℓ and ℓ', as shown in Fig. 2.1. The mutual inductance M between these two loops is a coefficient relating a magnetic flux Φ through a loop ℓ due to a current I' in loop ℓ',

$$\Phi = \iint_S \boldsymbol{B}' \cdot \boldsymbol{n}\, ds, \tag{2.18}$$

where S is a smooth surface bounded by the loop ℓ, $\boldsymbol{B}'$ is the magnetic flux created by the current in the loop ℓ', and $\boldsymbol{n}$ is a unit vector normal to the surface element ds. Substituting $\boldsymbol{B'} = \nabla \times \boldsymbol{A'}$ and using Stokes's theorem, the loop flux is expressed as

[2] Matrix entities are denoted with bold roman symbols, such as $\mathbf{L}$.

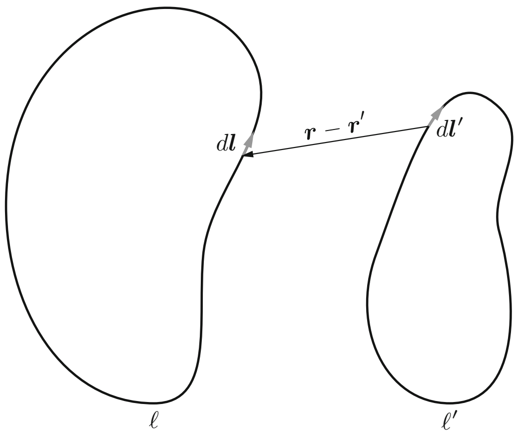

Fig. 2.1 Two complete current loops. The relative position of two differential loop elements $\boldsymbol{dl}$ and $\boldsymbol{dl'}$ is determined by the vector $\boldsymbol{r}-\boldsymbol{r'}$

$$\Phi = \iint_S (\nabla \times \boldsymbol{A'}) \cdot \boldsymbol{n}\, ds = \oint_\ell \boldsymbol{A'}\, \boldsymbol{dl}, \tag{2.19}$$

where $\boldsymbol{A}'$ is the vector potential created by the current $\boldsymbol{I}'$ in the loop ℓ'. The magnetic vector potential of the loop ℓ' $\boldsymbol{A}'$ is

$$\boldsymbol{A}'(\boldsymbol{r}) = \frac{\mu}{4\pi} \int_V \frac{\boldsymbol{J}'(\boldsymbol{r}')\, d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = I' \frac{\mu}{4\pi} \oint_{\ell'} \frac{\boldsymbol{dl'}}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{2.20}$$

where $|\boldsymbol{r}-\boldsymbol{r}'|$ is the distance between the loop element $\boldsymbol{dl}'$ and the point of interest $\boldsymbol{r}$. Substituting (2.20) into (2.19) yields

$$\Phi = I' \frac{\mu}{4\pi} \oint_\ell \oint_{\ell'} \frac{\boldsymbol{dl}\, \boldsymbol{dl'}}{|\boldsymbol{r}-\boldsymbol{r}'|} = MI', \tag{2.21}$$

where

$$M \equiv \frac{\mu}{4\pi} \oint_\ell \oint_{\ell'} \frac{\boldsymbol{dl}\, \boldsymbol{dl'}}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{2.22}$$

is a mutual inductance between the loops ℓ and ℓ'. As follows from the derivation, the integration in (2.20), (2.21), and (2.22) is performed in the direction of the current flow. The mutual inductance (2.22) and associated magnetic flux (2.21) can therefore be either positive or negative, depending on the relative direction of the current flow in the two loops.

Note that the finite cross-sectional dimensions of the loop conductors are neglected in the transition between the general volume integration to a more constrained but simpler contour integration in (2.20). An entire loop current is therefore confined to an infinitely thin filament.

The thin filament approximation of a mutual inductance is acceptable where the cross-sectional dimensions of the conductors are much smaller than the distance $|\boldsymbol{r} - \boldsymbol{r}'|$ between any two points on loop ℓ and loop ℓ'. This approximation becomes increasingly inaccurate as the two loops are placed closer together. More importantly, the thin filament approach cannot be used to determine a self-inductance by assuming ℓ to be identical to ℓ', as the integral (2.22) diverges at the points where $\boldsymbol{r} = \boldsymbol{r}'$.

To account for the finite cross-sectional dimensions of the conductors, both (2.19) and (2.20) are amended to include an explicit integration over the conductor cross-sectional area a,

$$\Phi = \frac{1}{I} \oint_{\ell} \int_{a} \boldsymbol{A}' \, J d\boldsymbol{l} \, da, \tag{2.23}$$

and

$$\boldsymbol{A}' = \frac{\mu}{4\pi} \oint_{\ell'} \int_{a'} \frac{J' \, d\boldsymbol{l'} \, da'}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{2.24}$$

where a and a' are the cross sections of the elemental loop segments $d\boldsymbol{l}$ and $d\boldsymbol{l'}$, da and da' are the differential elements of the respective cross sections, $|\boldsymbol{r} - \boldsymbol{r}'|$ is the distance between da and da', and J is a current density distribution over the wire cross section a, $d\boldsymbol{J} = J \, d\boldsymbol{l} \, da$, and $I = \int_a J \, da$. These expressions are more general than (2.19) and (2.20); the only constraint on the current flow imposed by formulations (2.23) and (2.24) is that the current flow has the same direction across the cross-sectional areas a and a'. This condition is satisfied in relatively thin conductors without sharp turns. Formulas (2.23) and (2.24) can be significantly simplified assuming a uniform current distribution (i.e., $J = \text{const}$ and $I = aJ$) and a constant cross-sectional area a,

$$\Phi = \frac{1}{a} \oint_{\ell} \int_{a} \boldsymbol{A}' \, d\boldsymbol{l} \, da, \tag{2.25}$$

and

$$\boldsymbol{A}' = \frac{\mu}{4\pi} \frac{I'}{a'} \oint_{\ell'} \int_{a'} \frac{d\boldsymbol{l'} \, da'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{2.26}$$

The magnetic flux through loop ℓ is transformed into

$$\Phi = \frac{\mu}{4\pi} \frac{I'}{a\,a'} \oint_{\ell} \oint_{\ell'} \int_{a} \int_{a'} \frac{da\,da'\,\mathbf{dl}\,\mathbf{dl'}}{|\mathbf{r}-\mathbf{r}'|} = MI'. \tag{2.27}$$

The mutual loop inductance is therefore defined as

$$M_{\ell\ell'} \equiv \frac{\mu}{4\pi} \frac{1}{a\,a'} \oint_{\ell} \oint_{\ell'} \int_{a} \int_{a'} \frac{da\,da'\,\mathbf{dl}\,\mathbf{dl'}}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.28}$$

The loop self-inductance L_ℓ is a special case of the mutual loop inductance where the loop ℓ is the same as loop ℓ',

$$L_\ell \equiv M_{\ell\ell} = \frac{\mu}{4\pi} \frac{1}{a^2} \oint_{\ell} \oint_{\ell} \int_{a} \int_{a} \frac{da\,da'\,\mathbf{dl}\,\mathbf{dl'}}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.29}$$

While straightforward and intuitive, the definition of the loop inductance as expressed by (2.17) cannot be applied to most practical circuits. Only the simplest circuits consist of a single current loop. In practical circuits with branch points, the current is not constant along the circumference of the conductor loops, as shown in Fig. 2.2. This difficulty can be circumvented by employing Kirchhoff's voltage law and including an inductive voltage drop within each loop equation. For example, two independent current loops carrying circular currents I_A and I_B can be identified in the circuit shown in Fig. 2.2. The inductive voltage drops V_A and V_B in loops A and B are

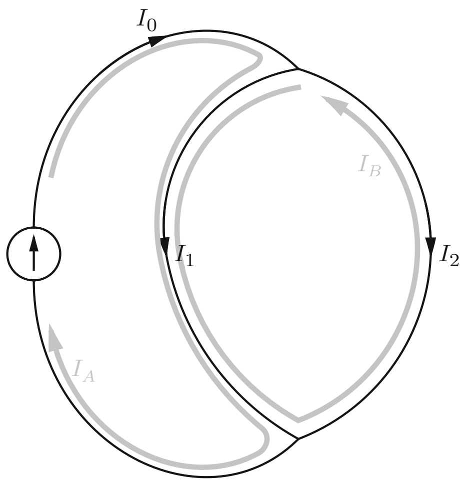

Fig. 2.2 A circuit with branch points. The current in each loop is not uniform along the circumference of the loop

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} L_{AA} & L_{AB} \\ L_{AB} & L_{BB} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix}. \tag{2.30}$$

The magnetic energy of the system is, analogous to (2.16),

$$W_{\mathrm{m}} = \frac{1}{2}\boldsymbol{I}^T\mathbf{L}\boldsymbol{I} = \frac{1}{2}\begin{bmatrix} I_A & I_B \end{bmatrix} \begin{bmatrix} L_{AA} & L_{AB} \\ L_{AB} & L_{BB} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix}. \tag{2.31}$$

Note that in a circuit with branch points, two current loops can share common parts, as illustrated in Fig. 2.2. The inductance between these two loops is therefore a hybrid between the self- and mutual loop inductance, as defined by (2.28) and (2.29).

The flux formulation of the inductive characteristics, as expressed by (2.29) and (2.31), is a special case of the field formulation, as expressed by (2.15) and (2.16). The magnetic field expressions (2.16) and (2.31) are the same, while the definition of the loop inductance as expressed by (2.29) is obtained from (2.15) by assuming that the current flows in well formed loops; the thin filament definition of the mutual inductance (2.22) is the result of further simplification of (2.15). The magnetic energy and field flux derivations of the inductance are equivalent; both (2.15) and (2.29) can be obtained from either the energy formulation expressed by (2.31) or the flux formulation expressed by (2.22).

The loop inductance approach provides a more convenient description of the magnetic properties of the circuit with little loss of accuracy and generality, as compared to the field formulation as expressed by (2.16). Nevertheless, significant disadvantages remain. In the magnetic flux formulation of the circuit inductance, the basic inductive element is a closed loop. This aspect presents certain difficulties for a traditional circuit analysis approach. In circuit analysis, the impedance characteristics are described in terms of the circuit elements connecting two circuit nodes. Circuit analysis tools also use a circuit representation based on two-terminal elements. Few circuit elements are manufactured in a loop form. In a strict sense, a physical inductor is also a two terminal element. The current flowing through a coil does not form a complete loop, therefore, the definition of the loop inductance does not apply. The loop formulation does not provide a direct link between the impedance characteristics of the circuit and the impedance of the comprising two terminal circuit elements.

It is therefore of practical interest to examine how the inductive characteristics can be described by a network of two terminal elements with self- and mutual impedances, without resorting to a multiple loop representation. This topic is the subject of the next section.

2.1.3 Partial Inductance

The loop inductance, as defined by (2.28), can be deconstructed into more basic elements if the two loops are broken into segments, as shown in Fig. 2.3. The loop ℓ is broken into N segments $S_1, \ldots, S_N$ and loop ℓ' is broken into N' segments $S'_1, \ldots, S'_{N'}$. The definition of the loop inductance (2.28) can be rewritten as

$$M_{\ell\ell'} = \sum_{i=1}^{N}\sum_{j=1}^{N'} \frac{\mu}{4\pi}\frac{1}{a_i\, a'_j}\oint_{S_i}\oint_{S'_j}\int_{a_i}\int_{a'_j}\frac{da_i\, da'_j\, \boldsymbol{dl}\, \boldsymbol{dl'}}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{i=1}^{N}\sum_{j=1}^{N'} L_{ij}, \tag{2.32}$$

where

$$L_{ij} \equiv \frac{\mu}{4\pi}\frac{1}{a_i\, a'_j}\oint_{S_i}\oint_{S'_j}\int_{a_i}\int_{a'_j}\frac{da_i\, da'_j\, \boldsymbol{dl}\, \boldsymbol{dl'}}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{2.33}$$

The integration along segments S_i and S'_j in (2.32) and (2.33) is performed in the direction of the current flow.

Equation (2.33) defines the mutual partial inductance between two arbitrary segments S_i and S'_j. Similar to the loop inductance, the mutual partial inductance can be either positive or negative, depending on the direction of the current flow in the two segments. In the special case where S_i is identical to S'_j, (2.33) defines the partial self-inductance of S_i. The partial self-inductance is always positive.

The partial inductance formulation, as defined by (2.33), is more suitable for circuit analysis as the basic inductive element is a two terminal segment of interconnect. Any circuit can be decomposed into a set of interconnected two terminal elements. For example, the circuit shown in Fig. 2.2 can be decomposed

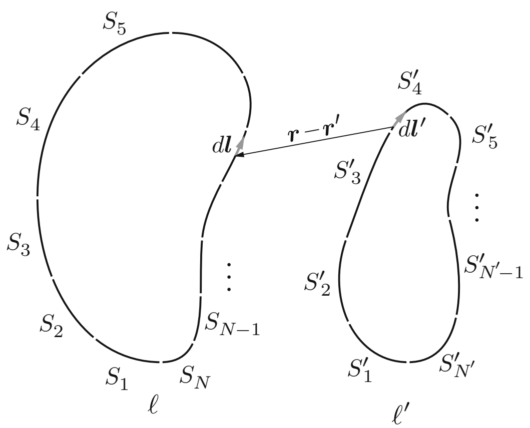

Fig. 2.3 Two complete current loops broken into segments

into three linear segments instead of two loops as in the case of a loop analysis. The magnetic properties of the circuit are described by a partial inductance matrix $\mathbf{L} = \{L_{ij}\}$. Assigning to each element S_i a corresponding current I_i, the vector of magnetic electromotive forces $\boldsymbol{V}$ across each segment is

$$\boldsymbol{V} = \mathbf{L}\frac{d\boldsymbol{I}}{dt}. \tag{2.34}$$

The magnetic energy of the circuit in terms of the partial inductance is determined, analogously to the loop inductance formulation (2.31), as

$$W_\mathrm{m} = \frac{1}{2}\boldsymbol{I}^T\mathbf{L}\boldsymbol{I} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}I_iI_j. \tag{2.35}$$

The partial inductance has another practical advantage. If the self- and mutual partial inductance of a number of basic segment shapes is determined as a function of the segment dimensions and orientations, the partial inductance matrix of any circuit composed of these basic shapes can be readily constructed according to the segment connectivity, permitting the efficient analysis of the magnetic properties of the circuit. In this regard, the partial inductance approach is more flexible than the loop inductance approach, as loops exhibit a greater variety of shapes and are difficult to precharacterize in an efficient manner.

For the purposes of circuit characterization, it is convenient to separate the sign and the absolute magnitude of the inductance. During precharacterization, the absolute magnitude of the mutual partial inductance L_{ij}^{abs} between basic conductor shapes (such as straight segments) is determined. During the process of analyzing a specific circuit structure, the absolute magnitude is multiplied by a sign function s_{ij}, resulting in the partial inductance as defined by (2.33), $L_{ij} = s_{ij}L_{ij}^{\mathrm{abs}}$. The sign function equals either 1 or −1, depending upon the sign of the scalar product of the segment currents: $s_{ij} = \mathrm{sign}\,(\boldsymbol{I}_i \cdot \boldsymbol{I}_j')$.

The case of a straight wire is of particular practical importance. A conductor of any shape can be approximated by a number of short straight segments. The partial self-inductance of a straight round wire is [44]

$$L_{\mathrm{line}} = \frac{\mu l}{2\pi}\left(\ln\frac{2l}{r} - \frac{3}{4}\right), \tag{2.36}$$

where l is the length of the wire and r is the radius of the wire cross section, as shown in Fig. 2.4. The precise analytic expressions for the partial inductance are generally not available for straight conductors with a radially asymmetric cross section. The partial inductance of a straight line with a square cross section can be evaluated with good accuracy using approximate analytic expressions augmented with tables of correction coefficients [44], or expressions suitable for efficient numerical evaluation [45].

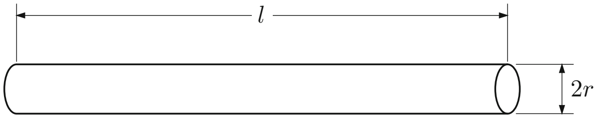

Fig. 2.4 A straight round wire

The partial self-inductance, as expressed by (2.33), depends only on the shape of the conductor segment. It is therefore possible to assign a certain partial self-inductance to an individual segment of the conductor. It should be stressed, however, that the partial self-inductance of the comprising conductors by itself provides no information on the inductive properties of the circuit. For example, a loop of wire can have a loop inductance that is much greater than the sum of the partial self-inductance of the comprising segments (where the wire is coiled) or much smaller than the sum of the comprising partial self-inductances (where the wire forms a narrow long loop). The inductive properties of a circuit are described by *all* partial inductances in the circuit, necessarily including all mutual partial inductances between all pairs of elements, as expressed in (2.32) for the specific case of a current loop.

Unlike the loop inductance, the partial inductance cannot be measured experimentally. The partial inductance is, essentially, a convenient mathematical construct used to describe the inductive properties of a circuit. This point is further corroborated by the fact that the partial inductance is not uniquely defined. An electromagnetic field is described by an infinite number of vector potentials. If a specific field is described by a vector potential $\boldsymbol{A}$, any vector potential $\boldsymbol{A}'$ differing from $\boldsymbol{A}$ by a gradient of an arbitrary scalar function Ψ, i.e., $\boldsymbol{A}' = \boldsymbol{A} + \nabla\Psi$, also describes the field.[3] The magnetic field is determined through the curl operation of the vector potential and is not affected by the $\nabla\Psi$ term, $\nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{A'}$ as $\nabla \times \nabla\Psi = 0$. The choice of a specific vector potential is inconsequential. The vector potential definition (2.11) is therefore not unique. The choice of a specific vector potential is also immaterial in determining the loop inductance as expressed by (2.28), as the integration of a gradient of any function over a closed contour yields a null value. The choice of the vector potential, however, affects the value of the partial inductance, where the integration is performed over a conductor segment. Equation (2.33) therefore defines only one of many possible partial inductance matrices. This ambiguity does not present a problem as long as all of the partial inductances in the circuit are consistently determined using the same vector potential. The contributions of the function gradient to the partial inductance cancel out, where the partial inductances are combined to describe the loop currents.

In the case of straight line segments, the partial inductance definition expressed by (2.33) has an intuitive interpretation. For a straight line segment, the partial

[3] This property of the electromagnetic field is referred to in electrodynamics as gauge invariance.

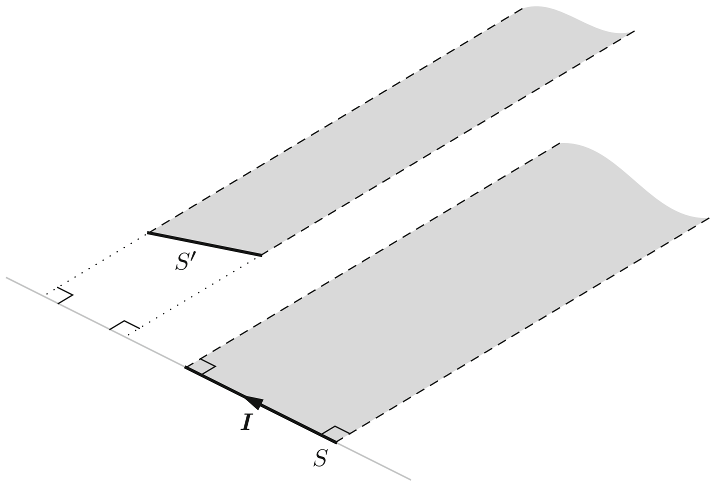

Fig. 2.5 Self- and mutual partial inductance of a straight segment of wire. The partial self-inductance of a segment S, as described by Rosa [46], is determined using the magnetic flux created by current $\boldsymbol{I}$ in segment S through an infinite contour formed by wire segment S (the *bold arrow*) and two rays perpendicular to the segment (the *dashed lines*). Similarly, the partial mutual inductance with another wire segment S' is determined using the flux created by current $\boldsymbol{I}$ through the contour formed by the segment S' and straight lines originating from the ends of the segment S' and perpendicular to segment S

self-inductance is a coefficient of proportionality between the segment current and the magnetic flux through the infinite loop formed by a line segment S and two rays perpendicular to the segment, as illustrated in Fig. 2.5.

This flux is henceforth referred to as a partial flux. This statement can be proved as follows. The flux through the aforementioned infinite loop is determined by integrating the vector potential $\boldsymbol{A}$ along the loop contour, according to (2.25). The magnetic vector potential $\boldsymbol{A}$ of a straight segment, as determined by (2.11), is parallel to the segment. The integration of the vector potential along the rays perpendicular to the segment is zero. The integration of the vector potential along the segment completing the loop at infinity is also zero as the vector potential decreases inversely proportionally with distance. Similarly, the mutual partial inductance between segments S and S' can be interpreted in terms of the magnetic flux through the infinite loop formed by segment S' and two rays perpendicular to the segment S, as illustrated in Fig. 2.5.

This interpretation of the partial inductance in terms of the partial flux is in fact the basis for the original introduction of the partial inductance by Rosa in 1908 in application to linear conductors [46]. Attempts to determine the inductance of a straight wire segment using the total magnetic flux were ultimately unsuccessful as the total flux of a segment is infinite. Rosa made an intuitive argument that only the partial magnetic flux, as illustrated in Fig. 2.5, should be associated with the self-inductance of the segment. The concept of partial inductance proved useful and was utilized in the inductance calculation formulæ and tables developed by Rosa and Cohen [47], Rosa and Grover [48], and Grover [44]. A rigorous theoretical treatment of the subject was first developed by Ruehli in [45], where a general definition of the partial inductance of an arbitrarily shaped conductor (2.33) is derived. Ruehli also coined the term "partial inductance."

Connections between the loop and partial inductance can also be established in terms of the magnetic flux. The magnetic flux through a specific loop is a sum of all of the partial fluxes of the comprising segments. The contribution of a magnetic field created by a specific loop segment to the loop flux is also the sum of all of the partial inductances of this segment with respect to all segments of the loop. This relationship is illustrated in Fig. 2.6.

2.1.4 Net Inductance

The inductance of a circuit without branch points (i.e., where the current flowing in all conductor segments is the same) can also be expressed in a form with no explicit mutual inductances. Consider a current loop consisting of N segments. As discussed in the previous section, the loop inductance L_{loop} can be described in terms of the partial inductances L_{ij} of the segments,

$$L_{\text{loop}} = \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij}. \tag{2.37}$$

This sum can be rearranged as

$$L_{\text{loop}} = \sum_{i=1}^{N} L_i^{\text{eff}}, \tag{2.38}$$

where

$$L_i^{\text{eff}} \equiv \sum_{i,j=1}^{N} L_{ij}. \tag{2.39}$$

The inductance L_i^{eff}, as defined by (2.39), is often referred to as the *net* inductance [49–51]. The net inductance also has an intuitive interpretation in terms of the magnetic flux. As illustrated in Fig. 2.6, a net inductance (i.e., the partial self-inductance plus the partial mutual inductances with all other segments) of the segment determines the contribution of the segment current to the overall magnetic flux through the circuit.

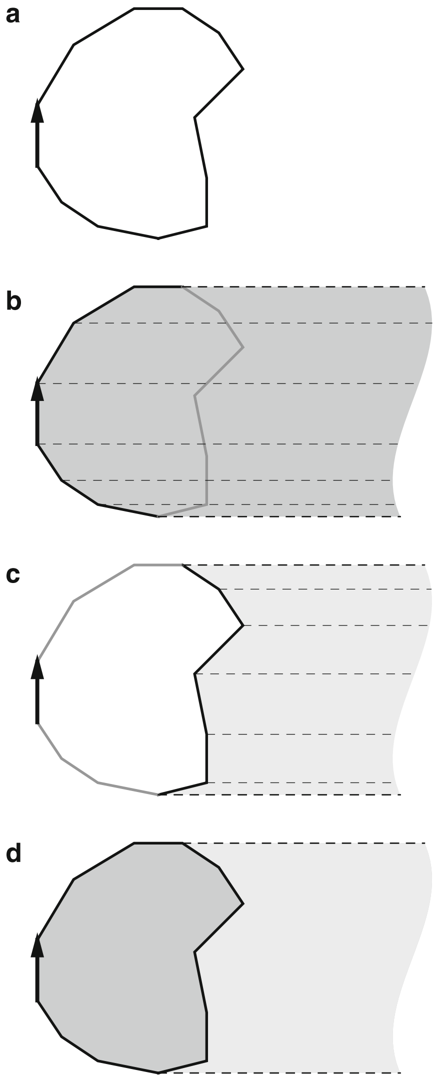

Fig. 2.6 The contribution of a current in a specific loop segment (shown with a *bold arrow*) to the total flux of the current loop is composed of the partial flux of this segment with all other segments of the loop; (**a**) a piecewise linear loop, (**b**) partial flux of the segment with all other segments carrying current in the same direction (i.e., the scalar product of the two segment vectors is positive)—this flux is positive, (**c**) the partial flux of the segment with all other segments carrying current in the opposite direction (i.e., the scalar product of the two segment vectors is negative)—this flux is negative, (**d**) the sum of the positive and negative fluxes, shown in (**b**) and (**c**) (i.e., the geometric difference between the contours (**b**) and (**c**)), is the overall contribution of the segment to the magnetic flux of the loop—this contribution is expressed as the net inductance of the segment

The net inductance describes the behavior of coupled circuits without using explicit mutual inductance terms, simplifying the circuit analysis process. For example, consider a current loop consisting of a signal current path with inductance L_{sig} and return current path with inductance L_{ret}, as shown in Fig. 2.7. The mutual inductance between the two paths is M. The net inductance of the two paths is $L_{\text{sig}}^{\text{eff}} = L_{\text{sig}} - M$ and $L_{\text{ret}}^{\text{eff}} = L_{\text{ret}} - M$. The loop inductance in terms of the net inductance is $L_{\text{loop}} = L_{\text{sig}}^{\text{eff}} + L_{\text{ret}}^{\text{eff}}$. The inductive voltage drop along the return current path is $V_{\text{ret}} = L_{\text{ret}}^{\text{eff}} \frac{dI}{dt}$.

The net inductance has another desirable property. Unlike the partial inductance, the net inductance is independent of the choice of the magnetic vector potential,

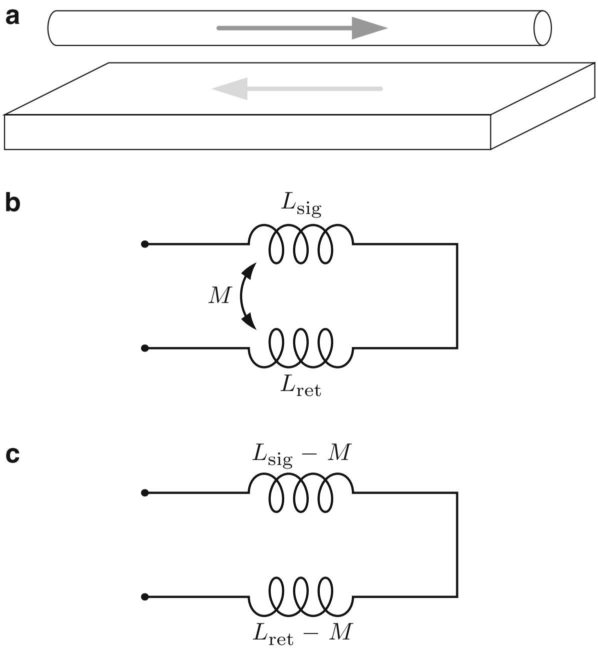

Fig. 2.7 The signal and return current paths. (**a**) The physical structure of the current loop. (**b**) The equivalent partial inductance model. (**c**) The equivalent net inductance model

because, similar to the loop inductance, the integration of the vector potential is performed along a complete loop, as implicitly expressed by (2.39). The net inductance is therefore uniquely determined.

Note that the net inductance of a conductor depends on the structure of the overall circuit as indicated by the mutual partial inductance terms in (2.39). Modifying the shape of a single segment in a circuit changes the net inductance of *all* of the segments. The net inductance is, in effect, a specialized form of the partial inductance and should only be used in the specific circuit where the net inductance terms are determined according to (2.39).

2.2 Variation of Inductance with Frequency

A circuit inductance, either loop or partial, depends upon the current distribution across the cross section of the conductors, as expressed by (2.23) and (2.24). The current density is assumed constant across the conductor cross sections in the inductance formulas described in Sect. 2.1, as is commonly assumed in practice. This assumption is valid where the magnetic field does not appreciably change the path of the current flow. The conditions where this assumption is accurate are discussed in Sect. 2.2.1. Where the effect of the magnetic field on the current path is significant, the current density becomes non-uniform and the magnetic properties of the circuit vary significantly with frequency. The mechanisms causing the inductance to vary with frequency are described in Sect. 2.2.2. A circuit analysis of the variation of inductance with frequency is performed in Sect. 2.2.3 based on a simple circuit model. The section concludes with a discussion of the relative significance of the different inductance variation mechanisms.

2.2.1 Uniform Current Density Approximation

The effect of the magnetic field on the current distribution can be neglected in two general cases. First, the current density is uniform where the magnetic impedance $L\,dI/dt$ is much smaller than the resistive impedance R of the interconnect structure. Under this condition, however, the magnetic properties of the circuit do not significantly affect the circuit behavior and are typically of little practical interest. The second case is of greater practical importance, where the magnetic impedance to the current flow, although greater than R, is uniform across the cross section of a conductor. This condition is generally satisfied where the separation between conductors is significantly greater that the cross-sectional dimensions. It can be shown by inspecting (2.11) that at a distance d much greater than the conductor cross-sectional dimension a, a non-uniform current distribution within the conductor contributes only a second order correction to the magnetic vector potential $\boldsymbol{A}$. The significance of this correction as compared to the primary term decreases with distance as a/d.

Where the separation of two conductors is comparable to the cross-sectional dimensions, the magnetic field significantly affects the current distribution within the conductors. The current density distribution across the cross section becomes non-uniform and varies with the signal frequency. In this case, the magnetic properties of an interconnect structure cannot be accurately represented by a constant value. Alternatively stated, the inductance varies with the signal frequency.

The frequency variation of the current density distribution and, consequently, of the conductor inductance can be explained from a circuit analysis point of view if the impedance characteristics of different paths *within the same conductor* are considered, as described in Sect. 2.2.2. The resistive properties of alternative parallel paths within the same conductors are identical, provided the conductivity of the conductor material is uniform. The magnetic properties of the paths however can be significantly different. At low frequencies, the impedance of the current paths is dominated by the resistance. The current density is uniform across the cross section, minimizing the overall impedance of the conductor. At sufficiently high frequencies, the impedance of the current paths is dominated by the inductive reactance. As the resistive impedance becomes less significant (as compared to the inductive impedance) at higher frequencies, the distribution of the current density asymptotically approaches the density profile that yields the minimum overall inductance of the interconnect structure. The inductance of the on-chip interconnect structures can therefore decrease significantly with signal frequency.

2.2.2 Inductance Variation Mechanisms

As discussed, the variation of inductance is the result of the variation of the current density distribution. The variation of the current distribution with frequency can be loosely classified into several categories.

Skin Effect

With the onset of the skin effect, the current becomes increasingly concentrated near the line surface, causing a decrease in the magnetic field within the line core, as illustrated in Fig. 2.8. The magnetic field outside the conductor is affected relatively little. It is therefore convenient to divide the circuit inductance into "internal" and "external" parts, $L = L_{\text{internal}} + L_{\text{external}}$, where L_{external} is the inductance associated with the magnetic field outside the conductors and L_{internal} is the inductance associated with the magnetic field inside the conductors. In these terms, a well developed skin effect produces a significant decrease in the internal inductance L_{internal}. For a round wire at low frequency (where the current distribution is uniform across the line cross section), the internal inductance is $0.05\,\frac{\text{nH}}{\text{mm}}$, independent of the radius (see the derivation in [52]). The external inductance of the round wire is unaffected by the skin effect.

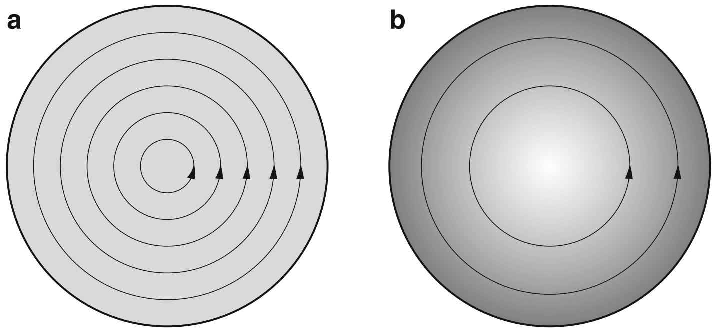

Fig. 2.8 Internal magnetic flux of a round conductor; (**a**) at low frequencies, the current density, as shown by the *shades of gray*, is uniform, resulting in the maximum magnetic flux inside the conductor, as shown by the *circular arrows*, and the associated internal inductance, (**b**) at high frequencies, the current flow is redistributed to the surface of the conductor, reducing the magnetic flux inside the conductor

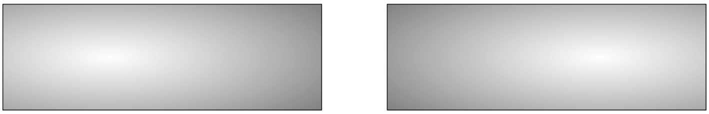

Fig. 2.9 Proximity effect in two closely spaced lines. Current density distribution in the cross section of two closely spaced lines at high frequencies is shown in *shades of gray*. *Darker shades of gray* indicate higher current densities. In lines carrying current in the same direction (parallel currents), the current concentration is shifted away from the parallel current. In lines carrying current in opposite directions (antiparallel currents), the current concentrates toward the antiparallel current, minimizing the circuit inductance

Proximity Effect

The current distribution also varies with frequency due to the proximity effect. At high frequencies, the current in the line concentrates along the side of the line facing an adjacent current return path, thereby reducing the effective area of the current loop and thus the loop inductance, as illustrated in Fig. 2.9.

The skin and proximity effects are closely related. These effects represent basically the same phenomenon—the tendency of the current to move closer to the current return path in order to minimize the interconnect inductance at high frequencies. When a conductor is surrounded by several alternative current return paths, leading to a relatively symmetric current distribution at high frequency, the effect is typically referred to as the skin effect. The classical example of such an interconnect structure is a coaxial cable, where the shield provides equivalent current return paths along all sides of the core conductor. In the case where the current distribution is significantly asymmetric due to the close proximity of a dominant return path, the effect is referred to as the proximity effect.

Multi-path Current Redistribution

The concept of current density redistribution within a conductor can be extended to redistribution of the current among several separate parallel conductors. This mechanism is henceforth referred to as *multi-path current redistribution*. For example, in standard single-ended digital logic, the forward current path is typically composed of a single line. No redistribution of the forward current occurs. The current return path, though, is not explicitly specified (although local shielding for particularly sensitive nets is becoming more common [53, 54]). Adjacent signal lines, power lines, and the substrate provide several alternative current return paths. A significant redistribution of the return current among these return paths can occur as signal frequencies increase. At low frequencies, the line impedance $Z(\omega) = R(\omega) + j\omega L(\omega)$ is dominated by the interconnect resistance R. In this case, the distribution of the return current over the available return paths is determined by the path resistance, as shown in Fig. 2.10a. The return current spreads out far

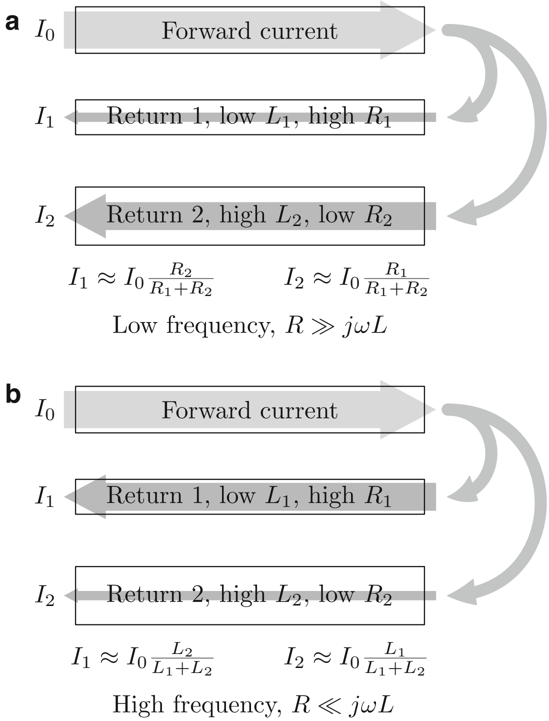

Fig. 2.10 Current loop with two alternative current return paths. The forward current I_0 returns both through return path one with resistance R_1 and inductance L_1, and return path two with resistance R_2 and inductance L_2. In this structure, $L_1 < L_2$ and $R_1 > R_2$. At low frequencies (**a**), the path impedance is dominated by the line resistance and the return current is distributed between two return paths according to the resistance of the lines. Thus, at low frequencies, most of the return current flows through the return path of lower resistance, path two. At high frequencies (**b**), however, the path impedance is dominated by the line inductance and the return current is distributed between two return paths according to the inductance of the lines. Most of the return current flows through the path of lower inductance, path one, minimizing the overall inductance of the circuit

from the signal line to reduce the resistance of the return path and, consequently, the impedance of the current loop. At high frequencies, the line impedance $Z(\omega) = R(\omega) + j\omega L(\omega)$ is dominated by the reactive component $j\omega L(\omega)$. The minimum impedance path is primarily determined by the inductance $L(\omega)$, as shown in Fig. 2.10b. Multi-path current redistribution, as described in Fig. 2.10, is essentially a proximity effect extended to several separate lines connected in parallel. In power grids, both the forward and return currents undergo multi-path redistribution as both the forward and return paths consist of multiple conductors connected in parallel.

The general phenomenon underlying these three mechanisms is, as viewed from a circuit perspective, the same. Where several parallel paths with significantly different electrical properties are available for current flow, the current is distributed among the paths so as to minimize the total impedance. As the frequency increases, the circuit inductance changes from the low frequency limit, determined by the ratio of the resistances of the parallel current paths, to the high frequency value, determined by the inductance ratios of the current paths. At high signal frequencies, the inductive reactance dominates the interconnect impedance; therefore, the path of minimum inductance carries the largest share of the current, minimizing the overall impedance (see Fig. 2.10). Note that parallel current paths can be formed either by several physically distinct lines, as in multi-path current redistribution, or by different paths within the same line, as in skin and proximity effects, as shown in Fig. 2.11. The difference is merely in the physical structure, the electrical behavior is fully analogous. A thick line can be thought of as being composed of multiple thin lines bundled together in parallel. The skin and proximity effects in such a thick line can be considered as a special case of current redistribution among multiple thin lines forming a thick line.

2.2.3 Simple Circuit Model

A simple model of current redistribution provides deeper insight into the process of inductance variation. This approach can be used to estimate the relative significance of the different current distribution mechanisms in various interconnect structures as well as the frequency characteristics of the inductance. Consider a simple case of two current paths with different inductive properties (for example, as shown in Fig. 2.11). The impedance characteristics are represented by the circuit diagram shown in Fig. 2.12, where the inductive coupling between the two paths is neglected for simplicity. Assume that $L_1 < L_2$ and $R_1 > R_2$.

For the purpose of evaluating the variation of inductance with frequency, the electrical properties of the interconnect are characterized by the inductive time

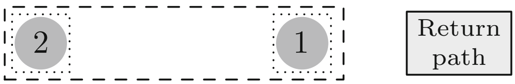

Fig. 2.11 A cross-sectional view of two parallel current paths (*gray circles*) sharing the same current return path (*gray rectangle*). The path closest to the return path, path 1, has a lower inductance than the other path, path 2. The parallel paths can be either two physically distinct lines, as shown by the *dotted line*, or two different paths within the same line, as shown by the *dashed line*

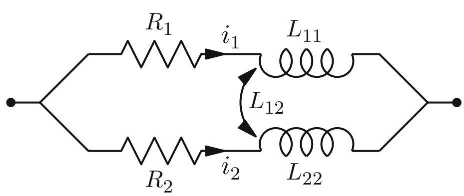

Fig. 2.12 A circuit model of two current paths with different inductive properties

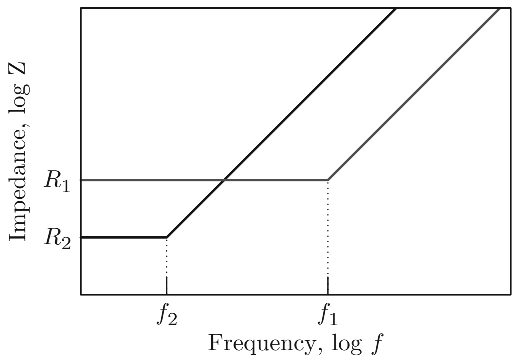

Fig. 2.13 Impedance magnitude versus frequency for two paths with dissimilar impedance characteristics

constant $\tau = L/R$. The impedance magnitude of these two paths is schematically shown in Fig. 2.13. The impedance of the first path is dominated by the inductive reactance above the frequency $f_1 = \frac{1}{2\pi}\frac{R_1}{L_1} = \frac{1}{2\pi\tau_1}$. The impedance of the second path is predominantly inductive above the frequency $f_2 = \frac{1}{2\pi}\frac{R_2}{L_2} = \frac{1}{2\pi\tau_2}$, such that $f_2 < f_1$. At low frequencies, i.e., from DC to the frequency f_1, the ratio of the two impedances is constant. The effective inductance at low frequencies is therefore also constant, determining the low frequency inductance limit. At high frequencies, i.e., frequencies exceeding f_2, the ratio of the impedances is also constant, determining the high frequency inductance limit, $\frac{L_1L_2}{L_1+L_2}$. At intermediate frequencies from f_1 to f_2, the impedance ratio changes, resulting in a variation of the overall inductance from the low frequency limit to the high frequency limit. The frequency range of inductance variation is therefore determined by the two time constants, τ_1 and τ_2. The magnitude of the inductance variation depends upon both the difference between the time constants τ_1 and τ_2 and on the inductance ratio L_1/L_2. Analogously, in the case of multiple parallel current paths, the frequency range and the magnitude of the variation in inductance is determined by the minimum and maximum time constants as well as the difference in inductance among the current paths.

The decrease in inductance begins when the inductive reactance $j\omega L$ of the path with the lowest R/L ratio becomes comparable to the path resistance R, $R \sim j\omega L$. The inductance, therefore, begins to decrease at a lower frequency if the minimum R/L ratio of the current paths is lower.

Due to this behavior, the proximity effect becomes significant at higher frequencies than the frequencies at which multi-path current redistribution becomes significant. Significant proximity effects occur in conductors containing current paths with significantly different inductive characteristics. That is, the inductive coupling of one edge of the line to the "return" current (i.e., the current in the opposite direction) is substantially different from the inductive coupling of the other edge of the line to the same "return" current. In geometric terms, this characteristic means that the line width is larger than or comparable to the distance between the line and the return current. Consequently, the line with significant proximity effects

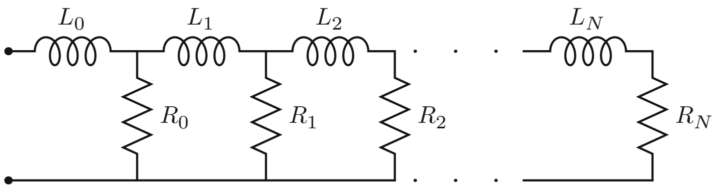

Fig. 2.14 An *RL* ladder circuit describing the variation of inductance with frequency

is typically the immediate neighbor of the current return line. A narrower current loop is therefore formed with the current return path as compared to the other lines participating in the multi-path current redistribution. A smaller loop inductance L results in a higher R/L ratio. Referring to Fig. 2.10, current redistribution between paths one and two develops at frequencies lower than the onset frequency of the proximity effect in path one.

Efficient and accurate lumped element models are necessary to incorporate skin and proximity effects into traditional circuit simulation tools. Developing such models is an area of ongoing research [55–61]. The resistance and internal inductance of conductors are typically modeled with *RL* ladder circuits [55], as shown in Fig. 2.14. The sections of the *RL* ladder represent the resistance and inductance of the conductor parts at different distances from the current return path. Different methods for determining the value of the *R* and *L* elements have been developed [56–58]. Analogously, *RL* ladders can also be extended to describe multi-path current redistribution [59, 60]. Techniques for reducing the order of a transfer function of an interconnect structure have also been described [61].

2.3 Inductive Behavior of Circuits

The strict meaning of the term "inductance" is the *absolute inductance*, as defined in Sect. 2.1. The absolute inductance is measured in henrys. Sometimes, however, the same term "inductance" is loosely used to denote the *inductive behavior* of a circuit; namely, overshoots, ringing, signal reflections, etc. The inductive behavior of a circuit is characterized by such quantities as a damping factor and the magnitude of the overshoot and reflections of the signals. While any circuit structure carrying an electrical current has a finite absolute inductance, as defined in Sect. 2.1, not every circuit exhibits inductive behavior. Generally, a circuit exhibits inductive behavior if the absolute inductance of the circuit is sufficiently high. The relationship between the inductive behavior and the absolute inductance is, however, circuit specific and no general metrics for the onset of inductive behavior have been developed.

Specific metrics have been developed to evaluate the onset of inductive behavior in high speed digital circuits [62–64]. A digital signal that is propagating in an underdriven uniform lossy transmission line exhibits significant inductive effects if the line length l satisfies the following condition [63],

$$\frac{t_\mathrm{r}}{2\sqrt{LC}} < l < \frac{2}{R}\sqrt{\frac{L}{C}}, \tag{2.40}$$

where R, L, and C are the resistance, inductance, and capacitance per line length, respectively, and t_r is the rise time of the signal waveform.

The two inequalities comprising condition (2.40) have an intuitive circuit interpretation. The velocity of the electromagnetic signal propagation along a line is $v_\mathrm{c} = \frac{1}{\sqrt{LC}}$. The left inequality of (2.40) therefore transforms into

$$t_\mathrm{r} < \frac{2l}{v_\mathrm{c}}, \tag{2.41}$$

i.e., the signal rise time should be smaller than the round trip time of flight. Alternatively stated, the line length l should be a significant fraction of the shortest wavelength of significant signal frequencies λ_r. The spectral content of the signal with rise time t_r rolls off at -20 dB/decade above the frequency $f_\mathrm{r} = 1/\pi t_\mathrm{r}$. The shortest effective wavelength of the signal is therefore $\lambda_\mathrm{r} = v_c/f_\mathrm{r} = \pi v_c t_\mathrm{r}$. The condition (2.41) can be rewritten as

$$\frac{l}{\lambda_\mathrm{r}} > \frac{1}{2\pi}. \tag{2.42}$$

The dimensionless ratio of the physical size of a circuit to the signal wavelength, l/λ, is referred to as the *electrical size* in high speed interconnect design [51, 65]. Circuits with an electrical size much smaller than unity are commonly called electrically small (or short), otherwise circuits are called electrically large (or long) [51, 65]. Electrically small circuits belong to the realm of classical circuit analysis and are well described by lumped circuits. Electrically large circuits require distributed circuit models and belong to the domain of high speed interconnect analysis techniques. The left inequality of condition (2.40) therefore restricts significant inductive effects to electrically long lines.

With the notion that the damping factor of the transmission line is $\zeta = \frac{R_0}{2}\sqrt{\frac{C_0}{L_0}}$, where $R_0 = Rl$, $L_0 = Ll$, and $C_0 = Cl$ are the total resistance, inductance, and capacitance of the line, respectively, the right inequality in condition (2.40) transforms into

$$\zeta < 1, \tag{2.43}$$

constraining the damping factor to be sufficiently small. Given a line with a specific R, L, and C, the inductive behavior is confined to a certain range of line length, as shown in Fig. 2.15. The upper bound of this range is determined by the damping factor of the line, while the lower bound is determined by the electrical size of the line.

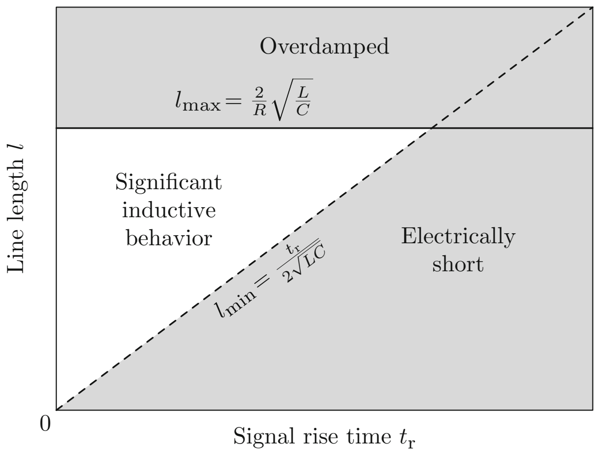

Fig. 2.15 The range of transmission line length where the signal propagation exhibits significant inductive behavior. The area of inductive behavior (the *unshaded area*) is bounded by the conditions of large electrical size (the *dashed line*) and insufficient damping (the *solid line*), as determined by (2.40). In the region where either of these conditions is not satisfied (the *shaded area*), the inductive effects are insignificant

Alternatively, condition (2.40) can be interpreted as a bound on the overall line inductance $L_0 = Ll$. The signal transmission exhibits inductive characteristics if the overall line inductance satisfies both of the following conditions,

$$L_0 > \frac{t_r^2}{4C_0} \tag{2.44}$$

and

$$L_0 > \frac{1}{4} R_0^2 C_0 \,. \tag{2.45}$$

Conditions (2.44) and (2.45) thereby quantify the term "inductance sufficiently large to cause inductive behavior" as applied to transmission lines. The design space for a line inductance with the region of inductive behavior, as determined by (2.44) and (2.45), is illustrated in Fig. 2.16.

2.4 Inductive Properties of On-Chip Interconnect

The distinctive feature of on-chip interconnect structures is the small cross-sectional dimensions and, consequently, a relatively high line resistance. For example, the resistance of a copper line with a $1 \times 3\ \mu m$ cross section is approximately $7\ \Omega/mm$.

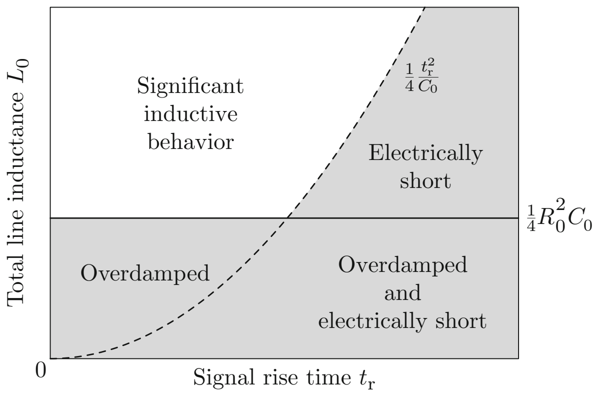

Fig. 2.16 The design space characterizing the overall transmission line inductance is divided into a region of inductive behavior and a region where inductive effects are insignificant. The region of inductive behavior (the *unshaded area*) is bounded by the conditions of large electrical size (the *dashed line*) and low damping (the *solid line*), as determined by (2.44) and (2.45). In the region where either of these conditions is not satisfied (the *shaded area*), the inductive effects are insignificant

The loop inductance of on-chip lines is typically between 0.4 nH/mm and 1 nH/mm. At frequencies lower than several gigahertz, the magnetic characteristics do not significantly affect the behavior of on-chip circuits.

As the switching speed of digital integrated circuits increases with technology scaling, the magnetic properties have become essential for accurately describing on-chip circuit operation. The density and complexity of the on-chip interconnect structures preclude exploiting commonly assumed circuit simplifications, rendering the accurate analysis of inductive properties particularly challenging. Large integrated circuits contain many tens of millions of interconnect segments while the segment spacing is typically either equal to or less than the cross-sectional dimensions. Accurate treatment of magnetic coupling in these conditions is especially important. Neither the loop nor the partial inductance formulation can be directly applied to an entire circuit as the size of the resulting inductance matrices makes the process of circuit analysis computationally infeasible. Simplifying the inductive properties of a circuit is also difficult. Simply omitting relatively small partial inductance terms can significantly change the circuit behavior, possibly causing instability in an originally passive circuit. Techniques to simplify the magnetic characteristics so as to allow an accurate analysis of separate circuit parts is currently an area of focused research [66–69].

The problem is further complicated by the significant variation of inductance with frequency. As discussed in Sect. 2.2, the inductance variation can be described in terms of the skin effect, proximity effect, and multi-path current redistribution. For a line with a rectangular cross section, the internal inductance is similar to the internal inductance of a round line, i.e., 0.05 nH/mm, decreasing with the aspect ratio of the cross section. Over the frequency range of interest, up to 100 GHz, the skin effect reduces the internal inductance by only a small fraction. The reduction in the internal inductance due to the skin effect is, therefore, relatively insignificant, as compared to the overall inductance. Due to the relatively high resistance of on-chip interconnect, the proximity effect is significant only in immediately adjacent wide lines that carry high frequency current. Where several parallel lines are available for current flow, redistribution of the current among the lines is typically the primary cause in integrated circuits of the decrease in inductance with frequency. The proximity effect and multi-path current redistribution are therefore two mechanisms that can produce a significant change in the on-chip interconnect inductance with signal frequency.

Note that the statement "sufficiently high inductance causes inductive behavior" does not necessarily mean "any change in the interconnect physical structure that increases the line inductance increases the inductive behavior of the line." In fact, the opposite is often the case in an integrated circuit environment, where varying a single physical interconnect characteristic typically affects many electrical characteristics. The relationship between the physical structure of interconnect and the inductive behavior of a circuit is highly complex.

Consider a 3 mm long copper line with a $1 \times 1\,\mu$m cross section. The resistance, inductance, and capacitance per length of the current loop (including both the line itself and the current return path) are, respectively, $R = 25\,\Omega/\text{mm}$, $L = 0.8\,\text{nH/mm}$, and $C = 100\,\text{fF/mm}$. The velocity of the electromagnetic wave propagation along the line is 0.11 mm/ps. This velocity is somewhat smaller than the speed of light, 0.15 mm/ps, in the media with an assumed dielectric constant of 4 and is due to the additional capacitive load of the orthogonal lines in the lower layer. For a signal with a 30 ps rise time, the line is electrically long. The line damping factor, however, is $\zeta = \frac{Rl}{2}\sqrt{\frac{C}{L}} = 1.33 > 1$. The line is therefore overdamped and, according to the metrics expressed by (2.44) and (2.45), does not exhibit inductive behavior, as shown in Fig. 2.17a.

Assume now that the line width is $4\,\mu$m and the resistance, inductance, and capacitance of the line change, respectively, to $R = 10\,\Omega/\text{mm}$, $L = 0.65\,\text{nH/mm}$, and $C = 220\,\text{fF/mm}$. The decrease in the loop resistance and inductance are primarily due to the smaller resistance and partial self-inductance of the line. The increase in the line capacitance is primarily due to the greater parallel plate capacitance between the signal line and the perpendicular lines in the lower layer. This capacitive load becomes more significant, as compared to the capacitance between the line and the return path, further slowing the velocity of the electromagnetic wave propagation to 0.084 mm/ps. For the same signal with a 30 ps transition time, the signal line becomes underdamped, $\zeta = 0.87 < 1$, and exhibits significant inductive behavior, as shown in Fig. 2.17b.

The *inductive behavior* has become significant while the *absolute inductance* has *decreased* from $3\,\text{mm} \times 0.8\frac{\text{nH}}{\text{mm}} = 2.40$ to 1.95 nH. The reason for this seeming contradiction is that the inductance is a weak function of the cross-sectional dimensions, as compared to the resistance and capacitance. In integrated circuits, the signal lines that exhibit inductive behavior are the lowest resistance lines, i.e., the

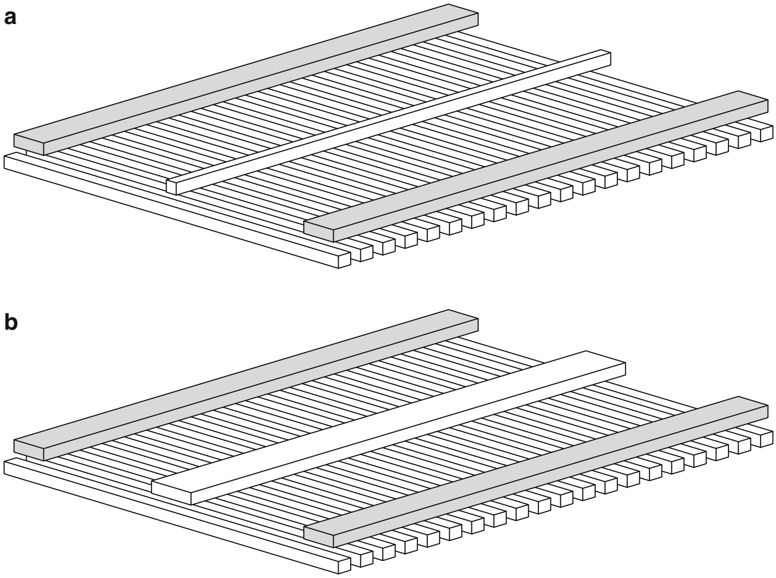

Fig. 2.17 A signal line within an integrated circuit. The power and ground lines (*shaded gray*) parallel to the signal line serve as a current return path. The lines in the lower metal layer increase the capacitive load of the line. The inductive behavior of a wide line, as shown in (**b**), is more significant as compared to a narrow line, as shown in (**a**)

wide lines in the thick upper metalization layers. These lines typically have a lower absolute inductance than other signal lines. It would therefore be misleading to state that the inductive behavior of on-chip interconnect has become important due to the increased inductance. This trend is due to the shorter signal transition times and longer line lengths, while maintaining approximately constant the resistive properties of the upper metal layers.

2.5 Summary

The preceding discussion of the inductive characteristics of electric circuits and different ways to represent these characteristics can be summarized as follows.

- The thin filament approximation is valid only for determining the mutual inductance of relatively thin conductors
- The partial inductance formulation is better suited to describe the inductive properties of circuits with branch points
- The partial inductance is a mathematical construct, not a physically observable property, and should only be used as part of a complete description of the circuit inductance

- The circuit inductance varies with frequency due to current redistribution within the circuit conductors. The current redistribution mechanisms can be classified as the skin effect, proximity effect, and multi-path current redistribution
- Signal propagation along a transmission line exhibits inductive behavior if the line is both electrically long and underdamped
- Characterizing on-chip inductance in both an efficient and accurate manner is difficult due to the density and complexity of on-chip interconnect structures
- The relationship between the physical structure of on-chip interconnect and the inductive behavior of a circuit is complex, as many electrical properties can be affected by changing a specific physical characteristic of an interconnect line