Chapter 5
Thermal Conductivity in Harmonic Lattices with Random Collisions

Giada Basile, Cédric Bernardin, Milton Jara, Tomasz Komorowski, and Stefano Olla

Abstract We review recent rigorous mathematical results about the macroscopic behaviour of harmonic chains with the dynamics perturbed by a random exchange of velocities between nearest neighbor particles. The random exchange models the effects of nonlinearities of anharmonic chains and the resulting dynamics have similar macroscopic behaviour. In particular there is a superdiffusion of energy for unpinned acoustic chains. The corresponding evolution of the temperature profile is governed by a fractional heat equation. In non-acoustic chains we have normal diffusivity, even if momentum is conserved.

5.1 Introduction

Lattice systems of coupled anharmonic oscillators have been widely used in order to understand the macroscopic transport of the energy, in particular the superdiffusive behavior in one and two dimensional unpinned chains. While a lot of numerical experiments and heuristic considerations have been made (cf. [30, 31, 41] and many

G. Basile
Dipartimento di Matematica, Università di Roma La Sapienza, Roma, Italy
e-mail: basile@mat.uniroma.it

C. Bernardin
Laboratoire J.A. Dieudonné UMR CNRS 7351, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
e-mail: cbernard@unice.fr

M. Jara
IMPA, Rio de Janeiro, Brazil
e-mail: mjara@impa.br

T. Komorowski
Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland
e-mail: komorow@hektor.umcs.lublin.pl

S. Olla (✉)
Ceremade, UMR CNRS 7534, Université Paris Dauphine, 75775 Paris Cedex 16, France
e-mail: olla@ceremade.dauphine.fr

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contributions in the present volume), very few mathematical rigourous scaling limits have been obtained until now.

For harmonic chains it is possible to perform explicit computations, even in the stationary state driven by thermal boundaries (cf. [38]). But since these dynamics are completely integrable, the energy transport is purely ballistic and they do not provide help in understanding the diffusive or superdiffusive behavior of anharmonic chains.

The *scattering* effect of the non-linearities can be modeled by stochastic perturbations of the dynamics such that they conserve total momentum and total energy, like a random exchange of the velocities between the nearest neighbor particles. We will describe the results for the one-dimensional chains, and we will mention the results in the higher dimensions in Sect. 5.9. In particular we will prove how the transport through the fractional Laplacian, either asymmetric or symmetric, emerges from microscopic models.

The *infinite dynamics* is described by the velocities and positions $\{(p_x, q_x) \in \mathbb{R}^2\}_{x\in\mathbb{Z}}$ of the particles. The formal Hamiltonian is given by

$$\mathscr{H}(p,q) := \frac{1}{2m}\sum_x p_x^2 + \frac{1}{2}\sum_{x,x'} \alpha_{x-x'} q_x q_{x'}, \tag{5.1}$$

where we assume that the masses are equal to 1 and that α is symmetric with a finite range or at most exponential decay $|\alpha_x| \le Ce^{-c|x|}$. We define the Fourier transform of a function $f : \mathbb{Z} \to \mathbb{R}$ as $\hat{f}(k) = \sum_x e^{-2\pi ixk} f(x)$ for $k \in \mathbb{T}$ the unitary length torus. We assume that $\hat{\alpha}(k) > 0$ for $k \neq 0$. The function $\omega(k) = \sqrt{\hat{\alpha}(k)}$ is called the *dispersion relation* of the chain.

5.1.1 Unpinned Chains

We are particularly interested in the unpinned chain, i.e. $\hat{\alpha}(0) = 0$, when the total momentum is conserved even under the stochastic dynamics described below. Then, the infinite system is translation invariant under shift in q, and the correct coordinates are the interparticle distances (also called stretches, or strains):

$$r_x = q_x - q_{x-1}, \qquad x \in \mathbb{Z}. \tag{5.2}$$

When $\hat{\alpha}''(0) > 0$ we say that the chain is *acoustic* (i.e. there is a non-vanishing sound speed). We will see that this is a crucial condition for the superdiffusivity of the energy in one dimension. For unpinned acoustic chains we have that $\omega(k) \sim |k|$ as $k \to 0$.

5.1.2 Pinned Chains

When $\hat{\alpha}(0) > 0$, the system is pinned and translation invariance is broken. In this case $\omega'(k) \sim k$ as $k \to 0$. This is also the case for unpinned non-acoustic chains, where $\hat{\alpha}(0) = \hat{\alpha}''(0) = 0$. This fact is responsible for the diffusive behavior of the energy, cf. Sect. 5.7.

5.1.3 Dynamics with Stochastic Collisions

To the Hamiltonian dynamics we add random elastic collisions, where momenta of the nearest-neighbor particles are exchanged. This happens at independent random exponential times: each couple of particles labeled $x, x+1$ exchange their velocities p_x and p_{x+1} at exponential independent random times of intensity γ. Equivalently there are independent Poisson processes $\{N_{x,x+1}(t), \quad x \in \mathbb{Z}\}$ of intensity γ, independent from the positions and velocities of all particles. The evolution of the system is described by the stochastic differential equations

$$\begin{aligned} \dot{q}_x &= p_x \\ \dot{p}_x &= -(\alpha * q(t))_x + (p_{x+1}(t^-) - p_x(t^-))\,\dot{N}_{x,x+1}(t) + (p_{x-1}(t^-) - p_x(t^-))\,\dot{N}_{x-1,x}(t) \end{aligned} \tag{5.3}$$

where $\dot{N}_{x,x+1}(t) = \sum_j \delta(t - T_{x,x+1}(j))$, with $\{T_{x,x+1}(j)\}$ the random times when $N_{x,x+1}$ jumps, and $p_x(t^-)$ is the velocity of the particle x just before time t, i.e. $\lim_{s \downarrow 0} p_x(t-s)$.

The evolution of the probability density on the configurations then follows the Fokker-Planck equations:

$$\partial_t f(t,p,q) = (A + S) f(t,p,q) \tag{5.4}$$

where A is the Hamiltonian operator

$$A = \sum_x \left(p_x \partial_{q_x} - (\partial_{q_x} \mathscr{H}) \partial_{p_x} \right), \tag{5.5}$$

while S is the generator of the random exchanges

$$Sf(p,q) = \gamma \sum_x \left(f(p^{x,x+1}, q) - f(p,q) \right), \tag{5.6}$$

where $p^{x,x+1}$ is the configuration obtained exchanging p_x and p_{x+1}.

This stochastic perturbation of the Hamiltonian dynamics has the property to conserve the total energy, and in the unpinned case the resulting dynamics conserves also the total momentum ($\sum_x p_x$), and the *volume* or *strain* of the chain ($\sum_x r_x$). It

has also the property that these are the only conserved quantities. In this sense it gives the necessary ergodicity to the dynamics [13, 23].

We have also considered other type of conservative random dynamics, like a continuous random exchange of the momenta of each triplets $\{p_{x-1}, p_x, p_{x+1}\}$. The intersection of the kinetic energy sphere $p_{x-1}^2 + p_x^2 + p_{x+1}^2 = C$ with the plane $p_{x-1} + p_x + p_{x+1} = C'$ gives a one dimensional circle. Then we define a dynamics on this circle by a standard Wiener process on the corresponding angle. This perturbation is locally more mixing, but it gives the same results for the macroscopic transport.

5.1.4 Equilibrium Stationary Measures

Due to the harmonicity of the interactions, the Gibbs equilibrium stationary measures are Gaussians. Positions and momenta are independent and the distribution is parametrized, according to the rules of statistical mechanics, by the temperature $T = \beta^{-1} > 0$. In the pinned case, they are formally given by

$$\nu_\beta(dq, dp) \sim \frac{e^{-\beta \mathscr{H}(p,q)}}{\mathscr{Z}}\, dq\, dp.$$

In the unpinned case, the correct definition should involve the r_x variables. Then the distribution of the r_x's is Gaussian and becomes uncorrelated in the case of the nearest neighbor interaction. For *acoustic unpinned chains* the Gibbs measures are parameterized by

$$\boldsymbol{\lambda} = (\beta^{-1}(\text{temperature}), \bar{p}(\text{velocity}), \tau(\text{tension})),$$

and are given formally by

$$\nu_{\boldsymbol{\lambda}}(dr, dp) \sim \frac{e^{-\beta\,[\mathscr{H}(p,q) - \bar{p}\sum_x p_x - \tau \sum_x r_x]}}{\mathscr{Z}}\, dr\, dp. \tag{5.7}$$

Non-acoustic chains are tensionless and the equilibrium measures have a different parametrization, see Sect. 5.7.

5.1.5 Macroscopic Space-Time Scales

We will mostly concentrate on the acoustic unpinned case (except Sect. 5.7). In this case the total Hamiltonian can be written as

$$\mathscr{H}(p, q) := \sum_x \frac{p_x^2}{2} - \frac{1}{4} \sum_{x,x'} \alpha_{x-x'} (q_x - q_{x'})^2\,, \tag{5.8}$$

where $q_x - q_{x'} = \sum_{y=x'+1}^{x} r_y$ for $x > x'$. We define the energy per atom as:

$$e_x(r,p) := \frac{p_x^2}{2} - \frac{1}{4}\sum_{x'} \alpha_{x-x'}(q_x - q_{x'})^2. \tag{5.9}$$

There are three conserved (also called *balanced*) fields: the energy $\sum_x e_x$, the momentum $\sum_x p_x$ and the strain $\sum_x r_x$. We want to study the macroscopic evolution of the spatial distribution of these fields in a large space-time scale. We introduce a scale parameter $\varepsilon > 0$ and, for any smooth test function $J : \mathbb{R} \to \mathbb{R}$, define the empirical distribution

$$\varepsilon \sum_x J(\varepsilon x)\mathbf{U}_x(\varepsilon^{-a}t), \qquad \mathbf{U}_x = (r_x, p_x, e_x). \tag{5.10}$$

We are interested in the limit as $\varepsilon \to 0$. The parameter $a \in [1, 2]$ corresponds to different possible scalings. The value $a = 1$ corresponds to the hyperbolic scaling, while $a = 2$ corresponds to the diffusive scaling. The intermediate values $1 < a < 2$ pertain to the superdiffusive scales.

The interest of the unpinned model is that there are three different macroscopic space-scales where we observe non-trivial behaviour of the chain: $a = 1, \frac{3}{2}, 2$.

5.2 Hyperbolic Scaling: The Linear Wave Equation

Let us assume that the dynamics starts with a random initial distribution $\mu_\varepsilon = \langle\cdot\rangle_\varepsilon$ of finite energy of size ε^{-1}, i.e. for some positive constant E_0:

$$\varepsilon \sum_x \langle e_x(0)\rangle_\varepsilon \le E_0. \tag{5.11}$$

where we denote $e_x(t) = e_x(p(t), q(t))$. We will also assume that some smooth macroscopic initial profiles

$$\mathfrak{u}_0(y) = (\mathfrak{r}_0(y), \mathfrak{p}_0(y), \mathfrak{e}_0(y))$$

are associated with the initial distribution, in the sense that:

$$\lim_{\varepsilon\to 0} \mu_\varepsilon \left(\left| \varepsilon \sum_x J(\varepsilon x)\mathbf{U}_x(0) - \int J(y)\mathfrak{u}_0(y)dy \right| > \delta \right) = 0, \qquad \forall \delta > 0, \tag{5.12}$$

for any test function J.

Then it can be proven [27] that these initial profiles are governed by the linear wave equation in the following sense

$$\lim_{\varepsilon\to 0}\varepsilon\sum_x J(\varepsilon x)\mathbf{U}_x(\varepsilon^{-1}t)=\int J(y)\mathfrak{u}(y,t)dy \tag{5.13}$$

where $\mathfrak{u}(y,t)=(\mathfrak{r}(y,t),\mathfrak{p}(y,t),\mathfrak{e}(y,t))$ is the solution of:

$$\partial_t\mathfrak{r}=\partial_y\mathfrak{p},\qquad \partial_t\mathfrak{p}=\tau_1\partial_y\mathfrak{r},\qquad \partial_t\mathfrak{e}=\tau_1\partial_y(\mathfrak{p}\mathfrak{r}) \tag{5.14}$$

with $\tau_1=\frac{\hat{\alpha}''(0)}{8\pi^2}$ (the square of the speed of sound of the chain). Notice that in the non-acoustic case $\hat{\alpha}''(0)=0$, there is no evolution at the hyperbolic scale.

Observe that the evolution of the fields of strain $\mathfrak{r}$ and momentum $\mathfrak{p}$ is autonomous of the energy field. Furthermore we can define the macroscopic *mechanical* energy as

$$\mathfrak{e}_{mech}(y,t)=\frac{1}{2}\left(\tau_1\mathfrak{r}(y,t)^2+\mathfrak{p}(y,t)^2\right) \tag{5.15}$$

and the temperature profile or *thermal* energy as

$$T(y,t)=\mathfrak{e}(y,t)-\mathfrak{e}_{mech}(y,t). \tag{5.16}$$

It follows immediately from (5.14) that $T(y,t)=T(y,0)$, i.e. the temperature profile does not change on the hyperbolic space-time scale.

There is a corresponding decomposition of the energy of the random initial configurations: long wavelengths (invisible for the exchange noise in the dynamics) contribute to the mechanical energy and they will evolve in this hyperbolic scale following the linear equations (5.14). This energy will eventually disperse to infinity at large time (in this scale). Because of the noise dynamics, short wavelength will contribute to the variance (temperature) of the distribution, and the corresponding profile does not evolve in this hyperbolic scale. See [27] for the details of this decomposition.

5.3 Superdiffusive Evolution of the Temperature Profile

As we have seen in the previous section, for acoustic chains the mechanical part of the energy evolves ballistically in the hyperbolic scale and eventually it will disperse to infinity. Consequently when we look at the larger superdiffusive time scale $\varepsilon^{-a}t$, $a>1$, we start only with the thermal profile of energy, while the strain and momentum profiles are equal to 0. It turns out that, for acoustic chains, the

temperature profile evolves at the time scale corresponding to $a = 3/2$:

$$\lim_{\varepsilon\to 0} \varepsilon \sum_x J(\varepsilon x)\ \langle e_x(\varepsilon^{-3/2}t)\rangle_\varepsilon = \int J(y)\mathfrak{T}(y,t)dy, \qquad t>0, \tag{5.17}$$

where $\mathfrak{T}(y,t)$ solves the fractional heat equation

$$\partial_t \mathfrak{T} = -c|\Delta_y|^{3/4}\mathfrak{T}, \qquad \mathfrak{T}(y,0) = T(y,0) \tag{5.18}$$

where $c = \hat{\alpha}''(0)^{3/4}2^{-9/4}(3\gamma)^{-1/2}$. This is proven in [26]. We also have that the profiles for the other conserved quantities remain flat:

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon \sum_x J(\varepsilon x)\ \langle r_x(\varepsilon^{-3/2}t)\rangle_\varepsilon = 0,\\ &\lim_{\varepsilon\to 0} \varepsilon \sum_x J(\varepsilon x)\ \langle p_x(\varepsilon^{-3/2}t)\rangle_\varepsilon = 0, \quad t>0. \end{aligned} \tag{5.19}$$

Clearly the null value is due to the finite energy assumption, otherwise it will be the corresponding constants, i.e. $\lim_{\varepsilon\to 0}\varepsilon\sum_x\langle r_x(0)\rangle_\varepsilon$ and $\lim_{\varepsilon\to 0}\varepsilon\sum_x\langle p_x(0)\rangle_\varepsilon$.

In finite volume with given boundary conditions (periodic or else), the mechanical energy will persist, oscillating in linear waves. At the larger superdiffusive time scale, waves will oscillate fast giving a weak convergence for the initial profiles of strain and momentum to constant values.

5.4 The Diffusive Behavior of the Phonon-Modes

We have seen that (5.19) holds at the superdiffusive time scale and consequently at any larger time scale. But if we recenter the evolution of the strain and momentum around the propagation of the wave equation we see Gaussian fluctuations at the diffusive space time scale. For this purpose it is useful to introduce a microscopic approximation of the Riemann invariants (normal modes) of the wave equation:

$$f_x^{\pm}(t) \ = \ p_x(t) \ \pm \ \tau_1^{1/2}\left(r_x(t) \pm \frac{3\gamma-1}{2}(r_{x+1}(t)-r_x(t))\right). \tag{5.20}$$

Once $f_x^{\pm}(t)$ are recentred on the Riemann invariants of the wave equation, they diffuse on the proper space-time scale, more precisely:

$$\varepsilon\sum_x J(\varepsilon x \mp \tau_1^{1/2}\varepsilon^{-1}t)\, f_x^{\pm}(\varepsilon^{-2}t) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}} J(y)\ \bar{\mathfrak{f}}^{\pm,d}(y,t)dy,$$

$$\partial_t\, \bar{\mathfrak{f}}^{\pm,d} = \frac{3\gamma}{2}\,\partial_y^2\, \bar{\mathfrak{f}}^{\pm,d}.$$

For a proof see [27].

5.5 Equilibrium Fluctuations

If we start with an equilibrium stationary measure corresponding to a certain temperature T, momentum $\bar{p}$ and strain $\bar{r}$, then of course there will be no evolution of the empirical fields defined by (5.10). But we shall observe the evolution of the equilibrium time correlations defined by

$$S^{\ell,\ell'}(x,t) = < u_x^\ell(t)\, u_0^{\ell'}(0) > - < u_0^\ell(0) >< u_0^{\ell'}(0) >, \quad u_x^1 = r_x,\ u_x^2 = p_x,\ u_x^3 = e_x, \tag{5.21}$$

where $\langle\cdot\rangle$ denotes the expectation with respect to the dynamics in the corresponding equilibrium. Let us assume for simplicity of notation that $\bar{r} = 0 = \bar{p}$, otherwise we have to shift x along the characteristics of the linear wave equation. At time $t = 0$, it is easy to compute the limit (in a distributional sense):

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} S([\varepsilon^{-1}y], 0) = \delta(y) \begin{pmatrix} T & 0 & \tilde{\alpha}T \\ 0 & T & 0 \\ \tilde{\alpha}T & 0 & (\frac{1}{2} + \alpha^2)T^2 \end{pmatrix} \tag{5.22}$$

where $\tilde{\alpha}$ is a constant depending only on the interaction.

In the hyperbolic time scale $a = 1$ this correlation matrix evolves deterministically, i.e.

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} S([\varepsilon^{-1}y], \varepsilon^{-1}t) = \bar{S}(y,t) \tag{5.23}$$

where

$$\partial_t \bar{S}^{11}(y,t) = \partial_y \bar{S}^{22}(y,t), \qquad \partial_t \bar{S}^{22}(y,t) = \tau_1 \partial_y \bar{S}^{11}(y,t), \tag{5.24}$$

while for the energy correlations

$$\partial_t \bar{S}^{33}(y,t) = 0. \tag{5.25}$$

In particular if $\bar{p} = \bar{r} = 0$, energy fluctuations do not evolve at the hyperbolic time scale. As for the energy profile out of equilibrium, the evolution is at a further time scale. By a duality argument (cf. [26]), the evolution of the energy correlations occurs at the superdiffusive time scale with $a = 3/2$, namely

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} S^{33}([\varepsilon^{-1}y], \varepsilon^{-3/2}t) = \tilde{S}^{33}(y,t), \tag{5.26}$$

where $\tilde{S}^{33}(y,t)$ is the solution of:

$$\partial_t \tilde{S}^{33} = -c|\Delta_y|^{3/4} \tilde{S}^{33}. \tag{5.27}$$

5.6 The Phonon Boltzmann Equation

A way to understand the energy superdiffusion in the one-dimensional system is to analyze its kinetic limit, i.e. a limit for weak noise where the number of stochastic collisions per unit time remains bounded. In the non-linear case it corresponds to a weak non-linearity limit as proposed first in his seminal paper [37] in 1929 by Peierls. He intended to compute thermal conductivity for insulators in analogy with the kinetic theory of gases. The main idea is that at low temperatures the lattice vibrations responsible for the energy transport can be described as a gas of interacting particles (phonons) characterized by a wave number k. The time-dependent distribution function of phonons solves a Boltzmann type equation. Over the last years, starting from the work of Spohn [40], several papers are devoted to achieve phononic Boltzmann type equations from microscopic dynamics of oscillators.

A rigorous derivation can be achieved for the chain of harmonic oscillators perturbed by a stochastic exchange of velocities [7]. The main tool is the introduction of a Wigner function, which describes the energy density of the phonons. Let $\hat{\psi}$ be the complex field

$$\hat{\psi}(k) = \frac{1}{\sqrt{2}}\big(\omega(k)\hat{q}(k) + i\hat{p}(k)\big), \qquad k \in \mathbb{T},$$

where $\hat{p}$, $\hat{q}$ are the Fourier transform of the variables p, q and $\omega(k)$ is the dispersion relation of chain. The energy of the chain can be expressed in terms of the fields $\hat{\psi}$, namely $\mathscr{H} = \int_{\mathbb{T}} dk\, |\hat{\psi}(k)|^2$. The evolution of the field $\hat{\psi}$ due to the pure harmonic Hamiltonian without noise ($\gamma = 0$) reads

$$\partial_t \hat{\psi}(k,t) = -i\omega(k)\hat{\psi}(k,t),$$

therefore the quantities $|\hat{\psi}(k)|^2$ are preserved by the harmonic dynamics.

The Wigner distribution is defined in analogy to the usual one in quantum mechanics

$$\begin{aligned} W^\varepsilon(y,k,t) &= (\varepsilon/2)\int_{\mathbb{R}} e^{2\pi i\xi y}\widehat{W}^\varepsilon(\xi,k,t)d\xi, \\ \widehat{W}^\varepsilon(\xi,k,t) &:= \langle \hat{\psi}(k-\varepsilon\xi/2,t)^*\ \hat{\psi}(k+\varepsilon\xi/2,t)\rangle_\varepsilon \end{aligned} \tag{5.28}$$

where $\langle\cdot\rangle_\varepsilon$ denotes the expectation value with respect to the initial measure, chosen in such a way that the average of the total energy, is of order ε^{-1}, i.e. $\varepsilon\,\langle\mathscr{H}\rangle_\varepsilon \le E_0$, see (5.11). We also require that the averages of all p_x and q_x are zero.

The Wigner distribution $W^\varepsilon(y,k,t)$ defined by (5.28) gives a different energy distribution from the one considered in the previous sections, i.e. $< e_{[\varepsilon^{-1}y]}(t) >$. But in the macroscopic limit, as $\varepsilon \to 0$, they are equivalent [26].

We look at the evolution of the Wigner function on a time scale $\varepsilon^{-1}t$, with the strength of the noise of order ε, i.e. we consider the dynamics defined by (5.3) with

γ replaced by $\varepsilon\gamma$, like in a Boltzmann-Grad limit. This evolution is not autonomous and is given by

$$\begin{aligned}\partial_t \hat{W}^\varepsilon(\xi,k,t) = &- i\xi\,\omega'(k)\hat{W}^\varepsilon(\xi,k,t) + \gamma\,\mathscr{C}\hat{W}^\varepsilon(\xi,k,t)\\ &- \frac{\gamma}{2}\,\mathscr{C}\hat{Y}^\varepsilon(\xi,k,t) - \frac{\gamma}{2}\,\mathscr{C}\hat{Y}^\varepsilon(\xi,k,t)^* + \mathscr{O}(\varepsilon),\end{aligned} \tag{5.29}$$

where $\hat{Y}^\varepsilon$ is the field $\hat{Y}^\varepsilon(\xi,k,t) = \langle\hat{\psi}(k+\varepsilon\xi/2,t)\hat{\psi}(k-\varepsilon\xi/2,t)\rangle_\varepsilon$ and $\mathscr{C}$ is a linear operator. The transport term is due to the harmonic Hamiltonian, while the "collision" operator $\mathscr{C}$ is related to the stochastic noise.

It turns out that the field $\hat{Y}^\varepsilon$ is the Wigner distribution associated to the *difference* between the kinetic energy and the potential energy. This is fast oscillating on the time scale ε^{-1} and in the limit $\varepsilon \to 0$ it disappears after time integration. Therefore in the limit $\varepsilon \to 0$ the Wigner function W^ε weakly converges to the solution of the following linear Boltzmann equation

$$\partial_t W(y,k,t) + \frac{1}{2\pi}\omega'(k)\,\partial_y W(y,k,t) = \mathscr{C}W(y,k,t). \tag{5.30}$$

It describes the evolution of the energy density distribution, over the physical space $\mathbb{R}$, of the phonons, characterized by a wave number k and traveling with velocity $\omega'(k)$. We remark that for the unpinned acoustic chains $\omega'(k)$ remains strictly positive for small k. The collision term has the following expression

$$\mathscr{C}f(k) = \int_{\mathbb{T}} dk' R(k,k')[f(k') - f(k)],$$

where the kernel R is positive and symmetric. One can write an exact expression on R, nevertheless its crucial feature is that R behaves like k^2 for small k, due to the fact that the noise preserves the total momentum. Naïvely, it means that phonons with small wave numbers travel with a finite velocity, but they have low probability to be scattered, thus their mean free paths have a macroscopic length (ballistic transport). This intuitive picture has an exact statement in the probabilistic interpretation of (5.30). The equation describes the evolution of the probability density of a Markov process $(K(t), Y(t))$ on $\mathbb{T}\times\mathbb{R}$, where $K(t)$ is a reversible jump process and $Y(t)$ is an additive functional of K, namely $Y(t) = \int_0^t \omega'(K_s)ds$. A phonon with the wave number k waits in its state an exponentially distributed random time $\tau(k)$ with mean value $\sim k^{-2}$ for small k. Then it jumps to another state k' with probability $\sim k'^2 dk'$. The additive functional $Y(t)$ describes the position of the phonon and can be expressed as

$$Y(t) = \sum_{i=1}^{\mathscr{N}_t} \tau(X_i)\omega'(X_i),$$

where $\{X_i\}_{i\geq 1}$ is the Markov chain given by the sequence of the states visited by the process $K(t)$. Here $\mathcal{N}_t$ denotes the number of jumps up to the time t. The tail distribution of the random variables $\{\tau(X_i)\omega'(X_i)\}$ with respect to the stationary measure π of the chain behaves like

$$\pi\left[|\tau(X_i)\omega'(X_i)| > \lambda\right] \sim \frac{1}{\lambda^{3/2}}.$$

Therefore the variables $\tau(X_i)\omega'(X_i)$ have an infinite variance with respect to the stationary measure. We remark that the variance is exactly the expression of the thermal conductivity obtained in [6]. The rescaled process $N^{-2/3}Y(Nt)$ converges in distribution to a stable symmetric Lévy process with index $3/2$ [3, 25]. As a corollary, the rescaled solution of the Boltzmann equation $W(N^{2/3}y, k, Nt)$ converges, as $N \to +\infty$, to the solution of the fractional diffusion equation

$$\partial_t u(y,t) = -\left|\Delta\right|^{3/4} u(y,t).$$

A different, more analytic approach can be found in [34].

In the pinned or the non-acoustic cases, $\omega' \sim k$ for small k, $\tau(X_i)\omega'(X_i)$ has finite variance with respect to the stationary measure. In particular this variance coincides with the thermal diffusivity computed by the Green-Kubo formula [6]. Then one can prove that the rescaled solution $W(N^{1/2}y, k, Nt)$ converges to the heat equation

$$\partial_t u(y,t) = D\Delta u(y,t),$$

with D given by the thermal diffusivity.

The results described above give a two step approach to the diffusion or superdiffusion of the energy: first a kinetic limit where the Boltzmann phonon equation is obtained in the weak noise limit, then a superdiffusive or diffusive rescaling of the solution of this equation. The results described in Sect. 5.3 concern a simple space-time rescaling, without any weak noise approximation.[1] Still Boltzmann equation helps to understand the proof of (5.17), that goes under the following lines.

Let us just consider the superdiffusive case and consider the evolution of the Wigner distribution at time $\varepsilon^{-3/2}t$ and for the noise of intensity γ. Equation (5.29) becomes

$$\begin{aligned}\partial_t \hat{W}^\varepsilon(\xi,k,t) =& -i\xi\,\omega'(k)\varepsilon^{-1/2}\hat{W}^\varepsilon(\xi,k,t) + \gamma\varepsilon^{-3/2}\mathscr{C}\hat{W}^\varepsilon(\xi,k,t)\\ &- \frac{\gamma}{2}\varepsilon^{-3/2}\mathscr{C}\left(\hat{Y}^\varepsilon(\xi,k,t) + \hat{Y}^\varepsilon(\xi,k,t)^*\right) + \mathscr{O}(\varepsilon),\end{aligned} \tag{5.31}$$

[1] In the non-linear cases we cannot expect that the two step approach would give the same result of the direct rescaling of the dynamics. In the β-FPU the kinetic limit seems to give a different superdiffusion scaling than the direct limit [33, 41].

This looks like a very singular limit. Still due to fast oscillations the $\hat{Y}^\varepsilon$ terms disappears after time integration. Furthermore, because the number of collisions per unit time tends to infinity, the limit of the Wigner distribution *homogenizes* in the variable k, i.e. its limit becomes a function $W(y,t)$ independent of variable k. Assume that the above facts have been proven, and consider the case of the simple random exchange of the velocities, that give a scattering rate of the form: $R(k,k') = R(k)R(k')$, with $\int R(k)dk = 1$. The argument below, that follows the line of Mellet et al. [34], can be generalized to various rate functions [26]. The Laplace transform w_ε in time of the Wigner distribution satisfies the equation:

$$\begin{aligned}\left(\lambda + 2\gamma\varepsilon^{-3/2}R(k) + i\omega'(k)\xi\varepsilon^{-1/2}\right) w_\varepsilon(\lambda,\xi,k)\\ = \widehat{W}_\varepsilon(\xi,k,0) + \varepsilon^{-3/2}2\gamma R(k)\int R(k')dk' + \mathscr{O}(\varepsilon).\end{aligned} \tag{5.32}$$

By dividing the above expression by $D_\varepsilon(\lambda,\xi,k) = \varepsilon^{3/2}\lambda + 2\gamma R(k) + i\omega'(k)\xi\varepsilon$, multiplying by $2\gamma R(k)$ and integrating in k one obtains

$$a_\varepsilon \int w_\varepsilon(\lambda,\xi,k)R(k)dk - \int \frac{2\gamma R(k)\widehat{W}_\varepsilon(\xi,k,0)}{D_\varepsilon(\lambda,\xi,k)}dk = \mathscr{O}(\varepsilon) \tag{5.33}$$

where, since $R(k) \sim k^2$ and due to the assumptions made on $\omega'(k)$:

$$a_\varepsilon = 2\gamma\varepsilon^{-3/2}\left(1 - \int \frac{2\gamma R^2(k)}{D_\varepsilon(\lambda,\xi,k)}dk\right) \underset{\varepsilon\to 0}{\longrightarrow} \lambda + c|\xi|^{3/2}.$$

Thanks to the homogenization property of $\widehat{W}_\varepsilon$

$$\int w_\varepsilon(\lambda,\xi,k)R(k)dk \underset{\varepsilon\to 0}{\longrightarrow} w(\lambda,\xi).$$

Furthermore

$$\frac{2\gamma R(k)}{D_\varepsilon(\lambda,\xi,k)} \underset{\varepsilon\to 0}{\longrightarrow} 1,$$

and we conclude that the limit function $w(\lambda,\xi)$ satisfies the equation

$$\left(\lambda + c|\xi|^{3/2}\right) w(\lambda,\xi) = \int \widehat{W}(\xi,k,0)dk \tag{5.34}$$

where $\widehat{W}(\xi,k,0)$ is the limit of the initial condition. Equation (5.34) is the Laplace-Fourier transform of the fractional heat equation. Assuming instead that $\omega'(k) \sim k$, a similar argument gives the normal heat equation [26].

5.7 Non-acoustic Chains: Beam Dynamics

We have seen in the previous sections that the ballistic behaviour at the hyperbolic scale and the superdiffusive behavior of the energy strictly depends on the positivity of the sound velocity τ_1. In the case $\hat{\alpha}(0) = \hat{\alpha}''(0) = 0$ all the coefficients in these evolutions are null and in fact the limit of the energy follows a regular diffusion. Notice that the dynamics is still momentum conserving.

A typical example is given by the following choice of the interaction:

$$\alpha_0 = 6\alpha, \quad \alpha_1 = -4\alpha, \quad \alpha_2 = \alpha, \quad \alpha_x = 0, \quad x > 2 \tag{5.35}$$

(then $\hat{\alpha}(k) = \alpha 4 \sin^4(\pi k)$) that corresponds to the Hamiltonian

$$\mathscr{H} = \sum_x \left[\frac{p_x^2}{2} + \frac{\alpha}{2} (q_{x+1} - 2q_x + q_{x-1})^2 \right]. \tag{5.36}$$

Notice that the expression (5.7) is not defined for any value of the tension τ, this is why we also call these chains tensionless. Basically, the equilibrium energy does not change by *pulling* the chain. It does change by *bending* it, this is why the relevant quantities are defined by the local *curvatures* or *deflections*:

$$\kappa_x = -\Delta q_x = 2q_x - q_{x+1} - q_{x-1}. \tag{5.37}$$

The relevant balanced quantities are now (κ_x, p_x, e_x).

The invariant equilibrium measures are formally given by

$$\frac{e^{-\beta\left[\mathscr{H} - \bar{p}\sum_x p_x - \mathfrak{L}\sum_x \kappa_x\right]}}{\mathscr{Z}}\, d\kappa\, dp \tag{5.38}$$

where the parameter $\mathfrak{L}$ is called *load*. Notice that these measures would be non-translation invariant in the coordinates r_x's.

Under these conditions the sound velocity τ_1 is always null, and there is no ballistic evolution of the chain. In fact it turns out that the three quantities $(\mathfrak{k}(t,y), \mathfrak{p}(t,y), \mathfrak{e}(t,y))$ evolve on the diffusive space-time scale. By defining

$$\mathfrak{e}_{mech}(t,y) = \frac{1}{2}\left(\mathfrak{p}(t,y)^2 + \alpha\mathfrak{k}^2\right) \quad \text{and} \quad \mathfrak{T}(t,y) = \mathfrak{e}(t,y) - \mathfrak{e}_{mech}(t,y),$$

after the corresponding space-time scaling we obtain [28]:

$$\begin{cases} \partial_t \mathfrak{k} &= -\partial_y^2 \mathfrak{p}, \\ \partial_t \mathfrak{p} &= \alpha \partial_y^2 \mathfrak{k} + \gamma \partial_y^2 \mathfrak{p}, \\ \partial_t \mathfrak{T} &= D_\gamma \partial_y^2 \mathfrak{T} + \gamma \left(\partial_y \mathfrak{p}\right)^2. \end{cases} \tag{5.39}$$

The first two equations are the damped Euler-Bernoulli beam equations. The third one describes the diffusive behavior of the energy. The thermal diffusivity $D_\gamma = \frac{C\alpha}{\gamma} + \gamma$, where C is an explicit constant independent from γ and α. In particular for constant initial values of $\mathfrak{k}$ and $\mathfrak{p}$, the thermal energy (i.e. temperature) profile follows a normal heat equation with the thermal diffusivity that can be computed explicitly (cf. [28]). Notice also that a gradient of the macroscopic velocity induce a local increase of the temperature.

These models provide rigorous counter-examples to the usual conjecture that the momentum conservation in one dimension always implies superdiffusivity of the energy (cf. [19, 31]). The presence of a non-vanishing sound velocity seems a necessary condition.

5.8 A Simpler Model with Two Conserved Quantities

In this section we consider the nearest neighbors unpinned harmonic chain (with mass 1 and coupling forces $\alpha_1 = \alpha_{-1} = -\frac{\alpha_0}{2}$) but we add different stochastic collisions with the properties that they conserve the total energy and some extra quantity (that we call "volume") but no longer momentum and stretch. By defining $a = \sqrt{\alpha_0}$ and the field $\{\eta_x \in \mathbb{R} \ ; \ x \in \mathbb{Z}\}$ by $\eta_{2x} = ar_x$ and $\eta_{2x+1} = p_x$, the Hamiltonian equations are reduced to

$$\dot{\eta_x} = a(\eta_{x+1} - \eta_{x-1}).$$

The stochastic collisions are such that at random times given by independent Poisson clocks $N_{x,x+1}(t)$ of intensity γ the kinetic energy at site x is exchanged with the corresponding potential energy. The simplest way to do it is to exchange the variable η_x with η_{x+1}. Because of the form of the noise the total energy $\sum_x \frac{\eta_x^2}{2m}$ and the "volume" $\sum_x \eta_x = \sum_x (p_x + ar_x)$ are the only conserved quantities of the dynamics [14]. By reducing the number of conserved quantities from 3 (energy, momentum, stretch) to 2 (energy, volume) we expect to see easily the influence of the other conserved quantity on the superdiffusion of energy. The nature of the superdiffusion for models with two conserved quantities are studied in the nonlinear fluctuating hydrodynamics framework by Spohn and Stoltz in [43].

In the hyperbolic time scaling, starting from an initial distribution associated to some smooth macroscopic initial volume-energy profiles $(v_0(y), \mathfrak{e}_0(y))$, we can prove that these initial profiles evolve following the linear wave equation $(v(t, y), e(t, y))$ which is solution of

$$\partial_t v = 2a\partial_y v, \quad \partial_t \mathfrak{e} = a\partial_y(v^2).$$

As in Sect. 5.2 we can introduce the *mechanical energy* $\mathfrak{e}_{mech}(y, t) = \frac{v^2(y,t)}{2}$ and the thermal energy $T(y, t) = \mathfrak{e}(y, t) - \mathfrak{e}_{mech}(y, t)$. The later remains constant in time.

Mutatis mutandis the discussion of Sects. 5.3–5.5 can be applied to this model with two conserved quantities with very similar conclusions[2] [15, 16]. The interesting difference is that in (5.18) and (5.27) the fractional Laplacian has to be replaced by the *skew fractional Laplacian*:

$$\partial_t \mathfrak{T} = -c \{ |\Delta_y|^{3/4} - \nabla_y |\Delta_y|^{1/4} \} \mathfrak{T}, \qquad \mathfrak{T}(y,0) = T(y,0) \tag{5.40}$$

for a suitable explicit constant $c > 0$. The skewness is produced here by the interaction of the (unique) sound mode with the heat mode. In the models of Sect. 5.1 which conserve three quantities, there are two sound modes with opposite velocities. The skewness produced by each of them is exactly counterbalanced by the other one so that it is not seen in the final equations (5.18) and (5.27).

5.8.1 The Extension Problem for the Skew-Fractional Laplacian

For the model with two conserved quantities introduced in this section, the derivation of the skew fractional heat equation, at least at the level of the fluctuations in equilibrium as defined in Sect. 5.5, can be implemented by means of the so-called *extension problem* for the fractional Laplacian [17, 44]. As we will see, this extension problem does not only provides a different derivation, but it also clarifies the role of the other (fast) conservation law (i.e. the volume). It can be checked that for any $\beta > 0$ and any $\rho \in \mathbb{R}$, the product measure with Gaussian marginals of mean ρ and variance (temperature) β^{-1} are stationary under the dynamics of $\{\eta_x(t); x \in \mathbb{Z}\}$. Let us assume that the dynamics starts from a stationary state. For simplicity we assume $\rho = 0$ and $a = 1$. The space-time energy correlation function $S_\varepsilon(x,t)$ is defined here by

$$S_\varepsilon(x,t) = \left\langle (\eta_x(\varepsilon^{-3/2}t)^2 - \beta^{-1})\, (\eta_0(0)^2 - \beta^{-1}) \right\rangle .$$

It turns out that the energy fluctuations are driven by volume correlations. Therefore it makes sense to define the *volume* correlation function as

$$G_\varepsilon(x,y,t) = \left\langle \eta_x(\varepsilon^{-3/2}t)\eta_y(\varepsilon^{-3/2}t)\, (\eta_0(0)^2 - \beta^{-1}) \right\rangle .$$

Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a smooth, regular function and for each $t \in [0,T]$, let $u_t : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be the solution of the boundary-value problem

$$\begin{cases} -\partial_x u + \gamma \partial_y^2 u = 0 \\ \qquad \partial_y u(x,0) = \partial_x f(x,t). \end{cases}$$

[2] In [15] only the equilibrium fluctuations are considered but the methods developed in [26, 27] can be applied also to the models considered in this section.

It turns out that $\partial_x u_t(x,0) = \mathscr{L} f_t(x)$, where $\mathscr{L}$ is the skew fractional Laplacian defined in (5.40).

For test functions $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ define

$$\langle S_\varepsilon(t), f\rangle_\varepsilon = \sum_{x\in\mathbb{Z}} S_\varepsilon(x,t) f(\varepsilon x),$$

$$\langle G_\varepsilon(t), u\rangle_\varepsilon = \sum_{\substack{x,y\in\mathbb{Z}\\ x<y}} G_\varepsilon(x,y,t)\; u\big(\tfrac{\varepsilon}{2}(x+y), \sqrt{\varepsilon}(y-x)\big).$$

After an explicit calculation we have:

$$\frac{d}{dt}\Big\{\langle S_\varepsilon(t) f_t\rangle_\varepsilon - \frac{\varepsilon^{1/2}}{2\gamma}\langle G_\varepsilon(t), u_t\rangle_\varepsilon + \varepsilon\langle S_\varepsilon(t), u_t(\cdot,0)\rangle_\varepsilon\Big\} = \langle S_\varepsilon(t), \partial_x f_t\rangle_\varepsilon$$

plus error terms that vanish as $\varepsilon \to 0$. From this observation it is not very difficult to obtain that, for any smooth function f of compact support,

$$\lim_{\varepsilon\to 0}\langle S_\varepsilon(t), f\rangle_\varepsilon = \int P(t,x) f(x)\,dx,$$

where $P(t,x)$ is the fundamental solution of the skew fractional heat equation (5.40).

Let us explain in more details why the introduction of the test function u_t solves the equation for the energy correlation function $S_\varepsilon(x,t)$. It is reasonable to parametrize volume correlations by its distance to the diagonal $x = y$. The microscopic current associated to the energy $\eta_x(t)^2$ is equal to $\eta_x(t)\eta_{x+1}(t)$, which can be understood as the volume correlations around the diagonal $x = y$. Volume evolves in the hyperbolic scale with speed 2. Fluctuations around this transport evolution appear in the diffusive scale and are governed by a diffusion equation. This means that at the hyperbolic scale ε^{-1}, fluctuations are of order $\varepsilon^{-1/2}$, explaining the non-isotropic space scaling introduced in the definition of $\langle G_\varepsilon(t), u\rangle_\varepsilon$. It turns out that the couple $(S_\varepsilon, G_\varepsilon)$ satisfies a closed system of equations, which can be checked to be a semidiscrete approximation of the system

$$\begin{cases} \sqrt{\varepsilon}\partial_t u_t = -\partial_x u_t + \gamma\,\partial_y^2 u_t \\ \partial_y u_t(x,0) = \partial_x f_t(x) \\ \partial_t f_t = \partial_x u_t(x,0). \end{cases}$$

Therefore, the volume serves as a fast variable for the evolution of the energy, which corresponds to a slow variable. The extension problem plays the role of the cell problem for the homogenization of this fast-slow system of evolutions.

5.9 The Dynamics in Higher Dimension

One of the interesting features of the harmonic dynamics with energy and momentum conservative noise is that they reproduce, at least qualitatively, the expected behavior of the non-linear dynamics, also in higher dimensions. In particular in the

three or higher dimension the thermal conductivity, computed by the Green-Kubo formula, is finite, while it diverges logarithmically in two dimension (always for non-acoustic systems), cf. [5, 6].

In dimension $d \geq 3$ it can be also proven that equilibrium fluctuations evolve diffusively, i.e. that the asymptotic correlation $\tilde{S}^{33}(y,t)$, defined in Sect. 5.5 but with a diffusive scaling, satisfies [4]:

$$\partial_t \tilde{S}^{33} = D\partial_y^2 \tilde{S}^{33} \tag{5.41}$$

where D can be computed explicitly by Green-Kubo formula in terms of ω and scattering rate R [5, 6]. Similar finite diffusivity and diffusive evolution of the fluctuations are proven for pinned models ($\hat{\alpha}(0) > 0$), see [4]. In two dimensions, while the logarithmic divergence of the Green-Kubo expression of the thermal conductivity is proven in [6], the corresponding diffusive behaviour at the logarithmic corrected time scale is still an open problem. For the result obtained from the kinetic equation see [2]. The two-dimensional model is particularly interesting in light of the large thermal conductivity measured experimentally on graphene [46], an essentially a two-dimensional material.

5.10 Thermal Boundary Conditions and the Non-equilibrium Stationary States

Traditionally the problem of thermal conductivity has been approached by considering the stationary non-equilibrium state for a finite system in contact with heat bath at different temperatures (cf.[31, 38]). This set-up is particularly suitable for numerical simulations and convenient because it gives an straight operational definition of the thermal conductivity in terms of the stationary flux of energy, avoiding to specify the macroscopic evolution equation. But for the theoretical understanding and corresponding mathematical proofs of the thermal conductivity phenomena, this is much harder than the non-stationary approach described in the previous section. This because the stationary state conceals the space-time scale.

The finite system consists of $2N+1$ atoms, labelled by $x = -N, \dots, N$, with end points connected to two heat baths at temperature T_l and T_r. These baths are modeled by Langevin stochastic dynamics, so that the evolution equations are given by

$$\begin{aligned}
\dot{q}_x &= p_x, \qquad\qquad x = -N, \dots, N, \\
\dot{p}_x &= -(\alpha_N * q(t))_x + (p_{x+1}(t^-) - p_x(t^-))\, \dot{N}_{x,x+1}(t) + (p_{x-1}(t^-) - p_x(t^-))\, \dot{N}_{x-1,x}(t) \\
&\quad + \delta_{x,-N} \left(-p_{-N} + \sqrt{2T_l}\, dw_{-N}(t) \right) + \delta_{x,N} \left(-p_N + \sqrt{2T_r}\, dw_N(t) \right),
\end{aligned} \tag{5.42}$$

where $w_{-N}(t), w_N(t)$ are two independent standard Brownian motions, and the coupling α_N is properly defined in order to take into account the boundary conditions. For this finite dynamics there is a (non-equilibrium) unique stationary state, where the energy flow from the hot to the cold side. Observe that because of the exchange noise between the atoms, the stationary state is not Gaussian, unlike in the case studied in [38].

Denoting the stationary energy flux by J_N, the thermal conductivity of the finite chain is defined as

$$\kappa_N = \lim_{|T_l - T_r| \to 0} \frac{(2N+1)J_N}{T_l - T_r}. \tag{5.43}$$

For a finite N it is not hard to prove that κ_N can be expressed in terms of the corresponding Green-Kubo formula. For the periodic boundary unpinned acoustic case this identification gives $\kappa_N \sim N^{1/2}$ (cf. [5]). For the noise that conserves only the energy, but not momentum (like independent random flips of the signs of the momenta), the system has a finite thermal diffusivity and the limit $\kappa_N \to \kappa$, as $N \to +\infty$, can be computed explicitly, as proven in [11].

The natural question is about the macroscopic evolution of the temperature profile in a non-stationary situation, and the corresponding stationary profile. It turns out [7] that this macroscopic equation is given by a fractional heat equation similar to (5.17) with a proper definition of the fractional Laplacian $|\Delta|^{3/4}$ on the interval $[-1, 1]$ subject to the boundary conditions $\mathfrak{T}(-1) = T_l, \mathfrak{T}(1) = T_r$. This is defined by using the following orthonormal basis of functions on the interval $[-1, 1]$:

$$u_n(y) = \cos\left(n\pi(y+1)/2\right), \qquad n = 0, 1, \dots \tag{5.44}$$

Any continuous function $f(y)$ on $[-1, 1]$ can be expressed in terms of a series expansion in u_n. Then we define $|\Delta|^s u_n(y) = (n\pi/2)^{2s} u_n(y)$. Observe that for $s = 1$ we recover the usual definition of the Laplacian. For $s \neq 1$ this is *not* equivalent to other definitions of the fractional Laplacian in a bounded interval, e.g. [47].

Correspondingly the stationary profile is given by

$$\begin{aligned} |\Delta|^s \mathfrak{T} &= 0, \text{ in } (-1, 1) \\ \mathfrak{T}(-1) &= T_l, \ \mathfrak{T}(1) = T_r, \end{aligned} \tag{5.45}$$

namely $\mathfrak{T}(y) = \frac{1}{2}(T_l + T_r) + \frac{1}{2}(T_l - T_r)\theta(y)$, with

$$\theta(y) = c_s \sum_{m \text{ odd}} \frac{1}{m^{2s}} \cos\left(m\pi(y+1)/2\right),$$

c_s such that $\theta(-1) = 1$. This expression corresponds with the formula computed directly in [32] using a continuous approximation of the covariance matrix of the stationary state for the dynamics with fixed boundaries.

5.11 The Non-linear Chain

From the above rigorous results on the harmonic chain with the random collision dynamics, and the arguments of Spohn from fluctuating hydrodynamics and mode couplings (cf. [41, 42]), we can conjecture the corresponding behaviour in the anharmonic case.

Let us consider just nearest neighbor interaction given by the potential energy $V(q_x - q_{x-1})$ of an anharmonic spring. We assume $V : \mathbb{R} \to (0, +\infty)$ is smooth and that it grows quadratically at infinity. Define the energy of the oscillator x as

$$e_x(r, p) := \frac{p_x^2}{2} + V(r_x) \tag{5.46}$$

The dynamics is defined as the solution of the Newton equations

$$\dot{q}_x(t) = p_x, \quad \dot{p}_x(t) = -\left(V'(r_x) - V'(r_{x+1})\right), \quad x \in \mathbb{Z} \tag{5.47}$$

plus a random exchange of velocities as in the previous sections, regulated by an intensity γ. The equilibrium Gibbs measures are parametrized by

$$\boldsymbol{\lambda} = (\beta^{-1} \text{ (temperature)}, \bar{p} \text{ (velocity)}, \tau \text{ (tension)}),$$

and are given explicitly by

$$d\nu_{\boldsymbol{\lambda}} = \prod_x e^{-\beta(V(r_x) - \tau r_x + \frac{(p_x - \bar{p})^2}{2}) - \mathscr{G}(\boldsymbol{\lambda})}\, dr_x\, dp_x \tag{5.48}$$

When a random exchange of velocity is present ($\gamma > 0$) it can be proven that these are the only *regular* translation invariant stationary measures [13, 23]. We have

$$\nu_\lambda(p_x) = \bar{p}, \quad \nu_\lambda(r_x) = -\frac{1}{\beta}\partial_\tau \mathscr{G} := \bar{r}, \quad \nu_\lambda(e_x) = \frac{1}{\beta} - \partial_\beta \mathscr{G} - \frac{\tau}{\beta}\partial_\tau \mathscr{G} := \bar{e}.$$

These thermodynamical relations can be inverted to express the parameters $(\beta^{-1}, \bar{p}, \tau)$ in terms of $(\bar{p}, \bar{r}, \bar{e})$. It turns out that the tension is then equal to a nonlinear function $\bar{\tau}(\bar{r}, \bar{u})$ of the average stretch $\bar{r}$ and the average internal energy $\bar{u} = \bar{e} - \bar{p}^2/2$.

After the hyperbolic rescaling of the dynamics, we expect that the empirical distribution of the balanced quantities converge to the system of hyperbolic equations:

$$\begin{aligned} \partial_t \bar{r} &= \partial_y \bar{p} \\ \partial_t \bar{p} &= \partial_y \bar{\tau} \\ \partial_t \bar{e} &= \partial_y (\bar{p}\bar{\tau}) . \end{aligned} \tag{5.49}$$

This limit can be proven, under certain condition on the boundaries, in the smooth regime, if the microscopic dynamics is perturbed by a random exchange of velocities between nearest neighbors particles by using relative entropy methods [13, 22, 36].

The limit should be still valid after shocks develop, with the limit profile given by an entropic weak solution. This is a main open problem in hydrodynamic limits.

After a long time, the (entropy) solution of (5.49) should converge (maybe in a weak sense) to some mechanical equilibrium characterized by:

$$\bar{p}(y) = p_0, \qquad \bar{\tau}(\bar{r}(y), \bar{u}(y)) = \tau_0, \quad \text{for some constants } p_0, \tau_0. \tag{5.50}$$

It is very hard to characterize all possible stationary solutions that satisfy (5.50). Probably they are generically very irregular. But certainly if we start with a smooth initial condition that satisfies (5.50), they do not move. Also by the relative entropy methods, it is possible to prove that starting with such initial profiles, the empirical distribution of the balanced quantities will converge at the hyperbolic space-time scale to such a stationary solution at any time.

Still we do know that the microscopic dynamics will converge to a global equilibrium, so this implies that there exists a larger time scale such that these profiles will evolve and eventually reach also thermal equilibrium.

There is a numerical evidence and heuristic arguments about the divergence of the Green-Kubo formula defining the thermal diffusivity for such one dimensional systems, so we expect that the larger time scale at which these profiles evolve is superdiffusive.

From the nonlinear fluctuation hydrodynamics [41], one can conjecture the following: the space-time scale is $(\varepsilon^{-1}x, \varepsilon^{-2a}t)$, and the temperature $T(x,t) = \beta^{-1}(x,t)$ evolves following some fractional heat equation, possibly non-linear. If V is symmetric and $\tau_0 = 0$, then $a = 3/4$, and in all other cases $a = 5/6$.

5.12 The Disordered Chain

The effect of disorder on transport and phonons localization properties in chains of oscillators has attracted a lot of interest [19, 31]. Randomness can appear at the level of the masses of particles or at the level of the potentials. We consider only the case of random masses with non random potential V or the case of non random masses and non random interaction potential V with random harmonic pinnings. In the first case, the Hamiltonian is then given by

$$\mathscr{H}(p,q) := \sum_x \frac{p_x^2}{2m_x} + \sum_{x,x'} V(q_x, q'_x)$$

where $\{m_x\}$ are positive random variables, while in the second case, the Hamiltonian is given by

$$\mathscr{H}(p,q) := \sum_x \frac{p_x^2}{2m} + \sum_{x,x'} V(q_x, q'_x) + \sum_x \nu_x q_x^2$$

where $\{v_x\}$ are positive random variables, $m > 0$ being the mass of the particles. The presence of randomness is relevant for the thermal properties of the system but the fact that randomness affect potentials or masses is not.

For one dimensional unpinned disordered harmonic chains it is known that the behavior of the conductivity is very sensitive to the boundary conditions since it can diverge as $\sqrt{N}$ or vanish as $1/\sqrt{N}$ with the systems length N [1, 18, 39, 45]. If harmonic pinning is added localization of normal modes leads an exponential decay of the heat current and a zero conductivity. The situation in higher dimensions, even in the case of harmonic interactions, is still under debate but it is expected that conductivity is finite in dimension $d \geq 3$ if disorder is sufficiently weak [29]. About the effect of nonlinearities, numerical evidences suggest that a very small amount of anharmonicity in pinned chains is sufficient to restore a diffusive regime with a positive finite value of the conductivity [20]. However it is a challenging open question to decide if the transition from an insulator to a conductor occurs at zero or some finite small value of anharmonicity [8, 20, 35].

It is suggestive to think that a stochastic noise could affect transport properties of harmonic chains in some rough sense similar to the addition of nonlinearities. This question has been first address in [9] in the Green-Kubo formula setting, revisited in [21] from the non equilibrium stationary state point of view (see Sect. 5.10) and extended in [10, 24] to incorporate weakly nonlinear chains. In these papers, the authors consider a disordered harmonic chain, or weakly nonlinear in [10, 24], with a stochastic noise which consists to flip, independently for each particle, at independent random exponential times of mean $1/\lambda$, $\lambda > 0$, the velocity of the particle. Notice that this energy conserving noise is very different from the noise considered in the rest of the paper since it does not conserve momentum. In particular, for ordered pinned and unpinned nonlinear chains, this noise is sufficient to provide a finite conductivity κ [12]. However, it turns out that for an ordered harmonic chain, $\kappa \sim \lambda^{-1}$ and, as expected, increases to infinity as the strength of the noise $\lambda \downarrow 0$. In [10, 24] it is proved that localization effects persist: $\kappa = \mathcal{O}(\lambda)$ for a pinned disordered chain with a small anharmonic potential, and $\kappa \sim \lambda$ for a pinned harmonic chain. As far as we know disordered chains with energy-momentum conserving noise have never been investigated.

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