Chapter 1
Illustration of Key Concepts in Dimension $d = 1$

Abstract This introductory chapter presents and illustrates many of the key concepts developed in this work on a simple example, namely the Su-Schriefer-Heeger model [205] of a conducting polymer. This model has a chiral symmetry and non-trivial topology, given by a non-commutative winding number which is remarkably stable against perturbations like a random potential [139]. Hence this is a relatively simple example of a topological insulator. Here the focus is on the bulk-boundary correspondence in this model, which connects the winding number to the number of edge states weighted by their chirality. This connection will be explained in a K-theoretic manner. These arguments constitute a rather mathematical introduction to the bulk-edge correspondence and the physical motivations and insights will be given in the following chapters.

1.1 Periodic Hamiltonian and Its Topological Invariant

As a general rule, the topology in topological insulators is always inherited from periodic models and this topology can be shown in many instances to be stable under perturbations which also break the periodicity. It is therefore instructive to start out with a detailed analysis of the periodic models and to identify their topological invariants. The one-dimensional periodic Hamiltonian H considered here acts on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$ and is given by

$$H \;=\; \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \otimes S + \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \otimes S^* + m\,\sigma_2 \otimes \mathbf{1}_N \otimes \mathbf{1}\;, \qquad (1.1)$$

where $\mathbf{1}_N$ and $\mathbf{1}$ are the identity operators on $\mathbb{C}^N$ and $\ell^2(\mathbb{Z})$ and the 2×2 Pauli matrices are

$$\sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 \;=\; \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and S is the right shift on $\ell^2(\mathbb{Z})$ while $m \in \mathbb{R}$ is the mass term. The component $\mathbb{C}^2 \otimes \mathbb{C}^N$ of the Hilbert space will be referred to as the fiber. This Hamiltonian goes

E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Mathematical Physics Studies,
DOI 10.1007/978-3-319-29351-6_1

back to Su et al. [205] and its physical origin will be discussed in Sect. 2.3.2. It has a chiral symmetry w.r.t. the real unitary $J = \sigma_3 \otimes \mathbf{1}_N \otimes \mathbf{1}$ squaring to the identity

$$J^* H J \;=\; -H \;. \tag{1.2}$$

The Fermi level μ is always assumed positioned at 0 for chiral symmetric systems, see Chap. 2. Note that a model with chiral symmetry can display a spectral gap at $\mu = 0$ only if the fiber has even dimension, which is obviously the case here.

The discrete Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ defined by

$$(\mathcal{F}\phi)(k) \;=\; (2\pi)^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}} \phi_x \, e^{-\mathrm{i}\langle x|k\rangle} \;,$$

partially diagonalizes the Hamiltonian to $\mathcal{F}H\mathcal{F}^* = \int_{\mathbb{S}^1}^{\oplus} dk \; H_k$ with

$$H_k \;=\; \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \, e^{-\mathrm{i}k} \;+\; \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \, e^{\mathrm{i}k} \;+\; m\,\sigma_2 \otimes \mathbf{1}_N$$

or

$$H_k \;=\; \begin{pmatrix} 0 & e^{-\mathrm{i}k} - \mathrm{i}m \\ e^{\mathrm{i}k} + \mathrm{i}m & 0 \end{pmatrix} \otimes \mathbf{1}_N \;.$$

Also the chiral symmetry operator diagonalizes $\mathcal{F}J\mathcal{F}^* = \int_{\mathbb{S}^1}^{\oplus} dk \; J_k$, even with constant fibers $J_k = \sigma_3 \otimes \mathbf{1}_N$. The two eigenvalues of H_k are

$$E_{\pm}(k) \;=\; \pm\sqrt{m^2 + 1 - 2m\sin(k)} \;,$$

and both are N-fold degenerate. Their symmetry around 0 reflects the chiral symmetry $J_k H_k J_k = -H_k$ which, as for any Hamiltonian with chiral symmetry, implies $\sigma(H_k) = -\sigma(H_k)$. The central gap around 0 is $\Delta = [-E_g, E_g]$ with $E_g = \big||m| - 1\big|$. Hence it is open as long as $m \notin \{-1, 1\}$. Let us also note that for $m = 0$, one has $E_{\pm}(k) = \pm 1$ for all k, namely the two bands are flat. In fact, one readily checks that the eigenfunctions of H are supported on two neighboring sites each.

In the mean-field approximation, which will be assumed throughout, the electron ground state is encoded in the Fermi projection $P_F = \chi(H \leq \mu)$ and we recall that in the chiral symmetric models one fixes $\mu = 0$ to ensure the charge neutrality of the system. Since we are in dimension one, this projection cannot be used to define a topological invariant (other then the electron density), and we should rather look for a unitary operator. Note that $JP_FJ = \mathbf{1} - P_F$ and therefore the so-called flat band Hamiltonian

$$Q \;=\; \mathbf{1} - 2P_F \;=\; \mathrm{sgn}(H)$$

satisfies again $J^*QJ = -Q$. It also satisfies $Q^2 = \mathbf{1}$, hence its spectrum consists of only two eigenvalues, 1 and -1, which are both infinitely degenerate. The chiral symmetry combined with $Q^2 = \mathbf{1}$ implies the existence of a unitary U_F on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z})$ such that

$$Q \;=\; \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix} \;. \tag{1.3}$$

In analogy with the Fermi projection, this unitary operator U_F will be called the Fermi unitary operator. The existence of the Fermi unitary operator is a generic characteristic of chiral symmetric gapped Hamiltonians. Note that U_F can be constructed entirely from the electron ground state and, reciprocally, the electron ground state can be reconstructed entirely from U_F. Also, note that in the physics literature and in our previous work [171] U_F and U_F^* are interchanged. The choice in (1.3) will prove more convenient here, especially when computing the index map, see below.

For the Hamiltonian (2.24), one readily calculates $\mathcal{F}Q\mathcal{F}^* = \int_{\mathbb{S}^1}^{\oplus} dk\, Q_k$, with

$$Q_k \;=\; \begin{pmatrix} 0 & \frac{e^{-\imath k}+\imath m}{|e^{-\imath k}+\imath m|} \\ \frac{e^{\imath k}+\imath m}{|e^{\imath k}+\imath m|} & 0 \end{pmatrix} \otimes \mathbf{1}_N \;.$$

In general, every flat band Hamiltonian of a periodic chiral Hamiltonian with open central gap is fibered as

$$Q_k \;=\; \begin{pmatrix} 0 & U_k^* \\ U_k & 0 \end{pmatrix} \;,$$

with some unitary matrix $U_k \in M_N(\mathbb{C})$ acting on $\mathbb{C}^N$ which is supposed to be differentiable in k. It is now natural to consider the winding number associated to the Fermi unitary operator, which for reasons explained further below will be called the first odd Chern number:

$$\mathrm{Ch}_1(U_F) \;=\; \imath \int_{\mathbb{S}^1} \frac{dk}{2\pi}\, \mathrm{tr}\big(U_k^* \partial_k U_k\big) \;. \tag{1.4}$$

For the Hamiltonian (2.24) one finds

$$\mathrm{Ch}_1(U_F) \;=\; \begin{cases} -N \,, & m \in (-1,1) \,, \\ 0 \,, & m \notin [-1,1] \,. \end{cases}$$

This integer $\mathrm{Ch}_1(U_F)$ is the bulk invariant associated to the ground state of Hamiltonian (1.1). The term *invariant* reflects the fact that $\mathrm{Ch}_1(U_F)$ does not change for sufficiently small perturbations of the Hamiltonian, even though U_F itself does change. In particular, the following perturbations are of interest:

(i) Next nearest hopping terms.
(ii) A random potential or random hopping elements.
(iii) Terms breaking the chiral symmetry (1.2).

The perturbations (i) and (iii) can be dealt with in the framework of periodic operators where a Bloch Floquet transform is applicable. If the chiral symmetry is broken, then the flat-band Hamiltonian is not described as in (1.3) by a unitary anymore, but it may still have invertible off-diagonal entries of which a winding number is well-defined as well. For the random perturbations in (ii) one is forced out of the realm of Bloch theory. One of the main points to be developed further down is to show how this can be accomplished. Of course, another question addressed is to find the adequate replacement for $\mathrm{Ch}_1(U_F)$ for higher dimensions.

1.2 Edge States and Bulk-Boundary Correspondence

In this section, an edge or boundary for the one-dimensional periodic Hamiltonian (1.1) is introduced. This can be achieved by simply restricting (1.1) to the half-space Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{N})$, e.g. by imposing the Dirichlet boundary condition

$$\widehat{H} \;=\; \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \otimes \widehat{S} \;+\; \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) \otimes \mathbf{1}_N \otimes \widehat{S}^* \;+\; m\,\sigma_2 \otimes \mathbf{1}_N \otimes \mathbf{1}\,.$$

All half-space operators will carry a hat from now on. For example, $\widehat{S}$ above is the unilateral right shift on $\ell^2(\mathbb{N})$ and there is the half-space chirality operator $\widehat{J} = \sigma_3 \otimes \mathbf{1}_N \otimes \mathbf{1}$. The half-space Hamiltonian still has the chiral symmetry $\widehat{J}\,\widehat{H}\,\widehat{J} = -\widehat{H}$. Again the chiral symmetry implies that the spectrum satisfies $\sigma(\widehat{H}) = -\sigma(\widehat{H})$. Furthermore, the direct sum of two copies of $\widehat{H}$ is a finite dimensional perturbation of H. Hence the essential spectra coincide $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(\widehat{H})$, but $\widehat{H}$ may have additional point spectrum, corresponding to the edge states which are also called bound or boundary states.

Example 1.2.1 Let us consider the Hamiltonian $\widehat{H}$ for $m = 0$. It takes the form

$$\widehat{H} \;=\; \begin{pmatrix} 0 & \mathbf{1}_N \otimes \widehat{S} \\ \mathbf{1}_N \otimes \widehat{S}^* & 0 \end{pmatrix}.$$

The spectrum is now $\sigma(\widehat{H}) = \{-1, 0, 1\}$ with infinitely degenerate eigenvalues ± 1 having compactly supported eigenstates on two neighboring sites, and a kernel of multiplicity N containing vectors supported in the upper entry over the boundary site 0. They result from the fact that $|0\rangle \in \ell^2(\mathbb{N})$ lies in the kernel of the unilateral left shift $\widehat{S}^*$. For $N = 1$, this zero mode is simple and perturbations of the Hamiltonian $\widehat{H}$ within the class of half-sided chiral Hamiltonians cannot remove it since the symmetry of the spectrum has to be conserved and a simple eigenvalue cannot split into two by perturbation theory. The same stability actually holds for $N > 1$ because the signature of $\widehat{J}$ on the kernel is N and also this signature is conserved during a homotopy of chiral Hamiltonians. Note also that the signature is equal to $N = -\mathrm{Ch}_1(U_F)$. Due to the stability of both quantities, the equality $\mathrm{Ch}_1(U_F) = -\mathrm{Sig}(\widehat{J}|_{\mathrm{Ker}(\widehat{H})})$ holds also in a neighborhood of the Hamiltonian $\widehat{H}$ with $m = 0$. ◇

Now let us go on with a more structural analysis of the edge states which is not as tightly linked to the special model under consideration. Suppose that $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{N})$ is such a normalized bound state with energy E, namely $\widehat{H}\psi = E\psi$. Then $\widehat{H}\widehat{J}\psi = -E\widehat{J}\psi$, which implies that the span $\mathcal{E}$ of all eigenvectors with eigenvalues in $[-\delta, \delta] \subset \Delta$ is invariant under J. Therefore $\widehat{J}$ can be diagonalized on $\mathcal{E}$ leading to a splitting $\mathcal{E} = \mathcal{E}_+ \otimes \mathcal{E}_-$ such that $\widehat{J}$ is ± 1 on $\mathcal{E}_\pm$. Accordingly, the spectral projection $\widetilde{P}(\delta) = \chi(|\widehat{H}| \leq \delta)$ can be decomposed into an orthogonal sum $\widetilde{P}(\delta) = \widetilde{P}_+(\delta) + \widetilde{P}_-(\delta)$ and $\widehat{J}\,\widetilde{P}(\delta) = \widetilde{P}_+(\delta) - \widetilde{P}_-(\delta)$. The difference of the dimensions of $\mathcal{E}_\pm$ spaces is the boundary invariant of the system

$$\mathrm{Tr}(\widehat{J}\,\widetilde{P}(\delta)) \;=\; N_+ - N_- \,, \qquad N_\pm \;=\; \dim(\mathcal{E}_\pm) \,.$$

This invariant is also equal to the signature of $\widehat{J}|_{\mathcal{E}}$ and such signatures are again well-known to be homotopy invariants, as already pointed out in the example above. The invariant is independent of the choice of $\delta > 0$ as long as δ lies in the gap of H, hence its value must be determined entirely by the spectral subspace of the zero eigenvalue, known also as the space of the zero modes. Zero modes in $\mathcal{E}_+$ and $\mathcal{E}_-$ are said to have positive and negative chirality, respectively. The following result now connects the bulk invariant $\mathrm{Ch}_1(U_F)$ to the boundary invariant $\mathrm{Tr}(\widehat{J}\,\widetilde{P}(\delta))$.

Theorem 1.2.2 *Consider the Hamiltonian H on $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$ given by* (1.1) *and let $\widehat{H}$ be its half-space restriction. If U_F is the Fermi unitary operator defined via* (1.3) *and if its winding number is defined by* (1.4)*, then the bulk-edge correspondence in the following form holds*

$$\mathrm{Ch}_1(U_F) \;=\; -\,\mathrm{Tr}(\widehat{J}\,\widetilde{P}(\delta)) \,. \tag{1.5}$$

This result can be proved by various means (see the above example and [64, 65], but likely there are other references). However, in the following, a detailed K-theoretic proof will be provided. Such a structural argument stresses the robust nature of the above equality. In particular, stability under the perturbations listed at the end of Sect. 1.1 will be covered. Furthermore, it will be possible to extend the structural argument to higher dimensional systems.

1.3 Why Use K-Theory?

There have been numerous works that use K-theory for topological condensed matter systems. Pioneering were the papers by Bellissard on the integer quantum Hall effect [17, 18], which were reviewed and extended to the regime of dynamical Anderson localization in [20]. K-theory can be used to obtain gap labelling [17]. Starting with the Kitaev's paper [115], K-theory and KR-theory (which is K-theory in presence of symmetries) were more recently used as a tool to classify topological insulators [54, 68, 111, 143, 203, 207] or define topological invariants in the absence of periodicity [85, 134]. Here the main objective is a different one:

- Use the connecting maps of K-theory to relate different invariants.

This was first achieved in [107, 109, 197] for integer quantum Hall systems, where the equality of bulk and edge Hall conductivity was proved using the exponential map of K-theory. There are other connecting maps in K-theory though, in particular the index map, the suspension map and the Bott periodicity map. In this work it will be shown how they can be put to work as well and produce interesting identities. In this introductory section on the one-dimensional Su-Schrieffer-Heeger model, the K-theoretic index map of the so-called Toeplitz extension will be used to prove Theorem 1.2.2. Along the lines, quite a few things about K-theory will said and used without proof. These are all standard facts that are well-known in the mathematics community and can be found in the introductory books on K-theory [187, 222] or the more advanced textbook [28], but for the convenience of the reader they will be briefly reviewed in Sect. 4.1 of Chap. 4.

The Toeplitz extension is at the very heart of K-theory. The reader familiar with all this can jump directly to Proposition 1.3.1. The Toeplitz extension is the following short exact sequence of C*-algebras:

$$0 \longrightarrow \mathcal{K} \overset{i}{\longrightarrow} T\big(C(\mathbb{S}^1)\big) \cong C^*(\widehat{S}) \overset{\text{ev}}{\to} C(\mathbb{S}^1) \cong C^*(S) \longrightarrow 0 \tag{1.6}$$

Here $\mathcal{K}$ denotes the algebra of compact operators on $\ell^2(\mathbb{N})$, $C(\mathbb{S}^1)$ is the algebra of continuous functions over the unit circle which, by the discrete Fourier transform, is isomorphic with the algebra generated by the shift operator S on $\ell^2(\mathbb{Z})$, and $T(C(\mathbb{S}^1))$ is the algebra of Toeplitz operators. The latter can be presented as the C*-algebra of operators on $\ell^2(\mathbb{N})$ which can be approximated in operator norm by polynomials in $\widehat{S}$ and $\widehat{S}^*$, that is, by finite sums

$$\sum_{n,m\geq 0} a_{n,m}(\widehat{S})^n(\widehat{S}^*)^m \;.$$

Since

$$\widehat{S}^*\widehat{S} \;=\; \mathbf{1} \quad \text{and} \quad \widehat{S}\,\widehat{S}^* \;=\; \mathbf{1} - \widetilde{P}\;, \tag{1.7}$$

where $\widetilde{P} = |0\rangle\langle 0|$ is the one-dimensional projection on the state $|0\rangle \in \ell^2(\mathbb{N})$ at the boundary, the operators from $T(C(\mathbb{S}^1))$ can be uniquely expressed as:

$$\sum_{n\geq 0} a_n(\widehat{S})^n \;+\; \sum_{n<0} a_n(\widehat{S}^*)^{-n} \;+\; \sum_{n,m\geq 0} c_{n,m}(\widehat{S})^n\widetilde{P}(\widehat{S}^*)^m \;. \tag{1.8}$$

One can now see explicitly the connection between the Toeplitz operators and the half-line observables. Indeed, the first two terms in (1.8) represent the restriction of the bulk operator $\sum_{n\in\mathbb{Z}} a_n S^n$ to the half-line via the Dirichlet boundary condition, while the third term redefines the boundary condition. The latter is just a compact operator on $\ell^2(\mathbb{N})$, hence $\mathcal{K}$ is a sub-algebra of $T(C(\mathbb{S}^1))$ and i in (1.6) denotes the associated inclusion map. The second morphism in (1.6) is defined by $\mathrm{ev}(\widehat{S}) = e^{-\mathrm{i}k}$ and $\mathrm{ev}(\widehat{S}^*) = e^{\mathrm{i}k}$, or equivalently $\mathrm{ev}(\widehat{S}) = S$ and $\mathrm{ev}(\widehat{S}^*) = S^*$. Since $\widetilde{P} = \widehat{S}^*\widehat{S} - \widehat{S}\widehat{S}^*$, one has $\mathrm{ev}(\widetilde{P}) = 0$ which means that the compact operators are sent to zero by the second morphism. As a consequence, the sequence (1.6) is exact, namely the image of each of the three maps is equal to the kernel of the following map.

All the operators appearing above lie in matrix algebras over one of the algebras in the Toeplitz extension (1.6). Indeed, the Hamiltonian H as well as P_F and Q belong to the algebra $M_{2N}(C(\mathbb{S}^1)) \cong \mathbb{C}^{2N\times 2N} \otimes C(\mathbb{S}^1)$ of $2N \times 2N$ matrices with coefficients in $C(\mathbb{S}^1)$, and the half-line Hamiltonian $\widehat{H}$ is an element of $M_{2N}(T(C(\mathbb{S}^1)))$. Actually, $\widehat{H}$ is a so-called lift of H, namely, one has $\mathrm{ev}(\widehat{H}) = H$. The Fourier transform of the Fermi unitary operator U_F lies in $M_N(C(\mathbb{S}^1))$. Finally, the finite dimensional projections $\widetilde{P}(\delta)$ and $\widetilde{P}_\pm(\delta)$ are projections in $M_{2N}(\mathcal{K})$.

Warning: *From here on, K-theoretic concepts will be used and only explained on an intuitive level. Details are found in Chap. 4.*

The proof of Theorem 1.2.2 will show how the equality (1.5) results from a K-theoretic index theorem associated to the Toeplitz extension. The definitions of K-groups and of the index map are recalled in Sect. 4.1. Roughly stated, for each C*-algebra $\mathcal{A}$ there exist two groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ given by homotopy classes of projections and unitaries, respectively, in the matrix algebras over $\mathcal{A}$. The group operation in $K_0(\mathcal{A})$ is given by the direct sum of projections, while in $K_1(\mathcal{A})$ by the multiplication of unitaries. The K-groups of all algebras in the Toeplitz extension (1.6) are well-known: $K_0(\mathcal{K}) \cong \mathbb{Z}$ generated by the rank one projection $\widetilde{P} = |0\rangle\langle 0|$, $K_0(T(C(\mathbb{S}^1))) \cong \mathbb{Z}$ and $K_0(C(\mathbb{S}^1)) \cong \mathbb{Z}$ both generated by the identity, $K_1(\mathcal{K}) = 0$ and $K_1(T(C(\mathbb{S}^1))) = 0$, and finally $K_1(C(\mathbb{S}^1)) \cong \mathbb{Z}$ generated by e^{-ik} (or S) which is a function with unit winding number. The elements $K_1(C(\mathbb{S}^1))$ can be uniquely labeled by their winding number, namely the first odd Chern number. It is also worth pointing out that the class $[\widetilde{P}]_0$ in $K_0(T(C(\mathbb{S}^1)))$ is trivial because the isometry $\widehat{S}$ satisfies $\widehat{S}^*\widehat{S} = \mathbf{1}$ and $\widehat{S}\widehat{S}^* = \mathbf{1} - \widetilde{P}$. Hence $\mathbf{1}$ and $\mathbf{1} - \widetilde{P}$ are Murray-von Neumann equivalent and are therefore in the same K_0-class, to that $[\widetilde{P}]_0 = [\mathbf{1}]_0 - [\mathbf{1} - \widetilde{P}]_0 = 0$. On the other hand, in $K_0(\mathcal{K})$ the projection $\widetilde{P}$ defines a non-trivial class which is actually the generator of $K_0(\mathcal{K})$.

The central result of K-theory used for the bulk-boundary correspondence is that, for every exact sequence of C*-algebras, there is a 6-term exact sequence of the 6 associated K-groups. For the Toeplitz extension, this sequence is

$$\begin{array}{ccccc} K_0(\mathcal{K}) = \mathbb{Z} & \xrightarrow{i_*} & K_0(T(C(\mathbb{S}^1))) = \mathbb{Z} & \xrightarrow{\mathrm{ev}_*} & K_0(C(\mathbb{S}^1)) = \mathbb{Z} \\ \mathrm{Ind}\uparrow & & & & \downarrow\mathrm{Exp} \\ K_1(C(\mathbb{S}^1)) = \mathbb{Z} & \xleftarrow{\mathrm{ev}_*} & K_1(T(C(\mathbb{S}^1))) = 0 & \xleftarrow{i_*} & K_1(\mathcal{K}) = 0 \end{array} \tag{1.9}$$

Here the maps i_* and ev_* are push-forward maps naturally induced by the maps in (1.6). Interesting are the so-called boundary maps Exp and Ind. The exponential map Exp has to be trivial for the Toeplitz extension as $K_1(\mathcal{K}) = 0$. Focus will therefore be on the index map Ind, which has to be an isomorphism. First of all, note that it maps classes of unitaries from the bulk algebra $C(\mathbb{S}^1)$ to projections in the boundary algebra $\mathcal{K}$. Hence it establishes a link between the topology of the bulk and the boundary, which is precisely what we are looking for. Let us first recall the general definition of the index map (as already pointed out, more details and a more stringent formulation using utilizations are given in Sect. 4.1) and then evaluate it explicitly. Given a class $[U]_1 \in K_1(C(\mathbb{S}^1))$ associated to a unitary $U \in M_N(C(\mathbb{S}^1))$, one first constructs a unitary lift

$$\widehat{W} \;=\; \text{Lift}\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \;\in\; M_{2N}(T(C(\mathbb{S}^1)))\;,$$

which is by definition a unitary satisfying $\text{ev}(\widehat{W}) = \text{diag}(U, U^*)$, and then defines

$$\text{Ind}([U]_1) \;=\; \left[\widehat{W}\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix}\widehat{W}^*\right]_0 - \left[\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix}\right]_0 . \tag{1.10}$$

In general, it can be shown that the lift exists and that the r.h.s. of (1.10) really specifies an element in $K_0(\mathcal{K})$ and not in $K_0(T(C(\mathbb{S}^1)))$, as one may think at first sight.

Let us first calculate $\text{Ind}([S^n]_1)$ for the bilateral left shift S^n by n sites. These unitaries generate $K_1(C(\mathbb{S}^1)) = \{[S^n]_1 \,|\, n \in \mathbb{Z}\}$. A unitary lift for $n \geq 0$ is

$$\text{Lift}\begin{pmatrix} S^n & 0 \\ 0 & (S^n)^* \end{pmatrix} \;=\; \begin{pmatrix} \widehat{S}^n & \widetilde{P}_n \\ 0 & (\widehat{S}^n)^* \end{pmatrix}\;,$$

where as above $\widehat{S}$ is the unilateral right shift and $\widetilde{P}_n = \sum_{k=1}^{n} |k\rangle\langle k|$ is the projection on the n states localized at the boundary of $\ell^2(\mathbb{N})$. Hence $\widehat{S}^n(\widehat{S}^*)^n = \mathbf{1} - \widetilde{P}_n$ and $\widetilde{P}_n\widehat{S}^n = 0$. Evaluating (1.10) now shows

$$\text{Ind}([S^n]_1) \;=\; \left[\begin{pmatrix} \widehat{S}^n(\widehat{S}^*)^n & 0 \\ 0 & 0 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \;=\; -[\widetilde{P}_n]_0\;, \tag{1.11}$$

which is the explicit form of the isomorphism between $K_1(C(\mathbb{S}^1))$ and $K_0(\mathcal{K})$. This concludes our description of the K-theory associated to the Toeplitz extension (1.6).

Now let us come to the application to the model (2.24). First of all, the Fermi unitary U_F in (1.3) defines a class in $K_1(C(\mathbb{S}^1))$, and the finite dimensional projections $\widetilde{P}(\delta)$ and $\widetilde{P}_\pm(\delta)$ specify classes in $K_0(\mathcal{K})$. Hence they lie in the l.h.s. of the six term exact sequence (1.9) for the Toeplitz extension (1.6) and they are connected via the index map. In fact, the following holds.

Proposition 1.3.1 *Let $U_F \in M_N(C(\mathbb{S}^1))$ be given by (1.3). Further let us choose an odd and non-decreasing smooth function $f_{\rm Ind} : \mathbb{R} \to [-1, 1]$ such that $f_{\rm Ind}(E) = -1$ for $E \le -E_g$ and $f_{\rm Ind}(E) = 1$ for $E \ge E_g$. Then*

$$\mathrm{Ind}([U_F]_1) \;=\; \left[e^{-\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix} e^{\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \right]_0 \;-\; \left[\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix} \right]_0 . \qquad (1.12)$$

Proof For the evaluation of the index map (1.10) one needs the lift

$$\widehat{W} \;=\; \mathrm{Lift}\begin{pmatrix} U_F & 0 \\ 0 & U_F^* \end{pmatrix} \;=\; \mathrm{Lift}\left(\begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix} \right) \;=\; \begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} \mathrm{Lift}(Q) \, .$$

Now recall that $Q = \mathrm{sgn}(H)$ is a self-adjoint unitary that will now be expressed as a smooth function of H with values on the unit circle. Actually, with the function $f_{\rm Ind}$ defined in the proposition, one has $Q = \mathrm{i} e^{-\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(H)}$. Hence a lift is given by

$$\mathrm{Lift}(Q) \;=\; \mathrm{i}\, e^{-\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \, .$$

As it is obtained by smooth functional calculus from $\widehat{H}$, it follows that $\mathrm{Lift}(Q) \in M_{2N}(T(C(\mathbb{S}^1)))$ as required. We arrived at

$$\widehat{W} \;=\; \mathrm{i} \begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} e^{-\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \, .$$

Plugging into the definition (1.10) of the index map

$$\mathrm{Ind}([U_F]_1) = \left[\begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} e^{-\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix} e^{\mathrm{i}\frac{\pi}{2} f_{\rm Ind}(\widehat{H})} \begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} \right]_0 - \left[\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix} \right]_0 ,$$

and the projection appearing in the first term is homotopic to the projection appearing in the statement. □

The previous argument did not require the presence of any spectral gaps in the spectrum of $\widehat{H}$ and will therefore also apply to higher dimensional models, see Proposition 4.3.2. In presence of spectral gaps, however, one can further refine the argument.

Proposition 1.3.2 *Let $U_F \in M_N(C(\mathbb{S}^1))$ be given by (1.3). Then for $0 < \delta < E_g$*

$$\mathrm{Ind}([U_F]_0) \;=\; [\widetilde{P}_+(\delta)]_0 \;-\; [\widetilde{P}_-(\delta)]_0 \, . \qquad (1.13)$$

Proof Let $f_{\rm Ind}$ be as in Proposition 1.3.1 and, moreover, let it be such that $f_{\rm Ind}(E) \in \{-1, 0, 1\}$ for any $E \in \sigma(\widehat{H})$. For sake of concreteness, suppose $f_{\rm Ind}(E) = 0$ only for $E = 0$ and no other $E \in \sigma(\widehat{H})$. Recall that, in dimension $d = 1$, the spectrum of $\widehat{H}$

is discrete inside $[-E_g, E_g]$. Now,

$$e^{-\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\,\mathrm{diag}(\mathbf{1}_N, 0_N)\,e^{\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})} \;=\; e^{-\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\,\tfrac{1}{2}\,(\widehat{J}+\mathbf{1}_{2N})\,e^{\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\;.$$

The chiral symmetry of $\widehat{H}$ combined with $f_{\rm Ind}(-E) = -f_{\rm Ind}(E)$, for $E \in \sigma(\widehat{H})$, implies

$$e^{-\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\,\widehat{J} \;=\; \widehat{J}\,e^{\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\;,$$

so that

$$e^{-\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})}\,\tfrac{1}{2}\,(\widehat{J}+\mathbf{1}_{2N})\,e^{\imath\frac{\pi}{2}f_{\rm Ind}(\widehat{H})} \;=\; \tfrac{1}{2}\,\widehat{J}(e^{\imath\pi f_{\rm Ind}(\widehat{H})}+\mathbf{1}_{2N}) + \mathrm{diag}(0_N, \mathbf{1}_N).$$

With the choice made for $f_{\rm Ind}$ one has $e^{\imath\pi f_{\rm Ind}(\widehat{H})} + \mathbf{1}_{2N} = 2\widetilde{P}(\delta)$, so that

$$\tfrac{1}{2}\,\widehat{J}\,(e^{\imath\pi f_{\rm Ind}(\widehat{H})} + \mathbf{1}_{2N}) \;=\; \widehat{J}\,\widetilde{P}(\delta) \;=\; \widetilde{P}_+(\delta) \;-\; \widetilde{P}_-(\delta).$$

Then, by noticing that $\widetilde{P}_+(\delta)$ and $\mathrm{diag}(0_N, \mathbf{1}_N) - \widetilde{P}_-(\delta)$ are orthogonal projections and that $\mathrm{diag}(\mathbf{1}_N, 0_N)$ and $\mathrm{diag}(0_N, \mathbf{1}_N)$ are homotopic,

$$\begin{aligned}\mathrm{Ind}([U_F]_1) \;&=\; [\widetilde{P}_+(\delta) \;+\; \mathrm{diag}(0_N, \mathbf{1}_N) \;-\; \widetilde{P}_-(\delta)]_0 \;-\; [\mathrm{diag}(\mathbf{1}_N, 0_N)]_0\\ &=\; [\widetilde{P}_+(\delta)] \;+\; [\mathrm{diag}(0_N, \mathbf{1}_N) \;-\; \widetilde{P}_-(\delta)]_0 \;-\; [\mathrm{diag}(0_N, \mathbf{1}_N)]_0\;.\end{aligned}$$

The statement now follows from the rule 3. of the standard characterization of the K_0 group, listed in Sect. 4.1.1. □

1.4 Why Use Non-commutative Geometry?

Theorem 1.2.2 results by extracting a numerical identity from the K-theoretic identity (1.13). This is done via a pairing of the K-groups with adequate cohomology theory, which is the cyclic cohomology developed by Connes since the early 1980s [46, 47]. This was at the heart of the early developments of non-commutative geometry. Actually, it could also be referred to as non-commutative differential topology as topological invariants are calculated by tools of non-commutative differential and integral calculus. In the simple framework of periodic models, the relevant pairings of K-theory with cyclic cohomology are established by the two maps

$$\widetilde{\mathrm{Ch}}_0 \;:\; K_0(\mathcal{K}) \to \mathbb{Z}\;, \qquad \widetilde{\mathrm{Ch}}_0([\widetilde{P}]_0 - [\widetilde{P}']_0) \;=\; \mathrm{Tr}(\widetilde{P}) - \mathrm{Tr}(\widetilde{P}')\;, \tag{1.14}$$

$$\mathrm{Ch}_1 \;:\; K_1(C(\mathbb{S}^1)) \to \mathbb{Z}\;, \qquad \mathrm{Ch}_1([U]_1) \;=\; \imath\int_{\mathbb{S}^1}\frac{dk}{2\pi}\,\mathrm{tr}\big(U(k)^*\partial_k U(k)\big)\;, \tag{1.15}$$

where in the second line it is supposed that $k \mapsto U(k)$ is differentiable. Any continuous path $k \mapsto U(k)$ can be approximated by a differentiable one, which means that any K-theory class in $K_1(C(\mathbb{S}^1))$ has differentiable representatives simply because the smooth functions $C^\infty(\mathbb{S}^1)$ are dense in $C(\mathbb{S}^1)$. Such arguments are always needed in differential topology, and also in non-commutative differential topology, where it is necessary to work with dense subalgebras (of smooth elements) of C*-algebras. This issue will be discussed in detail in Sect. 3.3.3. The term pairing expresses the fact that $\widetilde{\mathrm{Ch}}_0([\widetilde{P}]_0)$ and $\mathrm{Ch}_1([U]_1)$ do not depend on the choice of the representative of the two classes. The following result now connects the two pairings.

Proposition 1.4.1 *The maps* $\widetilde{\mathrm{Ch}}_0$ *and* Ch_1 *are well-defined group homomorphisms into the additive group* $\mathbb{Z}$*, and*

$$\mathrm{Ch}_1([U]_1) \;=\; -\,\widetilde{\mathrm{Ch}}_0(\mathrm{Ind}([U]_1))\;. \tag{1.16}$$

Proof Neither of the pairings depends on the representatives, namely, norm continuous paths of projections and unitaries, respectively, have constant pairings. Furthermore, $\widetilde{\mathrm{Ch}}_0([\widetilde{P}]_0 + [\widetilde{P}']_0) = \widetilde{\mathrm{Ch}}_0([\widetilde{P}]_0) + \widetilde{\mathrm{Ch}}_0([\widetilde{P}']_0)$ holds by definition and elementary properties of the winding number imply $\mathrm{Ch}_1([UU']_1) = \mathrm{Ch}_1([U]_1) + \mathrm{Ch}_1([U']_1)$. Finally the equality (1.16) follows once it is verified for every class. But

$$\mathrm{Ch}_1([S^n]_1) \;=\; n \;=\; \mathrm{Tr}(\widetilde{P}_n) \;=\; \widetilde{\mathrm{Ch}}_0([\widetilde{P}_n]) \;=\; -\,\widetilde{\mathrm{Ch}}_0(\mathrm{Ind}([S^n]_1))\;,$$

where in the last equality (1.11) was used. Actually, it would have been sufficient to check the above equality for the (sole) generator $n = 1$. □

Proof of Theorem 1.2.2. This follows by combining Propositions 1.3.1 and 1.4.1. □

1.5 Disordered Hamiltonian

The next step is to add a random perturbation to the Hamiltonian (2.24), just as in [139]. Let $\omega'_x, \omega''_x \in [-\frac{1}{2}, \frac{1}{2}]$ be independent and uniformly distributed random variables and define a disorder configuration in the Tychonov space $\Omega = ([-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}])^{\mathbb{Z}}$ by $\omega = (\omega'_x, \omega''_x)_{x\in\mathbb{Z}}$. The probability measure on Ω is just the product measure. The associated Hamiltonian H_ω for two coupling constants $\lambda', \lambda'' \geq 0$ is still acting on $\ell^2(\mathbb{Z}, \mathbb{C}^2 \otimes \mathbb{C}^N)$ and is given by

$$\begin{aligned} H_\omega \;=\; &\sum_{x\in\mathbb{Z}} \tfrac{1}{2}(1 + \lambda'\omega'_x)\big((\sigma_1 + \imath\sigma_2)\,|x\rangle\langle x+1| \;+\; (\sigma_1 - \imath\sigma_2)\,|x+1\rangle\langle x|\big) \\ &+\; m(1 + \lambda''\omega''_x)\,\sigma_2\;|x\rangle\langle x|\;. \end{aligned} \tag{1.17}$$

For $\omega = 0$ or $\lambda' = \lambda'' = 0$, the Hamiltonian H_ω is exactly the same as (2.24). From now on, the letter H will be used for the full family $H = \{H_\omega\}_{\omega\in\Omega}$ of random Hamiltonians. The spectra $\sigma(H_\omega)$ of these operators are known to be almost surely and given by $\sigma(H_\omega) = \sigma(H_{\omega=0}) + [-\lambda', \lambda'] + [-\lambda'', \lambda'']$.

As we have already seen, the periodic model exhibits a non-trivial topological phase and, according to [3, 4, 139, 171, 202], this phase is stable against disorder. This means that the trivial and topological phases continue, in the presence of disorder, to be separated by a sharp phase boundary where a localization-delocalization transition must occur. This phase boundary is characterized by a divergence of the Anderson localization length and it can be mapped using transport experiments. The existence of such sharp phase boundary can be established by an analytical calculation, which we reproduce below from [139]. To simplify notations, let us use $t_x = (1 + \lambda'\omega'_x)$ and $m_x = m(1 + \lambda''\omega''_x)$, in which case the Schrödinger equation at the Fermi level $E = 0$ for (1.17) reads

$$\begin{pmatrix} 0 & t_x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_{x+1,+1} \\ \psi_{x+1,-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t_x & 0 \end{pmatrix}\begin{pmatrix} \psi_{x-1,+1} \\ \psi_{x-1,-1} \end{pmatrix} + \mathrm{i}\begin{pmatrix} 0 & -m_x \\ m_x & 0 \end{pmatrix}\begin{pmatrix} \psi_{x,+1} \\ \psi_{x,-1} \end{pmatrix} = 0\,.$$

On the components, $t_x\psi_{x-\alpha,\alpha} + \mathrm{i}\alpha m_x\psi_{x,\alpha} = 0$, $\alpha = \pm 1$, hence the solution is

$$\psi_{\xi_\alpha+x,\alpha} = \mathrm{i}^x \prod_{j=1}^{x}\left(\frac{t_j}{m_j}\right)^{\alpha} \psi_{\xi_\alpha,\alpha}\,,$$

where $\xi_\alpha = 0, 1$ for $\alpha = \pm 1$, respectively. The inverse of Anderson localization length is given by

$$\Lambda^{-1} = \max_{\alpha=\pm 1}\Big[-\lim_{x\to\infty}\frac{1}{x}\log|\psi_{\xi_\alpha+x,\alpha}|\Big] = \left|\lim_{x\to\infty}\frac{1}{x}\sum_{j=1}^{x}(\ln|t_j| - \ln|m_j|)\right|\,.$$

Using Birkhoff's ergodic theorem [27] on the last expression,

$$\Lambda^{-1} = \left|\int_{-1/2}^{1/2} d\omega' \int_{-1/2}^{1/2} d\omega''\,(\ln|1+\lambda'\omega'| - \ln|m + \lambda''\omega''|)\right|\,.$$

The integrations can be performed explicitly and, in the regime of large λ's where the arguments of the logarithms (inside the absolute values) take negative to positive values as ω's are varied, the result is

$$\Lambda^{-1} = \left|\ln\left[\frac{|2+\lambda'|^{\frac{1}{\lambda'}+\frac{1}{2}}}{|2-\lambda'|^{\frac{1}{\lambda'}-\frac{1}{2}}}\,\frac{|2m-\lambda''|^{\frac{m}{\lambda''}-\frac{1}{2}}}{|2m+\lambda''|^{\frac{m}{\lambda''}+\frac{1}{2}}}\right]\right|\,. \tag{1.18}$$

One can now check that, indeed, the Anderson localization length diverges for certain values of λ' and λ''. A plot of the manifold where this occurs can be found in [139] and there one can see that the topological phase is indeed fully enclosed by this manifold. In other words, the only way to cross from the topological to the trivial phase is to go through a localization-delocalization quantum transition. As we shall see, it is exactly this divergence of the localization length which triggers an abrupt change in the quantized values of the bulk topological invariant.

While the bulk analysis, just by itself, can be carried in the regime of strong disorder, the bulk-boundary correspondence will be established under the following assumption:

Bulk Gap Hypothesis $E_g = \inf \sigma(H_\omega) \cap \mathbb{R}_{\geq 0}$ *is positive, namely* $0 \notin \sigma(H_\omega)$.

Each H_ω still has the chiral symmetry (1.2), that is $JH_\omega J = -H_\omega$, and therefore also the flat band Hamiltonian $Q_\omega = \mathbf{1} - 2P_\omega = \operatorname{sgn}(H_\omega)$ satisfies $JQ_\omega J = -Q_\omega$ and $Q_\omega^2 = \mathbf{1}$. This implies as in (1.3)

$$Q_\omega \;=\; \begin{pmatrix} 0 & U_\omega^* \\ U_\omega & 0 \end{pmatrix} \;, \tag{1.19}$$

with a unitary operator U_ω on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$. The aim in the following is to show that Theorem 1.2.2 remains valid provided that the disorder does not close the gap and the invariant $\mathrm{Ch}_1(U)$ is adequately defined.

Neither of the operators H_ω, U_ω and Q_ω is periodic anymore, but this lack is replaced by the so-called covariance relation, explained next. First of all, on Ω one has an $\mathbb{Z}$-action $\tau : \mathbb{Z} \times \Omega \to \Omega$ given by

$$\omega \;=\; (\omega'_x, \omega''_x)_{x\in\mathbb{Z}} \;\;\mapsto\;\; \tau\omega \;=\; (\omega'_{x-1}, \omega''_{x-1})_{x\in\mathbb{Z}} \;,$$

and with this action one has

$$S\, H_\omega\, S^* \;=\; H_{\tau\omega} \;. \tag{1.20}$$

Similar covariance relation applies to any function of the Hamiltonian (such as Q_ω) or to operators extracted from such functions (such as U_ω).

1.6 Why Use Operator Algebras?

A fruitful point of view [17] is to consider the whole C*-algebra $\mathcal{A}_1$ of one-dimensional covariant operator families on $\ell^2(\mathbb{Z})$, which is constructed as follows. One starts with the set $\mathcal{A}_{1,0}$ of families $a = \{A_\omega\}_{\omega\in\Omega}$ of operators on $\ell^2(\mathbb{Z})$ satisfying the covariance relation $SA_\omega S^* = A_{\tau\omega}$ as well as the finite range condition $\langle x|A_\omega|y\rangle = 0$ for all $|x-y| > C$ for some $C < \infty$. Then $\mathcal{A}_{1,0}$ is a $*$-algebra because the product and adjoint of finite range covariant operator families is again such a family. A C*-norm on $\mathcal{A}_{1,0}$ is defined by

$$\|a\| \;=\; \sup_{\omega\in\Omega} \;\|A_\omega\| \;,$$

where on the right we have the standard operator norm. Then $\mathcal{A}_1$ is the C*-algebra given by the closure of $\mathcal{A}_{1,0}$ under this norm. Elements in $\mathcal{A}_1$ are covariant families of bounded operators having decaying off-diagonal matrix elements and will still be denoted by $a = \{A_\omega\}_{\omega\in\Omega}$. Note the lower case notation, which will be use throughout for elements of the algebras, while the upper case letters will be reserved for operators on the physical Hilbert space. While $\mathcal{A}_1$ was defined as algebra of covariant operator families with certain decay conditions, it is isomorphic to the C*-algebraic (reduced) crossed product algebra $C(\Omega) \rtimes_\alpha \mathbb{Z}$ of $C(\Omega)$ w.r.t. the $\mathbb{Z}$-action $\alpha(f)(\omega) = f(\tau^{-1}\omega)$ on $C(\Omega)$. The isomorphism is

$$\{A_\omega\}_{\omega\in\Omega} \;\mapsto\; a \in C(\Omega \times \mathbb{Z}) \;, \qquad a(\omega, x) \;=\; \langle 0|A_\omega|x\rangle \;,$$

which associates a continuous function over $\Omega \times \mathbb{Z}$ to every covariant operator family. This identification of $\mathcal{A}_1$ with the crossed product algebra will tacitly be used below, and further stressed and explored in the higher dimensional cases. The Hamiltonian $h = \{H_\omega\}_{\omega\in\Omega}$, the flat band Hamiltonian $q = \{Q_\omega\}_{\omega\in\Omega}$ and the Fermi unitary $u_F = \{U_\omega\}_{\omega\in\Omega}$ are all elements of matrix algebras over $\mathcal{A}_1$. One crucial fact is that the 1-periodic (or translation invariant) operators are also covariant, and actually identified with those covariant operator families which do not depend on ω. Hence the algebra of periodic operators $C(\mathbb{S}^1)$ (in its Fourier transformed representation) is a (closed) subalgebra of $\mathcal{A}_1$. This implies that the generators of the K-groups of $C(\mathbb{S}^1)$ also specify elements of the K-groups of $\mathcal{A}_1$. In fact, even more holds, namely the K-groups coincide.

Proposition 1.6.1 *The K-groups of $\mathcal{A}_1$ are*

$$K_0(\mathcal{A}_1) \;=\; \mathbb{Z} \;, \qquad K_1(\mathcal{A}_1) \;=\; \mathbb{Z} \;,$$

and the generators are the same as those of $C(\mathbb{S}^1)$, namely $\mathbf{1}$ *and S respectively.*

Proof We will check that $C(\mathbb{S}^1)$ is a deformation retract of $\mathcal{A}_1 = C(\Omega) \rtimes \mathbb{Z}$ and this implies that $K_j(\mathcal{A}_1) = K_j(C(\mathbb{S}^1))$ [222, Sect. 6.4]. The key for this is the contractibility of Ω to one point which we choose to be $0 = (0, 0)_{x\in\mathbb{Z}}$. Indeed, $\gamma_\lambda : \mathcal{A}_1 \to \mathcal{A}_1$ defined by

$$(\gamma_\lambda a)(\omega, x) \;=\; a(\lambda\omega, x) \;, \qquad \lambda \in [0, 1] \;, \tag{1.21}$$

is a continuous family (in λ) of continuous morphisms which connects $\gamma_1 = \mathrm{id}_{\mathcal{A}_1}$ to a right inverse $\gamma_0 : \mathcal{A}_1 \to C(\mathbb{S}^1)$ of the inclusion map $i : C(\mathbb{S}^1) \to \mathcal{A}_1$ by a continuous path. □

The algebra $\mathcal{A}_1$ (and matrix algebras over it) contains covariant operator families on $\ell^2(\mathbb{Z})$. The edge algebra is now $\mathcal{E}_1 = C(\Omega) \otimes \mathcal{K}$ and the half-space algebra is $\widehat{\mathcal{A}}_1 = \mathcal{A}_1 \oplus \mathcal{E}_1$ as a direct sum of vector spaces, but not algebras. Operators in $\widehat{\mathcal{A}}_1$ are concretely given by the sum of a half-space restriction of a covariant operator in

$\mathcal{A}_1$ and a compact operator in $\mathcal{E}_1$, namely

$$\hat{a} \;=\; (a, \tilde{k}) \;=\; \{\Pi A_\omega \Pi^* + K_\omega\}_{\omega\in\Omega} \;,$$

if $a = \{A_\omega\}_{\omega\in\Omega} \in \mathcal{A}_1$ and $\tilde{k} = \{K_\omega\}_{\omega\in\Omega} \in \mathcal{E}_1$, and where $\Pi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ denotes the partial isometry with $\Pi\Pi^* = \mathbf{1}_{\ell^2(\mathbb{N})}$ and projection $\Pi^*\Pi$ in $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$. The product and adjoint in $\widehat{\mathcal{A}}_1$ and $\mathcal{E}_1$ are naturally inherited from the operator product on $\ell^2(\mathbb{N})$. Exactly as in (1.6), one has an exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{E}_1 \stackrel{i}{\longrightarrow} \widehat{\mathcal{A}}_1 \stackrel{\mathrm{ev}}{\longrightarrow} \mathcal{A}_1 \longrightarrow 0 \tag{1.22}$$

The detailed construction of these algebras will be given in Chap. 3. Again, various operators constructed from the disordered Hamiltonian $h = \{H_\omega\}_{\omega\in\Omega} \in \mathcal{A}_1$ are in this sequence. The half-space restriction $\widehat{h} = \{\widehat{H}_\omega\}_{\omega\in\Omega}$ is an element of a matrix algebra over the Toeplitz extension $\widehat{\mathcal{A}}_1$ as is the lift of $q = \{Q_\omega\}_{\omega\in\Omega} \in \mathcal{A}_1$. Furthermore, the projections $\tilde{p}_\pm(\delta) = \{\widetilde{P}_{\pm,\omega}(\delta)\}_{\omega\in\Omega}$ on bound states, constructed for every ω just as in Sect. 1.1 by splitting $\widetilde{P}_\omega(\delta) = \chi(H_\omega \in [-\delta, \delta])$ with $\delta < E_g$ into ± 1 eigenspaces of $\widehat{J}$, lie in $\mathcal{E}_1 = C(\Omega) \otimes \mathcal{K}$, and they define a class in the K_0-group of this C*-algebra. It is worth pointing out that both projections $\widetilde{P}_{\pm,\omega}(\delta)$ are indeed continuous and, in particular, do not change dimension. On the other hand, the covariant family of Fermi unitaries $u_F = \{U_\omega\}_{\omega\in\Omega}$ defined in (1.19) specify a class in $K_1(\mathcal{A}_1)$. Now the index map of the K-theoretic exact sequence associated with (1.22) connects these two classes, namely by exactly the same proof as given for (1.13), one shows the following.

Proposition 1.6.2 *Let $u_F = \{U_\omega\}_{\omega\in\Omega} \in M_N(\mathcal{A}_1)$ be given by (1.19) and $\tilde{p}_\pm(\delta) = \{\widetilde{P}_{\pm,\omega}(\delta)\}_{\omega\in\Omega}$ the projections on the zero energy bound states of positive and negative chirality, respectively. Then, with the K-theoretic index map associated to the exact sequence (1.22),*

$$\mathrm{Ind}([u_F]_1) \;=\; [\tilde{p}_+(\delta)]_0 - [\tilde{p}_-(\delta)]_0 \;. \tag{1.23}$$

1.7 Why Use Non-commutative Analysis Tools?

The equivalent of Theorem 1.2.2, namely Theorem 1.8.2 below, will again follow by extracting numbers from the K-theoretic identity (1.23). For this purpose, one has to extend the definitions (1.14) and (1.15) of the cyclic cocycles $\widetilde{\mathrm{Ch}}_0$ and Ch_1 to the operator algebra $\mathcal{A}_1$ describing disordered systems. The generalization of Ch_0 is

$$\widetilde{\mathrm{Ch}}_0([\tilde{p}]_0 - [\tilde{p}']_0) \;=\; \int \mathbb{P}(d\omega)\,\big(\mathrm{Tr}(\widetilde{P}_\omega) \;-\; \mathrm{Tr}(\widetilde{P}'_\omega)\big)\;. \tag{1.24}$$

Actually, by continuity, the map $\omega \mapsto \mathrm{Tr}(\widetilde{P}_\omega) \in \mathbb{Z}$ is constant and therefore the average $\mathbb{P}$ over the disorder is not necessary. As to Ch_1, the definition (1.15) involves differentiation in Fourier space and this now has to be replaced by non-commutative differentiation. For any finite range operator $a = \{A_\omega\}_{\omega\in\Omega} \in \mathcal{A}_{1,0}$, one defines its derivative $\partial a \in \mathcal{A}_{1,0}$ by

$$\partial a(\omega, x) \;=\; -\,\mathrm{i}\, x\, a(\omega, x)\;.$$

This definition can be extended to so-called differentiable operators $a \in \mathcal{A}_1$ as long as the r.h.s. defines an element in $\mathcal{A}_1$. The set of differentiable operators is denoted by $\mathrm{C}^1(\mathcal{A}_1)$. By iteration one defines $\mathrm{C}^n(\mathcal{A}_1)$, and then $\mathrm{C}^\infty(\mathcal{A}_1) = \bigcap_{n\geq 1} C^n(\mathcal{A}_1)$. The latter is a Fréchet algebra, clearly dense in $\mathcal{A}_1$, that is invariant under holomorphic functional calculus. It follows [75] that the algebraic K-groups $K_j(C^\infty(\mathcal{A}_1))$ are equal to the topological K-groups $K_j(\mathcal{A}_1)$ for $j = 0, 1$. Operators in this sub-algebra are sufficiently regular for differential topology. Apart from differentiation, a non-commutative integration tool is needed. A state $\mathcal{T}$ on $\mathcal{A}_1$ is defined by

$$\mathcal{T}(a) \;=\; \int \mathbb{P}(d\omega)\; \langle 0|A_\omega|0\rangle \;=\; \int \mathbb{P}(d\omega)\; a(\omega, 0)\;, \qquad a = \{A_\omega\}_{\omega\in\Omega}\;.$$

In fact, it is a trace that is invariant under ∂, as shows the following lemma.

Lemma 1.7.1 *The following holds.*

(i) *For $a, b \in \mathcal{A}_1$, one has $\mathcal{T}(ab) = \mathcal{T}(ba)$.*
(ii) *For $a \in C^1(\mathcal{A}_1)$, one has $\mathcal{T}(\partial a) = 0$.*
(iii) *For $a, b \in C^1(\mathcal{A}_1)$, one has $\mathcal{T}(\partial a\, b) = -\mathcal{T}(a\, \partial b)$.*
(iv) *For a translation invariant $a \in \mathcal{A}_1$ with Fourier transform $k \in \mathbb{S}^1 \mapsto a(k)$, one has $\mathcal{T}(a) = \int_{\mathbb{S}^1} \frac{dk}{2\pi}\, a(k)$.*
(v) *For a translation invariant $a \in C^1(\mathcal{A}_1)$, one has $(\partial a)(k) = \partial_k a(k)$ where $k \in \mathbb{S}^1 \mapsto a(k)$ and $k \in \mathbb{S}^1 \mapsto (\partial a)(k)$ are the Fourier transforms.*

The straightforward proof is left to the reader. Finally, one can introduce

$$\mathrm{Ch}_1(u) \;=\; \mathrm{i}\; \mathcal{T}(u^{-1}\, \partial u)\;, \qquad u \in C^1(\mathcal{A})\;. \tag{1.25}$$

Let us point out that, for translation invariant u, this reduces precisely to (1.4).

Proposition 1.7.2 Ch_1 *is a homotopy invariant, namely for any continuous path $\lambda \in [0, 1] \mapsto u(\lambda) \in C^1(\mathcal{A})$ the number* $\mathrm{Ch}_1(u(\lambda))$ *is constant.*

Proof First of all, $u \mapsto \mathrm{Ch}_1(u)$ is continuous and therefore the path $\lambda \in [0, 1] \mapsto u(\lambda)$ can be approximated by a differentiable one. For such a differentiable path,

$$\begin{aligned} -\mathrm{i}\; \partial_\lambda\; \mathrm{Ch}_1(u(\lambda)) &= \; \mathcal{T}(\partial_\lambda u^{-1}\, \partial u) \;+\; \mathcal{T}(u^{-1}\, \partial \partial_\lambda u) \\ &= \; -\,\mathcal{T}(u^{-1}\partial_\lambda u\, u^{-1}\, \partial u) \;-\; \mathcal{T}(\partial u^{-1}\, \partial_\lambda u)\;, \end{aligned}$$

where in the second equality Lemma 1.7.1(iii) was used. As $\partial u^{-1} = -u^{-1}\partial u\, u^{-1}$ one concludes that $\partial_\lambda\, \mathrm{Ch}_1(u(\lambda)) = 0$ and this completes the proof. □

The physical model is defined over $\mathbb{C}^{2N} \otimes \ell^2(\mathbb{Z})$ rather than just $\ell^2(\mathbb{Z})$ and U_ω is actually defined over $\mathbb{C}^N \otimes \ell^2(\mathbb{Z})$. As one can see, most of the time we will be dealing with the matrix algebras over $\mathcal{A}_1$. The non-commutative calculus can be trivially extended to cover these cases, by replacing $\mathcal{T}$ by $\mathcal{T} \otimes \mathrm{tr}$, where tr is the trace over the fiber. We now can finally define the bulk invariant for the disordered chiral system, as $\mathrm{Ch}_1(u_F)$. Based on the above result, we can state at once that, if $h(\lambda)$ is a smooth deformation of h such that its central spectral gap remains open, then $u_F(\lambda)$ varies smoothly in $M_N(\mathbb{C}) \otimes C^1(\mathcal{A}_1)$ and, consequently, $\mathrm{Ch}_1(u_F)$ remains unchanged.

1.8 Why Prove an Index Theorem?

Proposition 1.7.2 implies that Ch_1 only depends on the K_1-class of its argument so that one may write $\mathrm{Ch}_1(u) = \mathrm{Ch}_1([u]_1)$. The homotopy invariance can, in particular, be applied to the homotopy $u_\lambda = \gamma_\lambda(u)$ with γ_λ defined in (1.21). This implies $\mathrm{Ch}_1(u) = \mathrm{Ch}_1(u_0)$ for $u \in C^1(\mathcal{A}_1)$. Now $u_0 \in C^1(\mathcal{A}_1)$ is translation invariant and therefore $\mathrm{Ch}_1(u_0)$ can be calculated by (1.4) as a winding number. In particular, this shows that $\mathrm{Ch}_1(u) \in \mathbb{Z}$. An alternative way to verify the integrality of $\mathrm{Ch}_1(u)$ is to prove an index theorem. This has the advantage that one can also prove that the pairing is well-defined and integral in the regime of a mobility bulk gap, namely, when the Fermi level lies in a region of the essential spectrum which is dynamically Anderson localized. This type of extension is crucial for the understanding of the quantum Hall effect [20] and will be discussed further in Chap. 6, which also applies to the present one-dimensional example.

Theorem 1.8.1 *Let $\Pi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ be the surjective partial isometry as above. For a unitary $u = \{U_\omega\}_{\omega\in\Omega} \in C^1(\mathcal{A}_1)$, the operators $\Pi U_\omega \Pi^*$ are Fredholm operators with an almost sure index given by*

$$\mathrm{Ch}_1(u) \;=\; -\,\mathrm{Ind}(\Pi U_\omega \Pi^*)\ .$$

This is an extension of the Noether-Gohberg-Krein index theorem to covariant operators and its proof can be found in [107] as well as [171]. It assures us that the bulk invariant $\mathrm{Ch}_1(u_F)$ remains stable and quantized in the regime where the spectral gap of h is replaced by a mobility gap. After all these preparations, the disordered version of Theorem 1.2.2 can finally be stated and proved.

Theorem 1.8.2 *Consider the element $h = \{H_\omega\}_{\omega\in\Omega} \in \mathcal{A}_1$ associated to the Hamiltonian (1.17) and let $\hat{h} = \{\widehat{H}_\omega\}_{\omega\in\Omega} \in \widehat{\mathcal{A}}_1$ be a restriction to the half-space given by an arbitrary chiral symmetric boundary condition. Assume h to have a central spectral gap and let u_F be the Fermi unitary element as well as $N_{\omega,\pm} = \mathrm{Tr}(\widehat{P}_{\pm,\omega}(\delta))$. Then, for all ω,*

$$\mathrm{Ch}_1(u_F) \;=\; -\,N_{\omega,+} \;+\; N_{\omega,-}\ . \tag{1.26}$$

Proof Set $h(\lambda) = \gamma_\lambda(h)$ with the homotopy γ_λ given in (1.21), which induces a smooth deformation $u_F(\lambda)$. By homotopy invariance, $\mathrm{Ch}_1(u_F(\lambda))$ is constant, in particular, $\mathrm{Ch}_1(u_F) = \mathrm{Ch}_1(u_F(0))$. Furthermore, the projections supplied by the index map define a homotopy of projections and, since the pairing $\widetilde{\mathrm{Ch}}_0([p]_0) = \int \mathbb{P}(d\omega)\,\mathrm{Tr}(P_\omega) = \mathrm{Tr}(P_\omega)$ is homotopy invariant, it and can be can be computed at $\lambda = 0$. Consequently, the equality (1.26) follows from the equality at $\lambda = 0$, which was already proved in Theorem 1.2.2. □

Second Proof of Theorem 1.8.2, based merely on Theorem 1.8.1. First of all, the chiral symmetry $JH_\omega J = -H_\omega$ implies that there exists an invertible operator A_ω such that, in the grading of J,

$$H_\omega \;=\; \begin{pmatrix} 0 & A_\omega^* \\ A_\omega & 0 \end{pmatrix}\,. \tag{1.27}$$

By homotopy invariance of the index,

$$\mathrm{Ind}(\Pi U_\omega \Pi^*) \;=\; \mathrm{Ind}(\Pi A_\omega \Pi^*) \;=\; \dim(\mathrm{Ker}(\Pi A_\omega \Pi^*)) \;-\; \dim(\mathrm{Ker}(\Pi A_\omega^* \Pi^*))\,.$$

But $\mathrm{Ker}(\Pi H_\omega \Pi^*) = \big(\mathrm{Ker}(\Pi A_\omega^* \Pi^*) \oplus 0\big) \oplus \big(0 \oplus \mathrm{Ker}(\Pi A_\omega \Pi^*)\big)$, and $\widehat{J}$ is positive definite on the first and negative definite on the second summand. Therefore

$$\mathrm{Ind}(\Pi U_\omega \Pi^*) \;=\; \mathrm{Sig}\big(\widehat{J}|_{\mathrm{Ker}(\Pi H_\omega \Pi^*)}\big)\,,$$

where the signature is calculated of the (finite dimensional non-degenerate) quadratic form obtained by restriction of $\widehat{J}$ to $\mathrm{Ker}(\Pi H_\omega \Pi^*)$. But this signature is up to a sign precisely the r.h.s. of (1.26). □

Another thing that becomes apparent in the above proof is how to address the stability of the invariants under terms which break chiral symmetry, see Sect. 1.1. Indeed, such terms lead to non-vanishing diagonal entries in the Hamiltonian in the form (1.27). If, however, the off-diagonal entry A_ω remains invertible, then one can still define its winding number via the pairing with Ch_1. Such systems are called approximately chiral and are further described in Sect. 2.4.2

1.9 Can the Invariants be Measured?

Of course, it is interesting to link the invariants to quantities that can potentially be measured. The best know example is the quantum Hall effect in which an invariant is linked to the Hall conductance. For the present one-dimensional chiral models the so-called chiral polarization is connected to the bulk invariant $\mathrm{Ch}_1(u_F)$ as is discussed in Sect. 7.3. One of the things that is always true is that the bulk invariant determines the boundary invariant, which is here the chirality of the bound states. This boundary invariant can in principle be measured.