Descriptor Fractional Discrete-Time Linear System and Its Solution—Comparison of Three Different Methods

Łukasz Sajewski

Abstract Descriptor fractional discrete-time linear systems are addressed. Three different methods for finding the solution to the state equation of the descriptor fractional linear system are considered. The methods are based on: Shuffle algorithm, Drazin inverse of the matrices and Weierstrass-Kronecker decomposition theorem. Effectiveness of the methods is demonstrated on simple numerical example.

Keywords Descriptor · Fractional · Solution · Method

1 Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–8]. First definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [9, 10], another on was proposed in 20th century by Caputo [11] and next one in present times by Caputo-Fabrizio [12]. This idea has been used by engineers for modeling different processes [13, 14]. Mathematical fundamentals of fractional calculus are given in the monographs [9–11, 15]. Solution of the state equations of descriptor fractional discrete-time linear systems with regular pencils have been given in [7, 16] and for continuous-time in [5, 6]. Reduction and decomposition of descriptor fractional discrete-time linear systems has been considered in [17]. Application of the Drazin inverse method to analysis of descriptor fractional discrete-time and continuous-time linear systems have been given in [18, 19]. Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional has been introduced in [8].

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In this paper three different methods for finding the solution to descriptor fractional discrete-time linear systems will be considered and illustrated on single example.

The paper is organized as follows. In Sect. 2 the basic informations on the descriptor fractional discrete-time linear systems are recalled. Shuffle algorithm method is described in Sect. 3. Drazin inverse method is given in Sect. 4. Section 5 reccals Weierstrass-Kronecker decomposition method. In Sect. 6 single numerical example, illustrating three methods is presented. Concluding remarks are given in Sect. 7.

The following notation will be used: $\mathfrak{R}$—the set of real numbers, $\mathfrak{R}^{n\times m}$—the set of $n\times m$ real matrices, Z_+—the set of nonnegative integers, I_n—the $n\times n$ identity matrix.

2 Preliminaries

Consider the descriptor fractional discrete-time linear system described by the state equation

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, i\in Z_+ = \{0, 1, \ldots\}, \quad (2.1)$$

where, $x_i\in\mathfrak{R}^n$, $u_i\in\mathfrak{R}^m$ are the state and input vectors, $A\in\mathfrak{R}^{n\times n}$, $E\in\mathfrak{R}^{n\times n}$, $B\in\mathfrak{R}^{n\times m}$, and the fractional difference of the order α is defined by

$$\Delta^\alpha x_i = \sum_{k=0}^{i} c_k x_{i-k}, \quad 0<\alpha<1, \quad (2.2a)$$

where

$$c_k = (-1)^k \binom{\alpha}{k} = (-1)^k \begin{cases} 1 & \text{for} \quad k=0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,\ldots \end{cases} \quad (2.2b)$$

It is assumed that $\det E = 0$ and the pencil of the system (2.1) is regular, that is $\det[Ez - A] \neq 0$ for some $z\in C$ (the field of complex numbers). To find the solution of the system (2.1) at least three different methods can be used. These methods are the Shuffle algorithm method [17], the Drazin inverse method [18] and the Weierstrass-Kronecker decomposition method [7]. These methods was previously used to find the solution of the descriptor standard discrete-time linear systems and was extended to fractional systems. The question arise, does the order α has influence on the solution computed by the use of these methods?

In the next section, three different approaches to finding the solution to the state Eq. (2.1) of the descriptor fractional discrete-time linear systems will be given.

3 Shuffle Algorithm Method

First method is based on row and column elementary operations [20] and use the Shuffle algorithm to determine the solution [17].

By substituting (2.2a) into (2.1) we can write the state equation in the form

$$\sum_{k=0}^{i+1} Ec_k x_{i-k+1} = Ax_i + Bu_i, i \in Z_+, \tag{3.1}$$

where c_k is given by (2.2b). Applying the row elementary operations to (3.1) we obtain

$$\sum_{k=0}^{i+1} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} c_k x_{i-k+1} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, i \in Z_+, \tag{3.2}$$

where $E_1 \in \Re^{n_1 \times n}$ is full row rank and $A_1 \in \Re^{n_1 \times n}$, $A_2 \in \Re^{(n-n_1) \times n}$, $B_1 \in \Re^{n_1 \times m}$, $B_2 \in \Re^{(n-n_1) \times m}$. The Eq. (3.2) can be rewritten as

$$\sum_{k=0}^{i+1} E_1 c_k x_{i-k+1} = A_1 x_i + B_1 u_i \tag{3.3a}$$

and

$$0 = A_2 x_i + B_2 u_i. \tag{3.3b}$$

Substituting in (3.3b) i by $i + 1$ we obtain

$$A_2 x_{i+1} = -B_2 u_{i+1}. \tag{3.4}$$

The Eqs. (3.3a) and (3.4) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_1 - c_1 E_1 \\ 0 \end{bmatrix} x_i - \begin{bmatrix} c_2 E_1 \\ 0 \end{bmatrix} x_{i-1} - \cdots - \begin{bmatrix} c_{i+1} E_1 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1}. \tag{3.5}$$

If the matrix

$$[\,E_1^T \quad A_2^T\,]^T \tag{3.6}$$

is singular then applying the row operations to (3.5) we obtain

$$\begin{bmatrix} E_2 \\ 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{20} \\ \bar{A}_{20} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ \bar{A}_{21} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{2,i} \\ \bar{A}_{2,i} \end{bmatrix} x_0 + \begin{bmatrix} B_{20} \\ \bar{B}_{20} \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ \bar{B}_{21} \end{bmatrix} u_{i+1}, \tag{3.7}$$

where $E_2 \in \Re^{n_2 \times n}$ is full row rank with $n_2 \geq n_1$ and $A_{2,j} \in \Re^{n_2 \times n}$, $\bar{A}_{2,j} \in \Re^{(n-n_2) \times n}$, $j = 0, 1, \ldots, i$ $B_{2,k} \in \Re^{n_2 \times m}$, $\bar{B}_{2,k} \in \Re^{(n-n_2) \times m}$, $k = 0, 1$. From (3.7) we have

$$0 = \bar{A}_{20} x_i + \bar{A}_{21} x_{i-1} + \cdots + \bar{A}_{2,i} x_0 + \bar{B}_{20} u_i + \bar{B}_{21} u_{i+1}. \tag{3.8}$$

Substituting in (3.8) i by $i + 1$ (in state vector x and in input u) we obtain

$$\bar{A}_{20} x_{i+1} = -\bar{A}_{21} x_i - \cdots - \bar{A}_{2,i} x_1 - \bar{B}_{20} u_{i+1} - \bar{B}_{21} u_{i+2}. \tag{3.9}$$

From (3.7) and (3.9) we have

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ -\bar{A}_{22} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix} u_{i+2} \tag{3.10}$$

If the matrix

$$[E_2^T \quad \bar{A}_{20}^T]^T \tag{3.11}$$

is singular then we repeat the procedure.

Continuing this procedure after finite number of steps p we obtain

$$\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{p,0} \\ -\bar{A}_{p,1} \end{bmatrix} x_i + \begin{bmatrix} A_{p,1} \\ -\bar{A}_{p,2} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{pi} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_{p,0} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{p,1} \\ -\bar{B}_{p,0} \end{bmatrix} u_{i+1} + \cdots + \begin{bmatrix} 0 \\ -\bar{B}_{p,p-1} \end{bmatrix} u_{i+p-1} \tag{3.12}$$

where $E_p \in \Re^{n_p \times n}$ is full row rank, $A_{pj} \in \Re^{n_p \times n}$, $\bar{A}_{pj} \in \Re^{(n-n_p) \times n}$, $j = 0, 1, \ldots, p$ and $B_{pk} \in \Re^{n_p \times m}$, $\bar{B}_{pk} \in \Re^{(n-n_p) \times m}$, $k = 0, 1, \ldots, p-1$ with nonsingular matrix

$$[E_p^T \quad \bar{A}_{p.0}^T]^T \in \Re^{n \times n}. \tag{3.13}$$

In this, case premultiplying Eq. (3.12) by $\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1}$, we obtain the standard system

$$x_{i+1} = \hat{A}_0 x_i + \hat{A}_1 x_{i-1} + \cdots + \hat{A}_i x_0 + \hat{B}_0 u_i + \hat{B}_1 u_{i+1} + \cdots + \hat{B}_{p-1} u_{i+p-1} \tag{3.14}$$

with the matrices

$$\hat{A}_0 = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} A_{p,0} \\ -\bar{A}_{p,1} \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} A_{p,1} \\ -\bar{A}_{p,2} \end{bmatrix}, \ldots, \quad \hat{A}_i = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} A_{pi} \\ 0 \end{bmatrix},$$
$$\hat{B}_0 = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} B_{p,0} \\ 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} B_{p,1} \\ -\bar{B}_{p,0} \end{bmatrix}, \ldots, \quad \hat{B}_{p-1} = \begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{p,p-1} \end{bmatrix}. \tag{3.15}$$

Eventually, we reduce the descriptor system to standard system with delays. To compute the solution x_i of (3.14), now we can use methods given for standard discrete-time linear systems with delays, e.g. iterative approach (initial conditions are needed).

4 Drazin Inverse Method

Second method use the Drazin inverses of the matrices $\bar{E}$ and $\bar{F}$ [18].

Definition 4.1 [18] A matrix $\bar{E}^D$ is called the Drazin inverse of $\bar{E} \in \Re^{n \times n}$ if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \tag{4.1}$$

where q is the smallest nonnegative integer, satisfying condition rank $\bar{E}^q =$ rank $\bar{E}^{q+1}$ and it is called the index of $\bar{E}$.

The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exist and is unique [1]. If $\det \bar{E} \neq 0$ then $\bar{E}^D = \bar{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [20].

Lemma 4.1 [18] The matrices $\bar{E}$ and $\bar{F}$ satisfy the following equalities:

1. $$\bar{F}\bar{E} = \bar{E}\bar{F} \text{ and } \bar{F}^D\bar{E} = \bar{E}\bar{F}^D, \bar{E}^D\bar{F} = \bar{F}\bar{E}^D, \bar{F}^D\bar{E}^D = \bar{E}^D\bar{F}^D, \tag{4.2a}$$

2. $$\ker \bar{F}_1 \cap \ker \bar{E} = \{0\}, \tag{4.2b}$$

3. $$\bar{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}, \bar{F} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}T^{-1}, \bar{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \det T \neq 0, \tag{4.2c}$$

   $J \in \Re^{n_1 \times n_1}$, is nonsingular, $N \in \Re^{n_2 \times n_2}$ is nilpotent, $A_1 \in \Re^{n_1 \times n_1}$, $A_2 \in \Re^{n_2 \times n_2}$, $n_1 + n_2 = n$,

4. $$(I_n - \bar{E}\bar{E}^D)\bar{F}\bar{F}^D = I_n - \bar{E}\bar{E}^D \text{ and } (I_n - \bar{E}\bar{E}^D)(\bar{E}\bar{F}^D)^q = 0. \tag{4.2d}$$

Similar as in previous case, substitution of (2.2a) into (2.1) yields

$$Ex_{i+1} = Fx_i - \sum_{k=1}^{i} Ec_{k+1}x_{i-k} + Bu_i, i \in Z_+, \tag{4.3a}$$

where

$$F = A - Ec_1 \text{ and } \det[Ec - F] \neq 0 \text{ for some } c \in \mathbb{C}. \tag{4.3b}$$

Premultiplying (4.3a) by $[Ec - F]^{-1}$ we obtain

$$\bar{E}x_{i+1} = \bar{F}x_i - \sum_{k=1}^{i} \bar{E}c_{k+1}x_{i-k} + \bar{B}u_i, \tag{4.4a}$$

where

$$\bar{E} = [Ec - F]^{-1}E, \bar{F} = [Ec - F]^{-1}F, \bar{B} = [Ec - F]^{-1}B. \tag{4.4b}$$

Theorem 4.1 The solution to the Eq. (4.4a) with an admissible initial condition x_0, is given by

$$x_i = (\bar{E}^D\bar{F})^i\bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1} \bar{E}^D(\bar{E}^D\bar{F})^{i-k-1}[\bar{B}u_k - \sum_{j=1}^{k} \bar{E}c_{j+1}x_{k-j}] + (\bar{E}\bar{E}^D - I_n)\sum_{k=0}^{q-1} (\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_{i+k} \tag{4.5}$$

where q is the index of $\bar{E}$. Proof is given in [18].

From (4.5) for $i = 0$ we have

$$x_0 = \bar{E}^D\bar{E}x_0 + (\bar{E}\bar{E}^D - I_n)\sum_{k=0}^{q-1} (\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_k. \tag{4.6}$$

In practical case, for $u_i = 0$, $i \in Z_+$ we have $x_0 = \bar{E}^D\bar{E}x_0$. Thus, the equation $\bar{E}x_{i+1} = Ax_i$ has a unique solution if and only if $x_0 \in \text{Im}\bar{E}\bar{E}^D$, where Im denotes the image.

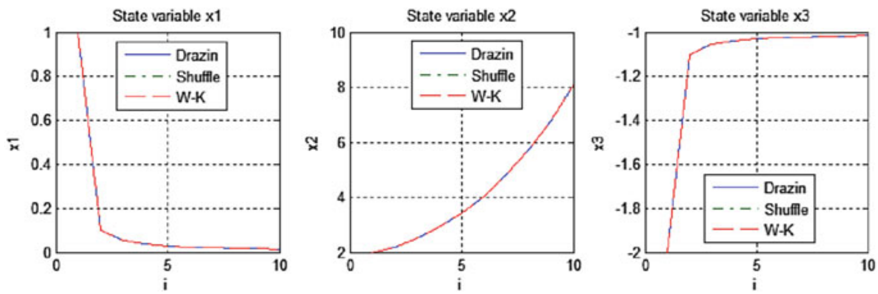

Fig. 1 Solution for $\alpha = 0.1$

5 Weierstrass-Kronecker Decomposition Method

Third method use the following Lemma, upon which the solution to the state equation will be derived.

Lemma 5.1 [7, 20, 21] If (2.3) holds, then there exist nonsingular matrices $P, Q \in \Re^{n \times n}$ such that

$$PEQ = \mathrm{diag}(I_{n_1}, N),\ \ PAQ = \mathrm{diag}(A_1, I_{n_2}), \tag{5.1}$$

where $N \in \Re^{n_2 \times n_2}$ is nilpotent matrix with the index μ (i.e. $N^{\mu} = 0$ and $N^{\mu-1} \neq 0$), $A_1 \in \Re^{n_1 \times n_1}$ and n_1 is equal to degree of the polynomial

$$\det[Es - A] = a_{n_1} z^{n_1} + \cdots + a_1 z + a_0, n_1 + n_2 = n. \tag{5.2}$$

A method for computation of the matrices P and Q has been given in [22].

Premultiplying the Eq. (2.1) by the matrix $P \in \Re^{n \times n}$ and introducing new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} = Q^{-1} x_i, \bar{x}_i^{(1)} \in \Re^{n_1}, \bar{x}_i^{(2)} \in \Re^{n_2}, i \in Z_+, \tag{5.3}$$

we obtain

$$PEQQ^{-1} \Delta^{\alpha} x_{i+1} = PEQ \Delta^{\alpha} Q^{-1} x_{i+1} = PAQQ^{-1} x_i + PBu_i. \tag{5.4}$$

Applying (5.1) and (5.3) to (5.4) we have

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \Delta^{\alpha} \begin{bmatrix} \bar{x}_{i+1}^{(1)} \\ \bar{x}_{i+1}^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, i \in Z_+, \tag{5.5}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, B_1 \in \Re^{n_1 \times m}, B_2 \in \Re^{n_2 \times m}. \tag{5.6}$$

Taking into account (2.2a), from (5.5) we obtain

$$\bar{x}_{i+1}^{(1)} = -\sum_{k=1}^{i+1} (-1)^k \binom{\alpha}{k} \bar{x}_{i-k+1}^{(1)} + A_1 \bar{x}_i^{(1)} + B_1 u_i = A_{1\alpha} \bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{\alpha}{k} \bar{x}_{i-k+1}^{(1)} + B_1 u_i \tag{5.7}$$

and

$$N\left[\bar{x}_{i+1}^{(2)}+\sum_{k=1}^{i+1}(-1)^k\binom{\alpha}{k}\bar{x}_{i-k+1}^{(2)}\right]=\bar{x}_i^{(2)}+B_2u_i, A_{1\alpha}=A_1+I_{n_1}\alpha. \tag{5.8}$$

The solution $\bar{x}_i^{(1)}$ to the Eq. (5.7) is well-known [20] and it is given by the following theorem.

Theorem 5.1 [7, 20] The solution $\bar{x}_i^{(1)}$ of the Eq. (5.7) is given by the formula

$$\bar{x}_i^{(1)}=\Phi_i\bar{x}_0^{(1)}+\sum_{k=0}^{i-1}\Phi_{i-k-1}B_1u_k, i\in Z_+, \tag{5.9}$$

where the matrices Φ_i are determined by the equation

$$\Phi_{i+1}=\Phi_iA_{1\alpha}+\sum_{k=2}^{i+1}(-1)^{k-1}\binom{\alpha}{k}\Phi_{i-k+1} \quad \Phi_0=I_{n_1}. \tag{5.10}$$

To find the solution $\bar{x}_i^{(2)}$ of the Eq. (5.8) for $N\neq 0$ nilpotent (e.g. for $N=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ we have two equations with two unknown elements) we simple start with solving the equation related to zero row and then continue solving the rest of the equations, see e.g. [7, 20].

If $N=0$ then from (5.8) we have

$$\bar{x}_i^{(2)}=-B_2u_i, i\in Z_+. \tag{5.11}$$

From (5.3), for known $\bar{x}_i^{(1)}$ and $\bar{x}_i^{(2)}$, we can find the desired solution of the Eq. (2.1).

6 Example

Main goal of this chapter as well as whole paper, is to show, how to use presented methods, for computation of the solution of the fractional discrete-time linear system described by the Eq. (2.1). The following example will be used to describe the procedure for computation of the solution.

Find the solution x_i of the descriptor fractional discrete-time linear system (2.1) with the matrices

$$E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & -1\end{bmatrix}, \quad B=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix} \tag{6.1}$$

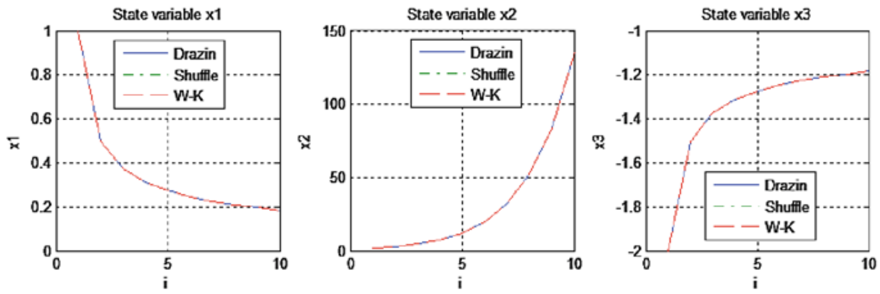

Fig. 2 Solution for $\alpha = 0.5$

for $\alpha = 0.5$, $u_i = u = 1$, $i \in Z_+$ and $x_0 = [1 \quad 2 \quad -2]^T$ (T denotes the transpose).

In this case, $\det E = 0$ and the pencil of the system (2.1) witch (6.1) is regular since

$$\det[Ez - A] = \begin{vmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = z(z-1). \tag{6.2}$$

6.1 *Case of Shuffle Algorithm Method*

Following Chap. 3, we compute

$$[E \quad A \quad B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} \tag{6.3}$$

and the Eqs. (3.3a) and (3.3b) has the form

$$\sum_{k=0}^{i+1} c_k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_{i-k+1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i, \tag{6.4a}$$

$$0 = [-1 \quad 0 \quad -1] x_i - u_i. \tag{6.4b}$$

Using (2.2b) we obtain $c_1 = -0.5$, $c_2 = 1/8$, …, $c_{i+1} = (-1)^{i+1} \left. \frac{\alpha(\alpha-1)\ldots(\alpha-i)}{(i+1)!} \right|_{\alpha=0.5}$ and the Eq. (3.5) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} x_{i+1} = \begin{bmatrix} 1.5 & 0 & 1 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i - \frac{1}{8} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_{i-1} - \cdots - c_{i+1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{i+1}. \tag{6.5}$$

The matrix $\begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix}^{-1}$ is nonsingular and the solution to the state Eq. (2.1) has the form

$$x_{i+1} = \hat{A}_0 x_i + \hat{A}_1 x_{i-1} + \cdots + \hat{A}_i x_0 + \hat{B}_0 u_i + \hat{B}_1 u_{i+1}, \tag{6.6}$$

where

$$\hat{A}_0 = \begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix}^{-1} \begin{bmatrix} 1.5 & 0 & 1 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_1 = \frac{1}{8} \begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots,$$
$$\hat{A}_i = (-1)^{i-1} \frac{0.5(-0.5)\ldots(0.5-i)}{(i+1)!} \begin{bmatrix} E_1 \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_0 = \begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} E_1 \\ \bar{A}_{10} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.7}$$

The desired solution of the descriptor fractional system (2.1) with (6.1) has the form

$$x_i = \sum_{k=0}^{i-1} \hat{A}_k x_{i-k-1} + \sum_{k=0}^{p} \hat{B}_k u_{i-k-1}. \tag{6.8}$$

6.2 Case of Drazin Inverse Method

Following this chapter, we compute

$$F = A - Ec_1 = A + E\alpha = \begin{bmatrix} 1+\alpha & 0 & 1 \\ 0 & 1+\alpha & 0 \\ -1 & 0 & -1 \end{bmatrix} \text{ and } q = 1. \tag{6.9}$$

For $c = 1$ the matrices (4.4b) have the form

$$\bar{E} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.667 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}. \tag{6.10}$$

Using e.g. formula $\bar{E}^D = V[W\bar{E}V]^{-1}W$ where $\bar{E} = VW = \begin{bmatrix} -2 & 0 \\ 0 & -0.667 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we compute

$$\bar{E}^D = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \text{ and } \bar{F}^D = \bar{F}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{6.11}$$

since $\det \bar{F} = 0.187 \neq 0$. Taking into account that

$$\bar{E}^D \bar{F} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}, \quad \bar{E}\bar{E}^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{6.12}$$

the desired solution for the descriptor fractional system (2.1) with (6.1) has the form

$$x_i = (\bar{E}^D \bar{F})^i \bar{E}^D \bar{E} x_0 + \sum_{k=0}^{i-1} \bar{E}^D (\bar{E}^D \bar{F})^{i-k-1} [\bar{B} u_k - \sum_{j=1}^{k} \bar{E} c_{j+1} x_{k-j}] + (\bar{E}\bar{E}^D - I_n) \sum_{k=0}^{q-1} (\bar{E}\bar{F}^D)^k \bar{F}^D \bar{B} u_{i+k} \tag{6.13}$$

where the coefficients c_j are defined by (2.2b). From (6.13) for $i = 0$ we have

$$x_0 = \bar{E}^D \bar{E} x_0 + (\bar{E}\bar{E}^D - I_3)\bar{F}^D \bar{B} u_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u_0. \tag{6.14}$$

Hence, for given $u_0 = u = 1$, the initial condition $x_0 = [1 \quad 2 \quad -2]^T$ satisfy (6.14) and their are admissible.

6.3 *Case of Weierstrass-Kronecker Decomposition Method*

In this case the for (6.1) matrices P and Q have the form

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{6.15}$$

and

$$PEQ=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad PAQ=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad PB=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad A_{1\alpha}=\begin{bmatrix}1.5&0\\0&0.5\end{bmatrix},$$
$$(n_1=2, n_2=1). \tag{6.16}$$

The Eqs. (3.5) and (3.6) have the form

$$\bar{x}_{i+1}^{(1)}=\begin{bmatrix}1.5&0\\0&0.5\end{bmatrix}\bar{x}_i^{(1)}+\sum_{k=2}^{i+1}(-1)^{k-1}\binom{0.5}{k}\bar{x}_{i-k+1}^{(1)}, i\in Z_+, \tag{6.17a}$$

$$\bar{x}_i^{(2)}=-B_2u_i=-u_i, i\in Z_+. \tag{6.17b}$$

The solution $\bar{x}_i^{(1)}$ of the Eq. (6.17a) has the form

$$\bar{x}_i^{(1)}=\Phi_i\bar{x}_0^{(1)}+\sum_{k=0}^{i-1}\Phi_{i-k-1}B_1u_k, i\in Z_+, \tag{6.18}$$

where

$$\Phi_0=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \Phi_1=\begin{bmatrix}1.5&0\\0&0.5\end{bmatrix},\quad \Phi_2=\begin{bmatrix}2.125&0\\0&0.125\end{bmatrix},\ldots. \tag{6.19}$$

From (5.3) for $i=0$ we have

$$\bar{x}_0=Q^{-1}x_0=\begin{bmatrix}0&1&0\\1&0&0\\1&0&1\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix},\quad \bar{x}_0^{(1)}=\begin{bmatrix}2\\1\end{bmatrix},\quad \bar{x}_0^{(2)}=0. \tag{6.20}$$

The desired solution of the descriptor fractional system (2.1) with (6.1) is given by

$$x_i=Q\bar{x}_i=\begin{bmatrix}0&1&0\\1&0&0\\0&-1&1\end{bmatrix}\begin{bmatrix}\bar{x}_i^{(1)}\\\bar{x}_i^{(2)}\end{bmatrix}, \tag{6.21}$$

where $\bar{x}_i^{(1)}$ and $\bar{x}_i^{(2)}$ are determined by (6.18) and (6.17b), respectively.

6.4 Comparison of the Results

Using Matlab/Simulink computing environment, the solution for 10 first steps have been calculated and shown on the Figs. 1, 2 and 3, where Fig. 1 represent solution

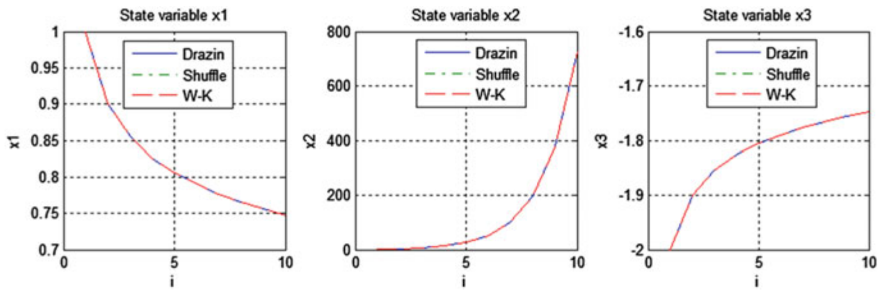

Fig. 3 Solution for $\alpha = 0.9$

for order α = 0.1, Fig. 2 represent solution for order α = 0.5 and Fig. 3 represent solution for order α = 0.9. Additionally *solid line (blue)* represent solution obtained by Drazin inverse method, *dash-dot line (green)* represent solution obtained by Shuffle algorithm method and *dash-dash line (red)* represent solution obtained by Weierstrass-Kronecker decomposition method.

All three methods gives coherent result. Smaller order α, results in faster response stabilization (see state variable x_1, x_3). The greatest disadvantage of the Weierstrass-Kronecker decomposition method is its first step, that is decomposition, which is difficult for numeric implementation. Similar problem occurs in Shuffle algorithm method, where elementary row and column operation need to be applied. Finally, the Drazin inverse method, where most difficult part is computation of the Drazin inverse of the matrix E. In author opinion, this method suits best for numerical implementation, since computation of the Drazin inverse is easy for numerical implementation.

7 Concluding Remarks

The descriptor fractional discrete-time linear systems have been recalled. Three different methods for finding the solution to the state equation of the descriptor fractional discrete-time linear system have been considered. Comparison of computation efforts of the methods has been demonstrated on single numerical example. Iterative approach have been used to compute the desired solution of the systems.

In Drazin inverse method admissible initial conditions should be applied. In Shuffle algorithm method admissible initial conditions as well as future inputs should be known. The weak point of Weierstrass-Kronecker decomposition approach is computation of the P and Q matrices, where elementary row and column operations method is recommended. The same method is used for Shuffle algorithm. In summary, the Drazin inverse method seems to be most suitable for numerical implementation. An open problem is extension of these considerations to the system with different fractional orders.

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