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Reinhard Kahle
Thomas Strahm
Thomas Studer
Editors

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Advances in Proof Theory

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Preface

Advances in proof theory was the title of a symposium organized on the occasion of the 60th birthday of Gerhard Jäger. The meeting took place on December 13 and 14, 2013, at the University of Bern, Switzerland.

The aim of this symposium was to bring together some of the best specialists from the area of proof theory, constructivity, and computation and discuss recent trends and results in these areas. Some emphasis was put on ordinal analysis, reductive proof theory, explicit mathematics and type-theoretic formalisms, as well as abstract computations.

Gerhard Jäger has devoted his research to these topics and has substantially advanced and shaped our knowledge in these fields.

The program of the symposium was as follows:

Friday, December 13

Wolfram Pohlers: *From Subsystems of Classical Analysis to Subsystems of Set Theory: A personal account*

Wilfried Buchholz: *On the Ordnungszahlen in Gentzen's First Consistency Proof*

Andrea Cantini: *About Truth, Explicit Mathematics and Sets*

Peter Schroeder-Heister: *Proofs That, Proofs Why, and the Analysis of Paradoxes*

Roy Dyckhoff: *Intuitionistic Decision Procedures since Gentzen*

Grigori Mints: *Two Examples of Cut Elimination for Non-Classical Logics*

Rajeev Goré: *From Display Calculi to Decision Procedures via Deep Inference for Full Intuitionistic Linear Logic*

Pierluigi Minari: *Transitivity Elimination: Where and Why*

Saturday, December 14

Per Martin-Löf: *Sample Space-Event Time*

Anton Setzer: *Pattern and Copattern Matching*

Helmut Schwichtenberg: *Computational Content of Proofs Involving Coinduction*

Michael Rathjen: *When Kripke-Platek Set Theory Meets Powerset*

Stan Wainer: *A Miniaturized Predicativity*

Peter Schuster: *Folding Up*

Solomon Feferman: *The Operational Perspective*

This volume comprises contributions of most of the speakers and represents the wide spectrum of Gerhard Jäger's interests. We deeply miss Grisha Mints who planned to contribute to this Festschrift.

We acknowledge gratefully the financial support of Altonaer Stiftung für philosophische Grundlagenforschung, Burgergemeinde Bern, Swiss Academy of Sciences, Swiss National Science Foundation, and Swiss Society for Logic and Philosophy of Science. We further thank the other members of the program committee, namely Roman Kuznets, George Metcalfe, and Giovanni Sommaruga.

For the production of this volume, we thank the editors of the *Progress in Computer Science and Applied Logic (PCS)* Series, the staff members of Birkhäuser/Springer Basel, and the reviewers of the papers of this volume.

We dedicate this Festschrift to Gerhard Jäger and thank him for his great intellectual inspiration and friendship.

Lisbon	Reinhard Kahle
Bern	Thomas Strahm
Bern	Thomas Studer
December 2015	

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A Survey on Ordinal Notations Around the Bachmann-Howard Ordinal

Wilfried Buchholz

Dedicated to Gerhard Jäger on the occasion of his 60th birthday.

Abstract Various ordinal functions which in the past have been used to describe ordinals not much larger than the Bachmann-Howard ordinal are set into relation.

1 Introduction

In recent years a renewed interest in ordinal notations around the Bachmann-Howard ordinal $\phi_{\varepsilon_{\Omega+1}}(0)$ has evolved, amongst others caused by Gerhard Jäger's metapredicativity program. Therefore it seems worthwile to review some important results of this area and to present detailed and streamlined proofs for them. The results in question are mainly comparisons of various functions which in the past have been used for describing ordinals not much larger than the Bachmann-Howard ordinal. We start with a treatment of the Bachmann hierarchy $\left(\phi_\alpha\right)_{\alpha\leq\Gamma_{\Omega+1}}$ from [3]. This hierarchy consists of normal functions $\phi_\alpha : \Omega \to \Omega$ $(\alpha \leq \Gamma_{\Omega+1})$ which are defined by transfinite recursion on α referring to previously defined fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$ (with $\tau_\alpha \leq \Omega$). The most important new concept in Bachmann's approach is the systematic use of ordinals $\alpha > \Omega$ as indices for functions from Ω into Ω. Bachmann describes his approach as a generalization of a method introduced by Veblen in [22]; according to him the initial segment $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ is just a modified presentation of a system of normal functions defined by Veblen. But actually this connection is not so easy to see. At the end of Sect. 2 we will establish the connection between $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ and Schütte's Klammersymbols [19] for which the relation to [22] is clear

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cf. [19, footnote 4]. In Sect. 3 we give an alternative characterization of the Bachmann hierarchy which instead of fundamental sequences $\big(\alpha[\xi]\big)_{\xi<\tau_\alpha}$ uses finite sets $K\alpha \subseteq \Omega$ of *coefficients* ("Koeffizienten"). For $\alpha < \varepsilon_{\Omega+1}$, $K\alpha$ is almost identical to the set $C(\alpha)$ of *constituents* (i.e., ordinals $< \Omega$ which occur in the complete base Ω Cantor normal form of α) in [15], where it was shown how to construct a recursive system of ordinal notations on the basis of Bachmann's functions.

In the 1960s, the Bachmann method for generating hierarchies of normal functions on Ω was extended by Pfeiffer [17] and, much further, by Isles [16]. These extensions were highly complex; especially the Isles approach was so complicated that it was practically unusable for proof-theoretic applications. Therefore Feferman, in unpublished work around 1970, proposed an entirely different and much simpler method for generating hierarchies of normal functions θ_α ($\alpha \in On$) (see e.g. [14]). Aczel (in [1]) showed how the θ_α ($\alpha < \Gamma_{\Omega+1}$) correspond to Bachmann's ϕ_α. (Independently, Weyhrauch [23] established the same results for $\alpha < \varepsilon_{\Omega+1}$.) In addition, Aczel generalized Feferman's definition and conjectured that the generalized hierarchy (θ_α) matches up with the Isles functions. This conjecture was proved by Bridge in [4, 5]. In Sect. 4 of the present paper we show how Feferman's functions θ_α ($\alpha < \Gamma_{\Omega+1}$) can also be defined by use of the $K\alpha$'s. Together with the content of Sect. 3 this leads to an easy comparison of the hierarchies $\big(\phi_\alpha\big)_{\alpha<\Gamma_{\Omega+1}}$ and $\big(\theta_\alpha\big)_{\alpha<\Gamma_{\Omega+1}}$ which becomes particularly simple if one switches to the fixed-point-free versions: $\overline{\phi}_\alpha(\beta) = \overline{\theta}_\alpha(\beta)$ for all $\alpha < \Gamma_{\Omega+1}$, $\beta < \Omega$ (Theorem 4.7).

In Sects. 5, 6 we deal with the unary functions $\vartheta : \varepsilon_{\Omega+1} \to \Omega$ and $\psi : \varepsilon_{\Omega+1} \to \Omega$ which play an important rôle in [18]. We show that $\overline{\theta}_{1+\alpha}(\beta) = \vartheta(\Omega\alpha + \beta)$ (for $\alpha < \varepsilon_{\Omega+1}$, $\beta < \Omega$) and refine a result from [18] on the relationship between ϑ and ψ. In Sect. 7, largely following [23], we show how the Bachmann hierarchy below $\varepsilon_{\Omega+1}$ can be defined by means of functionals of finite higher types.

A nice survey on the history of the subject can be found in [13].

Preliminaries. The letters $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta$ always denote ordinals. *On* denotes the class of all ordinals and *Lim* the class of all limit ordinals. We are working in ZFC. So, every ordinal α is identical to the set $\{\xi \in On : \xi < \alpha\}$, and we have $\beta < \alpha \Leftrightarrow \beta \in \alpha$ and $\beta \leq \alpha \Leftrightarrow \beta \subseteq \alpha$. For $X \subseteq On$ we define: $X <(\leq)\, \alpha :\Leftrightarrow \forall x \in X\, (x <(\leq)\, \alpha)$ and $\alpha \leq X :\Leftrightarrow \exists x \in X (\alpha \leq x)$, i.e., $X < \alpha \Leftrightarrow X \subseteq \alpha$ and $\alpha \leq X \Leftrightarrow \neg(X < \alpha)$. By $\mathbb{H}$ we denote the class $\{\gamma \in On : \forall \alpha, \beta < \gamma(\alpha + \beta < \gamma)\} = \{\omega^\alpha : \alpha \in On\}$ of all *additive principal numbers* (*Hauptzahlen*), and by $\mathbb{E}$ the class $\{\alpha \in On : \omega^\alpha = \alpha\} = \{\varepsilon_\alpha : \alpha \in On\}$ of all *epsilon-numbers*. A *normal function* is a strictly increasing continuous function $F : On \to On$. The normal functions $\varphi_\alpha : On \to On$ ($\alpha \in On$) are defined by: $\varphi_0(\beta) := \omega^\beta$, and $\varphi_\alpha :=$ ordering (or enumerating) function of $\{\beta : \forall \xi < \alpha(\varphi_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The family $(\varphi_\alpha)_{\alpha \in On}$ is called *the Veblen hierarchy over* $\lambda\xi.\omega^\xi$. An ordinal α is called *strongly critical* iff $\varphi_\alpha(0) = \alpha$. The class of all strongly critical ordinals is denoted by SC, and its enumerating function by $\lambda\alpha.\Gamma_\alpha$. It is well-known that $\lambda\alpha.\Gamma_\alpha$ is again a normal function, and that $\Gamma_\Omega = \Omega$, where Ω is the least regular ordinal $>\omega$.

2 Fundamental Sequences and the Bachmann Hierarchy

The following stems from Bachmann's seminal paper [3], but in some minor details we deviate from that paper. We start by assigning to each limit number $\alpha \le \Gamma_{\Omega+1}$ a fundamental sequence $(\alpha[\xi])_{\xi<\tau_\alpha}$ with $\tau_\alpha \le \Omega$. The definition of $\alpha[\xi]$ is based on the normal form representation of α in terms of $0, +, \cdot, F$, where $(F_\alpha)_{\alpha\in On}$ is the Veblen hierarchy over $\lambda x.\Omega^x$, i.e., $F_0(\beta) := \Omega^\beta$, and $F_\alpha :=$ ordering function of $\{\beta : \forall\xi < \alpha(F_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The relationship between F_α and φ_α for $\alpha > 0$ is given by

$$F_\alpha(\beta) = \varphi_\alpha(\widetilde{\alpha} + \beta) \text{ with } \widetilde{\alpha} := \begin{cases} \Omega{+}1 & \text{if } 0 < \alpha < \Omega, \\ 1 & \text{if } \alpha = \Omega, \\ 0 & \text{if } \Omega < \alpha. \end{cases}$$

From this it follows that $\Gamma_{\Omega+1}$ is the least fixed point of $\lambda\alpha.F_\alpha(0)$.
For completeness note, that $F_0(\beta) = \varphi_0(\Omega\beta)$.

Abbreviations

1. $\Lambda := \Gamma_{\Omega+1} \ = \min\{\alpha : F_\alpha(0) = \alpha\}$.
2. $\alpha|\gamma \ :\Leftrightarrow \ \exists\xi(\gamma = \alpha{\cdot}\xi)$.
3. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \ :\Leftrightarrow \ \alpha = \gamma + \Omega^\beta\eta \ \& \ 0 < \eta < \Omega \ \& \ \Omega^{\beta+1}|\gamma$.
4. $\gamma =_{\mathrm{NF}} F_\alpha(\beta) \ :\Leftrightarrow \ \alpha, \beta < \gamma = F_\alpha(\beta)$.

Proposition

(a) For each $0 < \delta < \Lambda$ *there are unique* γ, β, η *such that* $\delta =_{\mathrm{NF}} \gamma + \Omega^\beta\eta$.
(b) For each $\delta \in \mathrm{ran}(F_0) \cap \Lambda$ *there are unique* α, β *such that* $\delta =_{\mathrm{NF}} F_\alpha(\beta)$.
(c) $\delta < \Lambda \ \Rightarrow \ (\delta =_{\mathrm{NF}} F_\alpha(\beta) \Leftrightarrow \beta < \delta = F_\alpha(\beta))$.

Definition of a fundamental sequence $(\lambda[\xi])_{\xi<\tau_\lambda}$ for each limit number $\lambda \le \Lambda$

1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$:
1.1. $\eta \in$ *Lim*: $\tau_\lambda := \eta$ and $\lambda[\xi] := \gamma + \Omega^\beta{\cdot}(1{+}\xi)$.
1.2. $\eta = \eta_0{+}1$: $\tau_\lambda := \tau_{\Omega^\beta}$ and $\lambda[\xi] := \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi]$.
2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$:
2.1. $\beta \in$ *Lim*: $\tau_\lambda := \tau_\beta$ and $\lambda[\xi] := F_\alpha(\beta[\xi])$.
2.2. $\beta \notin$ *Lim*: Let $\lambda^- := \begin{cases} 0 & \text{if } \beta = 0, \\ F_\alpha(\beta_0){+}1 & \text{if } \beta = \beta_0{+}1. \end{cases}$
2.2.0. $\alpha = 0$: Then $\beta = \beta_0{+}1$. $\tau_\lambda := \Omega$ and $\lambda[\xi] := \Omega^{\beta_0}{\cdot}(1{+}\xi)$.
2.2.1. $\alpha = \alpha_0{+}1$: $\tau_\lambda := \omega$ and $\lambda[n] := F_{\alpha_0}^{(n+1)}(\lambda^-)$.
2.2.2. $\alpha \in$ *Lim*: $\tau_\lambda := \tau_\alpha$ and $\lambda[\xi] := F_{\alpha[\xi]}(\lambda^-)$.
3. $\tau_\Lambda := \omega$ and $\Lambda[0] := 1$, $\Lambda[n{+}1] := F_{\Lambda[n]}(0)$.

Definition
For each limit $\lambda \leq \Lambda$ we set $\lambda[\tau_\lambda] := \lambda$.
Further $\tau_0 := 0$, $0[\xi] := 0$ and $\tau_{\alpha+1} := 1$, $(\alpha{+}1)[\xi] := \alpha$.

Lemma 2.1 $\lambda =_{\mathrm{NF}} F_\alpha(\beta) < \Lambda \ \&\ \beta \in \mathit{Lim} \ \&\ 1 \leq \xi < \tau_\beta \ \Rightarrow$ $\lambda[\xi] =_{\mathrm{NF}} \ F_\alpha(\beta[\xi])$.

Proof Cf. Appendix.

Lemma 2.2 *Let* $\lambda \in \mathit{Lim} \cap (\Lambda{+}1)$.

(a) $\xi < \eta \leq \tau_\lambda \ \Rightarrow\ \lambda[\xi] < \lambda[\eta]$.
(b) $\lambda = \sup_{\xi<\tau_\lambda} \lambda[\xi]$.
(c) $\eta \in \mathit{Lim} \cap (\tau_\lambda + 1) \ \Rightarrow\ \lambda[\eta] \in \mathit{Lim} \ \&\ \tau_{\lambda[\eta]} = \eta \ \&\ \forall \xi < \eta(\lambda[\eta][\xi] = \lambda[\xi])$.
(d) $\xi < \tau_\lambda \ \&\ \lambda[\xi] < \delta \leq \lambda[\xi{+}1] \ \Longrightarrow\ \lambda[\xi] \leq \delta[1]$.

The proof of (a), (b), (c) is left to the reader. The proof of (d) will be given in the Appendix.

We now introduce a binary relation $\underline{\ll}$ which corresponds to Bachmann's $\rightarrow$ (cf. [3] p. 123, 130) and is essential for proving the basic properties of the Bachmann hierarchy. The advantage of $\ll$ over $\rightarrow$ is that its definition does not refer to the functions ϕ_α but only to the fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$.

Definition of $\ll^1$, $\ll$ and $\underline{\ll}$

1. $\beta \ll^1 \alpha \ :\Leftrightarrow\ \alpha \leq \Lambda \ \&\ \beta \in \{\alpha[\xi] : \xi < \tau_\alpha^\circ\}$, where $\tau_\alpha^\circ := \begin{cases} \omega & \text{if } \tau_\alpha = \Omega, \\ \tau_\alpha & \text{otherwise.} \end{cases}$
2. $\ll$ ($\underline{\ll}$) is the transitive (transitive and reflexive) closure of $\ll^1$.

Lemma 2.3 *Let* $\alpha \leq \Lambda$.

(a) $\alpha \in \mathit{Lim} \ \&\ \xi{+}1 < \tau_\alpha \ \Rightarrow\ \alpha[\xi]{+}1 \ \underline{\ll}\ \alpha[\xi{+}1]$.
(b) $\alpha \in \mathit{Lim} \ \&\ \xi < \eta < (\tau_\alpha{+}1) \cap \Omega \ \Rightarrow\ \alpha[\xi] \ll \alpha[\eta]$.
(c) $\beta \ll \alpha \ \Rightarrow\ \beta{+}1 \ \underline{\ll}\ \alpha$.
(d) $n < \omega \ \&\ n \leq \alpha \ \Rightarrow\ n \ \underline{\ll}\ \alpha$.

Proof
(a) By induction on δ we prove: $\alpha[\xi] < \delta \leq \alpha[\xi{+}1] \Rightarrow \alpha[\xi] + 1 \ \underline{\ll}\ \delta$.

1. $\delta = \delta_0{+}1$ with $\alpha[\xi] \leq \delta_0$: Then either $\alpha[\xi]{+}1 = \delta$ or $\alpha[\xi]{+}1 \overset{\mathrm{IH}}{\underline{\ll}} \delta_0 \ll^1 \delta$.
2. $\delta \in \mathit{Lim}$:

By Lemma 2.2a, d, $\alpha[\xi] < \delta[2] < \alpha[\xi{+}1]$. Hence $\alpha[\xi]{+}1 \overset{\mathrm{IH}}{\underline{\ll}} \delta[2] \ll^1 \delta$.

(b) Induction on η:

1. $\eta = \eta_0{+}1 < \tau_\alpha$: $\alpha[\xi] \overset{\text{IH}}{\underline{\ll}} \alpha[\eta_0] \ll^1 \alpha[\eta_0] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\eta]$.
2. $\eta \in \mathit{Lim}$: Then $\tau_{\alpha[\eta]} = \eta$ and $\alpha[\xi] = \alpha[\eta][\xi] \ll^1 \alpha[\eta]$.

(c) We may assume $\beta \ll^1 \alpha$, i.e. $\beta = \alpha[\xi]$ with $\xi < \tau^\circ_\alpha$.

Then either $\tau^\circ_\alpha = 1 \;\&\; \beta{+}1 = \alpha$ or $\tau^\circ_\alpha \in \mathit{Lim} \;\&\; \alpha[\xi] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\xi{+}1] \ll^1 \alpha$.

(d) Induction on n:

1. Using Lemma 2.2a we get $0 \underline{\ll} \alpha$ by transfinite induction on α.
2. $n{+}1 \le \alpha \;\Rightarrow\; n < \alpha \;\&\; n \overset{\text{IH}}{\underline{\ll}} \alpha \;\Rightarrow\; n \ll \alpha \overset{\text{(c)}}{\Rightarrow} n{+}1 \underline{\ll} \alpha$.

Definition
An Ω-*normal function* is a strictly increasing continuous function $f : \Omega \to \Omega$.
A set $M \subseteq \Omega$ is Ω-*club* (*closed and unbounded in* Ω) iff

$$\forall X \subseteq M(X \neq \emptyset \;\&\; \sup(X) < \Omega \Rightarrow \sup(X) \in M) \text{ and } \forall \alpha < \Omega \exists \beta \in M(\alpha < \beta).$$

It is well-known that $M \subseteq \Omega$ is Ω-club if, and only if, M is the range of some Ω-normal function. Hence the ordering function of any Ω-club set is Ω-normal.

The collection of Ω-club sets has the following closure properties:

1. If f is Ω-normal then $\{\beta \in \Omega : f(\beta) = \beta\}$ is Ω-club.
2. If $(M_\xi)_{\xi<\alpha}$ is a sequence of Ω-club sets with $0 < \alpha < \Omega$ then $\bigcap_{\xi<\alpha} M_\xi$ is Ω-club.
3. If $(M_\xi)_{\xi<\Omega}$ is a sequence of Ω-club sets then also $\{\alpha \in \Omega : \alpha \in \bigcap_{\xi<\alpha} M_\xi\}$ is Ω-club.

Drawing upon 1.–3. and upon the above assignment of fundamental sequences we now define Bachmann's hierarchy of Ω-normal functions ϕ_α ($\alpha \le \Lambda$).

Definition $\phi_\alpha : \Omega \to \Omega$ is the ordering function of the Ω-club set R_α, where R_α is defined by recursion on α as follows:

$$R_0 := \mathbb{H} \cap \Omega,$$
$$R_{\alpha+1} := \{\beta \in \Omega : \phi_\alpha(\beta) = \beta\}\,,$$
$$R_\alpha := \begin{cases} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} & \text{if } \tau_\alpha \in \Omega \cap \mathit{Lim}, \\ \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} & \text{if } \tau_\alpha = \Omega. \end{cases}$$

Notes

1. In Lemma 2.5d we will show that $R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$ if $\tau_\alpha = \Omega$.
2. As mentioned above, our definition of the Bachmann hierarchy (and of F_α) diverges in some minor points from [3]. As a consequence of this, Bachmann's ordinals $H(1) = \varphi_{F_\Omega(1)+1}(1)$ and $\varphi_{F_{\omega_2+1}(1)}(1)$ are $\phi_{F_\Omega(0)}(0)$ and $\phi_\Lambda(0)$, respectively, in the present paper. For more details cf. [2, Note on p. 35].

Lemma 2.4

(a) $\alpha_0 \underline{\ll} \alpha \;\Rightarrow\; R_\alpha \subseteq R_{\alpha_0}$.
(b) $\alpha_0 \ll \alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_\alpha(0)$.
(c) $n < \alpha \cap \omega \;\&\; \beta \in R_\alpha \;\Rightarrow\; \omega{\cdot}n < \beta \in \mathit{Lim}$.

Proof

(a) It suffices to prove $R_\alpha \subseteq R_{\alpha_0}$ for $\alpha_0 \ll^1 \alpha$.

1. $\alpha = \alpha_0{+}1$: Then $R_\alpha = \{\beta \in \Omega : \phi_{\alpha_0}(\beta) = \beta\} \subseteq R_{\alpha_0}$.
2. $\tau_\alpha \in \Omega \cap \mathit{Lim}$: Then $\alpha_0 \in \{\alpha[\xi] : \xi < \tau_\alpha\}$ and thus $R_\alpha = \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$.
3. $\tau_\alpha = \Omega$: $\beta \in R_\alpha \Rightarrow \omega \le \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]} \Rightarrow \beta \in \bigcap_{\xi<\omega} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$, since $\alpha_0 \in \{\alpha[\xi] : \xi < \omega\}$.

(b) 1. $\alpha = \alpha_0{+}1$: $\beta := \phi_\alpha(0) \in R_\alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_{\alpha_0}(\beta) = \beta$.

2. $\alpha_0{+}1 < \alpha$: $\alpha_0 \ll \alpha \overset{2.3c}{\Rightarrow} \alpha_0{+}1 \underline{\ll} \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{\alpha_0+1} \;\Rightarrow\; \phi_{\alpha_0}(0) \overset{1.}{<} \phi_{\alpha_0+1}(0) \le \phi_\alpha(0)$.

(c) We have $1 \le \phi_0(0) < \phi_1(0) < \cdots$ and $\phi_{k+1}(0) \in \mathit{Lim}$. Hence $\omega{\cdot}n < \phi_{n+1}(0)$. Further: $n{<}\alpha \overset{2.3d}{\Rightarrow} n{+}1 \underline{\ll} \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{n+1} \subseteq \{\beta : \phi_{n+1}(0) \le \beta \in \mathit{Lim}\}$.

Lemma 2.5 *For each $\alpha \in \mathit{Lim} \cap (\Lambda{+}1)$ the following holds:*

(a) $\xi < \eta < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\eta]} \subseteq R_{\alpha[\xi]}\ \&\ \phi_{\alpha[\xi]}(0) < \phi_{\alpha[\eta]}(0)$.
(b) $\xi < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; \xi \le \phi_{\alpha[\xi]}(0)$.
(c) $\lambda \in \mathit{Lim} \cap (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.
(d) $\tau_\alpha = \Omega \;\Rightarrow\; R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.
(e) $n < \omega \;\Rightarrow\; \phi_{\alpha[n]}(0) < \phi_\alpha(0)$.

Proof

(a) follows from Lemmata 2.3b, 2.4a, b.
(b) follows from (a).
(c) By Lemma 2.2c we have $\tau_{\alpha[\lambda]} = \lambda$ and $\alpha[\lambda][\xi] = \alpha[\xi]$. Hence, by definition, $R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.
(d) $R_\alpha = \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} \overset{(c)}{=} \{\beta \in \Omega : \beta \in R_{\alpha[\beta]}\} \overset{(b)}{=} \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.
(e) follows from Lemma 2.4b.

Schütte's Klammersymbols

In [19], building on [22], Schütte introduced a system of ordinal notations based on so-called 'Klammersymbols'. A Klammersymbol is a matrix $\begin{pmatrix}\xi_0 & \cdots & \xi_n \\ \alpha_0 & \ldots & \alpha_n\end{pmatrix}$ with $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_n < \Omega$ and $\xi_0, \ldots, \xi_n < \Omega$. Two Klammersymbols are defined to be equal if they are identical after deleting all columns of the form $\binom{0}{\alpha_i}$. This means that one can identify the Klammersymbol $\begin{pmatrix}\xi_0 & \cdots & \xi_n \\ \alpha_0 & \ldots & \alpha_n\end{pmatrix}$ with the ordinal $\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0$. Under this identification the $<$-relation between ordinals induces a well-ordering $\prec$ on the Klammersymbols. To each Ω-normal function f and each Klammersymbol A an ordinal $fA < \Omega$ is assigned by $\prec$-recursion: $f\binom{\xi}{0} := f(\xi)$, and for $\xi_1 > 0$, the function $\lambda x.f\begin{pmatrix}x & \xi_1 & \ldots & \xi_n \\ 0 & \alpha_1 & \ldots & \alpha_n\end{pmatrix}$ is the ordering function of the set $\{\beta \in \Omega : \forall\xi < \xi_1 \forall\alpha_0 < \alpha_1 [f \begin{pmatrix} \beta & \xi & \xi_2 & \cdots & \xi_n \\ \alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} = \beta]\}$. In this subsection we will locate the values $\phi_0 A$ within the Bachmann hierarchy, i.e., we will prove $\phi_0 \begin{pmatrix} \beta & \xi_0 & \cdots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n \end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0}(\beta)$.

Lemma 2.6 *Assume* $\alpha =_{\mathrm{NF}} \gamma + \Omega^{\delta_1}\xi_1$ *with* $\delta_1 < \Omega$.

(a) $\xi < \xi_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + 1 \underline{\ll} \gamma + \Omega^{\delta_1}(\xi{+}1) \underline{\ll} \alpha$.
(b) $\xi < \xi_1 \;\&\; \delta_0 < \delta_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \;\underline{\ll}\; \alpha$.
(c) $\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1[\,\phi_{\gamma+\Omega^{\delta_1}\xi}(\beta) = \beta \;\&\; \forall\delta_0 < \delta_1(\phi_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0}\beta}(0) = \beta)\,]$.

Proof
(a) Let $\hat{\alpha} := \gamma + \Omega^{\delta_1+1}$, $\eta := -1 + (\xi + 1)$, and $\eta_1 := -1 + \xi_1$. Then $\hat{\alpha}[\eta] = \gamma + \Omega^{\delta_1}(\xi{+}1)$, $\hat{\alpha}[\eta_1] = \gamma + \Omega^{\delta_1}\xi_1 = \alpha$, and $\eta \le \eta_1 < \tau_{\hat{\alpha}}$. Hence $\gamma + \Omega^{\delta_1}(\xi{+}1) \underline{\ll} \alpha$ by Lemma 2.3b. For the first inequality one needs the following auxiliary lemma (to be proved by induction on δ_1): $\Omega^{\delta_1}|\gamma_1 \;\Rightarrow\; \gamma_1 + 1 \underline{\ll} \gamma_1 + \Omega^{\delta_1}$.

(b) $\gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \overset{(*)}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_1} = \gamma + \Omega^{\delta_1}(\xi{+}1) \overset{\text{(a)}}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi_1 = \alpha$.

$(*)$ Let $\gamma_1 := \gamma + \Omega^{\delta_1}\xi$. We have $\delta_1 = \delta + n$ with ($\delta_0 < \delta \in$ *Lim* or $\delta = \delta_0 + 1$).

Further, $\gamma_1 + \Omega^{\delta_0+1} \underline{\ll} \gamma_1 + \Omega^{\delta} \ll \gamma_1 + \Omega^{\delta+1} \ll \cdots \ll \gamma_1 + \Omega^{\delta+n}$.

(c) We have to show:
$\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1[\,\beta \in R_{\gamma+\Omega^{\delta_1}\xi+1} \;\&\; \forall\delta_0 < \delta_1(\beta \in R_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0+1}})\,]$.
"$\Rightarrow$": Cf. Lemma 2.4a and (a), (b).
"$\Leftarrow$": We distinguish the following cases:
1. $\xi_1 \in$ *Lim*: $\beta \in \bigcap_{\xi<\xi_1} R_{\gamma+\Omega^{\delta_1}(1+\xi)} = R_\alpha$.
2. $\xi_1 = \xi_0{+}1$:
2.1. $\delta_1 = 0$: Then $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+1} = R_\alpha$.
2.2. $\delta_1 = \delta_0{+}1$: $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\delta_0+1}} = R_\alpha$.
2.3. $\delta_1 \in$ *Lim*: Since $\delta_1 < \Omega$, we then have $\tau_\alpha = \delta_1$ and $\alpha[\xi] = \gamma + \Omega^{\delta_1}\xi_0 + \Omega^{1+\xi}$.
From $\forall\xi < \delta_1(\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\xi+1}})$ we get $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi+1]} \overset{2.5\text{a}}{\subseteq} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.

Definition Due to the fact that every ordinal can be uniquely represented in the form $\Omega\alpha + \beta$ with $\beta < \Omega$ it is possible to code the binary function $(\alpha, \beta) \mapsto \phi_\alpha(\beta)$ $(\alpha \le \Lambda, \beta < \Omega)$ into a unary one by $\phi\langle\Omega\alpha + \beta\rangle := \phi_\alpha(\beta)$ $(\alpha \le \Lambda, \beta < \Omega)$.

Using $\phi\langle\cdot\rangle$, the values of the Klammersymbols can be presented in a particularly nice way (cf. Theorem 2.8a below).

Lemma 2.7 *Assume* $\tilde{\alpha} =_{\mathrm{NF}} \gamma_1 + \Omega^{\alpha_1}\xi_1$ *with* $0 < \alpha_1 < \Omega$.

(a) $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle$ *enumerates*
$Q := \{\beta \in \Omega : \forall\xi < \xi_1 \forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

(b) *If* $\alpha_1 = \alpha_0 + 1$ *then* $Q = \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

Proof There are δ_1 and γ such that $\alpha_1 = 1 + \delta_1$ and $\gamma_1 = \Omega\gamma$. Let $\alpha := \gamma + \Omega^{\delta_1}\xi_1$. From (the proof of) Lemma 2.6c we get

$$\begin{aligned} R_\alpha &= \{\beta \in \Omega : \forall\xi < \xi_1[\, \phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \beta\rangle = \beta \;\& \\ &\qquad\qquad \forall\delta_0 < \delta_1(\phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \Omega^{1+\delta_0}\beta\rangle = \beta)\,]\} \\ &= \{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}, \text{ and} \\ R_\alpha &= \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}, \text{ if } \alpha_1 = \alpha_0{+}1. \end{aligned}$$

On the other side, $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle = \lambda x.\phi\langle\Omega\alpha + x\rangle$ enumerates R_α.

Theorem 2.8 *For* $\alpha_0 < \cdots < \alpha_n < \Omega$ *and* $\xi_0, \ldots, \xi_n < \Omega$:

(a) $\phi_0\begin{pmatrix}\xi_0 & \cdots & \xi_n \\ \alpha_0 & \ldots & \alpha_n\end{pmatrix} = \phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0\rangle$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \cdots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0}(\beta)$.

Proof
(a) W.l.o.g. $\alpha_0 = 0$.

1. $n = 0$: $\phi\langle\Omega^0\xi_0\rangle = \phi\langle\Omega{\cdot}0 + \xi_0\rangle = \phi_0(\xi_0) = \phi_0\begin{pmatrix}\xi_0 \\ 0\end{pmatrix}$.
2. $n > 0$: W.l.o.g. $\xi_1 > 0$.

By Lemma 2.7a, $\lambda x.\phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_1}\xi_1 + x\rangle$ is the ordering function of $\{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

Combining this with the above given definition of $\phi_0 A$ (for Klammersymbols A) the assertion is established by induction on $\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \cdots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} \overset{\text{(a)}}{=} \phi\langle\Omega^{1+\alpha_n}\xi_n + \cdots + \Omega^{1+\alpha_0}\xi_0 + \Omega^0\beta\rangle =$

$$= \phi\langle\Omega{\cdot}(\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0) + \beta\rangle.$$

Lemma 2.9 *For* $\xi_0, \ldots, \xi_n < \Omega$ *let* $\varphi^{n+1}(\xi_n, \ldots, \xi_0) := \phi\langle\Omega^n\xi_n + \cdots + \Omega^0\xi_0\rangle$. *Then the following holds:*

(i) $\varphi^{n+1}(0, \ldots, 0, \beta) = \phi_0(\beta)$.

(ii) *If* $0 < k \leq n$ *and* $\xi_k > 0$, *then* $\lambda x.\varphi^{n+1}(\xi_n, \ldots, \xi_k, 0, \ldots, 0, x)$ *enumerates* $\{\beta \in \Omega : \forall\xi < \xi_k(\varphi^{n+1}(\xi_n, \ldots, \xi_{k+1}, \xi, \beta, 0, \ldots, 0) = \beta)\}$.

Proof of (ii):
By definition, $\varphi^{n+1}(\xi_n, \ldots, \xi_k, \vec{0}, x) = \phi\langle\gamma + \Omega^k\xi_k + \Omega^0 x\rangle$ with $\gamma := \Omega^n\xi_n + \cdots + \Omega^{k+1}\xi_{k+1}$.

Therefore by Lemma 2.7a, b, $\lambda x.\varphi^{n+1}(\xi_n,\dots,\xi_k,\vec{0},x)$ enumerates $\{\beta\in\Omega : \forall\xi<\xi_k[\phi\langle\gamma+\Omega^k\xi+\Omega^{k-1}\beta\rangle=\beta]\}$.

Note
φ^{n+1} $(n\geq 1)$ is known as the $n{+}1$-ary Veblen function.
Usually it is *defined* by (i), (ii).

3 Characterization of ϕ_α via $K\alpha$

In [15] the Bachmann hierarchy (ϕ_α) restricted to $\alpha<\varepsilon_{\Omega+1}$ is studied, and thereby, as a technical tool, the sets $C(\alpha)$ and $ND(\alpha)$ (of *constituents* and *nondistinguished constituents* of α) are defined. From Lemmata 4.1, 4.2 and Theorems 3.1, 3.3 of this paper one can derive the following interesting result which provides an alternative definition of the Bachmann hierarchy not referring to fundamental sequences:

$$R_\alpha=\{\gamma\in R_0 : C(\alpha)\leq\gamma\ \&\ ND(\alpha)<\gamma\ \&\ \forall\xi<\alpha(C(\xi)<\gamma\Rightarrow\phi_\xi(\gamma)=\gamma)\}\ (\alpha<\varepsilon_{\Omega+1}). \tag{G}$$

In the following we will directly prove an analogue of (G), namely Theorem 3.4, and then exemplarily derive Gerber's Theorems 5.1, 4.3 (our Lemmas 3.7, 3.8) from that.

Definition of $K\alpha$ for $\alpha\leq\Lambda$

1. $K\alpha:=\begin{cases}\emptyset & \text{if } \alpha\in\{0,\Omega\},\\ \{\alpha\} & \text{if } \alpha\in \mathit{Lim}\cap\Omega,\\ K\alpha_0 & \text{if } \alpha=\alpha_0{+}1<\Omega.\end{cases}$
2. $\Omega<\alpha=_{\mathrm{NF}}\gamma+\Omega^\beta\eta\notin\mathrm{ran}(F_0)$: $K\alpha:=K\gamma\cup K\beta\cup K\eta$.
3. $\Omega<\alpha=_{\mathrm{NF}}F_\xi(\eta)<\Lambda$: $K\alpha:=K'\xi\cup K\eta$ with $K'\xi:=\begin{cases}\emptyset & \text{if } \xi=0,\\ \{\omega\}\cup K\xi & \text{if } \xi>0.\end{cases}$
4. $K\Lambda:=\{\omega\}$.

Remark $K(\alpha_0{+}1)=K\alpha_0$.

Lemma 3.1 $\lambda\in \mathit{Lim}\ \&\ 1\leq\xi\leq\tau_\lambda\ \Rightarrow\ K\lambda[\xi]=K\lambda[1]\cup K\xi$.

Proof

1. $\lambda=_{\mathrm{NF}}\gamma+\Omega^\beta\eta\notin\mathrm{ran}(F_0)$:

1.1. $\eta\in \mathit{Lim}$: $\tau_\lambda=\eta$ and $\lambda[\xi]=\gamma+\Omega^\beta(1{+}\xi)$.
$\xi\leq\eta\ \Rightarrow\ K\lambda[\xi]=K\gamma\cup K\beta\cup K\xi$.

1.2. $\eta=\eta_0{+}1$: $\tau_\lambda=\tau_{\Omega^\beta}$ and $\lambda[\xi]=\gamma+\Omega^\beta\eta_0+\Omega^\beta[\xi]$.
$K\lambda[\xi]=K\gamma\cup K(\Omega^\beta\eta_0)\cup K(\Omega^\beta[\xi])\overset{\mathrm{IH}}{=}K\gamma\cup K(\Omega^\beta\eta_0)\cup K(\Omega^\beta[1])\cup K\xi$.

2. $\lambda=_{\mathrm{NF}}F_\alpha(\beta)$:

2.1. $\beta\in \mathit{Lim}$: Then by Lemma 2.1, $\lambda[\xi]=_{\mathrm{NF}}F_\alpha(\beta[\xi])$ and thus $K\lambda[\xi]=K'\alpha\cup K(\beta[\xi])\overset{\mathrm{IH}}{=}K'\alpha\cup K\beta[1]\cup K\xi=K\lambda[1]\cup K\xi$.

2.2. $\beta \notin Lim$: Then $K\lambda^- = \begin{cases} K'\alpha \cup K\beta \text{ if } \beta = \beta_0{+}1 \;\&\; \beta_0 < F_\alpha(\beta_0), \\ K\beta \qquad\qquad \text{otherwise.} \end{cases}$
Hence $K\lambda = K'\alpha \cup K\beta = K'\alpha \cup K\lambda^-$.

2.2.0. $\alpha = 0$: Then $\lambda = \Omega^{\beta_0+1}$, $\tau_\lambda = \Omega$ and $\lambda[\xi] = \Omega^{\beta_0}(1{+}\xi)$.
Hence $K\lambda[\xi] = K\beta_0 \cup K\xi$.

2.2.1. $\alpha = \alpha_0{+}1$: Then $\tau_\lambda = \omega$ and, for $\xi < \omega$, $K\lambda[\xi] = K(F_{\alpha_0}^{(\xi+1)}(\lambda^-)) = K'\alpha \cup K\lambda^-$ and $K\xi = \emptyset$.
Further $K\lambda[\omega] = K\lambda = K'\alpha \cup K\lambda^- = K'\alpha \cup K\lambda^- \cup K\omega$.

2.2.2. $\alpha \in Lim$: For $\xi < \tau_\lambda = \tau_\alpha$ we have $K\lambda[\xi] = KF_{\alpha[\xi]}(\lambda^-) = K\alpha[\xi] \cup \{\omega\} \cup K\lambda^- \overset{\text{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\xi$.
Further $K\lambda = K\alpha \cup \{\omega\} \cup K\lambda^- \overset{\text{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\tau_\alpha$.

3. $\lambda = \Lambda$: For $1 \le \xi \le \omega$ we have $K\Lambda[\xi] = \{\omega\}$, whence $K\Lambda[\xi] = K\Lambda[1] \cup K\xi$.

Lemma 3.2

(a) $\alpha \in Lim \;\&\; \alpha[\xi] \le \delta \le \alpha[\xi{+}1] \;\Rightarrow\; K\alpha[\xi] \subseteq K\delta$.
(b) $\delta < \alpha \;\&\; K\delta < \xi \in Lim \cap \tau_\alpha \;\Rightarrow\; \delta < \alpha[\xi]$.

Proof

(a) Induction on δ:

1. $\delta = \alpha[\xi]$: trivial.
2. $\delta = \delta_0 + 1$ with $\alpha[\xi] \le \delta_0$: Then $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta_0 = K\delta$.
3. $\alpha[\xi] < \delta \in Lim$: Then, by Lemma 2.2d, $\alpha[\xi] \le \delta[1]$. Hence $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta[1] \overset{3.1}{\subseteq} K\delta$.

(b) Assume $\alpha[0] \le \delta$. Then by Lemma 2.2a, b, c there exists $\zeta < \tau_\alpha$ such that $\alpha[\zeta] \le \delta < \alpha[\zeta{+}1]$. By (a) and Lemma 3.1 we get $K\zeta \subseteq K\alpha[\zeta] \subseteq K\delta < \xi \in Lim$. Hence $\delta < \alpha[\zeta{+}1] < \alpha[\xi]$.

Definition
$\mathsf{k}(\alpha) := \max(K\alpha \cup \{0\})$. $\mathsf{k}^+(\alpha) := \max\{\mathsf{k}(\alpha[1]){+}1, \mathsf{k}(\alpha)\}$.

Lemma 3.3

(a) $\mathsf{k}(\alpha) \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1$;
(b) $\mathsf{k}^+(\alpha{+}1) = \mathsf{k}(\alpha) + 1$;
(c) $\mathsf{k}^+(\alpha) \le \phi_\alpha(0)$.

Proof

(a) By Lemma 3.1, $\mathsf{k}(\alpha) = \max\{\mathsf{k}(\alpha[1]), \mathsf{k}(\tau_\alpha)\}$ and thus
$\mathsf{k}^+(\alpha) = \max\{\mathsf{k}(\alpha[1]) + 1, \mathsf{k}(\tau_\alpha)\}$ $(*)$.
(b) $\mathsf{k}^+(\alpha{+}1) = \max\{\mathsf{k}(\alpha){+}1, \mathsf{k}(\alpha{+}1)\} = \mathsf{k}(\alpha){+}1$.
(c) Induction on α:

1. $\mathsf{k}^+(0) = 1 \le \phi_0(0)$.

2. $\alpha > 0$: By IH and Lemma 2.5e, $\mathsf{k}(\alpha[1]) \leq \phi_{\alpha[1]}(0) < \phi_\alpha(0)$. By Lemma 2.5b, $\mathsf{k}(\tau_\alpha) \leq \phi_\alpha(0)$. Hence $\mathsf{k}^+(\alpha) \overset{(*)}{=} \max\{\mathsf{k}(\alpha[1]) + 1, \mathsf{k}(\tau_\alpha)\} \leq \phi_\alpha(0)$.

Theorem 3.4 $R_\alpha = \{\beta \in R_0 : \mathsf{k}^+(\alpha) \leq \beta \ \&\ \forall \xi < \alpha(K\xi < \beta \Rightarrow \phi_\xi(\beta) = \beta)\}$.

Proof "$\subseteq$": Assume $\beta \in R_\alpha$. By Lemmata 2.4a, 3.3a, c we get $\mathsf{k}^+(\alpha) \leq \beta \in R_0$. The second part is proved by induction on α. So let $\delta < \alpha \ \&\ K\delta < \beta \in R_\alpha$.

1. $\alpha = \delta + 1$: $\beta \in R_{\delta+1}$ implies $\phi_\delta(\beta) = \beta$.
2. $\alpha = \alpha_0 + 1 \ \&\ \delta < \alpha_0$: From $\delta < \alpha_0 \ \&\ K\delta < \beta \in R_\alpha \subseteq R_{\alpha_0}$ we obtain $\phi_\delta(\beta) = \beta$ by IH.
3. $\alpha \in Lim \ \&\ \tau_\alpha < \Omega$: Then $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]}$ and $\delta < \alpha$. From this we get $\exists \xi < \tau_\alpha(\beta \in R_{\alpha[\xi]} \ \&\ \delta < \alpha[\xi])$ and then $\phi_\delta(\beta) = \beta$ by IH.
4. $\tau_\alpha = \Omega$: By Lemmata 2.4c, 2.5c we get $\beta \in Lim \cap R_{\alpha[\beta]}$. From $\delta < \alpha$ and $K\delta < \beta \in Lim \cap \tau_\alpha$ we get $\delta < \alpha[\beta]$ by Lemma 3.2b. Now we have $\beta \in R_{\alpha[\beta]}$ and $\delta < \alpha[\beta] < \alpha \ \&\ K\delta < \beta$ which by IH yields $\phi_\delta(\beta) = \beta$.

"$\supseteq$": Assume (1) $\mathsf{k}^+(\alpha) \leq \beta \in R_0$, and (2) $\forall \delta < \alpha(K\delta < \beta \Rightarrow \beta \in R_{\delta+1})$. From $\mathsf{k}^+(\alpha) \leq \beta$ we get (3) $K\alpha[1] < \beta$.

1. $\alpha = 0$: trivial.
2. $\alpha = \alpha_0+1$: From $\alpha_0 < \alpha \ \&\ K\alpha_0 = K\alpha[1] < \beta$ by (2) we obtain $\beta \in R_{\alpha_0+1} = R_\alpha$.
3. $\alpha \in Lim \ \&\ \tau_\alpha < \Omega$: By Lemma 3.1 and (1) we have $\tau_\alpha \leq \mathsf{k}(\alpha) \leq \beta$. From $0 < \xi < \tau_\alpha \leq \beta$ by Lemma 3.1 and (3) we conclude $\alpha[\xi] < \alpha \ \&\ K\alpha[\xi] \subseteq K\alpha[1] \cup K\xi < \beta$, and then by (2), $\beta \in R_{\alpha[\xi]+1}$. Hence $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.
4. $\tau_\alpha = \Omega$: From $0 < \alpha \ \&\ K0 = \emptyset < \beta$ by (2) we get $\beta \in R_1$, thence $\beta \in Lim$. Similarly as above we obtain $\beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}$. Hence $\beta \in R_\alpha$.

The Fixed-point-free Functions $\overline{\phi}_\alpha$

Definition

$\overline{\phi}_\alpha(\beta) := \phi_\alpha(\beta + \tilde{\iota}\alpha\beta)$ where

$$\tilde{\iota}\alpha\beta := \begin{cases} 1 & \text{if } \beta = \beta_0 + n \text{ with } \phi_\alpha(\beta_0) \in K\alpha \cup \{\beta_0\}, \\ 0 & \text{otherwise.} \end{cases}$$

$\overline{R}_\alpha := \mathrm{ran}(\overline{\phi}_\alpha)$.

Notation. From now on we mostly write $\phi\alpha\beta$, $\overline{\phi}\alpha\beta$ for $\phi_\alpha(\beta)$, $\overline{\phi}_\alpha(\beta)$.

Theorem 3.5

(a) $\overline{\phi}_\alpha$ *is order preserving.*

(b) $\overline{R}_\alpha = \{\phi\alpha\beta : K\alpha \cup \{\beta\} < \phi\alpha\beta\} = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$.

(c) $\overline{\phi}\alpha\beta = \min\{\gamma \in R_\alpha : \forall \eta < \beta(\overline{\phi}\alpha\eta < \gamma) \ \&\ K\alpha \cup \{\beta\} < \gamma\}$.

Proof

(a) If $\beta_1 < \beta_2$ then $\beta_1 + \tilde{\iota}\alpha\beta_1 < \beta_2$ or $\beta_1 + \tilde{\iota}\alpha\beta_1 = \beta_2$.
In the latter case $\tilde{\iota}\alpha\beta_2 = \tilde{\iota}\alpha\beta_1 = 1$.

(b) The first equation follows immediately from the definition, since $\mathsf{k}(\alpha) \leq \phi\alpha 0$ and $\eta{+}1 < \phi\alpha(\eta{+}1)$ for all $\eta < \Omega$. The second equation follows from the first, since $\phi\alpha\beta \in R_{\alpha+1} \Leftrightarrow \beta = \phi\alpha\beta$.

(c) Let $X := \{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\}$. By (a) and (b) we have $\overline{\phi}\alpha\beta \in X$. It remains to prove $\forall\gamma \in X(\overline{\phi}\alpha\beta \leq \gamma)$. So let $\gamma \in X$, i.e. $\gamma = \phi\alpha\delta$ with $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\imath}\alpha\eta) < \phi\alpha\delta)\ \&\ K\alpha \cup \{\beta\} < \phi\alpha\delta$ $(*)$.

To prove: $\overline{\phi}\alpha\beta \leq \phi\alpha\delta$, i.e. $\beta + \tilde{\imath}\alpha\beta \leq \delta$.
From $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\imath}\alpha\eta) < \phi\alpha\delta)$ we get $\beta \leq \delta$. Therefore if $\beta < \delta$ or $\tilde{\imath}\alpha\beta = 0$, we are done.
Assume now $\beta = \delta\ \&\ \tilde{\imath}\alpha\beta = 1$. Then $\delta = \beta = \beta_0 + n$ with $\phi\alpha\beta_0 \in K\alpha \cup \{\beta_0\}$.

1. $0 < n$: Then $\eta := \beta_0 + (n{-}1) < \beta = \eta + 1$ and therefore $\beta = \eta + \tilde{\imath}\alpha\eta \overset{(*)}{<} \delta = \beta$. Contradiction.
2. $n = 0$: Then $\phi\alpha\beta \in K\alpha \cup \{\beta\} \overset{(*)}{<} \phi\alpha\delta = \phi\alpha\beta$. Contradiction.

Corollary 3.6

(a) $\xi < \alpha\ \&\ K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta\ \Rightarrow\ \overline{\phi}\xi\eta < \overline{\phi}\alpha\beta$.
(b) $K\alpha \cup \{\beta\} < \overline{\phi}\alpha\beta$.

Proof

(a) $\xi < \alpha\ \&\ K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta \in R_\alpha\ \Rightarrow\ \overline{\phi}\xi\eta \leq \phi\xi(\eta{+}1) < \phi\xi\overline{\phi}\alpha\beta \overset{3.4}{=} \overline{\phi}\alpha\beta$.
(b) follows immediately from Theorem 3.5c.

Lemma 3.7 *Let* $\gamma_i = \overline{\phi}\alpha_i\beta_i$ *(*$i = 1, 2$*).*

(a) $\gamma_1 < \gamma_2$ *if, and only if, one of the following holds:*

(i) $\alpha_1 < \alpha_2\ \&\ K\alpha_1 \cup \{\beta_1\} < \gamma_2$;
(ii) $\alpha_1 = \alpha_2\ \&\ \beta_1 < \beta_2$;
(iii) $\alpha_2 < \alpha_1\ \&\ \gamma_1 \leq K\alpha_2 \cup \{\beta_2\}$.

(b) $\gamma_1 = \gamma_2\ \Rightarrow\ \alpha_1 = \alpha_2\ \&\ \beta_1 = \beta_2$.

Proof
(a) Let $Q(\alpha_1, \beta_1, \alpha_2, \beta_2) :\equiv (i) \vee (ii) \vee (iii)$.
To prove: $\gamma_1 < \gamma_2\ \Leftrightarrow\ Q(\alpha_1, \beta_1, \alpha_2, \beta_2)$.
From Theorem 3.5a and Corollary 3.6 we get the implications
(1) $Q(\alpha_1, \beta_1, \alpha_2, \beta_2)\ \Rightarrow\ \gamma_1 < \gamma_2$ and (2) $Q(\alpha_2, \beta_2, \alpha_1, \beta_1)\ \Rightarrow\ \gamma_2 < \gamma_1$.
Obviously,
(3) $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2)\ \Rightarrow\ Q(\alpha_2, \beta_2, \alpha_1, \beta_2) \vee (\alpha_1 = \alpha_2\ \&\ \beta_1 = \beta_2)$.
From (2) and (3) we get: $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2)\ \Rightarrow\ \neg(\gamma_1 < \gamma_2)$.
(b) Proof by contradiction. Assume $\gamma_1 = \gamma_2\ \&\ \alpha_1 < \alpha_2$. Then by Corollary 2.6b we have $\alpha_1 < \alpha_2\ \&\ K\alpha_1 \cup \{\beta_1\} < \gamma_1 = \gamma_2$. Hence $\gamma_1 < \gamma_2$ by Corollary 2.6a.

Lemma 3.8 *For each* $\gamma \in R_0 \cap \phi_\Lambda(0)$ *there exists* $\alpha < \Lambda$ *such that* $\gamma \in \overline{R}_\alpha$.

Proof
Assume $\omega < \gamma$. Then $K\Lambda < \gamma \notin R_\Lambda$. Let α_1 be the least ordinal such that $K\alpha_1 < \gamma \notin R_{\alpha_1}$. Then by Theorem 3.4 there exists $\alpha < \alpha_1$ such that $K\alpha < \gamma \notin R_{\alpha+1}$. By minimality of α_1 we get $\gamma \in R_\alpha$. Hence $\gamma \in \overline{R}_\alpha$ by Theorem 3.5b.

The following will prove useful in Sect. 5.

Theorem 3.9 *Let* $\overline{\phi}\langle\Omega\alpha+\beta\rangle := \overline{\phi}\alpha\beta$ *(*$\alpha \le \Lambda$, $\beta < \Omega$*). Then for all* $\alpha < \Lambda + \Omega$, $\overline{\phi}\langle\alpha\rangle = \min\{\gamma \in R_0 : \forall\xi < \alpha(K\xi < \gamma \Rightarrow \overline{\phi}\langle\xi\rangle < \gamma)\ \&\ K\alpha < \gamma\}$.

Proof

$$\begin{array}{l}
\overline{\phi}\langle\Omega\alpha+\beta\rangle = \overline{\phi}\alpha\beta \overset{3.5c}{=} \\
\min\{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\} \overset{3.4}{=} \\
\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup \{\eta\} < \gamma \Rightarrow \overline{\phi}\xi\eta < \gamma)\ \& \\
\hfill \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\} \overset{(*)}{=} \\
\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\xi+\eta\rangle < \gamma)\ \& \\
\hfill \forall\eta < \beta(K\alpha \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\alpha+\eta\rangle < \gamma)\ \&\ K\alpha \cup K\beta < \gamma\} = \\
\min\{\gamma \in R_0 : \forall\zeta < \Omega\alpha+\beta(K\zeta < \gamma \Rightarrow \overline{\phi}\langle\zeta\rangle < \gamma)\ \&\ K(\Omega\alpha+\beta) < \gamma\}.
\end{array}$$

$(*)$ For $\alpha = \beta = 0$ the equation is trivial. Otherwise it follows from the fact that for $1 < \gamma \in R_0$ we have $\forall\eta < \Omega(K\eta < \gamma \Leftrightarrow \eta < \gamma)$.

4 Comparison of ϕ_α, $\overline{\phi}_\alpha$ with θ_α, $\overline{\theta}_\alpha$

In this section we will compare the Bachmann functions ϕ_α with Feferman's functions θ_α. We will prove that $\phi\alpha\beta = \theta\alpha(\widehat{\alpha}+\beta)$ for all $\alpha \le \Lambda$, $\beta < \Omega$, where $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \le \theta\alpha\eta\}$. This result is already stated in [1], Theorem 3[1] and, for $\alpha < \varepsilon_{\Omega+1}$, proved in [23].

Before we can turn to the proper subject of this section we have to do some elementary ordinal arithmetic.

Definition $E_\Omega(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in \{0, \Omega\}, \\ \{\alpha\} & \text{if } \alpha \in \mathbb{E} \setminus \{\Omega\}, \\ \bigcup_{i \le n} E_\Omega(\alpha_i) & \text{if } \alpha = \omega^{\alpha_0} \# \cdots \# \omega^{\alpha_n} \notin \mathbb{E}. \end{cases}$

Definition A set $C \subseteq On$ is *nice* iff
$0 \in C\ \&\ \forall n\forall\alpha_0, \ldots, \alpha_n(\omega^{\alpha_0}\#\cdots\#\omega^{\alpha_n} \in C \Leftrightarrow \{\alpha_0, \ldots, \alpha_n\} \subseteq C)$.

Lemma 4.1

(a) $E_\Omega(\Omega+\alpha) = E_\Omega(\Omega\cdot\alpha) = E_\Omega(\Omega^\alpha) = E_\Omega(\alpha)$.
(b) $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \ \Rightarrow\ E_\Omega(\alpha) = E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta)$.
(c) *If* C *is nice and* $\Omega \in C$ *then* $\forall\alpha(\alpha \in C \Leftrightarrow E_\Omega(\alpha) \subseteq C)$.

---

[1] Actually Aczel's Theorem 3 looks somewhat different, but it implies the above formulated result. A proof of Theorem 3 can be extracted from the proof of Theorem 3.5 in [5].

(d) $\alpha < \varepsilon_{\Omega+1} \,\&\, \delta \in \mathbb{E} \;\Rightarrow\; (E_{\Omega}(\alpha) < \delta \Leftrightarrow K\alpha < \delta)$.

Proof

(a) Let $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ with $\alpha_1 \geq \cdots \geq \alpha_n$.

1. $E_{\Omega}(\Omega + \alpha) = \begin{cases} E_{\Omega}(\alpha) & \text{if } \Omega < \alpha_0, \\ E_{\Omega}(\Omega) \cup E_{\Omega}(\alpha) & \text{if } \Omega \geq \alpha_0. \end{cases}$

2. $E_{\Omega}(\Omega{\cdot}\alpha) = E_{\Omega}(\omega^{\Omega+\alpha_0} + \cdots + \omega^{\Omega+\alpha_n}) = \bigcup_{i \leq n} E_{\Omega}(\Omega + \alpha_i) \overset{1.}{=} \bigcup_{i \leq n} E_{\Omega}(\alpha_i) = E_{\Omega}(\alpha)$.

3. $E_{\Omega}(\Omega^{\alpha}) = E_{\Omega}(\omega^{\Omega\cdot\alpha}) = E_{\Omega}(\Omega \cdot \alpha) \overset{2.}{=} E_{\Omega}(\alpha)$.

(b) Let $\eta = \omega^{\eta_0} + \cdots + \omega^{\eta_m}$ with $\eta_0 \geq \cdots \geq \eta_m$.
Then $\Omega^{\beta}\eta = \omega^{\Omega\cdot\beta} \cdot (\omega^{\eta_0} + \cdots + \omega^{\eta_m}) = \omega^{\Omega\beta+\eta_0} + \cdots + \omega^{\Omega\beta+\eta_m}$.
Hence $E_{\Omega}(\Omega^{\beta}\eta) = \bigcup_{i \leq m} E_{\Omega}(\Omega\beta + \eta_0) = \bigcup_{i \leq m}(E_{\Omega}(\beta) \cup E_{\Omega}(\eta_i)) = E_{\Omega}(\beta) \cup \bigcup_{i \leq m} E_{\Omega}(\eta_i) = E_{\Omega}(\beta) \cup E_{\Omega}(\eta)$.

(c) 1. $\alpha \in \{0, \Omega\}$: $E_{\Omega}(\alpha) = \emptyset \subseteq C$ and $\alpha \in C$.
2. $\alpha \in \mathbb{E}$: $E_{\Omega}(\alpha) = \{\alpha\}$.
3. $\alpha = \omega^{\alpha_0} \,\#\, \cdots \,\#\, \omega^{\alpha_n} \notin \mathbb{E}$: $E_{\Omega}(\alpha) = E_{\Omega}(\alpha_0) \cup \cdots \cup E_{\Omega}(\alpha_n)$ and therefore:
$E_{\Omega}(\alpha) \subseteq C \;\Leftrightarrow\; \forall i \leq n(E_{\Omega}(\alpha_i) \subseteq C) \overset{\text{IH}}{\Leftrightarrow} \forall i \leq n(\alpha_i \in C) \overset{C\,\text{nice}}{\Leftrightarrow} \alpha \in C$.

(d) 1. $\alpha \in \{0, \Omega\}$: $E_{\Omega}(\alpha) = \emptyset = K\alpha$.
2. $\alpha < \Omega$: $E_{\Omega}(\alpha) < \delta \Leftrightarrow \alpha < \delta \Leftrightarrow K\alpha < \delta$.
3. $\Omega < \alpha =_{\text{NF}} \gamma + \Omega^{\beta}\eta$: $E_{\Omega}(\alpha) < \delta \overset{\text{(b)}}{\Leftrightarrow} E_{\Omega}(\gamma) \cup E_{\Omega}(\beta) \cup E_{\Omega}(\eta) < \delta \overset{\text{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta < \delta \Leftrightarrow K\alpha < \delta$.

Basic Properties of the Functions θ_{α}
The functions $\theta_{\alpha} : On \to On$ and sets $C(\alpha, \beta) \subseteq On$ are defined simultaneously by recursion on α (cf. [5], p. 174, [7], p. 6, [20], p. 225). Instead of giving this definition we present a list of basic properties which are sufficient for proving Theorems 4.6, 4.7 below.—*Notation:* $\theta\alpha\beta := \theta_{\alpha}(\beta)$.

(θ1) $\theta_{\alpha} : On \to On$ is a normal function and $\text{In}_{\alpha} := \text{ran}(\theta_{\alpha})$.
(θ2) (i) $\text{In}_0 = \mathbb{H}$,
(ii) $\text{In}_{\alpha+1} = \{\beta \in \text{In}_{\alpha} : \alpha \in C(\alpha, \beta) \Rightarrow \beta = \theta\alpha\beta\}$,
(iii) $\text{In}_{\alpha} = \bigcap_{\xi<\alpha} \text{In}_{\xi}$ if $\alpha \in Lim$.
(θ3) $\theta\alpha\Omega = \Omega$.
(θ4) $\text{In}_{\alpha} \cap \Omega = \{\beta \in \Omega : C(\alpha, \beta) \cap \Omega \subseteq \beta\}$.
(θ5) $\{0\} \cup \beta \subseteq C(\alpha, \beta)$, and if $\alpha > 0$ then $C(\alpha, \beta)$ is nice and $\Omega \in C(\alpha, \beta)$.
(θ6) $\xi < \alpha \leq \Lambda \,\&\, \Omega < \eta < \theta\xi\eta \;\Rightarrow\; (\xi, \eta \in C(\alpha, \beta) \Leftrightarrow \theta\xi\eta \in C(\alpha, \beta))$.

Remark (θ4)–(θ6) are only needed for the proof of Lemma 4.3c (via Lemmas 4.2 and 4.3a, b). Having established Lemma 4.3c we will make use only of (θ1)–(θ3) with (θ2*ii*) replaced by Lemma 4.3c.

Lemma 4.2

(a) $\alpha < \theta\alpha(\Omega{+}1) \ \&\ \Omega \le \beta \ \Rightarrow\ (\beta \in \mathrm{In}_{\alpha+1} \Leftrightarrow \beta = \theta\alpha\beta)$.
(b) $0 < \alpha \le \Lambda \ \Rightarrow\ F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta)$.

Proof
(a) "$\Leftarrow$": immediate consequence of $(\theta 2ii)$ (and $(\theta 1)$).
"$\Rightarrow$": Assume $\beta \in \mathrm{In}_\alpha$ and $(\alpha \in C(\alpha,\beta) \Rightarrow \beta = \theta\alpha\beta)$. For $\beta = \Omega$ the claim follows directly from $(\theta 3)$. Otherwise:

$$\theta\alpha\Omega \overset{(\theta 3)}{=} \Omega < \beta \in \mathrm{In}_\alpha \Rightarrow \theta\alpha(\Omega{+}1) \le \beta \Rightarrow \alpha < \beta \overset{(\theta 5)}{\Rightarrow} \alpha \in C(\alpha,\beta) \Rightarrow \beta = \theta\alpha\beta.$$

(b) Let $J := \{\beta : \Omega < \beta\}$. We prove $\mathrm{ran}(F_\alpha) = \mathrm{In}_\alpha \cap J$ which is equivalent to the claim $\forall\beta(F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta))$.

The proof proceeds by induction on α.

1. $\alpha = 1$: $\mathrm{ran}(F_1) = \{\beta : \beta = \Omega^\beta\} = \{\beta : \Omega < \beta = \omega^\beta\} \overset{(\theta 2)}{=} \mathrm{In}_1 \cap J$.
2. $\alpha = \alpha_0{+}1$ with $1 \le \alpha_0$: $\mathrm{ran}(F_\alpha) = \{\beta : \beta = F_{\alpha_0}(\beta)\} \overset{\mathrm{IH}}{=}$
   $\{\beta : \beta = \theta\alpha_0(\Omega{+}1{+}\beta)\} = \{\beta : \Omega < \beta = \theta\alpha_0\beta\} \overset{(*)}{=} \mathrm{In}_\alpha \cap J$.
   $(*)$ $\alpha_0{<}\Lambda \Rightarrow \alpha_0{<}F_{\alpha_0}(0) \overset{\mathrm{IH}}{=} \theta_{\alpha_0}(\Omega{+}1) \overset{\text{(a)}}{\Rightarrow} \forall\beta > \Omega(\beta = \theta\alpha_0\beta \Leftrightarrow \beta \in \mathrm{In}_\alpha)$.
3. $\alpha \in \mathit{Lim}$: $\mathrm{ran}(F_\alpha) = \bigcap_{\xi<\alpha} \mathrm{ran}(F_\xi) \overset{\mathrm{IH}}{=} \bigcap_{\xi<\alpha} \mathrm{In}_\xi \cap J \overset{(\theta 2\mathrm{iii})}{=} \mathrm{In}_\alpha \cap J$.

Lemma 4.3 *For $\alpha < \Lambda$ we have:*

(a) $\xi < \alpha \ \&\ \eta < F_\xi(\eta) < \Lambda \ \Rightarrow\ (\xi, \eta \in C(\alpha,\beta) \ \Leftrightarrow\ F_\xi(\eta) \in C(\alpha,\beta))$.
(b) $\forall\delta \le \alpha(\delta \in C(\alpha,\beta) \ \Leftrightarrow\ K\delta \subseteq C(\alpha,\beta))$.
(c) $\mathrm{In}_{\alpha+1} = \{\beta \in \mathrm{In}_\alpha : K\alpha < \beta \Rightarrow \beta = \theta\alpha\beta\}$.

Proof
(a) For $\xi = 0$ the claim follows from Lemma 4.1a, c and $(\theta 5)$.
Assume now $\xi > 0$ and let $\gamma := F_\xi(\eta)$.
Then $\xi, \eta_1 < \gamma = \theta\xi\eta_1$ with $\eta_1 := \Omega{+}1{+}\eta$.
By $(\theta 5)$ and Lemma 4.1a, c we have $(\eta \in C(\alpha,\beta) \Leftrightarrow \eta_1 \in C(\alpha,\beta))$.
Hence: $\xi, \eta \in C(\alpha,\beta) \Leftrightarrow \xi, \eta_1 \in C(\alpha,\beta) \overset{(\theta 6)}{\Leftrightarrow} \gamma \in C(\alpha,\beta)$.
(b) Induction on δ: Assume $\delta \le \alpha$, and let $C := C(\alpha,\beta)$.

1. $\delta \in \{0, \Omega\}$: $\delta \in C \ \&\ K\delta = \emptyset$.
2. $\delta = \delta_0{+}1$: $\delta \in C \Leftrightarrow \delta_0 \in C$, and $K\delta = K\delta_0$.
3. $\delta \in \mathit{Lim} \cap \Omega$: $K\delta = \{\delta\}$.
4. $\delta =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$: $\delta \in C \overset{4.1\mathrm{c}}{\Leftrightarrow} E_\Omega(\delta) \subseteq C \overset{4.1\mathrm{b}}{\Leftrightarrow} E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta) \subseteq C \overset{4.1\mathrm{c}}{\Leftrightarrow} \gamma, \beta, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta \subseteq C \Leftrightarrow K\delta \subseteq C$.
5. $\delta =_{\mathrm{NF}} F\xi\eta$: $\delta \in C \overset{\text{(a)}}{\Leftrightarrow} \xi, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\xi \cup K\eta \subseteq C \overset{(*)}{\Leftrightarrow} K\delta \subseteq C$.

$(*)$ $\omega = \theta 01 \in C$.
(c) follows from $(\theta 2ii)$, $(\theta 4)$, (b) and the fact that $K\alpha \subseteq \Omega$.

Theorem 4.4 $\alpha \le \Lambda \;\Rightarrow\; \mathrm{In}_\alpha = \{\beta \in \mathbb{H} : \forall\xi < \alpha(K\xi < \beta \;\Rightarrow\; \theta\xi\beta = \beta)\}$.

Proof by induction on α

1. $\alpha = 0$: By $(\theta 2i)$ we have $\mathrm{In}_0 = \mathbb{H}$.
2. $\alpha = \alpha_0{+}1$: $\mathrm{In}_\alpha \overset{4.3c}{=} \{\beta \in \mathrm{In}_{\alpha_0} : K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta\} \overset{\mathrm{IH}}{=}$ $\{\beta \in \mathbb{H} : \forall\xi < \alpha_0(K\xi < \beta \Rightarrow \beta = \theta\xi\beta) \;\&\; (K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta)\}$.
3. $\alpha \in \mathit{Lim}$: Then, by $(\theta 2iii)$, $\mathrm{In}_\alpha = \bigcap_{\xi<\alpha} \mathrm{In}_\xi$ and the assertion follows immediately from the IH.

Definition $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \le \theta\alpha\eta\}$.

Lemma 4.5 $\alpha \le \Lambda \;\&\; K\alpha < \theta\alpha\beta \;\Rightarrow\; (\theta\alpha(\widehat{\alpha} + \beta) = \beta \Leftrightarrow \theta\alpha\beta = \beta)$.

Proof
"$\Rightarrow$": This follows from $\beta \le \theta\alpha\beta \le \theta\alpha(\widehat{\alpha} + \beta)$.
"$\Leftarrow$": If $K\alpha < \beta = \theta\alpha\beta$ then $\widehat{\alpha} \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 < \beta \in \mathbb{H}$ and thus $\widehat{\alpha} + \beta = \beta$.

Theorem 4.6 *If* $\alpha \le \Lambda$, *then* $R_\alpha = \{\gamma \in \Omega : \mathsf{k}^+(\alpha) \le \gamma \in \mathrm{In}_\alpha\}$, *and thus* $\forall\beta < \Omega\,(\phi\alpha\beta = \theta\alpha(\widehat{\alpha} + \beta))$.

Proof by induction on α:
For $\beta < \Omega$ we have:
$\beta \in R_\alpha \overset{3.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathbb{H} \;\&\; \forall\xi < \alpha(K\xi < \beta \Rightarrow \phi\xi\beta = \beta) \overset{IH+4.5}{\Leftrightarrow} \mathsf{k}^+(\alpha)$ $\le \beta \in \mathbb{H} \;\&\; \forall\xi < \alpha(K\xi < \beta \Rightarrow \theta\xi\beta = \beta) \overset{4.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathrm{In}_\alpha$.

The Functions $\overline{\theta}_\alpha$
In [7] the fixed-point-free functions $\overline{\theta}_\alpha$ are introduced, which are more suitable for proof-theoretic applications than the θ_α's. By definition, $\overline{\theta}_\alpha$ is the $<$-isomorphism from $\{\eta \in \mathit{On} : S\mu(\alpha) \le \eta\}$ onto $\overline{\mathrm{In}}_\alpha$ where $\overline{\mathrm{In}}_\alpha := \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$, $\mu(\alpha) := \min\{\eta : \theta\alpha\eta \in \overline{\mathrm{In}}_\alpha\}$, $S\mu(\alpha) := \min\{\Omega_\xi : \mu(\alpha) < \Omega_{\xi+1}\}$ where $\Omega_0 := 0$.

As we will show in a moment, $S\mu(\alpha) = 0$ for all $\alpha < \Lambda$, and therefore, if $\alpha < \Lambda$ then $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha$. On the other side, by Theorem 3.5, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$. Using Theorem 4.6 one easily sees that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. So we arrive at the following theorem.

Theorem 4.7 $\overline{\phi}\alpha\beta = \overline{\theta}\alpha\beta$ *for all* $\alpha < \Lambda$, $\beta < \Omega$.

Proof
I. From $\alpha < \Lambda$ by Lemma 4.3c and $(\theta 3)$ we obtain $\forall\beta \in \Omega\,(\mathsf{k}(\alpha) \le \beta \Rightarrow \theta\alpha(\beta{+}1) \in \overline{\mathrm{In}}_\alpha \cap \Omega)$. Hence $S\mu(\alpha) = 0$, and $\overline{\mathrm{In}}_\alpha \cap \Omega$ is unbounded in Ω. This implies that $\overline{\theta}_\alpha{\restriction}\Omega$ is the ordering function of $\overline{\mathrm{In}}_\alpha \cap \Omega$.
II. As mentioned above, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha$. So it remains to prove that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. First note that

(1) $\mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 = \mathsf{k}^+(\alpha{+}1)$ and (2) $\forall\gamma \in \overline{\mathrm{In}}_\alpha(\mathsf{k}(\alpha) < \gamma)$ (by Lemma 4.3c).

Then for $\gamma < \Omega$ we get: $\gamma \in \overline{R}_\alpha \Leftrightarrow \mathsf{k}(\alpha) < \gamma \in R_\alpha \;\&\; \gamma \notin R_{\alpha+1} \overset{4.6,(1)}{\Leftrightarrow} \mathsf{k}(\alpha)$ $< \gamma \in \mathrm{In}_\alpha \;\&\; (\mathsf{k}(\alpha) < \gamma \Rightarrow \gamma \notin \mathrm{In}_{\alpha+1}) \overset{(2)}{\Leftrightarrow} \gamma \in \overline{\mathrm{In}}_\alpha$.

5 The Unary Functions $\vartheta^{\mathbb{X}}$ and $\psi^{\mathbb{X}}$

As we have seen above, $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha = \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$ (if $\alpha < \Lambda$).
From this together with ($\theta 2ii$) and ($\theta 4$) one easily derives the following equation
(1) $\overline{\theta}\alpha 0 = \min\{\beta : C(\alpha,\beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in C(\alpha,\beta)\}$
which motivates the definition of $\vartheta\alpha$ in [18]:
(2) $\vartheta\alpha := \min\{\beta : \tilde{C}(\alpha,\beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in \tilde{C}(\alpha,\beta)\}$ $(\alpha < \varepsilon_{\Omega+1})$
where $\tilde{C}(\alpha,\beta)$ is the closure of $\{0,\Omega\} \cup \beta$ under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$.
On the other side, by Theorems 4.7, 3.9 we have:
(3) $\overline{\theta}\alpha 0 = \overline{\phi}\langle\Omega\alpha\rangle = \min\{\beta \in \mathbb{H} : \forall\xi < \Omega\alpha(K\xi < \beta \Rightarrow \overline{\phi}\langle\xi\rangle < \beta) \;\&\; K\alpha < \beta\}$.

In the light of (1)–(3) the following theorem suggests itself.

Theorem 5.1
$\alpha < \varepsilon_{\Omega+1} \;\Rightarrow\; \vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; K\alpha < \beta\}$.

Proof
I. From [18], Lemma 2.1 and 2.2(1)–(4) we obtain
$\vartheta\alpha \in \mathbb{E} \;\&\; \forall\xi < \alpha(E_\Omega(\xi) < \vartheta\alpha \Rightarrow \vartheta\xi < \vartheta\alpha) \;\&\; E_\Omega(\alpha) < \vartheta\alpha$.
II. Assume $\beta \in \mathbb{E} \;\&\; \forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; E_\Omega(\alpha) < \beta$.
We will prove that $\vartheta\alpha \le \beta$.

For this let $Q := \{\gamma : E_\Omega(\gamma) \subseteq \beta\}$. Since $\beta \in \mathbb{E}$, we have $Q \subseteq \beta$. Moreover, as one easily sees, $\{0,\Omega\} \subseteq Q$ and Q is closed under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$. Hence $\tilde{C}(\alpha,\beta) \subseteq Q$ and thus $\tilde{C}(\alpha,\beta) \cap \Omega \subseteq Q \cap \Omega \subseteq \beta$. It remains to show that $\alpha \in \tilde{C}(\alpha,\beta)$. But this follows immediately from $E_\Omega(\alpha) \subseteq \beta \subseteq \tilde{C}(\alpha,\beta)$ and [18, 1.2(4)].

From I. and II. we get
$\vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; E_\Omega(\alpha) < \beta\}$,
which together with Lemma 4.1 d yields the claim.

Relativization
Comparing the recursion equations for $\vartheta\alpha$ and $\overline{\phi}\langle\alpha\rangle$ in Theorems 5.1, 3.9 one notices that these equations are almost identical. The only difference is that in the equation for $\vartheta\alpha$ there appears $\mathbb{E}$ where in the equation for $\overline{\phi}\langle\alpha\rangle$ we have R_0 (i.e. $\mathbb{H}$). In order to establish the exact relationship between ϑ and $\overline{\phi}$ we go back to the definition of the Bachmann hierarchy in Sect. 2 and replace the initial clause "$R_0 := \mathbb{H} \cap \Omega$" of this definition by "$R_0 := \mathbb{X} \cap \Omega$" where here and in the sequel $\mathbb{X}$ *always denotes a subclass of* $\{1\} \cup$ *Lim such that* $\mathbb{X} \cap \Omega$ is Ω-*club*. Then the whole of Sects. 2, 3 remains valid as it stands. To make the dependency on $\mathbb{X}$ visible we write $R^{\mathbb{X}}_\alpha, \overline{R}^{\mathbb{X}}_\alpha, \phi^{\mathbb{X}}_\alpha, \overline{\phi}^{\mathbb{X}}_\alpha, \phi^{\mathbb{X}}\langle\alpha\rangle, \overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$ instead of $R_\alpha, \overline{R}_\alpha, \ldots$.

Remark
Theorems 5.1, 3.9 yield $\vartheta\alpha = \overline{\phi}^{\mathbb{E}}\langle\alpha\rangle$ and $\vartheta(\Omega\alpha + \beta) = \overline{\phi}^{\mathbb{E}}_\alpha(\beta)$ $(\alpha < \varepsilon_{\Omega+1}, \beta < \Omega)$
The previous explanations motivate the following definition.

Definition
$\vartheta^{\mathbb{X}}\alpha := \min\{\beta \in \mathbb{X} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \vartheta^{\mathbb{X}}\xi < \beta)\,\&\, K\alpha < \beta\}$ $(\alpha \le \Lambda)$.
Theorem 5.1 now reads: $\vartheta\alpha = \vartheta^{\mathbb{E}}\alpha$ for $\alpha < \varepsilon_{\Omega+1}$.
Further, by Theorem 3.9 we have
$(\vartheta 0)$ $\vartheta^{\mathbb{X}}(\Omega\alpha + \beta) = \overline{\phi}^{\mathbb{X}}_{\alpha}(\beta)$, if $\beta < \Omega$.

Therefore, properties of $\vartheta^{\mathbb{X}}$ can be proved by deriving them from corresponding properties of $\overline{\phi}$. But for various reasons it is also advisable to work directly from the above definition.

Let us first mention that for $\beta < \Omega$ the set $\{\xi < \alpha : K\xi < \beta\}$ is countable too, and therefore $\vartheta^{\mathbb{X}}\alpha < \Omega$. Moreover, directly from the definition of $\vartheta^{\mathbb{X}}$ we obtain:
$(\vartheta 1)$ $K\alpha < \vartheta^{\mathbb{X}}\alpha \in \mathbb{X}$,
$(\vartheta 2)$ $\alpha_0 < \alpha \;\&\; K\alpha_0 < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \vartheta^{\mathbb{X}}\alpha_0 < \vartheta^{\mathbb{X}}\alpha$,
$(\vartheta 3)$ $\beta \in \mathbb{X} \;\&\; K\alpha < \beta < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \exists\xi < \alpha(K\xi < \beta \le \vartheta^{\mathbb{X}}\xi)$,
and then
$(\vartheta 4)$ $\vartheta^{\mathbb{X}}\alpha_0 = \vartheta^{\mathbb{X}}\alpha_1 \Rightarrow \alpha_0 = \alpha_1$ [from $(\vartheta 1)$, $(\vartheta 2)$],
$(\vartheta 5)$ $\beta \in \mathbb{X} \;\&\; \beta < \vartheta^{\mathbb{X}}\Lambda \;\Rightarrow\; \exists\xi < \Lambda(\beta = \vartheta^{\mathbb{X}}\xi)$.
Proof of $(\vartheta 5)$: If $\beta \le \omega$ then $\beta \in \{\vartheta 0, \vartheta 1\}$. Otherwise we have $K\Lambda < \beta < \vartheta^{\mathbb{X}}\Lambda$, and the assertion follows by transfinite induction from $(\vartheta 3)$.

Note on Klammersymbols. As we mentioned above, Sects. 2, 3 remain valid if ϕ is replaced by $\phi^{\mathbb{X}}$. So by Theorem 2.8, for $A = \begin{pmatrix} \xi_0 & \dots & \xi_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ and $\alpha = \Omega^{\alpha_n}\xi_n + \dots + \Omega^{\alpha_0}\xi_0$ we have $\phi^{\mathbb{X}}_0 A = \phi^{\mathbb{X}}\langle\alpha\rangle$ from which one easily derives $\overline{\phi^{\mathbb{X}}_0} A = \overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$,[2] whence (by Theorem 3.9) $\overline{\phi^{\mathbb{X}}_0} A = \vartheta^{\mathbb{X}}\alpha$. Via Theorem 5.1 this fits together with Schütte's result $\overline{\phi^{\mathbb{E}}_0} A = \vartheta\alpha$ in [21].

The Function $\psi^{\mathbb{X}}$
In [9] (actually already in [8]) the author introduced the functions $\psi_\sigma : On \to \Omega_{\sigma+1}$ and proved, via an ordinal analysis of ID_ν, that $\psi_0\varepsilon_{\Omega_\nu+1} = \theta_{\varepsilon_{\Omega_\nu+1}}(0)$. In [12] ordinal analyses of several impredicative subsystems of 2nd order arithmetic are carried out by means of the ψ_σ's. The definition of ψ_σ in [12] differs in some minor respects from that in [9]; for example, $\lambda\xi.\omega^\xi$ is a basic function in [12] but not in [9]. In [18] Rathjen and Weiermann compare their ϑ with $\psi_0{\restriction}\varepsilon_{\Omega+1}$ from [12] which they abbreviate by ψ. In Sect. 6 we will present a refinement of this comparison which is based on Schütte's definition of the Veblen function φ (below Γ_0) in terms of ψ, given in Sect. 7 of [12].

Similarly as Theorem 5.1 one can prove

$\psi\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \psi\xi < \beta)\}$, for $\alpha < \varepsilon_{\Omega+1}$.

---

[2] $\overline{\varphi}A$ is the 'fixed-point-free version' of φA defined in [19, Sect. 4].

This motivates the following

Definition of $\psi^{\mathbb{X}}\alpha$ for $\alpha \le \Lambda{+}1$

$\psi^{\mathbb{X}}\alpha := \min\{\beta \in \mathbb{X} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \psi^{\mathbb{X}}\xi < \beta)\}$.

For the rest of this section we assume $\mathbb{X}$ to be fixed, and write ϑ, ψ for $\vartheta^{\mathbb{X}}$, $\psi^{\mathbb{X}}$.

Remark Immediately from the definitions it follows that $\psi\alpha \le \vartheta\alpha$.

Before turning to the announced exact comparison of ϑ and ψ, we prove a somewhat weaker (but still very useful) result which can be obtained with much less effort. This corresponds to [18, p. 64] which in turn stems from [10, 11].

Lemma 5.2 *For $\alpha \le \Lambda$.*

(a) $\alpha_0 \le \alpha \;\Rightarrow\; \psi\alpha_0 \le \psi\alpha$.
(b) $\alpha_0 < \alpha \;\&\; K\alpha_0 < \psi\alpha \;\Rightarrow\; \psi\alpha_0 < \psi\alpha$.
(c) $\psi\alpha < \psi(\alpha{+}1) \;\Leftrightarrow\; K\alpha < \psi\alpha$.
(d) $\alpha \in Lim \;\Rightarrow\; \psi\alpha = \sup_{\xi<\alpha}\psi\xi$.
(e) $\psi\alpha = \min\{\gamma \in \mathbb{X} : \forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)\}$.

Proof
(a), (b) follow directly from the definition.
(c) "$\Rightarrow$": Assume $\neg(K\alpha < \psi\alpha)$. Then from $\psi\alpha \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \psi\alpha \Rightarrow \psi\xi < \psi\alpha)$ we conclude $\psi\alpha \in \mathbb{X} \;\&\; \forall\xi < \alpha{+}1(K\xi < \psi\alpha \Rightarrow \psi\xi < \psi\alpha)$, and thus $\psi(\alpha{+}1) \le \psi\alpha$.
"$\Leftarrow$": From $\alpha < \alpha{+}1 \;\&\; K\alpha < \psi\alpha \le \psi(\alpha{+}1)$ we conclude $\psi\alpha < \psi(\alpha{+}1)$ by (b).
(d) By (a) we have $\gamma := \sup_{\xi<\alpha}\psi\xi \le \psi\alpha$. Assume $\gamma < \psi\alpha$. Then $\gamma \in \mathbb{X} \cap \psi\alpha$, and therefore by definition of $\psi\alpha$ there exists $\xi < \alpha$ with $K\xi < \gamma \le \psi\xi$. Hence by (c), $\exists\xi < \alpha(\gamma < \psi(\xi{+}1))$. Contradiction.
(e) 1. We have $\psi\alpha \in \mathbb{X}$ and, by (a), (b), $\forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \psi\alpha)$.
2. Notice that $(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)$ implies $(K\xi < \gamma \Rightarrow \psi\xi < \gamma)$. Therefore, if $\gamma \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)$ then $\gamma \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \gamma \Rightarrow \psi\xi < \gamma)$ which yields $\psi\alpha \le \gamma$.

Definition
Let $\alpha \le \Lambda$ with $K\alpha < \psi\Lambda$.

Then by Lemma 5.2d there exists $\xi < \Lambda$ such that $K\alpha < \psi\xi$, and we can define

$\tilde{\mathsf{g}}(\alpha) := \min\{\xi < \Lambda : K\alpha < \psi\xi\}$,
$\mathsf{g}(\alpha) := \tilde{\mathsf{g}}(\alpha)\dot{-}1$, where $\beta\dot{-}1 := \begin{cases}\beta_0 & \text{if } \beta = \beta_0{+}1,\\ \beta & \text{otherwise.}\end{cases}$
$\mathsf{h}(\alpha) := \mathsf{g}(\alpha) + \Omega^{\alpha}$. (Note that $\mathsf{h}(\alpha) \le \Lambda$.)

Lemma 5.3 *Assume $\alpha \le \Lambda \;\&\; K\alpha < \psi\Lambda$.*

(a) $\psi 0 \le K\alpha \;\Rightarrow\; \psi\mathsf{g}(\alpha) \le K\alpha < \psi(\mathsf{g}(\alpha){+}1)$.
(b) $K\mathsf{g}(\alpha) < \psi\mathsf{g}(\alpha)$.
(c) $K\mathsf{h}(\alpha) < \psi\mathsf{h}(\alpha)$.
(d) $\alpha_0 < \alpha \;\&\; K\alpha_0 < \psi\mathsf{h}(\alpha) \;\Rightarrow\; \psi\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha)$.

Proof

(a) From $\psi 0 \leq K\alpha$ and Lemma 5.2d it follows that $0 < \tilde{\mathsf{g}}(\alpha) \notin \mathit{Lim}$. Therefore $\tilde{\mathsf{g}}(\alpha) = \mathsf{g}(\alpha){+}1$, which yields the assertion.

(b) Follows from (a) and Lemma 5.2c.

(c) $K(\mathsf{g}(\alpha) + \Omega^\alpha) \subseteq K\mathsf{g}(\alpha) \cup K\alpha \overset{(b),(a)}{<} \psi(\mathsf{g}(\alpha) + 1) \leq \psi(\mathsf{g}(\alpha) + \Omega^\alpha)$.

(d) From $\alpha_0 < \alpha \;\&\; K\alpha_0 < \psi\mathsf{h}(\alpha)$ by (a) we obtain $\alpha_0 < \alpha \;\&\; \mathsf{g}(\alpha_0) < \mathsf{h}(\alpha) = \mathsf{g}(\alpha) + \Omega^\alpha$ and then $\mathsf{h}(\alpha_0) = \mathsf{g}(\alpha_0) + \Omega^{\alpha_0} < \mathsf{h}(\alpha)$. This together with $K\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha_0)$ (cf. (c)) yields $\psi\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha)$ by Lemma 5.2a, b.

Theorem 5.4 $\alpha \leq \Lambda \;\&\; K\alpha < \psi\Lambda \;\Rightarrow\; \vartheta\alpha \leq \psi\mathsf{h}(\alpha)$.

Proof by induction on α:

By Lemma 5.3a, d, $K\alpha{<}\psi\mathsf{h}(\alpha) \in \mathbb{X} \;\&\; \forall\xi{<}\alpha(K\xi{<}\psi\mathsf{h}(\alpha) \Rightarrow \psi\mathsf{h}(\xi){<}\psi\mathsf{h}(\alpha))$. Hence by IH, $K\alpha < \psi\mathsf{h}(\alpha) \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \psi\mathsf{h}(\alpha) \Rightarrow \vartheta\xi < \psi\mathsf{h}(\alpha))$ which yields $\vartheta\alpha \leq \psi\mathsf{h}(\alpha)$.

Corollary 5.5

(a) $\alpha = \Omega^\alpha \leq \Lambda \;\&\; K\alpha < \psi\alpha \;\Rightarrow\; \vartheta\alpha = \psi\alpha$.

(b) $\vartheta\varepsilon_{\Omega+1} = \psi\varepsilon_{\Omega+1} \;\&\; \vartheta\Lambda = \psi\Lambda$.

Proof

(a) $K\alpha < \psi\alpha \;\&\; \alpha = \Omega^\alpha \;\Rightarrow\; \mathsf{g}(\alpha) < \alpha = \Omega^\alpha \;\Rightarrow\; \mathsf{h}(\alpha) = \mathsf{g}(\alpha) + \Omega^\alpha = \alpha \overset{5.4}{\Rightarrow} \vartheta\alpha \leq \psi\alpha \leq \vartheta\alpha$.

(b) are instances of (a).

Note In the appendix of [6] it is shown that $\psi^{\mathrm{SC}}\Lambda$ equals Bachmann's $\varphi_{F_{\omega_2+1}(1)}(1)$. In the present context this equation can be derived as follows

$\psi^{\mathrm{SC}}\Lambda \overset{\text{Cor.5.5}}{=} \vartheta^{\mathrm{SC}}\Lambda \overset{(\vartheta 0)}{=} \overline{\phi}^{\mathrm{SC}}_{\Lambda}(0) = \phi^{\mathrm{SC}}_{\Lambda}(0) \overset{\text{L.5.6}}{=} \phi^{\mathbb{H}}_{\Lambda}(0) = \text{‘}\varphi_{F_{\omega_2+1}(1)}(1)\text{’}$.

Lemma 5.6

(a) $K\gamma = \emptyset \;\&\; \mathbb{Y} \cap \Omega = R^{\mathbb{X}}_{\gamma} \;\Rightarrow\; \phi^{\mathbb{Y}}_{\alpha} = \phi^{\mathbb{X}}_{\gamma+\alpha}$.

(b) $\mathrm{SC} \cap \Omega = R^{\mathbb{H}}_{\Omega}$.

Proof

(a) Induction on α using Theorem 3.4 and the fact that $K\gamma = \emptyset$ implies $K(\gamma + \alpha) = K\alpha$ and $\mathsf{k}^+(\gamma + \alpha) = \mathsf{k}^+(\alpha)$ for all α.

(b) By definition we have $\forall\alpha < \Omega(\phi^{\mathbb{H}}_{\alpha} = \varphi_\alpha)$, which together with Lemma 2.5d yields $\mathrm{SC} \cap \Omega = \{\alpha \in \Omega : \phi^{\mathbb{H}}_{\alpha}(0) = \alpha\} = R^{\mathbb{H}}_{\Omega}$.

Corollary 5.7

(i) $K\gamma = \emptyset \;\&\; \mathbb{Y} \cap \Omega = R^{\mathbb{X}}_{\gamma} \;\Rightarrow\; \overline{\phi}^{\mathbb{Y}}_{\alpha} = \overline{\phi}^{\mathbb{X}}_{\gamma+\alpha}$ *and* $\vartheta^{\mathbb{Y}}\alpha = \vartheta^{\mathbb{X}}(\Omega\gamma + \alpha)$.

(ii) $\phi^{\mathbb{E}}_{\alpha} = \phi^{\mathbb{H}}_{1+\alpha}$, $\phi^{\mathrm{SC}}_{\alpha} = \phi^{\mathbb{H}}_{\Omega+\alpha}$, $\vartheta^{\mathbb{E}}\alpha = \vartheta^{\mathbb{H}}(\Omega + \alpha)$, *and* $\vartheta^{\mathrm{SC}}\alpha = \vartheta^{\mathbb{H}}(\Omega^2 + \alpha) = \vartheta^{\mathbb{E}}(\Omega^2 + \alpha)$.

Proof

(i) follows from Lemma 5.6a by Theorem 3.5b and ($\vartheta 0$).

(ii) follows from Lemma 5.6, (i), and $\mathbb{E} \cap \Omega = R^{\mathbb{H}}_{1}$.

6 Exact Comparison of ϑ and ψ

Let $\mathbb{X} \subseteq \mathbb{H}$ be fixed such that $\mathbb{X} \cap \Omega$ is Ω-club. As before we write ϑ, ψ for $\vartheta^{\mathbb{X}}$, $\psi^{\mathbb{X}}$.

In this section we always assume $\alpha < \Lambda$ and $K\alpha \cup \{\beta\} < \psi\Lambda$.

Lemma 6.1

(a) $\alpha_0 < \alpha \;\&\; \forall\xi(\alpha_0 \le \xi < \alpha \to \psi\xi = \psi(\xi{+}1)) \Rightarrow \psi\alpha_0 = \psi\alpha$.
(b) $\psi\alpha_0 < \psi\alpha \;\Rightarrow\; \exists\alpha_1(\alpha_0 \le \alpha_1 < \alpha \;\&\; K\alpha_1 < \psi\alpha_1 = \psi\alpha_0)$.

Proof
(a) follows from Lemma 5.2a, d by induction on α.
(b) From $\psi\alpha_0 < \psi\alpha$ by Lemmata 5.2a, 6.1a we obtain $\exists\xi(\alpha_0 \le \xi < \alpha \;\&\; \psi\xi < \psi(\xi{+}1))$. Let $\alpha_1 := \min\{\xi \ge \alpha_0 : \psi\xi < \psi(\xi{+}1)\}$. Then $\alpha_0 \le \alpha_1 < \alpha$ and, by (a) and Lemma 5.2c, $K\alpha_1 < \psi\alpha_1 = \psi\alpha_0$.

Lemma 6.2

(a) $\psi\alpha < \gamma \in \mathbb{X} \;\Rightarrow\; \psi(\alpha{+}1) \le \gamma$.
(b) $\gamma \in \mathbb{X} \cap \psi\Lambda \;\Rightarrow\; \exists\alpha(K\alpha < \psi\alpha = \gamma)$.
(c) $\Omega^\alpha|\gamma \;\&\; \delta < \Omega^\alpha \;\&\; K(\gamma+\delta) < \psi(\gamma+\delta) \;\Rightarrow\; K\gamma < \psi\gamma$.
(d) $\Omega^\alpha|\gamma \;\&\; \psi\gamma < \psi(\gamma+\Omega^\alpha) \;\Rightarrow\; K\gamma < \psi\gamma$.

Proof
(a) $\mathbb{X} \ni \gamma < \psi(\alpha+1) \;\Rightarrow\; \exists\xi < \alpha{+}1(K\xi < \gamma \le \psi\xi) \;\Rightarrow\; \gamma \le \psi\alpha$.
(b) By Lemma 5.2a, d it follows that $\psi\alpha \le \gamma < \psi(\alpha{+}1)$ for some $\alpha < \Lambda$. By (a) it follows that $\psi\alpha = \gamma$.
(c) Induction on δ: Since $\Omega^\alpha|\gamma \;\&\; \delta < \Omega^\alpha$ we have $K\gamma \subseteq K(\gamma+\delta)$. Therefore, if $\psi\gamma = \psi(\gamma+\delta)$ then $K\gamma < \psi\gamma$. If $\psi\gamma < \psi(\gamma+\delta)$, then by Lemma 6.1b there exists $\delta_0 < \delta$ such that $K(\gamma+\delta_0) < \psi(\gamma+\delta_0)$; thence, by IH, $K\gamma < \psi\gamma$.
(d) By Lemma 6.1b there exists $\delta < \Omega^\alpha$ such that $K(\gamma+\delta) < \psi(\gamma+\delta)$. Hence $K\gamma < \psi\gamma$ by (c).

Lemma 6.3 $\delta =_{\mathrm{NF}} \gamma + \Omega^\alpha\xi \;\;\&\;\; K(\Omega^\alpha\xi) < \psi(\gamma+\Omega^{\alpha+1}) \;\;\Rightarrow\;\; K(\Omega^\alpha\xi) < \psi\delta$.

Proof For $\psi\delta = \psi(\gamma+\Omega^{\alpha+1})$ the claim is trivial. Otherwise, by Lemma 6.1b there exists δ_1 with $\delta \le \delta_1 < \gamma+\Omega^{\alpha+1}$ and $K\delta_1 < \psi\delta_1 = \psi\delta$. Then $\delta_1 = \gamma + \Omega^\alpha\beta + \delta_2$ with $\xi \le \beta < \Omega$ and $\delta_2 < \Omega^\alpha$. Hence $K(\Omega^\alpha\beta) \subseteq K\delta_1 < \psi\delta$. Now assume $\beta > 0$. Then $K\alpha \cup K\beta = K(\Omega^\alpha\beta) < \psi\delta$ which together with $\xi \le \beta < \Omega$ yields $K(\Omega^\alpha\xi) \subseteq K\alpha \cup K\xi < \psi\delta$.

Definition
1. $\dot{\psi}\alpha := \begin{cases} 0 & \text{if } \alpha = 0, \\ \psi\alpha & \text{if } \alpha > 0. \end{cases}$
2. If $\alpha \le \beta$ then $-\alpha+\beta$ denotes the unique γ such that $\alpha+\gamma = \beta$.

The following definition is an extension and modification of the corresponding definition on p. 26 of [12].

Definition of $[\alpha, \beta] < \Lambda$

By Lemma 5.2a, d there exists $\eta < \Lambda$ such that
$\dot\psi(\Omega^{\alpha+1}\eta) \le K\alpha \cup \{\beta\} < \psi(\Omega^{\alpha+1}(\eta{+}1))$.

Let $\gamma := \Omega^{\alpha+1}\eta$. Then $\dot\psi\gamma \le K\alpha \cup \{\beta\} < \psi(\gamma + \Omega^{\alpha+1})$.

If $\Omega\alpha + \beta < \omega$ then $[\alpha, \beta] := \beta$, else
$[\alpha, \beta] := \gamma + \Omega^{\alpha}(1{+}\xi)$ with $\xi := \begin{cases} -\dot\psi\gamma + \beta & \text{if } K\alpha < \dot\psi\gamma, \\ \beta & \text{otherwise.} \end{cases}$

Remark $\omega \le \Omega\alpha + \beta \;\Rightarrow\; \omega \le [\alpha, \beta]$.

Lemma 6.4 *(a)* $K[\alpha, \beta] < \psi[\alpha, \beta]$ *; (b)* $K(\Omega\alpha + \beta) < \psi[\alpha, \beta]$.

Proof
Assume $\omega \le \Omega\alpha + \beta$ (otherwise $K[\alpha, \beta] = \emptyset$ and $K(\Omega\alpha + \beta) = \emptyset$). Then $[\alpha, \beta]$ $=_{\mathrm{NF}} \gamma + \Omega^{\alpha}(1{+}\xi)$ with $\psi\gamma \le K\alpha \cup \{\beta\} < \psi(\gamma + \Omega^{\alpha+1})$ and $\xi \le \beta \le \dot\psi\gamma + \xi$.

(a) By Lemmata 6.2d and 5.2a, b we obtain $K\gamma < \psi\gamma < \psi[\alpha, \beta]$.

$K(\Omega^{\alpha}(1{+}\xi)) = K\alpha \cup K\xi \;\&\; \xi \le \beta < \Omega \;\&\; K\alpha \cup \{\beta\} < \psi(\gamma + \Omega^{\alpha+1}) \;\Rightarrow$
$K(\Omega^{\alpha}(1{+}\xi)) < \psi(\gamma + \Omega^{\alpha+1})$.
$[\alpha, \beta] =_{\mathrm{NF}} \gamma + \Omega^{\alpha}(1{+}\xi) \;\&\; K(\Omega^{\alpha}(1{+}\xi)) < \psi(\gamma + \Omega^{\alpha+1}) \overset{6.3}{\Rightarrow}$
$K(\Omega^{\alpha}(1{+}\xi)) < \psi[\alpha, \beta]$.

(b) By (proof of) (a) we have $K\alpha \cup \{\xi\} \subseteq K[\alpha, \beta] < \psi[\alpha, \beta]$ and $\psi\gamma < \psi[\alpha, \beta]$. From this together with $\beta \le \dot\psi\gamma + \xi$ and $\psi[\alpha, \beta] \in \mathbb{X} \subseteq \mathbb{H}$, we obtain $K(\Omega\alpha + \beta) = K\alpha \cup \{\beta\} < \psi[\alpha, \beta]$.

Lemma 6.5
$\Omega\alpha_0 + \beta_0 < \Omega\alpha_1 + \beta_1 \;\&\; K(\Omega\alpha_0 + \beta_0) < \psi[\alpha_1, \beta_1] \;\Rightarrow\; [\alpha_0, \beta_0] < [\alpha_1, \beta_1]$.

Proof
1. $\Omega\alpha_1 + \beta_1 < \omega$: Then $[\alpha_0, \beta_0] = \beta_0 < \beta_1 = [\alpha_1, \beta_1]$.
2. $\Omega\alpha_0 + \beta_0 < \omega \le \Omega\alpha_1 + \beta_1$: Then $[\alpha_0, \beta_0] = \beta_0 < \omega \le [\alpha_1, \beta_1]$.
3. $\omega \le \Omega\alpha_0 + \beta_0$: Then $[\alpha_i, \beta_i] =_{\mathrm{NF}} \gamma_i + \Omega^{\alpha_i}(1 + \xi_i)$ $(i = 0, 1)$, and $\dot\psi\gamma_0 \le K\alpha_0 \cup \{\beta_0\} < \psi[\alpha_1, \beta_1]$.
3.1. $\alpha := \alpha_0 = \alpha_1 \;\&\; \beta_0 < \beta_1$:
3.1.1. $\gamma_0 < \gamma_1$: Then $[\alpha_0, \beta_0] = \gamma_0 + \Omega^{\alpha}(1{+}\xi_0) < \gamma_0 + \Omega^{\alpha+1} \le \gamma_1 \le [\alpha_1, \beta_1]$.
3.1.2. $\gamma := \gamma_0 = \gamma_1$: To prove $\xi_0 < \xi_1$. We have $\xi_i = \begin{cases} -\dot\psi\gamma + \beta_i & \text{if } K\alpha < \dot\psi\gamma, \\ \beta_i & \text{otherwise.} \end{cases}$
Hence $\xi_0 < \xi_1$ follows from $\beta_0 < \beta_1$.
3.2. $\alpha_0 < \alpha_1$: From $\dot\psi\gamma_0 < \psi[\alpha_1, \beta_1]$ and $0 < \alpha_1$ we get $\gamma_0 < [\alpha_1, \beta_1] = \gamma_1 + \Omega^{\alpha_1}(1{+}\xi_1)$, and then $\gamma_0 + \Omega^{\alpha_1} \le [\alpha_1, \beta_1]$. Further we have $[\alpha_0, \beta_0] = \gamma_0 + \Omega^{\alpha_0}$ $(1{+}\xi_0) < \gamma_0 + \Omega^{\alpha_0+1} \le \gamma_0 + \Omega^{\alpha_1}$.

Lemma 6.6 $\vartheta(\Omega\alpha+\beta) \le \psi([\alpha,\beta]) < \psi\Lambda$.

Proof by induction on $\Omega\alpha+\beta$:
Let $\gamma_0 := \psi[\alpha,\beta]$.
To prove: $\gamma_0 \in \mathbb{X}$ & $K(\Omega\alpha+\beta) < \gamma_0$ & $\forall\zeta < \Omega\alpha+\beta(K\zeta < \gamma_0 \Rightarrow \vartheta\zeta < \gamma_0)$.
1. By definition of ψ and Lemma 6.4b we have $\gamma_0 \in \mathbb{X}$ & $K(\Omega\alpha+\beta) < \gamma_0$.
2. Assume $\Omega\xi+\eta < \Omega\alpha+\beta$ & $K(\Omega\xi+\eta) < \gamma_0$. Then, by Lemma 6.5, $[\xi,\eta] < [\alpha,\beta]$. From this by Lemmata 6.4a, 5.2a, b and the IH we obtain $\vartheta(\Omega\xi+\eta) \le \psi[\xi,\eta] < \psi[\alpha,\beta] = \gamma_0$.

Definition of $\overline{\delta} < \Lambda$ for $\delta < \Lambda$[3]
1. If $\delta < \omega$ then $\overline{\delta} := \delta$.
2. If $\omega \le \delta =_{\mathrm{NF}} \gamma + \Omega^{\alpha}(1+\xi)$ then $\overline{\delta} := \Omega\alpha+\beta$ with $\beta := \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma, \\ \xi & \text{otherwise.} \end{cases}$

Remark $\omega \le \delta \;\Rightarrow\; \omega \le \overline{\delta}$.

Lemma 6.7 $\overline{[\alpha,\beta]} = \Omega\alpha+\beta$.

Proof

$$[\alpha,\beta] =_{\mathrm{NF}} \gamma + \Omega^{\alpha}(1{+}\xi) \text{ with } \xi = \begin{cases} -\dot{\psi}\gamma+\beta & \text{if } K\alpha < \dot{\psi}\gamma, \\ \beta & \text{otherwise.} \end{cases}$$

Hence $\overline{[\alpha,\beta]} = \Omega\alpha+\tilde{\beta}$ with $\tilde{\beta} := \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma, \\ \xi & \text{otherwise.} \end{cases}$ Obviously $\tilde{\beta} = \beta$.

Lemma 6.8 *Let* $\delta, \delta' < \Lambda$.

(a) $K\delta < \psi\delta$ & $\overline{\delta} = \Omega\alpha+\beta \;\Rightarrow\; \delta = [\alpha,\beta]$.
(b) $K\delta < \psi\delta \;\Rightarrow\; \vartheta\overline{\delta} \le \psi\delta$.
(c) $K\delta < \psi\delta$ & $K\delta' < \psi\delta'$ & $\overline{\delta} = \overline{\delta'} \;\Rightarrow\; \delta = \delta'$.

Proof
(a) 1. $\delta < \omega$: Then $\Omega\alpha+\beta = \overline{\delta} = \delta < \omega$ and thus $[\alpha,\beta] = \beta = \delta$.
2. Otherwise: Then $\omega \le \delta =_{\mathrm{NF}} \gamma + \Omega^{\alpha}(1{+}\xi)$ with $\beta = \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma, \\ \xi & \text{otherwise.} \end{cases}$

The latter yields $\dot{\psi}\gamma \le K\alpha \cup \{\beta\}$. From $K\delta < \psi\delta$ by Lemma 6.2c we get $K\gamma < \psi\gamma$ and then $\psi\gamma < \psi\delta$. Now we have $K\alpha \cup K\xi \subseteq K\delta < \psi\delta \in \mathbb{H}$ & $\psi\gamma < \psi\delta$ which implies $K\alpha \cup \{\beta\} < \psi\delta \le \psi(\gamma + \Omega^{\alpha+1})$.

$$\text{It follows that } [\alpha,\beta] = \gamma + \Omega^{\alpha}(1{+}\tilde{\xi}) \text{ where } \tilde{\xi} := \begin{cases} -\dot{\psi}\gamma+\beta & \text{if } K\alpha < \dot{\psi}\gamma, \\ \beta & \text{otherwise.} \end{cases}$$

Obviously $\tilde{\xi} = \xi$ and therefore $[\alpha,\beta] = \delta$.

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[3] This definition is closely related to clause 5 in Definition 3.6 of [18]. But be aware that $\overline{\delta}$ there has a different meaning than here.

(b) Take α, β such that $\overline{\delta} = \Omega\alpha + \beta$. Then by Lemma 6.6 and (a) we obtain $\vartheta\overline{\delta} = \vartheta(\Omega\alpha + \beta) \leq \psi[\alpha, \beta] = \psi\delta$.
(c) By (a) there are $\alpha, \beta, \alpha', \beta'$ such that $\overline{\delta} = \Omega\alpha + \beta$ & $\delta = [\alpha, \beta]$ and $\overline{\delta}' = \Omega\alpha' + \beta'$ & $\delta' = [\alpha', \beta']$. Therefore from $\overline{\delta} = \overline{\delta'}$ one concludes $\alpha = \alpha'$ & $\beta = \beta'$ and then $\delta = \delta'$.

Theorem 6.9 $\delta < \Lambda$ & $K\delta < \psi\delta \Rightarrow \vartheta\overline{\delta} = \psi\delta$.

Proof by induction on δ:
By Lemma 6.8b we have $\vartheta\overline{\delta} \leq \psi\delta$. Assumption: $\vartheta\overline{\delta} < \psi\delta$. Then by Lemma 6.2b there exists γ s.t. $K\gamma < \psi\gamma = \vartheta\overline{\delta} < \psi\delta$. Hence $\gamma < \delta$ and therefore, by IH, $\psi\gamma = \vartheta\overline{\gamma}$. From $\vartheta\overline{\delta} = \psi\gamma = \vartheta\overline{\gamma}$ & $K\delta < \psi\delta$ & $K\gamma < \psi\gamma$ by (ϑ4) and Lemma 6.8c we obtain $\delta = \gamma$. Contradiction.

Corollary 6.10

(a) $\vartheta(\Omega\alpha + \beta) = \psi[\alpha, \beta]$.
(b) $K\alpha < \psi\Omega^{\alpha+1} \Rightarrow \vartheta(\Omega\alpha) = \psi\Omega^{\alpha}$.

Proof
(a) Let $\delta := [\alpha, \beta]$. Then by Lemma 6.4a $K\delta < \psi\delta$, and therefore $\vartheta(\Omega\alpha + \beta) \overset{\text{L.6.7}}{=} \vartheta\overline{\delta} = \psi\delta = \psi[\alpha, \beta]$.
(b) $\alpha < \Lambda$ & $K\alpha < \psi\Omega^{\alpha+1} \Rightarrow \vartheta(\Omega\alpha) = \psi[\alpha, 0] = \psi(\Omega^{\alpha}(1 + 0)) = \psi\Omega^{\alpha}$.

7 Defining the Bachmann Hierarchy by Functionals of Higher Type

This section is based on [23, (3.2.9)–(3.2.11), (3.2.15)].

Convention. n ranges over natural numbers ≥ 1.

Definition Let M be an arbitrary nonempty set.
1. $\mathsf{M}^1 := \mathsf{M}$. 2. $\mathsf{M}^{n+1} :=$ set of all functions $F : \mathsf{M}^n \to \mathsf{M}^n$.

Notation If $1 \leq m < n$ and $F_i \in \mathsf{M}^i$ for $m \leq i \leq n$, then $F_n F_{n-1} \ldots F_m := F_n(F_{n-1}) \ldots (F_m)$.

Abbreviation $\mathrm{Id}^{n+1} := \mathrm{Id}_{\mathsf{M}^n} \in \mathsf{M}^{n+1}$.

Assumption
∇ is an operation such that for every family $(X_\xi)_{\xi<\alpha}$ with $0 < \alpha \leq \Omega$ the following holds: $\forall \xi < \alpha (X_\xi \in \mathsf{M}^1) \Rightarrow \nabla_{\xi<\alpha} X_\xi \in \mathsf{M}^1$.

Definition If $n > 1$ and $\forall \xi < \alpha (F_\xi \in \mathsf{M}^{n+1})$ then
$\nabla_{\xi<\alpha} F_\xi \in \mathsf{M}^{n+1}$ is defined by $(\nabla_{\xi<\alpha} F_\xi) G := \nabla_{\xi<\alpha} (F_\xi G)$.

Lemma 7.1 *If* $0 < \alpha \leq \Omega$ & $\forall \xi < \alpha (F_\xi \in \mathsf{M}^{n+1})$ & $H \in \mathsf{M}^{n+1}$, *then*
$(\nabla_{\xi<\alpha} F_\xi) \circ H = \nabla_{\xi<\alpha} (F_\xi \circ H)$.

Proof
For each $G \in \mathsf{M}^n$ we have
$((\nabla_{\xi<\alpha} F_\xi) \circ H)G = (\nabla_{\xi<\alpha} F_\xi)(HG) = \nabla_{\xi<\alpha}(F_\xi(HG)) = \nabla_{\xi<\alpha}((F_\xi \circ H)G) = (\nabla_{\xi<\alpha}(F_\xi \circ H))G$.

Definition For $F \in \mathsf{M}^{n+1}$ and $\alpha \leq \Omega$ we define $F^{(\alpha)} \in \mathsf{M}^{n+1}$ by (i) $F^{(0)} := \mathrm{Id}^{n+1}$; (ii) $F^{(\alpha+1)} := F \circ F^{(\alpha)}$; (iii) $F^{(\alpha)} := \nabla_{\xi<\alpha} F^{(1+\xi)}$ if $\alpha \in \mathit{Lim}$.

Definition

(i) Let $\mathrm{I}_2 \in \mathsf{M}^2$ be given;
(ii) For $m \geq 2$ we define $\mathrm{I}_{m+1} \in \mathsf{M}^{m+1}$ by $\mathrm{I}_{m+1} F := F^{(\Omega)}$.

Definition of $\llbracket\alpha\rrbracket_m$
For $m \geq 2$ and $\alpha < \varepsilon_{\Omega+1}$ we define $\llbracket\alpha\rrbracket_m \in \mathsf{M}^m$ by recursion on α:
(i) $\llbracket 0\rrbracket_m := \mathrm{Id}^m$; (ii) If $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta \eta$, then $\llbracket\alpha\rrbracket_m := (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta)} \circ \llbracket\gamma\rrbracket_m$.

Lemma 7.2 *For $m \geq 2$ and $\alpha < \varepsilon_{\Omega+1}$:*

(a) $\llbracket\alpha+1\rrbracket_m = \mathrm{I}_m \circ \llbracket\alpha\rrbracket_m$*;*
(b) $\alpha \in \mathit{Lim} \;\Rightarrow\; \llbracket\alpha\rrbracket_m = \nabla_{\xi<\tau(\alpha)} \llbracket\alpha[\xi]\rrbracket_m$.

Proof
(a) $\llbracket\gamma + \Omega^0(\eta+1)\rrbracket_m = (\llbracket 0\rrbracket_{m+1}\mathrm{I}_m)^{(\eta+1)} \circ \llbracket\gamma\rrbracket_m = \mathrm{I}_m^{(\eta+1)} \circ \llbracket\gamma\rrbracket_m = \mathrm{I}_m \circ (\mathrm{I}_m^{(\eta)} \circ \llbracket\gamma\rrbracket_m) = \mathrm{I}_m \circ \llbracket\gamma + \Omega^0 \cdot \eta\rrbracket_m$.
(b) Induction on α:
1. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta \eta$ with $\eta \in \mathit{Lim}$: Then $\tau(\alpha) = \eta$ and $\alpha[\xi] = \gamma + \Omega^\beta(1+\xi)$.
$\llbracket\alpha\rrbracket_m = (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta)} \circ \llbracket\gamma\rrbracket_m = (\nabla_{\xi<\eta}(\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(1+\xi)}) \circ \llbracket\gamma\rrbracket_m = \nabla_{\xi<\eta}((\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(1+\xi)} \circ \llbracket\gamma\rrbracket_m) = \nabla_{\xi<\eta}\llbracket\alpha[\xi]\rrbracket_m$.
2. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta(\eta+1)$ with $\beta = \beta_0+1$: Then $\tau(\alpha) = \Omega$ and $\alpha[\xi] = \gamma + \Omega^\beta \eta + \Omega^{\beta_0}(1+\xi)$.
$\llbracket\alpha\rrbracket_m = (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta+1)} \circ \llbracket\gamma\rrbracket_m = (\llbracket\beta_0+1\rrbracket_{m+1}\mathrm{I}_m) \circ (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta)} \circ \llbracket\gamma\rrbracket_m \overset{(a)}{=} (\mathrm{I}_{m+1}(\llbracket\beta_0\rrbracket_{m+1}\mathrm{I}_m)) \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m = (\nabla_{\xi<\Omega}(\llbracket\beta_0\rrbracket_{m+1}\mathrm{I}_m)^{(1+\xi)}) \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m = \nabla_{\xi<\Omega}((\llbracket\beta_0\rrbracket_{m+1}\mathrm{I}_m)^{(1+\xi)} \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m) = \nabla_{\xi<\Omega}\llbracket\alpha[\xi]\rrbracket_m$.
3. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta(\eta+1)$ with $\beta \in \mathit{Lim}$:
Then $\tau(\alpha) = \tau(\beta)$ and $\alpha[\xi] = \gamma + \Omega^\beta\eta + \Omega^{\beta[\xi]}$.
$\llbracket\alpha\rrbracket_m = (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta+1)} \circ \llbracket\gamma\rrbracket_m = (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m) \circ (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m)^{(\eta)} \circ \llbracket\gamma\rrbracket_m = (\llbracket\beta\rrbracket_{m+1}\mathrm{I}_m) \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m \overset{\mathrm{IH}}{=} (\nabla_{\xi<\tau(\beta)}(\llbracket\beta[\xi]\rrbracket_{m+1}\mathrm{I}_m)) \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m = \nabla_{\xi<\tau(\beta)}((\llbracket\beta[\xi]\rrbracket_{m+1}\mathrm{I}_m) \circ \llbracket\gamma + \Omega^\beta\eta\rrbracket_m) = \nabla_{\xi<\tau(\alpha)}\llbracket\alpha[\xi]\rrbracket_m$.

Corollary 7.3 *For $X \in \mathsf{M}^1$ and $\alpha < \varepsilon_{\Omega+1}$ the following holds:*

(i) $\llbracket 0\rrbracket_2 X = X$*;*
(ii) $\llbracket\alpha+1\rrbracket_2 X = \mathrm{I}_2(\llbracket\alpha\rrbracket_2 X)$*;*
(iii) $\llbracket\alpha\rrbracket_2 X = \nabla_{\xi<\tau(\alpha)}(\llbracket\alpha[\xi]\rrbracket_2 X)$ *if $\alpha \in \mathit{Lim}$.*

Now we fix M, I_2 and ∇ as follows:

1. $\mathsf{M} :=$ set of all Ω-club subsets of Ω.
2. $\mathrm{I}_2 : \mathsf{M} \to \mathsf{M}, I_2(X) := \{\beta \in \Omega : \mathrm{en}_X(\beta) = \beta\}$, where en_X is the ordering function of X.
3. If $\forall \xi < \alpha (X_\xi \in \mathsf{M})$ then $\nabla_{\xi<\alpha} X_\xi := \begin{cases} \bigcap_{\xi<\alpha} X_\xi & \text{if } \alpha < \Omega, \\ \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} X_\xi\} & \text{if } \alpha = \Omega. \end{cases}$

Then by transfinite induction on α from the above Corollary and the definition of R_α^X we conclude

Theorem 7.4 $R_\alpha^X = [[\alpha]]_2 X$, *for all* $\alpha < \varepsilon_{\Omega+1}$ *and* $X \in \mathsf{M}$.

Appendix

This appendix is devoted to the proof of Lemmata 2.1, 2.2d.

Lemma A1

(a) $\lambda \in \mathit{Lim} \;\Rightarrow\; 0 < \lambda[0]$.
(b) $\gamma + \Omega^\beta < \Omega^\alpha \;\&\; \eta < \Omega \;\Rightarrow\; \gamma + \Omega^\beta \eta < \Omega^\alpha$.
(c) $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \notin \mathrm{ran}(F_0) \;\&\; \Omega^\alpha < \lambda \;\Rightarrow\; \Omega^\alpha \leq \lambda[0]$.

Proof of (c):
From $\Omega^\alpha < \lambda = \gamma + \Omega^\beta \eta$ by (b) we get $\Omega^\alpha \leq \gamma + \Omega^\beta$. If $\eta \in \mathit{Lim}$ then $\lambda[0] = \gamma + \Omega^\beta$. If $1 < \eta = \eta_0{+}1$ then $\gamma + \Omega^\beta \leq \gamma + \Omega^\beta \eta_0 \leq \lambda[0]$. If $\eta = 1$ then $0 < \gamma$ (since $\lambda \notin \mathrm{ran}(F_0)$) and therefore $\Omega^{\beta+1} \leq \gamma$ which together with $\Omega^\alpha < \lambda = \gamma + \Omega^\beta$ yields $\Omega^\alpha \leq \gamma \leq \lambda[0]$.

Lemma A2 $\lambda =_{\mathrm{NF}} F_\alpha(\beta) \;\&\; 0 < \beta \;\Rightarrow\; F_\alpha(\beta[n]) \leq \lambda[n]$.

Proof
1. $\beta \in \mathit{Lim}$: $F_\alpha(\beta[n]) = \lambda[n]$.
2. $\beta = \beta_0{+}1$:
2.1. $\alpha = 0$: $F_\alpha(\beta[n]) = \Omega^{\beta_0} \leq \Omega^{\beta_0} \cdot (1{+}n) = \lambda[n]$.
2.2. $\alpha > 0$: $F_\alpha(\beta[n]) = F_\alpha(\beta_0) < \lambda^- \leq \lambda[n]$.

Lemma A3 $F_\zeta(\mu) < \lambda \leq F_\zeta(\mu{+}1) \;\Rightarrow\; F_\zeta(\mu) \leq \lambda[0]$.

Proof
0. $\lambda = F_\zeta(\mu + 1)$:
0.1. $\zeta = 0$: $\lambda = \Omega^{\mu+1}$, $\lambda[\xi] = \Omega^\mu(1{+}\xi)$, $\lambda[0] = F_0(\mu)$.
0.2. $\zeta > 0$: $F_\zeta(\mu) < \lambda^- < F_{\zeta[0]}(\lambda^-) = \lambda[0]$.
1. $\lambda < F_\zeta(\mu + 1)$:
1.1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \notin \mathrm{ran}(F_0)$:
$F_\zeta(\mu) \in \mathrm{ran}(F_0) \;\&\; F_\zeta(\mu) < \lambda \overset{\mathrm{L.A1c}}{\Rightarrow} F_\zeta(\mu) \leq \lambda[0]$.
1.2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$: Then $\alpha < \zeta$ and thus $F_\alpha(F_\zeta(\mu)) = F_\zeta(\mu) < F_\alpha(\beta)$. Hence $F_\zeta(\mu) < \beta$ and therefore $F_\zeta(\mu) \overset{\mathrm{IH}}{\leq} \beta[0] \leq F_\alpha(\beta[0]) \overset{\mathrm{A2}}{\leq} \lambda[0]$.

Definition $r(\gamma) := \begin{cases} -1 & \text{if } \gamma \notin \mathrm{ran}(F_0), \\ \alpha & \text{if } \gamma =_{\mathrm{NF}} F_\alpha(\beta), \\ \gamma & \text{if } \gamma = \Lambda. \end{cases}$

Lemma A4

(a) $r(F_\alpha(\beta)) = \max\{\alpha, r(\beta)\}$.
(b) $\lambda[0] < \delta < \lambda \;\Rightarrow\; r(\delta) \le r(\lambda)$.
(c) $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \notin \mathit{Lim}$ & $\lambda^- < \eta < \lambda \;\Rightarrow\; \lambda^- \le \eta[1]$.

Proof
(a) 1. $\beta < F_\alpha(\beta)$:
Then $r(F_\alpha(\beta)) = \alpha$ and ($r(\beta) = -1$ or $\beta =_{\mathrm{NF}} F_{\beta_0}(\beta_1)$ with $\beta_0 \le \alpha$).
2. $\beta = F_\alpha(\beta)$: Then $\beta =_{\mathrm{NF}} F_{\beta_0}(\beta_1)$ with $\alpha < \beta_0 = r(\beta) = r(F_\alpha(\beta))$.
(b) 1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$:
1.1. $\eta \in \mathit{Lim}$: $\gamma + \Omega^\beta = \lambda[0] < \delta < \gamma + \Omega^\beta\eta \overset{\mathrm{A1b}}{\Rightarrow} \delta \notin \mathrm{ran}(F_0)$.
1.2. $\eta = \eta_0{+}1$:
$\gamma + \Omega^\beta\eta_0 < \lambda[0] < \delta < \gamma + \Omega^\beta(\eta_0{+}1) \notin \mathrm{ran}(F_0)$ & $\Omega^{\beta+1}|\gamma \;\Rightarrow\; \delta \notin \mathrm{ran}(F_0)$.
2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$: If $\lambda < F_{\alpha+1}(0)$ then also $\delta < F_{\alpha+1}(0)$ and thus $r(\delta) \le \alpha = r(\lambda)$. Otherwise there exists μ such that $F_{\alpha+1}(\mu) < \lambda < F_{\alpha+1}(\mu{+}1)$. Then by Lemma A3 we get $F_{\alpha+1}(\mu) \le \lambda[0] < \delta < F_{\alpha+1}(\mu{+}1)$ and thus $\delta \notin \mathrm{ran}(F_{\alpha+1})$, i.e. $r(\delta) \le \alpha = r(\lambda)$.
3. $\lambda = \Lambda$: $r(\delta) < \Lambda = r(\Lambda)$.
(c) For $\beta = 0 \vee \eta = \eta_0{+}1$ the claim is trivial. Assume now $\beta = \beta_0{+}1$ & $\eta \in \mathit{Lim}$.
$F_\alpha(\beta_0) < \eta < F_\alpha(\beta_0{+}1) \overset{\mathrm{L.A3}}{\Rightarrow} \lambda^- = F_\alpha(\beta_0){+}1 \le \eta[0] + 1 \le \eta[1]$.

Lemma 2.1 $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \in \mathit{Lim}$ & $1 \le \xi < \tau_\beta \;\Rightarrow\; \lambda[\xi] =_{\mathrm{NF}} F_\alpha(\beta[\xi])$.

Proof
We have $\lambda[\xi] = F_\alpha(\beta[\xi])$ & $\beta[0] < \beta[\xi] < \beta$. By Lemma A4b this yields $\lambda[\xi] = F_\alpha(\beta[\xi])$ & $r(\beta[\xi]) \le r(\beta) \le \alpha$, whence $\lambda[\xi] =_{\mathrm{NF}} F_\alpha(\beta[\xi])$.

Lemma 2.2d $\xi{+}1 < \tau_\lambda$ & $\lambda[\xi] < \delta \le \lambda[\xi{+}1] \;\Rightarrow\; \lambda[\xi] \le \delta[1]$.

Proof by induction on $\delta \,\#\, \lambda$:
If $r(\delta) < r(\lambda[\xi])$ then, by Lemma A4b, $\lambda[\xi] \le \delta[0]$. (Proof: $\delta[0] < \lambda[\xi] < \delta$ $\overset{\mathrm{L.A4b}}{\Rightarrow} r(\lambda[\xi]) \le r(\delta)$).

Assume now that $r(\lambda[\xi]) \le r(\delta)$ (†).
1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$.
1.1. $\eta \in \mathit{Lim}$:
$\gamma + \Omega^\beta(1{+}\xi) = \lambda[\xi] < \delta < \lambda[\xi{+}1] = \gamma + \Omega^\beta(1{+}\xi) + \Omega^\beta \;\Rightarrow\; \lambda[\xi] \le \delta[0]$.
1.2. $\eta = \eta_0{+}1$: $\gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi] = \lambda[\xi] < \delta \le \lambda[\xi{+}1] = \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi{+}1]$ $\Rightarrow \delta = (\gamma + \Omega^\beta\eta_0) + \delta_0$ with $\Omega^\beta[\xi] < \delta_0 \le \Omega^\beta[\xi{+}1] \;\Rightarrow \delta[0] = \gamma + \Omega^\beta\eta_0 + \delta_0[0]$ with $\Omega^\beta[\xi] \overset{\mathrm{IH}}{\le} \delta_0[0] \;\Rightarrow\; \lambda[\xi] \le \delta[0]$.

2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \in \mathit{Lim}$: Then (1) $\lambda[\xi] = F_\alpha(\beta[\xi])$, and (2) $\lambda[\xi] < \delta < \lambda$. From $\alpha \overset{(1)}{\le} r(\lambda[\xi]) \overset{(\dagger)}{\le} r(\delta) \overset{(2),\mathrm{L.A4b}}{\le} r(\lambda) = \alpha$ we get $r(\delta) = \alpha$, i.e. $\delta =_{\mathrm{NF}} F_\alpha(\eta)$ for some η. Now from $\lambda[\xi] < \delta \le \lambda[\xi{+}1]$ we conclude $\beta[\xi] < \eta \le \beta[\xi{+}1]$ and then, by IH, $\beta[\xi] \le \eta[0]$. Hence $\lambda[\xi] \le F_\alpha(\eta[0]) \overset{\mathrm{L.A2}}{\le} \delta[0]$.

3. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \notin \mathit{Lim}$:

3.1. $\alpha = 0$: Then $\beta = \beta_0{+}1$, and $\lambda[\xi] = \Omega^{\beta_0}(1{+}\xi) < \delta \le \Omega^{\beta_0}(1{+}\xi) + \Omega^{\beta_0}$ implies $\lambda[\xi] \le \delta[0]$.

3.2. $\alpha = \alpha_0{+}1$: Then $\lambda[\xi] = F_{\alpha_0}^{\xi+1}(\lambda^-)$. Hence, by (†), $\delta =_{\mathrm{NF}} F_\zeta(\eta)$ with $\alpha_0 \le \zeta$.

3.2.1. $\alpha_0 < \zeta$: $\lambda^- < F_\zeta(\eta) \;\Rightarrow\; \lambda[\xi{+}1] = F_{\alpha_0}^{\xi+2}(\lambda^-) < F_\zeta(\eta)$. Contradiction.

3.2.2. $\zeta = \alpha_0$: Then from $F_{\alpha_0}^{\xi+1}(\lambda^-) = \lambda[\xi] < \delta = F_{\alpha_0}(\eta) \le \lambda[\xi{+}1]$ we conclude $F_{\alpha_0}^{\xi}(\lambda^-) < \eta \le \lambda[\xi]$. As we will show, this implies $F_{\alpha_0}^{\xi}(\lambda^-) \le \eta[1]$, thence $F_{\alpha_0}^{\xi+1}(\lambda^-) \le F_\zeta(\eta[1]) \overset{\mathrm{L.A2}}{\le} \delta[1]$.

Proof of $F_{\alpha_0}^{\xi}(\lambda^-) \le \eta[1]$:

(i) $\xi = n{+}1$: Then the claim follows by IH from $\lambda[n] = F_{\alpha_0}^{\xi}(\lambda^-) < \eta \le \lambda[n{+}1]$.

(ii) $\xi = 0$: $\lambda^- < \eta < \lambda \overset{\mathrm{L.A4c}}{\Rightarrow} \lambda^- \le \eta[1]$.

3.3. $\alpha \in \mathit{Lim}$: $\lambda[\xi] = F_{\alpha[\xi]}(\lambda^-)$, and by (†) we have $\delta =_{\mathrm{NF}} F_\zeta(\eta)$ with $\alpha[\xi] \le \zeta$.

3.3.1. $\alpha[\xi{+}1] < \zeta$: $\lambda^- < F_\zeta(\eta) \;\Rightarrow\; F_{\alpha[\xi+1]}(\lambda^-) < F_\zeta(\eta) = \delta$. Contradiction.

3.3.2. $\alpha[\xi] < \zeta \le \alpha[\xi{+}1]$:

(i) $\eta \in \mathit{Lim}$: Then $\lambda^- < \delta[1] = F_\zeta(\eta[1])$ (for $\beta = 0$, $\lambda^- = 0$. If $\beta = \beta_0{+}1$, then $F_\alpha(\beta_0) < \delta < F_\alpha(\beta_0{+}1)$ and thus, by Lemma A3, $F_\alpha(\beta_0) \le \delta[0]$). $\alpha[\xi] < \zeta$ & $\lambda^- < \delta[1] \;\Rightarrow\; \lambda[\xi] = F_{\alpha[\xi]}(\lambda^-) < \delta[1]$.

(ii) $\eta \notin \mathit{Lim}$: By IH $\alpha[\xi] \le \zeta[1]$. Further $\lambda^- \le \delta^-$.

Proof of $\lambda^- \le \delta^-$: Assume $\beta = \beta_0{+}1$.

$F_\alpha(\beta_0) < \delta = F_\zeta(\eta)$ & $\zeta < \alpha \Rightarrow 0 < \eta \Rightarrow \eta = \eta_0{+}1$.

$F_\alpha(\beta_0) < F_\zeta(\eta_0{+}1)$ & $\zeta < \alpha \;\Rightarrow\; F_\alpha(\beta_0) \le F_\zeta(\eta_0)$.

From $\alpha[\xi] \le \zeta[1]$ and $\lambda^- \le \delta^-$ we conclude $\lambda[\xi] = F_{\alpha[\xi]}(\lambda^-) \le F_{\zeta[1]}(\delta^-) \le \delta[1]$.

3.3.3. $\zeta = \alpha[\xi]$: This case is similar to 3.2.2(ii):

$\lambda[\xi] = F_\zeta(\lambda^-) < F_\zeta(\eta) < F_\alpha(\beta) \;\Rightarrow\; \lambda^- < \eta < F_\alpha(\beta) \;\Rightarrow \lambda[\xi] = F_\zeta(\lambda^-)$
$\overset{\mathrm{L.A4c}}{\le} F_\zeta(\eta[1]) \overset{\mathrm{L.A2}}{\le} \delta[1]$.

4. $\lambda = \Lambda$: This case is very similar to 3.3, but considerably simpler.

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About Truth and Types

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Dedicated to Gerhard Jäger on occasion of his 60th birthday.

Abstract We investigate a weakening of the classical theory of Frege structures and extensions thereof which naturally interpret (predicative) theories of explicit types and names à la Jäger.

1 Introduction

Non-extensionality is a basic feature of the so-called systems of *explicit mathematics* (EM in short), be they formalized in the style of Feferman [9] or in the framework of *types and names* à la Jäger [18]. Instead of *functions* in set theoretic sense, EM assumes the notion of *rule* or *algorithm* as basic. Similarly, a fundamental tenet is that a collection X always comes equipped with an *explicit presentation*, i.e. by specifying a *defining property* given by a. It follows that the membership predicate for stating that a given object x is a member of X is naturally interpreted by means of satisfaction or predicate application: $x \in X$ iff *a truly applies* to x, or x satisfies the defining property (presented by) a of X, a being termed a *name* of X (see [18]). This calls for a ground applicative structure $\mathcal{M}$ with a primitive application operation (a applies to x) and a truth predicate T which applies to elements of $\mathcal{M}$. It is a crucial point: T is not a metamathematical predicate in the standard sense, that applies to

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(the Gödel numbers of) formulas of a given formal language. Abstract truth applies in the present context to suitable objects—termed *propositions*—as distinguished from sentences: sentences may represent propositions, but it is in general open whether every sentence represents a proposition, and whether every proposition is explicitly represented by a sentence.

This leads to the motivation of the present paper: we like to investigate some connections between axiomatic theories of truth in the general setting explained above, and explicit mathematics. In this direction Jäger and his school have offered important contributions, which range from the field of (meta)predicativity to the field of feasible applicative systems, as witnessed e.g. by [20, 22] or [8].

As to the contents, we survey four different truth-theoretic extensions of a basic applicative theory **TON**, the core of explicit mathematics. Section 2 describes a compositional theory **CT** of propositions and truth, corresponding to Jäger's **EET**, the elementary theory of types and names, which is related to the system of arithmetical analysis, and ought to be compared with axiomatic theories of truth over Peano arithmetic, as investigated by Halbach [17]. We then consider two incompatible extensions of **CT**. Section 3 deals with an extension **AT**, where the very predicate of being a proposition does define a propositional function. **AT** justifies theories of types and names, where elementary types are closed under a weak form of power type operation. In Sect. 4 we describe a theory **PT** of propositions and truth, in which propositions and truth interact up to a certain point, and the collection of types is also closed under the so-called join axiom. Thus **PT** interprets **EETJ**, i.e. **EET** with the join principle. The conclusive Sect. 5 explores a strengthening of axiomatic abstract self-referential truth via generalized induction principles; this matches the theory **NEM** of **EM**, where a principle of name induction is assumed besides join and elementary comprehension.

Some results are only stated and proofs sketched. A detailed development is left to a subsequent paper. Nevertheless, there should be little somewhat novel points in comparison to the extant literature, e.g. Theorems 5–6 of Sect. 2.4, Theorem 38 of Sect. 4.3, and Theorem 46 in Sect. 5.2.

2 Truth and Types I

2.1 Abstract Truth Over Combinatory Structures

The picture to have in mind corresponds to a structure $\mathcal{M} = \langle M, \cdot_M, \mathcal{P}^M, \mathcal{T}^M \rangle$, such that:

1. M is an *expanded combinatory algebra* with binary total application $\cdot_M$ and distinguished elements representing (i) basic combinators; (ii) basic number theoretic constructors; (iii) logic constructors and predicates; as usual $|M|$ is the underlying non-empty set of M (*domain* of M);
2. M contains an isomorphic copy N_M of the standard natural numbers;

3. $\mathcal{P}^M$, $\mathcal{T}^M$ are distinguished subsets of $|M|$ containing (objects representing) *elementary propositions* and *true* elementary propositions.

   $\mathcal{T}^M$ acts as a *classifier* over $|M|$: there is a natural map $\mathcal{R}_M$ such that

$$a \in M \mapsto \mathcal{R}_M(a) := \{c \in M | \mathcal{T}^M(ac)\}. \qquad (1)$$

$\mathcal{R}_M$ yields a connection with Jäger's approach to Explicit Mathematics. The set $\mathcal{R}_M(a)$ corresponds to *the type named by a in the sense of explicit mathematics.*

The collection of explicit types over $\mathcal{M}$ depends on the closure conditions on $\mathcal{P}^M$ and $\mathcal{T}^M$. And this is a reason to investigate abstract notions of truth with non-trivial closure properties. $\mathcal{P}^M$ may be absent and defined via $\mathcal{T}^M$.

2.2 Ground System: Language and Notations

The *basic first order language* $\mathcal{L}_T$ includes: predicate symbols $=$, P (proposition), T (true), N (natural number), the binary function symbol App (application), combinators K, S, the constant 0, successor SUC, predecessor PR, definition by cases on numbers $\mathsf{D_N}$, pairing PAIR, with projections LEFT, RIGHT; certain additional constants for representing predicate and logical constructors, namely: $\dot{=}$, $\dot{N}$, $\dot{P}$, $\dot{T}$, $\dot{\wedge}$, $\dot{\neg}$, $\dot{\forall}$. The constants $\dot{\vee}$, $\dot{\exists}$ and $\dot{\rightarrow}$ are also freely used (when needed), as being defined according to their classical definitions.[1]

$\mathcal{L}_P$ is the T-free sublanguage of $\mathcal{L}_T$ (i.e. T does not occur in $\mathcal{L}_P$-formulas); $\mathcal{L}_{op}$ is the P-free sublanguage of $\mathcal{L}_P$ (i.e. P does not occur in $\mathcal{L}_P$-formulas).

As usual, terms are inductively generated from variables and individual constants via application (for which we adopt the notation $ts := App(t, s)$). Formulas are inductively defined from atoms of the form $t = s$, $P(t)$, $T(t)$, $N(t)$ by means of sentential operators and quantifiers.

We further introduce *a map from formulas into terms*, preserving free variables:

$$A \mapsto [A].$$

E.g. $[t = s] := \dot{=}ts$, $[P(t)] := \dot{P}t$; $[T(t)] := \dot{T}t$; $[\forall x A] := \dot{\forall}(\lambda x.[A])$, etc.

Henceforth we also agree to use *infix notation* whenever we have constants representing binary operators or predicates, that is, we write $t \dot{\rightarrow} s$, $t \dot{\wedge} s$, etc. instead of the proper $\dot{\rightarrow} ts$, $\dot{\wedge} ts$, etc.

Abstraction terms can then be defined by lambda abstraction, i.e. if A is an arbitrary formula, $\{x|A\} := \lambda x.[A]$. We need some defined predicates:

---

[1] E.g. $\dot{\exists} f := \dot{\neg}(\dot{\forall}(\lambda x.\dot{\neg}(fx)))$; $\dot{\rightarrow}$ is assumed as primitive in Sect. 4.

1. $PF(f) := \forall x.P(fx), f$ is a *propositional function*;
2. $DF(f) := \forall x D(fx)$, where

$$D(a) := T(a) \vee T(\dot{\neg} a). \tag{2}$$

   $D(a)$ can be read as: a is *determinate* or *meaningful* (see [10, 13]), while $DF(f)$ means that f has *determinate truth values*.
3. $t =_e s := \forall u(T(tu) \leftrightarrow T(su))$; $=_e$ represents *extensional equality*.
4. $x\eta y := T(yx)$ is used for predicate application (or intensional membership).

2.3 Ground System: Applicative and Compositional Axioms

All systems we consider include a ground theory $\mathbf{TON}^-$ of total operations and numbers, i.e. equations for the standard combinators K and S, pairing, projections, closure axioms for the predicate N (natural numbers) and basic operations of successor SUC, predecessor PR, the constant 0, definition by cases $\mathsf{D_N}$ on N. For details, see [21, 22, 28].

In addition, we tacitly include in the ground system $\mathbf{TON}^-$ the following *independence conditions*: if b_0, b_1 (c, d) are distinct (arbitrary) constants among $\dot{=}$, $\dot{N}$, $\dot{P}$, $\dot{T}$, $\dot{\wedge}$, $\dot{\neg}$, $\dot{\forall}$:

$$b_0 \neq b_1, \tag{3}$$

$$cx = dy \to c = d \wedge x = y. \tag{4}$$

We also require some form of number theoretic induction; below we need a taxonomy of induction principles about numbers and propositional objects. We then state the basic axioms on propositions and we describe the core system.

Number-theoretic induction The schema $\mathcal{L}_T$-IND_N: if A is any arbitrary formula of $\mathcal{L}_T$, and $x + 1$ stands for the applicative term SUCx,

$$A(0) \wedge (\forall x \in N)(A(fx) \to A(f(x+1))) \to \forall x(N(x) \to A(fx)). \tag{5}$$

Besides $\mathcal{L}_T$-IND_N, we shall also consider other versions of the induction principle for N (in decreasing strength).

- The schema $\mathcal{L}_P$-IND_N: as $\mathcal{L}_T$-IND_N except that A is any formula of $\mathcal{L}_P$.
- The schema $\mathcal{L}_{op}$-IND_N: as $\mathcal{L}_T$-IND_N except that A is any formula of $\mathcal{L}_{op}$.
- The axiom PF-IND_N for propositional functions: add the hypothesis $PF(f)$ to (5) while replacing $A(-)$ by $T(f-)$.
- N-induction for determinate conditions D-IND_N: replace $PF(f)$ by $DF(f)$ in PF-IND_N.

Axioms for atomic propositions P-axioms

$$T(x) \to P(x), \tag{6}$$

$$P([x = y]) \wedge (T([x = y]) \leftrightarrow x = y), \tag{7}$$

$$P([N(x)]) \wedge (T([N(x)]) \leftrightarrow N(x)). \tag{8}$$

Axioms for classical compositional truth T-axioms

$$P(x) \quad \to \quad P(\dot{\neg}x) \wedge (T(\dot{\neg}x) \leftrightarrow \neg T(x)), \tag{9}$$

$$P(x) \wedge P(y) \quad \to \quad P(x \mathbin{\dot{\wedge}} y) \wedge (T(x \mathbin{\dot{\wedge}} y) \leftrightarrow T(x) \wedge T(y)), \tag{10}$$

$$\forall x P(fx) \quad \to \quad P(\dot{\forall} f) \wedge (T(\dot{\forall} f) \leftrightarrow \forall x T(fx)). \tag{11}$$

Axioms for strictness S-axioms:

$$P(\dot{\neg}x) \to P(x), \tag{12}$$

$$P(x \mathbin{\dot{\wedge}} y) \to P(x) \wedge P(y), \tag{13}$$

$$P(f) \to \forall x P(fx). \tag{14}$$

2.3.1 Simple Consequences of the Core System

Definition 1

1. **CT** is $\mathbf{TON}^-$ with the truth axioms for atomic propositions, classical compositional truth, strictness and the schema of number theoretic induction $\mathcal{L}_T$-IND_N for arbitrary formulas of $\mathcal{L}_T$ (5).
2. **CT**$\lceil$ = **CT** with formula N-induction replaced by PF-IND_N.

Notation. In general, given any formal theory SF, SF^- is the theory obtained from SF by omitting N-induction (of any sort).

Proposition 2 (provably in $\mathbf{CT}^-$)

(i) Propositional objects are exactly the determinate ones in the sense of (2)*:*

$$\forall x(P(x) \leftrightarrow T(x) \vee T(\dot{\neg}x)). \tag{15}$$

Hence

$$\mathsf{PF}\text{-}\mathsf{IND}_N \leftrightarrow \mathsf{D}\text{-}\mathsf{IND}_N.$$

(ii) If A is a formula of $\mathcal{L}_{op}$ with free variables in the list $\vec{x} = x_1, \ldots, x_n$,

$$\forall x_1 \ldots \forall x_n (P([A(\vec{x})])). \tag{16}$$

(iii) Moreover, under the same assumption as (ii):

$$\forall x_1 \dots \forall x_n (T([A(\vec{x})]) \leftrightarrow A(\vec{x})). \tag{17}$$

Proof Ad (i), (15): apply (6), (12) from right to left, and (9) from left to right.

Ad (ii)–(iii): easy induction on the definition of A, applying the closure conditions on the predicate P. ☐

Remark 1 (i) **CT** and $\mathbf{CT}\lceil$ are neutral as to the internal status of P, i.e. we do not have in general $T([P(x)]) \vee T([\neg P(x)])$. In order to conclude $T([\neg P(x)])$ from $\neg T([P(x)])$, we need that $P(x)$ *is a proposition*: this is an axiom of the next system to be considered.

(ii) If we interpret $P(x)$ trivially, i.e. we assume that *everything is a proposition*, all axioms involving P go through except the one for negation: for then we should have to postulate $T(a) \vee T(\dot{\neg} a)$, which leads to inconsistency.

***P*-induction** A natural assumption on the set of propositions is to assume that it is inductively generated according to clauses embodied by the axioms on atomic propositions and compositional truth.

Definition 3 Let the corresponding positive elementary operator be given by the formula

$$\begin{aligned}\mathcal{C}(u, X) \quad \Leftrightarrow \quad & \exists x \exists y [(u = [x = y] \vee u = [N(x)] \vee \\ & \vee (u = \dot{\neg} x \wedge X(x)) \vee (u = (x \dot{\wedge} y) \wedge X(x) \wedge X(y)) \vee \\ & \vee (u = \dot{\forall} x \wedge \forall z X(xz)))].\end{aligned}$$

(i) Then $\mathcal{L}_T$-IND_P is the schema of induction on propositions:

$$\forall x (\mathcal{C}(x, B) \to B(x)) \to \forall x (P(x) \to B(x)) \tag{18}$$

where $B(x)$ is an arbitrary formula of $\mathcal{L}_T$.

(ii) If we restrict B to be a formula of $\mathcal{L}_P$ (respectively of $\mathcal{L}_{op}$), we have the corresponding schema $\mathcal{L}_P$-IND_P($\mathcal{L}_{op}$-IND_P). Similarly, PF-IND_P(D-IND_P) is the P-induction axiom restricted to propositional (determinate) functions (choose $B(x) := T(fx)$ and assume fx to be a proposition).

Lemma 4 *The strictness axioms for propositions become provable in* $\mathbf{CT}^-$ *with* $\mathcal{L}_P$-IND_P *restricted to formulas which are positive in* P.

(Simply choose $B(x) := \mathcal{C}(x, P)$ in $\mathcal{L}_P$-IND_P and apply the independence axioms (3), (4)).

2.4 *Conservativity and Upper Bound*

First of all, a preliminary remark. We assume that the reader is acquainted with the relationship between formal systems: $\mathcal{S}$ *is proof-theoretically reducible to* $\mathcal{R}$, in short $\mathcal{S} \leq \mathcal{R}$ [12, 13]. It is understood that the relation holds modulo some fixed class of formulas in the common language and a fixed metatheory U, where the reduction proof can be carried out. Usually the reduction consists of an effective method for transforming proofs in $\mathcal{S}$ into proofs in $\mathcal{R}$, which is shown to converge in U and to preserve formulas of the given class Φ. As standard choice, U is primitive recursive arithmetic **PRA**, while Φ includes at least the formulas expressing the totality of operations from N to N. We also say that $\mathcal{S}$ is *proof-theoretically equivalent* to $\mathcal{R}$ (in short $\mathcal{S} \equiv \mathcal{R}$) iff $\mathcal{S} \leq \mathcal{R}$ and $\mathcal{R} \leq \mathcal{S}$; $\leq$ is so defined that it is reflexive and transitive, and hence $\equiv$ is an equivalence.

We further assume acquaintance with the relation: $\mathcal{S}$ *is relatively interpretable in* $\mathcal{R}$.

With this in mind we pass to consider the following problem: is **CT**$\lceil$ conservative with respect to **TON**?

Theorem 5 **CT**$\lceil$ + $\mathcal{L}_T$-IND_P *is proof-theoretically reducible to* **TON** *(and actually conservative over* **TON***).*

Proof This can be shown by interpreting **CT**$\lceil$ in the least fixed point model of the abstract Kripke-Feferman theory, which is mirrored axiomatically by the system **KF**$\lceil$**+GID** and whose upper bound is **TON** (see [3, 6]). To this aim, we assume that the reader has skipped just a moment to Sect. 5 where the relevant interpreting theory **KF**$\lceil$**+GID** is described.

First of all, the applicative language of **CT**$\lceil$ may be regarded as a sublanguage of the language of **KF**$\lceil$**+GID**. Indeed, let * be the map which preserves application, it is the identity on the **TON**-constants (combinators, number-theoretic operations) and is defined as follows on the special constants[2]:

$$
\begin{aligned}
\mathsf{eq}^* &= \lambda x \lambda y.\langle 1, \langle x, y\rangle\rangle, \\
\mathsf{nat}^* &= \lambda x.\langle 2, x\rangle, \\
\dot{\wedge}^* &= \lambda x \lambda y.\langle 3, \langle x, y\rangle\rangle, \\
\dot{\neg}^* &= \lambda x.\langle 4, x\rangle, \\
\dot{\forall}^* &= \lambda x.\langle 5, x\rangle.
\end{aligned}
$$

Choose P^* by the fixed point theorem for predicates in **KF**$\lceil$, so that $\forall x(P^*(x) \leftrightarrow \mathcal{C}(x, P^*))$ where $\mathcal{C}(x, -)$ is the positive elementary operator of (18). We argue in in **KF**$\lceil$**+GID** *by generalized induction on* P^* and show that every proposition in the sense of P^* has a determinate truth value, i.e.

---

[2] We use the standard abbreviations $\langle t, s\rangle := \mathsf{PAIR}ts$; $(t)_0 := \mathsf{LEFT}t$, $(t)_1 := \mathsf{RIGHT}t$. Below 1, 2, … stand for the corresponding numerals.

$$\forall x(P^*(x) \to T(x) \vee T(\dot{\neg}x)). \tag{19}$$

We then extend the star map to a translation $A \mapsto A^*$ of the language of **CT**$\lceil$ into the language of **KF**$\lceil$+**GID**, such that

$$(P(x))^* \quad := \quad P^*(x), \tag{20}$$

$$(T(x))^* \quad := \quad T(x) \wedge P^*(x). \tag{21}$$

The essential step is (19), which requires a special instance of **GID** which is positive in T. Also observe that, again by (19), the axiom of N-induction for *propositional functions* is sent onto an instance of N-induction for determinate conditions. Hence the *-translations of all theorems of **CT**$\lceil$ are transformed into theorems of **KF**$\lceil$+**GID**. Moreover, the *-translation of $\mathcal{L}_T$-IND_P becomes provable by **GID** and every theorem of **CT**$\lceil$ in the applicative part of the language is also provable in **KF**$\lceil$+**GID**. But **KF**$\lceil$+**GID** is proof-theoretically reducible to and conservative over **TON** by [3, 6] whence the claim. ☐

Remark 2 (i) We *cannot prima facie directly embed* **CT**$\lceil$ into **KF**$\lceil$ by identifying $P(x)$ with $T(x) \vee T(\dot{\neg}x)$ because of the strictness conditions; this is the reason we have to pass through the inductive definition of P in **KF**$\lceil$+**GID**.

(ii) On the surface, one is tempted to consider **CT** as an abstraction from and an analogue to Halbach's system **CT**[**PA**] of compositional truth over **PA** (see [17]). And it is known that **CT**[**PA**] is conservative over **PA** [24]. However, the analogy is only superficial, as the notion of sentence in the case of **CT**[**PA**] is arithmetically fixed and does not depend on an inductive definition.

Theorem 6 **CT** + $\mathcal{L}_T$-IND_P *and arithmetical analysis* **ACA** *are mutually interpretable.*

Proof

(i) Lower bound: it follows from Theorem 7 (see below).

(ii) Upper bound. First of all, we fix an arithmetization of the open term model TER for **TON**, induced by the underlying expanded combinatory logic; this can be done within theories of strength at most **PA** (for details see e.g. [21, 28]). Then we devise *an arithmetical interpretation* $\mathcal{P}^{TM}$ for the predicate P by means of a *recursively enumerable* derivability relation $\vdash$. The axioms of $\vdash$ have the form

$$\frac{}{\vdash [t = s]} \qquad \frac{}{\vdash [N(t)]}$$

for the basic atomic formulas with $=$ and N, where t, s range over elements of TM. The inference rules corresponding to negation, conjunction and universal quantification have the form:

$$\frac{\vdash t}{\vdash \dot{\neg}t} \qquad \frac{\vdash t \quad \vdash s}{\vdash (t \mathbin{\dot{\wedge}} s)}$$

The clause for $\forall$ is rephrased as a *finitary* inference:

$$\frac{\vdash\ tx}{\vdash\ \dot{\forall} t}$$

provided x is not free in t.

It is then easy to check that the derivability relation is *closed under substitution*, that is, for arbitrary terms t, s:

$$\vdash t\ \Rightarrow \vdash t[x := s]. \tag{22}$$

Each statement $\vdash t$ can be naturally attached a number k such that $\vdash^k t$ means that the deduction tree for t has depth k. Then define

$$t \in \mathcal{P}^{TM} :\Leftrightarrow \exists k(\vdash^k\ t \wedge t \in TM). \tag{23}$$

We can verify:

(i) $\mathcal{P}^{TM}$ satisfies the P-axioms in **ACA**.
(ii) $\mathcal{P}^{TM}$ satisfies (the interpretation of) $\mathcal{L}_T$-IND_P in the open term model.

As to (ii), it amounts to check by number theoretic induction on k and (22) that $\forall k \forall t(\vdash^k\ t \to B^{TM}(t))$ holds under the assumption that the subset defined by $B(x)$ in TM is closed under the clauses generating propositions. Finally, we can define an interpretation for T by simulating the canonical definition of the truth predicate for arithmetic in the subsystem **ACA** of second order arithmetic, based upon arithmetical comprehension. $\square$

2.5 *Elementary Types*

The language $\mathcal{L}_\nu$ for the elementary theory **EET** of types and names consists of:

(i) predicate constants $=$ (identity predicate for individuals), $=_t$ (identity predicate for types), $\in$ (membership), $\mathcal{R}$ (naming relation), N (natural numbers);
(ii) besides the usual individual constants for the extended combinatory algebras, one has naming operations for types: nat, id, co, int, inv, dom;
(iii) countably many individual variables $(x, y, z \ldots)$ and type variables $X, Y, Z \ldots$.

EET obviously includes $\mathbf{TON}^-$, independence conditions for constructors[3] and the number theoretic induction schema. $\mathbf{EET}\lceil$ is the corresponding subsystem with *the axiom of N-induction for types*:

---

[3] Among them nat, id, co, int, inv, dom; we leave the obvious statement to the reader, in analogy to (3), (4).

$$0 \in X \wedge (\forall x \in N)(x \in X \to (x+1) \in X) \to \forall x(N(x) \to x \in X). \tag{24}$$

EET: representation axioms and extensionality

R1 $\forall X \exists y \mathcal{R}(y, X)$;
R2 $\mathcal{R}(a, X) \wedge \mathcal{R}(a, Y) \to X =_t Y$;
Ext $\forall x(x \in X \leftrightarrow x \in Y) \to X =_t Y$.

In sum: there exists a partial surjection $\mathcal{R}$ from the universe V onto $TYPE$:

$$\mathcal{R} : W \twoheadrightarrow TYPE,$$

for some $W \subseteq V$.

EET: existence of elementary types

nat $\exists X(\mathcal{R}(\mathsf{nat}, X) \wedge \forall x(x \in X \leftrightarrow N(x)))$;
id $\exists X(\mathcal{R}(\mathsf{id}, X) \wedge \forall x(x \in X \leftrightarrow \exists y(x = \mathsf{PAIR} yy)))$;
co $\mathcal{R}(y, Y) \to \exists X(\mathcal{R}(\mathsf{co} y, X) \wedge \forall x(x \in X \leftrightarrow \neg x \in Y))$;
int $\mathcal{R}(y, Y) \wedge \mathcal{R}(z, Z) \to \exists X(\mathcal{R}(\mathsf{int} yz, X) \wedge \forall x(x \in X \leftrightarrow x \in Y \wedge x \in Z))$;
inv $\mathcal{R}(y, Y) \to \exists X(\mathcal{R}(\mathsf{inv} fy, X) \wedge \forall x(x \in X \leftrightarrow fx \in Y))$;
dom $\mathcal{R}(y, Y) \to \exists X(\mathcal{R}(\mathsf{dom} y, X) \wedge \forall x(x \in X \leftrightarrow \exists v(\mathsf{PAIR} xv \in Y)))$.

CT and EET

Theorem 7

(i) **EET**$\lceil$ *is interpretable into* **CT**$\lceil$.
(ii) **EET** *is interpretable into* **CT**.

Proof We define a suitable embedding. □

Definition 8 The translation $A \mapsto A^*$.

(i) * preserves application, it is the identity transform on **TON**-constants (combinators, number-theoretic operations) and variables.[4] Moreover the *-transform of the special constants is defined as follows:

1. $\mathsf{id}^* = \lambda x[\exists y(x = \langle y, y \rangle)]$,
2. $\mathsf{nat}^* = \lambda x[N(x)]$,
3. $\mathsf{dom}^* = \lambda y \lambda u \dot{\exists} v(y(\langle u, v \rangle))$,
4. $\mathsf{int}^* = \lambda x \lambda y \lambda u.((xu) \dot{\wedge} (yu))$,
5. $\mathsf{co}^* = \lambda x \lambda u. \dot{\neg}(xu)$,
6. $\mathsf{inv}^* = \lambda y \lambda f \lambda x. y(fx)$.

---

[4] This makes sense, since we can identify individual variables of **EET** with **CT**-variables with odd indices, and type variables of **EET** with **CT**-variables with even indices.

(ii) Atomic formulas:

$$
\begin{aligned}
\mathcal{R}(a, X)^* &:= (a =_e X) \wedge PF(a) \wedge PF(X), && (25)\\
(X =_t Y)^* &:= X =_e Y, && (26)\\
(a \in X)^* &:= T(X(a)), && (27)\\
(t = s)^* &:= (t^* = s^*), && (28)\\
N(t)^* &:= N(t^*). && (29)
\end{aligned}
$$

(For $=_e$, $PF(x)$ in (25) above, see Sect. 2.2.)

(iii) $A \mapsto A^*$ is extended to the non-atomic formulas by requiring that it preserves sentential connectives, individual quantification. In addition:

$$
\begin{aligned}
(\forall XA)^* &:= \forall x(PF(x) \to A^*[X := x]), && (30)\\
(\exists XA)^* &:= \exists x(PF(x) \wedge A^*[X := x]). && (31)
\end{aligned}
$$

It is enough to check that the translation of the axioms of **EET**$\lceil$ is provable in **CT**$\lceil$.

(i) R1–R2, Ext: the translations are tautologies

(ii) nat-id-co-int-inv-dom: apply the corresponding axioms for atomic propositions and classical compositional truth. For instance, in the case of inverse image, if y is a propositional function, then $\mathsf{inv}^*fy := \lambda x.y(fx)$ is a propositional function of x given that y is, and it satifies the *-transform of

$$\exists X(\mathcal{R}(\mathsf{inv}fy, X) \wedge \forall x(x \in X \leftrightarrow fx \in Y)).$$

3 Truth and Types II

3.1 Strengthening CT*: The Theory* AT

Does the notion of proposition define a propositional function, i.e. is $\lambda x.[P(x)]$ a propositional function? We consider theories which yield a positive answer.

Definition 9

$$P([P(x)]) \wedge (T([P(x)]) \leftrightarrow P(x)). \tag{32}$$

1. **AT** := **CT** + (32),
2. **AT**$\lceil$:= **CT**$\lceil$ + (32).

Lemma 10

(i) *Let S be such that $S = [P(S)]$. Then* $\mathbf{AT}^{-} \vdash P(S) \wedge T(S)$.
(ii) *Let L be such that $L = [\neg P(L)]$. Then* $\mathbf{AT}^{-} \vdash P(L) \wedge T(\dot{\neg} L)$.
(iii) $\mathbf{AT}^{-}$ *proves:*

$$\forall x (T([P(x)]) \vee T([\neg P(x)])), \tag{33}$$

$$\forall x (T(x) \to T([P(x)])), \tag{34}$$

$$\forall x T([P([P(x)])]). \tag{35}$$

Proof

(i): by the axiom (32), $P([P(S)])$; hence $P(S)$, so $T([P(S)]))$, hence $T(S)$.
(ii): by (32) and (12) we obtain $P(L)$. Were $T(L)$, then $T([\neg P(L)])$ and hence $\neg T([P(L)])$, i.e. $\neg P(L)$: contradiction. Hence $\neg T(L)$, i.e. $T(\dot{\neg} L)$.
(iii): by axioms (6), (9), (32). □

Informally, (i)–(ii) above can be rendered as: there is a true proposition saying "I am a proposition", there is a false proposition saying "I am not a proposition".

Lemma 11 *Proposition 2 holds for* $\mathbf{AT}^{-}$*: the Tarski schema (see* (16)*,* (17)*) holds for every formula A of $\mathcal{L}_P$. In particular, every formula A of $\mathcal{L}_P$ with free variables in the list $\vec{x} = x_1, \ldots, x_n$ defines a propositional function:*

$$\forall x_1 \ldots \forall x_n (P([A(\vec{x})])). \tag{36}$$

Remark 3 Under the same label **AT** a related system was introduced in [5]; the original system included $\mathbf{CT}^{-}$ with (37)–(39):

$$P([T(x)]) \leftrightarrow P(x), \tag{37}$$

$$T([T(x)]) \leftrightarrow T(x), \tag{38}$$

$$T([\neg P(x)]). \tag{39}$$

It is consistent to expand the present **AT** with (37), (38): simply put $[T(x)] := x$. Also, the old system proved $P([P(x)])$ but *it is incompatible* with the present one.

3.2 Generating an AT*-Model: Propositions*

As usual, if $\mathcal{M}$ is a structure for a given language $\mathcal{L}$ (among those described), $\mathcal{L}^M$ is $\mathcal{L}$ expanded with distinct constants for distinct elements of the domain $|M|$. For the sake of simplicity, we keep using the same notation a for an element of $|M|$ and the corresponding constant. If t is a closed term of the expanded language, t^M is the unique element of $|M|$ denoted by t in $\mathcal{M}$.

Definition 12 We recursively define the collection of propositional objects uniformly in any given expanded combinatory algebra.

- Initial clause: if $A := (a = b),\ Na,\ Pa$,

$$\mathcal{P}_0^M = \{[A]^M \mid a,\ b \in M\}.$$

- Successor clause:

$$\begin{aligned}\mathcal{P}_{\alpha+1}^M \quad = \quad & \mathcal{P}_\alpha^M \cup \{(\dot{\neg} a)^M | a \in \mathcal{P}_\alpha^M\} \cup \\ & \cup \{(b \mathbin{\dot{\wedge}} c)^M | b \in \mathcal{P}_\alpha^M,\ c \in \mathcal{P}_\alpha^M\} \cup \\ & \cup \{(\dot{\forall} f)^M | \text{for all } c \text{ in } |M|, fc^M \in \mathcal{P}_\alpha^M\}.\end{aligned}$$

- If λ is a limit,

$$\mathcal{P}_\lambda^M = \bigcup \{\mathcal{P}_\beta^M \mid \beta < \lambda\}.$$

- Let

$$\mathcal{P}^M = \bigcup \{\mathcal{P}_\beta^M \mid \beta < card(M)^+\}.$$

Lemma 13

(i) For every α, β,

$$\alpha \leq \beta \Rightarrow \mathcal{P}_\alpha^M \subseteq \mathcal{P}_\beta^M. \tag{40}$$

(ii) If $a,\ b \in |M|$ and $A := (a = b),\ Na,\ Pa$, then $([A])^M \in \mathcal{P}^M$.
(iii) Moreover:

$$a \in \mathcal{P}^M \Leftrightarrow (\dot{\neg} a)^M \in \mathcal{P}^M \tag{41}$$

$$(a \mathbin{\dot{\wedge}} b)^M \in \mathcal{P}^M \Leftrightarrow a \in \mathcal{P}^M \wedge b \in \mathcal{P}^M \tag{42}$$

$$\forall c \in |M| (ac)^M \in \mathcal{P}^M \Leftrightarrow (\dot{\forall} a)^M \in \mathcal{P}^M. \tag{43}$$

Proof Trivial transfinite induction on ordinals using the closure properties embodied in the definitions of $\mathcal{P}^M$. □

Incidentally the construction ensures that there are models of an auxiliary theory of operations and propositions **TON**$_P$ formalized in the T-free language.

Definition 14 **TON**$_P$ includes, besides the applicative axioms of **TON**:

1. the axioms concerning the predicate P, i.e. the T-free part of (7)–(14) and (32)[5];
2. N-induction and P-induction schemata restricted to $\mathcal{L}_P$-formulas.

---

[5] Concerning (32), we are thus left only with $\forall x P([P(x)])$.

Lemma 15 **TON**$_P$ *is interpretable into* **TON**.

The lemma holds since the open term model for **TON**$_P$ can be formalized in **TON** (for the proof, see Theorem 6).

Definition 16 If M is a model of **TON**$^-$, a structure $\langle M, \mathcal{P}^M\rangle$ for the language $\mathcal{L}_P$ which is a model of **TON**$_P$ is N-standard iff the denotation N^M of N is isomorphic to the structure of natural numbers; if, in addition, $\mathcal{P}^M$ is the least fixed point of the positive elementary operator $\mathcal{C}_p(x, -)$[6] inductively generating the notion of proposition over M according to Sect. 3.2, $\langle M, \mathcal{P}^M\rangle$ is called N, P-standard.

Trivially:

Lemma 17 *If M is N-standard and $\mathcal{P}^M$ is defined as above (see Definition 12), then $\langle M, \mathcal{P}^M\rangle$ is N, P-standard and $\langle M, \mathcal{P}^M\rangle \models$* **TON**$_P$.

3.3 Generating an **AT**-*Model: Truth*

A N, P-standard model of **TON**$_P$ can be expanded to a model of **AT**. Indeed, we produce a sequence $\{\mathcal{T}^M_\alpha\}$ approximating the truth set, uniformly in $\mathcal{P}^M$:

- Initial clause:

$$\begin{aligned}
\mathcal{T}^M_0 \quad = \quad & \{[a = b]^M | M \models (a = b)\} \cup \\
& \cup \{[\neg a = b]^M | M \models \neg(a = b)\} \cup \\
& \cup \{[N(a)]^M | M \models N(a)\} \cup \\
& \cup \{[\neg N(a)]^M | M \models \neg N(a)\} \cup \\
& \cup \{[P(a)]^M | a \in \mathcal{P}^M\} \cup \\
& \cup \{[\neg P(a)]^M | a \notin \mathcal{P}^M\}.
\end{aligned}$$

- Successor clause:

$$\begin{aligned}
\mathcal{T}^M_{\alpha+1} \quad = \quad & \mathcal{T}^M_\alpha \cup \{(\dot\neg\dot\neg b)^M | b \in \mathcal{T}^M_\alpha\} \cup \\
& \cup \{(b \,\dot\wedge\, c)^M | b \in \mathcal{T}^M_\alpha \wedge c \in \mathcal{T}^M_\alpha\} \cup \\
& \cup \{(\dot\neg(b \,\dot\wedge\, c))^M | ((\dot\neg b)^M \in \mathcal{T}^M_\alpha \wedge (\dot\neg c)^M \in \mathcal{P}^M) \vee \\
& \vee ((\dot\neg c)^M \in \mathcal{T}^M_\alpha \wedge (\dot\neg b)^M \in \mathcal{P}^M)\} \cup \\
& \cup \{(\dot\forall f)^M | \text{ for all } c \text{ in } |M| \; (fc)^M \in \mathcal{T}^M_\alpha\} \cup \\
& \cup \{(\dot\neg\dot\forall f)^M | \text{ for some } c \text{ in } |M| \; (\dot\neg fc)^M \in \mathcal{T}^M_\alpha \wedge \\
& \wedge \; \text{ for all } c \text{ in } |M| (\dot\neg fc)^M \in \mathcal{P}^M)\}.
\end{aligned}$$

---

[6] NB: this operator is distinct from $\mathcal{C}(x, -)$ of Remark 1 and schema (18), since it embodies the initial condition ensuring $P[(P(x))]$.

- If λ is a limit,

$$\mathcal{T}^M_\lambda = \bigcup\{\mathcal{T}^M_\beta \mid \beta < \lambda\}.$$

Lastly, define

$$\mathcal{T}^M = \bigcup\{\mathcal{T}^M_\beta \mid \beta < card(M)^+\}.$$

Remark 4 The internal truth value of $P(a)$ at stage 0 is determined *in an impredicative way*, i.e. by assuming the set of propositional objects as completed.

Lemma 18 *If $a \in |M|$, then for every α, β*

$$\begin{aligned} \alpha \le \beta \quad &\Rightarrow \quad \mathcal{T}^M_\alpha \subseteq \mathcal{T}^M_\beta, && (44)\\ a \in \mathcal{T}^M_\alpha \quad &\Rightarrow \quad a \in \mathcal{P}^M, && (45)\\ a \in \mathcal{P}^M_\alpha \quad &\Rightarrow \quad a \in \mathcal{T}^M_\alpha \vee (\dot\neg a)^M \in \mathcal{T}^M_\alpha, && (46)\\ a \in \mathcal{P}^M_\alpha \quad &\Rightarrow \quad a \notin \mathcal{T}^M_\alpha \vee (\dot\neg a)^M \notin \mathcal{T}^M_\alpha. && (47) \end{aligned}$$

Hence completeness and consistency hold:

$$a \in \mathcal{P}^M \Rightarrow (a \in \mathcal{T}^M \ \vee\ (\dot\neg a)^M \in \mathcal{T}^M) \wedge (a \notin \mathcal{T}^M \ \vee\ (\dot\neg a)^M \notin \mathcal{T}^M). \tag{48}$$

Proof Transfinite induction on ordinals using the closure properties embodied in the definitions of $\mathcal{T}^M$, $\mathcal{P}^M$. Let us only deal with (46).

Ad (46): assume $a \in \mathcal{P}^M_0$: if $a = [P(b)]^M$, then either $b \in \mathcal{P}^M$ and hence $[P(b)]^M \in \mathcal{T}^M_0$, or else $b \notin \mathcal{P}^M$ and hence by definition $[\neg P(b)]^M \in \mathcal{T}^M_0$. The other atomic cases are immediate.

Let $a = (\dot\neg b)^M \in \mathcal{P}^M_{\alpha+1}$. Then $b \in \mathcal{P}^M_\alpha$ and by IH, $b \in \mathcal{T}^M_\alpha \vee (\dot\neg b)^M \in \mathcal{T}^M_\alpha$. Hence by (44) and closure conditions on truth, $(\dot\neg b)^M \in \mathcal{T}^M_{\alpha+1} \vee (\dot\neg\dot\neg b)^M \in \mathcal{T}^M_{\alpha+1}$.

Let $a = (b \mathbin{\dot\wedge} c)^M \in \mathcal{P}^M_{\alpha+1}$, i.e. $b \in \mathcal{P}^M_\alpha$ and $c \in \mathcal{P}^M_\alpha$. Then by IH,

$$\begin{aligned} b \in \mathcal{T}^M_\alpha \vee (\dot\neg b)^M \in \mathcal{T}^M_\alpha, && (49)\\ c \in \mathcal{T}^M_\alpha \vee (\dot\neg c)^M \in \mathcal{T}^M_\alpha. && (50) \end{aligned}$$

If $b \in \mathcal{T}^M_\alpha$ and $c \in \mathcal{T}^M_\alpha$, then by the closure conditions on truth, $(b \mathbin{\dot\wedge} c)^M \in \mathcal{T}^M_{\alpha+1}$. If $c \in \mathcal{T}^M_\alpha$ but $(\dot\neg b)^M \in \mathcal{T}^M_\alpha$, then by (45) we have $c \in \mathcal{P}^M$ and $(\dot\neg b)^M \in \mathcal{T}^M_\alpha$, whence by the closure conditions on truth, $(\dot\neg(b \mathbin{\dot\wedge} c))^M \in \mathcal{T}^M_{\alpha+1}$. The symmetric case is similar.

Let $a = (\dot\forall f)^M \in \mathcal{P}^M_{\alpha+1}$. Then for all $c \in |M|$, $fc \in \mathcal{P}^M_\alpha$, whence by IH, for all $c \in |M|$:

$$fc \in \mathcal{T}^M_\alpha \vee (\dot\neg fc)^M \in \mathcal{T}^M_\alpha \tag{51}$$

which implies either $(\dot\forall f)^M \in \mathcal{T}^M_{\alpha+1}$ or, for some $c \in |M|$, $(\dot\neg fc)^M \in \mathcal{T}^M_\alpha$, i.e. since $(\dot\forall f)^M \in \mathcal{P}^M$, $(\dot\forall f)^M \in \mathcal{T}^M_{\alpha+1} \vee (\dot\neg\dot\forall f)^M \in \mathcal{T}^M_{\alpha+1}$.

If $a \in \mathcal{P}^M_\lambda$ with λ limit, apply IH and (40). □

Hence:

Proposition 19 *If* $\langle M, \mathcal{P}^M \rangle \models$ **TON**$_P$ *is a* N, P*-standard model of* **TON**$_P$*, then*

$$\langle M, \mathcal{P}^M, \mathcal{T}^M \rangle \models \mathbf{AT}. \tag{52}$$

3.4 On the Strength of AT

Theorem 20 **AT** *is proof theoretically equivalent to* **ACA**.

Proof

(i) Lower bound: obvious since **AT** extends **CT**.

(ii) Upper bound: by a straightforward extension of the proof of Theorem 6. Indeed, we first slightly modify the open term model TER for **TON** by adding to the derivability relation $\vdash$ a clause

$$\frac{}{\vdash [P(t)]}$$

The meaning of Pt becomes as before $\vdash^k\ t$, for some natural number k. Then we inductively define a sequence $\{\mathcal{T}^M_k\}$ satisfying the same closure conditions as those given in Sect. 3.3. The truth predicate is then interpreted as the union over the corresponding countable sequence and the finitary interpretation of P allows us to replace *arbitrary ordinals* by finite ones, to the extent that the whole construction can be carried out in **ACA** by a suitable non-arithmetical instance of number-theoretic induction. □

Conjecture 1 **AT**$\lceil$ *is proof-theoretically reducible to* **PA**.

3.5 Adding a Weak Power Type Operation

3.5.1 Axioms for Explicit Types: Weak Power

EET$^\pi$ (**EET**$^\pi\lceil$) is the extension of **EET** (**EET**$\lceil$) with the axiom (U)

$$\exists X(\mathcal{R}(\mathsf{cl}, X) \wedge \forall x(x \in X \leftrightarrow \exists Y(\mathcal{R}(x, Y))).$$

cl is the object representing the type of all names; intensionally (U) states the existence of *the type of all types*.

cl exists if the axiom *everything is a name* is assumed (and hence $\mathcal{R}$ is a surjection defined on V).

Proposition 21 (Uniform weak power type axiom [9]) *There exists a term π such that* $\mathbf{EET}^{\pi}\lceil$ *proves:*

$$\begin{aligned}&\mathcal{R}(x, X) \to \exists Y(\mathcal{R}(\pi x, Y) \wedge \\ &\wedge \forall Z(Z \subseteq X \to \exists v(v \in Y \wedge \mathcal{R}(v, Z)) \\ &\wedge \forall v(v \in Y \to \exists Z(Z \subseteq X \wedge \mathcal{R}(v, Z))))) \end{aligned} \tag{53}$$

For more information about power types in explicit mathematics, see also [4, 19].

3.5.2 Embedding $\mathbf{EET}^{\pi}$ into AT

We extend the translation $A \mapsto A^*$ of Definition 8 with an additional clause:

$$\mathsf{cl}^* = \lambda u[PF(u)].$$

Theorem 22 $\mathbf{EET}^{\pi} \vdash A \ \Rightarrow \mathbf{AT} \vdash A^*$ *and the same holds for the pair* $\mathbf{EET}^{\pi}\lceil$ *and* $\mathbf{AT}\lceil$.

Proof By Theorem 7 it is enough to verify the *-transform of the axiom (U)

$$\exists X(\mathcal{R}(\mathsf{cl}, X) \wedge \forall x(x \in X \leftrightarrow \exists Y(\mathcal{R}(x, Y))$$

holds. Therefore we check

$$\begin{aligned}&PF(\lambda x.[PF(x)]),\\ &\forall u(T(\lambda x[PF(x)])u \leftrightarrow \exists Y(PF(Y) \wedge PF(u) \wedge u =_e Y)).\end{aligned}$$

The *-transform of the second condition is trivial (choose $Y = u$). As to the first one, we have by the first part of axiom (32), the definition of propositional function, β-conversion and (11):

$$\begin{aligned}&\Rightarrow \quad \forall v P([P(v)])\\ &\Rightarrow \quad \forall u \forall x P([P(ux)])\\ &\Rightarrow \quad \forall u P([\forall x P(ux)])\\ &\Rightarrow \quad \forall u P([PF(u)])\\ &\Rightarrow \quad \forall u P((\lambda x.[PF(x)])u)\\ &\Rightarrow \quad PF(\lambda x.[PF(x)]).\end{aligned}$$

□

Theorem 23 $\mathbf{EET}^{\pi}\lceil$ *(*$\mathbf{EET}^{\pi}$*) is proof-theoretically reducible to* **PA** *(*$\mathbf{ACA}$*).*

This is known to be true by [16]. It follows by Theorem 22 if Conjecture 1 is true.

4 Truth and Types III

4.1 Strengthening **CT***: The System* **PT**

We introduce a theory of propositions and truth, where the constant $\dot{\to}$ is a primitive symbol.[7]

Definition 24 (i) **PT**:=**CT** + (54) + (55) + (56), where

$$
\begin{aligned}
P(x) &\leftrightarrow P([P(x)]), && (54)\\
P(a \dot{\to} b) &\leftrightarrow P(a) \wedge (T(a) \to P(b)), && (55)\\
P(a \dot{\to} b) &\to (T(a \dot{\to} b) \leftrightarrow (T(a) \to T(b))). && (56)
\end{aligned}
$$

(ii) $\mathbf{PT}\lceil$ is **PT** with number-theoretic induction *restricted to propositional functions.*
(iii) $a \odot b := a \dot{\wedge} (a \dot{\to} b)$.[8]

Remark 5 Essentially the same system (also labelled **PT**) was presented to the Russell conference in München in 2001 (see [5]) in the context of the discussion of a Russellian paradox about truth and propositions. If we compare it with systems available in the literature, **PT** and its variants are closely related to Aczel's theory of Frege structures (see [1]). Aczel's axioms do not include (54) and (56) while (55) is only stated from right to left and also the strictness conditions for propositions are not included. **PT** can be regarded as abstract version of *Feferman's theory* **DT** *of determinate truth* (see [10, 13], p. 318) over Peano Arithmetic. Here **PA** is replaced by the applicative theory **TON**, Feferman's determinatess predicate D is interpreted as P and assumed as primitive.[9]

Lemma 25 (provably in $\mathbf{PT}^{-}$)

(i) Assume that a is a proposition and that b is a proposition provided a is true. Then

$$P(a \odot b) \wedge (T(a \odot b) \leftrightarrow T(a) \wedge T(b)). \qquad (57)$$

(ii) Assume that f is a family of propositional functions indexed by the propositional function a. Choose $t(u) := [u = (u_0, u_1)]$, $s(u) := (au_0) \odot (fu_0)u_1$, and define $\mathsf{j}(a, f) := \lambda u.(t(u) \dot{\wedge} s(u))$. Then $\mathsf{j}(a, f)$ is a propositional function, such that

$$T(\mathsf{j}(a, f)u) \leftrightarrow T(au_0) \wedge T((fu_0)u_1). \qquad (58)$$

---

[7] Hence (3), (4) are expanded so as to include $\dot{\to}$.

[8] This is a sequential conjunction introduced by Aczel in [1].

[9] Indeed Feferman [10], noting that Aczel's approach is based on λ-calculus which allows for more general interpretations, adds that "further work on systems like **DT** might usefully incorporate similar features."

Proof (i): apply (10), (55), (56). As to (ii), simply apply (i). ☐

Remark 6 **PT**$^-$ can be conservatively extended by

$$P([T(x)]) \leftrightarrow P(x), \tag{59}$$

$$T([T(x)]) \leftrightarrow T(x), \tag{60}$$

$$T([\neg T(x)]) \leftrightarrow T(\dot{\neg} x). \tag{61}$$

Simply define $[Tt] := t$.

The negation axioms for P and T are redundant once we assume an implication operator as primitive. Define $\dot{\neg} a := (a \dot{\to} \bot)$, where $\bot := [K = S]$. Then since **PT**$^-$ without axioms on $\dot{\neg}$ derives $P(\bot)$ and $\neg T(\bot)$, **PT**$^-$ without axioms for $\dot{\neg}$ derives the negation axioms for propositions and truth.

4.2 Generating PT*-Models*

Can we produce **PT**-models by generalized inductive definition? The difficulty is that the clause for introducing implication makes use (negatively) of the collection of truths. But we can adapt to our case a trick of Aczel [1].

Definition 26 Fix a model M of **TON**$^-$ and let X, Y range over subsets of the domain $|M|$ of M. X is called *suitable*, if $X_1 \subseteq X_0$ and for every u, if $u \in X$, then either $u = \langle 0, (u)_1\rangle$ or $u = \langle 1, (u)_1\rangle$.[10] We put $\mathcal{S} := \{X | X \text{ is suitable}\}$.

We use the following abbreviations:

1. $u \in X_0 := \langle 0, u\rangle \in X$ and $u \in X_1 := \langle 1, u\rangle \in X$.
2. A suitable X is determined by an ordered pairing $\langle X_0, X_1\rangle$ of sets by letting $X_i := \{u | \langle i, u\rangle \in X\}$ where $i = 0, 1$.
3. We also define a partial ordering on suitable sets:

$$X \le Y := X_0 \subseteq Y_0 \wedge \forall u(u \in X_0 \to (u \in X_1 \leftrightarrow u \in Y_1)). \tag{62}$$

4. If $\Gamma : \mathcal{S} \longrightarrow \mathcal{S}$, we define: $\Gamma_i(X) := \{u | \langle i, u\rangle \in \Gamma(X)\}$; Γ is $\le$-monotone iff $X \le Y$ implies $\Gamma(X) \le \Gamma(Y)$.
5. An operator Γ is *suitable* if $\Gamma : \mathcal{S} \longrightarrow \mathcal{S}$ and Γ is $\le$-monotone.

Hence by standard facts (see e.g. [7]) we can state the

---

[10] Recall footnote 2 of Sect. 2.4: think of $\langle a, b\rangle$, $(u)_0$, $(u)_1$ as values of the terms $\mathsf{PAIR}ab$, $\mathsf{LEFT}u$, $\mathsf{RIGHT}u$.

Lemma 27

(i) The structure $\langle \mathcal{S}, \leq \rangle$ is a partial ordering in which every $\leq$-increasing sequence of elements of $\mathcal{S}$ has a $\leq$-least upper bound in $\mathcal{S}$.

(ii) Every suitable operator Γ has fixed points (in particular there exists the $\leq$-least suitable X with $X = F(X)$).

We now define a suitable operator whose fixed points provide **PT**-models. This operator will be given by two separate operators described by means of elementary formulas in the language of **TON**, possibly expanded by parameters for naming subsets of M.

(i) $\mathcal{S}_0(u, X)$ is the formula

$$\begin{aligned}
&\exists x \exists y((u = [x = y] \vee u = [N(x)]) \vee \\
&\vee (u = x \dot{\to} y \wedge X_0(x) \wedge (X_1(x) \to X_0(y))) \vee \\
&\vee (u = x \dot{\wedge} y \wedge X_0(x) \wedge X_0(y)) \vee \\
&\vee (u = [P(x)] \wedge X_0(x)) \vee \\
&\vee (u = \dot{\forall} x \wedge \forall z X_0(xz)))).
\end{aligned}$$

(ii) $\mathcal{S}_1(u, X)$ is the formula

$$\begin{aligned}
&\exists x \exists y((u = [x = y] \wedge x = y) \vee \\
&\vee (u = [N(x)] \wedge N(x)) \vee \\
&\vee (u = [P(x)] \wedge X_0(x)) \vee \\
&\vee (u = x \dot{\to} y \wedge X_0(u) \wedge (X_1(x) \to X_1(y))) \vee \\
&\vee (u = (x \dot{\wedge} y) \wedge X_0(u) \wedge X_1(x) \wedge X_1(y)) \vee \\
&\vee (u = (\dot{\forall} x) \wedge X_0(u) \wedge \forall u X_1(xu))).
\end{aligned}$$

(iii) Finally, $\mathcal{S}(u, X)$—the disjoint sum of the two operators—is an elementary operator in the applicative language of **TON** with a new predicate variable X, which formalizes the clauses inductively generating the interpretation of T and P. Explicitly:

$$\mathcal{S}(t, X) \Leftrightarrow \exists i \exists y[t = (i, y) \wedge ((i = 0 \wedge \mathcal{S}_0(y, X)) \vee (i = 1 \wedge \mathcal{S}_1(y, X))].$$

$\mathcal{S}(u, B)$ is the formula, which result from $\mathcal{S}(u, Y))$ by replacing each subformula of the form $Y(t)$ with $B[x := t]$).

NB. Just to avoid further notational overloading, we identify operators with their defining formulas, and we leave the dependence on a fixed ground structure M implicit.

Lemma 28 *The operator defined by $\mathcal{S}$ is suitable (with respect to any M such that $M \models$ **TON**$^-$).*

Proof Clearly by definition the image of a suitable subset under $\mathcal{S}$ is suitable. Let us check that it preserves the ordering $\leq$. We argue informally. Let X, Y be suitable and $X \leq Y$. Then $X_0 \subseteq Y_0$ and hence, since $\mathcal{S}_0(a, X)$ is positive in X_0, $\mathcal{S}_0(a, Y)$. Thus it is enough to check

$$\mathcal{S}_0(a, X) \to (\mathcal{S}_1(a, X) \leftrightarrow \mathcal{S}_1(a, Y)). \tag{63}$$

There are several cases according to the form of a and all can be dealt with by standard arguments. Let us consider three cases.

1. Let $a = [P(x)]$, for some x. Then if we assume $\mathcal{S}_0(a, X)$, by definition of $\mathcal{S}_0$ we have $X_0(x)$, which trivially implies (63), since $\mathcal{S}_1(a, X) \equiv X_0(x)$, $\mathcal{S}_1(a, Y) \equiv Y_0(x)$ and $X_0 \subseteq Y_0$.
2. $a = x \dot{\to} y$. Assume $\mathcal{S}_0(a, X)$. Then $X_0(x)$ and $X_1(x) \to X_0(y)$. We want:
    - $\mathcal{S}_1(x \dot{\to} y, X) \to \mathcal{S}_1(x \dot{\to} y, Y)$;
    - $\mathcal{S}_1(x \dot{\to} y, Y) \to \mathcal{S}_1(x \dot{\to} y, X)$.

   As to the first implication, from the antecedent it follows $Y_0(x)$ since $X_0 \subseteq Y_0$. On the other hand if $X_1(x)$, X_1 and Y_1 coincide for elements of X_0; hence we conclude $Y_1(x)$, that is, we have shown $\mathcal{S}_1(x \dot{\to} y, Y)$. The second implication is similar.
3. Let $a = \dot{\forall} x$ and assume $\mathcal{S}_0(\dot{\forall} x, X)$, which implies $\forall u X_0(xu)$. By definition of the operator $\mathcal{S}_1$, we have to check $\forall u(X_1(xu)) \leftrightarrow \forall u(Y_1(xu))$. By logic, it is enough to prove $\forall u(X_1(xu) \leftrightarrow Y_1(xu))$. But if we choose any u with $X_0(xu)$, since $X \leq Y$, $X_1(xu) \leftrightarrow Y_1(xu)$. ☐

Theorem 29 *Let M be a model of* **TON**$^-$. *If X is a fixed point of the operator $\mathcal{S}$, then $\langle M, X\rangle \models$* **PT**$^-$. *If M is N-standard, $\langle M, X\rangle \models$* **PT**.

Proof Assume X satisfies

- $X_0(x) \leftrightarrow \mathcal{S}_0(x, X)$;
- $X_1(x) \leftrightarrow \mathcal{S}_1(x, X)$.

We have to show that every **PT**-axiom is satisfied, whenever we interpret $P(a)$, $T(a)$ by $X_0(a)$, $X_1(a)$ (in the given order).

Let us check the interpretation of $T(a) \to P(a)$. So assume $X_1(a)$; since X is a fixed point, $\mathcal{S}_1(a, X)$. There are several cases according to the form of a. If $a = [x = y]$ or $a = [N(x)]$, by definition of $\mathcal{S}_0$, we have $\mathcal{S}_0(a, X)$, and hence $X_0(a)$. Let $a = [P(x)]$; then $X_0(x)$, i.e. again by definition of $\mathcal{S}_0$, $\mathcal{S}_0([P(x)], X)$ whence, since X is a fixed point, $X_0([P(x)])$. The converse is similar. In all other cases, by inspection of $\mathcal{S}_1(a, X)$, $X_0(a)$ follows.

Consider the interpretation of $T(P(a)) \leftrightarrow P(a) \leftrightarrow P([P(x)])$. Indeed, by definition and fixed point property:

$$\begin{aligned} X_1([P(x)]) &\Leftrightarrow \mathcal{S}_1([P(x)], X) \\ &\Leftrightarrow X_0(x) \\ &\Leftrightarrow \mathcal{S}_0([P(x)], X) \\ &\Leftrightarrow X_0([P(x)]). \end{aligned} \tag{64}$$

Consider the interpretation of $P(x \dot{\to} y) \leftrightarrow P(x) \wedge (T(x) \to P(y))$. Indeed by fixed point and definition of $\mathcal{S}_0$, $\mathcal{S}_1$:

$$\begin{aligned} X_0(x \dot{\to} y) &\Leftrightarrow \mathcal{S}_0(x \dot{\to} y, X) \\ &\Leftrightarrow X_0(x) \wedge (X_1(x) \to X_0(y)). \end{aligned} \tag{65}$$

Let us check the soundness of $\forall u T(xu) \to T(\dot{\forall} x)$. Then:

$$\begin{aligned} \forall u X_1(xu) &\Rightarrow \forall u \mathcal{S}_1(xu, X) \\ &\Rightarrow \mathcal{S}_1(\dot{\forall} x, X) \\ &\Rightarrow X_1(\dot{\forall} x). \end{aligned} \tag{66}$$

The remaining cases are also straightforward. □

4.3 Upper Bounds for **PT⌈** *and* **PT**

In order to classify the proof-theoretic strength of **PT**, **PT⌈**, we consider variants, which serve for proof theoretic investigations. We formalize the new systems in the sublanguage $\mathcal{L}_t$ of $\mathcal{L}_T$ without the predicate P, but we adopt the obvious abbreviation

$$P(x) := T(x) \vee T(\dot{\neg} x) \tag{67}$$

so that we can identify the new language with the language of **PT**.

The basic *positive atoms* have the form: $t = s$, $N(t)$, $T(t)$. The *negative atoms* are obtained by negating the positive ones; an *atom* is simply a positive or a negative atom and we stipulate that $\neg\neg A := A$ (A atom). Formulas are inductively generated from atoms by closing under disjunction, conjunction, unbounded quantification. If A is an arbitrary formula, $\neg A$ is the formula which results from the negation normal form of $\neg A$ by erasing each even sequence of occurrences of negation in front of atoms.

If $Q := P, T$, a formula A of $\mathcal{L}_T$ is *Q-positive* (*Q-negative*) if every occurrence of Q in A occurs within positive (negative) atoms of the form $Q(t)$ ($\neg Q(t)$). A formula

A is *Q-separated* if A is Q-positive or Q-negative. A formula A is *Q-free* if Q does not occur in A. A Q-free formula can be regarded as both Q-positive and Q-negative.

The *rank* of a formula over its Q-separated formulas is assigned as follows: a) if A is Q-separated, $rk(A) = 0$; b) else, if A or B is not Q-separated, $rk(A \circ B) = \max(rk(A), rk(B)) + 1$ ($\circ$ is a conjunction or a disjunction); $rk(QsA) = rk(A) + 1$ (where Qs is an unbounded quantifier).

Definition 30

(i) $\mathcal{T}_w(u, Y)$ is an elementary positive operator[11] in the applicative language of **TON**, expanded with a new predicate variable Y; $\mathcal{T}_w(u, Y)$ formalizes the clauses inductively generating the interpretation of T, and Y occurs positively in it. Explicitly:

$$\begin{aligned}
\mathcal{T}_w(t, Y) \quad \Leftrightarrow \quad & \exists x \exists y((t = [x = y] \wedge x = y) \vee \\
& \vee (t = [\neg x = y] \wedge \neg x = y) \vee \\
& \vee (t = [N(x)] \wedge N(x)) \vee \\
& \vee (t = [\neg N(x)] \wedge \neg N(x)) \vee \\
& \vee (t = \dot{\neg}(\dot{\neg} x) \wedge Y(x)) \vee \\
& \vee (t = (x \dot{\wedge} y) \wedge Y(x) \wedge Y(y)) \vee \\
& \vee (t = \dot{\neg}(x \dot{\wedge} y) \wedge ((Y(\dot{\neg} x) \wedge Y(y)) \vee \\
& \vee (Y(\dot{\neg} y) \wedge Y(x)) \vee (Y(\dot{\neg} x) \wedge Y(\dot{\neg} y))) \vee \\
& \vee (t = x \dot{\rightarrow} y \wedge ((Y(x) \wedge Y(y)) \vee Y(\dot{\neg} x))) \vee \\
& \vee (t = \dot{\neg}(x \dot{\rightarrow} y) \wedge ((Y(x) \wedge Y(\dot{\neg} y)))) \vee \\
& \vee (t = (\dot{\forall} x) \wedge \forall u Y(xu)) \vee \\
& \vee (t = \dot{\neg}(\dot{\forall} x) \wedge \exists u Y(\dot{\neg}(xu)) \wedge \forall u (Y(xu) \vee Y(\dot{\neg}(xu))))).
\end{aligned}$$

$\mathcal{T}_w(u, B)$ is the formula, which result from $\mathcal{T}_w(u, Y))$ by replacing each subformula of the form $Y(t)$ with $B[x := t]$).

Definition 31 **FL**$\lceil$[12] is consists of

1. logical axioms of the form

$$\Gamma, \neg A, A$$
$$\Gamma, \neg t = s, A[x := t], A[x := s]$$

   where A is an atom (according to the previous definitions);
2. axioms of the form Γ, Δ where Δ is an e-atom or a finite set of e-atoms; Δ formalizes the standard axioms for extended combinatory, logic, natural numbers;

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[11] In the standard sense, see [25].

[12] **FL** is reminiscent of Feferman's logic.

3. standard logical rules for introducing $\wedge$, $\vee$, $\forall$, $\exists$ and the cut rule

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}$$

4. T-consistency:

$$\Gamma, \neg T(t), \neg T(\dot{\neg} t)$$

5. N-induction rule for *determinate functions* (see Sect. 2.2):

$$\frac{\Gamma, PF(f) \quad \Gamma, T(f0) \quad \Gamma, \forall x(N(x) \rightarrow (T(fx) \rightarrow T(f(x+1))))}{\Gamma, \neg N(t), T(ft)}$$

6. T-closure:

$$\frac{\Gamma, \mathcal{T}_w(t, T)}{\Gamma, T(t)}$$

7. T-soundness:

$$\frac{\Gamma, \neg \mathcal{T}_w(t, T)}{\Gamma, \neg T(t)}$$

FL is the calculus which is obtained from $\mathbf{FL}\lceil$ by replacing N-induction rule for *determinate functions* by full N-induction rule, that is, if $A(x)$ is an arbitary formula of $\mathcal{L}_t$,

$$\frac{\Gamma, A(0) \quad \Gamma, \forall x(N(x) \rightarrow (A(x) \rightarrow A(x+1)))}{\Gamma, \neg N(t), A(t)}$$

Remark 7 The label **FL** hints at a compositional theory of truth introduced by Kentaro Fujimoto in [13] under the label **FKF**. The original theory **FKF** is a variant of Kripke-Feferman **KF** over Peano arithmetic, where one assumes for the predicate T Feferman's logic for determinate truth [10].

Definition 32 If $S := \mathbf{FL}\lceil, \mathbf{FL}, \mathbf{TON}$, we inductively define a derivability relation $S \vdash^m_n \Gamma$ as follows:

(i) If Γ is an axiom, $S \vdash^m_n \Gamma$;
(ii) assume Γ is inferred by means of a finitary rule (but not a cut) from the set $\{\Gamma_i | i \leq j\}$; if $S \vdash^{m_i}_{n_i}$ where $m_i < m$, $n_i \leq n$, $i \leq j$, then also $S \vdash^m_n \Gamma$;
(iii) assume Γ is inferred by means of a cut from the premises Γ, A and $\Gamma, \neg A$. If $S \vdash^w_u \Gamma, A$ and $S \vdash^p_q \Gamma, \neg A$ where $w, p < m$ and $R(A) + 1, u, q \leq n$, $S \vdash^m_n \Gamma$.

Lemma 33 (Partial Cut-elimination) *Let* $\mathbf{FL}\lceil \vdash^m_{k+2} \Gamma$. *Then* $\mathbf{FL}\lceil \vdash^{2^m}_{k+1} \Gamma$. *Hence every* $\mathbf{FL}\lceil$*-derivation* $\mathcal{D}$ *can be effectively transformed into a* $\mathbf{FL}\lceil$*-derivation of the same end-sequent, where cut-formulas are* T*-separated and have rank* 0.

Proof The argument is standard; it essentially depends on the fact that the active formulas in the axioms and in the conclusions of the mathematical inferences are T-separated. ☐

Lemma 34 *The system* **FL**$^-$ *proves*

$$\begin{array}{rcl}
T(x) & \rightarrow & P(x),\\
P([x = y]) & \wedge & P([N(x)]),\\
P(x) & \rightarrow & \neg(T(x) \wedge T(\dot{\neg}x)),\\
T(\dot{\neg}\dot{\neg}x) & \leftrightarrow & T(x),\\
T(x \mathbin{\dot{\wedge}} y) & \leftrightarrow & T(x) \wedge T(y),\\
T(\dot{\neg}(x \mathbin{\dot{\wedge}} y)) & \leftrightarrow & (T(\dot{\neg}x) \wedge T(\dot{\neg}y))\vee,\\
 & & \vee\ (T(\dot{\neg}x) \wedge T(y)) \vee (T(x) \wedge T(\dot{\neg}y)),\\
T(x \mathbin{\dot{\rightarrow}} y) & \leftrightarrow & (T(x) \wedge T(y)) \vee T(\dot{\neg}x),\\
T(\dot{\neg}(x \mathbin{\dot{\rightarrow}} y)) & \leftrightarrow & (T(x) \wedge T(\dot{\neg}y)),\\
T(\dot{\forall}f) & \leftrightarrow & \forall x T(fx),\\
T(\dot{\neg}\dot{\forall}f) & \leftrightarrow & \exists x T(\dot{\neg}(fx)) \wedge \forall x P(fx).
\end{array} \qquad (68)$$

Proof Apply T-consistency, and T-closure, T-soundness rules. ☐

Lemma 35 **PT**$^-$ $\vdash A$ *iff* **FL**$^-$ $\vdash A$.

Proof This amounts to check that the axioms ruling implication internally are interdeducible. But this is an exercise in formal deducibility. ☐

4.3.1 Approximating Truth by Its Finite Levels

Is it possible to approximate truth by its finite levels and hence to eliminate truth?

Definition 36 Let $\bot = \mathsf{K} = \mathsf{S}$; then

$$T^0(t) = \bot, \qquad (69)$$

$$T^{m+1}(t) = \mathcal{T}_w(t, T^m)). \qquad (70)$$

Clearly each formula in the sequence belongs to the language $\mathcal{L}_{op}$.

If A is any formula in negation normal form, let $A[m, n]$ be obtained from A by replacing each atom of the form $T(t)$ $(\neg T(t))$ by $T^n(t)$ $(\neg T^m(t))$. Clearly $T^m(t)$, $\neg T^n(t)$ are T-free formulas of $\mathcal{L}_{op}$.

Lemma 37 *If* $0 < m_2 \leq m_1 \leq k_1 \leq k_2$, Γ *is a set of formulas such that* **TON** $\vdash^n_p$ $\Gamma[m_1, k_1], \Delta$, *then* **TON** $\vdash^n_p$ $\Gamma[m_2, k_2], \Delta$.

Theorem 38 *Let* $\mathcal{D}$ *be a* **FL**$\lceil$*-derivation of* Γ *with height k, where cut-formulas are T-separated. Then, provably in* **TON**, *for every* $m > 0$,[13] *if* $H(m) := m + 2^k$:

$$\Gamma[m, H(m)]. \qquad (71)$$

---

[13] In general, if $\Gamma := \{A_1, \ldots, A_q\}$, $\Gamma[m, n] := \{A_1[m, n], \ldots, A_q[m, n]\}$.

Proof The restriction to propositional functions has the effect that N-induction for $\mathcal{L}_{op}$-formulas is enough. The asymmetric interpretation relies on the fact that cut formulas are always T-separated. Of course, note that the transform $A \mapsto A[m, n]$ is the identity function on formulas of $\mathcal{L}_{op}$.

1. Logical axioms and rules: by persistence.
2. Cut: use partial cut elimination and a standard substitution argument.
3. Consistency: verify by outer induction on m

$$\forall x(\neg T^m(x) \vee \neg T^m(\dot\neg x)). \tag{72}$$

4. N-induction rule for *determinate functions*: assume that, provably in **TON**, for some g, given any $m > 0$, we have $m \leq g(m)$ and

$$T^{g(m)}(f0), \tag{73}$$
$$\forall x(N(x) \rightarrow (T^m(fx) \rightarrow T^{g(m)}(f(x+1))), \tag{74}$$
$$\forall x(T^{g(m)}(fx) \vee T^{g(m)}(\dot\neg(fx))). \tag{75}$$

Then by $\mathcal{L}_{op}$-IND_N it is enough to verify:

$$\forall x(N(x) \rightarrow (T^{g(m)}(fx) \rightarrow T^{g(m)}(f(x+1))).$$

Assume $N(x)$, $T^{g(m)}(fx)$: then by (74),

$$T^{g(g(m))}(f(x+1)). \tag{76}$$

Moreover by (75), consistency and downward persistence, we have

$$T^{g(m)}(f(x+1)) \vee T^{g(m)}(\dot\neg(f(x+1)), \tag{77}$$
$$\neg T^{g(g(m))}(f(x+1)) \vee \neg T^{g(m)}(\dot\neg(f(x+1))). \tag{78}$$

whence by cut between (77) and (78)

$$T^{g(m)}(f(x+1)) \vee \neg T^{g(g(m))}(f(x+1)). \tag{79}$$

The conclusion follows again by cut between (79) and (76).

5. T-closure (T-soundness): straightforward. □

Corollary 39 *If* $\mathbf{PT}\lceil\vdash A$ *and* $A \in \mathcal{L}_{op}$, *then* $\mathbf{TON} \vdash A$.

Proof Apply Theorem 38. □

Let $\Pi^0_1 - CA_{<\varepsilon_0}$ be the theory of iterated jump up to any $\alpha < \varepsilon_0$.[14]

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[14] For a precise definition, see [11].

Theorem 40

(i) $\mathbf{PA} \equiv \mathbf{PT}\lceil \equiv \mathbf{CT}\lceil$.
(ii) $\Pi^0_1 - CA_{<\varepsilon_0} \equiv \mathbf{PT}$.

Proof

(i) Ad $\mathbf{PA} \ \leq \mathbf{PT}\lceil$: obvious.
(ii) Ad $\mathbf{PT}\lceil \leq \mathbf{PA}$: by Theorem 38 since $\mathbf{CT}\lceil$ is a subtheory of $\mathbf{PT}\lceil$(even if $\dot{\neg}$ is omitted, see by Remark 6).
(iii) Ad $\Pi^0_1 - CA_{<\varepsilon_0} \leq \mathbf{PT}$: consider the ω-model consisting—as range of second order variables—of the collection PF_N of propositional functions which define subsets of N. Then the jump hierarchy up to any $\alpha < \varepsilon_0$ can be shown to exist in PF_N by applying Lemma 25.
(iv) Ad $\mathbf{PT} \leq \Pi^0_1 - CA_{<\varepsilon_0}$: by lifting the previous method to the case of systems with full number theoretic induction. In this case we rely upon a standard embedding into systems with ω-rule. ☐

Remark 8 We can strengthen the theorem above by adding generalized induction schemata over truth, e.g. in the form of the following *inference rule of T-induction*: if $B(x)$ is any *T-positive* $\mathcal{L}_T$-formula,

$$\frac{\Gamma, \forall x(\mathcal{T}_w(x, B) \rightarrow B(x))}{\Gamma, \neg T(t), B(t)} \tag{80}$$

Then **GID**-induction can be eliminated in favour of a suitable infinitary rule $\mathbf{T}^\infty$:

$$\frac{\Gamma, \neg T^n(t) \quad \text{for each } n \in \omega}{\Gamma, \neg T(t)} \tag{81}$$

The rule $\mathbf{T}^\infty$ allows to show each instance of the schema of generalized on truth (argue by induction on $n \in \omega$). Moreover the system based on (81)—with N-induction restricted to determinate functions—satisfies partial cut elimination theorem. Of course, derivation trees are now infinitary and the interpretation theorem holds with $m \mapsto H(m)$, where H is an α-recursive function,[15] for some $\alpha < \varepsilon_0$.

Remark 9 Feferman [10] stated a conjecture about the strength of the theory of determinate truth over **PA**, and the conjecture has been settled by Fujimoto [13] by employing the so-called relative truth definability. The construction above can be adapted to the system **DT** over **PA**, in order to produce an alternative proof of the conjecture.

---

[15] See [26, 27].

Remark 10 In view of Theorems 6 and 40, **PT** is strictly stronger than **CT** and hence the proof theoretic strength of **PT** over **CT** resides in the implication axioms (55) and (56). This should be contrasted with the situation in [13], Theorem 50.

4.4 Adding the Join Operator

4.4.1 Axioms for Explicit Types: Join

Define:

- $\mathcal{R}(a) := \exists Y \mathcal{R}(a, Y)$.
- $s \mathrel{\dot{\in}} t := \exists Y(\mathcal{R}(t, Y) \wedge s \in Y)$.
- Join is the principle $(\mathcal{J})$:

$$\begin{aligned} &\mathcal{R}(a) \wedge \forall x(x \mathrel{\dot{\in}} a \to \mathcal{R}(fx)) \to \mathcal{R}(\mathsf{j}\,(a,f)) \wedge \\ &\quad \wedge \forall u, v(\langle u, v\rangle \mathrel{\dot{\in}} \mathsf{j}\,(a,f) \leftrightarrow u \mathrel{\dot{\in}} a \wedge v \mathrel{\dot{\in}} fu)), \end{aligned} \tag{82}$$

- **EETJ**:= **EET** with $(\mathcal{J})$.
- **EETJ**$\lceil$: as **EETJ** except that it now includes only type induction for numbers (24).

Theorem 41 **EETJ**(**EETJ**$\lceil$) *is interpretable in* **PT** *(***PT**$\lceil$*).*

Proof Apply Lemma 25 and the previous interpretability results about **EET**, since **PT**$^-$ contain **CT**$^-$. □

5 Truth and Types IV

5.1 Abstract 'Kripke-Feferman'

We outline an abstract version of the Kripke-Feferman system over **PA**, which can also be regarded as the theory of *classical* Frege structures (see [6, 14, 22]).

Definition 42 **KF**$\lceil$ comprises the base theory **TON**$^-$, and the fixed point axiom $(\mathcal{T})$ for abstract truth:

$$\mathcal{T}(x, T) \leftrightarrow T(x). \tag{83}$$

Here $\mathcal{T}(x, T)$ is a formula encoding the closure properties:

$$\frac{a = b}{T[a = b]} \qquad \frac{\neg(a = b)}{T[\neg(a = b)]} \qquad \frac{N(a)}{T[N(a)]} \qquad \frac{\neg N(a)}{T[\neg N(a)]}$$

for the basic atomic formulas with = and N. Further, the following additional clauses for the compound formulas:

$$\frac{T(a)}{T(\dot{\neg}\dot{\neg}a)} \qquad \frac{T(a) \quad T(b)}{T(a \mathbin{\dot{\wedge}} b)} \qquad \frac{T(\dot{\neg}a)\ [\ or\ T(\dot{\neg}b)]}{T(\dot{\neg}(a \mathbin{\dot{\wedge}} b))}$$

$$\frac{\forall x\, T(ax)}{T(\dot{\forall}a)} \qquad \frac{\exists x\, T(\dot{\neg}ax)}{T(\dot{\neg}\dot{\forall}a)}$$

Finally $\mathbf{KF}\lceil$ includes:

1. Consistency axiom: $\neg(T(x) \wedge T(\dot{\neg}x))$.
2. the axiom $\mathsf{D\text{-}IND}_N$ of induction on natural numbers N for functions with determinate truth values (see Sect. 2.3).

5.1.1 Recursion-Theoretic Structure

For each formula A, if $\mathbf{Y}$ is the Curry fixed point combinator, define

$$\mathbf{I}(A) \ := \ \mathbf{Y}(\lambda v.\{x : A(x, v)\}).$$

Hence by β-conversion $\mathbf{I}(A) = \{x : A(x, \mathbf{I}(A))\}$, and a general *second recursion theorem* for predicates holds in $\mathbf{KF}^-$:

Lemma 43 *If A is T-positive*

$$\forall x\, (T(\mathbf{I}(A)x) \ \leftrightarrow \ A(x, \mathbf{I}(A))). \tag{84}$$

Theorem 44 *Let $\mathcal{M}$ be a model of* **TON** *and let* MIN_M *be the least fixed point model of* $\mathbf{KF}_\mu\lceil$ *expanding $\mathcal{M}$. Then* $\mathbf{I}(A)$ *represents the least fixed-point of the monotone operator defined by A in* MIN_M.[16]

This suggests the schema **GID**, ensuring the minimality of the fixed points: if $A(x, v)$ is a positive operator

$$\mathbf{Clos}_A(B) \rightarrow \forall x\, (T(\mathbf{I}(A)x) \rightarrow B(x))$$

with $\mathbf{Clos}_A(B) \ := \ \forall x\, (A(x, B) \rightarrow B(x))$.

Theorem 45 ([6])

(i) $\mathbf{KF}\lceil + \mathbf{GID}$ *is proof-theoretically equivalent to* **PA**.
(ii) $\mathbf{KF} + \mathbf{GID}$ *is proof-theoretically equivalent to* $\mathbf{ID_1}$.

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[16] This means: the set of all $a \in \mathcal{M}$ satisfying $T(I(A)x$ in MIN_M is the least fixed point of the operator defined by A in MIN_M.

5.2 Partial Truth with Minimality

$\mathbf{KF}_\mu$ (Truth with minimality, see Burgess [2]): it is the fragment of **KF**[17] with

(i) only the *composition principles*, e.g. $\forall x(\mathcal{T}(x, T) \to T(x))$;
(ii) the schema: if B is an arbitrary formula,

$$\forall x(\mathcal{T}(x, B) \to B(x)) \to \forall x(T(x) \to B(x)). \tag{85}$$

Then $\mathbf{KF}_\mu$ proves the decomposition axioms $\forall x(T(x)) \to \mathcal{T}(x, T))$ and the consistency axiom. Also, it explicitly refutes statements that fail in the least fixed point model, e.g. the so-called Truth-Teller S such that, provably in $\mathbf{KF}^-$, $S = \dot{T}S$. Indeed, choose $T_S(x) := T(x) \wedge x \neq S$. Then it's easy to check by independence and T-closure that $\forall x(\mathcal{T}(x, T_S) \to T_S(x)$, and hence $T(x)$ implies $T_S(x)$, for arbitrary x. Therefore $T(S)$ implies $T_S(S)$, whence $\neg T(S)$ by logic.

Clearly $\mathbf{KF}_\mu\lceil$ has an inner model in $\mathbf{KF}\lceil+\mathbf{GID}$.

As to the upper bound, we can adapt the direct proof theoretic analysis of $\mathbf{KF}\lceil+\mathbf{GID}$ (see [3]) to $\mathbf{KF}_\mu\lceil$.

As to the lower bound, let $\mathbf{ID}_\mathbf{1}^\mathbf{acc}$ be the theory of accessibility inductive definitions over **PA**, which is known to be proof-theoretically equivalent to the theory of elementary inductive definitions. Let $\prec$ be a binary relation encoded by a propositional function,[18] and let $\mathit{Field}(\prec) := \{u | \exists v(\langle u, v\rangle \eta \prec \vee \langle v, u\rangle \eta \prec)\}$. Let

$$\Phi := \lambda a \lambda x.[\forall y(y \prec x \to ay)]$$

where $[\forall y(y \prec u \to ay)]$ is the applicative term $\dot{\forall}(\lambda y.(\dot{\to})[y \prec x](ay))$. Then we can choose by (84) a term $W(\prec)$, such that—using the notations of Sect. 2.2—$\mathbf{KF}\lceil$ proves

$$\begin{array}{rcl} W(\prec)x & = & [\forall y(y \prec x \to W(\prec)y)], \\ x\eta W(\prec) & \leftrightarrow & x\eta \mathit{Field}(\prec) \wedge \forall y(y \prec x \to y\eta W(\prec)). \end{array}$$

The schema of transfinite induction along $\prec$ has the form

$$\mathit{TI}(\prec, A) := \mathit{Progr}(\prec, A) \to \forall x(x\eta W(\prec) \to A(x)). \tag{86}$$

where

$$\mathit{Progr}(\prec, A) := (\forall x \eta \mathit{Field}(\prec))(\forall y(y \prec x \to A(y)) \to A(x)).$$

---

[17] As for **KF**, a warning: we keep using the same label of [2] for a theory $\mathbf{KF}_\mu$, which is *not* an extension of Peano Arithmetic.

[18] So $\prec$ is determinate in the sense of (2).

Then we lift to the present context the (classical form of so-called) *bar-induction* schema, which goes back to Kreisel; the proof below takes inspiration from an analogous result of [15], also exploited by [2]:

Theorem 46 *If A is an arbitrary formula, $\prec$ is a (propositional function encoding a) binary relation, then the schema of transfinite induction on the largest well-founded part $W(\prec)$ of $\prec$ holds, provably in* $\mathbf{KF}_\mu$.

Proof For simplicity, it is convenient to work in a definitional extension of $\mathbf{KF}_\mu$, where $T(\dot{\neg}x) \vee T(y) \to T(x \dot{\to} y)$ is provable. Fix any A, $\prec$, such that $\prec$ is a propositional function. Assume that A is $\prec$-progressive and $u\eta W(\prec)$. It is sufficient to verify $A(u)$. For the sake of semplification, we further assume that the field of $\prec$ is the whole universe (so that we can avoid to make explicit reference to it). Define

$$\begin{aligned} T_A(x) :\Leftrightarrow\ & T(x) \wedge \forall u(x = W(\prec)u \to A(u)) \wedge \\ & \wedge \forall u \forall v(x = [v \prec u \to W(\prec)v] \wedge v \prec u \to A(v)). \end{aligned} \tag{87}$$

Assume that we have shown

$$\forall x(\mathcal{T}(x, T_A) \to T_A(x)). \tag{88}$$

Then by the schema (85) we can conclude $\forall x(T(x) \to T_A(x))$. Hence by assumption, for $x := W(\prec)u$, we have $T_A(W(\prec)u)$, which immediately implies $A(u)$.

Hence it remains to check (88). But this follows with the closure axioms of T and with the independence conditions ruling the dotted constants $\dot{N}$, $\dot{=}$, etc. Let us consider three cases.

1. Assume $\mathcal{T}([a = b], T_A)$; then $a = b$ holds and hence $T([a = b])$. Since $W(\prec)u = \dot{\forall}f$ for a suitable f, then $W(\prec)u \neq [a = b]$ and

$$[v \prec u \to W(\prec)v] \neq [a = b]$$

by (3). So we can trivially conclude $T_A([a = b])$.

2. Assume $\mathcal{T}(\dot{\forall}f, T_A)$; then for all x, $T_A(fx)$. Then by definition of T_A, we have $\forall x T(fx)$ whence $T(\dot{\forall}f)$. It remains to check:

$$\forall u(\dot{\forall}f = W(\prec)u \to A(u)), \tag{89}$$

$$\forall u \forall v(\dot{\forall}f = [v \prec u \to W(\prec)v] \wedge v \prec u \to A(v)). \tag{90}$$

The second is trivially true by independence. As to the first condition, assuming $\dot{\forall}f = W(\prec)u$, we must prove $A(u)$. But $\dot{\forall}f = W(\prec)u$ implies

$$f = \lambda v[v \prec u \to W(\prec)v],$$

whence, for arbitrary v:

$$fv = [v \prec u \to W(\prec)v]. \tag{91}$$

Since for all x, $T_A(fx)$, we have, for $x := v$

$$\forall y \forall z (fv = [y \prec z \to W(\prec)y] \wedge y \prec z \to A(y)).$$

Hence, for $y := v$, $z := u$

$$fv = [v \prec u \to W(\prec)v] \wedge v \prec u \to A(v).$$

By (91), for arbitrary v :

$$v \prec u \to A(v).$$

But A is $\prec$-progressive and hence $A(u)$. It follows $T_A(\dot{\forall} f)$.

3. Assume $T_A(\dot{\neg} a) \vee T_A(b)$. We check $T_A(a \dot{\to} b)$. By assumption $T(\dot{\neg} a) \vee T(b)$ and hence $T(a \dot{\to} b)$. So it is enough to check

$$\forall u (a \dot{\to} b = W(\prec)u \to A(u)), \tag{92}$$
$$\forall u \forall v (a \dot{\to} b = [v \prec u \to W(\prec)v] \wedge v \prec u \to A(v)). \tag{93}$$

(92) is trivial by independence (since $\dot{\to} \neq \dot{\forall}$). As to (93), assume $v \prec u$ and $a \dot{\to} b = [v \prec u \to W(\prec)v]$. Then $a = [v \prec u]$ and $b = W(\prec)v$. Were $T_A(\dot{\neg} a)$, then $T([\neg v \prec u])$; but $\prec$ is a propositional function and hence $\neg v \prec u$, contradiction! Hence $T_A(b)$, and, in particular:

$$\forall u \forall v (b = W(\prec)v \wedge v \prec u \to A(v)).$$

By separation $A(v)$, and we have checked (93). □

Hence by the previous lemma:

Corollary 47 $\mathbf{ID}_1^{\mathbf{acc}}$ *is interpretable in* $\mathbf{KF}_\mu$.

Conjecture 2 $\mathbf{KF}\lceil + \mathbf{GID}$ *is interpretable in* $\mathbf{KF}_\mu \lceil$.

Note that, according to [2], the conjecture holds for the ordinary formal system formalized in the language of Peano arithmetic.

5.3 Explicit Types and Name Induction

There is a simple extension of **EETJ** which corresponds to truth minimality in $\mathbf{KF}_\mu$. The idea (see [23]) is that *names are inductively generated from the basic constructors only* (identity id, natural numbers nat, inverse image inv, domain dom, complement co, intersection int, join j). Therefore, if $B(x)$ satisfies the same closure conditions $\mathcal{N}(x, -)$[19] as the name constructors, B contains all names:

$$\forall x(\mathcal{N}(x, B) \to B(x)) \to \forall x(\mathcal{R}(x) \to B(x)). \tag{94}$$

NEM := **EETJ** + (94).

Theorem 48 $\mathbf{ID_1}$, **NEM**, $\mathbf{KF}_\mu$, **KF** + **GID** *are proof-theoretically equivalent.*

As to the proof, the crucial step is Theorem 1 in [23], that is, if $\prec$ is a binary relation on X, the largest well-founded part $W(\prec, X)$ of $\prec$ can be assigned a name, uniformly in any given name for X and $\prec$. Hence $\mathbf{ID}_\mathbf{1}^{\mathbf{acc}}$ is interpretable in **NEM**. On the other hand, it is straighforward to check:

Lemma 49 **NEM** *is interpretable into* **KF** + **GID**.

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[19] We will not spell them explicitly for the sake of brevity.

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Lindenbaum's Lemma via Open Induction

Francesco Ciraulo, Davide Rinaldi and Peter Schuster

Abstract With Raoult's Open Induction in place of Zorn's Lemma, we do a perhaps more perspicuous proof of Lindenbaum's Lemma for not necessarily countable languages of first-order predicate logic. We generally work for and with classical logic, but say what can be achieved for intuitionistic logic, which prompts the natural generalizations for distributive and complete lattices.

1 Introduction

It is not uncommon in mathematics that a concrete theorem admits an elegant but highly abstract proof by some transfinite method, typically Zorn's Lemma (ZL) combined with a proof by contradiction: under the hypothesis that there is any counterexample at all, by ZL there exists a maximal or minimal counterexample, which helps to the desired contradiction. Unfortunately, one thus virtually loses the computational information given by the input data, and the proof fails to produce an algorithm for computing the output data.

Some theorems of this kind [1, 5, 6, 10, 12, 18, 24] have already proved to follow in a direct and elementary way from (a variant of) the principle of Open Induction (OI) distinguished by Raoult [18]. Although fully-fledged OI is equivalent to ZL with classical logic [12], using OI rather than ZL has some advantage at least when the original statement is sufficiently concrete. In this case there is in fact some evidence

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that—in Hilbert's terms—the ideal objects characteristic of any invocation of ZL can be eliminated, and that one can get by with finite means only. Under sufficiently concrete circumstances, for example, OI can be reduced to ordinary mathematical induction or even, by fixing the size of the data under consideration, an entirely first-order proof can be obtained.

As a classical equivalent of ZL, OI cannot be considered a theorem from any constructive perspective; whence we prefer to call OI rather a principle than a lemma. Since, however, OI is a form of induction, it not only looks less harmful than ZL already at first glance, but also allows for a computational interpretation. Moreover, one can extract the computational content from proofs in which OI is used in combination with intuitionistic logic, which in turn can be made possible just by moving away from the notorious proofs by contradiction with ZL.[1]

In logic, semantic arguments based on Gödel's Completeness Theorem (CT) give short and elegant proofs of purely syntactical results. To prove, say, a conservation theorem, with CT at hand it suffices to show that every model of the base theory can be extended to a model of the extended theory. Although this proof technique is non-constructive a priori, similar arguments are valid constructively [4]. Constructive completeness theorems for intuitionistic logic have been proved and applied e.g. in [2, 3, 7, 8, 22].

The aforementioned uses of OI suggest the possibility of getting some form of CT by means of OI. In this paper we re-prove Lindenbaum's Lemma (LL) [26] in contrapositive form [23], the novelty being that we work with OI in place of ZL. We then discuss how to carry this over to intuitionistic logic, which leads us to the natural generalizations to distributive and complete lattices.

In spite of the method of proof by cases we need to employ, which is essentially classical, our approach with OI appears to be somewhat more direct than the usual one with ZL. Thanks to the move from ZL to OI, we indeed expect that, as in the case studies mentioned above, one will eventually be able to do with finite methods only, at least when dealing with some concrete application of CT or LL. Evidence for the latter claim is yet to be given, though there already is strong motivation: while the conservation of extensions of Peano Arithmetic by a Tarskian truth predicate was first dealt with from a semantic point of view [9, 14], a syntactical proof has become possible [15] by adapting just those semantic methods [14].

Conventions Let X be a set. The *complement* of a subset P of X is denoted by $-P$, that is,

$$-P = \{x \in X : x \notin P\}.$$

We say that a subset P of X is *inhabited* (rather than non-empty) if P has an element.[2]

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[1] The authors are most grateful to the anonymous referee for hinting at this issue.

[2] This and other choices of terminology typical for constructive settings are made to prepare for Sect. 3.3.

Adopting a handy notation used by Sambin, we denote by $M \between N$ (rather than $M \cap N \neq \emptyset$) that the intersection of the subsets M, N of X is inhabited, that is,

$$M \between N \iff \exists x \in X \, (x \in M \wedge x \in N).$$

For example, $M \between M$ says that M is inhabited; $\{x, y\} \between N$ means that either $x \in N$ or $y \in N$; and $\neg(M \between N)$ is tantamount to each of $M \subseteq -N$ and $N \subseteq -M$. Denoting the subset relation by $\subseteq$, we only write $M \subset N$ if M is a *proper* subset of N, that is, $-M \between N$. We sometimes view a subset P of X as a unary predicate, and write $P(x)$ in place of $x \in P$ for any $x \in X$.

Last but not least, if X is a poset, then every subset of X is tacitly endowed with the induced order; and a *chain* in X is a totally ordered, inhabited subset. Let us stress that we need to require that every chain be inhabited, to have that the supersets of a fixed set be closed under forming unions of chains.

2 Open Induction

Raoult's principle of Open Induction is essentially the logical contrapositive of *Zorn's Lemma* (ZL). To briefly explain this, let $(X, \leqslant)$ be a poset; and recall that X is *chain-complete* if every chain $C \subseteq X$ has a least upper bound $\bigvee C \in X$. The form of ZL we have in mind reads as follows:

Let X be chain-complete. If X is inhabited, then X has a maximal element.

We cannot avoid requiring X to be inhabited, for in this paper the empty set cannot be admitted as a chain (see above). With classical logic we can rewrite ZL in an equivalent contrapositive form:

Let X be chain-complete. A subset Q of X is empty whenever

1. Q has no maximal elements and
2. Q is closed, that is, $C \subseteq Q \Rightarrow \bigvee C \in Q$ for every chain C.

In fact, the form of ZL we have given first can be relativised equivalently as follows:

If X is chain-complete, and $P \subseteq X$ inhabited and closed, then P has a maximal element.

Conditions 1 and 2 above have the following well-known dual forms (think of Q as $-P$):

Definition 1 Let X be a chain-complete poset. A subset P of X is

1. *progressive* if $(\forall x \in X)(x > a \Rightarrow x \in P) \Rightarrow a \in P$ for every $a \in X$,
2. *open* if $\bigvee C \in P \Rightarrow C \between P$ for every chain $C \subseteq X$.

With classical logic, in fact, P is progressive precisely when $-P$ has no maximal elements, and P is open precisely when $-P$ is closed. Note also that if P is progressive, then P is satisfied by every maximal element of X, and thus by the greatest element of X whenever this exists.

Open predicates form a topology. We pause to recall this standard fact, if only for the reader's convenience. We write $\uparrow x$ for the set of all y with $x \leqslant y$.

Lemma 2 *Let A and C be an open subset and a chain, respectively, of a chain-complete poset. If $\bigvee C \in A$, then $C \cap \uparrow c \subseteq A$ for some $c \in C$.*

Proof Suppose, towards a contradiction, that for every $c \in C$ there exists $c' \in C$ such that $c \leqslant c'$ and $c' \notin A$. This gives $\bigvee(C - A) = \bigvee C$. So $\bigvee(C - A) \in A$ and hence $(C - A) \between A$, a contradiction. □

While the proof involves classical logic, the lemma itself holds trivially if A is even *Scott open*, that is, open and upwards closed.

Proposition 3 *The open subsets of a chain-complete poset are closed under finite intersections and arbitrary unions.*

Proof Closure under union is clear. Now let A and B be open and assume that $\bigvee C \in A \cap B$ for a given chain C. By the previous lemma, there exists $c \in C$ such that $C \cap \uparrow c \subseteq A$. Clearly, $C \cap \uparrow c$ is a chain and $\bigvee(C \cap \uparrow c) = \bigvee C$. So $\bigvee(C \cap \uparrow c) \in B$ and hence $(C \cap \uparrow c) \between B$. Hence $C \between (A \cap B)$. □

After this digression on topology, we recall from Raoult [18] the principle we use in place of ZL:

Open Induction (OI) *Let X be a chain-complete poset. If P is a progressive and open predicate on X, then $P(x)$ for all $x \in X$.*

With classical logic (taking P and Q as complements of each other), OI is equivalent to one of the equivalents ("if Q is a closed subset of a chain-complete poset X, and Q has no maximal elements, then Q is empty") of ZL we have displayed before; whence OI is equivalent to ZL [12].

In the sequel, as in [24], by *induction for P and X* we mean the following principle:

If P is progressive, then $(\forall x \in X) P(x)$.

Classically, induction holds for *every* P precisely when X is *well-founded* in the sense that every inhabited predicate on X has a maximal element. This is known as *Transfinite Induction* (TI), and is implied by OI. In fact, if the poset X is well-founded, then every chain in X has a greatest element; whence X is chain-complete, and every predicate P on X is open. Unlike OI, TI is provable in **ZF**.

In some important cases, moreover, induction is provable by mathematical induction only. For instance, induction holds whenever X is

1. a tree (with the root as top element) or, more generally, a forest;
2. a finite poset.

In either case there is no need to have that X be well-founded or chain-complete, let alone that P be open. To see this, let P be a progressive predicate on a poset X. Recall first that $P(y)$ whenever y is a maximal element of X. If X is a forest, to prove $P(x)$ for any $x \in X$ one can do induction on the distance from x to the root y of the tree of X to which x belongs, for which $P(y)$ as this y is maximal. If X is finite and inhabited, then X has a maximal element y, for which $P(y)$ and which y we thus may remove from X; whence induction on the size of X applies.

The second instance above also was the outcome of a reduction of OI in a concrete situation [24]. The following is yet another special case of induction provable by mathematical induction.

Example 4 Let P be a progressive predicate on $\mathbb{N}$. If $P \between I$ for every infinite $I \subseteq \mathbb{N}$, then $P = \mathbb{N}$.

Proof First, $-P$ must be finite, since otherwise, by hypothesis, we would have $P \between -P$, which is impossible. We now can show that $-P$ is empty. If $-P$ were inhabited, then $-P$ would have a greatest element m, for which $n \in P$ for all $n > m$. But P is progressive and thus $m \in P$, a contradiction. $\square$

This example can also be seen as a case of OI, as follows. Consider the chain-complete poset $X = \mathbb{N} \cup \{\infty\}$, that is, the ordinal $\omega + 1$. Given P as above, the property $Q = P \cup \{\infty\}$ is progressive, because so is P, and also open. To check this, let C be a chain in X with $\bigvee C \in Q$. If $\bigvee C \in \mathbb{N}$, then C is finite and hence $\bigvee C \in C$; therefore $C \between Q$. On the other hand, if $\bigvee C = \infty$, then either $\infty \in C$, and hence $C \between Q$, or C is an infinite subset of $\mathbb{N}$, in which case $C \between P$ by hypothesis.

3 Lindenbaum's Lemma

Let $\mathcal{L}$ be a first-order language, which need not be countable. We write $\mathcal{S}_\mathcal{L}$ for the set of sentences over $\mathcal{L}$. Let $\vdash$ denote the deducibility relation of classical predicate logic.[3]

The *deductive closure* of $\Gamma \subseteq \mathcal{S}_\mathcal{L}$ is

$$\overline{\Gamma} = \{\varphi \in \mathcal{S}_\mathcal{L} \mid \Gamma \vdash \varphi\}. \tag{1}$$

We note in passing that $\Gamma \mapsto \overline{\Gamma}$ defines a closure operator on the subsets of $\mathcal{S}_\mathcal{L}$. This means that (i) $\Gamma \subseteq \overline{\Gamma}$, (ii) $\Gamma \subseteq \Gamma'$ implies $\overline{\Gamma} \subseteq \overline{\Gamma'}$ and (iii) $\overline{\overline{\Gamma}} = \overline{\Gamma}$, for all $\Gamma, \Gamma' \subseteq \mathcal{S}_\mathcal{L}$. Note that the latter says that $\overline{\Gamma} \vdash \varphi$ is tantamount to $\Gamma \vdash \varphi$. This closure operator, however, is not topological, that is, it does not preserve finite unions. In fact, $\overline{\emptyset}$ is not empty and $\overline{\Gamma_1 \cup \Gamma_2}$ is, in general, greater than $\overline{\Gamma_1} \cup \overline{\Gamma_2}$.

3.1 Types of Theories

A *theory* is a set of sentences that equals its deductive closure. Conversely, with the concept of theory at hand one can characterise the one of deductive closure. In fact,

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[3] We could equally have worked for and with propositional logic, with arbitrary formulas in place of sentences.

$\overline{\Gamma}$ is the smallest theory in the language $\mathcal{L}$ which contains the given set $\Gamma \subseteq \mathcal{S}_{\mathcal{L}}$, which is to say that $\overline{\Gamma}$ equals the intersection of all theories in the language $\mathcal{L}$ which contain Γ. In view of the impredicative character of all this, we have preferred to define $\overline{\Gamma}$ by (1), from which one can of course prove these characterisations.

We next recollect a few well-known features of theories, which are scattered across the literature.

Remark 5 A set of sentences Γ is a theory if and only if the following hold for all $\varphi, \psi \in \mathcal{S}_{\mathcal{L}}$:

1. $\top \in \Gamma$;
2. if $\{\varphi, \psi\} \subseteq \Gamma$, then $\varphi \wedge \psi \in \Gamma$;
3. if $\varphi \in \Gamma$ and $\varphi \vdash \psi$, then $\psi \in \Gamma$.

So theories correspond to filters of the Lindenbaum algebra, the quotient of $\mathcal{S}_{\mathcal{L}}$ modulo equivalence.

As usual, a set of sentences Γ is *consistent* if $\Gamma \nvdash \bot$. A theory Γ is consistent if and only if it is *proper*: that is, $\Gamma \subset \mathcal{S}_{\mathcal{L}}$. So consistent theories correspond to proper filters of the Lindenbaum algebra.

We further recall that a theory Γ is

- *complete* if $\{\varphi, \neg\varphi\} \between \Gamma$ for every $\varphi \in \mathcal{S}_{\mathcal{L}}$;
- *prime* if $\varphi \vee \psi \in \Gamma$ implies $\{\varphi, \psi\} \between \Gamma$ for every $\{\varphi, \psi\} \subseteq \mathcal{S}_{\mathcal{L}}$.

Although the next lemma is well-known, we give a proof for the sake of later inspection (Sect. 3.3).

Lemma 6 *The complete consistent theories are exactly the proper prime theories. More precisely,*

1. *if a consistent theory* Γ *is complete, then* Γ *is a prime theory;*
2. *every prime theory* Γ *is complete.*

Proof As for part 1, let Γ be a consistent theory. Assume that Γ is complete. To show that Γ is prime, let $\varphi \vee \psi \in \Gamma$. Since Γ is complete, we have as required either $\varphi \in \Gamma$ or $\psi \in \Gamma$. In fact, if otherwise both $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$, that is, $\neg(\varphi \vee \psi) \in \Gamma$, then $\bot \in \Gamma$, which is impossible in view of Γ being consistent. As for part 2, every prime theory Γ is complete, simply because $\varphi \vee \neg\varphi \in \Gamma$. □

In other words, the ultrafilters of the Lindenbaum algebra are just its proper prime filters, which is no surprise as this is a Boolean algebra. As a digression, we next recall that these notions of theories are equivalent to yet another one—of which, however, we will not make any use in the sequel.

Remark 7 Let Γ be a consistent set of sentences. The following conditions are equivalent:

1. Γ is a complete theory.
2a. For every sentence ψ, if $\Gamma \cup \{\psi\}$ is consistent, then $\psi \in \Gamma$.

2b. For every sentence ψ, if $\psi \notin \Gamma$, then $\Gamma \cup \{\psi\}$ is inconsistent.
3a. If Γ' is a consistent theory with $\Gamma' \supseteq \Gamma$, then $\Gamma' = \Gamma$.
3b. If Γ' is a theory with $\Gamma' \supset \Gamma$, then Γ' is inconsistent.

In all, a set of sequences Γ is a complete consistent theory precisely when Γ is a *maximal consistent* set of sequences, i.e. maximal among the consistent sets of sentences. Here maximality may be understood in the sense of any of the conditions 2a–b, 3a–b of Remark 7. In particular, the maximal consistent sets of sentences correspond to the ultrafilters of the Lindenbaum algebra [19].

A typical example of a maximal consistent theory is the theory $\mathrm{Th}(\mathcal{M})$ of a model $\mathcal{M}$. By the Completeness Theorem, this is the only type of maximal consistent theory. In fact, if Γ is a consistent theory, then it has a model $\mathcal{M}$. So $\Gamma \subseteq \mathrm{Th}(\mathcal{M})$, and if Γ is maximal consistent, then $\Gamma = \mathrm{Th}(\mathcal{M})$.

3.2 Lindenbaum's Lemma with Open Induction

We now show how to prove Lindenbaum's Lemma with OI in place of ZL.

Theorem 8 *(OI) For each $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}_{\mathcal{L}}$ the following are equivalent:*

(i) $\Gamma \vdash \varphi$;
(ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ *for every proper prime theory* $\Delta \subseteq \mathcal{S}_{\mathcal{L}}$.

Proof The non-trivial implication using OI is the one from (ii) to (i). To this end, define

$$X = \{\Delta \subseteq \mathcal{S}_{\mathcal{L}} \mid \Delta \text{ is a theory and } \Gamma \subseteq \Delta\}.$$

Clearly $\overline{\Gamma}$ is the least element of X, partially ordered by inclusion.

Claim 1: $(X, \subseteq)$ is chain-complete. To see this, let $\{\Delta_i \mid i \in I\}$ be a chain in X. We claim that $\Delta = \bigcup_{i \in I} \Delta_i$ belongs to X. Since $\Gamma \subseteq \Delta$, as I is inhabited, we only need to verify that Δ is a theory. If $\Delta \vdash \psi$, then $K \vdash \psi$ for some finite $K \subseteq \Delta$. So there is $k \in I$ such that $K \subseteq \Delta_k$. Hence $\Delta_k \vdash \psi$ and so $\psi \in \Delta_k \subseteq \Delta$. Note that the join of a chain of theories is its union.
Now let P be the predicate on X defined by

$$P(\Delta) \iff \varphi \in \Delta.$$

Claim 2: P is open. In fact, if $\varphi \in \bigcup_{i \in I} \Delta_i$, then $\varphi \in \Delta_i$ for some i.

Claim 3: P is progressive. Given $\Delta \in X$, we have to deduce $\varphi \in \Delta$ from the induction hypothesis that $\varphi \in \Delta'$ for every $\Delta' \in X$ such that $\Delta' \supset \Delta$. To this end we distinguish the following cases.

Case I: Δ is both proper and prime. In this case, (ii) applies to Δ; whence $\varphi \in \Delta$.

Case II: the negation of Case I, is split into two subcases, as follows.

Subcase IIa: Δ is not proper, i.e. $\Delta = \mathcal{S}_{\mathcal{L}}$, in which case trivially $\varphi \in \Delta$.

Subcase IIb: Δ is not prime, i.e. there are $\psi_1, \psi_2 \in \mathcal{S}_\mathcal{L}$, both outside Δ, such that $\psi_1 \vee \psi_2 \in \Delta$. For $i = 1, 2$ we then have $\Delta \cup \{\psi_i\} \supset \Delta$ and thus $P(\overline{\Delta \cup \{\psi_i\}})$, that is, $\Delta, \psi_i \vdash \varphi$, by induction hypothesis. Hence $\Delta, \psi_1 \vee \psi_2 \vdash \varphi$ by disjunction elimination, and so eventually $\varphi \in \Delta$ because $\psi_1 \vee \psi_2 \in \Delta$.

With OI at hand we can conclude that $P(\Delta)$ for every $\Delta \in X$. In particular, P holds for the least element of X, that is, $P(\overline{\Gamma})$. By definition of P, this means $\Gamma \vdash \varphi$. $\square$

In view of Lemma 6, Theorem 8 can equivalently be put as follows.

Corollary 9 (OI) *For each* $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}_\mathcal{L}$ *the following are equivalent:*

(i) $\Gamma \vdash \varphi$;
(ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ *for every complete consistent theory* $\Delta \subseteq \mathcal{S}_\mathcal{L}$.

To prepare for Sect. 3.3, we sketch how a proof of Corollary 9 can be obtained from the above proof of Theorem 8. The only modifications are required in Case II of the proof of Claim 3, which again is to be split into two subcases.

Subcase IIa': Δ is inconsistent, that is, $\bot \in \Delta$. Hence again $\Delta = \mathcal{S}_\mathcal{L}$ and thus trivially $\varphi \in \Delta$.

Subcase IIb': Δ is not complete, i.e. there is $\psi \in \mathcal{S}_\mathcal{L}$ such that both ψ and $\neg\psi$ lie outside Δ. Hence $\Delta \cup \{\psi\} \supset \Delta$ and $\Delta \cup \{\neg\psi\} \supset \Delta$, and thus $P(\overline{\Delta \cup \{\psi\}})$ and $P(\overline{\Delta \cup \{\neg\psi\}})$ by induction hypothesis, which is to say that $\Delta, \psi \vdash \varphi$ and $\Delta, \neg\psi \vdash \varphi$. In all, $\Delta, \psi \vee \neg\psi \vdash \varphi$ by disjunction elimination, and so $\varphi \in \Delta$ simply because $\psi \vee \neg\psi \in \Delta$, as Δ is a classical theory.

3.3 *Intuitionistic Logic*

As usual, $\bot$ stands for absurdity, $\neg\varphi$ for $\varphi \to \bot$ and $\top$ for $\neg\bot$. We write EFQ and TND for the axioms *ex falso quodlibet* $\bot \to \psi$ and *tertium non datur* $\psi \vee \neg\psi$ restricted to sentences ψ.

Convention *If we say that a statement holds in a certain logic, then we mean that this statement can be proved for deducibility within this logic and with the very same logic used in the meta-language. Likewise, if we say that we use EFQ and/or TND, then we mean this in the meta-language, too.*

3.3.1 Excluded Middle

Remark 5 holds in intuitionistic logic, too. This also is the case for some but not all implications within Remark 7—in which, however, we are not interested, for we have focussed on proper prime and complete consistent theories rather than maximal consistent sets of sentences.

Let us turn our attention to Lemma 6 instead. Trivially, every consistent theory is proper. By EFQ every proper theory is consistent; and again by EFQ every complete consistent theory is prime. In intuitionistic logic, in particular, Corollary 9 implies Theorem 8. In intuitionistic logic, however, one cannot prove that every prime theory is complete, for which—as in the proof of Lemma 6—one needs TND. Here is a characteristic example.

The smallest theory $\overline{\emptyset}$ is consistent and prime in intuitionistic logic, thanks to Gentzen's *Hauptsatz* and the disjunction property, respectively. In particular, Theorem 8 holds for $\Gamma = \emptyset$. On the other hand, $\overline{\emptyset}$ cannot be proved to be complete in intuitionistic logic, as this would just mean to postulate TND.

This example also shows that intuitionistic logic has not enough complete theories to make Corollary 9 hold for $\Gamma = \emptyset$, which would indeed entail TND. In fact, if ψ is any sentence, then $\varphi \equiv \psi \vee \neg\psi$ belongs to every complete theory Δ, since either $\psi \in \Delta$ or $\neg\psi \in \Delta$. So Corollary 9 is definitely too strong to hold in intuitionistic logic.

Before we study the status of Theorem 8 in intuitionistic logic, let us make a digression in the spirit of constructive reverse mathematics [13]. We say that a theory Γ is *Boolean* if $\psi \vee \neg\psi \in \Gamma$ for every sentence ψ. In classical logic every theory is Boolean, for which TND suffices. Independent of that, every complete theory is Boolean, as we have just stated—even in intuitionistic logic where conversely every Boolean theory is complete provided that it is prime. In intuitionistic logic, in particular, the complete consistent theories are exactly the Boolean proper prime theories. Now we can sum up.

Proposition 10 *In intuitionistic logic the following are equivalent.*

1. *TND holds.*
2. *Every theory is Boolean.*
3. *Every prime theory is complete.*
4. *The smallest theory $\overline{\emptyset}$ is complete.*
5. *Theorem 8 implies Corollary 9.*
6. *Corollary 9 holds for $\Gamma = \emptyset$.*

We recommend to proceed alongside the following paths: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 6$ and $3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$.

3.3.2 Proof by Cases

We next inspect the proofs of Theorem 8 and Corollary 9. In the proof of Corollary 9 sketched before, we have used EFQ and TND to settle Subcase IIa' and Subcase IIb', respectively. This parallels our use of these principles, noticed above, during the proof of the parts of Lemma 6 that are relevant for deducing Corollary 9 from Theorem 8.

In the proof of Theorem 8, on the other hand, we did not need EFQ and TND to settle Subcase IIa and Subcase IIb. Yet in the proof of Theorem 8, and likewise in the one of Corollary 9, we had to distinguish Case I from Case II, actually from Subcase

IIa and Subcase IIb. To control this instance of a proof by cases, which is essentially classical, we make it explicit following [16].

To this end we say that a finitely axiomatisable theory Γ has a *strong primality test* if for every theory Δ with $\Delta \supseteq \Gamma$ one can do a proof by cases of the type incriminated above. By this we mean that one can tell whether Δ is proper and prime, and if this is not the case, then one either knows that $\Delta = \mathcal{S}_{\mathcal{L}}$ or else one has at hand $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{L}}$, both lying outside Δ, such that $\psi_1 \vee \psi_2 \in \Delta$.

Hence the following variant of Theorem 8 equally holds for intuitionistic logic.

Theorem 11 (OI) *If a finitely axiomatisable theory* $\Gamma \subseteq \mathcal{S}_{\mathcal{L}}$ *admits a strong primality test, then for each* $\varphi \in \mathcal{S}_{\mathcal{L}}$ *the following are equivalent:*

(i) $\varphi \in \Gamma$*;*
(ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ *for every proper prime theory* $\Delta \subseteq \mathcal{S}_{\mathcal{L}}$*.*

Two choices have made possible to move to intuitionistic logic in the foregoing: first, to work with proper prime theories rather than complete consistent theories; secondly, to assume a strong primality test. Note that neither choice would make any difference for classical logic.

Although the assumption of a strong primality test may somehow look like cheating, it is of interest to review the situation in commutative algebra by which it has been inspired:

1. The appropriate analogue of a strong primality test is constructively provable for a sufficiently rich class of rings, the so-called fully Lasker-Noether rings [16].
2. If Krull's Lemma (KL), the counterpart of Theorem 8, is used to give a short and elegant proof of a certain theorem of concrete character, by reduction to the case of integral domains, then a constructive proof of the theorem is possible [17, 24] under decidability assumptions that are
    (a) more elementary than a strong primality test and
    (b) can be eliminated by basic proof-theoretic means.

For the time being, however, we do not know whether any part of this method can be carried over from algebra to logic. But KL and LL together, in the form of Theorem 8, have given rise to a universal Krull-Lindenbaum Theorem that equally follows from OI [20], which in turn has prompted a fairly general, constructive and syntactical conservation theorem for abstract entailment relations [21].

4 Related Results for Lattices

4.1 Distributive Lattices

The proof given above of Theorem 8—which, as we have seen, in fact proves Theorem 11—can be generalized from the Lindenbaum algebra to arbitrary distributive lattices.

Proposition 12 (OI) *In a distributive lattice, every filter is the intersection of the (proper) prime filters above.*

Proof Let $(S, \leqslant)$ be a distributive lattice. We show that every filter $A \subseteq S$ is the intersection of all proper prime filters U with $A \subseteq U$. In other words, given $b \in S$, we show that $b \in A$ follows from the assumption that $A \subseteq U \Rightarrow b \in U$ for every proper prime filter U.

To this end, we consider

$$X = \{B \subseteq S \mid B \text{ is a filter and } A \subseteq B\}.$$

Partially ordered by inclusion, this X is chain-complete: e.g., the union of a chain of filters is a filter.

Let P be the predicate on X defined by

$$P(B) \iff b \in B.$$

Note that P is open, as joins of chains in X are given by unions.

We now prove that P is progressive, that is, $P(B)$ follows from the assumption that $P(B')$ holds for all $B' \in X$ with $B' \supset B$. To this end we distinguish two cases. If B is proper and prime, then we use the hypothesis $A \subseteq U \Rightarrow b \in U$ for $U = B$. Otherwise, either B is improper, i.e. $B = S$, in which case $b \in B$ anyway; or B is not prime. In the latter case there must be $c, d \notin B$ such that $c \vee d \in B$. Consider the filters B_c and B_d generated by $B \cup \{c\}$ and $B \cup \{d\}$, respectively. Since these filters are strictly larger than B, by assumption both $P(B_c)$ and $P(B_d)$, that is, $b \in B_c \cap B_d$. This means that either $b \in B$, and we are done, or there are b_c and b_d in B with $b \geqslant b_c \wedge c$ and $b \geqslant b_d \wedge d$. Hence

$$b = b \vee b \geqslant (b_c \wedge c) \vee (b_d \wedge d) = (b_c \vee b_d) \wedge (b_c \vee d) \wedge (c \vee b_d) \wedge (c \vee d) \in B,$$

because $b_c, b_d, c \vee d \in B$ and B is a filter. So $b \in B$ as well, that is, $P(B)$.

By OI, the predicate P holds for all $U \in X$. In particular, $P(A)$, that is, $b \in A$. □

Corollary 13 *For all $a, b \in S$,*

$$a \leqslant b \textit{ if and only if } a \in U \Rightarrow b \in U \textit{ for all (proper) prime filters } U \subseteq S.$$

Proof Apply the previous proposition to the principal filter $\uparrow a$. □

4.2 Complete Lattices

In the case of complete lattices, it is natural to try to replace prime filters with completely-prime filters. Recall that a filter U in a complete lattice S is *completely-prime* if $\bigvee T \in U$ implies $T \between U$ for every $T \subseteq S$. One cannot, however, expect a similar proposition to hold true in general. For instance, in the case of locales (frames), the statement "every filter is an intersection of completely-prime filters" would imply the so-called property of spatiality, which is simply not true in general. Recall that a locale is spatial when $a \leqslant b$ holds if and only if every "point" in a also belongs to b, where the notion of a point for a locale is equivalent to that of a completely-prime filter. The regular open sets of, say, the real line form a locale in which joins are interiors of closures of unions. Such a locale has no point (though having many filters, of course: e.g. the principal ones).

For complete lattices we must content ourselves with the following. Note that Proposition 12 remains true, by a similar proof, when distributivity is dropped, but filters are replaced by upward closed sets (*upsets*). The same result extends to the complete case as well.

Proposition 14 (OI) *Every upset in a complete lattice is the intersection of the (proper) completely-prime upsets above.*

As for the proof of Proposition 12, the crucial case is the one in which the upset B is not completely-prime, i.e. in which there is $\{c_i : i \in I\} \subseteq S$, disjoint from B, with $\bigvee_{i \in I} c_i \in B$. Now if I is empty, then $B = S$ and so $b \in B$ for the $b \in S$ under consideration. If I is inhabited, for each $i \in I$ the upset $B_i = \uparrow (B \cup \{c_i\})$ is strictly larger than B and hence contains b by hypothesis. So either $b \in B$ or $b \geqslant c_i$ for all $i \in I$, that is, $b \geqslant \bigvee_{i \in I} c_i \in B$, and thus $b \in B$.

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Ordinal Analysis of Intuitionistic Power and Exponentiation Kripke Platek Set Theory

Jacob Cook and Michael Rathjen

Abstract Until the 1970s, proof theoretic investigations were mainly concerned with theories of inductive definitions, subsystems of analysis and finite type systems. With the pioneering work of Gerhard Jäger in the late 1970 s and early 1980s, the focus switched to set theories, furnishing ordinal-theoretic proof theory with a uniform and elegant framework. More recently it was shown that these tools can even sometimes be adapted to the context of strong axioms such as the powerset axiom, where one does not attain complete cut elimination but can nevertheless extract witnessing information and characterize the strength of the theory in terms of provable heights of the cumulative hierarchy. Here this technology is applied to intuitionistic Kripke-Platek set theories **IKP**$(\mathcal{P})$ and **IKP**$(\mathcal{E})$, where the operation of powerset and exponentiation, respectively, is allowed as a primitive in the separation and collection schemata. In particular, **IKP**$(\mathcal{P})$ proves the powerset axiom whereas **IKP**$(\mathcal{E})$ proves the exponentiation axiom. The latter expresses that given any sets A and B, the collection of all functions from A to B is a set, too. While **IKP**$(\mathcal{P})$ can be dealt with in a similar vein as its classical cousin, the treatment of **IKP**$(\mathcal{E})$ posed considerable obstacles. One of them was that in the infinitary system the levels of terms become a moving target as they cannot be assigned a fixed level in the formal cumulative hierarchy solely based on their syntactic structure. As adumbrated in an earlier paper, the results of this paper are an important tool in showing that several intuitionistic set theories with the collection axiom possess the existence property, i.e., if they prove an existential theorem then a witness can be provably described in the theory, one example being intuitionistic Zermelo-Fraenkel set theory with bounded separation.

Dedicated to Gerhard Jäger on the occasion of his 60th birthday.

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1 Introduction

In his early work, Gerhard Jäger laid the foundations for a direct proof-theoretic treatment of set theories (cf. [11, 12]) which then began to a large extent to supplant earlier work on theories of inductive definitions and subsystems of analysis. By and large, ordinal analyses for set theories are more uniform and transparent than for the latter theories. The primordial example of a set theory amenable to ordinal analysis is Kripke-Platek set theory, **KP**. It is an important theory for various reasons, one being that a great deal of set theory requires only the axioms of **KP**. Another reason is that admissible sets, the transitive models of **KP**, have been a major source of interaction between model theory of infinitary languages, recursion theory and set theory (cf. [4]). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to bounded formulae. Many of the familiar subsystems of second order arithmetic can be viewed as reduced versions of set theories based on the notion of admissible set. This applies for example to a fairly strong theory like Δ^1_2-**CA** plus bar induction which is of the same strength as **KP** augmented by an axiom saying that every set is contained in an admissible set, whose ordinal analysis is due to Jäger and Pohlers [15]. By restricting or completely omitting induction principles in theories of admissible sets, Jäger was also able to give a unified proof-theoretic treatment of many predicative theories in [14]. Systematic accounts and surveys of admissible proof theory can be found in [7, 13, 14, 16, 20] and other places.

Ordinal analyses of ever stronger theories have been obtained over the last 20 years. The strongest systems for which proof-theoretic ordinals have been determined are extensions of **KP** augmented by Σ_1-separation that correspond to subsystems of second-order arithmetic with comprehension restricted to Π^1_2 comprehension or iterations thereof (cf. [3, 19, 23]). Thus it appears that it is currently impossible to furnish an ordinal analysis of any set theory which has the power set axiom among its axioms as such a theory would dwarf the strength of second-order arithmetic. It is, however, possible to relativize the techniques of ordinal analysis developed for Kripke-Platek set theory to obtain useful information about Power Kripke-Platek set theory as shown in [27]. The kind of information one can extract concerns bounds for the transfinite iterations of the power set operation that are provable in the latter theory. In this paper the method is applied to intuitionistic Kripke-Platek set theories **IKP**$(\mathcal{P})$ and **IKP**$(\mathcal{E})$, where the operation of powerset, respectively, exponentiation, is allowed as a primitive in the separation and collection schemata. In particular, **IKP**$(\mathcal{P})$ proves the powerset axiom whereas **IKP**$(\mathcal{E})$ proves the exponentiation axiom. The latter expresses that given any sets A and B, the collection of all functions from A to B is a set, too. While **IKP**$(\mathcal{P})$ can be dealt with in a similar vein as its classical cousin in [27], the treatment of **IKP**$(\mathcal{E})$ posed considerable obstacles, one of them being that in the infinitary system terms cannot be assigned a fixed level in the formal cumulative hierarchy.

It was outlined in [25] that the results of this paper are an important tool for showing that several intuitionistic set theories with the collection axiom possess the existence property, i.e., if they prove an existential theorem then a witness can be provably described in the theory. One example for such a theory is intuitionistic Zermelo-Fraenkel set theory with bounded separation. Details will be presented in [28].

1.1 Intuitionistic Set Theories and the Existence Property

Intuitionistic theories are known to often possess very pleasing metamathematical properties such as the disjunction property and the numerical existence property. While it is fairly easy to establish these properties for arithmetical theories and theories with quantification over sets of natural numbers or Baire space (e.g. second order arithmetic and function arithmetic), set theories with their transfinite hierarchies of sets of sets and the extensionality axiom can pose considerable technical challenges.

Definition 1.1 Let T be a theory whose language, $L(T)$, encompasses the language of set theory. T has the *existence property*, **EP**, if whenever $T \vdash \exists x A(x)$ holds for a formula $A(x)$ having at most the free variable x, then there is a formula $C(x)$ with exactly x free, so that

$$T \vdash \exists! x\, [C(x) \wedge A(x)].$$

A theory that does not have the existence property is intuitionistic Zermelo-Fraenkel set theory, **IZF**, formulated with Collection, as was shown in [10]. Since the version of **IZF** with replacement in lieu of collection has the existence property, collection is clearly implicated in the failure of **EP**. This prompted Beeson in [5, IX.1] to ask the following question:

Does any reasonable set theory with collection have the existence property?

An important theory that is closely related to Martin-Löf type theory is Constructive Zermelo-Fraenkel set theory, **CZF** (cf. [1, 2]). It has been shown in [30] that **CZF** lacks the **EP**. While the proof is quite difficult, the failure of **EP** is perhaps not that surprising since **CZF** features an axiom, Subset Collection, that follows from a combination of exponentiation and a choice principle called the presentation axiom.[1] However, in [25] it was shown that three perhaps more natural versions of **CZF** possess the weak existence property, which requires a provably definable inhabited set of witnesses for every existential theorem. Tellingly, neither **IZF** nor **CZF** has the weak existence property (see [25, Proposition 1.3] and [30]). The three versions of **CZF** shown to have the weak existence property are **CZF** without subset collection (**CZF**$^-$), **CZF** with exponentiation instead of subset collection (**CZF**$_{\mathcal{E}}$), and **CZF** augmented by the powerset axiom (**CZF**$_{\mathcal{P}}$). Rathjen [25] also provided reductions of

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[1] Although sometimes even set theories with strong choice principles can have the **EP** (see [21]).

these three theories to pertaining versions of intuitionistic Kripke-Platek set theories in such a way that if the latter theories possessed the existence property for pertaining syntactically restricted classes of existential theorems, then the former would possess the full **EP**. This gave rise to the strategy of first embedding these extended theories of intuitionistic Kripke-Platek set theory into infinitary proof systems and use techniques of ordinal analysis to remove those inferences which embody collection. The second step, then, consists in showing that the infinitary systems have the term existence property, i.e., for each provable existential theorem there is a witnessing term. It will then ensue from the fact that the numerical existence property holds for $\mathbf{CZF}^{-}$, $\mathbf{CZF}_{\mathcal{E}}$, and $\mathbf{CZF}_{\mathcal{P}}$ that the existence property holds for these theories, too. The numerical existence property was verified in [22] and also holds for **CZF**, even when augmented by various choice principles [24].

1.2 Intuitionistic Power and Exponentiation Kripke-Platek Set Theories

We call a formula *bounded* or Δ_0 if all its quantifiers are of the form $\forall x \in a$ and $\exists y \in b$. The axioms of classical **KP** consist of *Extensionality, Pair, Union, Infinity, Bounded Separation*

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u \in a \ \wedge \ A(u))]$$

for all bounded formulae $A(u)$, *Bounded Collection*

$$\forall x \in a\, \exists y\, B(x, y) \ \rightarrow \ \exists z\, \forall x \in a\, \exists y \in z\, B(x, y)$$

for all bounded formulae $B(x, y)$, and *Set Induction*

$$\forall x \ [(\forall y \in x\, C(y)) \rightarrow C(x)] \ \rightarrow \ \forall x\, C(x)$$

for all formulae $C(x)$.

We denote by **IKP** the version of **KP** where the underlying logic is intuitionistic logic.

We use subset bounded quantifiers $\exists x \subseteq y \ \ldots$ and $\forall x \subseteq y \ \ldots$ as abbreviations for $\exists x (x \subseteq y \ \wedge \ \ldots)$ and $\forall x (x \subseteq y \rightarrow \ldots)$, respectively.

We call a formula of $\mathcal{L}_{\in}$ $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Q\, x \subseteq y$ or $Q\, x {\in} y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables.

Let $\mathrm{Fun}(f, x, y)$ be a acronym for the bounded formula expressing that f is a function with domain x and co-domain y. We use exponentiation bounded quantifiers $\exists f \in {}^{x}y \ \ldots$ and $\forall f \in {}^{x}y \ \ldots$ as abbreviations for $\exists f (\mathrm{Fun}(f, x, y) \ \wedge \ \ldots)$ and $\forall x (\mathrm{Fun}(f, x, y) \rightarrow \ldots)$, respectively.

Definition 1.2 The $\Delta_0^{\mathcal{P}}$-formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

The $\Delta_0^{\mathcal{E}}$-formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall f \in {}^a b,\ \exists f \in {}^a b.$$

Definition 1.3 **IKP**($\mathcal{E}$) has the same language and logic as **IKP**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Exponentiation, Set Induction, $\Delta_0^{\mathcal{E}}$-Separation and $\Delta_0^{\mathcal{E}}$-Collection.

IKP($\mathcal{P}$) has the same language and logic as **IKP**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, Set Induction, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

The transitive classical models of **IKP**($\mathcal{P}$) have been termed **power admissible** sets in [9]. There is also a close connection between **IKP**($\mathcal{P}$) and versions of Martin-Löf type theory with an impredicative type of propositions and the calculus of constructions (see [26]).

Remark 1.4 Alternatively, **IKP**($\mathcal{P}$) can be obtained from **IKP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbols to the language and the axiom

$$\forall y\,[y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0-Separation and Collection to the Δ_0-formulae of this new language.

Likewise, **IKP**($\mathcal{E}$) can be obtained from **IKP** by adding a primitive function symbol $\mathcal{E}$ for the exponentiation and the pertaining axioms.

Definition 1.5 The class of Σ-formulae in the strict sense, denoted by strict-Σ, is the smallest class of formulae containing the Δ_0-formulae closed under $\wedge$, $\vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \exists x.$$

The strict-$\Sigma^{\mathcal{P}}$-formulae are the smallest class of formulae containing the $\Delta_0^{\mathcal{P}}$-formulae closed under $\wedge$, $\vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a,\ \exists x.$$

The strict-$\Sigma^{\mathcal{E}}$-formulae are the smallest class of formulae containing the $\Delta_0^{\mathcal{E}}$-formulae closed under $\wedge$, $\vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall f \in {}^a b,\ \exists f \in {}^a b,\ \exists x.$$

Later on we shall have occasion to introduce the wider class of Σ-formulae. A crucial property that singles out the strict-Σ-formulae is that every such formula is equivalent to a Σ_1-formula provably in **IKP**, where a Σ_1-formula is a formula that starts with a single existential quantifier and thereafter continues with a Δ_0-formula. A similar characterization holds for the strict-$\Sigma^{\mathcal{P}}$ and strict-$\Sigma^{\mathcal{E}}$-formulae. It is standard to show that **IKP**($\mathcal{P}$) proves strict-$\Sigma^{\mathcal{P}}$-Collection and **IKP**($\mathcal{E}$) proves strict-$\Sigma^{\mathcal{E}}$-Collection (for details in the case of **IKP** see [1, 2]).

1.3 Outline of the Paper

The main objective of the paper is an in-depth presentation of the ordinal analyses of the three theories **IKP**, **IKP**($\mathcal{P}$), and **IKP**($\mathcal{E}$). In the case of **IKP** this means characterising the *proof theoretic ordinal* in the sense of [20], this is done in such a way that we can also extract witness terms from cut-free derivations of existential statements in the infinitary system. In the cases of **IKP**($\mathcal{P}$) and **IKP**($\mathcal{E}$) we present a type of relativised ordinal analysis similar to that given in [27], we characterise the number of iterations of the relevant hierarchy of sets (Von Neumann and Exponentiation respectively) that can be proven to exist within the theory. These details cannot be found in the existing research literature. They are needed for term extraction and, moreover, have to be shown to be formalizable in **CZF**$^-$, i.e., constructive Zermelo-Fraenkel set theory without subset collection. Naturally, the first system to be analyzed is the simplest. Section 2 treats **IKP**, so is basically a detailed rendering of the intuitionistic version of the classical analysis of **KP**. However, this section is also used later since several parts of it are modular and thus transfer to the stronger systems without any changes. Section 3 deals with **IKP**($\mathcal{P}$) and adapts the machinery of classical [27] to the intuitionistic case. The system **IKP**($\mathcal{E}$) is the most difficult to handle and is addressed in Sect. 4. A particular challenge is provided by the problem of assigning a rank to the formal terms of the infinitary system. Ultimately this turned out to be impossible and we had to deal with it in a new way by allowing for rank declarations as extra hypothesis.

A further reason to document these ordinal analyses in the literature is that the second author fosters hopes that this framework can be used to analyze much stronger theories such as ones with full negative separation. It appears that in striking difference to classical theories, negative separation does not block the methods of ordinal analysis that bring about the elimination of collection inferences in intuitionistic derivations.

2 The Case of IKP

This section provides a detailed rendering of the ordinal analysis of Kripke-Platek set theory formulated with intuitionistic logic, **IKP**. This is done in such a way that we are able to extract witness terms from the resulting cut-free derivations of Σ sentences in the infinitary system. This results in a proof that **IKP** has the existence property for Σ sentences, which in conjunction with results in [25] verifies that **CZF**$^{-}$ has the full existence property. Many of the arguments in this section are modular and transfer over to the stronger systems analysed in subsequent sections with minimal changes.

2.1 A Sequent Calculus Formulation of IKP

Definition 2.1 The language of **IKP** consists of free variables $a_0, a_1, \ldots,$ bound variables $x_0, x_1, \ldots$, the binary predicate symbol $\in$ and the logical symbols $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\forall$, $\exists$ as well as parentheses),

The atomic formulas are those of the form $a \in b$.

The formulas of **IKP** are defined inductively by:

- (i) All atomic formulas are formulas.
- (ii) If A and B are formulas then so are $\neg A$, $A \vee B$, $A \wedge B$ and $A \rightarrow B$.
- (iii) If $A(b)$ is a formula in which the bound variable x does not occur, then $\forall x A(x)$, $\exists x A(x)$, $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are also formulas.

Quantifiers of the form $\exists x$ and $\forall x$ will be called unbounded and those of the form $(\exists x \in a)$ and $(\forall x \in a)$ will be referred to as bounded quantifiers.

A Δ_0-formula is one in which no unbounded quantifiers appear.

The expression $a = b$ is to be treated as an abbreviation for $(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$.

The derivations of **IKP** take place in a two-sided sequent calculus. The sequents derived are *intuitionistic sequents* of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas and Δ contains at most one formula. The intended meaning of $\Gamma \Rightarrow \Delta$ is that the conjunction of formulas in Γ implies the formula in Δ, or if Δ is empty, a contradiction. The expressions $\Rightarrow \Delta$ and $\Gamma \Rightarrow$ are shorthand for $\emptyset \Rightarrow \Delta$ and $\Gamma \Rightarrow \emptyset$ respectively. Γ, A stands for $\Gamma \cup \{A\}$, $\Gamma \Rightarrow A$ stands for $\Gamma \Rightarrow \{A\}$, etc.

The axioms of **IKP** are:

Logical axioms:	$\Gamma, A \Rightarrow A$ for every Δ_0 formula A.
Extensionality:	$\Gamma \Rightarrow a = b \wedge B(a) \rightarrow B(b)$ for every Δ_0 formula $B(a)$.
Pair:	$\Gamma \Rightarrow \exists z(a \in z \wedge b \in z)$.
Union:	$\Gamma \Rightarrow \exists z(\forall y \in z)(\forall x \in y)(x \in z)$.
Δ_0-*Separation*:	$\Gamma \Rightarrow \exists y[(\forall x \in y)(x \in a \wedge B(x)) \wedge (\forall x \in a)(B(x) \rightarrow x \in y)]$ for every Δ_0-formula $B(a)$.
Set Induction:	$\Gamma \Rightarrow \forall x[(\forall y \in x F(y) \rightarrow F(x)] \rightarrow \forall x F(x)$ for any formula $F(a)$.
Infinity:	$\Gamma \Rightarrow \exists x[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$.
Δ_0-*Collection*:	$\Gamma \Rightarrow (\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for any Δ_0-formula G.

The rules of inference are

$$(\wedge L)\ \frac{\Gamma, C \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \text{for } C \in \{A, B\}. \qquad (\wedge R)\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$(\vee L)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow A \vee B}\ \text{for } C \in \{A, B\}.$$

$$(\neg L)\ \frac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow} \qquad (\neg R)\ \frac{\Gamma, A \Rightarrow}{\Gamma \Rightarrow \neg A} \qquad (\bot)\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}$$

$$(\rightarrow L)\ \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad (\rightarrow R)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$(b\exists L)\ \frac{\Gamma, a \in b \wedge F(a) \Rightarrow \Delta}{\Gamma, (\exists x \in b)F(x) \Rightarrow \Delta} \qquad (b\exists R)\ \frac{\Gamma \Rightarrow a \in b \wedge F(a)}{\Gamma \Rightarrow (\exists x \in b)F(x)}$$

$$(b\forall L)\ \frac{\Gamma, a \in b \rightarrow F(a) \Rightarrow \Delta}{\Gamma, (\forall x \in b)F(x) \Rightarrow \Delta} \qquad (b\forall R)\ \frac{\Gamma \Rightarrow a \in b \rightarrow F(a)}{\Gamma \Rightarrow (\forall x \in b)F(x)}$$

$$(\exists L)\ \frac{\Gamma, F(a) \Rightarrow \Delta}{\Gamma, \exists x F(x) \Rightarrow \Delta} \qquad (\exists R)\ \frac{\Gamma \Rightarrow F(a)}{\Gamma \Rightarrow \exists x F(x)}$$

$$(\forall L)\ \frac{\Gamma, F(a) \Rightarrow \Delta}{\Gamma, \forall x F(x) \Rightarrow \Delta} \qquad (\forall R)\ \frac{\Gamma \Rightarrow F(a)}{\Gamma \Rightarrow \forall x F(x)}$$

$$(\text{Cut})\ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

In each of the inferences $(b\exists L)$, $(\exists L)$ $(b\forall R)$ and $(\forall R)$ the variable a is forbidden from occurring in the conclusion. Such a variable is known as the *eigenvariable* of the inference.

The *minor formulae* of an inference are those rendered prominently in its premises, the other formulae in the premises will be referred to as *side formulae*. The *principal* formula of an inference is the one rendered prominently in the conclusion. Note that in inferences where the principal formula is on the left, the principal formula can also be a side formula of that inference, when this happens we say that there has been a *contraction*.

2.2 An Ordinal Notation System

Given below is a very brief description of how to carry out the construction of a primitive recursive ordinal notation system for the Bachmann-Howard ordinal.

Definition 2.2 Let Ω be a 'big' ordinal, eg. $\aleph_1$. (In fact we could have chosen ω_1^{CK}, see [18].) We define the sets $B^{\Omega}(\alpha)$ and ordinals $\psi_{\Omega}(\alpha)$ by transfinite recursion on α as follows

$$B^{\Omega}(\alpha) \;=\; \begin{cases} \textit{closure of}\ \{0, \Omega\}\ \textit{under:} \\ +, (\xi, \eta \mapsto \varphi\xi\eta), \\ (\xi \longmapsto \psi_{\Omega}(\xi))_{\xi<\alpha}. \end{cases} \tag{1}$$

$$\psi_{\Omega}(\alpha) \;\simeq\; \min\{\rho < \Omega : \ \rho \notin B(\alpha)\}. \tag{2}$$

It can be shown that $\psi_{\Omega}(\alpha)$ is always defined and thus $\psi_{\Omega}(\alpha) < \Omega$. Moreover, it can also be shown that $B_{\Omega}(\alpha) \cap \Omega = \psi_{\Omega}(\alpha)$.

Let $\varepsilon_{\Omega+1}$ be the least orinal $\eta > \Omega$ such that $\omega^{\eta} = \eta$. The set $B^{\Omega}(\varepsilon_{\Omega+1})$ gives rise to a primitive recursive ordinal notation system [6, 17]. The ordinal $\psi_{\Omega}(\varepsilon_{\Omega+1})$ is known as the Bachmann-Howard ordinal. There are many slight variants in the specific ordinal functions used to build up a notation system for this ordinal, for example rather than 'closing off' under the φ function at each stage, we could have chosen ω-exponentiation, all the systems turn out to be equivalent, in that they eventually 'catch-up' with one another and the specific ordinal functions used can be defined in terms of one another. Here the functions φ and ψ are chosen as primitive since they correspond to the ordinal operations arising from the two main cut elimination theorems of the next section.

2.3 The Infinitary System $\mathbf{IRS}_{\Omega}$

The purpose of this section is to define an intuitionistic style infinitary system $\mathbf{IRS}_{\Omega}$ within which we will be able to embed **IKP** and then extract useful information about **IKP** derivations.

Henceforth all ordinals will be assumed to belong to the primitive recursive ordinal representation system arising from $B^{\Omega}(\varepsilon_{\Omega+1})$.

The system is based around the constructible hierarchy up to level Ω.

$$\begin{aligned} L_0 &:= \emptyset, \\ L_{\alpha+1} &:= \{X \subseteq L_{\alpha} | X \text{ is definable over } L_{\alpha} \text{ in the language of } \mathbf{IKP} \text{ with parameters}\}, \\ L_{\lambda} &:= \bigcup_{\xi<\lambda} L_{\xi} \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Definition 2.3 We inductively define the terms of $\mathbf{IRS}_\Omega$. To each term t we also assign an ordinal level $|t|$.

(i) For each $\alpha < \Omega$, $\mathbb{L}_\alpha$ is a term with $|\mathbb{L}_\alpha| := \alpha$.
(ii) If $F(a, b_1, \dots, b_n)$ is a formula of **IKP** with all free variables indicated and $s_1, \dots, s_n$ are $\mathbf{IRS}_\Omega$ terms with levels less than α, then

$$[x \in \mathbb{L}_\alpha \mid F(x, s_1, \dots, s_n)^{\mathbb{L}_\alpha}]$$

is a term of level α. Here $F^{\mathbb{L}_\alpha}$ indicates that all unbounded quantifiers in F are restricted to $\mathbb{L}_\alpha$.

The formulae of $\mathbf{IRS}_\Omega$ are of the form $F(s_1, \dots, s_n)$ where $F(a_1, \dots, a_n)$ is a formula of **IKP** with all free variables displayed and $s_1, \dots, s_n$ are $\mathbf{IRS}_\Omega$-terms.

Note that the system $\mathbf{IRS}_\Omega$ does not contain free variables. We can think of the universe made up of $\mathbf{IRS}_\Omega$-terms as a formal, syntactical version of L_Ω, unbounded quantifiers in $\mathbf{IRS}_\Omega$-formulas can be thought of as ranging over L_Ω.

For the remainder of this section $\mathbf{IRS}_\Omega$-terms and $\mathbf{IRS}_\Omega$-formulae will simply be referred to as terms and formulae.

A formula is said to be Δ_0 if it contains no unbounded quantifiers.

We inductively (and simultaneously) define the class of Σ-formulae and the class of Π-formulae by the following clauses:

(i) Every Δ_0-formula is a Σ and a Π-formula.
(ii) If A and B are Σ-formulae (Π-formulae) then so are $A \lor B$, $A \land B$, $(\forall x \in s)A$, and $(\exists x \in s)A$.
(iii) If A is a Σ-formula (Π-formula) then so is $\exists x A$ ($\forall x A$).
(iv) If A is Π-formula and B is a Σ-formula, then $A \to B$ and $\neg A$ are Σ-formulae while $B \to A$ and $\neg B$ are Π-formulae.

The strict Σ-formulae of Definition 1.5 are Σ-formulae but the latter form a larger collection. It's perhaps worth noting that in classical **KP** every Σ-formula is equivalent to a Σ_1-formula and every Π-formula is equivalent to a Π_1-formula, and therefore both are equivalent to strict versions. This, however, does not extend to **IKP**. These formulae, though, satisfy well-known persistence properties.

Lemma 2.4 *For a formula C and free variable a, let C^a be the result of replacing each* unbounded *quantifier $\forall x$ and $\exists y$ in C by $\forall x \in a$ and $\exists y \in a$, respectively. Suppose A is a Σ-formula and B is a Π-formula. Then the following are provable in **IKP**:*

(i) $a \subseteq b \land A^a \to A^b$,
(ii) $a \subseteq b \land B^b \to B^a$.

Proof Straightforward by simultaneously induction on the buildup of A and B. □

Abbreviation 2.5 For $\diamond$ a binary propositional connective, A a formula and s, t terms with $|\, s \,| < |\, t \,|$ we define the following abbreviation:

$$\begin{aligned} s \dot\in t \diamond A &:= A && \text{if } t \text{ is of the form } \mathbb{L}_\alpha, \\ &:= B(s) \diamond A && \text{if } t \text{ is of the form } [x \in \mathbb{L}_\alpha \mid B(x)]. \end{aligned}$$

Like in **IKP**, derivations in $\mathbf{IRS}_\Omega$ take place in a two sided sequent calculus. Intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ are derived, where Γ and Δ are finite sets of formulae and at most one formula occurs in Δ. $\Gamma, \Delta, \Lambda, \ldots$ will be used as meta variables ranging over finite sets of formulae.

$\mathbf{IRS}_\Omega$ has no axioms, although note that some of the rules can have an empty set of premises. The inference rules are as follows:

$$(\in L)_\infty \quad \frac{\Gamma,\ p \dot\in t \wedge r = p \Rightarrow \Delta \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma, r \in t \ \Rightarrow \Delta}$$

$$(\in R) \quad \frac{\Gamma \Rightarrow s \dot\in t \ \wedge \ r = s}{\Gamma \Rightarrow s \in t} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(b\forall L) \quad \frac{\Gamma,\ s \dot\in t \to A(s) \Rightarrow \Delta}{\Gamma, (\forall x \in t)A(x) \ \Rightarrow \Delta} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(b\forall R)_\infty \quad \frac{\Gamma \Rightarrow p \dot\in t \to A(p) \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma \Rightarrow (\forall x \in t)A(x)}$$

$$(b\exists L)_\infty \quad \frac{\Gamma,\ p \dot\in t \wedge A(p) \Rightarrow \Delta \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma,\ (\exists x \in t)A(x) \ \Rightarrow \Delta}$$

$$(b\exists R) \quad \frac{\Gamma \Rightarrow s \dot\in t \wedge A(s)}{\Gamma \Rightarrow (\exists x \in t)A(x)} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(\forall L) \quad \frac{\Gamma,\ A(s) \Rightarrow \Delta}{\Gamma, \forall x\, A(x) \ \Rightarrow \Delta}$$

$$(\forall R)_\infty \quad \frac{\Gamma \Rightarrow A(p) \ \text{ for all } p}{\Gamma \Rightarrow \forall x A(x)}$$

$$(\exists L)_\infty \quad \frac{\Gamma,\ A(p) \Rightarrow \Delta \ \text{ for all } p}{\Gamma,\ \exists x A(x) \ \Rightarrow \Delta}$$

$$(\exists R) \quad \frac{\Gamma \Rightarrow A(s)}{\Gamma \Rightarrow \exists x A(x)}$$

$$(\Sigma\text{-Ref}_\Omega) \quad \frac{\Gamma \ \Rightarrow \ A}{\Gamma \Rightarrow \exists z A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

as well as the rules $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\to L)$, $(\to R)$ and (Cut) which are defined identically to the rules of the same name in **IKP**.

In general we are unable to remove cuts from $\mathbf{IRS}_\Omega$ derivations, one of the main obstacles to full cut elimination comes from $(\Sigma\text{-Ref}_\Omega)$ since it breaks the symmetry of the other rules. However we can still perform cut elimination on certain derivations, provided they are of a very uniform kind. Luckily, certain embedded proofs from **IKP** will be of this form. In order to express uniformity in infinite proofs we draw on [7], where Buchholz developed a powerful method of describing such uniformity, called *operator control*.

Definition 2.6 Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called an **operator** if $\mathcal{H}$ satisfies the following conditions for all $X \in P(ON)$:

1. $X \subseteq Y \Rightarrow \mathcal{H}(X) \subseteq \mathcal{H}(Y)$ (monotone).
2. $X \subseteq \mathcal{H}(X)$ (inclusive).
3. $\mathcal{H}(\mathcal{H}(X)) = \mathcal{H}(X)$ (idempotent).
4. $0 \in \mathcal{H}(X)$ and $\Omega \in \mathcal{H}(X)$.
5. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then

$$\alpha \in \mathcal{H}(X) \quad \text{iff} \quad \alpha_1, \ldots, \alpha_n \in \mathcal{H}(X).$$

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^{\sigma}$, and decomposition of its members into additive and multiplicative components.

From now on $\alpha \in \mathcal{H}$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{H}$ will be considered shorthand for $\alpha \in \mathcal{H}(\emptyset)$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{H}(\emptyset)$ respectively.

Definition 2.7 If A is a formula let

$$k(A) := \{\alpha \in ON \ : \text{ the symbol } \mathbb{L}_{\alpha} \text{ occurs in } A, \text{ subterms included}\}.$$

Likewise we define

$$k(\{A_1, \ldots, A_n\}) := k(A_1) \cup \cdots \cup k(A_n) \quad \text{and} \quad k(\Gamma \Rightarrow \Delta) := k(\Gamma) \cup k(\Delta).$$

Now for $\mathcal{H}$ an arbitrary operator, s a term and $\mathfrak{X}$ a formula, set of formulae or a sequent we define

$$\begin{aligned} \mathcal{H}[s](X) &:= \mathcal{H}(X \cup \{|s|\}), \\ \mathcal{H}[\mathfrak{X}](X) &:= \mathcal{H}(X \cup k(\mathfrak{X})). \end{aligned}$$

Lemma 2.8 *Let $\mathcal{H}$ be an operator, s a term and $\mathfrak{X}$ a formula, set of formulae or sequent.*

(i) For any $X, X' \in P(ON)$, if $X' \subseteq X$ then $\mathcal{H}(X') \subseteq \mathcal{H}(X)$].
(ii) $\mathcal{H}[s]$ and $\mathcal{H}[\mathfrak{X}]$ are operators.
(iii) If $k(\mathfrak{X}) \subseteq \mathcal{H}(\emptyset)$ then $\mathcal{H}[\mathfrak{X}] = \mathcal{H}$.
(iv) If $|\, s \,| \in \mathcal{H}$ then $\mathcal{H}[s] = \mathcal{H}$.

Proof This result is demonstrated in full in [17]. □

We also need to keep track of the complexity of cuts appearing in derivations.

Definition 2.9 The *rank* of a term or formula is determined by

1. $rk(\mathbb{L}_\alpha) := \omega \cdot \alpha$,
2. $rk([x \in \mathbb{L}_\alpha \mid F(x)]) := \max\{\omega \cdot \alpha + 1, rk(F(\mathbb{L}_0)) + 2\}$,
3. $rk(s \in t) := \max\{rk(s) + 6, rk(t) + 1\}$,
4. $rk(A \wedge B) = rk(A \vee B) = rk(A \rightarrow B) := \max\{rk(A) + 1, rk(B) + 1\}$,
5. $rk(\neg A) := rk(A) + 1$,
6. $rk((\exists x \in t)A(x)) = rk((\forall x \in t)A(x)) := \max\{rk(t), rk(F(\mathbb{L}_0)) + 2\}$,
7. $rk(\exists x A(x)) = rk(\forall x A(x)) := \max\{\Omega, rk(F(\mathbb{L}_0)) + 1\}$.

Observation 2.10 (i) $rk(s) = \omega \cdot |s| + n$ for some $n < \omega$.
(ii) If A is Δ_0, $rk(A) = \omega \cdot \max(k(A)) + m$ for some $m < \omega$.
(iii) If A contains unbounded quantifiers $rk(A) = \Omega + m$ for some $m < \omega$.
(iv) $rk(A) < \Omega$ if and only if A is Δ_0.

There is plenty of leeway in defining the actual rank of a formula, basically we need to make sure the following lemma holds.

Lemma 2.11 *In every rule of* ***IRS***$_\Omega$ *other than* (Σ*-Ref*$_\Omega$) *and (Cut), the rank of the minor formulae is strictly less than the rank of the principal formula.*

Proof This result is demonstrated for a different set of propositional connectives in [17], the adapted proof to the intuitionistic system is similar. □

Definition 2.12 (*Operator controlled derivability for* **IRS**$_\Omega$) Let $\mathcal{H}$ be an operator and $\Gamma \Rightarrow \Delta$ an intuitionistic sequent of **IRS**$_\Omega$, we define the relation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$ by recursion on α.

We require always that $k(\Gamma \Rightarrow \Delta) \cup \{\alpha\} \subseteq \mathcal{H}$, this condition will not be repeated in the inductive clauses for each of the inference rules of **IRS**$_\Omega$ below. The column on the right gives the ordinal requirements for each of the inference rules.

$$(\in L)_\infty \quad \frac{\mathcal{H}[r] \overset{\alpha_r}{\underset{\rho}{\vdash}} \Gamma, r \dot{\in} t \wedge r = s \Rightarrow \Delta \text{ for all } |r| < |t|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \in t \Rightarrow \Delta} \quad |r| \leq \alpha_r < \alpha$$

$$(\in R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow r \dot{\in} t \wedge r = s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t} \quad \begin{array}{r} \alpha_0 < \alpha \\ |r| < |t| \\ |r| < \alpha \end{array}$$

$$(b\forall L) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, s \mathrel{\dot{\in}} t \to A(s) \Rightarrow \Delta}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |s| < |t| \\ |s| < \alpha \end{array}$$

$$(b\forall R)_\infty \quad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma \Rightarrow s \mathrel{\dot{\in}} t \to F(s) \text{ for all } |s| < |t|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in t)F(x)} \quad |s| \leq \alpha_s < \alpha$$

$$(b\exists L)_\infty \quad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, s \mathrel{\dot{\in}} t \wedge F(s) \Rightarrow \Delta \text{ for all } |s| < |t|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in t)F(x) \Rightarrow \Delta} \quad |s| \leq \alpha_s < \alpha$$

$$(b\exists R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \mathrel{\dot{\in}} t \wedge A(s)}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |s| < |t| \\ |s| < \alpha \end{array}$$

$$(\forall L) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, F(s) \Rightarrow \Delta}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |s| < \alpha \end{array}$$

$$(\forall R)_\infty \quad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma \Rightarrow F(s) \text{ for all } s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \forall x F(x)} \quad |s| < \alpha_s + 1 < \alpha$$

$$(\exists L)_\infty \quad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, F(s) \Rightarrow \Delta \text{ for all } s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \exists x F(x) \Rightarrow \Delta} \quad |s| < \alpha_s + 1 < \alpha$$

$$(\exists R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow F(s)}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \Rightarrow \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |s| < \alpha \end{array}$$

$$(\text{Cut}) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, B \Rightarrow \Delta \qquad \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow B}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\Sigma\text{-Ref}_\Omega) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow A}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma\text{-formula} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\to L)$ or $(\to R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

Lemma 2.13 (Weakening and Persistence for $\mathbf{IRS}_\Omega$)

(i) *If* $\Gamma_0 \subseteq \Gamma$, $k(\Gamma) \subseteq \mathcal{H}$, $\alpha_0 \leq \alpha \in \mathcal{H}$, $\rho_0 \leq \rho$ *and* $\mathcal{H} \overset{\alpha_0}{\underset{\rho_0}{\vdash}} \Gamma_0 \Rightarrow \Delta$ *then*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta\,.$$

(ii) *If* $\beta \geq \gamma \in \mathcal{H}$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x) \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\gamma)$ $\vdash A(x) \Rightarrow \Delta$.

(iii) *If* $\beta \geq \gamma \in \mathcal{H}$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\beta)A(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$.

(iv) *If* $\gamma \in \mathcal{H}$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x A(x) \Rightarrow \Delta$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x) \Rightarrow \Delta$.

(v) *If* $\gamma \in \mathcal{H}$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \forall x A(x)$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$.

Proof We show (i), (ii) and (v).

(i) is proved by an easy induction on α.
(ii) Is also proved using induction on α, suppose $\beta \geq \gamma \in \mathcal{H}(\emptyset)$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x) \Rightarrow \Delta$. If $(\exists x \in \mathbb{L}_\beta)A(x)$ was not the principal formula of the last inference or the last inference was not $(b\exists L)_\infty$ then we may apply the induction hypotheses to it's premises followed by the same inference again. So suppose $(\exists x \in \mathbb{L}_\beta)A(x)$ was the principal formula of the last inference which was $(b\exists L)_\infty$, so we have

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x), A(s) \Rightarrow \Delta \quad \text{for all } |s| < \beta, \text{ with } \alpha_s < \alpha.$$

From the induction hypothesis we obtain

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x), A(s) \Rightarrow \Delta \quad \text{for all } |s| < \beta, \text{ with } \alpha_s < \alpha$$

but since $\beta \geq \gamma$ this also holds for all $|s| < \gamma$. So by another application of $(b\exists L)_\infty$ we get

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x) \Rightarrow \Delta$$

as required.

For (v) suppose $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \forall x A(x)$. The interesting case is where $\forall x A(x)$ was the principal formula of the last inference, which was $(\forall R)_\infty$, in this case we have

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A(s) \quad \text{for all } s, \text{ with } |\, s\,| < \alpha_s + 1 < \alpha.$$

So taking just the cases where $|\, s\,| < \gamma$ and noting that in these cases $A(s) \equiv s \,\dot{\in}\, \mathbb{L}_\gamma \rightarrow A(s)$, we may apply $(b\forall R)$ to obtain

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$$

as required.

The proofs of (iii) and (iv) may be carried out in a similar manner to those above. □

2.4 Cut Elimination for $\mathbf{IRS}_\Omega$

Lemma 2.14 (Inversions of $\mathbf{IRS}_\Omega$)

(i) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$.

(ii) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A \wedge B$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A$ *and* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow B$.

(iii) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$ *and* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, B \Rightarrow \Delta$.

(iv) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \rightarrow B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, B \Rightarrow \Delta$.

(v) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A \rightarrow B$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow B$.

(vi) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \neg A$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow$.

(vii) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, r \in t \Rightarrow \Delta$ *then* $\mathcal{H}[s] \vdash^{\alpha}_{\rho} \Gamma, s \dot{\in} t \wedge r = s \Rightarrow \Delta$ *for all* $|s| < |t|$.

(viii) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ *then* $\mathcal{H}[s] \vdash^{\alpha}_{\rho} \Gamma, s \dot{\in} t \wedge A(s) \Rightarrow \Delta$ *for all* $|s| < |t|$.

(ix) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in t)A(x)$ *then* $\mathcal{H}[s] \vdash^{\alpha}_{\rho} \Gamma \Rightarrow s \dot{\in} t \rightarrow A(s)$ *for all* $|s| < |t|$.

(x) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \exists x A(x) \Rightarrow \Delta$ *then* $\mathcal{H}[s] \vdash^{\alpha}_{\rho} \Gamma, A(s) \Rightarrow \Delta$ *for all* s.

(xi) *If* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \Rightarrow \forall x A(x)$ *then* $\mathcal{H}[s] \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A(s)$ *for all* s.

Proof All proofs are by induction on α, we treat three of the most interesting cases, (iv), (vi) and (x).

(iv) Suppose $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \rightarrow B \Rightarrow \Delta$, If the last inference was not $(\rightarrow L)$ or the principal formula of that inference was not $A \rightarrow B$ we may apply the induction hypothesis to the premises of that inference, followed by the same inference again. Now suppose $A \rightarrow B$ was the principal formula of the last inference, which was $(\rightarrow L)$. Thus, with the possible use of weakening, we have

$$\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, B, A \rightarrow B \Rightarrow \Delta \qquad \text{for some } \alpha_0 < \alpha, \tag{1}$$

$$\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma, A \rightarrow B \Rightarrow A \qquad \text{for some } \alpha_1 < \alpha. \tag{2}$$

Applying the induction hypothesis to (1) yields $\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, B \Rightarrow \Delta$ from which we may obtain the desired result by weakening.

(vi) Now suppose $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \neg A$. If $\neg A$ was the principal formula of the last inference which was $(\neg R)$ then we have $\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, A \Rightarrow$ for some $\alpha_0 < \alpha$, from which we may obtain the desired result by weakening. If the last inference was $(\bot)$ then $\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow$ for some $\alpha_0 < \alpha$, from which we also obtain the desired result by weakening. If the last inference was different to $(\neg R)$ and $(\bot)$ we may apply the induction hypothesis to the premises of that inference followed by the same inference again.

(x) Finally suppose $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \exists x A(x) \Rightarrow \Delta$. If $\exists x A(x)$ was the principal formula of the last inference which was $(\exists L)_\infty$ then we have

$$\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, \exists x A(x), A(s) \Rightarrow \Delta \quad \text{with } \alpha_s < \alpha \text{ for each } s.$$

Applying the induction hypothesis yields

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, A(s) \Rightarrow \Delta$$

from which we get the desired result by weakening. If $\exists x A(x)$ was not the principal formula of the last inference or the last inference was not $(\exists L)_\infty$ then we may apply the induction hypothesis to the premises of that inference followed by the same inference again. □

Lemma 2.15 (Reduction for $\mathbf{IRS}_\Omega$) *Let* $\rho := rk(C) \neq \Omega$.

$$\textit{If}\ \ \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta \quad \textit{and}\ \ \mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C \quad \textit{then}\ \ \mathcal{H} \overset{\alpha \# \alpha \# \beta \# \beta}{\underset{\rho}{\vdash}} \Gamma, \Xi \Rightarrow \Delta .$$

Proof The proof is by induction on $\alpha \# \alpha \# \beta \# \beta$. Assume that

$$\rho := rk(C) \neq \Omega, \tag{1}$$

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta\,, \tag{2}$$

$$\mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C. \tag{3}$$

If C was not the principal formula of the last inference in both derivations then we may simply use the induction hypothesis on the premises and then the final inference again.

So suppose C was the principal formula of the last inference in both (2) and (3). Note also that (1) gives us immediately that the last inference in (3) was *not* (Σ-Ref$_\Omega$).

We treat three of the most interesting cases.

Case 1. Suppose $C \equiv r \in t$, thus we have

$$\mathcal{H}[p] \overset{\alpha_p}{\underset{\rho}{\vdash}} \Gamma, C, p \mathrel{\dot{\in}} t \wedge r = p \Rightarrow \Delta \quad \text{for all } |\,p\,| < |\,t\,| \text{ with } \alpha_p < \alpha$$

and

$$\mathcal{H} \overset{\beta_0}{\underset{\rho}{\vdash}} \Xi \Rightarrow s \mathrel{\dot{\in}} t \wedge r = s \quad \text{for some } |\,s\,| < |\,t\,| \text{ with } \beta_0 < \beta. \tag{4}$$

Now from (5) we know that $|\,s\,| \in \mathcal{H}$ and thus from (4) we have

$$\mathcal{H} \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, C, s \mathrel{\dot{\in}} t \wedge r = s \Rightarrow \Delta\,. \tag{5}$$

Applying the induction hypothesis to (6) and (3) yields

$$\mathcal{H} \overset{\alpha_s \# \alpha_s \# \beta \# \beta}{\underset{\rho}{\vdash}} \Xi, \Gamma, s \mathrel{\dot{\in}} t \wedge r = s \Rightarrow \Delta\,. \tag{6}$$

Finally a (Cut) applied to (5) and (7) yields

$$\mathcal{H} \;\Big|\!\frac{\alpha \,\#\, \alpha \,\#\, \beta \,\#\, \beta}{\rho}\; \Xi, \Gamma \Rightarrow \Delta \tag{7}$$

as required.

Case 2. Now suppose $C \equiv (\forall x \in t) F(x)$ so we have

$$\mathcal{H} \;\Big|\!\frac{\alpha_0}{\rho}\; \Gamma, C, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta \quad \text{for some } |\, s \,| < |\, t \,| \text{ with } \alpha_0, |\, s \,| < \alpha \tag{8}$$

and

$$\mathcal{H}[p] \;\Big|\!\frac{\beta_p}{\rho}\; \Xi \Rightarrow p \mathrel{\dot{\in}} t \to F(p) \quad \text{for all } |\, p \,| < |\, t \,| \text{ with } \beta_p < \beta. \tag{9}$$

Now (8) gives $s \in \mathcal{H}$ and thus from (9) we have

$$\mathcal{H} \;\Big|\!\frac{\beta_s}{\rho}\; \Xi \Rightarrow s \mathrel{\dot{\in}} t \to F(s)\,. \tag{10}$$

Applying the induction hypothesis to (3) and (8) gives

$$\mathcal{H} \;\Big|\!\frac{\alpha_0 \,\#\, \alpha_0 \,\#\, \beta \,\#\, \beta}{\rho}\; \Gamma, \Xi, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta\,. \tag{11}$$

Finally (Cut) applied to (10) and (11) yields the desired result.

Case 3. Now suppose $C \equiv A \to B$ so we have

$$\mathcal{H} \;\Big|\!\frac{\alpha_0}{\rho}\; \Gamma, C \Rightarrow A \quad \text{with } \alpha_0 < \alpha, \tag{12}$$

$$\mathcal{H} \;\Big|\!\frac{\alpha_1}{\rho}\; \Gamma, C, B \Rightarrow \Delta \quad \text{with } \alpha_1 < \alpha, \tag{13}$$

$$\mathcal{H} \;\Big|\!\frac{\beta_0}{\rho}\; \Xi, A \Rightarrow B \quad \text{with } \beta_0 < \beta. \tag{14}$$

The induction hypothesis applied to (12) and (3) gives

$$\mathcal{H} \;\Big|\!\frac{\alpha_0 \,\#\, \alpha_0 \,\#\, \beta \,\#\, \beta}{\rho}\; \Gamma, \Xi \Rightarrow A\,. \tag{15}$$

Now an application of (Cut) to (15) and (14) gives

$$\mathcal{H} \;\Big|\!\frac{\alpha_0 \,\#\, \alpha \,\#\, \beta \,\#\, \beta}{\rho}\; \Gamma, \Xi \Rightarrow B\,. \tag{16}$$

Inversion (Lemma 2.14 (iv)) applied to (13) gives

$$\mathcal{H} \;\Big|\!\frac{\alpha_1}{\rho}\; \Gamma, B \Rightarrow \Delta\,. \tag{17}$$

Finally a single application of (Cut) to (16) and (17) yields the desired result. □

Theorem 2.16 (Predicative Cut Elimination for $\mathbf{IRS}_\Omega$) *Suppose* $\mathcal{H} \mid\frac{\alpha}{\rho+\omega^\beta} \Gamma \Rightarrow \Delta$, *where* $\Omega \notin [\rho, \rho+\omega^\beta)$ *and* $\beta \in \mathcal{H}$*, then*

$$\mathcal{H} \mid\frac{\varphi\beta\alpha}{\rho} \Gamma \Rightarrow \Delta\,.$$

Provided $\mathcal{H}$ *is an operator closed under* φ*.*

Proof The proof is by main induction on β and subsidiary induction on α.

If the last inference was anything other than (Cut) or was a cut of rank $< \rho$ then we may apply the subsidiary induction hypothesis to the premises and then re-apply the final inference. So suppose the last inference was (Cut) with cut-formula C and $rk(C) \in [\rho, \rho+\omega^\beta)$. So we have

$$\mathcal{H} \mid\frac{\alpha_0}{\rho+\omega^\beta} \Gamma, C \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha, \tag{1}$$

$$\mathcal{H} \mid\frac{\alpha_1}{\rho+\omega^\beta} \Gamma \Rightarrow C \quad \text{with } \alpha_1 < \alpha. \tag{2}$$

First applying the subsidiary induction hypothesis to (1) and (2) gives

$$\mathcal{H} \mid\frac{\varphi\beta\alpha_0}{\rho} \Gamma, C \Rightarrow \Delta, \tag{3}$$

$$\mathcal{H} \mid\frac{\varphi\beta\alpha_1}{\rho} \Gamma, \Rightarrow C\,. \tag{4}$$

Now if $rk(C) = \rho$ then one application of the Reduction Lemma 2.15 gives the desired result (once it is noted that $\varphi\beta\alpha_0 \# \varphi\beta\alpha_0 \# \varphi\beta\alpha_1 \# \varphi\beta\alpha_1 < \varphi\beta\alpha$ since $\varphi\beta\alpha$ is additive principal.)

Now let us suppose that $\beta > 0$ and $rk(C) \in (\rho, \rho+\omega^\beta)$. Since $rk(C) < \rho+\omega^\beta$ we can find some $\beta_0 < \beta$ and some $n < \omega$ such that

$$rk(C) < \rho + n \cdot \omega^{\beta_0}.$$

Thus applying (Cut) to (3) and (4) gives

$$\mathcal{H} \mid\frac{\varphi\beta\alpha}{\rho+n\cdot\omega^{\beta_0}} \Gamma \Rightarrow \Delta\,.$$

Now by the main induction hypothesis we obtain

$$\mathcal{H} \mid\frac{\varphi\beta_0(\varphi\beta\alpha)}{\rho+(n-1)\cdot\omega^{\beta_0}} \Gamma \Rightarrow \Delta.$$

But by definition $\varphi\beta\alpha$ is a fixed point of the function $\varphi\beta_0(\cdot)$ i.e. $\varphi\beta_0(\varphi\beta\alpha) = \varphi\beta\alpha$, so we have

$$\mathcal{H} \mid\frac{\varphi\beta\alpha}{\rho+(n-1)\cdot\omega^{\beta_0}} \Gamma \Rightarrow \Delta.$$

From here a further $(n - 1)$ applications of the main induction hypothesis yields the desired result. ☐

Lemma 2.17 (Boundedness for $\mathbf{IRS}_\Omega$) *If A is a Σ-formula, B is a Π-formula, $\alpha \leq \beta < \Omega$ and $\beta \in \mathcal{H}$ then*

(i) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A^{\mathbb{L}_\beta}$.

(ii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B^{\mathbb{L}_\beta} \Rightarrow \Delta$.

Proof Suppose that $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A$. We prove (i) and (ii) simultaneously by induction on α.

First we look at (i). If A was not the principal formula of the last inference then we can simply use the induction hypothesis. If A was the principal formula of the last inference and is of the form $\neg C$, $C \wedge D$, $C \vee D$, $C \to D$, $(\exists x \in t)C(x)$ or $(\forall x \in t)C(x)$, then again the result follows immediately from the induction hypothesis.

Note that the last inference cannot have been $(\forall R)_\infty$ or $(\Sigma\text{-Ref}_\Omega)$ since A is a Σ formula and $\alpha < \Omega$.

So suppose $A \equiv \exists x C(x)$ and

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow C(s)$$

for some $\alpha_0, |s| < \alpha$. By induction hypothesis we obtain

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow C(s)^{\mathbb{L}_\beta}.$$

Which may be written as

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \mathrel{\dot{\in}} \mathbb{L}_\beta \wedge C(s)^{\mathbb{L}_\beta}.$$

Now an application of $(b\exists R)$ yields the desired result.

As part (ii) is proved in a similar manner, we shall confine ourselves to the case when the last inference was $(\to L)$ with principal formula B. So suppose $B \equiv C \to D$ and

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow C \quad \text{and} \quad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, D \Rightarrow \Delta$$

for some $\alpha_0, |s| < \alpha$. By induction hypothesis we obtain

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow C^{\mathbb{L}_\beta} \text{ and } \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, D^{\mathbb{L}_\beta} \Rightarrow \Delta.$$

Now an application of $(\to L)$ yields the desired result. ☐

Definition 2.18 For each η we define

$$\mathcal{H}_\eta : \mathcal{P}(B^\Omega(\varepsilon_{\Omega+1})) \longrightarrow \mathcal{P}(B^\Omega(\varepsilon_{\Omega+1})),$$
$$\mathcal{H}_\eta(X) := \bigcap \{B^\Omega(\alpha) \ : \ X \subseteq B^\Omega(\alpha) \text{ and } \eta < \alpha\}.$$

Lemma 2.19 *(i)* $\mathcal{H}_\eta$ *is an operator for each* η.
(ii) $\eta < \eta' \implies \mathcal{H}_\eta(X) \subseteq \mathcal{H}_{\eta'}(X)$.
(iii) If $\xi \in \mathcal{H}_\eta(X)$ *and* $\xi < \eta + 1$ *then* $\psi_\Omega(\xi) \in \mathcal{H}_\eta(X)$.

Proof This is proved in [7]. □

Lemma 2.20 *Suppose* $\eta \in \mathcal{H}_\eta$ *and let* $\hat{\beta} := \eta + \omega^{\Omega+\beta}$.

(i) If $\alpha \in \mathcal{H}_\eta$ *then* $\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}$.
(ii) If $\alpha_0 \in \mathcal{H}_\eta$ *and* $\alpha_0 < \alpha$ *then* $\psi_\Omega(\hat{\alpha_0}) < \psi_\Omega(\hat{\alpha})$.

Proof (i) From $\alpha, \eta \in \mathcal{H}_\eta = B^\Omega(\eta + 1)$ we get $\hat{\alpha} \in B^\Omega(\eta + 1)$ and hence $\hat{\alpha} \in B^\Omega(\hat{\alpha})$ by Lemma 2.19(ii). Thus $\psi_\Omega(\hat{\alpha}) \in B^\Omega(\hat{\alpha} + 1) = \mathcal{H}_{\hat{\alpha}}(\emptyset)$.
(ii) Suppose that $\alpha > \alpha_0 \in \mathcal{H}_\eta$. By the argument above we get $\psi_\Omega(\hat{\alpha_o}) \in B^\Omega(\hat{\alpha_0} + 1) \subseteq B^\Omega(\hat{\alpha})$, thus $\psi_\Omega(\hat{\alpha_0}) < \psi_\Omega(\hat{\alpha})$. □

Theorem 2.21 (Collapsing for $\mathbf{IRS}_\Omega$) *Suppose that* $\eta \in \mathcal{H}_\eta$, Δ *is a set of at most one* Σ*-formula and* Γ *a finite set of* Π*-formulae. Then*

$$\mathcal{H}_\eta \vdash^{\alpha}_{\Omega+1} \Gamma \Rightarrow \Delta \quad \textit{implies} \quad \mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha})}_{\psi_\Omega(\hat{\alpha})} \Gamma \Rightarrow \Delta\,.$$

Proof We proceed by induction on α. If the last inference was propositional then the assertion follows easily from the induction hypothesis.

Case 1. Suppose the last inference was $(b\forall R)_\infty$, then $\Delta = \{(\forall x \in t)F(x)\}$ and

$$\mathcal{H}_\eta[p] \vdash^{\alpha_p}_{\Omega+1} \Gamma \Rightarrow p \,\dot{\in}\, t \to F(p) \quad \text{for all } |\,p\,| < |\,t\,| \text{ with } \alpha_p < \alpha.$$

Since $k(t) \subseteq \mathcal{H}_\eta$, we know that $|\,t\,| \in B(\eta + 1)$ and thus $|\,t\,| < \psi_\Omega(\eta + 1)$. Thus $k(p) \subseteq \mathcal{H}_\eta$ for all $|\,p\,| < |\,t\,|$, so $\mathcal{H}_\eta[p] = \mathcal{H}_\eta$ for all such p. Since $p \,\dot{\in}\, t \to F(p)$ is also a Σ-formula we can invoke the induction hypothesis to give

$$\mathcal{H}_{\hat{\alpha_p}} \vdash^{\psi_\Omega(\hat{\alpha_p})}_{\psi_\Omega(\hat{\alpha_p})} \Gamma, p \,\dot{\in}\, t \Rightarrow F(p)\,.$$

Since $\psi_\Omega(\hat{\alpha_p}) + 1 < \psi_\Omega(\hat{\alpha})$ for all p, we may apply $(\to R)$ and then $(b\forall R)_\infty$ to obtain the desired result.

Case 2. Suppose the last inference was $(b\forall L)$ so $(\forall x \in t)F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma, s \,\dot{\in}\, t \to F(s) \Rightarrow \Delta \quad \text{for some } |\,s\,| < |\,t\,| \text{ with } \alpha_0 < \alpha.$$

Noting that $s \,\dot{\in}\, t \to F(s)$ is itself a Π-formula, we may apply the induction hypothesis to give

$$\mathcal{H}_{\hat{\alpha_0}} \vdash^{\psi_\Omega\hat{\alpha_0}}_{\psi_\Omega\hat{\alpha_0}} \Gamma, s \,\dot{\in}\, t \to F(s) \Rightarrow \Delta$$

from which we obtain the desired result using one application of $(b\forall L)$.

Case 3. $(b\exists L)_\infty$ and $(b\exists R)$ are similar to cases 1 and 2.

Case 4. Suppose the last inference was $(\exists R)$, so $\Delta = \{\exists x F(x)\}$ and

$$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1} \Gamma \Rightarrow F(s) \quad \text{for some } |\, s \,| < \alpha \text{ and } \alpha_0 < \alpha.$$

Since $F(s)$ is Σ we may immediately apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi_\Omega \hat{\alpha}_0}{\psi_\Omega \hat{\alpha}_0} \Gamma \Rightarrow F(s)\,.$$

Now since $|\, s \,| \in \mathcal{H}_\eta = B(\eta + 1)$ we know that $|\, s \,| < \psi_\Omega(\eta + 1) < \psi_\Omega \hat{\alpha}$, so we may apply $(\exists R)$ to obtain the desired result.

Case 5. If the last inference was $(\forall L)$ we may argue in a similar fashion to Case 4.

It cannot be the case that the last inference was $(\exists L)$ or $(\forall R)$ since Γ contains only Π formulae and Δ only Σ formulae.

Case 6. Suppose the last inference was (Σ-Ref$_\Omega$), so $\Delta = \{\exists z F^z\}$ for some Σ formula F and

$$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1} \Gamma \Rightarrow F\,.$$

The induction hypothesis yields

$$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi_\Omega \hat{\alpha}_0}{\psi_\Omega \hat{\alpha}_0} \Gamma \Rightarrow F.$$

Now applying Boundedness Lemma 2.17 yields

$$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi_\Omega \hat{\alpha}_0}{\psi_\Omega \hat{\alpha}_0} \Gamma \Rightarrow F^{\mathbb{L}_{\psi_\Omega(\hat{\alpha}_0)}}.$$

From which one application of $(\exists R)$ yields the desired result.

Case 7. Finally suppose the last inference was (Cut), then there is a formula C with $rk(C) \leq \Omega$ and $\alpha_0 < \alpha$ such that

$$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1} \Gamma, C \Rightarrow \Delta, \tag{1}$$

$$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1} \Gamma \Rightarrow C. \tag{2}$$

7.1 If $rk(C) < \Omega$ then C contains only bounded quantification and as such is both Σ and Π, thus we may apply the induction hypothesis to both (1) and (2) to give

$$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi_\Omega(\hat{\alpha}_0)}{\psi_\Omega(\hat{\alpha}_0)} \Gamma, C \Rightarrow \Delta, \tag{3}$$

$$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi_\Omega(\hat{\alpha}_0)}{\psi_\Omega(\hat{\alpha}_0)} \Gamma \Rightarrow C\,. \tag{4}$$

Since $k(C) \subseteq \mathcal{H}_\eta$ and $rk(C) < \Omega$, we have $rk(C) < \psi_\Omega(\eta + 1)$, so we may apply (Cut) to (3) and (4) to obtain the desired result.

7.2 If $rk(C) = \Omega$ then $C \equiv \exists x F(x)$ or $C \equiv \forall x F(x)$ with $F(\mathbb{L}_0)$ a Δ_0 formula. The two cases are similar so for simplicity just the case where $C \equiv \exists x F(x)$ is considered.

We can begin by immediately applying the induction hypothesis to (2) since C is a Σ formula, giving

$$\mathcal{H}_{\hat{\alpha_0}} \Big|\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\, \Gamma \Rightarrow C\,.$$

Now applying Boundedness Lemma 2.17 yields

$$\mathcal{H}_{\hat{\alpha_0}} \Big|\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\, \Gamma \Rightarrow C^{\mathbb{L}_{\psi_\Omega(\hat{\alpha_0})}}\,. \tag{5}$$

Since $\psi_\Omega(\hat{\alpha_0}) \in \mathcal{H}_{\hat{\alpha_0}}$ we may apply Lemma 2.13(ii) to (1) to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \Big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, (\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x) \Rightarrow \Delta\,.$$

Now $(\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x)$ is bounded and hence Π so by the induction hypothesis we obtain

$$\mathcal{H}_{\hat{\alpha_1}} \Big|\!\frac{\psi_\Omega(\alpha_1}{\psi_\Omega(\alpha_1)}\, \Gamma, (\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x) \Rightarrow \Delta\,. \tag{6}$$

where $\alpha_1 := \hat{\alpha_0} + \omega^{\Omega+\alpha_0}$. Since $\alpha_1 < \eta + \omega^{\Omega+\alpha} := \hat{\alpha}$ and $rk((\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x)) < \psi_\Omega(\alpha)$ we may apply (Cut) to (5) and (6) to complete the proof. $\square$

2.5 Embedding IKP into IRS$_\Omega$

In this section we show how **IKP** derivations can be carried out in a very uniform manner within **IRS**$_\Omega$. First some preparatory definitions.

Definition 2.22 (i) Given ordinals $\alpha_1, \ldots, \alpha_n$. The expression $\omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}$ denotes the ordinal

$$\omega^{\alpha_{p(1)}} + \cdots + \omega^{\alpha_{p(n)}}$$

where $p:\{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ such that $\alpha_{p(1)} \geq \cdots \geq \alpha_{p(n)}$. More generally $\alpha \# 0 := 0 \# \alpha := 0$ and if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ then $\alpha \# \beta := \omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n} \# \omega^{\beta_1} \# \cdots \# \omega^{\beta_m}$.

(ii) If A is any **IRS**$_\Omega$-formula then $no(A) := \omega^{rk(A)}$ and if $\Gamma \Rightarrow \Delta$ is an **IRS**$_\Omega$-sequent containing formulas $\{A_1, \ldots, A_n\}$, then $no(\Gamma \Rightarrow \Delta) := no(A_1) \# \cdots \# no(A_n)$.

(iii) $\Vdash \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \Big|\!\frac{no(\Gamma\Rightarrow\Delta)}{0}\, \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

(iv) $\Vdash^{\xi}_{\rho} \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \; \Big|\frac{no(\Gamma \Rightarrow \Delta) \,\#\, \xi}{\rho} \; \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

We would like to be able to use $\Vdash$ as a calculus since it dispenses with a lot of superfluous notation, luckily under certain conditions this is possible.

Lemma 2.23 *(i) If* $\Gamma \Rightarrow \Delta$ *follows from premises* $\Gamma_i \Rightarrow \Delta_i$ *by an inference other than (Cut) or* (Σ*-Ref*$_\Omega$) *and without contractions then*

$$\textit{if } \Vdash^{\alpha}_{\rho} \Gamma_i \Rightarrow \Delta_i \quad \textit{then} \quad \Vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta .$$

(ii) If $\Vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$ *then* $\Vdash^{\alpha}_{\rho} \Gamma, A \wedge B \Rightarrow \Delta$.

Proof The first part follows from the additive principal nature of ordinals of the form ω^{α} and Lemma 2.11.

For the second part suppose $\Vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$ which means we have

$$\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] \; \Big|\frac{no(\Gamma \Rightarrow \Delta) \,\#\, no(A) \,\#\, no(B) \,\#\, \alpha}{\rho} \; \Gamma, A, B \Rightarrow \Delta \, .$$

Two applications of $(\wedge L)$ yields

$$\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] \; \Big|\frac{no(\Gamma \Rightarrow \Delta) \,\#\, no(A) \,\#\, no(B) \,\#\, \alpha{+}2}{\rho} \; \Gamma, A \wedge B \Rightarrow \Delta \, .$$

It remains to note that $\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] = \mathcal{H}[\Gamma, A \wedge B \Rightarrow \Delta]$ and

$$no(A) \,\#\, no(B) + 2 = \omega^{rk(A)} \,\#\, \omega^{rk(B)} + 2 < \omega^{rk(A \wedge B)} = no(A \wedge B)$$

to complete the proof. □

Lemma 2.24 *For any* ***IRS***$_\Omega$ *formulas* A, B *and terms* s, t *we have*

- *(i)* $\Vdash \Gamma, A \Rightarrow A$.
- *(ii)* $\Vdash s \in s \Rightarrow$.
- *(iii)* $\Vdash \Rightarrow s \subseteq s$ *here* $s \subseteq s$ *is shorthand for* $(\forall x \in s)(x \in s)$.
- *(iv)* $\Vdash \Rightarrow s \dot{\in} t \rightarrow s \in t$ *and* $\Vdash s \dot{\in} t \Rightarrow s \in t$*, for* $|s| < |t|$.
- *(v)* $\Vdash s = t \Rightarrow t = s$.
- *(vi) If* $\Vdash \Gamma, A \Rightarrow B$ *then* $\Gamma, s \dot{\in} t \wedge A \Rightarrow s \dot{\in} t \wedge B$ *for* $|s| < |t|$.
- *(vii) If* $\Vdash \Gamma, A, B \Rightarrow \Delta$ *then* $\Vdash \Gamma, s \dot{\in} t \rightarrow A, s \dot{\in} t \wedge B \Rightarrow \Delta$ *for* $|s| < |t|$.
- *(viii) If* $|s| < \beta$ *then* $\Vdash \Rightarrow s \in \mathbb{L}_{\beta}$.

Proof (i) By induction of $rk(A)$. We split into cases based on the form of the formula A.

Case 1. If $A \equiv (r \in t)$ then by the induction hypothesis we have

$$\Vdash \Gamma, s \dot{\in} t \wedge r = s \Rightarrow s \dot{\in} t \wedge r = s \quad \text{for all } |s| < |t| .$$

The following is a template for $\mathbf{IRS}_\Omega$ derivations.

$$(\in L)_\infty\,\dfrac{(\in R)\,\dfrac{\Vdash s \mathrel{\dot{\in}} t \wedge r = s \Rightarrow s \mathrel{\dot{\in}} t \wedge r = s \quad \text{for all } |s| < |t|}{\Vdash s \mathrel{\dot{\in}} t \wedge r = s \Rightarrow r \in t \quad \text{for all } |s| < |t|}}{\Vdash r \in t \Rightarrow r \in t}$$

Case 2. If $A \equiv (\exists x \in t)F(x)$ then by the induction hypothesis we have

$$\Vdash s \mathrel{\dot{\in}} t \wedge F(s) \Rightarrow s \mathrel{\dot{\in}} t \wedge F(s) \quad \text{for all } |s| < |t|.$$

We have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(b\exists L)_\infty\,\dfrac{(b\exists R)\,\dfrac{\Vdash s \mathrel{\dot{\in}} t \wedge F(s) \Rightarrow s \mathrel{\dot{\in}} t \wedge F(s) \quad \text{for all } |s| < |t|}{\Vdash s \mathrel{\dot{\in}} t \wedge F(s) \Rightarrow (\exists x \in t)F(x) \quad \text{for all } |s| < |t|}}{\Vdash (\exists x \in t)F(x) \Rightarrow (\exists x \in t)F(x)}$$

Case 3. All remaining cases can be proved in a similar fashion to above.

(ii) The proof is by induction on $rk(s)$, inductively we get $\Vdash r \in r \Rightarrow$ for all $|r| < |s|$. Now if s is of the form $\mathbb{L}_\alpha$, then $r \in r \equiv r \mathrel{\dot{\in}} s \rightarrow r \in r$ and we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(\in L)_\infty\,\dfrac{(\wedge L)\,\dfrac{(b\forall L)\,\dfrac{\Vdash r \in r \Rightarrow \quad \text{for all } |r| < |s|}{\Vdash (\forall x \in s)(x \in r) \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash s = r \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash s \in s \Rightarrow}$$

Now if $s \equiv [x \in \mathbb{L}_\alpha \mid B(x)]$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$(\in L)_\infty\,\dfrac{\text{Lemma 2.23(ii)}\,\dfrac{(\wedge L)\,\dfrac{(b\forall L)\,\dfrac{(\rightarrow L)\,\dfrac{\dfrac{(i)}{\Vdash B(r) \Rightarrow B(r) \quad \text{for all } |r| < |s|} \qquad \dfrac{\text{Induction Hypothesis}}{\Vdash r \in r \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash B(r), B(r) \rightarrow r \in r \Rightarrow}}{\Vdash B(r), (\forall x \in s)(x \in r) \Rightarrow}}{\Vdash B(r), r = s \Rightarrow}}{\Vdash B(r) \wedge r = s \Rightarrow}}{\Vdash s \in s \Rightarrow}$$

(iii) Again we use induction on $rk(s)$. Inductively we have $\Vdash \Rightarrow r \subseteq r$ for all $|r| < |s|$. If $s \equiv [x \in \mathbb{L}_\alpha \mid B(x)]$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{(\in R)\,\dfrac{(\wedge R)\,\dfrac{\dfrac{(i)}{\Vdash B(r) \Rightarrow B(r) \quad \text{for all } |r| < |s|} \qquad (\wedge R)\,\dfrac{\dfrac{\text{Induction Hypothesis}}{\Vdash B(r) \Rightarrow r \subseteq r \quad \text{for all } |r| < |s|}}{\Vdash B(r) \Rightarrow r = r}}{\Vdash B(r) \Rightarrow B(r) \wedge r = r}}{\Vdash B(r) \Rightarrow r \in s}}{\Vdash \Rightarrow r \mathrel{\dot{\in}} s \rightarrow r \in s}}{\Vdash \Rightarrow (\forall x \in s)(x \in s)}$$

If $s \equiv \mathbb{L}_\alpha$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$(b\forall R)_\infty\ \dfrac{(\in R)\ \dfrac{(\wedge R)\ \dfrac{\dfrac{\text{Induction Hypothesis}}{\Vdash\Rightarrow r \subseteq r \quad \text{for all } |\, r\,| < |\, s\,|}}{\Vdash\Rightarrow r = r}}{\Vdash\Rightarrow r \in s}}{\Vdash\Rightarrow (\forall x \in s)(x \in s)}$$

(iv) Was shown whilst proving (iii).
(v) The following is a template for $\mathbf{IRS}_\Omega$ derivations

$$(\wedge R)\ \dfrac{(\wedge L)\ \dfrac{\dfrac{(i)}{\Vdash s \subseteq t \Rightarrow s \subseteq t}}{\Vdash s = t \Rightarrow s \subseteq t} \qquad (\wedge L)\ \dfrac{\dfrac{(i)}{\Vdash t \subseteq s \Rightarrow t \subseteq s}}{\Vdash s = t \Rightarrow t \subseteq s}}{\Vdash s = t \Rightarrow t = s}$$

(vi) Trivial for $t \equiv \mathbb{L}_\alpha$, now if $t \equiv [x \in \mathbb{L}_\alpha \mid C(x)]$ then we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(\wedge R)\ \dfrac{(\wedge L)\ \dfrac{\Vdash \Gamma, A \Rightarrow B}{\Vdash \Gamma, C(s) \wedge A \Rightarrow B} \qquad (\wedge L)\ \dfrac{\Vdash \Gamma, C(s) \Rightarrow C(s)}{\Vdash \Gamma, C(s) \wedge A \Rightarrow C(s)}}{\Vdash \Gamma, C(s) \wedge A \Rightarrow C(s) \wedge B}$$

(vii) This is also trivial for $t \equiv \mathbb{L}_\alpha$ so suppose $t \equiv [x \in \mathbb{L}_\alpha \mid C(x)]$ and we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(\to L)\ \dfrac{(\wedge L)\ \dfrac{\Vdash \Gamma, C(s) \Rightarrow C(s)}{\Vdash \Gamma, C(s) \wedge B \Rightarrow C(s)} \qquad (\wedge L)\ \dfrac{\Vdash \Gamma, A, B \Rightarrow \Delta}{\Vdash \Gamma, A, C(s) \wedge B \Rightarrow \Delta}}{\Vdash \Gamma, C(s) \to A, C(s) \wedge B \Rightarrow \Delta}$$

(viii) Suppose $|\, s\,| < \beta$ then we have the following template for derivations in IRS_Ω.

$$(\in R)\ \dfrac{\dfrac{(iii)}{\Vdash\Rightarrow s = s}}{\Vdash\Rightarrow s \in \mathbb{L}_\beta}$$

□

Lemma 2.25 *For any terms* $s_1, \ldots, s_n, t_1, \ldots, t_n$ *and any formula* $A(s_1, \ldots, s_n)$ *we have*

$$\Vdash [s_1 = t_1], \ldots, [s_n = t_n], A(s_1, \ldots, s_n) \Rightarrow A(t_1, \ldots, t_n)$$

where $[s_i = t_i]$ *is shorthand for* $s_i \subseteq t_i, t_i \subseteq s_i$.

Proof We proceed by induction on $rk(A(s_1, \ldots, s_n)) \,\#\, rk(A(t_1, \ldots, t_n))$.

Case 1. Suppose $A(x_1, x_2) \equiv (x_1 \in x_2)$, then for all $|\, s\,| < |\, s_2\,|$ and $|\, t\,| < |\, t_2\,|$ we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$\begin{array}{rl}
 & \Vdash [s_1 = t_1], [t = s], s_1 = s \Rightarrow t_1 = t \\ \cline{2-2}
\text{Lemma 2.23(ii)} & \Vdash [s_1 = t_1], t = s, s_1 = s \Rightarrow t_1 = t \\ \cline{2-2}
\text{2.24(vi)} & \Vdash [s_1 = t_1], t \mathrel{\dot{\in}} t_2 \wedge t = s, s_1 = s \Rightarrow t \mathrel{\dot{\in}} t_2 \wedge t_1 = t \\ \cline{2-2}
(\in R) & \Vdash [s_1 = t_1], t \mathrel{\dot{\in}} t_2 \wedge t = s, s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\in L)_\infty & \Vdash [s_1 = t_1], s \in t_2, s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
\text{2.24(vii)} & \Vdash [s_1 = t_1], s \mathrel{\dot{\in}} s_2 \to s \in t_2, s \mathrel{\dot{\in}} s_2 \wedge s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\forall L) & \Vdash [s_1 = t_1], (\forall x \in s_2)(x \in t_2), s \mathrel{\dot{\in}} s_2 \wedge s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\in L)_\infty & \Vdash [s_1 = t_1], (\forall x \in s_2)(x \in t_2), s_1 \in s_2 \Rightarrow t_1 \in t_2 \\ \cline{2-2}
\text{Lemma 2.13(i)} & \Vdash [s_1 = t_1], [s_2 = t_2], s_1 \in s_2 \Rightarrow t_1 \in t_2
\end{array}$$

Case 2. If $A(x_1, x_2) \equiv x_1 \in x_1$ then the assertion follows by Lemma 2.24(ii) and weakening.

Case 3. Suppose $A(x_1, \ldots, x_n) \equiv (\exists y \in x_i)B(y, x_1, \ldots, x_n)$, for simplicity let us suppose that $i = 1$. Inductively for all $|\, r \,| < |\, s_1 \,|$ we have

$$\begin{array}{rl}
 & \Vdash [s_1 = t_1], \ldots, [s_n = t_n], r \mathrel{\dot{\in}} s_1 \wedge B(r, s_1, \ldots, s_n) \Rightarrow r \mathrel{\dot{\in}} t_1 \wedge B(r, t_1, \ldots, t_n) \\ \cline{2-2}
(b\exists R) & \Vdash [s_1 = t_1], \ldots, [s_n = t_n], r \mathrel{\dot{\in}} s_1 \wedge B(r, s_1, \ldots, s_n) \Rightarrow (\exists y \in s_1)B(y, t_1, \ldots, t_n) \\ \cline{2-2}
(b\exists L)_\infty & \Vdash [s_1 = t_1], \ldots, [s_n = t_n], (\exists y \in s_1)B(y, t_1, \ldots, t_n) \Rightarrow (\exists y \in s_1)B(y, t_1, \ldots, t_n)
\end{array}$$

Case 4. The bounded universal quantification case is dual to the bounded existential one.

Case 5. If $A(x_1, \ldots, x_n) \equiv \exists y B(y, x_1, \ldots, x_n)$ then inductively for all terms r we have

$$\Vdash [s_1 = t_1], \ldots, [s_n = t_n], B(r, s_1, \ldots, s_n) \Rightarrow B(r, t_1, \ldots, t_n)$$

subsequently applying $(\exists R)$ followed by $(\exists L)_\infty$ yields the desired result.

Case 6. The unbounded universal quantification case is dual to the unbounded existential one.

Case 7. All propositional cases follow immediately from the induction hypothesis.

□

Corollary 2.26 (Equality) *For any* $\boldsymbol{IRS}_\Omega$*-formula* $A(s_1, \ldots, s_n)$

$$\Vdash \Rightarrow s_1 = t_1 \wedge \cdots \wedge s_n = t_n \wedge A(s_1, \ldots, s_n) \to A(t_1, \ldots, t_n).$$

Lemma 2.27 (Set Induction) *For any formula* F

$$\Vdash^{\omega^{rk(A)}}_0 \Rightarrow \forall x[(\forall y \in x)F(y) \to F(x)] \to \forall x F(x).$$

where $A := \forall x[(\forall y \in x)F(y) \to F(x)]$.

Proof First we verify the following claim:

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|+1}}_0 A \Rightarrow F(s) \quad \text{for all } s. \tag{*}$$

The claim is verified by induction on $|\, s \,|$, inductively suppose that

$$\mathcal{H}[A, t] \vdash^{\omega^{rk(A)} \# \omega^{|t|+1}}_0 A \Rightarrow F(t) \quad \text{holds for all } |\, t \,| < |\, s \,|.$$

If necessary we may apply ($\to R$) to obtain

$$\mathcal{H}[A, t, s] \vdash^{\omega^{rk(A)} \# \omega^{|t|+1}+1}_{0} A \Rightarrow t \mathrel{\dot{\in}} s \to F(t) .$$

Next applying $(b\forall R)_\infty$ yields

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|}+2}_{0} A \Rightarrow (\forall y \in s)F(y) .$$

Also by Lemma 2.24(i) we have

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(F(s))} \# \omega^{rk(F(s))}}_{0} F(s) \Rightarrow F(s) .$$

Now one may note that $\omega^{rk(F(s))} \# \omega^{rk(F(s))} \leq \omega^{rk(F(s))+1} \leq \omega^{\max(\Omega, rk(F(\mathbb{L}_0))+3)} = \omega^{rk(A)}$ to see that by weakening we can conclude

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|}+2}_{0} F(s) \Rightarrow F(s) .$$

Hence using one application of ($\to L$) we get

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|}+3}_{0} A, (\forall y \in s)F(y) \to F(s) \Rightarrow F(s) .$$

Applying $(b\forall L)$ yields

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|}+4}_{0} A \Rightarrow F(s) .$$

Thus the claim (*) is verified. A single application of $(\forall R)_\infty$ to (*) furnishes us with

$$\mathcal{H}[A] \vdash^{\omega^{rk(A)} \# \Omega}_{0} A \Rightarrow \forall x F(x) .$$

Finally applying ($\to R$) gives

$$\Vdash^{\omega^{rk(A)}}_{0} \Rightarrow A \to \forall x F(x)$$

as required. □

Lemma 2.28 (Infinity) *For any ordinal $\alpha > \omega$ we have*

$$\Vdash \Rightarrow (\exists x \in \mathbb{L}_\alpha)[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)].$$

Proof The following is a template for derivations in $\mathbf{IRS}_\Omega$:

$$(b\exists R)\ \dfrac{(\wedge R)\ \dfrac{(b\exists R)\ \dfrac{\dfrac{\text{Lemma 2.24(viii)}}{\Vdash\Rightarrow \mathbb{L}_0 \in \mathbb{L}_\omega}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(z \in \mathbb{L}_\omega)} \qquad (b\forall R)_\infty\ \dfrac{(b\exists R)\ \dfrac{\dfrac{\text{Lemma 2.24(viii)}}{\Vdash\Rightarrow s \in \mathbb{L}_\alpha \quad \text{for all } |s| < \alpha < \omega}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(s \in z) \quad \text{for all } |s| < \omega}}{\Vdash\Rightarrow (\forall y \in \mathbb{L}_\omega)(\exists z \in \mathbb{L}_\omega)(y \in z)}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(z \in \mathbb{L}_\omega) \wedge (\forall y \in \mathbb{L}_\omega)(\exists z \in \mathbb{L}_\omega)(y \in z)}}{\Vdash\Rightarrow (\exists x \in \mathbb{L}_\alpha)[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]}$$

□

Lemma 2.29 (Δ_0-Separation) *Suppose* $|s|, |t_1|, \dots, |t_n| < \lambda$ *where* λ *is a limit ordinal and* $A(a, b_1, \dots, b_n)$ *is a* Δ_0*-formula of* ***IKP*** *with all free variables displayed, then*

$$\Vdash\Rightarrow (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \wedge A(x, t_1, \dots, t_n)) \wedge (\forall x \in s)(A(x, t_1, \dots, t_n) \to x \in y)].$$

Proof First let $\beta := \max\{|s|, |t_1|, \dots, |t_n|\} + 1$ and note that $\beta < \lambda$ since λ is a limit. Now let

$$t := [u \in \mathbb{L}_\beta \mid u \in s \wedge A(u, t_1, \dots, t_n)].$$

Let $B(x) := A(x, t_1, \dots, t_n)$, in what follows r ranges over terms with $|r| < |t|$ and p ranges over terms with $|p| < |s|$. We have the following two templates for derivations in $\mathbf{IRS}_\Omega$:

Derivation (1)

$$(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{\dfrac{\text{Lemma 2.24(i)}}{\Vdash r \in s \wedge B(r) \Rightarrow r \in s \wedge B(r)}}{\Vdash\Rightarrow r \mathrel{\dot{\in}} t \to (r \in s \wedge B(r))}}{\Vdash\Rightarrow (\forall x \in t)(x \in s \wedge B(x))}$$

Derivation (2)

$$(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(\to R)\ \dfrac{(\in R)\ \dfrac{\dfrac{(\wedge R)\ \dfrac{\dfrac{\text{Lemma 2.24(iv)}}{\Vdash p \mathrel{\dot{\in}} s, B(p) \Rightarrow p \in s} \qquad \dfrac{\text{Lemma 2.24(i)}}{\Vdash p \mathrel{\dot{\in}} s, B(p) \Rightarrow B(p)}}{\Vdash p \mathrel{\dot{\in}} s, B(p) \Rightarrow p \in s \wedge B(p)} \qquad (\wedge R)\ \dfrac{\dfrac{\text{Lemma 2.24(iii)}}{\Vdash\Rightarrow p \subseteq p}}{\Vdash\Rightarrow p = p}}{\Vdash p \mathrel{\dot{\in}} s, B(p) \Rightarrow (p \in s \wedge B(p)) \wedge p = p}}{\Vdash p \mathrel{\dot{\in}} s, B(p) \Rightarrow p \in t}}{\Vdash p \mathrel{\dot{\in}} s \Rightarrow B(p) \to p \in t}}{\Vdash\Rightarrow p \mathrel{\dot{\in}} s \to (B(p) \to p \in t)}}{\Vdash\Rightarrow (\forall x \in s)(B(x) \to x \in t)}$$

Now applying $(\wedge R)$ to the conclusions of derivations (1) and (2) we obtain

$$\Vdash\Rightarrow (\forall x \in t)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \to x \in t).$$

Finally note that $|t| = \beta < \lambda$ so we may apply $(b\exists R)$ to obtain

$$\Vdash\Rightarrow (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \to x \in y)]$$

as required. □

Lemma 2.30 (Pair) *If* λ *is a limit ordinal and* $|s|, |t| < \lambda$*, then*

$$\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(s \in z \wedge t \in z).$$

Proof Let $\delta := \max\{|s|, |t|\} + 1$ and note that $\delta < \lambda$ since λ is a limit. We have the following template for $\mathbf{IRS}_\Omega$ derivations:

$$(b\exists R)\;\dfrac{(\wedge R)\;\dfrac{\dfrac{\text{Lemma 2.24(viii)}}{\Vdash\Rightarrow s \in \mathbb{L}_\delta}\qquad\dfrac{\text{Lemma 2.24(viii)}}{\Vdash\Rightarrow t \in \mathbb{L}_\delta}}{\Vdash\Rightarrow (s \in \mathbb{L}_\delta \wedge t \in \mathbb{L}_\delta)}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(s \in z \wedge t \in z)}$$

□

Lemma 2.31 (Union) *If* λ *is a limit ordinal and* $|s| < \lambda$ *then*

$$\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z).$$

Proof Let $\alpha = |s|$, we have the following template for derivations in $\mathbf{IRS}_\Omega$:

$$(b\exists R)\;\dfrac{(b\forall R)_\infty\;\dfrac{(\to R)\;\dfrac{(b\forall R)_\infty\;\dfrac{(\to R)\;\dfrac{\dfrac{\text{Lemma 2.24(viii)}}{\Vdash r \mathrel{\dot\in} s, q \mathrel{\dot\in} r \Rightarrow q \in \mathbb{L}_\alpha \quad \text{for all } |q| < |r| < \alpha}}{\Vdash r \mathrel{\dot\in} s \Rightarrow q \mathrel{\dot\in} r \to q \in \mathbb{L}_\alpha}}{\Vdash r \mathrel{\dot\in} s \Rightarrow (\forall x \in r)(x \in \mathbb{L}_\alpha)}}{\Vdash\Rightarrow r \mathrel{\dot\in} s \to (\forall x \in r)(x \in \mathbb{L}_\alpha)}}{\Vdash\Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{L}_\alpha)}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z)}$$

□

Lemma 2.32 (Δ_0-Collection) *For any* Δ_0 *formula* $F(x, y)$*,*

$$\Vdash\Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y).$$

Proof Using Lemma 2.24 we have

$$\Vdash (\forall x \in s)\exists y F(x, y) \Rightarrow (\forall x \in s)\exists y F(x, y).$$

Now let $\bar{\mathcal{H}} := \mathcal{H}[(\forall x \in s)\exists y F(x, y)]$ and $\alpha := no((\forall x \in s)\exists y F(x, y) \Rightarrow (\forall x \in s)\exists y F(x, y))$, by applying $(\Sigma\text{-Ref}_\Omega)$ we obtain

$$\bar{\mathcal{H}} \,\frac{\;\alpha+1\;}{0}\, (\forall x \in s)\exists y F(x, y) \Rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)\,.$$

Applying ($\to R$) gives

$$\bar{\mathcal{H}} \left|\frac{\alpha+2}{0}\right. \Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y)\,.$$

It remains to note that

$$\begin{aligned}\alpha + 2 = \alpha =& no((\forall x \in s)\exists y F(x, y) \Rightarrow (\forall x \in s)\exists y F(x, y)) + 2\\ &< no(\Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y))\end{aligned}$$

and $\bar{\mathcal{H}} = \mathcal{H}[\Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y)]$ to complete the proof. □

Theorem 2.33 *If* ***IKP*** $\vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ *where* $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ *is an intuitionistic sequent containing exactly the free variables* $\bar{a} := a_1, \dots, a_n$*, then there is an* $m < \omega$ *(which we may compute from the* ***IKP****-derivation) such that*

$$\mathcal{H}[\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})] \left|\frac{\Omega\cdot\omega^m}{\Omega+m}\right. \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

for any ***IRS***$_\Omega$ *terms* $\bar{s} := s_1, \dots, s_n$ *and any operator* $\mathcal{H}$.

Proof Let A be any **IRS**$_\Omega$ formula, note that by Observation 2.10, we have $rk(A) \leq \Omega + l$ for some $l < \omega$. Therefore

$$no(A) = \omega^{rk(A)} \leq \omega^{\Omega+l} = \omega^\Omega \cdot \omega^l = \Omega \cdot \omega^l.$$

Thus for any choice of terms $\bar{s}$ we have

$$no(\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})) \leq \Omega \cdot \omega^m \quad \text{for some } m < \omega.$$

The remainder of the proof is by induction on the derivation **IKP** $\vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$.

If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ is an axiom of **IKP** then the assertion follows by Lemmas 2.26, 2.27, 2.28, 2.29, 2.30, 2.31 or 2.32. If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of a propositional inference then we may apply the induction hypothesis to the premises and then the corresponding inference in **IRS**$_\Omega$. In order to cut down on notation we make the following abbreviation, let

$$\bar{\mathcal{H}} := \mathcal{H}[\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})].$$

Case 1. Suppose that $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of the inference ($b\forall R$), then $\Delta(\bar{s}) = \{(\forall x \in s_i)F(x)\}$. The induction hypothesis furnishes us with an $k < \omega$ such that

$$\bar{\mathcal{H}}[p] \left|\frac{\Omega\cdot\omega^k}{\Omega+k}\right. \Gamma(\bar{s}) \Rightarrow p \in s_i \to F(p) \quad \text{for all } |\, p\,| < |\, s\,|_i.$$

Now by Lemma 2.14(v) we have

$$\bar{\mathcal{H}}[p] \left|\frac{\Omega\cdot\omega^k}{\Omega+k}\right. \Gamma(\bar{s}),\, p \in s_i \Rightarrow F(p).$$

Also by Lemma 2.24(iv) we have

$$\Vdash p \mathrel{\dot{\in}} s_i \Rightarrow p \in s_i .$$

Applying (Cut) to these two yields

$$\bar{\mathcal{H}}[p] \left|\frac{\Omega\cdot\omega^k+1}{\Omega+k}\right. \Gamma(\bar{s}),\, p \mathrel{\dot{\in}} s_i \Rightarrow F(p).$$

Now by $(\rightarrow R)$ we have

$$\bar{\mathcal{H}}[p] \left|\frac{\Omega\cdot\omega^k+2}{\Omega+k}\right. \Gamma(\bar{s}) \Rightarrow p \mathrel{\dot{\in}} s_i \rightarrow F(p)\,.$$

Hence by $(b\forall R)_\infty$ we have

$$\bar{\mathcal{H}} \left|\frac{\Omega\cdot\omega^{k+1}}{\Omega+k}\right. \Gamma(\bar{s}) \Rightarrow (\forall x \in s_i)F(x)$$

as required.

Case 2. Now suppose that $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of the inference $(b\forall L)$. So $(\forall x \in a_i)F(x) \in \Gamma(\bar{a})$ and we are in the following situation in **IKP**

$$(b\forall L)\ \frac{\Gamma(\bar{a}),\, c \in a_i \rightarrow F(c) \Rightarrow \Delta(\bar{a})}{\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})}$$

If c is not a member of $\bar{a}$ then by the induction hypothesis we have an $m < \omega$ such that

$$\bar{\mathcal{H}} \left|\frac{\Omega\cdot\omega^m}{\Omega+m}\right. \Gamma(\bar{s}),\, s_1 \in s_i \rightarrow F(s_1) \Rightarrow \Delta(\bar{s})\,. \tag{1}$$

Now if c is a member of $\bar{a}$, for simplicity let us suppose that $c = a_1$. Inductively we can find an $m < \omega$ such that (1) is also satisfied. First we verify the following claim:

$$\Vdash \Gamma,\, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \rightarrow F(s_1). \tag{2}$$

2.1 Suppose s_i is of the form $\mathbb{L}_\alpha$. The claim is verified by the following template for derivations in $\mathbf{IRS}_\Omega$, here r ranges over terms with $|r| < |s_i|$.

$$\begin{array}{rl}
 & \text{Lemma 2.25} \\ \cline{2-2}
 & \Vdash \Gamma,\, F(r), r \in s_i, r = s_1 \Rightarrow F(s_1) \\ \cline{2-2}
(b\forall L) & \Vdash \Gamma,\, (\forall x \in s_i)F(x), r \in s_i, r = s_1 \Rightarrow F(s_1) \\ \cline{2-2}
\text{Lemma 2.23(ii)} & \Vdash \Gamma,\, (\forall x \in s_i)F(x), r \in s_i \wedge r = s_1 \Rightarrow F(s_1) \\ \cline{2-2}
(\in L)_\infty & \Vdash \Gamma,\, (\forall x \in s_i)F(x), s_1 \in s_i \Rightarrow F(s_1) \\ \cline{2-2}
(\rightarrow R) & \Vdash \Gamma,\, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \rightarrow F(s_1)
\end{array}$$

2.2 Now suppose s_i is of the form $[x \in \mathbb{L}_\alpha \mid B(x)]$, we have the following template for derivations in $\mathbf{IRS}_\Omega$, where r and p range over terms with level below $|\, s_i \,|$.

$$(\to R)\;\dfrac{(\in L)_\infty\;\dfrac{\text{Lemma 2.23(ii)}\;\dfrac{(b\forall L)\;\dfrac{(\to L)\;\dfrac{(\in L)_\infty\;\dfrac{\text{Lemma 2.23(ii)}\;\dfrac{\dfrac{\text{Lemma 2.25}}{\Vdash p \mathrel{\dot\in} s_i, r = p, r = s_i \Rightarrow r \mathrel{\dot\in} s_i}}{\Vdash p \mathrel{\dot\in} s_i \wedge r = p, r = s_i \Rightarrow r \mathrel{\dot\in} s_i}}{\Vdash r \in s_i, r = s_i \Rightarrow r \mathrel{\dot\in} s_i} \qquad \dfrac{\text{Lemma 2.25}}{\Vdash F(r), r \in s_i, r = s_1 \Rightarrow F(s_1)}}{\Vdash \Gamma, r \mathrel{\dot\in} s_i \to F(r), r \in s_i, r = s_1 \Rightarrow F(s_1)}}{\Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i, r = s_1 \Rightarrow F(s_1)}}{\Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i \wedge r = s_1 \Rightarrow F(s_1)}}{\Vdash \Gamma, (\forall x \in s_i)F(x), s_1 \in s_i \Rightarrow F(s_1)}}{\Vdash \Gamma, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)}$$

Now that the claim is verified we may apply (Cut) to (1) and (2) to obtain

$$\bar{\mathcal{H}} \,\Big|\!\frac{\Omega\cdot\omega^{m'}}{\Omega+m'}\; \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

where $\Omega + m' := \max\{\Omega + m, rk(s_1 \in s_i \to F(s_1))\}$, which is the desired result.

All other quantification cases are similar to Cases 1 and 2.

Finally suppose $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of (Cut). So we are in the following situation in **IKP**.

$$\frac{\Gamma(\bar{a}), F(\bar{a}, \bar{c}) \Rightarrow \Delta(\bar{a}) \qquad \Gamma(\bar{a}) \Rightarrow F(\bar{a}, \bar{c})}{\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})}$$

where $\bar{c}$ are the free variables occurring in $F(\bar{a}, \bar{c})$ that are distinct from $\bar{a}$. By the induction hypothesis we can find $m_0, m_1 < \omega$ such that

$$\bar{\mathcal{H}} \,\Big|\!\frac{\Gamma\cdot\omega^{m_0}}{\Omega+m_0}\; \Gamma(\bar{s}), F(\bar{s}, \overline{\mathbb{L}_0}) \Rightarrow \Delta(\bar{s}),$$
$$\bar{\mathcal{H}} \,\Big|\!\frac{\Gamma\cdot\omega^{m_1}}{\Omega+m_1}\; \Gamma(\bar{s}) \Rightarrow F(\bar{s}, \overline{\mathbb{L}_0})\,.$$

Note that $k(F(\bar{s}, \overline{\mathbb{L}_0})) \subseteq \bar{\mathcal{H}}$ so we may apply (Cut) to finish the proof. □

2.6 An Ordinal Analysis of IKP

Lemma 2.34 *If A is a Σ-sentence and **IKP** $\vdash \Rightarrow A$, then there is some $m < \omega$, which we may compute explicitly from the derivation, such that*

$$\mathcal{H}_\gamma \;\Big|\!\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0} \;\Rightarrow\; A \quad \textit{where } \gamma := \omega_m(\Omega\cdot\omega^m).$$

Here $\omega_0(\alpha) := \alpha$ and $\omega_{k+1}(\alpha) := \omega^{\omega_k(\alpha)}$.

Proof Suppose that A is a Σ-sentence and that **IKP** $\vdash \Rightarrow A$, then by Theorem 2.33 there is some $1 \leq m < \omega$ such that

$$\mathcal{H}_0 \ \left|\frac{\Omega \cdot \omega^m}{\Omega + m}\right. \Rightarrow \ A\,. \tag{1}$$

By applying Predicative Cut Elimination Theorem 2.16 $(m-1)$ times we obtain

$$\mathcal{H}_0 \ \left|\frac{\omega_{m-1}(\Omega \cdot \omega^m)}{\Omega+1}\right. \Rightarrow \ A\,. \tag{2}$$

Applying Collapsing Theorem 2.21 to (2) gives

$$\mathcal{H}_\gamma \ \left|\frac{\psi_\Omega(\gamma)}{\psi_\Omega(\gamma)}\right. \Rightarrow \ A \quad \text{where } \gamma := \omega_m(\Omega \cdot \omega^m). \tag{3}$$

Finally by applying Predicative Cut Elimination Theorem 2.16 again we get

$$\mathcal{H}_\gamma \ \left|\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0}\right. \Rightarrow \ A$$

completing the proof. □

Theorem 2.35 *If* $A \equiv \exists x C(x)$ *is a* Σ*-sentence such that* ***IKP*** $\vdash \Rightarrow A$ *then there is an ordinal term* $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$*, which we may compute from the derivation, such that*

$$L_\alpha \models A.$$

Moreover, there is a specific ***IRS***$_\Omega$ *term* s*, with* $|\,s\,| < \alpha$*, which we may compute explicitly from the* ***IKP*** *derivation, such that*

$$L_\alpha \models C(s).$$

Proof Suppose **IKP** $\vdash \Rightarrow A$ for some Σ-sentence A, from Lemma 2.34 we may compute some $1 \leq m < \omega$ such that

$$\mathcal{H}_\gamma \ \left|\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0}\right. \Rightarrow \ A \quad \text{where } \gamma := \omega_m(\Omega \cdot \omega^m).$$

Let $\alpha := \varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))$, applying Boundedness Lemma 2.17 we obtain

$$\mathcal{H}_\gamma \ \left|\frac{\alpha}{0}\right. \Rightarrow \ A^{\mathbb{L}_\alpha}\,. \tag{2}$$

Since the derivation (2) contains no instances of (Cut) or (Σ-Ref$_\Omega$) and the correctness of the remaining rules within L_α is easily verified by induction on the derivation, it may be seen that

$$L_\alpha \models A.$$

For the second part of the theorem note that it must be the case that the final inference in (2) was $(b\exists R)$ and thus by the intuitionistic nature of $\mathbf{IRS}_\Omega$ there must be some s, with $|\, s \,| < \alpha$, such that

$$\mathcal{H}_\gamma \overset{\alpha}{\underset{0}{\vert}} \Rightarrow C(s)^{\mathbb{L}_\alpha}\,. \tag{3}$$

Thus

$$L_\alpha \models C(s). \tag{4}$$

The remainder of the proof is by checking that each part of the embedding and cut elimination of the previous two sections may be carried out effectively, details will appear in [28]. □

Remark 2.36 In fact Theorem 2.35 can be verified within **IKP**, this is not immediately obvious since we do not have access to induction up to $\psi_\Omega(\varepsilon_{\Omega+1})$. However one may observe that in an infinitary proof of the form (3) above, no terms of level higher than α are used. By carrying out the construction of $\mathbf{IRS}_\Omega$ just using ordinals from $B(\omega_{m+1}(\Omega \cdot \omega^m))$ we get a restricted system, but a system still capable of carrying out the embedding and cut elimination necessary for the particular derivation of the sentence A. This can be done inside **IKP** since we do have access to induction up to $\psi_\Omega(\omega_{m+1}(\Omega \cdot \omega^{m+1}))$. It follows that **IKP** has the set existence property for Σ sentences. More details will be found [28].

Finally it is also worth pointing out that we can improve on Theorem 2.33. Instead of just verifying Δ_0-Collection in the infinitary system (Lemma 2.32) we could have shown the embedding result for Σ-Reflection. As result we get a new conservativity result.

Theorem 2.37 *IKP and IKP + Σ-Reflection prove the same Σ-sentences. In particular if IKP $\vdash A$ with A a Σ-sentence, then IKP $\vdash \exists x\, A^x$.*

3 The Case of IKP($\mathcal{P}$)

This section provides a relativised ordinal analysis for intuitionistic power Kripke-Platek set theory **IKP**($\mathcal{P}$). The relativised ordinal analysis for the classical version of the theory, **KP**($\mathcal{P}$), was carried out in [27], the work in this section adapts the techniques from that paper to the intuitionistic case. We begin by defining an infinitary system $\mathbf{IRS}^{\mathcal{P}}_\Omega$, unlike in $\mathbf{IRS}_\Omega$ the terms in $\mathbf{IRS}^{\mathcal{P}}_\Omega$ can contain sub terms of a higher level, or from higher up the Von-Neumann hierarchy in the intended interpretation. This reflects the impredicativity of the power set operation. Next we prove some cut elimination theorems, allowing us to transform infinite derivations of Σ formulae into infinite derivations with only power-bounded cut formulae. The following section provides an embedding of **IKP**($\mathcal{P}$) into $\mathbf{IRS}^{\mathcal{P}}_\Omega$. The final section collates these results into a relativised ordinal analysis of **IKP**($\mathcal{P}$).

3.1 A Sequent Calculus Formulation of IKP($\mathcal{P}$)

Definition 3.1 The formulas of **IKP**($\mathcal{P}$) are the same as those of **IKP** except we also allow *subset bounded quantifiers* of the form

$$(\forall x \subseteq a)A(x) \quad \text{and} \quad (\exists x \subseteq a)A(x).$$

These are treated as quantifiers in their own right, not abbreviations. In contrast, the formula $a \subseteq b$ is still viewed as an abbreviation for the formula $(\forall x \in a)(x \in b)$

Quantifiers $\forall x$, $\exists x$ will still be referred to as unbounded, whereas the other quantifiers (including the subset bounded ones) will be referred to as bounded.

A $\Delta_0^{\mathcal{P}}$-formula of **IKP**($\mathcal{P}$) is one that contains no unbounded quantifiers.

As with **IKP**, the system **IKP**($\mathcal{P}$) derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where at most one formula can occur in Δ.

The axioms of **IKP**($\mathcal{P}$) are the following:

Logical axioms: $\Gamma, A \Rightarrow A$ for every $\Delta_0^{\mathcal{P}}$–formula A.
Extensionality: $\Gamma \Rightarrow a = b \wedge B(a) \rightarrow B(b)$ for every $\Delta_0^{\mathcal{P}}$-formula $B(a)$.
Pair: $\Gamma \Rightarrow \exists x[a \in x \wedge b \in x]$.
Union: $\Gamma \Rightarrow \exists x(\forall y \in a)(\forall z \in y)(z \in x)$.
$\Delta_0^{\mathcal{P}}$ *–Separation*: $\Gamma \Rightarrow \exists y[(\forall x \in y)(x \in a \wedge B(x)) \wedge (\forall x \in a)(B(x) \rightarrow x \in y)]$ for every $\Delta_0^{\mathcal{P}}$-formula $B(a)$.
$\Delta_0^{\mathcal{P}}$ *–Collection*: $\Gamma \Rightarrow (\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for every $\Delta_0^{\mathcal{P}}$-formula $G(a, b)$.
Set Induction: $\Gamma \Rightarrow \forall u\, [(\forall x \in u)\, G(x) \rightarrow G(u)] \rightarrow \forall u\, G(u)$ for every formula $G(b)$.
Infinity: $\Gamma \Rightarrow \exists x\, [(\exists y \in x)\, y \in x \;\wedge\; (\forall y \in x)(\exists z \in x)\, y \in z]$.
Power Set: $\Gamma \Rightarrow \exists z\, (\forall x \subseteq a) x \in z$.

The rules of **IKP**($\mathcal{P}$) are the same as those of **IKP** (extended to the new language containing subset bounded quantifiers), together with the following four rules:

$$(pb\exists L)\ \frac{\Gamma, a \subseteq b \wedge F(a) \Rightarrow \Delta}{\Gamma, (\exists x \subseteq b)F(x) \Rightarrow \Delta} \qquad (pb\exists R)\ \frac{\Gamma \Rightarrow a \subseteq b \wedge F(a)}{\Gamma \Rightarrow (\exists x \subseteq b)F(x)}$$

$$(pb\forall L)\ \frac{\Gamma, a \subseteq b \rightarrow F(a) \Rightarrow \Delta}{\Gamma, (\forall x \subseteq b)F(x) \Rightarrow \Delta} \qquad (pb\forall R)\ \frac{\Gamma \Rightarrow a \subseteq b \rightarrow F(a)}{\Gamma \Rightarrow (\forall x \subseteq b)F(x)}$$

As usual it is forbidden for the variable a to occur in the conclusion of the rules $(pb\exists L)$ and $(pb\forall R)$, such a variable is referred to as the eigenvariable of the inference.

3.2 The Infinitary System $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$

The purpose of this section is to introduce an infinitary proof system $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$. As before all ordinals will be assumed to be members of $B^{\Omega}(\varepsilon_{\Omega+1})$.

Definition 3.2 We define the $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms. To each $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term t we also assign its ordinal level, $|\,t\,|$.

1. For each $\alpha < \Omega$, $\mathbb{V}_\alpha$ is an $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term with $|\,\mathbb{V}_\alpha\,| = \alpha$.
2. For each $\alpha < \Omega$, we have infinitely many free variables $a_0^\alpha, a_1^\alpha, a_2^\alpha, \ldots$, with $|\,a_i^\alpha\,| = \alpha$.
3. If $F(x, \bar{y})$ is a $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{IKP}(\mathcal{P})$ (whose free variables are exactly those indicated) and $\bar{s} \equiv s_1, \ldots, s_n$ are $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms, then the formal expression $[x \in \mathbb{V}_\alpha \mid F(x, \bar{s})]$ is an $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term with $|\,[x \in \mathbb{V}_\alpha \mid F(x, \bar{s})]\,| := \alpha$.

The $\mathbf{IRS}_\Omega^{\mathcal{P}}$ formulae are of the form $A(s_1, \ldots, s_n)$, where $A(a_1, \ldots, a_n)$ is a formula of $\mathbf{IKP}(\mathcal{P})$ with all free variables indicated and $s_1, \ldots, s_n$ are $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms.

A formula $A(s_1, \ldots, s_n)$ of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ is $\Delta_0^{\mathcal{P}}$ if $A(a_1, \ldots, a_n)$ is a $\Delta_0^{\mathcal{P}}$ formula of $\mathbf{IKP}(\mathcal{P})$.

The $\Sigma^{\mathcal{P}}$ formulae of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are the smallest collection containing the $\Delta_0^{\mathcal{P}}$-formulae and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$, $(\forall x \subseteq s)A$, $(\exists x \subseteq s)A$, $\exists x A$, $\neg C$ and $C \to A$ whenever it contains A and B and C is a $\Pi^{\mathcal{P}}$-formula. The $\Pi^{\mathcal{P}}$-formulae are the smallest collection containing the $\Delta_0^{\mathcal{P}}$ formulae and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$, $(\forall x \subseteq s)A$, $(\exists x \subseteq s)A$, $\forall x A$, $\neg D$ and $D \to A$ whenever it contains A and B and D is a $\Sigma^{\mathcal{P}}$-formula.

The *axioms* of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are:

(A1) $\Gamma, A \Rightarrow A$ for A in $\Delta_0^{\mathcal{P}}$.
(A2) $\Gamma \Rightarrow t = t$.
(A3) $\Gamma, s_1 = t_1, \ldots, s_n = t_n, A(s_1, \ldots, s_n) \Rightarrow A(t_1, \ldots, t_n)$ for $A(s_1, \ldots, s_n)$ in $\Delta_0^{\mathcal{P}}$.
(A4) $\Gamma \Rightarrow s \in \mathbb{V}_\alpha$ if $|\,s\,| < \alpha$.
(A5) $\Gamma \Rightarrow s \subseteq \mathbb{V}_\alpha$ if $|\,s\,| \leq \alpha$.
(A6) $\Gamma, t \in [x \in \mathbb{V}_\alpha \mid F(x, \bar{s})] \Rightarrow F(t, \bar{s})$ for $F(t, \bar{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\,t\,| < \alpha$.
(A7) $\Gamma, F(t, \bar{s}) \Rightarrow t \in [x \in \mathbb{V}_\alpha \mid F(x, \bar{s})]$ for $F(t, \bar{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\,t\,| < \alpha$.

The *inference rules* of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are:

$$(b\forall L) \quad \frac{\Gamma,\ s \in t \to F(s) \Rightarrow \Delta}{\Gamma,\ (\forall x \in t)F(x) \ \Rightarrow \Delta} \ \text{ if } |\,s\,| < |\,t\,|$$

$$(b\forall R)_\infty \quad \frac{\Gamma \Rightarrow s \in t \to F(s) \ \text{ for all } |\,s\,| < |\,t\,|}{\Gamma \Rightarrow (\forall x \in t)F(x)}$$

$$(b\exists L)_\infty \quad \frac{\Gamma,\ s \in t \wedge F(s) \Rightarrow \Delta \ \text{ for all } |\,s\,| < |\,t\,|}{\Gamma,\ (\exists x \in t)F(x) \ \Rightarrow \Delta}$$

$$(b\exists R) \quad \frac{\Gamma \Rightarrow s \in t \wedge F(s)}{\Gamma \Rightarrow (\exists x \in t)F(x)} \ \text{ if } |\,s\,| < |\,t\,|$$

$$(pb\forall L) \quad \frac{\Gamma, s \subseteq t \to F(s) \Rightarrow \Delta}{\Gamma, (\forall x \subseteq t)F(x) \ \Rightarrow \Delta} \ \text{ if } |\,s\,| \leq |\,t\,|$$

$$(pb\forall R)_\infty \quad \frac{\Gamma \Rightarrow s \subseteq t \to F(s) \ \text{ for all } |\,s\,| \leq |\,t\,|}{\Gamma \Rightarrow (\forall x \subseteq t)F(x)}$$

$$(pb\exists L)_\infty \quad \frac{\Gamma,\ s \subseteq t \wedge F(s) \Rightarrow \Delta \ \text{ for all } |\,s\,| \leq |\,t\,|}{\Gamma, (\exists x \subseteq t)F(x) \ \Rightarrow \Delta}$$

$$(pb\exists R)\quad \frac{\Gamma \Rightarrow s \subseteq t \wedge F(s)}{\Gamma \Rightarrow (\exists x \subseteq t)F(x)} \quad \text{if } |\, s\,| \leq |\, t\,|$$

$$(\forall L)\quad \frac{\Gamma, F(s) \Rightarrow \Delta}{\Gamma,\ \forall x\, F(x)\ \Rightarrow \Delta}$$

$$(\forall R)_{\infty}\quad \frac{\Gamma \Rightarrow F(s)\ \text{ for all } s}{\Gamma \Rightarrow \forall x\, F(x)}$$

$$(\exists L)_{\infty}\quad \frac{\Gamma,\ F(s) \Rightarrow \Delta\ \text{ for all } s}{\Gamma,\ \exists x\, F(x)\ \Rightarrow \Delta}$$

$$(\exists R)\quad \frac{\Gamma \Rightarrow F(s)}{\Gamma \Rightarrow \exists x\, F(x)}$$

$$(\in L)_{\infty}\quad \frac{\Gamma, r \in t \wedge r = s \Rightarrow \Delta\ \text{ for all } |\, r\,| < |\, t\,|}{\Gamma, s \in t\ \Rightarrow \Delta}$$

$$(\in R)\quad \frac{\Gamma \Rightarrow r \in t\ \wedge\ r = s}{\Gamma, s \in t} \quad \text{if } |\, r\,| < |\, t\,|$$

$$(\subseteq L)_{\infty}\quad \frac{\Gamma,\ r \subseteq t \wedge r = s \Rightarrow \Delta\ \text{ for all } |\, r\,| \leq |\, t\,|}{\Gamma,\ s \subseteq t\ \Rightarrow \Delta}$$

$$(\subseteq R)\quad \frac{\Gamma \Rightarrow r \subseteq t\ \wedge\ r = s}{\Gamma \Rightarrow s \subseteq t} \quad \text{if } |\, r\,| \leq |\, s\,|$$

$$(\text{Cut})\quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow A}{\Gamma\ \Rightarrow \Delta}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref)\quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow\ \exists z\, A^{z}} \quad \text{if } A \text{ is a } \Sigma^{\mathcal{P}}\text{-formula,}$$

as well as the rules $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\rightarrow L)$, $(\rightarrow R)$ from **IRS**$_{\Omega}$. As usual A^{z} results from A by restricting all unbounded quantifiers to z.

Definition 3.3 The *rank* of a formula is determined as follows.

1. $rk(s \in t) := max\{|\, s\,| + 1, |\, t\,| + 1\}$.
2. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := max\{|\, t\,|, rk(F(\mathbb{V}_0)) + 2\}$.
3. $rk((\exists x \subseteq t)F(x)) := rk((\forall x \subseteq t)F(x)) := max\{|\, t\,| + 1, rk(F(\mathbb{V}_0)) + 2\}$.
4. $rk(\exists x\, F(x)) := rk(\forall x\, F(x)) := max\{\Omega, rk(F(\mathbb{V}_0)) + 2\}$.
5. $rk(A \wedge B) := rk(A \vee B) := rk(A \rightarrow B) := max\{rk(A), rk(B)\} + 1$.
6. $rk(\neg A) := rk(A) + 1$.

Note that the definition of rank for **IRS**$_{\Omega}^{\mathcal{P}}$ formulae is much less complex than for **IRS**$_{\Omega}$, this is because we are only aiming for partial cut-elimination for this system. In general it will not be possible to remove cuts with $\Delta_0^{\mathcal{P}}$ cut formulae. Note however that we still have $rk(A) < \Omega$ if and only if A is $\Delta_0^{\mathcal{P}}$.

We also have the following useful lemma.

Lemma 3.4 *If A is a formula of* **IRS**$_{\Omega}^{\mathcal{P}}$ *with $rk(A) \geq \Omega$ (i.e. A contains unbounded quantifiers), and A was the result of an* **IRS**$_{\Omega}^{\mathcal{P}}$ *inference other than $(\Sigma^{\mathcal{P}}\text{-}Ref)$ and (Cut) then the rank of the minor formulae of that inference is strictly less than $rk(A)$.*

Definition 3.5 (*Operator controlled derivability for* $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$) If $A(s_1, \dots, s_n)$ is a formula of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ then let

$$|\, A(s_1, \dots, s_n)\,| := \{|\, s_1\,|, \dots, |\, s_n\,|\}.$$

Likewise if $\Gamma \Rightarrow \Delta$ is an intuitionistic sequent of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ containing formulas $A_1, \dots, A_n$, we define

$$|\, \Gamma \Rightarrow \Delta\,| := |\, A_1\,| \cup \dots \cup |\, A_n\,|.$$

Definition 3.6 Let $\mathcal{H}$ be an operator and $\Gamma \Rightarrow \Delta$ an intuitionistic sequent of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ formulae. We define the relation $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta$ by recursion on α.

If $\Gamma \Rightarrow \Delta$ is an *axiom* and $|\, \Gamma \Rightarrow \Delta\,| \cup \{\alpha\} \subseteq \mathcal{H}$, then $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta$.

We require always that $|\, \Gamma \Rightarrow \Delta\,| \cup \{\alpha\} \subseteq \mathcal{H}$ where $\Gamma \Rightarrow \Delta$ is the sequent in the conclusion, this condition will not be repeated in the inductive clauses pertaining to the inference rules of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ given below. The column on the right gives the ordinal requirements for each of the inference rules.

$$(\in L)_\infty \qquad \frac{\mathcal{H}[r] \vdash^{\alpha_r}_{\rho} \Gamma, r \in t \wedge r = s \Rightarrow \Delta \text{ for all } |\, r\,| < |\, t\,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, s \in t \Rightarrow \Delta} \qquad |\, r\,| \le \alpha_r < \alpha$$

$$(\in R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \ \Rightarrow r \in t \wedge r = s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow s \in t} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, r\,| < |\, t\,| \\ |\, r\,| < \alpha \end{array}$$

$$(\subseteq L)_\infty \qquad \frac{\mathcal{H}[r] \vdash^{\alpha_r}_{\rho} \Gamma, r \subseteq t \wedge r = s \Rightarrow \Delta \text{ for all } |\, r\,| \le |\, t\,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, s \subseteq t \Rightarrow \Delta} \qquad |\, r\,| \le \alpha_r < \alpha$$

$$(\subseteq R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \ \Rightarrow r \subseteq t \wedge r = s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow s \subseteq t} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, r\,| \le |\, t\,| \\ |\, r\,| < \alpha \end{array}$$

$$(b\forall L) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, s \in t \rightarrow A(s) \Rightarrow \Delta}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in t) A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s\,| < |\, t\,| \\ |\, s\,| < \alpha \end{array}$$

$$(b\forall R)_\infty \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma \Rightarrow s \in t \rightarrow F(s) \text{ for all } |\, s\,| < |\, t\,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in t) F(x)} \qquad |\, s\,| \le \alpha_s < \alpha$$

$$(b\exists L)_\infty \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, s \in t \wedge F(s) \Rightarrow \Delta \text{ for all } |\, s\,| < |\, t\,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in t) F(x) \Rightarrow \Delta} \qquad |\, s\,| \le \alpha_s < \alpha$$

$$(b\exists R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \in t \wedge A(s)}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| < |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(pb\forall L) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, s \subseteq t \rightarrow A(s) \Rightarrow \Delta}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \subseteq t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| \leq |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(pb\forall R)_{\infty} \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma \Rightarrow s \subseteq t \rightarrow F(s) \text{ for all } |\, s \,| \leq |\, t \,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \subseteq t)F(x)} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(pb\exists L)_{\infty} \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, s \subseteq t \wedge F(s) \Rightarrow \Delta \text{ for all } |\, s \,| \leq |\, t \,|}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \subseteq t)F(x) \Rightarrow \Delta} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(pb\exists R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \subseteq t \wedge A(s)}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \subseteq t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| \leq |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(\forall L) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, F(s) \Rightarrow \Delta}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\, s \,| < \alpha \end{array}$$

$$(\forall R)_{\infty} \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma \Rightarrow F(s) \text{ for all } s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \forall x F(x)} \qquad |\, s \,| < \alpha_s + 1 < \alpha$$

$$(\exists L)_{\infty} \qquad \frac{\mathcal{H}[s] \vdash^{\alpha_s}_{\rho} \Gamma, F(s) \Rightarrow \Delta \text{ for all } s}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \exists x F(x) \Rightarrow \Delta} \qquad |\, s \,| < \alpha_s + 1 < \alpha$$

$$(\exists R) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow F(s)}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \Rightarrow \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\, s \,| < \alpha \end{array}$$

$$(\text{Cut}) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, B \Rightarrow \Delta \qquad \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow B}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow A}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \exists z\, A^{z}} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma^{\mathcal{P}}\text{-formula} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\rightarrow L)$ or $(\rightarrow R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \mid\frac{\alpha_0}{\rho} \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

3.3 Cut Elimination for $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$

Lemma 3.7 (Weakening and Persistence for $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$)

(i) If $\Gamma_0 \subseteq \Gamma$, $|\,\Gamma\,| \subseteq \mathcal{H}$, $\alpha_0 \leq \alpha \in \mathcal{H}$, $\rho_0 \leq \rho$ *and* $\mathcal{H} \mid\frac{\alpha_0}{\rho_0} \Gamma_0 \Rightarrow \Delta$ *then*

$$\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow \Delta .$$

(ii) If $\gamma \in \mathcal{H}$ *and* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, \exists x A(x) \Rightarrow \Delta$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, (\exists x \in \mathbb{V}_{\gamma})A(x) \Rightarrow \Delta$.

(iii) If $\gamma \in \mathcal{H}$ *and* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow \forall x A(x)$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow (\forall x \in \mathbb{V}_{\gamma})A(x)$.

Proof All proofs are by induction on α. We show (ii), suppose $\gamma \in \mathcal{H}$ and $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, \exists x A(x) \Rightarrow \Delta$. The interesting case is where $\exists x A(x)$ was the principal formula of the last inference which was $(\exists L)_{\infty}$, in this case we have $\mathcal{H}[s] \mid\frac{\alpha_s}{\rho} \Gamma, \exists x A(x), A(s) \Rightarrow \Delta$ for each term s with $|\,s\,| < \alpha_s + 1 < \alpha$ (If $\exists x A(x)$ was not a side formula we can use part (i) to make it one). By the induction hypothesis we obtain $\mathcal{H}[s] \mid\frac{\alpha_s}{\rho} \Gamma, (\exists x \in \mathbb{V}_{\gamma})A(x), A(s) \Rightarrow \Delta$ for all $|\,s\,| < \gamma$. By $(\wedge L)$ we get

$$\mathcal{H}[s] \mid\frac{\alpha_s+1}{\rho} \Gamma, (\exists x \in \mathbb{V}_{\gamma})A(x), s \in \mathbb{V}_{\gamma} \wedge A(s) \Rightarrow \Delta .$$

Hence we may apply $(b\exists L)_{\infty}$ to obtain $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, (\exists x \in \mathbb{V}_{\gamma})A(x) \Rightarrow \Delta$ as required. □

Lemma 3.8 (Inversions of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$)

(i) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \wedge B \Rightarrow \Delta$ *and* $rk(A \wedge B) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A, B \Rightarrow \Delta$.

(ii) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow A \wedge B$ *and* $rk(A \wedge B) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow A$ *and* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow B$.

(iii) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \vee B \Rightarrow \Delta$ *and* $rk(A \vee B) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \Rightarrow \Delta$ *and* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, B \Rightarrow \Delta$.

(iv) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \rightarrow B \Rightarrow \Delta$ *and* $rk(A \rightarrow B) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, B \Rightarrow \Delta$.

(v) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow A \rightarrow B$ *and* $rk(A \rightarrow B) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \Rightarrow B$.

(vi) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow \neg A$ *and* $rk(A) \geq \Omega$ *then* $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A \Rightarrow$.

(vii) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ *and* $rk(A(\mathbb{V}_0)) \geq \Omega$ *then* $\mathcal{H}[s] \mid\frac{\alpha}{\rho} \Gamma, s \in t \wedge A(s) \Rightarrow \Delta$ *for all* $|s| < |t|$.

(viii) If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma \Rightarrow (\forall x \in t)A(x)$ *and* $rk(A(\mathbb{V}_0)) \geq \Omega$ *then* $\mathcal{H}[s] \mid\frac{\alpha}{\rho} \Gamma \Rightarrow s \in t \rightarrow A(s)$ *for all* $|s| < |t|$.

(ix) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ *and* $rk(A(\mathbb{V}_0)) \geq \Omega$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \subseteq t \wedge A(s) \Rightarrow \Delta$ *for all* $|s| \leq |t|$.

(x) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \subseteq t)A(x)$ *and* $rk(A(\mathbb{V}_0)) \geq \Omega$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \subseteq t \rightarrow A(s)$ *for all* $|s| \leq |t|$.

(xi) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x A(x) \Rightarrow \Delta$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F(s) \Rightarrow \Delta$ *for all* s.

(xii) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \Rightarrow \forall x A(x)$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow F(s)$ *for all* s.

Proof The proof is by induction on α and many parts are standard for many intuitionistic systems of a similar nature. We show (viii) and (ix).

(viii) Suppose that $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in t)A(x)$ and $rk(A(\mathbb{V}_0)) \geq \Omega$. Since A must contain an unbounded quantifier, the sequent $\Gamma \Rightarrow (\forall x \in t)A(x)$ cannot be an axiom. If the last inference was not $(b\forall R)_\infty$ then we may apply the induction hypothesis to the premises of that inference, and then the same inference again. Finally suppose the last inference was $(b\forall R)_\infty$ so we have

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t \rightarrow A(s) \quad \text{for all } |s| < |t|, \text{ with } \alpha_s < \alpha.$$

Applying weakening completes the proof of this case.

(ix) Suppose that $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{V}_0)) \geq \Omega$. Since $A(x)$ contains an unbounded quantifier $\exists x \subseteq t)A(x)$ cannot be the *active part* of an axiom, thus if $\Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ is an axiom then so is $\Gamma, s \subseteq t \wedge A(x) \Rightarrow \Delta$ for any $|s| \leq |t|$. As in (viii) the remaining interesting case is where $(\exists x \subseteq t)A(x)$ was the principal formula of the last inference, which was $(pb\exists L)_\infty$. In this case we have

$$\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, (\exists x \subseteq t)A(x), s \subseteq t \wedge A(s) \Rightarrow \Delta \quad \text{for all } |s| \leq |t| \text{ with } \alpha_s < \alpha.$$

Now applying the induction hypothesis yields $\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, s \subseteq t \wedge A(s) \Rightarrow \Delta$, to which we may apply weakening to complete the proof of this case. □

Lemma 3.9 (Reduction) *If* $rk(C) := \rho > \Omega$, $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta$ *and* $\mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C$ *then*

$$\mathcal{H} \overset{\alpha\#\alpha\#\beta\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Xi \Rightarrow \Delta.$$

Proof The proof is by induction on $\alpha\#\alpha\#\beta\#\beta$. The interesting case is where C was the principal formula of both final inferences, notice that in this case the last inference cannot have been $(\Sigma^{\mathcal{P}}\text{-}Ref)$ since $rk(C) > \Omega$ and the conclusion of an application of $(\Sigma^{\mathcal{P}}\text{-}Ref)$ always has rank Ω. Thus the rest of the proof follows in the usual way by the symmetry of the rules and Lemmas 3.4 and 3.8, we treat the case where $C \equiv (\forall x \subseteq t)A(x)$ and C was the principal formula of both last inferences, so we have

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta. \tag{1}$$

$$\mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C. \tag{2}$$

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, C, s \subseteq t \to A(s) \Rightarrow \Delta \qquad \text{with } \alpha_0, |\, s \,| < \alpha \text{ and } |\, s \,| \leq |\, t \,|. \tag{3}$$

$$\mathcal{H}[p] \left|\frac{\beta_p}{\rho}\right. \Xi \Rightarrow p \subseteq t \to A(p) \qquad \text{for all } |\, p \,| \leq |\, t \,| \text{ with } |\, p \,| \leq \alpha_p < \alpha. \tag{4}$$

From (3) we know that $s \in \mathcal{H}$, so from (4) we get

$$\mathcal{H} \left|\frac{\beta_s}{\rho}\right. \Xi \Rightarrow s \subseteq t \to A(s)\,. \tag{5}$$

Applying the induction hypothesis to (2) and (3) yields

$$\mathcal{H} \left|\frac{\alpha_0 \# \alpha_0 \# \beta \# \beta}{\rho}\right. \Gamma, s \subseteq t \to A(s) \Rightarrow \Delta\,. \tag{6}$$

Finally by applying (Cut) to (5) and (6), whilst noting that by Lemma 3.4 $rk(s \subseteq t \to A(s)) < \rho$, we obtain

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta \# \beta}{\rho}\right. \Gamma, \Xi \Rightarrow \Delta$$

as required. □

Lemma 3.10 *If* $\mathcal{H} \left|\frac{\alpha}{\Omega+n+1}\right. \Gamma \Rightarrow \Delta$ *then* $\mathcal{H} \left|\frac{\omega^\alpha}{\Omega+n}\right. \Gamma \Rightarrow \Delta$ *for any* $n < \omega$.

Proof The proof is by induction on α, suppose $\mathcal{H} \left|\frac{\alpha}{\Omega+n+1}\right. \Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is an axiom there is nothing to show. If $\Gamma \Rightarrow \Delta$ was the result of an inference other that (Cut) or a cut with cut-rank $< \Omega + n$ then we may apply the induction hypothesis to the premises of that inference and then the same inference again. So suppose the last inference was (Cut) with cut-formula C, and that $rk(C) = \Omega + n$. So we have

$$\mathcal{H} \left|\frac{\alpha_0}{\Omega+n+1}\right. \Gamma, C \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha. \tag{1}$$

$$\mathcal{H} \left|\frac{\alpha_1}{\Omega+n+1}\right. \Gamma \Rightarrow C \quad \text{with } \alpha_1 < \alpha. \tag{2}$$

Applying the induction hypothesis to (1) and (2) gives

$$\mathcal{H} \left|\frac{\omega^{\alpha_0}}{\Omega+n}\right. \Gamma, C \Rightarrow \Delta. \tag{3}$$

$$\mathcal{H} \left|\frac{\omega^{\alpha_1}}{\Omega+n}\right. \Gamma \Rightarrow C. \tag{4}$$

Now applying the Reduction Lemma 3.9 to (3) and (4) provides us with

$$\mathcal{H} \left|\frac{\omega^{\alpha_0} \# \omega^{\alpha_0} \# \omega^{\alpha_1} \# \omega^{\alpha_1}}{\Omega+n}\right. .$$

It remains to note that $\omega^{\alpha_0} \# \omega^{\alpha_0} \# \omega^{\alpha_1} \# \omega^{\alpha_1} < \omega^\alpha$ since ω^α is additive principal, so we can complete the proof by weakening. □

Theorem 3.11 (Partial cut elimination for $\mathbf{IRS}_\Omega^{\mathcal{P}}$) *If* $\mathcal{H} \left|\frac{\alpha}{\Omega+n+1}\right. \Gamma \Rightarrow \Delta$ *then* $\mathcal{H} \left|\frac{\omega_n(\alpha)}{\Omega+1}\right. \Gamma \Rightarrow \Delta$ *where* $\omega_0(\beta) := \beta$ *and* $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

Proof The proof uses an easy induction on n and the previous Lemma. □

Note that Theorem 3.11 is much weaker than the full predicative cut elimination result we obtained for $\mathbf{IRS}_\Omega$ (Theorem 2.16), this is because in general we cannot eliminate cuts with $\Delta_0^{\mathcal{P}}$ cut-formulae from $\mathbf{IRS}_\Omega^{\mathcal{P}}$ derivations.

Lemma 3.12 (Boundedness) *If A is a $\Sigma^{\mathcal{P}}$-formula, B is a $\Pi^{\mathcal{P}}$-formula, $\alpha \leq \beta < \Omega$ and $\beta \in \mathcal{H}$ then*

(i) If $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow A$ then $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow A^{\mathbb{V}_\beta}$.
(ii) If $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B \Rightarrow \Delta$ then $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow \Delta$.

Proof The proofs are by induction on α, we show (ii), the proof of (i) is similar. As with Lemma 2.17 the only interesting case is where B was the principal formula of the last inference and B is of the form $\forall x C(x)$. So we have

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B, C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Using the induction hypothesis we obtain

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta}, C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Now since $\Gamma, B^{\mathbb{V}_\beta} \Rightarrow s \in \mathbb{V}_\beta$ is an axiom, we have $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow s \in \mathbb{V}_\beta$, so by $(\rightarrow L)$ we obtain

$$\mathcal{H} \left|\frac{\alpha_0+1}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta}, s \in \mathbb{V}_\beta \rightarrow C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Finally an application of $(b\forall L)$ yields

$$\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow \Delta$$

as required. □

Theorem 3.13 (Collapsing) *Suppose that $\eta \in \mathcal{H}_\eta$, Δ is a set of at most one $\Sigma^{\mathcal{P}}$-formula and Γ a set of $\Pi^{\mathcal{P}}$-formulae. Then*

$$\mathcal{H}_\eta \left|\frac{\alpha}{\Omega+1}\right. \Gamma \Rightarrow \Delta \quad \textit{implies} \quad \mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow \Delta.$$

Here $\hat{\beta} = \eta + \omega^{\Omega+\beta}$ and the operators $\mathcal{H}_\xi$ are those defined in Definition 2.18.

Proof Note first that from $\eta \in \mathcal{H}_\eta$ and Lemma 2.20 we obtain

$$\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}. \tag{1}$$

The proof is by induction on α.

Case 0. If $\Gamma \Rightarrow \Delta$ is an axiom then the result follows immediately from (1).

Case 1. If the last inference was propositional then the assertion follows easily by applying the induction hypothesis and then the same inference again.

Case 2. Suppose the last inference was $(pb\forall R)_\infty$, then $\Delta = \{(\forall x \subseteq t) F(x)\}$ and

$$\mathcal{H}_\eta[p] \mathrel{\Big|\!\!\frac{\alpha_p}{\Omega+1}} \Gamma \Rightarrow p \subseteq t \rightarrow F(p) \quad \text{for all } |\, p \,| \leq |\, t \,| \text{ with } \alpha_p < \alpha.$$

Since $|\, t \,| \in \mathcal{H}_\eta(\emptyset) = B^\Omega(\eta + 1)$ and $|\, t \,| < \Omega$, we have $|\, t \,| < \psi_\Omega(\eta + 1)$, thus $|\, p \,| \in \mathcal{H}_\eta$ for all $|\, p \,| \leq |\, t \,|$. So we have

$$\mathcal{H}_\eta \mathrel{\Big|\!\!\frac{\alpha_p}{\Omega+1}} \Gamma \Rightarrow p \subseteq t \rightarrow F(p)\,.$$

Since $p \subseteq t \rightarrow F(p)$ is also in $\Sigma^{\mathcal{P}}$ we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_p}} \mathrel{\Big|\!\!\frac{\psi_\Omega(\hat{\alpha_p})}{\psi_\Omega(\hat{\alpha_p})}} \Gamma \Rightarrow p \subseteq t \rightarrow F(p) \quad \text{for all } |\, p \,| \leq |\, t \,| \text{ with } \alpha_p < \alpha.$$

Now noting that $\psi_\Omega(\hat{\alpha_p}) + 1 < \psi_\Omega(\hat{\alpha})$, by applying $(pb\forall R)_\infty$ we obtain the desired result. The cases where the last inference was $(b\forall R)_\infty$, $(pb\exists L)_\infty$, $(b\exists L)_\infty$, $(\in L)_\infty$ or $(\subseteq L)_\infty$ are similar.

Case 3. Now suppose the last inference was $(pb\forall L)$, so $(\forall x \subseteq t) F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \mathrel{\Big|\!\!\frac{\alpha_0}{\Omega+1}} \Gamma, s \subseteq t \rightarrow F(s) \Rightarrow \Delta \quad \text{for some } |\, s \,| \leq |\, t \,| \text{ with } \alpha_0 < \alpha.$$

Noting that $s \subseteq t \rightarrow F(s)$ is in $\Pi^{\mathcal{P}}$, too, we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \mathrel{\Big|\!\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}} \Gamma, s \subseteq t \rightarrow F(s) \Rightarrow \Delta$$

to which we may apply $(pb\forall L)$ to complete this case. The cases where the last inference was $(b\forall L)$, $(pb\exists R)$, $(b\exists R)$, $(\in R)$ or $(\subseteq R)$ are similar.

Case 4. Now suppose the last inference was $(\forall L)$, so $\forall x A(x) \in \Gamma$ and

$$\mathcal{H}_\eta \mathrel{\Big|\!\!\frac{\alpha_0}{\Omega+1}} \Gamma, F(s) \Rightarrow \Delta \quad \text{for some } |\, s \,| < \alpha \text{ and } \alpha_0 < \alpha.$$

Since $F(s)$ is $\Pi^{\mathcal{P}}$ we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \mathrel{\Big|\!\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}} \Gamma, F(s) \Rightarrow \Delta\,.$$

Now since $|\, s \,| \in \mathcal{H}_\eta = B^\Omega(\eta + 1)$ we have $|\, s \,| < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$. So we may apply $(\forall L)$ to complete the case. The case where the last inference was $(\exists R)$ is similar.

The rest of the proof is completely analogous to that of Theorem 2.21, using boundedness for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ Lemma 3.12 instead of for $\mathbf{IRS}_\Omega$. $\square$

3.4 Embedding IKP($\mathcal{P}$) into $\mathbf{IRS}_\Omega^{\mathcal{P}}$

Definition 3.14 As in the embedding section for the case of **IKP**, $\Vdash \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \; \vert\!\frac{no(\Gamma \Rightarrow \Delta)}{0} \; \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

Also $\Vdash_\rho^\xi \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \; \vert\!\frac{no(\Gamma \Rightarrow \Delta)\,\#\,\xi}{\rho} \; \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

Only this time we are referring to operator controlled derivability in $\mathbf{IRS}_\Omega^{\mathcal{P}}$.

Lemma 3.15 *For any formula A*

$$\Vdash A \Rightarrow A.$$

Proof We proceed by induction on the complexity of A. If A is $\Delta_0^{\mathcal{P}}$ then this is axiom (A1) of $\mathbf{IRS}_\Omega^{\mathcal{P}}$.

Suppose A is of the form $\exists x F(x)$. Let $\alpha_s = |s| + no(F(s) \Rightarrow F(s))$ and $\alpha = no(A \Rightarrow A)$, note that $|s| < \alpha_s + 1 < \alpha_s + 2 < \alpha$ for all s. By the induction hypothesis we have

$$\mathcal{H}[F(s), s] \; \vert\!\frac{\alpha_s}{0} \; F(s) \Rightarrow F(s) \quad \text{for all terms } s \text{ and for an arbitrary operator } \mathcal{H}.$$

Now using $(\exists R)$ we get

$$\mathcal{H}[F(s), s] \; \vert\!\frac{\alpha_s + 1}{0} \; F(s) \Rightarrow \exists x F(x).$$

Finally since $\mathcal{H}[F(s), s](\emptyset) \subseteq \mathcal{H}[\exists x F(x)][s](\emptyset)$ we may apply $(\exists L)_\infty$ to obtain the desired result. The other cases are similar. $\square$

Lemma 3.16 (Extensionality) *For any formula A and any terms* $s_1, \ldots, s_n, t_1, \ldots, t_n$

$$\Vdash s_1 = t_1, \ldots, s_n = t_n, A(s_1, \ldots, s_n) \Rightarrow A(t_1, \ldots, t_n).$$

Proof If A is $\Delta_0^{\mathcal{P}}$ then this is an axiom. The remainder of the proof is by induction on $rk(A(s_1, \ldots, s_n))$, note that $rk(A(s_1, \ldots, s_n) = rk(A(t_1, \ldots, t_n)$ since A is not $\Delta_0^{\mathcal{P}}$.

Case 1. Suppose $A(s_1, \ldots, s_n) \equiv \exists x B(x, s_1, \ldots, s_n)$, we know that $rk(B(r, s_1, \ldots, s_n)) < rk(A(s_1, \ldots, s_n))$ for all r by Lemma 3.4, so by induction hypothesis we have

$$\Vdash s_1 = t_1, \ldots, s_n = t_n, B(r, s_1, \ldots, s_n) \Rightarrow B(r, t_1, \ldots, t_n) \quad \text{for all terms } r.$$

Now successively applying $(\exists R)$ and then $(\exists L)_\infty$ yields the desired result.

Case 2. Now suppose $A(s_1, \ldots, s_n) \equiv (\exists x \subseteq s_i)B(x, s_1, \ldots, s_n)$. Since A is not $\Delta_0^{\mathcal{P}}$, B must contain an unbounded quantifier, and thus by Lemma 3.4 $\Omega \leq rk(r \subseteq s_i \wedge B(r, s_1, \ldots, s_n)) < rk(A(s_1, \ldots, s_n))$ for any $|\, r\,| \leq |\, s_i\,|$, thus by induction hypothesis we have

$$\Vdash s_1 =t_1, \ldots, s_n = t_n, r \subseteq s_i \wedge B(r, s_1, \ldots, s_n) \Rightarrow r \subseteq t_i \wedge B(r, t_1, \ldots, t_n)$$
$$\text{for all } |\, r\,| \leq |\, s_i\,|.$$

Thus successively applying $(pb\exists R)$ and then $(pb\exists L)_\infty$ yields the desired result. The other cases are similar. □

Lemma 3.17 ($\Delta_0^{\mathcal{P}}$-Collection) *For any* $\Delta_0^{\mathcal{P}}$ *formula* F

$$\Vdash\Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y).$$

Proof Lemma 3.15 provides us with

$$\Vdash (\forall x \in s)\exists y F(x, y) \Rightarrow (\forall x \in s)\exists y F(x, y).$$

Noting that $(\forall x \in s)\exists y F(x, y)$ is a $\Sigma^{\mathcal{P}}$ formula and that $rk((\forall x \in s)\exists y F(x, y)) = \omega^{\Omega+2}$ we may apply $(\Sigma^{\mathcal{P}}\text{-}Ref)$ to obtain

$$\bar{\mathcal{H}} \left|\frac{\omega^{\Omega+2}\cdot 2+2}{0}\right. (\forall x \in s)\exists y F(x, y) \Rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)$$

where $\bar{\mathcal{H}} = \mathcal{H}[(\forall x \in s)\exists y F(x, y)]$ and $\mathcal{H}$ is an arbitrary operator. Now applying $(\to R)$ we get

$$\bar{\mathcal{H}} \left|\frac{\omega^{\Omega+2}\cdot 2+3}{0}\right. \Rightarrow (\forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y).$$

It remains to note that $\omega^{\Omega+2} \cdot 2 + 3 < \omega^{\Omega+3} = no((\Rightarrow \forall x \in s)\exists y F(x, y) \to \exists z(\forall x \in s)(\exists y \in z)F(x, y))$ to see that the result is verified. □

Lemma 3.18 (Set Induction) *For any formula* F

$$\Vdash\Rightarrow \forall x[(\forall y \in x)F(y) \to F(x)] \to \forall x F(x).$$

Proof Let $\mathcal{H}$ be an arbitrary operator and let $A := \forall x[(\forall y \in x)F(y) \to F(x)]$. First we prove the following

$$\text{Claim:} \quad \mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|+1}}_{0} A \Rightarrow F(s) \quad \text{for all terms } s.$$

The claim is proved by induction on $|s|$. By the induction hypothesis we have

$$\mathcal{H}[A, t] \vdash^{\omega^{rk(A)} \# \omega^{|t|+1}}_{0} A \Rightarrow F(t) \quad \text{for all } |t| < |s|.$$

Using weakening and then $(\to R)$ we get

$$\mathcal{H}[A, s, t] \vdash^{\omega^{rk(A)} \# \omega^{|t|+1}+1}_{0} A \Rightarrow t \in s \to F(t) \quad \text{for all } |t| < |s|.$$

Hence by $(b\forall R)_\infty$ we get

$$\mathcal{H}[A, s] \vdash^{\omega^{rk(A)} \# \omega^{|s|}+2}_{0} A \Rightarrow (\forall x \in s)F(x)$$

(the extra $+2$ is needed when $|s|$ is not a limit). Now let $\eta_s := \omega^{rk(A)} \# \omega^{|s|} + 2$. By Lemma 3.15 we have $\mathcal{H}[A, s] \vdash^{\eta_s}_{0} F(s) \Rightarrow F(s)$, so by $(\to L)$ we get

$$\mathcal{H}[A, s] \vdash^{\eta_s+1}_{0} A, (\forall y \in s)F(y) \to F(s) \Rightarrow F(s)\,.$$

Finally by applying $(\forall L)$ we get

$$\mathcal{H}[A, s] \vdash^{\eta_s+3}_{0} A \Rightarrow F(s)\,,$$

since $\eta_s + 3 < \omega^{rk(A)} \# \omega^{|s|+1}$ the claim is verified. Now by applying $(\forall R)_\infty$ we deduce from the claim that

$$\mathcal{H}[A] \vdash^{\omega^{rk(A)}+\Omega}_{0} A \Rightarrow \forall x F(x)\,.$$

Hence by $(\to R)$ we obtain the desired result. □

Lemma 3.19 (Infinity) *For any operator* $\mathcal{H}$ *we have*

$$\mathcal{H} \vdash^{\omega+2}_{0} \Rightarrow \exists x[(\exists y \in x)(y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)].$$

Proof First note that for any $|s| < \alpha$ we have $\mathcal{H} \vdash^{0}_{0} s \in \mathbb{V}_\alpha$ by virtue of axiom (A4). Let $|s| = n < \omega$, we have the following derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$
(\exists R)\ \dfrac{(\wedge R)\ \dfrac{(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(b\exists R)\ \dfrac{(\wedge R)\ \dfrac{\mathcal{H} \vdash^{0}_{0} \Rightarrow \mathbb{V}_{n+1}\in\mathbb{V}_\omega \qquad \mathcal{H}\vdash^{0}_{0} \Rightarrow s\in\mathbb{V}_{n+1}}{\mathcal{H}\vdash^{1}_{0} \Rightarrow \mathbb{V}_{n+1}\in\mathbb{V}_\omega\wedge s\in\mathbb{V}_{n+1}}}{\mathcal{H}\vdash^{n+2}_{0}\Rightarrow(\exists z\in\mathbb{V}_\omega)(s\in z)}}{\mathcal{H}\vdash^{n+3}_{0} \Rightarrow s\in\mathbb{V}_\omega\to(\exists z\in\mathbb{V}_\omega)(s\in z)}}{\mathcal{H}\vdash^{\omega}_{0}\Rightarrow(\forall y\in\mathbb{V}_\omega)(\exists z\in\mathbb{V}_\omega)(y\in z)} \qquad (b\exists R)\ \dfrac{(\wedge R)\ \dfrac{\mathcal{H}\vdash^{0}_{0}\Rightarrow\mathbb{V}_0\in\mathbb{V}_\omega}{\mathcal{H}\vdash^{1}_{0}\Rightarrow\mathbb{V}_0\in\mathbb{V}_\omega\wedge\mathbb{V}_0\in\mathbb{V}_\omega}}{\mathcal{H}\vdash^{2}_{0}\Rightarrow(\exists z\in\mathbb{V}_\omega)(z\in\mathbb{V}_\omega)}}{\mathcal{H}\vdash^{\omega+1}_{0}\Rightarrow (\forall y\in\mathbb{V}_\omega)(\exists z\in\mathbb{V}_\omega)(y\in z)\wedge(\exists z\in\mathbb{V}_\omega)(z\in\mathbb{V}_\omega)}}{\mathcal{H}\vdash^{\omega+2}_{0}\Rightarrow\exists x[(\forall y\in x)(\exists z\in x)(y\in z)\wedge(\exists z\in x)(z\in x)]}
$$

□

Lemma 3.20 ($\Delta_0^{\mathcal{P}}$-Separation) *If $A(a, b, c_1, \dots, c_n)$ is a $\Delta_0^{\mathcal{P}}$-formula of $\boldsymbol{IKP}(\mathcal{P})$ with all free variables indicated, $r, \bar{s} := s_1, \dots, s_n$ are $\boldsymbol{IRS}_\Omega^{\mathcal{P}}$ terms and $\mathcal{H}$ is an arbitrary operator then:*

$$\mathcal{H}[r, \bar{s}] \vdash^{\alpha+7}_{\rho} \Rightarrow \exists y[(\forall x \in y)(x \in r \wedge A(x, r, \bar{s})) \wedge (\forall x \in r)(A(x, r, \bar{s}) \to x \in y)]$$

where $\alpha := |\, r \,|$ and $\rho := max\{|\, r \,|, |\, s_1 \,|, \dots, |\, s_n \,|\} + \omega$.

Proof First we define

$$p := [x \in \mathbb{V}_\alpha \mid x \in r \wedge A(x, r, \bar{s})] \quad \text{and} \quad \bar{\mathcal{H}} := \mathcal{H}[r, \bar{s}].$$

For t any term with $|\, t \,| < \alpha$ the following are derivations in $\mathbf{IRS}_\Omega^{\mathcal{P}}$, first we have:

$$
(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(\to R)\ \dfrac{(\text{cut})\ \dfrac{(\wedge R)\ \dfrac{\dfrac{\text{Axiom (A1)}}{\bar{\mathcal{H}}\vdash^{0}_{0} t \in r \Rightarrow t \in r} \qquad \dfrac{\text{Axiom (A1)}}{\bar{\mathcal{H}}\vdash^{0}_{0} A(t, r, \bar{s}) \Rightarrow A(t, r, \bar{s})}}{\bar{\mathcal{H}}\vdash^{1}_{0} t \in r, A(t, r, \bar{s}) \Rightarrow t \in r \wedge A(t, r, \bar{s})} \qquad \dfrac{\text{Axiom (A7)}}{\bar{\mathcal{H}}\vdash^{0}_{0} t \in r \wedge A(t, r, \bar{s}) \Rightarrow t \in p}}{\bar{\mathcal{H}}\vdash^{2}_{\rho} t \in r, A(t, r, \bar{s}) \Rightarrow t \in p}}{\bar{\mathcal{H}}\vdash^{3}_{\rho} t \in r \Rightarrow A(t, r, \bar{s}) \to t \in p}}{\bar{\mathcal{H}}\vdash^{4}_{\rho} \Rightarrow t \in r \to (A(t, r, \bar{s}) \to t \in p)}}{\bar{\mathcal{H}}\vdash^{\alpha+5}_{\rho} \Rightarrow (\forall x \in r)(A(x, r, \bar{s}) \to x \in p)}
$$

Next we have:

$$
(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{\dfrac{\text{Axiom (A6)}}{\bar{\mathcal{H}}\vdash^{0}_{0} t \in p \Rightarrow t \in r \wedge A(t, r, \bar{s})}}{\bar{\mathcal{H}}\vdash^{1}_{0} \Rightarrow t \in p \to t \in r \wedge A(t, r, \bar{s})}}{\bar{\mathcal{H}}\vdash^{\alpha+2}_{0} \Rightarrow (\forall x \in p)(x \in r \wedge A(x, r, \bar{s}))}
$$

Now by applying $(\wedge R)$ followed by $(\exists R)$ to the conclusions of these two derivations we get

$$\bar{\mathcal{H}} \vdash^{\alpha+7}_{\rho} \Rightarrow \exists y[(\forall x \in y)(x \in r \wedge A(x, r, \bar{s})) \wedge (\forall x \in r)(A(x, r, \bar{s}) \rightarrow x \in y)]$$

as required. □

Lemma 3.21 (Pair) *For any operator $\mathcal{H}$ and any terms s and t we have*

$$\mathcal{H}[s, t] \vdash^{\alpha+2}_{0} \Rightarrow \exists z(s \in z \wedge t \in z)$$

where $\alpha := max(|\, s\,|, |\, t\,|) + 1$.

Proof The following is a derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$(\exists R)\,\dfrac{(\wedge R)\,\dfrac{\dfrac{\text{Axiom (A4)}}{\mathcal{H}[s, t] \vdash^{0}_{0} \Rightarrow s \in \mathbb{V}_\alpha} \qquad \dfrac{\text{Axiom (A4)}}{\mathcal{H}[s, t] \vdash^{0}_{0} \Rightarrow t \in \mathbb{V}_\alpha}}{\mathcal{H}[s, t] \vdash^{1}_{0} \Rightarrow s \in \mathbb{V}_\alpha \wedge t \in \mathbb{V}_\alpha}}{\mathcal{H}[s, t] \vdash^{\alpha+2}_{0} \Rightarrow \exists z(s \in z \wedge t \in z)}$$

□

Lemma 3.22 (Union) *For any operator $\mathcal{H}$ and any term s we have*

$$\mathcal{H}[s] \vdash^{\beta+5}_{0} \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z)$$

where $\beta = |\, s\,|$.

Proof Let r and t be terms such that $|\, r\,| < |\, t\,| < \beta$, we have the following derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$(\exists R)\,\dfrac{(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{\dfrac{\text{Axiom (A4)}}{\mathcal{H}[s, t, r] \vdash^{0}_{0} t \in s, r \in t \Rightarrow r \in \mathbb{V}_\beta}}{\mathcal{H}[s, t, r] \vdash^{1}_{0} t \in s \Rightarrow r \in t \rightarrow r \in \mathbb{V}_\beta}}{\mathcal{H}[s, t] \vdash^{\beta+2}_{0} t \in s \Rightarrow (\forall x \in t)(x \in \mathbb{V}_\beta)}}{\mathcal{H}[s, t] \vdash^{\beta+3}_{0} \Rightarrow t \in s \rightarrow (\forall x \in t)(x \in \mathbb{V}_\beta)}}{\mathcal{H}[s] \vdash^{\beta+4}_{0} \Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{V}_\beta)}}{\mathcal{H}[s] \vdash^{\beta+5}_{0} \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z)}$$

□

Lemma 3.23 (Powerset) *For any operator* $\mathcal{H}$ *and any term* s *we have*

$$\mathcal{H}[s] \vdash^{\alpha+3}_{0} \Rightarrow \exists z(\forall x \subseteq s)(x \in z)$$

where $\alpha = |\, s \,|$.

Proof Let t be any term with $|\, t \,| < \alpha$, we have the following derivation in $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$:

$$\begin{array}{rc}
 & \text{Axiom (A4)} \\ \cline{2-2}
 & \mathcal{H}[s,t] \vdash^{0}_{0} t \subseteq s \Rightarrow t \in \mathbb{V}_{\alpha+1} \\ \cline{2-2}
(\rightarrow R) & \mathcal{H}[s,t] \vdash^{1}_{0} \Rightarrow t \subseteq s \rightarrow t \in \mathbb{V}_{\alpha+1} \\ \cline{2-2}
(pb\forall R)_{\infty} & \mathcal{H}[s] \vdash^{\alpha+2}_{0} \Rightarrow (\forall x \subseteq s)(x \in \mathbb{V}_{\alpha+1}) \\ \cline{2-2}
(\exists R) & \mathcal{H}[s] \vdash^{\alpha+3}_{0} \Rightarrow \exists z(\forall x \subseteq s)(x \in z)
\end{array}$$

□

Theorem 3.24 *If* $\mathbf{IKP}(\mathcal{P}) \vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ *where* $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ *is an intuitionistic sequent containing exactly the free variables* $\bar{a} = a_1, \ldots, a_n$*, then there exists an* $m < \omega$ *(which we may calculate from the derivation) such that*

$$\mathcal{H}[\bar{s}] \vdash^{\Omega\cdot\omega^{m}}_{\Omega+m} \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

for any operator $\mathcal{H}$ *and any* $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ *terms* $\bar{s} = s_1, \ldots, s_n$.

Proof Note that the rank of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ formulas is always $< \Omega + \omega$ and thus the norm of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ sequents is always $< \omega^{\Omega+\omega} = \Omega \cdot \omega^{\omega}$. The proof is by induction on the $\mathbf{IKP}(\mathcal{P})$ derivation. If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{b})$ is an axiom of $\mathbf{IKP}(\mathcal{P})$ then the result follows by one of Lemmas 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22 and 3.23. Let $\bar{\mathcal{H}} := \mathcal{H}[\bar{s}]$.

Case 1. Suppose the last inference of the $\mathbf{IKP}(\mathcal{P})$ derivation was $(pb\exists L)$ then $(\exists x \subseteq a_i)F(x) \in \Gamma(\bar{a})$ and from the induction hypothesis we obtain a k such that

$$\bar{\mathcal{H}}[p] \vdash^{\Omega\cdot\omega^{k}}_{\Omega+k} \Gamma(\bar{s}), p \subseteq s_i \wedge F(p) \Rightarrow \Delta(\bar{s})$$

for all $|\, p \,| \leq |\, s_i \,|$ (using weakening if necessary). Thus we may apply $(pb\exists L)_{\infty}$ to obtain the desired result.

Case 2. Now suppose the last inference was $(pb\exists R)$ then $\Delta(\bar{a}) = \{(\exists x \subseteq a_i)F(x)\}$ and we are in the following situation in $\mathbf{IKP}(\mathcal{P})$:

$$(pb\exists R)\ \frac{\vdash \Gamma(\bar{a}) \Rightarrow c \subseteq a_i \wedge F(c)}{\vdash \Gamma(\bar{a}) \Rightarrow (\exists x \subseteq a_i)F(x)}$$

2.1 If c is not a member of $\bar{a}$ then by the induction hypothesis we have a $k < \omega$ such that

$$\bar{\mathcal{H}} \left|\frac{\Omega \cdot \omega^k}{\Omega + k}\right. \Gamma(\bar{s}) \Rightarrow \mathbb{V}_0 \subseteq s_i \wedge F(\mathbb{V}_0)\,.$$

Hence we can apply $(pb\exists R)$ to complete this case.

2.2 Now suppose c is a member of $\bar{a}$ for simplicity let us suppose that $c = a_1$. Inductively we can find a $k < \omega$ such that

$$\bar{\mathcal{H}} \left|\frac{\Omega \cdot \omega^k}{\Omega + k}\right. \Gamma(\bar{s}) \Rightarrow s_1 \subseteq s_i \wedge F(s_1). \tag{1}$$

Next we verify the following

$$\text{claim:} \quad \Vdash^{\omega} \Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i) F(x). \tag{2}$$

Owing to axiom (A1) we have

$$\bar{\mathcal{H}}[r] \left|\frac{0}{0}\right. r \subseteq s_i \Rightarrow r \subseteq s_i \quad \text{for all } |\, r \,| \leq |\, s_i \,|. \tag{3}$$

Also by Lemma 3.16 we have

$$\Vdash \Gamma[\bar{s}], r = s_1, F(s_1) \Rightarrow F(r) \quad \text{for all } |\, r \,| \leq |\, s_i \,|. \tag{4}$$

Now let $\gamma_r = no(\Gamma[\bar{s}], r = s_1, F(s_1) \Rightarrow F(r))$. Applying $(\wedge R)$ to (3) and (4) provides

$$\bar{\mathcal{H}}[r] \left|\frac{\gamma_r + 1}{0}\right. \Gamma(\bar{s}), r \subseteq s_i, r = s_1, F(s_1) \Rightarrow r \subseteq s_i \wedge F(r)\,.$$

Using $(pb\exists R)$ we may conclude

$$\bar{\mathcal{H}}[r] \left|\frac{\gamma_r + 2}{0}\right. \Gamma(\bar{s}), r \subseteq s_i, r = s_1, F(s_1) \Rightarrow (\exists x \subseteq s_i) F(x)\,.$$

Now two applications of $(\wedge L)$ gives us

$$\bar{\mathcal{H}}[r] \left|\frac{\gamma_r + 4}{0}\right. \Gamma(\bar{s}), r \subseteq s_i \wedge r = s_1, F(s_1) \Rightarrow (\exists x \subseteq s_i) F(x)\,.$$

Now applying $(\subseteq L)_\infty$ provides

$$\bar{\mathcal{H}} \left|\frac{\gamma + 5}{0}\right. \Gamma(\bar{s}), s_1 \subseteq s_i, F(s_1) \Rightarrow (\exists x \subseteq s_i) F(x)$$

where $\gamma = \sup_{|\, r \,| \leq |\, s_i \,|} \gamma_r$. Finally, by applying $(\wedge L)$ a further two times we can conclude

$$\bar{\mathcal{H}} \left|\frac{\gamma + 7}{0}\right. \Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i) F(x)\,.$$

Via some ordinal arithmetic it can be observed that

$$\gamma + 7 \leq no(\Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x)) \,\#\, \omega,$$

so the claim is verified.

To complete this case we may now apply (Cut) to (1) and (2).

All other cases are similar to those above, or may be treated in a similar manner to Theorem 2.33. □

3.5 A Relativised Ordinal Analysis of IKP($\mathcal{P}$)

A major difference to the case of **IKP** is that we don't immediately have the soundness of cut-reduced $\mathbf{IRS}_\Omega^{\mathcal{P}}$ derivations of $\Sigma^{\mathcal{P}}$-formulae within the appropriate segment of the Von-Neumann Hierarchy. This is partly due to the fact that we don't have a term for each element of the hierarchy (this can be seen from a simple cardinality argument). In fact we do still have soundness for certain derivations within $V_{\psi_\Omega(\varepsilon_{\Omega+1})}$, which is demonstrated in the next lemma, where we must make essential use of the free variables in $\mathbf{IRS}_\Omega^{\mathcal{P}}$. First we need the notion of an assignment. Let $VAR_{\mathcal{P}}$ be the set of free variables of $\mathbf{IRS}_\Omega^{\mathcal{P}}$. A variable assignment is a function

$$v : VAR_{\mathcal{P}} \longrightarrow V_{\psi_\Omega(\varepsilon_{\Omega+1})}$$

such that $v(a_i^\alpha) \in V_{\alpha+1}$ for each i. v canonically lifts to all terms as follows

$$\begin{aligned} v(\mathbb{V}_\alpha) &= V_\alpha, \\ v(\{x \in \mathbb{V}_\alpha \mid F(x, s_1, \ldots, s_n)\}) &= \{x \in V_\alpha \mid F(x, v(s_1), \ldots, v(s_n))\}. \end{aligned}$$

Moreover it can be seen that $v(s) \in V_{|s|+1}$ and thus $v(s) \in V_{\psi_\Omega(\varepsilon_{\Omega+1})}$ for all terms s.

Theorem 3.25 (Soundness for $\mathbf{IRS}_\Omega^{\mathcal{P}}$) *Suppose* $\Gamma[s_1, \ldots, s_n]$ *is a finite set of* $\Pi^{\mathcal{P}}$ *formulae with max*$\{rk(A) \mid A \in \Gamma\} \leq \Omega$, $\Delta[s_1, \ldots, s_n]$ *a set containing at most one* $\Sigma^{\mathcal{P}}$ *formula and*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}] \quad \textit{for some operator } \mathcal{H} \textit{ and some } \alpha, \rho < \Omega.$$

Then for any assignment v,

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(s_1), \ldots, v(s_n)] \rightarrow \bigvee \Delta[v(s_1), \ldots, v(s_n)]$$

where $\bigwedge \Gamma$ *and* $\bigvee \Delta$ *stand for the conjunction of formulas in* Γ *and the disjunction of formulas in* Δ *respectively, by convention* $\bigwedge \emptyset = \top$ *and* $\bigvee \emptyset = \bot$.

Proof The proof is by induction on α. Note that the derivation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ contains no inferences of the form $(\forall R)_\infty$, $(\exists L)_\infty$ or $(\Sigma^{\mathcal{P}}\text{-}Ref)$ and all cuts have $\Delta_0^{\mathcal{P}}$ cut formulae. All axioms of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ can be observed to be sound with respect to the interpretation.

First we treat the case where the last inference was $(pb\forall L)$ so we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}], t \subseteq s_i \to F(t, \bar{s}) \Rightarrow \Delta[\bar{s}] \quad \text{for some } \alpha_0, |\, t\,| < \alpha, \text{ with } |\, t\,| \leq |\, s_i\,|.$$

Since $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, it follows that $t \subseteq s_i \to F(t, \bar{s})$ is a $\Delta_0^{\mathcal{P}}$ formula. So we may apply the induction hypothesis to obtain

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \wedge [v(t) \subseteq v(s_i) \to F(v(t), v(\bar{s}))] \to \bigvee \Delta[v(\bar{s})],$$

where $v(\bar{s}) := v(s_1), \ldots, v(s_n)$. From here the desired result follows by regular logical semantics.

Now suppose the last inference was $(pb\forall R)_\infty$, so we have

$$\mathcal{H} \overset{\alpha_t}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow t \subseteq s_i \to F(t, \bar{s}) \quad \text{for all } |\, t\,| \leq |\, s_i\,| \text{ with } \alpha_t < \alpha. \tag{1}$$

In particular this means we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow a_j^\beta \subseteq s_i \to F(a_j^\beta, \bar{s}) \quad \text{for some } \alpha_0 < \alpha. \tag{2}$$

Here $\beta := |\, s_i\,|$ and j is chosen such that a_j^β does not occur in any of the terms $s_1, \ldots, s_n$. If F contains an unbounded quantifier we may use inversion for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ Lemma 3.8(v) to obtain

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}], a_j^\beta \subseteq s_i \Rightarrow F(a_j^\beta, \bar{s}) \quad \text{for some } \alpha_0 < \alpha. \tag{3}$$

So we may apply the induction hypothesis to get

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})], v(a_j^\beta) \subseteq v(s_i) \to F(v(a_j^\beta), v(\bar{s})) \tag{4}$$

for all variable assignments v. Thus by the choice of a_j^β we have

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \to (\forall x \subseteq v(s_i)) F(x, v(\bar{s})) \tag{5}$$

as required. If F is $\Delta_0^{\mathcal{P}}$ then we may immediately apply the induction hypothesis to (2) to obtain

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \to [v(a_j^\beta) \subseteq v(s_i) \to F(v(a_j^\beta), v(\bar{s}))] \tag{6}$$

for all variable assignments v, again by the choice of a_i^β we obtain the desired result. All other cases may be treated in a similar manner to the two above. □

Lemma 3.26 *Suppose* ***IKP***$(\mathcal{P}) \vdash \Rightarrow A$ *for some* $\Sigma^{\mathcal{P}}$ *sentence* A*, then there is an* $m < \omega$*, which we may compute from the derivation, such that*

$$\mathcal{H}_\sigma \;\Big|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\; \Rightarrow \; A \quad \textit{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Proof Suppose **IKP**$(\mathcal{P}) \vdash \Rightarrow A$ for some $\Sigma^{\mathcal{P}}$ sentence A, then by Theorem 3.24 we can explicitly find some $m < \omega$ such that

$$\mathcal{H}_0 \;\Big|\frac{\Omega\cdot\omega^m}{\Omega+m}\; \Rightarrow A\,.$$

Applying Partial cut elimination Theorem 3.11 we have

$$\mathcal{H}_0 \;\Big|\frac{\omega_{m-1}(\Omega\cdot\omega^m)}{\Omega+1}\; \Rightarrow A\,.$$

Now using Collapsing Theorem 3.13 we obtain

$$\mathcal{H}_\sigma \;\Big|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\; \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m)$$

completing the proof. □

Note that we cannot eliminate *all* cuts from the derivation since we don't have full predicative cut elimination for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ as we do for $\mathbf{IRS}_\Omega$.

Theorem 3.27 *If* A *is a* $\Sigma^{\mathcal{P}}$*-sentence and* ***IKP***$(\mathcal{P}) \vdash \Rightarrow A$ *then there is some ordinal term* $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$*, which we may compute from the derivation, such that*

$$V_\alpha \models A.$$

Proof From Lemma 3.26 we obtain some $m < \omega$ such that

$$\mathcal{H}_\sigma \;\Big|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\; \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m). \tag{1}$$

Let $\alpha := \psi_\Omega(\sigma)$. Applying Boundedness Lemma 3.12 to (1) we obtain

$$\mathcal{H}_\sigma \;\Big|\frac{\alpha}{\alpha}\; \Rightarrow A^{\mathbb{V}_\alpha}\,. \tag{2}$$

Now applying Theorem 3.25 to (2) we obtain

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A^{V_\alpha}$$

and thus

$$V_\alpha \models A$$

as required. □

Remark 3.28 Suppose $A \equiv \exists x C(x)$ is a $\Sigma^{\mathcal{P}}$ sentence and **IKP**$(\mathcal{P}) \vdash \Rightarrow A$. As well as the ordinal term α given by Theorem 3.27, it is possible to determine (making essential use of the intuitionistic nature of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$) a term s, with $|s| < \alpha$, such that

$$V_\alpha \models C(s).$$

This proof is somewhat more complex than in the case of **IKP** since the proof tree corresponding to (2) above can still contain cuts with $\Delta_0^{\mathcal{P}}$ cut formulae.

Moreover, in order to show that **IKP**$(\mathcal{P})$ has the existence property, the embedding and cut elimination for a given finite derivation of a $\Sigma^{\mathcal{P}}$ sentence, needs to be carried out *inside* **IKP**$(\mathcal{P})$. In order to do this it needs to be shown that from the finite derivation we can calculate some ordinal term $\gamma < \varepsilon_{\Omega+1}$ such that the embedding and cut elimination for that derivation can still be performed inside $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ with the term structure restricted to $B(\gamma)$.

These proofs will appear in [28].

Like in the case of **IKP** we also arrive at a conservativity result.

Theorem 3.29 *IKP*$(\mathcal{P}) + \Sigma^{\mathcal{P}}$*-Reflection is conservative over* *IKP*$(\mathcal{P})$ *for* $\Sigma^{\mathcal{P}}$*-sentences.*

4 The Case of IKP($\mathcal{E}$)

This final section provides a relativised ordinal analysis for intuitionistic exponentiation Kripke-Platek set theory **IKP**$(\mathcal{E})$. Given sets a and b, set-exponentiation allows the formation of the set ${}^{a}b$, of all functions from a to b. A problem that presents itself in this case is that it is not clear how to formulate a term structure in such a way that we can read off a terms level in the pertinent 'exponentiation hierarchy' from that terms syntactic structure. Instead we work with a term structure similar to that used in $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$, and a terms level becomes a dynamic property *inside* the infinitary system. Making this work in a system for which we can prove all the necessary embedding and cut-elimination theorems turned out to be a major technical hurdle. The end result of the section is a characterisation of **IKP**$(\mathcal{E})$ in terms of provable height of the exponentiation hierarchy, this machinery will also be used in a later paper by Rathjen [28], to show that $\mathbf{CZF}^{\mathcal{E}}$ has the full existence property.

4.1 A Sequent Calculus Formulation of IKP($\mathcal{E}$)

Definition 4.1 The formulas of **IKP**($\mathcal{E}$) are the same as those of **IKP** except we also allow *exponentiation bounded quantifiers* of the form

$$(\forall x \in {}^{a}b)A(x) \quad \text{and} \quad (\exists x \in {}^{a}b)A(x).$$

These are treated as quantifiers in their own right, not abbreviations. The formula "fun(x, a, b)" is defined below. It's intuitive meaning is "x is a function from a to b".

$$\begin{aligned}\mathrm{fun}(x, a, b) := x \subseteq a \times b \ \wedge \ (\forall y \in a)(\exists z \in b)((y, z) \in x)\\ \wedge\ (\forall y \in a)(\forall z_1 \in b)(\forall z_2 \in b)[((y, z_1) \in x \wedge (y, z_2) \in x) \rightarrow z_1 = z_2].\end{aligned}$$

Quantifiers $\forall x$, $\exists x$ will be referred to as unbounded, whereas the other quantifiers (including the exponentiation bounded ones) will be referred to as bounded.

A $\Delta_0^{\mathcal{E}}$-formula of **IKP**($\mathcal{E}$) is one that contains no unbounded quantifiers.

As with **IKP**, the system **IKP**($\mathcal{E}$) derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulae and Δ contains at most one formula.

The axioms of **IKP**($\mathcal{E}$) are given by:

Logical axioms: $\Gamma, A, \Rightarrow A$ for every $\Delta_0^{\mathcal{E}}$-formula A.
Extensionality: $\Gamma \Rightarrow a = b \wedge B(a) \rightarrow B(b)$. for every $\Delta_0^{\mathcal{E}}$-formula $B(a)$.
Pair: $\Gamma \Rightarrow \exists x[a \in x \wedge b \in x]$.
Union: $\Gamma \Rightarrow \exists x(\forall y \in a)(\forall z \in y)(z \in x)$.
Infinity: $\Gamma \Rightarrow \exists x\,[(\exists y \in x)\, y \in x \ \wedge\ (\forall y \in x)(\exists z \in x)\, y \in z]$.
$\Delta_0^{\mathcal{E}}$ *-Separation*: $\Gamma \Rightarrow \exists x((\forall y \in x)(y \in a \wedge A(y)) \wedge (\forall y \in a)(A(y) \rightarrow y \in x))$ for every $\Delta_0^{\mathcal{E}}$ formula $A(b)$.
$\Delta_0^{\mathcal{E}}$ *-Collection*: $\Gamma \Rightarrow (\forall x \in a)\exists y B(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)B(x, y)$ for every $\Delta_0^{\mathcal{E}}$ formula $B(b, c)$.
Set Induction: $\Gamma \Rightarrow \forall u\,[(\forall x \in u)\, G(x) \rightarrow G(u)] \rightarrow \forall u\, G(u)$ for every formula $G(b)$.
Exponentiation: $\Gamma \Rightarrow \exists z\,(\forall x \in {}^{a}b)(x \in z)$.

The rules of **IKP**($\mathcal{E}$) are the same as those of **IKP** (extended to the new language containing exponentiation bounded quantifiers), together with the following four rules:

$$(\mathcal{E}b\exists L)\ \frac{\Gamma, \mathrm{fun}(c, a, b) \wedge F(c) \Rightarrow \Delta}{\Gamma, (\exists x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\exists R)\ \frac{\Gamma \Rightarrow \mathrm{fun}(c, a, b) \wedge F(c)}{\Gamma \Rightarrow (\exists x \in {}^{a}b)F(x)}$$

$$(\mathcal{E}b\forall L)\ \frac{\Gamma, \mathrm{fun}(c, a, b) \rightarrow F(c) \Rightarrow \Delta}{\Gamma, (\forall x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\forall R)\ \frac{\Gamma \Rightarrow \mathrm{fun}(c, a, b) \rightarrow F(c)}{\Gamma \Rightarrow (\forall x \in {}^{a}b)F(x)}$$

As usual it is forbidden for the variable a to occur in the conclusion of the rules ($\mathcal{E}b\exists L$) and ($\mathcal{E}b\forall R$), such a variable is referred to as the eigenvariable of the inference.

4.2 The Infinitary System $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$

The purpose of this section is to introduce an infinitary system $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ within which we will be able to embed $\mathbf{IKP}(\mathcal{E})$. As with the von Neumann hierarchy built by iterating the power set operation through the ordinals, one may define an Exponentiation-hierarchy as follows

$$\begin{aligned}
E_0 &:= \emptyset,\\
E_1 &:= \{\emptyset\},\\
E_{\alpha+2} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\}\\
&\quad \cup \{f \mid \text{fun}(f, a, b) \text{ for some } a, b \in E_\alpha\},\\
E_\lambda &:= \bigcup_{\beta<\lambda} E_\beta \quad \text{for } \lambda \text{ a limit ordinal,}\\
E_{\lambda+1} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\} \quad \text{for } \lambda \text{ a limit ordinal.}
\end{aligned}$$

Lemma 4.2 *If* $y \in E_{\alpha+1}$ *and* $x \in y$ *then* $x \in E_\alpha$.

Proof The proof is by induction on α. If y is a set definable over $\langle E_\alpha, \in\rangle$ with parameters, the members of y, including x, must be members of E_α.

Now suppose $\alpha = \beta + 1$ and $y \in E_{\alpha+1}$ is a function $y : p \to q$ for two sets $p, q \in E_\beta$. Since $x \in y$, it follows that x is of the form (x_0, x_1) with $x_0 \in p$ and $x_1 \in q$, we use the standard definition of ordered pair so

$$(x_0, x_1) := \{\{x_0, x_1\}, \{x_0\}\}. \tag{1}$$

We must now verify the following claim:

$$\{x_0\}, \{x_1\}, \{x_0, x_1\} \in E_\beta. \tag{*}$$

If $\beta = \gamma + 1$ then by the induction hypothesis applied to $x_0 \in p \in E_\beta$ and $x_1 \in q \in E_\beta$ we get $x_0, x_1 \in E_\gamma$ and thus $\{x_0\}, \{x_1\}, \{x_0, x_1\} \in E_\beta$ as required.

If β is a limit then by the induction hypothesis and the construction of the E hierarchy at limit ordinals, we know that $s_0 \in E_{\beta_0}$ and $s_1 \in E_{\beta_1}$ for some $\beta_0, \beta_1 < \beta$, thus $\{s_0\}, \{s_1\}, \{s_0, s_1\} \in E_{\max(\beta_0,\beta_1)+1}$ which completes the proof of (*).

From (*) and (1) it is clear that $(s_0, s_1) \in E_{\beta+1}$ as required. □

The idea of $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ is to build an infinitary system for reasoning about the E hierarchy.

Definition 4.3 The terms of $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ are defined as follows

1. $\mathbb{E}_\alpha$ is an $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ term for each $\alpha < \Omega$.
2. a_i^α is an $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ term for each $\alpha < \Omega$ and each $i < \omega$, these terms will be known as free variables.

3. If $F(a, \bar{b})$ is a $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables indicated, and $t, \bar{s} := s_1, \ldots, s_n$ are $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms then
$$[x \in t \mid F(x, \bar{s})]$$
is also a term of $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

Observe that $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms do not come with 'levels' as in the other infinitary systems. This is because it is not clear how to immediately read off the location of a given term within the E hierarchy, just from the syntactic information available within that term.

The formulas of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ are of the form $F(s_1, \ldots, s_n)$, where $F(a_1, \ldots, a_n)$ is a formula of $\mathbf{IKP}(\mathcal{E})$ with all free variables indicated and $s_1, \ldots, s_n$ are $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms. The formula $A(s_1, \ldots, s_n)$ is said to be $\Delta_0^{\mathcal{E}}$ if $A(a_1, \ldots, a_n)$ is a $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$. The $\Sigma^{\mathcal{E}}$ formulae are the smallest collection containing the $\Delta_0^{\mathcal{E}}$ formulae such that $A \wedge B$, $A \vee B$, $(\forall x \in t)A$, $(\exists x \in t)A$, $(\exists x \in {}^a b)A$, $(\forall x \in {}^a b)A$, $\exists x A$, $\neg C$, and $C \rightarrow A$ are in $\Sigma^{\mathcal{E}}$ whenever A, B are in $\Sigma^{\mathcal{E}}$ and C is in $\Pi^{\mathcal{E}}$. Dually, the $\Pi^{\mathcal{E}}$ formulae are the smallest collection containing the $\Delta_0^{\mathcal{E}}$ formulae such that $A \wedge B$, $A \vee B$, $(\forall x \in t)A$, $(\exists x \in t)A$, $(\exists x \in {}^a b)A$, $(\forall x \in {}^a b)A$, $\forall x A$, $\neg C$, and $C \rightarrow A$ are in $\Pi^{\mathcal{E}}$ whenever A, B are in $\Pi^{\mathcal{E}}$ and C is in $\Sigma^{\mathcal{E}}$.

The axioms of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ are given by

(E1) $\Gamma, A \Rightarrow A$ for every $\Delta_0^{\mathcal{E}}$–formula A.
(E2) $\Gamma \Rightarrow t = t$ for every $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term t.
(E3) $\Gamma, \bar{s} = \bar{t}, B(\bar{s}) \Rightarrow B(\bar{t})$ for every $\Delta_0^{\mathcal{E}}$-formula $B(\bar{s})$.
(E4) $\Gamma \Rightarrow \mathbb{E}_\beta \in \mathbb{E}_\alpha$ for all $\beta < \alpha < \Omega$.
(E5) $\Gamma \Rightarrow a_i^\beta \in \mathbb{E}_\alpha$ for all $i \in \omega$ and $\beta < \alpha < \Omega$.
(E6) $\Gamma, t \in \mathbb{E}_\alpha, s \in t \Rightarrow s \in \mathbb{E}_\alpha$ for all $\alpha < \Omega$.
(E7) $\Gamma, t \in \mathbb{E}_{\alpha+1}, s \in t \Rightarrow s \in \mathbb{E}_\alpha$ for all $\alpha < \Omega$.
(E8) $\Gamma, s \in t, F(s, \bar{p}) \Rightarrow s \in [x \in t \mid F(x, \bar{p})]$.
(E9) $\Gamma, s \in [x \in t \mid F(x, \bar{p})] \Rightarrow s \in t \wedge F(s, \bar{p})$.
(E10) $\Gamma, s \in \mathbb{E}_\alpha, t \in \mathbb{E}_\beta, \mathrm{fun}(p, s, t) \Rightarrow p \in \mathbb{E}_\gamma$ for all $\gamma \geq \max(\alpha, \beta) + 2$.
(E11) $\Gamma, t \in \mathbb{E}_\beta, \bar{p} \in \mathbb{E}_{\bar{\alpha}} \Rightarrow [x \in t \mid F(x, \bar{p})] \in \mathbb{E}_\gamma$ for all $\gamma \geq \max(\beta, \bar{\alpha})$.

Definition 4.4 For a formula $A(a_1, \ldots, a_n)$ of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables $\bar{a} := a_1, \ldots, a_n$ and any $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms $\bar{s} := s_1, \ldots, s_n$, we define the *$\bar{\beta}$-rank* $\|A(\bar{s})\|_{\bar{\beta}}$ where $\bar{\beta} := \beta_1, \ldots, \beta_n$ are any ordinals $< \Omega$. The definition is made by recursion on the build up of the formula A.

(i) $\|s \in t\|_{\beta_1, \beta_2} := \max(\beta_1, \beta_2)$.
(ii) $\|(\exists x \in t)F(x, \bar{s})\|_{\gamma, \bar{\beta}} := \|(\forall x \in t)F(x, \bar{s})\|_{\gamma, \bar{\beta}} := \max(\gamma, \|F(\mathbb{E}_0, \bar{s})\|_{0, \bar{\beta}} + 2)$.
(iii) $\|(\exists x \in {}^s t)F(x, \bar{p})\|_{\gamma, \delta, \bar{\beta}} := \|(\forall x \in {}^s t)F(x, \bar{p})\|_{\gamma, \delta, \bar{\beta}}$
$:= \max(\gamma + \omega, \delta + \omega, \|F(\mathbb{E}_0, \bar{p})\|_{0, \bar{\beta}} + 2)$.
(iv) $\|\exists x F(x, \bar{s})\|_{\bar{\beta}} := \|\forall x F(x, \bar{s})\|_{\bar{\beta}} := \max(\Omega, \|F(\mathbb{E}_0, \bar{s})\|_{0, \bar{\beta}} + 2)$.
(v) $\|A \wedge B\|_{\bar{\beta}} := \|A \vee B\|_{\bar{\beta}} := \|A \rightarrow B\|_{\bar{\beta}} := \max(\|A\|_{\bar{\beta}}, \|B\|_{\bar{\beta}}) + 1$.
(vi) $\|\neg A\|_{\bar{\beta}} := \|A\|_{\bar{\beta}} + 1$.

We define the *rank* of $A(\bar{s})$ by

$$rk(A(\bar{s})) := \|A(\bar{s})\|_{\bar{0}}.$$

Observation 4.5

(i) $\|A(\bar{s})\|_{\bar{\beta}} < \Omega$ if and only if A is $\Delta_0^{\mathcal{E}}$.
(ii) If A contains unbounded quantifiers then $rk(A(\bar{s})) = \|A(\bar{s})\|_{\bar{\beta}}$ for all $\bar{s}$ and $\bar{\beta}$.

Definition 4.6 (*Operator Controlled Derivability in* $\mathbf{IRS}_\Omega^{\mathbb{E}}$) $\mathbf{IRS}_\Omega^{\mathbb{E}}$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ formulae and Δ contains at most one formula. For $\mathcal{H}$ an operator and α, ρ ordinals we define the relation $\mathcal{H} \,\vdash^{\alpha}_{\rho}\, \Gamma \Rightarrow \Delta$ by recursion on α.

If $\Gamma \Rightarrow \Delta$ is an axiom and $\alpha \in \mathcal{H}$ then $\mathcal{H} \,\vdash^{\alpha}_{\rho}\, \Gamma \Rightarrow \Delta$.

It is always required that $\alpha \in \mathcal{H}$, this requirement is not repeated for each inference rule below.

$$(\mathbb{E}\text{-Lim})_\infty \quad \frac{\mathcal{H}[\delta] \vdash^{\alpha_\delta}_{\rho} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \text{ for all } \delta < \gamma}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, s \in \mathbb{E}_\gamma \Rightarrow \Delta} \qquad \begin{array}{r} \gamma \text{ a limit} \\ \alpha_\delta < \alpha \\ \gamma \in \mathcal{H} \end{array}$$

$$(b\forall L) \quad \frac{\begin{array}{l} \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, s \in t \to A(s) \Rightarrow \Delta \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\beta \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \gamma < \alpha \\ \gamma \leq \beta \end{array}$$

$$(b\forall R)_\infty \quad \frac{\begin{array}{l} \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \in t \to F(s) \text{ for all } s \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\beta \end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in t)F(x)} \qquad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$(b\exists L)_\infty \quad \frac{\begin{array}{l} \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, s \in t \wedge F(s) \Rightarrow \Delta \text{ for all } s \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\beta \end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in t)F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$(b\exists R) \quad \frac{\begin{array}{l} \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \in t \wedge A(s) \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\beta \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \gamma < \alpha \\ \gamma \leq \beta \end{array}$$

$$(\mathcal{E}b\forall L)\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, \mathrm{fun}(p,s,t) \rightarrow A(p) \Rightarrow \Delta \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta} \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_{\gamma} \\ \mathcal{H} \vdash^{\alpha_3}_{\rho} \Gamma \Rightarrow p \in \mathbb{E}_{\delta}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in {}^{s}t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha \\ \beta, \gamma, \delta \in \mathcal{H} \\ \delta < \alpha \\ \delta \leq \max(\beta, \gamma) + 2\end{array}$$

$$(\mathcal{E}b\forall R)_{\infty}\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow \mathrm{fun}(p,s,t) \rightarrow F(p) \text{ for all } p \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta} \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_{\gamma}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in {}^{s}t)F(x)} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \max(\beta, \gamma) + 2 \leq \alpha\end{array}$$

$$(\mathcal{E}b\exists L)_{\infty}\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, \mathrm{fun}(p,s,t) \wedge F(p) \Rightarrow \Delta \text{ for all } p \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta} \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_{\gamma}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in {}^{s}t)F(x) \Rightarrow \Delta} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \max(\beta, \gamma) + 2 \leq \alpha\end{array}$$

$$(\mathcal{E}b\exists R)\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow \mathrm{fun}(p,s,t) \wedge A(p) \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta} \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_{\gamma} \\ \mathcal{H} \vdash^{\alpha_3}_{\rho} \Gamma \Rightarrow p \in \mathbb{E}_{\delta}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in {}^{s}t)A(x)} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha \\ \beta, \gamma, \delta \in \mathcal{H} \\ \delta < \alpha \\ \delta \leq \max(\beta, \gamma) + 2\end{array}$$

$$(\forall L)\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, F(s) \Rightarrow \Delta \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r}\alpha_0 + 3, \alpha_1 + 3 < \alpha \\ \beta < \alpha \\ \beta \in \mathcal{H}\end{array}$$

$$(\forall R)_{\infty}\quad \dfrac{\mathcal{H}[\beta] \vdash^{\alpha_{\beta}}_{\rho} \Gamma, s \in \mathbb{E}_{\beta} \Rightarrow F(s) \text{ for all } s \text{ and all } \beta < \Omega}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \forall x F(x)} \qquad \beta < \alpha_{\beta} + 3 < \alpha$$

$$(\exists L)_{\infty}\quad \dfrac{\mathcal{H}[\beta] \vdash^{\alpha_{\beta}}_{\rho} \Gamma, s \in \mathbb{E}_{\beta}, F(s) \Rightarrow \Delta \text{ for all } s \text{ and all } \beta < \Omega}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \forall x F(x)} \qquad \beta < \alpha_{\beta} + 3 < \alpha$$

$$(\exists R)\quad \dfrac{\begin{array}{l}\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow F(s) \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_{\beta}\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \exists x F(x)} \qquad \begin{array}{r}\alpha_0 + 3, \alpha_1 + 3 < \alpha \\ \beta < \alpha \\ \beta \in \mathcal{H}\end{array}$$

$$(\Sigma^{\mathcal{E}}\text{-}Ref)\ \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow A}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma^{\mathcal{E}}\text{-formula} \end{array}$$

$$(\text{Cut})\ \frac{\begin{array}{l} \mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, A(s_1, \dots, s_n) \Rightarrow \Delta \\ \mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow A(s_1, \dots, s_n) \\ \mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \ \ i = 1, \dots, n \end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \|A(\bar{s})\|_{\bar{\beta}} < \rho \\ \bar{\beta} \in \mathcal{H} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\rightarrow L)$ or $(\rightarrow R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

Convention 4.7 In cases where terms $\mathbb{E}_\alpha$ and a_i^α occur directly as witnesses in existential rules or in cut formulae we will omit the extra premise declaring the terms location in the $\mathbb{E}$ term hierarchy since

$$\mathbb{E}_\alpha \in \mathbb{E}_{\alpha+1} \quad \text{and} \quad a_i^\alpha \in \mathbb{E}_{\alpha+1}$$

are axioms (E4) and (E5) respectively. It must still be checked that $\alpha \in \mathcal{H}$ however.

4.3 Cut Elimination for $\mathbf{IRS}_{\mathbf{\Omega}}^{\mathbb{E}}$

Lemma 4.8 (Inversions of $\mathbf{IRS}_\Omega^{\mathbb{E}}$) *If* $max(rk(A), rk(B)) \geq \Omega$ *then we have the usual propositional inversions for intuitionistic systems:*

(i) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$.
(ii) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A \wedge B$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A$ *and* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow B$.
(iii) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$ *and* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, B \Rightarrow \Delta$.
(iv) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \rightarrow B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, B \Rightarrow \Delta$.
(v) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow A \rightarrow B$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow B$.

If $rk(A) \geq \Omega$ *we have the following additional inversions:*

(vi) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \neg A$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow$.
(vii) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in t)A(x)$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow s \in t \rightarrow A(s)$ *for all terms* s.
(viii) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, s \in t \wedge A(s) \Rightarrow \Delta$ *for all terms* s.
(ix) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in^s t)A(x)$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow fun(p, s, t) \rightarrow A(p)$ *for all terms* p.
(x) If $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in^s t)A(x) \Rightarrow \Delta$ *then* $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, fun(p, s, t) \wedge A(p) \Rightarrow \Delta$ *for all terms* p.

Finally we have the following persistence properties:

(xi) *If* $\gamma \in \mathcal{H} \cap \Omega$ *and* $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma \Rightarrow \forall x A(x)$ *then* $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma \Rightarrow (\forall x \in \mathbb{E}_\gamma)A(x)$.

(xii) *If* $\gamma \in \mathcal{H} \cap \Omega$ *and* $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, \exists x A(x) \Rightarrow \Delta$ *then* $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, (\exists x \in \mathbb{E}_\gamma)A(x) \Rightarrow \Delta$.

Proof All proofs are by induction on α, (i)–(vi) are standard for intuitionistic systems of this type.

For (viii) suppose that $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{E}_0)) \geq \Omega$. $(\exists x \in t)A(x)$ cannot have been the "active component" of an axiom, so if $\Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ is an axiom then so is $\Gamma, s \in t \wedge A(s) \Rightarrow \Delta$. Now if $(\exists x \in t)A(x)$ was not the principal formula of the last inference we may apply the induction hypothesis to the premises of that inference followed by the same inference again. Finally if $(\exists x \in t)A(x)$ was the principal formula of the last inference and the last inference was $(b\exists L)_\infty$ so we have

$$\mathcal{H} \mathrel{\substack{\alpha_0 \\ \vdash \\ \rho}} \Gamma, (\exists x \in t)A(x), s \in t \wedge A(s) \Rightarrow \Delta \quad \text{for all terms } s \text{ and for some } \alpha_0 < \alpha.$$

Applying the induction hypothesis followed by weakening yields

$$\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, s \in t \wedge A(s) \Rightarrow \Delta \quad \text{for all terms } s$$

as required. The proofs of (vii), (xi) and (x) are similar.

For (xi) suppose $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma \Rightarrow \forall x A(x)$ and $\gamma \in \mathcal{H} \cap \Omega$. $\Gamma \Rightarrow \forall x A(x)$ cannot be an axiom. If the last inference was not $(\forall R)_\infty$ then we may apply the induction hypothesis to its premises and then the same inference again. So suppose the last inference was $(\forall R)_\infty$ in which case we have the premise

$$\mathcal{H}[\delta] \mathrel{\substack{\alpha_\delta \\ \vdash \\ \rho}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow A(s) \quad \text{for all } s \text{ and all } \delta < \Omega, \text{ with } \delta < \alpha_\delta + 3 < \alpha.$$

In particular since $\gamma \in \mathcal{H}$ we have

$$\mathcal{H} \mathrel{\substack{\alpha_\gamma \\ \vdash \\ \rho}} \Gamma, s \in \mathbb{E}_\gamma \Rightarrow A(s) \quad \text{for all } s \text{ with } \gamma < \alpha_\gamma + 3 < \alpha.$$

So by $(\rightarrow R)$ we have

$$\mathcal{H} \mathrel{\substack{\alpha_\gamma + 1 \\ \vdash \\ \rho}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \rightarrow A(s) \quad \text{for all } s.$$

Now since $\Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_{\gamma+1}$ is an instance of axiom (E4), $\gamma \in \mathcal{H}$ and $\gamma < \alpha$ we may apply $(b\forall R)$ to obtain

$$\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma \Rightarrow (\forall x \in \mathbb{E}_\gamma)A(x)$$

as required. The proof of (xii) is similar. □

Lemma 4.9 (Reduction for $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$) *Suppose* $rk(C(\bar{s})) := \rho > \Omega$ *where* $C(\bar{a})$ *is an* ***IKP***$(\mathcal{E})$ *formula with all free variables displayed. If*

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow C(\bar{s}),$$
$$\mathcal{H} \mid\!\frac{\beta}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow \Delta,$$
$$\mathcal{H} \mid\!\frac{\gamma_i}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\eta_i} \quad \textit{with } \eta_i \in \mathcal{H} \cap \Omega \textit{ for each } 1 \leq i \leq n$$

then

$$\mathcal{H} \mid\!\frac{\alpha \,\#\, \alpha \,\#\, \beta \,\#\, \beta \,\#\, \gamma}{\rho}\, \Gamma \Rightarrow \Delta \quad \textit{where } \gamma := max_{i=1,\ldots,n}(\gamma_i).$$

Proof The proof is by induction on $\alpha \,\#\, \alpha \,\#\, \beta \,\#\, \beta \,\#\, \gamma$. Assume that

$$rk(C(\bar{s})) := \rho > \Omega, \tag{1}$$
$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow C(\bar{s}), \tag{2}$$
$$\mathcal{H} \mid\!\frac{\beta}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow \Delta, \tag{3}$$
$$\mathcal{H} \mid\!\frac{\gamma_i}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\eta_i} \quad \text{for each } 1 \leq i \leq n \text{ and for some } \eta_i \in \mathcal{H} \cap \Omega. \tag{4}$$

Since $rk(C(\bar{s})) := \rho > \Omega$, C cannot be the 'active part' of an axiom, hence if (2) or (3) are axioms of $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ then so is $\Gamma \Rightarrow \Delta$.

If $C(\bar{s})$ was not the principal formula of the last inference in either (2) or (3) then we may apply the induction hypothesis to the premises of that inference and then the same inference again.

So suppose $C(\bar{s})$ was the principal formula of the last inference in both (2) and (3). Since the conclusion of a $(\Sigma^{\mathcal{E}}\text{-}Ref)$ inference always has rank Ω and $rk(C(\bar{s})) := \rho > \Omega$ we may conclude that the last inference of (2) was not $(\Sigma^{\mathcal{E}}\text{-}Ref)$.

Case 1. Suppose $C(\bar{s}) \equiv (\exists x \in s_i)F(x, \bar{s})$, thus we have

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow r \in s_i \wedge F(r, \bar{s}) \quad \alpha_0 < \alpha, \tag{5}$$
$$\mathcal{H} \mid\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\delta} \quad \alpha_1 < \alpha \text{ and } \delta \in \mathcal{H}, \tag{6}$$
$$\mathcal{H} \mid\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_{\xi} \quad \xi, \alpha_2 < \alpha\ ,\ \xi \in \mathcal{H}(\emptyset) \text{ and } \xi \leq \delta, \tag{7}$$
$$\mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), p \in s_i \wedge F(p, \bar{s}) \Rightarrow \Delta \quad \text{for all } p \text{ and } \beta_0 < \beta, \tag{8}$$
$$\mathcal{H} \mid\!\frac{\beta_1}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_i \in \mathbb{E}_{\delta'} \quad \delta', \beta_1 < \beta \text{ and } \delta' \in \mathcal{H}(\emptyset). \tag{9}$$

From (8) we obtain

$$\mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), r \in s_i \wedge F(r, \bar{s}) \Rightarrow \Delta\,. \tag{10}$$

Applying the induction hypothesis to (2), (4) and (10) yields

$$\mathcal{H} \mid\!\frac{\alpha \,\#\, \alpha \,\#\, \beta_0 \,\#\, \beta_0 \,\#\, \gamma}{\rho}\, \Gamma, r \in s_i \wedge F(r, \bar{s}) \Rightarrow \Delta\,. \tag{11}$$

Note that

$$\begin{aligned}\Omega < rk(r \in s_i \wedge F(r,\bar{s})) &= rk(F(r,\bar{s})) + 1\\ &< rk(F(r,\bar{s})) + 2\\ &= rk(C(\bar{s})) := \rho.\end{aligned}$$

So we may apply (Cut) to (4), (5), (7) and (11) giving

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta\#\beta\#\gamma}{\rho}\, \Gamma \Rightarrow \Delta$$

as required. The case where $C(\bar{s}) \equiv (\forall x \in s_i)F(x,\bar{s})$ is similar.

Now suppose $C(\bar{s}) \equiv (\forall x \in^{s_i} s_j)F(x,\bar{s})$, so we have

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow \mathrm{fun}(p, s_i, s_j) \rightarrow F(p,\bar{s}) \quad \text{for all } p \text{ and } \alpha_0 < \alpha, \tag{12}$$

$$\mathcal{H} \mid\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_\delta \quad \alpha_1 < \alpha \text{ and } \delta \in \mathcal{H}(\emptyset), \tag{13}$$

$$\mathcal{H} \mid\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow s_j \in \mathbb{E}_{\delta'} \quad \alpha_2 < \alpha,\, \delta' \in \mathcal{H}(\emptyset) \text{ and } max(\delta,\delta') + 2 \leq \alpha, \tag{14}$$

$$\mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), \mathrm{fun}(r, s_i, s_j) \rightarrow F(r,\bar{s}) \Rightarrow \Delta \quad \beta_0 < \beta, \tag{15}$$

$$\mathcal{H} \mid\!\frac{\beta_1}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow r \in \mathbb{E}_\xi \quad \xi < \beta, \xi \in \mathcal{H}(\emptyset) \text{ and } \beta_1 < \beta, \tag{16}$$

$$\mathcal{H} \mid\!\frac{\beta_2}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_i \in \mathbb{E}_\zeta \quad \zeta \in \mathcal{H}(\emptyset) \text{ and } \beta_2 < \beta, \tag{17}$$

$$\mathcal{H} \mid\!\frac{\beta_3}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_j \in \mathbb{E}_{\zeta'} \quad \zeta' \in \mathcal{H}(\emptyset), \beta_3 < \beta \text{ and } \xi \leq \max(\zeta,\zeta') + 2. \tag{18}$$

As an instance of (12) we have

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow \mathrm{fun}(r, s_i, s_j) \rightarrow F(r,\bar{s})\,. \tag{19}$$

Applying the induction hypothesis to (2), (4) and (15) gives

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta_0\#\beta_0\#\gamma}{\rho}\, \Gamma, \mathrm{fun}(r, s_i, s_j) \rightarrow F(r,\bar{s}) \Rightarrow \Delta\,. \tag{20}$$

Furthermore the induction hypothesis applied to (2), (4) and (16) gives

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta_1\#\beta_1\#\gamma}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_\xi\,. \tag{21}$$

Note that

$$\begin{aligned}\Omega < rk(\mathrm{fun}(r, s_i, s_j) \rightarrow F(r,\bar{s})) &= rk(F(r,\bar{s})) + 1\\ &< rk(F(r,\bar{s})) + 2 = rk(C(\bar{s}))\end{aligned}$$

so we may apply (Cut) to (4), (19), (20), (21) to give

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta \# \beta \# \gamma}{\rho}\right. \Gamma \Rightarrow \Delta \tag{22}$$

as required.

The case where $C(\bar{s}) \equiv (\exists x \in^{s_i} s_j) F(x, \bar{s})$ is similar.

Case 3. Now suppose that $C(\bar{s}) \equiv \forall x F(x, \bar{s})$, so we have

$$\mathcal{H}[\delta] \left|\frac{\alpha_\delta}{\rho}\right. \Gamma, p \in \mathbb{E}_\delta \Rightarrow F(p, \bar{s}) \quad \text{for all } p \text{ and all } \delta < \Omega \text{ with } \alpha_\delta + 3 < \alpha, \tag{23}$$

$$\mathcal{H} \left|\frac{\beta_0}{\rho}\right. \Gamma, C(\bar{s}), F(r, \bar{s}) \Rightarrow \Delta \quad \text{with } \beta_0 + 3 < \beta, \tag{24}$$

$$\mathcal{H} \left|\frac{\beta_1}{\rho}\right. \Gamma, C(\bar{s}) \Rightarrow r \in \mathbb{E}_\xi \quad \text{with } \xi < \beta, \xi \in \mathcal{H}(\emptyset) \text{ and } \beta_1 + 3 < \beta. \tag{25}$$

Since $\xi \in \mathcal{H}(\emptyset)$, from (23) we obtain

$$\mathcal{H} \left|\frac{\alpha_\xi}{\rho}\right. \Gamma, r \in \mathbb{E}_\xi \Rightarrow F(r, \bar{s}). \tag{26}$$

Applying the induction hypothesis to (2), (4) and (24) gives

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta_0 \# \beta_0 \# \gamma}{\rho}\right. \Gamma, F(r, \bar{s}) \Rightarrow \Delta\,. \tag{27}$$

Again applying the induction hypothesis to (2), (4) and (25) gives

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta_1 \# \beta_1 \# \gamma}{\rho}\right. \Gamma \Rightarrow r \in \mathbb{E}_\xi\,. \tag{28}$$

Now a (Cut) applied to (26) and (28) yields

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta \# \beta_1 \# \gamma}{\rho}\right. \Gamma \Rightarrow F(r, \bar{s})\,. \tag{29}$$

Note that

$$\Omega \leq rk(F(r, \bar{s})) < rk(F(r, \bar{s})) + 2 = rk(C) = \rho.$$

So a (Cut) applied to (4), (27), (28) and (29) yields

$$\mathcal{H} \left|\frac{\alpha \# \alpha \# \beta \# \beta \# \gamma}{\rho}\right. \Gamma \Rightarrow \Delta \tag{30}$$

as required.

The case where $C(\bar{s}) \equiv \exists x F(x, \bar{s})$ is similar.

In the cases where $C \equiv A \wedge B$, $A \vee B$, $A \rightarrow B$ or $\neg A$ we may argue as with other intuitionistic systems of a similar nature. □

Theorem 4.10 (Cut Elimination I) *If* $\mathcal{H} \overset{\alpha}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\omega_n(\alpha)}{\underset{\Omega+1}{\vdash}} \Gamma \Rightarrow \Delta$ *for all* $n < \omega$,
where $\omega_0(\alpha) = \alpha$ *and* $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$.

Proof By main induction on n and subsidiary induction on α. The interesting case is where the last inference was (Cut), with cut formula $A(\bar{s})$ such that $rk(A(\bar{s})) = \Omega + n$ and $\bar{s} = s_1, \dots, s_m$ are the only terms occurring $A(\bar{s})$. In this case we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow A(\bar{s}) \quad \text{with } \alpha_0 < \alpha, \tag{1}$$

$$\mathcal{H} \overset{\alpha_1}{\underset{\Omega+n+1}{\vdash}} \Gamma, A(\bar{s}) \Rightarrow \Delta \quad \text{with } \alpha_1 < \alpha, \tag{2}$$

$$\mathcal{H} \overset{\alpha_2}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \quad \text{with } \alpha_2 < \alpha \text{ and } \beta_i \in \mathcal{H} \text{ for each } i = 1, \dots, m. \tag{3}$$

Applying the subsidiary induction hypothesis to (1), (2) and (3) gives

$$\mathcal{H} \overset{\omega^{\alpha_0}}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow A(\bar{s}) \quad \text{with } \alpha_0 < \alpha, \tag{4}$$

$$\mathcal{H} \overset{\omega^{\alpha_1}}{\underset{\Omega+n}{\vdash}} \Gamma, A(\bar{s}) \Rightarrow \Delta \quad \text{with } \alpha_1 < \alpha, \tag{5}$$

$$\mathcal{H} \overset{\omega^{\alpha_2}}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \quad \text{with } \alpha_2 < \alpha \text{ and } \beta_i \in \mathcal{H} \text{ for each } i = 1, \dots, m. \tag{6}$$

Now applying the Reduction Lemma 4.9 to (4), (5) and (6) gives

$$\mathcal{H} \overset{\omega^{\alpha_0} \# \omega^{\alpha_0} \# \omega^{\alpha_1} \# \omega^{\alpha_1} \# \omega^{\alpha_2}}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow \Delta\,. \tag{7}$$

Note that $\omega^{\alpha_0} \# \omega^{\alpha_0} \# \omega^{\alpha_1} \# \omega^{\alpha_1} \# \omega^{\alpha_2} < \omega^{\alpha}$ so by weakening we have

$$\mathcal{H} \overset{\omega^{\alpha}}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow \Delta\,. \tag{8}$$

Finally applying the main induction hypothesis gives

$$\mathcal{H} \overset{\omega_n(\alpha)}{\underset{\Omega+1}{\vdash}} \Gamma \Rightarrow \Delta$$

as required. □

Lemma 4.11 *If* $\gamma \leq \beta < \Omega$ *with* $\beta, \gamma \in \mathcal{H}(\emptyset)$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma$ *then*

$$\mathcal{H} \overset{\alpha+2}{\underset{\rho^*}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\beta$$

where $\rho^* := max(\rho, \beta + 1)$.

Proof If $\gamma = \beta$ the result follows by weakening, so suppose $\gamma < \beta$. Assume that

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma\,. \tag{1}$$

Now as instances of axioms (E4) and (E6) respectively we have

$$\mathcal{H} \vdash_{0}^{0} \Gamma \Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_\beta, \tag{2}$$

$$\mathcal{H} \vdash_{0}^{0} \Gamma, s \in \mathbb{E}_\gamma, \mathbb{E}_\gamma \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta\,. \tag{3}$$

Applying (Cut) to (2) and (3) yields

$$\mathcal{H} \vdash_{\beta+2}^{1} \Gamma, s \in \mathbb{E}_\gamma \Rightarrow s \in \mathbb{E}_\beta\,. \tag{4}$$

Now applying a second (Cut) to (1) and (4) supplies us with

$$\mathcal{H} \vdash_{\rho^*}^{\alpha+2} \Gamma \Rightarrow s \in \mathbb{E}_\beta$$

as required. □

Lemma 4.12 (Boundedness) *Suppose* $\alpha \leq \beta < \Omega$, $\beta \in \mathcal{H}$, *A is a* $\Sigma^{\mathcal{E}}$*-formula and B is a* $\Pi^{\mathcal{E}}$ *formula then*

(i) If $\mathcal{H} \vdash_{\rho}^{\alpha} \Gamma \Rightarrow A$ *then* $\mathcal{H} \vdash_{\rho^*}^{\alpha} \Gamma \Rightarrow A^{\mathbb{E}_\beta}$.

(ii) If $\mathcal{H} \vdash_{\rho}^{\alpha} \Gamma, B \Rightarrow \Delta$ *then* $\mathcal{H} \vdash_{\rho^*}^{\alpha} \Gamma, B^{\mathbb{E}_\beta} \Rightarrow \Delta$,

where $\rho^* := max(\rho, \beta + 1)$.

Proof By induction on α. The interesting case of (i) is where $A \equiv \exists x C(x)$ and A was the principal formula of the last inference which was $(\exists R)$. Note that since $\alpha < \Omega$ the last inference cannot have been $(\Sigma^{\mathcal{E}}\text{-}Ref)$. So we have

$$\mathcal{H} \vdash_{\rho}^{\alpha_0} \Gamma \Rightarrow C(r) \quad \text{with } \alpha_0 + 3 < \alpha, \tag{1}$$

$$\mathcal{H} \vdash_{\rho}^{\alpha_1} \Gamma \Rightarrow r \in \mathbb{E}_\gamma \quad \text{with } \alpha_1 < \alpha, \gamma \in \mathcal{H} \text{ and } \gamma < \alpha. \tag{2}$$

Since $\gamma < \alpha$ we also know that $\gamma < \beta$ so using Lemma 4.11 we get

$$\mathcal{H} \vdash_{\rho^*}^{\alpha_1+2} \Gamma \Rightarrow r \in \mathbb{E}_\beta\,. \tag{3}$$

Now by applying the induction hypothesis to (1) we get

$$\mathcal{H} \vdash_{\rho}^{\alpha_0} \Gamma \Rightarrow C(r)^{\mathbb{E}_\beta}\,. \tag{4}$$

$(\wedge R)$ applied to (3) and (4) yields

$$\mathcal{H} \vdash_{\rho^*}^{\max(\alpha_0+1,\alpha_1+3)} \Gamma \Rightarrow r \in \mathbb{E}_\beta \wedge C(r)^{\mathbb{E}_\beta}\,. \tag{5}$$

Now since $\Gamma \Rightarrow \mathbb{E}_\beta \in \mathbb{E}_{\beta+1}$ is an axiom we may apply $(b\exists R)$ to (2) and (5) giving

$$\mathcal{H} \,\Big|\frac{\alpha}{\rho^*}\, \Gamma \Rightarrow (\exists x \in \mathbb{E}_\beta)C(x)^{\mathbb{E}_\beta}$$

as required.

Now for (ii) the interesting case is where B was the principal formula of the last inference which was $(b\forall L)$, thus $B \equiv \forall x C(x)$. So we have

$$\mathcal{H} \,\Big|\frac{\alpha_0}{\rho}\, \Gamma, B, C(s) \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha, \tag{6}$$

$$\mathcal{H} \,\Big|\frac{\alpha_1}{\rho}\, \Gamma, B \Rightarrow s \in \mathbb{E}_\gamma \quad \text{with } \alpha_1 + 3 < \alpha, \gamma \in \mathcal{H} \text{ and } \gamma < \alpha. \tag{7}$$

Applying the induction hypothesis twice to (6) and once to (7) we get

$$\mathcal{H} \,\Big|\frac{\alpha_0}{\rho}\, \Gamma, B^{\mathbb{E}_\beta}, C(s)^{\mathbb{E}_\beta} \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha, \tag{8}$$

$$\mathcal{H} \,\Big|\frac{\alpha_1}{\rho}\, \Gamma, B^{\mathbb{E}_\beta} \Rightarrow s \in \mathbb{E}_\gamma \quad \text{with } \alpha_1 + 3 < \alpha, \gamma \in \mathcal{H} \text{ and } \gamma < \alpha. \tag{9}$$

Now since $\gamma < \alpha$ we also know that $\gamma < \beta$ so by applying Lemma 4.11 to (9) we get

$$\mathcal{H} \,\Big|\frac{\alpha_1+2}{\rho^*}\, \Gamma, B^{\mathbb{E}_\beta} \Rightarrow s \in \mathbb{E}_\beta\,. \tag{10}$$

Applying $(\rightarrow L)$ to (8) and (10) supplies us with

$$\mathcal{H} \,\Big|\frac{\max(\alpha_0+1, \alpha_1+3)}{\rho^*}\, \Gamma, B^{\mathbb{E}_\beta}, s \in \mathbb{E}_\beta \rightarrow C(s)^{\mathbb{E}_\beta} \Rightarrow \Delta\,. \tag{11}$$

Now applying $(b\forall L)$ to (11), (9) and $\Rightarrow \mathbb{E}_\beta \in \mathbb{E}_{\beta+1}$ which is an instance of axiom (E4), we obtain

$$\mathcal{H} \,\Big|\frac{\alpha}{\rho^*}\, \Gamma, B^{\mathbb{E}_\beta} \Rightarrow \Delta$$

completing the proof. □

Theorem 4.13 (Cut Elimination II; Collapsing) *Suppose $\eta \in \mathcal{H}_\eta$, Δ is a set of at most one $\Sigma^{\mathcal{E}}$ formula and Γ is a finite set of $\Pi^{\mathcal{E}}$ formulae. Then*

$$\mathcal{H}_\eta \,\Big|\frac{\alpha}{\Omega+1}\, \Gamma \Rightarrow \Delta \quad \textit{implies} \quad \mathcal{H}_{\hat{\alpha}} \,\Big|\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow \Delta\,,$$

where $\hat{\alpha} := \eta + \omega^\alpha$.

Proof The proof is by induction on α. Note that since $\eta \in \mathcal{H}_\eta$ we know from Lemma 2.20 that

$$\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}.$$

Case 1. If $\Gamma \Rightarrow \Delta$ is an axiom the result follows easily.

Case 2. If $\Gamma \Rightarrow \Delta$ was the result of a propositional inference we may apply the induction hypothesis to the premises of that inference, and then the same inference again.

Case 3. Suppose the last inference was ($\mathbb{E}$-Lim), then $s \in \mathbb{E}_\gamma$ is a formula in Γ for some limit ordinal γ and

$$\mathcal{H}_\eta[\delta] \vdash^{\alpha_\delta}_{\Omega+1} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Since $\gamma \in \mathcal{H}_\eta(\emptyset) = B^\Omega(\eta + 1)$ and $\gamma < \Omega$ we know that $\gamma < \psi_\Omega(\eta + 1)$ and thus $\delta \in \mathcal{H}_\eta$ for all $\delta < \gamma$. So we have

$$\mathcal{H}_\eta \vdash^{\alpha_\delta}_{\Omega+1} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Now applying the induction hypothesis provides

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\widehat{\alpha_\delta})}_{\psi_\Omega(\widehat{\alpha_\delta})} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Now since $\psi_\Omega(\widehat{\alpha_\delta}) < \psi_\Omega(\hat{\alpha})$ we may apply ($\mathbb{E}$-Lim) to get the desired result.

Case 4. Suppose the last inference was $(b\forall L)$, then $(\forall x \in t)F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma, s \in t \to F(s) \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha, \tag{1}$$

$$\mathcal{H}_\eta \vdash^{\alpha_1}_{\Omega+1} \Gamma \Rightarrow t \in \mathbb{E}_\beta \quad \beta \in \mathcal{H}_\eta(\emptyset) \text{ and } \alpha_1 < \alpha, \tag{2}$$

$$\mathcal{H}_\eta \vdash^{\alpha_2}_{\Omega+1} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \quad \gamma \in \mathcal{H}_\eta(\emptyset), \gamma, \alpha_2 < \alpha \text{ and } \gamma \leq \beta. \tag{3}$$

Since $s \in t \to F(s)$ is also a $\Pi^{\mathcal{E}}$-formula we may immediately apply the induction hypothesis to (1), (2) and (3) giving

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha}_0)}_{\psi_\Omega(\hat{\alpha})} \Gamma, s \in t \to F(s) \Rightarrow \Delta, \tag{4}$$

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha}_1)}_{\psi_\Omega(\hat{\alpha})} \Gamma \Rightarrow t \in \mathbb{E}_\beta, \tag{5}$$

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha}_2)}_{\psi_\Omega(\hat{\alpha})} \Gamma \Rightarrow s \in \mathbb{E}_\gamma. \tag{6}$$

Since $\gamma \in \mathcal{H}_\eta$ we know that $\gamma < \psi_\Omega(\eta + 1)$ and thus $\gamma \in \mathcal{H}_{\hat{\alpha}}$ and $\gamma < \psi_\Omega(\hat{\alpha})$. Moreover $\psi_\Omega(\alpha_i) < \psi_\Omega(\alpha)$ for $i = 0, 1, 2$ so we may apply $(b\forall L)$ to complete this case. The case where the last inference was $(b\exists R)$ is treated in a similar manner.

Case 5. Suppose the last inference was $(b\forall R)_\infty$, then $\Delta = \{(\forall x \in t)F(x)\}$ and

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma \Rightarrow s \in t \to F(s) \quad \text{for all } s, \text{ with } \alpha_0 < \alpha, \tag{7}$$

$$\mathcal{H}_\eta \vdash^{\alpha_1}_{\Omega+1} \Gamma \Rightarrow t \in \mathbb{E}_\beta \quad \text{with } \alpha_1, \beta < \alpha \text{ and } \beta \in \mathcal{H}_\eta. \tag{8}$$

Applying the induction hypothesis to (7) and (8) yields

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha_1})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow t \in \mathbb{E}_\beta, \tag{9}$$

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow s \in t \rightarrow F(s). \tag{10}$$

Note that since $\beta \in \mathcal{H}_\eta$ we know that $\beta < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$, thus applying $(b\forall R)_\infty$ to (10) (noting that $\psi_\Omega(\hat{\alpha_0}) + 1 < \psi_\Omega(\hat{\alpha})$) gives the desired result. The case where the last inference was $(b\exists L)_\infty$ is treated in a similar manner.

Case 6. Now suppose the last inference was $(\mathcal{E}b\exists L)_\infty$, so $(\exists x \in {}^s t)F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma, \mathrm{fun}(p, s, t) \wedge F(p) \Rightarrow \Delta \quad \text{for all } p, \text{ with } \alpha_0 < \alpha, \tag{11}$$

$$\mathcal{H}_\eta \left|\frac{\alpha_1}{\Omega+1}\right. \Gamma \Rightarrow s \in \mathbb{E}_\beta \quad \text{with } \beta \in \mathcal{H}_\eta \text{ and } \alpha_1 < \alpha, \tag{12}$$

$$\mathcal{H}_\eta \left|\frac{\alpha_2}{\Omega+1}\right. \Gamma \Rightarrow t \in \mathbb{E}_\gamma \quad \text{with } \alpha_2 < \alpha, \gamma \in \mathcal{H}_\eta \text{ and } \max(\beta, \gamma) + 2 \leq \alpha. \tag{13}$$

By assumption $\mathrm{fun}(p, s, t) \wedge F(p)$ is a $\Pi^{\mathcal{E}}$ formula so we may apply the induction hypothesis to (11), (12) and (13) giving

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma, \mathrm{fun}(p, s, t) \wedge F(p) \Rightarrow \Delta \quad \text{for all } p, \tag{14}$$

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha_1})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow s \in \mathbb{E}_\beta, \tag{15}$$

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha_2})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow t \in \mathbb{E}_\gamma . \tag{16}$$

Since $\psi_\Omega(\hat{\alpha_i}) < \psi_\Omega(\hat{\alpha})$ for $i = 0, 1, 2$ and $\beta, \gamma \in \mathcal{H}_\eta$ means that $\max(\beta, \gamma) + 2 < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$ we may apply $(\mathcal{E}b\exists L)_\infty$ to (14), (15) and (16) to complete this case. The case where the last inference was $(\mathcal{E}b\forall R)_\infty$ may be treated in a similar manner.

Case 7. Now suppose the last inference was $(\mathcal{E}b\forall R)$, so $\Delta = \{(\exists x \in {}^s t)F(x)\}$ and we have

$$\mathcal{H}_\eta \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma \Rightarrow \mathrm{fun}(p, s, t) \wedge F(p) \quad \text{for all } p \text{ with } \alpha_0 < \alpha, \tag{17}$$

$$\mathcal{H}_\eta \left|\frac{\alpha_1}{\Omega+1}\right. \Gamma \Rightarrow s \in \mathbb{E}_\beta \quad \text{with } \beta \in \mathcal{H}_\eta(\emptyset) \text{ and } \alpha_1 < \alpha, \tag{18}$$

$$\mathcal{H}_\eta \left|\frac{\alpha_2}{\Omega+1}\right. \Gamma \Rightarrow t \in \mathbb{E}_\gamma \quad \text{with } \gamma \in \mathcal{H}_\eta(\emptyset) \text{ and } \alpha_2 < \alpha, \tag{19}$$

$$\mathcal{H}_\eta \left|\frac{\alpha_3}{\Omega+1}\right. \Gamma \Rightarrow p \in \mathbb{E}_\delta \quad \alpha_3, \delta < \alpha, \delta \in \mathcal{H}_\eta(\emptyset) \text{ and } \delta \leq \max(\beta, \gamma) + 2. \tag{20}$$

Since $\mathrm{fun}(p, s, t) \wedge F(p)$ is a $\Sigma^{\mathcal{E}}$ formula we can apply the induction hypothesis to (17), (18), (19) and (20) followed by $(\mathcal{E}b\forall R)$, in a similar manner to Case 4. The case where the last inference was $(\mathcal{E}b\forall L)$ can also be treated in a similar manner.

Now suppose the last inference was $(\forall L)$, so $\forall x F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma, F(s) \Rightarrow \Delta \quad \text{with } \alpha_0 + 3 < \alpha, \tag{21}$$

$$\mathcal{H}_\eta \vdash^{\alpha_1}_{\Omega+1} \Gamma \Rightarrow s \in \mathbb{E}_\beta \quad \beta, \alpha_1 + 3 < \alpha \text{ and } \beta \in \mathcal{H}_\eta(\emptyset). \tag{22}$$

Since $F(s)$ is $\Pi^{\mathcal{E}}$ we may immediately apply the induction hypothesis to (21) and (22) giving

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha}_0)}_{\psi_\Omega(\hat{\alpha})} \Gamma, F(s) \Rightarrow \Delta, \tag{23}$$

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha}_1)}_{\psi_\Omega(\hat{\alpha})} \Gamma \Rightarrow s \in \mathbb{E}_\beta\,. \tag{24}$$

Now since $\beta \in \mathcal{H}_\eta$ we know that $\beta < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$ hence we may apply $(\forall L)$ to (23) and (24) to complete this case. The case where the last inference was $(\exists R)$ can be treated in a similar manner.

Case 9. Now suppose the last inference was $(\Sigma^{\mathcal{E}}\text{-}Ref)$, so $\Delta = \{\exists z A^z\}$ where A is a $\Sigma^{\mathcal{E}}$ formula and

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma \Rightarrow A \qquad\qquad \text{with } \alpha_0 + 1, \Omega < \alpha. \tag{25}$$

We may immediately apply the induction hypothesis to (25) giving

$$\mathcal{H}_{\hat{\alpha}_0} \vdash^{\psi_\Omega(\hat{\alpha}_0)}_{\psi_\Omega(\hat{\alpha}_0)} \Gamma \Rightarrow A\,. \tag{26}$$

Applying Boundedness Lemma 4.12(i) to (26) provides

$$\mathcal{H}_{\hat{\alpha}_0} \vdash^{\psi_\Omega(\hat{\alpha}_0)}_{\psi_\Omega(\hat{\alpha}_0)+2} \Gamma \Rightarrow A^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}}\,. \tag{27}$$

Now as an instance of axiom (E4) we have

$$\mathcal{H}_{\hat{\alpha}_0} \vdash^{0}_{0} \Rightarrow \mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)} \in \mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)+1}\,. \tag{28}$$

Since $\psi_\Omega(\hat{\alpha}_0) + 1 \in \mathcal{H}_{\hat{\alpha}}$ and $\psi_\Omega(\hat{\alpha}_0) + 1 < \psi_\Omega(\hat{\alpha})$ we may apply $(\exists R)$ to (27) and (28) to complete the case.

Now suppose the last inference was (Cut), so that we have

$$\mathcal{H}_\eta \vdash^{\alpha_0}_{\Omega+1} \Gamma \Rightarrow A(\bar{s}) \quad \text{with } \alpha_0 < \alpha, \tag{29}$$

$$\mathcal{H}_\eta \vdash^{\alpha_1}_{\Omega+1} \Gamma, A(\bar{s}) \Rightarrow \Delta \quad \text{with } \alpha_1 < \alpha, \tag{30}$$

$$\mathcal{H}_\eta \vdash^{\alpha_2}_{\Omega+1} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \quad \text{with } \alpha_2 < \alpha, \bar{\beta} \in \mathcal{H}_\eta \text{ and } \|A(\bar{s})\|_{\bar{\beta}} \leq \Omega. \tag{31}$$

Subcase 10.1: If $\|A(\bar{s})\|_{\bar{\beta}} < \Omega$ it follows from $\bar{\beta} \in \mathcal{H}_\eta = B^\Omega(\eta + 1)$ that $\|A(\bar{s})\|_{\bar{\beta}} \in B^\Omega(\eta + 1)$ and thus $\|A(\bar{s})\|_{\bar{\beta}} < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$. Also A is $\Delta_0^{\mathcal{E}}$, thus we may apply the induction hypothesis to (29), (30) and (31) followed by (Cut) to complete this (sub)case.

Subcase 10.2: Now suppose $\|A(\bar{s})\|_{\bar{\beta}} = \Omega$. Then either $A \equiv \forall x F(x)$ or $A \equiv \exists x F(x)$ with F a $\Delta_0^{\mathcal{E}}$ formula. The two cases are dual, we assume that the former is the case. Thus A is $\Pi^{\mathcal{E}}$, so we may apply the induction hypothesis to (30) giving

$$\mathcal{H}_{\hat{\alpha}_1} \vdash^{\psi_\Omega(\hat{\alpha}_1)}_{\psi_\Omega(\hat{\alpha}_1)} \Gamma, A(\bar{s}) \Rightarrow \Delta \,. \tag{32}$$

Applying Boundedness Lemma 4.12(ii) to (32) yields

$$\mathcal{H}_{\hat{\alpha}} \vdash^{\max(\psi_\Omega(\hat{\alpha}_0), \psi_\Omega(\hat{\alpha}_1))}_{\psi_\Omega(\hat{\alpha}_1)} \Gamma, A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}} \Rightarrow \Delta \,. \tag{33}$$

Now applying Lemma 4.8(xi) (persistence) to (29) gives

$$\mathcal{H}_{\hat{\alpha}_0} \vdash^{\alpha_0}_{\Omega+1} \Gamma \Rightarrow A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}} \,. \tag{34}$$

Noting that $A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}}$ is $\Delta_0^{\mathcal{E}}$ we may apply the induction hypothesis to (34) giving

$$\mathcal{H}_{\alpha^*} \vdash^{\psi_\Omega(\alpha^*)}_{\psi_\Omega(\alpha^*)} \Gamma \Rightarrow A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}} \,. \tag{35}$$

where $\alpha^* := \hat{\alpha}_0 + \omega^{\Omega+\alpha_0}$. Now applying the induction hypothesis to (31) gives

$$\mathcal{H}_{\hat{\alpha}_2} \vdash^{\psi_\Omega(\hat{\alpha}_2)}_{\psi_\Omega(\hat{\alpha}_2)} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \,. \tag{36}$$

Now as an instance of axiom (E4) we have

$$\mathcal{H}_{\hat{\alpha}} \vdash^{0}_{0} \Rightarrow \mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)} \in \mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)+1} \,. \tag{37}$$

Since $\bar{\beta} \in B^\Omega(\eta + 1)$ we get

$$\|A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha}_0)}}\|_{\bar{\beta}, \psi_\Omega(\hat{\alpha}_0)+1} = \psi_\Omega(\hat{\alpha}_0) + 1 < \psi_\Omega(\hat{\alpha}).$$

It remains to note that

$$\alpha^* = \eta + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \eta + \omega^{\Omega+\alpha} = \hat{\alpha}$$

and thus $\psi_\Omega(\alpha^*) < \psi_\Omega(\alpha)$. So we may apply (Cut) to (33), (35), (36) and (37) to conclude

$$\mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow \Delta$$

as required. □

4.4 Embedding IKP($\mathcal{E}$) into IRS$_{\Omega}^{\mathcal{P}}$

Definition 4.14 If $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ is an intuitionistic sequent of **IKP**($\mathcal{E}$) with exactly the free variables $\bar{a} = a_1, \ldots, a_n$ and containing the formulas $A_1(\bar{a}), \ldots, A_m(\bar{a})$ then

$$no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]) := \omega^{\|A_1\|_{\bar{\beta}}} \# \cdots \# \omega^{\|A_m\|_{\bar{\beta}}}.$$

For terms $\bar{s} := s_1, \ldots, s_n$ and ordinals $\bar{\beta} := \beta_1, \ldots, \beta_n$ the expression $\bar{s} \in \mathbb{E}_{\bar{\beta}}$ will be considered shorthand for $s_1 \in \mathbb{E}_{\beta_1}, \ldots, s_n \in \mathbb{E}_{\beta_n}$

The expression $\Vdash \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for

$$\mathcal{H}[\bar{\beta}] \left|\frac{no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}])}{0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

For any operator $\mathcal{H}$ and any ordinals $\bar{\beta} < \Omega$.

The expression $\Vdash^{\alpha}_{\rho} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for

$$\mathcal{H}[\bar{\beta}] \left|\frac{no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]) \# \alpha}{\rho}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

For any operator $\mathcal{H}$ and any ordinals $\bar{\beta} < \Omega$.

As might be expected $\Vdash^{\alpha} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ and $\Vdash_{\rho} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for $\Vdash^{\alpha}_{0} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ and $\Vdash^{0}_{\rho} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ respectively.

Lemma 4.15 *For any formula $A(\bar{a})$ of **IKP**($\mathcal{E}$) containing exactly the free variables displayed and any **IRS**$_{\Omega}^{\mathbb{E}}$ terms $\bar{s} = s_1, \ldots, s_n$*

$$\Vdash_{\Omega} A(\bar{s}) \Rightarrow A(\bar{s}).$$

Proof By induction on the construction of the formula A. If A is $\Delta_0^{\mathcal{E}}$ then this is an instance of axiom (E1).

Suppose $A(\bar{s}) \equiv \forall x F(x, \bar{s})$. For each $\gamma < \Omega$ we define

$$\alpha^{\gamma} := \gamma + no_{\gamma,\bar{\beta}}(F(t, \bar{s}) \Rightarrow F(t, \bar{s})),$$

note that

$$\gamma < \alpha^{\gamma} < \alpha^{\gamma} + 8 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s})).$$

By axiom (E1) we have

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash^{0}_{0} t \in \mathbb{E}_{\gamma} \Rightarrow t \in \mathbb{E}_{\gamma} \quad \text{for all } t \text{ and all } \gamma < \Omega. \tag{1}$$

Now from the induction hypothesis we have

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash^{\alpha^{\gamma}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\gamma}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \gamma < \Omega. \tag{2}$$

It is worth noting that this use of the induction hypothesis is where we really need cuts of $\bar{\beta}$-rank arbitrarily high in Ω. Applying $(\forall L)$ to (1) and (2) yields

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash^{\alpha^{\gamma}+4}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\gamma}, A(\bar{s}) \Rightarrow F(t, \bar{s})$$

to which we may apply $(\forall R)_{\infty}$ to get the desired result.

Case 2. Now suppose $A \equiv (\forall x \in s_i) F(x, \bar{s})$. From the induction hypothesis we have

$$\mathcal{H}[\delta, \bar{\beta}] \vdash^{(\omega^{\|F(t,\bar{s})\|_{\delta,\bar{\beta}}}) \cdot 2}_{\Omega} t \in \mathbb{E}_{\delta}, \bar{s} \in \mathbb{E}_{\bar{\beta}}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \delta < \Omega. \tag{3}$$

In particular when $\delta = \beta_i$ in (3) we have

$$\mathcal{H}[\delta, \bar{\beta}] \vdash^{\alpha_0}_{\Omega} t \in \mathbb{E}_{\beta_i}, \bar{s} \in \mathbb{E}_{\bar{\beta}}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \tag{4}$$

where $\alpha_0 := (\omega^{\|F(t,\bar{s})\|_{\beta_i,\bar{\beta}}}) \cdot 2$. Now as an instance of axiom (E6) we have

$$\mathcal{H}[\bar{\beta}] \vdash^{0}_{0} s_i \in \mathbb{E}_{\beta_i}, t \in s_i \Rightarrow t \in \mathbb{E}_{\beta_i} \,. \tag{5}$$

Now applying (Cut) to (4) and (5) yields

$$\mathcal{H}[\bar{\beta}] \vdash^{\alpha_0+1}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \,. \tag{6}$$

As an instance of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}] \vdash^{0}_{0} t \in s_i \Rightarrow t \in s_i \,. \tag{7}$$

Applying $(\rightarrow L)$ to (6) and (7) yields

$$\mathcal{H}[\bar{\beta}] \vdash^{\alpha_0+2}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, t \in s_i \rightarrow F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \,. \tag{8}$$

An application of $(b\forall L)$ to (5) and (8) provides

$$\mathcal{H}[\bar{\beta}] \vdash^{\alpha_0+3}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, (\forall x \in s_i) F(x, \bar{s}) \Rightarrow F(t, \bar{s}) \,.$$

Finally using $(\rightarrow R)$ followed by $(b\forall R)_\infty$ and noting that $\alpha_0 + 5 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s}))$ we get the desired result.

Case 3. Suppose that $A \equiv (\exists x \in^{s_i} s_j)F(x, \bar{s})$. From the induction hypothesis we know that

$$\mathcal{H}[\bar{\beta}, \delta] \left|\!\frac{(\omega^{\|F(t,\bar{s})\|_{\delta,\bar{\beta}}})\cdot 2}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\delta}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \delta < \Omega. \tag{9}$$

In particular when $\delta = \gamma := \max(\beta_i, \beta_j) + 2$ we have

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\gamma}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t, \tag{10}$$

where $\alpha_0 := (\omega^{\|F(t,\bar{s})\|_{\bar{\beta},\gamma}}) \cdot 2$. Now as an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{0}{0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \text{fun}(t, s_i, s_j) \Rightarrow t \in \mathbb{E}_{\gamma}\,. \tag{11}$$

Applying (Cut) to (10) and (11) gives

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0+1}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \text{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow F(t, \bar{s})\,. \tag{12}$$

As an instance of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{0}{0}\right. \text{fun}(t, s_i, s_j) \Rightarrow \text{fun}(t, s_i, s_j)\,. \tag{13}$$

Applying $(\wedge R)$ to (12) and (13) gives

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0+2}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \text{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow \text{fun}(t, s_i, s_j) \wedge F(t, \bar{s})\,. \tag{14}$$

Now applying $(\mathcal{E}b\exists R)$ to (11) and (14) yields

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0+3}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \text{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow (\exists x \in^{s_i} s_j)F(x, \bar{s})\,. \tag{15}$$

Two applications of $(\wedge L)$ gives

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0+5}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \text{fun}(t, s_i, s_j) \wedge F(t, \bar{s}) \Rightarrow (\exists x \in^{s_i} s_j)F(x, \bar{s})\,. \tag{16}$$

Finally using $(\mathcal{E}b\exists L)_\infty$ gives

$$\mathcal{H}[\bar{\beta}] \left|\!\frac{\alpha_0+6}{\Omega}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\exists x \in^{s_i} s_j)F(x, \bar{s}) \Rightarrow (\exists x \in^{s_i} s_j)F(x, \bar{s})\,. \tag{17}$$

It remains to note that $\alpha_0 + 6 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s}))$ to complete this case.

All other cases are either propositional, for which the proof is standard or may be regarded as dual to one of the three above. □

Lemma 4.16 (Extensionality) *For any formula $A(\bar a)$ of* $\mathbf{IKP}(\mathcal{E})$ *(not necessarily with all free variables displayed) and any* $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$ *terms* $\bar s := s_1, \ldots, s_n, \bar t := t_1, \ldots, t_n$ *we have*

$$\Vdash_{\Omega} \bar s = \bar t, A(\bar s) \Rightarrow A(\bar t)$$

where $\bar s = \bar t$ *is shorthand for* $s_1 = t_1, \ldots, s_n = t_n$.

Proof If $A(\bar s)$ is $\Delta_0^{\mathcal{E}}$ then this is an instance of axiom (E3). The remainder of the proof is by induction on $rk(A(\bar s))$, note that since A is assumed to contain an unbounded quantifier

$$rk(A) = \|A(\bar s)\|_{\bar\beta} \geq \Omega \quad \text{for any ordinals } \bar\beta < \Omega.$$

Case 1. Suppose $A(\bar s) \equiv \forall x F(x, \bar s)$. By the induction hypothesis we have

$$\mathcal{H}[\bar\beta, \bar\gamma, \delta] \vdash^{no_{\bar\beta,\bar\gamma,\delta}(\bar s = \bar t, F(r,\bar s) \Rightarrow F(r,\bar t))}_{\Omega} \bar s \in \mathbb{E}_{\bar\beta}, \bar t \in \mathbb{E}_{\bar\gamma}, r \in \mathbb{E}_{\delta}, \bar s = \bar t, F(r, \bar s) \Rightarrow F(r, \bar t)$$

for all r and all $\delta < \Omega$. For ease of reading we suppress the other terms possibly occurring in $F(r, \bar s)$ and the assumptions about their locations in the $\mathbb{E}$ hierarchy since these do not affect the proof. By virtue of axiom (E1) we have

$$\mathcal{H}[\bar\beta, \bar\gamma, \delta] \vdash^{0}_{0} r \in \mathbb{E}_{\delta} \Rightarrow r \in \mathbb{E}_{\delta}\,.$$

Hence we may apply $(\forall L)$ to obtain

$$\mathcal{H}[\bar\beta, \bar\gamma, \delta] \vdash^{\alpha_\delta}_{\Omega} \bar s \in \mathbb{E}_{\bar\beta}, t \in \mathbb{E}_{\bar\gamma}, \bar s = \bar t, r \in \mathbb{E}_{\delta}, \forall x F(x, \bar s) \Rightarrow F(r, \bar t)$$

where $\alpha_\delta := \delta + no_{\bar\beta,\bar\gamma,\delta}(\bar s = \bar t, F(r, \bar s) \Rightarrow F(r, \bar t)) + 1$. Note that

$$\alpha_\delta + 3 < no_{\bar\beta,\bar\gamma}(\bar s = \bar t, A(\bar s) \Rightarrow A(\bar t)) =: \alpha.$$

Hence we may apply $(\forall R)_\infty$ to obtain

$$\mathcal{H}[\bar\beta, \bar\gamma] \vdash^{\alpha}_{\Omega} \bar s \in \mathbb{E}_{\bar\beta}, \bar t \in \mathbb{E}_{\bar\gamma}, \bar s = \bar t, A(\bar s) \Rightarrow A(\bar t)$$

as required.

Case 2. Now suppose $A(\bar s) \equiv (\forall x \in^{s_i} s_j) F(x, \bar s)$. Using the induction hypothesis we have

$$\mathcal{H}[\bar\beta, \bar\gamma, \delta] \vdash^{\alpha_0}_{\Omega} \bar s \in \mathbb{E}_{\bar\beta}, \bar t \in \mathbb{E}_{\bar\gamma}, r \in \mathbb{E}_{\delta}, \bar s = \bar t, F(r, \bar s) \Rightarrow F(r, \bar t) \tag{1}$$

for any term r and any $\delta < \Omega$, where $\alpha_0 = no_{\bar\beta,\bar\gamma,\delta}(\bar s = \bar t, F(r, \bar s) \Rightarrow F(r, \bar t))$. At this point we set $\delta = \max(\beta_i, \beta_j) + 2$, note that $\delta \in \mathcal{H}[\bar\beta, \bar\gamma]$. By virtue of axiom (E1) we have

$$\mathcal{H}[\bar\beta, \bar\gamma] \vdash^{0}_{0} \mathrm{fun}(r, s_i, s_j) \Rightarrow \mathrm{fun}(r, s_i, s_j). \tag{2}$$

Hence by ($\to L$) we get

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{\alpha_0+1}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \bar{t}\in\mathbb{E}_{\bar{\gamma}}\ , r\in\mathbb{E}_{\delta}, \bar{s}=\bar{t}, \\ \text{fun}(r,s_i,s_j)\to F(r,\bar{s}), \text{fun}(r,s_i,s_j)\Rightarrow F(r,\bar{t}). \tag{3}$$

As an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{0}_{0} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \text{fun}(r,s_i,s_j)\Rightarrow r\in\mathbb{E}_{\delta}\ . \tag{4}$$

An application of (Cut) to (3) and (4) yields

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{\alpha_0+2}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \bar{t}\in\mathbb{E}_{\bar{\gamma}}\ , \bar{s}=\bar{t}, \\ \text{fun}(r,s_i,s_j)\to F(r,\bar{s}), \text{fun}(r,s_i,s_j)\Rightarrow F(r,\bar{t}). \tag{5}$$

Now applying ($\mathcal{E}b\forall L$) to (4) and (5) gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{\alpha_0+3}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \bar{t}\in\mathbb{E}_{\bar{\gamma}}, \bar{s}=\bar{t}, (\forall x\in^{s_i}s_j)F(x,\bar{s}), \text{fun}(r,s_i,s_j)\Rightarrow F(r,\bar{t}). \tag{6}$$

Note that $\alpha_0\geq\Omega$ since F is not $\Delta_0^{\mathcal{E}}$, so we don't have to worry about the condition $\delta<\alpha_0+3$. Now as an instance of axiom ($E3$) we have

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{0}_{0} \bar{s}=\bar{t}, \text{fun}(r,t_i,t_j)\Rightarrow \text{fun}(r,s_i,s_j)\ . \tag{7}$$

Also axiom (E10) gives rise to

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{0}_{0} \bar{t}\in\mathbb{E}_{\bar{\gamma}}, \text{fun}(r,t_i,t_j)\Rightarrow r\in\mathbb{E}_{\eta} \quad \text{where } \eta=\max(\gamma_i,\gamma_j)+2. \tag{8}$$

Applying (Cut) to (6), (7) and (8) gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{\alpha_0+4}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \bar{t}\in\mathbb{E}_{\bar{\gamma}}, \bar{s}=\bar{t}, (\forall x\in^{s_i}s_j)F(x,\bar{s}), \text{fun}(r,t_i,t_j)\Rightarrow F(r,\bar{t}). \tag{9}$$

Now ($\to R$) gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma}]\vdash^{\alpha_0+5}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}}, \bar{t}\in\mathbb{E}_{\bar{\gamma}}, \bar{s}=\bar{t}, (\forall x\in^{s_i}s_j)F(x,\bar{s})\Rightarrow \text{fun}(r,t_i,t_j)\to F(r,\bar{t}). \tag{10}$$

Finally we may apply $(\mathcal{E}b\forall R)_\infty$, noting that $\alpha_0+6<no_{\bar{\beta},\bar{\gamma}}(\bar{s}=\bar{t}, A(\bar{s})\Rightarrow A(\bar{t}))$ to complete this case.

Note that it could also be the case that $A(\bar{s})\equiv(\forall x\in^{p}q)F(x,\bar{s})$ where p and/or q is not a member of $\bar{s}$. The following case is an example of this kind of thing.

Case 3. Suppose $A(\bar{s})\equiv(\exists x\in p)F(x,\bar{s},p)$, where p is not present in $\bar{s}$. By the induction hypothesis we have

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{\alpha_0}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}},\bar{t}\in\mathbb{E}_{\bar{\gamma}},p\in\mathbb{E}_{\delta},r\in\mathbb{E}_{\delta},\bar{s}=\bar{t},F(r,\bar{s},p)\Rightarrow F(r,\bar{t},p) \tag{11}$$

where $\alpha_0 := no_{\bar{\beta},\bar{\gamma},\delta,\delta}(\bar{s}=\bar{t},F(r,\bar{s},p)\Rightarrow F(r,\bar{t},p))$. As an instance of axiom (E1) we have

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{0}_{0} r\in p\Rightarrow r\in p\,. \tag{12}$$

Applying $(\wedge R)$ to (11) and (12) yields

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{\alpha_0+1}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}},\bar{t}\in\mathbb{E}_{\bar{\gamma}},p\in\mathbb{E}_{\delta},r\in\mathbb{E}_{\delta},\bar{s}=\bar{t},F(r,\bar{s},p),r\in p\Rightarrow r\in p\wedge F(r,\bar{t},p)\,. \tag{13}$$

As an instance of axiom (E6) we have

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{0}_{0} p\in\mathbb{E}_{\delta},r\in p\Rightarrow r\in\mathbb{E}_{\delta}\,. \tag{14}$$

(Cut) applied to (12) and (13) gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{\alpha_0+2}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}},\bar{t}\in\mathbb{E}_{\bar{\gamma}},p\in\mathbb{E}_{\delta},\bar{s}=\bar{t},F(r,\bar{s},p),r\in p\Rightarrow r\in p\wedge F(r,\bar{t},p)\,. \tag{15}$$

Now $(b\exists R)$ gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{\alpha_0+3}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}},\bar{t}\in\mathbb{E}_{\bar{\gamma}},p\in\mathbb{E}_{\delta},\bar{s}=\bar{t},F(r,\bar{s},p),r\in p\Rightarrow A(\bar{s})\,. \tag{16}$$

Two applications of $(\wedge L)$ gives

$$\mathcal{H}[\bar{\beta},\bar{\gamma},\delta]\vdash^{\alpha_0+5}_{\Omega} \bar{s}\in\mathbb{E}_{\bar{\beta}},\bar{t}\in\mathbb{E}_{\bar{\gamma}},p\in\mathbb{E}_{\delta},\bar{s}=\bar{t},r\in p\wedge F(r,\bar{s},p)\Rightarrow A(\bar{s})\,. \tag{17}$$

To which we may apply $(b\exists L)$ to complete this case.

All other cases are similar to one of those above. □

Lemma 4.17 (Set induction) *For any formula $F(a)$ of **IKP**$(\mathcal{E})$ we have*

$$\Vdash_{\Omega}\Rightarrow\forall x[(\forall y\in x)F(y)\rightarrow F(x)]\rightarrow\forall x F(x).$$

Proof Let $\mathcal{H}$ be an arbitrary operator and let

$$A := \forall x[(\forall y\in x)F(y)\rightarrow F(x)].$$

Let $\bar{p}$ be the terms other than s that occur in $F(s)$, sub-terms not included. Let $\bar{\mathcal{H}} := \mathcal{H}[\bar{\beta}]$ where $\bar{\beta}$ is an arbitrary choice of ordinals $<\Omega$. In the remainder of the proof we shall just write $\bar{\mathcal{H}}\vdash^{\alpha}_{\rho}\Gamma\Rightarrow\Delta$ instead of $\mathcal{H}[\bar{\beta}]\vdash^{\alpha}_{\rho}\bar{p}\in\mathbb{E}_{\bar{\beta}},\Gamma\Rightarrow\Delta$, since $\bar{p}\in\mathbb{E}_{\bar{\beta}}$ will always remain a side formula in the derivation. We claim that

$$\bar{\mathcal{H}}[\gamma]\vdash^{\omega^{rk(A)}\#\omega^{\gamma+1}}_{\Omega} A,s\in\mathbb{E}_{\gamma}\Rightarrow F(s)\quad\text{for all }\gamma<\Omega\text{ and all terms }s. \tag{*}$$

Note that since A contains an unbounded quantifier $rk(A) = no_{\bar{\beta}}(A)$. We prove the claim by induction on γ. Thus the induction hypothesis supplies us with

$$\bar{\mathcal{H}}[\delta] \,\vdash^{\omega^{rk(A)} \# \omega^{\delta+1}}_{\Omega} A, t \in \mathbb{E}_{\delta} \Rightarrow F(t) \quad \text{for all } \delta < \gamma \text{ and all terms } t. \tag{1}$$

So by weakening we have

$$\bar{\mathcal{H}}[\gamma, \delta] \,\vdash^{\omega^{rk(A)} \# \omega^{\delta+1}}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s, t \in \mathbb{E}_{\delta} \Rightarrow F(t)\,. \tag{2}$$

Case 1. Suppose $\gamma = \gamma_0 + 1$, so a special case of (2) becomes

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma}}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s, t \in \mathbb{E}_{\gamma_0} \Rightarrow F(t)\,. \tag{3}$$

As an instance of axiom (E7) we have

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{0}_{0} s \in \mathbb{E}_{\gamma}, t \in s \Rightarrow t \in \mathbb{E}_{\gamma_0}\,. \tag{4}$$

Applying (Cut) to (3) and (4) yields

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma}+1}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s \Rightarrow F(t)\,. \tag{5}$$

$(\rightarrow R)$ followed by $(b\forall R)_{\infty}$ provides

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma}+3}_{\Omega} A, s \in \mathbb{E}_{\gamma} \Rightarrow (\forall x \in s)F(x)\,. \tag{6}$$

Now from Lemma 4.15 we have

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{no_{\bar{\beta},\gamma}(F(s) \Rightarrow F(s))}_{\Omega} s \in \mathbb{E}_{\gamma}, F(s) \Rightarrow F(s)\,. \tag{7}$$

Since $no_{\bar{\beta},\gamma}(F(s) \Rightarrow F(s)) < \omega^{rk(A)}$, by $(\rightarrow L)$ we get

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma}+4}_{\Omega} A, s \in \mathbb{E}_{\gamma}, (\forall x \in s)F(x) \rightarrow F(s) \Rightarrow F(s)\,. \tag{8}$$

To which we may apply $(\forall L)$ giving

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma+1}}_{\Omega} A, s \in \mathbb{E}_{\gamma} \Rightarrow F(s) \tag{9}$$

as required.

Case 2. Now suppose γ is a limit ordinal. Applying ($\mathbb{E}$-Lim) to (2) provides us with

$$\bar{\mathcal{H}}[\gamma] \,\vdash^{\omega^{rk(A)} \# \omega^{\gamma}}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s, t \in \mathbb{E}_{\gamma} \Rightarrow F(t)\,. \tag{10}$$

As an instance of axiom (E6) we have

$$\bar{\mathcal{H}}[\gamma]\left|\frac{0}{0}\right. s \in \mathbb{E}_\gamma, t \in s \Rightarrow t \in \mathbb{E}_\gamma \,. \tag{11}$$

An application of (Cut) to (10) and (11) yields

$$\bar{\mathcal{H}}[\gamma]\left|\frac{\omega^{rk(A)} \# \omega^\gamma + 1}{\Omega}\right. A, s \in \mathbb{E}_\gamma, t \in s \Rightarrow F(t) \,. \tag{12}$$

The remainder of this case can proceed exactly as in Case 1 from (5) onwards. Thus the claim (*) is verified.

Finally applying $(\forall R)_\infty$ to (*) gives

$$\bar{\mathcal{H}}\left|\frac{\omega^{rk(A)} \# \Omega}{\Omega}\right. A \Rightarrow \forall x F(x) \,.$$

Finally noting that $\omega^{rk(A)} \# \Omega < no_{\bar{\beta}}(A \to \forall x F(x))$ we may apply $(\to R)$ to complete the proof. □

Lemma 4.18 (Infinity) *For any operator $\mathcal{H}$ we have*

$$\mathcal{H}\left|\frac{\omega+4}{\omega}\right. \Rightarrow \exists x[(\forall y \in x)(\exists z \in x)(y \in z) \wedge (\exists y \in x)(y \in x)] \,.$$

Proof Firstly note that by Definition 2.6 1, $\omega \in \mathcal{H}$. We have the following derivation trees in $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$.

$$
(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(\mathbb{E}\text{-Lim})\ \dfrac{(b\exists R)\ \dfrac{(\wedge R)\ \dfrac{(\text{Cut})\ \dfrac{\dfrac{\text{Axiom (E6)}}{\mathcal{H}\left|\frac{0}{0}\right. s \in \mathbb{E}_n, \mathbb{E}_n \in \mathbb{E}_{n+1} \Rightarrow s \in \mathbb{E}_{n+1}} \quad \dfrac{\text{Axiom (E4)}}{\mathcal{H}\left|\frac{0}{0}\right. \Rightarrow \mathbb{E}_n \in \mathbb{E}_{n+1}}}{\mathcal{H}\left|\frac{1}{n+3}\right. s \in \mathbb{E}_n \Rightarrow s \in \mathbb{E}_{n+1}} \quad \dfrac{\text{Axiom (E4)}}{\mathcal{H}\left|\frac{0}{0}\right. \Rightarrow \mathbb{E}_{n+1} \in \mathbb{E}_\omega}}{\mathcal{H}\left|\frac{2}{n+3}\right. s \in \mathbb{E}_n \Rightarrow \mathbb{E}_{n+1} \in \mathbb{E}_\omega \wedge s \in \mathbb{E}_{n+1}}}{\mathcal{H}\left|\frac{n+3}{n+3}\right. s \in \mathbb{E}_n \Rightarrow (\exists z \in \mathbb{E}_\omega)(s \in z)}}{\mathcal{H}\left|\frac{\omega}{\omega}\right. s \in \mathbb{E}_\omega \Rightarrow (\exists z \in \mathbb{E}_\omega)(s \in z)}}{\mathcal{H}\left|\frac{\omega+1}{\omega}\right. \Rightarrow s \in \mathbb{E}_\omega \to (\exists z \in \mathbb{E}_\omega)(s \in z)}}{\mathcal{H}\left|\frac{\omega+2}{\omega}\right. \Rightarrow (\forall y \in \mathbb{E}_\omega)(\exists z \in \mathbb{E}_\omega)(y \in z)}
$$

$$
(b\exists R)\ \dfrac{(\wedge R)\ \dfrac{\dfrac{\text{Axiom (E4)}}{\mathcal{H}\left|\frac{0}{0}\right. \Rightarrow \mathbb{E}_0 \in \mathbb{E}_\omega}}{\mathcal{H}\left|\frac{1}{0}\right. \Rightarrow \mathbb{E}_0 \in \mathbb{E}_\omega \wedge \mathbb{E}_0 \in \mathbb{E}_\omega}}{\mathcal{H}\left|\frac{2}{0}\right. \Rightarrow (\exists y \in \mathbb{E}_\omega)(y \in \mathbb{E}_\omega)}
$$

Applying $(\wedge R)$ followed by $(b\exists R)$ to the conclusions of the two proof trees above yields the required result. □

Lemma 4.19 ($\Delta_0^{\mathcal{E}}$-Separation) *For any $\Delta_0^{\mathcal{E}}$ formula $A(a, \bar{b})$ of **IKP**$(\mathcal{E})$ containing exactly the free variables $a, \bar{b} = b_1, \ldots, b_n$, any **IRS**$_\Omega^{\mathbb{E}}$ terms $r, s_1, \ldots, s_n$ and any operator $\mathcal{H}$:*

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash_{0}^{\alpha+7} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow \exists x[(\forall y \in x)(y \in r \wedge A(y, \bar{s})) \wedge (\forall y \in r)(A(y, \bar{s}) \to y \in x)]$$

where $\alpha = max(\bar{\beta}, \gamma)$.

Proof First let

$$p := [x \in r \mid A(x, \bar{s})].$$

As an instance of axiom (E11) we have

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash_{0}^{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow p \in \mathbb{E}_\alpha\,. \tag{1}$$

Moreover we have the following derivations in **IRS**$_\Omega^{\mathbb{E}}$:

$$(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{\dfrac{\text{Axiom (E9)}}{\mathcal{H} \vdash_{0}^{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in p \Rightarrow t \in r \wedge A(t, \bar{s})}}{\mathcal{H} \vdash_{0}^{1} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow t \in p \to t \in r \wedge A(t, \bar{s})} \qquad (1)}{\mathcal{H} \vdash_{0}^{\alpha+2} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow (\forall y \in p)(y \in r \wedge A(y, \bar{s}))}$$

$$(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(\to R)\ \dfrac{\dfrac{\text{Axiom (E8)}}{\mathcal{H} \vdash_{0}^{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in r, A(t, \bar{s}) \Rightarrow t \in p}}{\mathcal{H} \vdash_{0}^{1} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in r \Rightarrow A(t, \bar{s}) \to t \in p}}{\mathcal{H} \vdash_{0}^{2} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow t \in r \to (A(t, \bar{s}) \to t \in p)}}{\mathcal{H} \vdash_{0}^{\gamma+3} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow (\forall y \in r)(A(y, \bar{s}) \to y \in p)}$$

Now applying ($\wedge R$) to (1) and the conclusions of the two proof trees above, followed by an application of ($\exists R$) yields the desired result. ☐

Lemma 4.20 (Pair) *For any operator $\mathcal{H}$, and **IRS**$_\Omega^{\mathbb{E}}$ terms s, t and any ordinals $\beta, \gamma < \Omega$:*

$$\mathcal{H}[\beta, \gamma] \vdash_{\alpha+2}^{\alpha+6} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(s \in z \wedge t \in z)$$

where $\alpha := max(\beta, \gamma)$.

Proof If $\beta = \gamma$ the proof is straightforward, without loss of generality let us assume $\beta > \gamma$. As instances of axioms (E6) and (E4) we have

$$\mathcal{H}[\beta,\gamma] \vdash^{0}_{0} t \in \mathbb{E}_\gamma, \mathbb{E}_\gamma \in \mathbb{E}_\beta \Rightarrow t \in \mathbb{E}_\beta, \tag{1}$$

$$\mathcal{H}[\beta,\gamma] \vdash^{0}_{0} \Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_\beta\,. \tag{2}$$

Applying (Cut) gives

$$\mathcal{H}[\beta,\gamma] \vdash^{1}_{\beta+2} t \in \mathbb{E}_\gamma \Rightarrow t \in \mathbb{E}_\beta\,. \tag{3}$$

By axiom (E1) we have

$$\mathcal{H}[\beta,\gamma] \vdash^{0}_{0} s \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta\,. \tag{4}$$

Applying $(\wedge R)$ to (3) and (4) provides

$$\mathcal{H}[\beta,\gamma] \vdash^{2}_{\beta+2} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta \wedge t \in \mathbb{E}_\beta\,, \tag{5}$$

to which we may apply $(\exists R)$ giving

$$\mathcal{H}[\beta,\gamma] \vdash^{\beta+6}_{\beta+2} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(s \in z \wedge t \in z)$$

as required. □

Lemma 4.21 (Union) *For any operator $\mathcal{H}$, $\boldsymbol{IRS}_\Omega^{\mathbb{E}}$ term s and any $\beta < \Omega$ we have*

$$\mathcal{H}[\beta] \vdash^{\beta+9}_{\beta+2} s \in \mathbb{E}_\beta \Rightarrow \exists z[(\forall y \in s)(\forall x \in y)(x \in z)]\,.$$

Proof We have the following template for derivations in $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

$$\text{(}\exists R\text{)}\,\dfrac{(b\forall R)_\infty\,\dfrac{(\to R)\,\dfrac{(b\forall R)_\infty\,\dfrac{(\to R)\,\dfrac{\text{(Cut)}\,\dfrac{\dfrac{\text{Axiom (E6)}}{\mathcal{H}[\beta] \vdash^{0}_{0} t \in \mathbb{E}_\beta, r \in t \Rightarrow r \in \mathbb{E}_\beta} \qquad \dfrac{\text{Axiom (E6)}}{\mathcal{H}[\beta] \vdash^{0}_{0} s \in \mathbb{E}_\beta, t \in s \Rightarrow t \in \mathbb{E}_\beta}}{\mathcal{H}[\beta] \vdash^{1}_{\beta+2} s \in \mathbb{E}_\beta, t \in s, r \in t \Rightarrow r \in \mathbb{E}_\beta}}{\mathcal{H}[\beta] \vdash^{2}_{\beta+2} s \in \mathbb{E}_\beta, t \in s \Rightarrow r \in t \to r \in \mathbb{E}_\beta}}{\mathcal{H}[\beta] \vdash^{\beta+3}_{\beta+2} s \in \mathbb{E}_\beta, t \in s \Rightarrow (\forall x \in t)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta] \vdash^{\beta+4}_{\beta+2} s \in \mathbb{E}_\beta \Rightarrow t \in s \to (\forall x \in t)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta] \vdash^{\beta+5}_{\beta+2} s \in \mathbb{E}_\beta \Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta] \vdash^{\beta+9}_{\beta+2} s \in \mathbb{E}_\beta \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z).}$$

□

Lemma 4.22 ($\Delta_0^{\mathcal{E}}$-Collection) *Let $F(a, b, \bar{c})$ be any $\Delta_0^{\mathcal{E}}$ formula of **IKP**$(\mathcal{E})$ containing exactly the free variables displayed then for any $\bar{s} = s_1, \ldots, s_n$*

$$\Vdash_{\Omega} \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \rightarrow \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s}).$$

Proof Since F is $\Delta_0^{\mathcal{E}}$ we have

$$no_{\bar{\beta}}((\forall x \in s_i)\exists y F(x, y, \bar{s})) = \omega^{\Omega+2}.$$

Hence by Lemma 4.15 we have

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\omega^{\Omega+2}\cdot 2}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\forall x \in s_i)\exists y F(x, y, \bar{s}) \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \, .$$

Applying ($\Sigma^{\mathcal{E}}$-*Ref*) gives

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\omega^{\Omega+2}\cdot 2+2}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\forall x \in s_i)\exists y F(x, y, \bar{s}) \Rightarrow \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s}) \, .$$

By ($\rightarrow R$) we get

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\omega^{\Omega+2}\cdot 2+3}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \rightarrow \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s}) \, .$$

Finally since $\omega^{\Omega+2} \cdot 2 + 3 < \omega^{\Omega+3}$ we may conclude

$$\Vdash_{\Omega} \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \rightarrow \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s})$$

as required. □

Lemma 4.23 (Exponentiation) *For any terms s, t any $\beta, \gamma < \Omega$ and any operator $\mathcal{H}$*

$$\mathcal{H}[\beta, \gamma] \,\Big|\!\frac{\delta+4}{\delta+3}\, s \in \mathbb{E}_{\beta}, t \in \mathbb{E}_{\gamma} \Rightarrow \exists z(\forall x \in {}^{s}t)(x \in z)$$

where $\delta := max(\beta, \gamma) + 2$.

Proof First let

$$p := [x \in \mathbb{E}_{\delta} \mid \text{fun}(x, s, t)].$$

As an instance of axiom (E10) we have

$$\mathcal{H}[\beta, \gamma] \,\Big|\!\frac{0}{0}\, s \in \mathbb{E}_{\beta}, t \in \mathbb{E}_{\gamma}, \text{fun}(q, s, t) \Rightarrow q \in \mathbb{E}_{\delta} \quad \text{for all } q. \tag{1}$$

Also axiom (E8) provides

$$\mathcal{H}[\beta, \gamma] \,\Big|\!\frac{0}{0}\, q \in \mathbb{E}_{\delta}, \text{fun}(q, s, t) \Rightarrow q \in p \quad \text{for all } q. \tag{2}$$

Applying (Cut) to (1) and (2) provides

$$\mathcal{H}[\beta, \gamma] \vdash^{1}_{\delta+2} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma, \mathrm{fun}(q, s, t) \Rightarrow q \in p \quad \text{for all } q. \tag{3}$$

Now by $(\rightarrow R)$ we have

$$\mathcal{H}[\beta, \gamma] \vdash^{2}_{\delta+2} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \mathrm{fun}(q, s, t) \rightarrow q \in p \quad \text{for all } q. \tag{4}$$

Thus we may use $(\mathcal{E}b\forall R)_\infty$ giving

$$\mathcal{H}[\beta, \gamma] \vdash^{\delta+1}_{\delta+2} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow (\forall x \in {}^{s}t)(x \in p) \quad \text{for all } q. \tag{5}$$

As instances of axioms (E11) and (E4) we also have

$$\mathcal{H}[\beta, \gamma] \vdash^{0}_{0} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma, \mathbb{E}_\delta \in \mathbb{E}_{\delta+1} \Rightarrow p \in \mathbb{E}_{\delta+1}, \tag{6}$$

$$\mathcal{H}[\beta, \gamma] \vdash^{0}_{0} \Rightarrow \mathbb{E}_\delta \in \mathbb{E}_{\delta+1}. \tag{7}$$

We may apply (Cut) to (6) and (7) to obtain

$$\mathcal{H}[\beta, \gamma] \vdash^{1}_{\delta+3} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow p \in \mathbb{E}_{\delta+1}. \tag{8}$$

Finally by applying $(\exists R)$ to (5) and (8) we get

$$\mathcal{H}[\beta, \gamma] \vdash^{\delta+4}_{\delta+3} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z (\forall x \in {}^{s}t)(x \in z)$$

as required. □

Theorem 4.24 *If* ***IKP***$(\mathcal{E}) \vdash \Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ *with* $\bar{a}$ *the only free variables occurring in the intuitionistic sequent* $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$. *Then there is a* $k < \omega$ *such that for any* ***IRS***$^{\mathbb{E}}_{\Omega}$ *terms* $\bar{s}$, *any* $\bar{\beta} < \Omega$ *and any operator* $\mathcal{H}$

$$\mathcal{H}[\bar{\beta}] \vdash^{\Omega \cdot \omega^k}_{\Omega+k} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

Proof The proof is by induction on the **IKP**$(\mathcal{E})$ derivation. If $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ is an axiom of **IKP**$(\mathcal{E})$ then the result follows by one of Lemmas 4.15, 4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22 and 4.23.

Case 1. Suppose the last inference was $(\mathcal{E}b\exists L)$, then $(\exists x \in^{a_i} a_j)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\mathcal{E}b\exists L)\ \frac{\Gamma[\bar{a}], \mathrm{fun}(b, a_i, a_j) \wedge F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

where b does not occur in $\bar{a}$. By the induction hypothesis we have a k_0 such that

$$\mathcal{H}[\bar{\beta}, \gamma] \left|\frac{\Omega \cdot \omega^{k_0}}{\Omega + k_0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \wedge F(p) \Rightarrow \Delta[\bar{s}] \tag{1}$$

for all p and all $\gamma < \Omega$. Let us choose the special case of (1) where $\gamma := \max(\beta_i, \beta_j) + 2$ and note that for this choice of γ, $\mathcal{H}[\bar{\beta}, \gamma] = \mathcal{H}[\bar{\beta}]$. Now $\mathrm{fun}(p, s_i, s_j) \Rightarrow \mathrm{fun}(p, s_i, s_j)$ is an axiom due to (E1) and by Lemma 4.15 we have $\Vdash_{\Omega} F(p) \Rightarrow F(p)$ so applying $(\wedge R)$ gives

$$\Vdash_{\Omega} \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \mathrm{fun}(p, s_i, s_j) \wedge F(p). \tag{2}$$

Applying (Cut) to (1) and (2) provides

$$\mathcal{H}[\bar{\beta}] \left|\frac{\Omega \cdot \omega^{k_1}}{\Omega + k_1}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \Delta[\bar{s}]\,. \tag{3}$$

Now as an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta}] \left|\frac{0}{0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(p, s_i, s_j) \Rightarrow p \in \mathbb{E}_{\gamma}\,. \tag{4}$$

So (Cut) to (3) and (4) gives

$$\mathcal{H}[\bar{\beta}] \left|\frac{\Omega \cdot \omega^{k_1} + 1}{\Omega + k_1}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \Delta[\bar{s}]\,. \tag{5}$$

To which we may apply $(\wedge L)$ twice followed by $(\mathcal{E}b\exists L)_{\infty}$ to complete the case.

Case 2. Suppose the last inference was $(\mathcal{E}b\exists R)$ then $\Delta[\bar{a}] = \{(\exists x \in {}^{a_i}a_j)F(x)\}$ and the final inference looks like

$$(\mathcal{E}b\exists R)\ \frac{\Gamma[\bar{a}] \Rightarrow \mathrm{fun}(b, a_i, a_j) \wedge F(b)}{\Gamma[\bar{a}] \Rightarrow (\exists x \in {}^{a_i}a_j)F(x)}$$

Suppose b is a member of $\bar{a}$, without loss of generality let us suppose that $b \equiv a_1$, so by the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \left|\frac{\Omega \cdot \omega^{k_0}}{\Omega + k_0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \wedge F(s_1). \tag{6}$$

If b is not a member of $\bar{a}$ we can also conclude (6) by the induction hypothesis. As an instance of axiom (E1) we have $\mathrm{fun}(s_1, s_i, s_j) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)$ to which we may apply $(\wedge L)$ giving

$$\mathcal{H}[\bar{\beta}] \left|\frac{1}{0}\right. \mathrm{fun}(s_1, s_i, s_j) \wedge F(s_1) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)\,. \tag{7}$$

Now applying (Cut) to (6) and (7) yields

$$\mathcal{H}[\bar{\beta}] \left|\frac{\Omega \cdot \omega^{k_0} + 1}{\Omega + k_0}\right. \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j). \tag{8}$$

Axiom (E10) gives us

$$\mathcal{H}[\bar{\beta}] \vdash^{0}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(s_1, s_i, s_j) \Rightarrow s_1 \in \mathbb{E}_{\delta} \quad \text{where } \delta := \max(\beta_i, \beta_j) + 2. \tag{9}$$

So applying (Cut) to (8) and (9) gives

$$\mathcal{H}[\bar{\beta}] \vdash^{\Omega\cdot\omega^{k_0}+1}_{\Omega+k_0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow s_1 \in \mathbb{E}_{\delta}. \tag{10}$$

Finally we may apply $(\mathcal{E}b\exists R)$ to (6) and (10) to complete this case.

Case 3. Now suppose the last inference was $(\mathcal{E}b\forall L)$, so $(\forall x \in {}^{a_i} a_j)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\mathcal{E}b\forall L)\ \frac{\Gamma[\bar{a}], \mathrm{fun}(b, a_i, a_j) \to F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}].}$$

If b is present in $\bar{a}$, without loss of generality let us suppose $b \equiv a_1$, regardless of whether b is present in $\bar{a}$, by the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \vdash^{\Omega\cdot\omega^{k_0}}_{\Omega+k_0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \to F(p) \Rightarrow \Delta[\bar{s}]\,. \tag{11}$$

The problem here is that β_1 may be greater than $\max(\beta_i, \beta_j) + 2$ meaning we cannot immediately apply $(\mathcal{E}b\forall L)$, moreover unlike in case 2 it is not possible to derive $\bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j)$. Instead we verify the following claim:

$$\Vdash_{\Omega} \Gamma[\bar{s}], (\forall x \in {}^{s_i} s_j)F(x) \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \to F(s_1). \tag{*}$$

To prove the claim we first note that as an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta}] \vdash^{0}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(s_1, s_i, s_j) \Rightarrow s_1 \in \mathbb{E}_{\delta} \quad \text{where } \delta := \max(\beta_i, \beta_j) + 2. \tag{12}$$

Then we have the following template for derivations in $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$.

$$(\mathcal{E}b\forall L)\ \dfrac{(\to L)\ \dfrac{\dfrac{\text{(E1)}}{\Vdash \mathrm{fun}(s_1, s_i, s_j) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)} \qquad \dfrac{\text{Lemma 4.15}}{\Vdash_{\Omega} F(s_1) \Rightarrow F(s_1)}}{\Vdash_{\Omega} \mathrm{fun}(s_1, s_i, s_j) \to F(s_1), \mathrm{fun}(s_1, s_i, s_j) \Rightarrow F(s_1)}}{(\to R)\ \dfrac{\Vdash_{\Omega} (\forall x \in {}^{s_i} s_j)F(x), \mathrm{fun}(s_1, s_i, s_j) \Rightarrow F(s_1)}{\Vdash_{\Omega} (\forall x \in {}^{s_i} s_j)F(x) \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \to F(s_1)}} \tag{12}$$

Thus the claim is verified. Now we may complete the case by applying (Cut) to (11) and (*).

Case 4. Now suppose the last inference was $(b\forall L)$, so $(\forall x \in a_i)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(b\forall L)\ \frac{\Gamma[\bar{a}], b \in a_i \to F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

If b does occur in $\bar{a}$, without loss of generality we may assume $b \equiv a_1$. Regardless of whether b is present in $\bar{a}$, by the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], s_1 \in s_i \to F(s_1) \Rightarrow \Delta[\bar{s}]. \tag{13}$$

We claim that

$$\Vdash_\Omega (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1). \tag{**}$$

To prove the claim we first note that by axiom (E6) we have

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, s_1 \in s_i \Rightarrow s_1 \in \mathbb{E}_{\beta_i}. \tag{14}$$

Then we have the following template for derivations in $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

$$(\to R)\;\dfrac{b\forall L)\;\dfrac{(\to L)\;\dfrac{\dfrac{\text{(E1)}}{\Vdash s_1 \in s_j \Rightarrow s_1 \in s_j} \qquad \dfrac{\text{Lemma 4.15}}{\Vdash_\Omega F(s_1) \Rightarrow F(s_1)}}{\Vdash_\Omega s_1 \in s_j \to F(s_1), s_1 \in s_j \Rightarrow F(s_1)} \qquad (14)}{\Vdash_\Omega (\forall x \in s_i)F(x), s_1 \in s_j \Rightarrow F(s_1)}}{\Vdash_\Omega (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)}$$

Finally we may apply (Cut) to (13) and (**) to complete this case.

Case 5. Now suppose the last inference was $(\forall L)$, so $\forall x F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\forall L)\;\frac{\Gamma[\bar{a}], F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

If b is a member of $\bar{a}$, without loss of generality let us assume $b \equiv a_1$. By the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}+1}{\Omega+k_0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], F(s_1) \Rightarrow \Delta[\bar{s}]. \tag{15}$$

If b is not a member of $\bar{a}$ we can in fact still conclude (15) from the induction hypothesis. Now as an instance of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow s_1 \in \mathbb{E}_{\beta_1}\ . \tag{16}$$

So applying $(\forall L)$ gives the desired result.

Case 6. Now suppose the last inference was $(\forall R)$, then $\{\forall x F(x)\} \equiv \Delta[\bar{a}]$ and the final inference looks like

$$(\forall L)\;\frac{\Gamma[\bar{a}] \Rightarrow F(b)}{\Gamma[\bar{a}] \Rightarrow \forall x F(x)}$$

with b not present in $\bar{a}$. By the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}, \gamma] \overset{\Omega\cdot\omega^{k_0}}{\underset{\Omega+k_0}{\vdash}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_\gamma, \Gamma[\bar{s}] \Rightarrow F(p)$$

for all p and all $\gamma < \Omega$. Applying $(\forall R)_\infty$ gives the desired result.

Case 7. Suppose the last inference was (Cut) then the derivation looks like

$$\frac{\Gamma[\bar{a}], B(\bar{a}, \bar{b}) \Rightarrow \Delta[\bar{a}] \qquad \Gamma[\bar{a}] \Rightarrow B(\bar{a}, \bar{b})}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

where each member of $\bar{b}$ is distinct from the members of $\bar{a}$. By the induction hypothesis we get $k_0, k_1 \in \omega$ such that

$$\mathcal{H}[\bar{\beta}] \overset{\Omega\cdot\omega^{k_0}}{\underset{\Omega+k_0}{\vdash}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{\mathbb{E}}_0 \in \mathbb{E}_1, \Gamma[\bar{s}], B(\bar{s}, \bar{\mathbb{E}}_0) \Rightarrow \Delta[\bar{s}], \tag{17}$$

$$\mathcal{H}[\bar{\beta}] \overset{\Omega\cdot\omega^{k_1}}{\underset{\Omega+k_1}{\vdash}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{\mathbb{E}}_0 \in \mathbb{E}_1, \Gamma[\bar{s}] \Rightarrow B(\bar{s}, \bar{\mathbb{E}}_0)\,. \tag{18}$$

Now since $\Rightarrow \mathbb{E}_0 \in \mathbb{E}_1$ is an instance of axiom (E4) and $\bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow s_i \in \mathbb{E}_{\beta_i}$ is an instance of axiom (E1) we may apply (Cut) to (17) and (18) giving

$$\mathcal{H}[\bar{\beta}] \overset{\Omega\cdot\omega^{k}}{\underset{\Omega+k}{\vdash}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathbb{E}_0 \in \mathbb{E}_1, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,. \tag{19}$$

Finally applying (Cut) to (19) and $\mathcal{H}[\bar{\beta}] \overset{0}{\underset{0}{\vdash}} \mathbb{E}_0 \in \mathbb{E}_1$ we can complete this case.

All other cases can be treated in a similar manner to one of those above. □

4.5 A Relativised Ordinal Analysis of IKP($\mathcal{E}$)

Analogously to with $\mathbf{IRS}_\Omega^{\mathcal{P}}$ we will prove a soundness theorem for certain $\mathbf{IRS}_\Omega^{\mathbb{E}}$ derivable sequents in $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$. Again we need the notion of an assignment. Let $VAR_{\mathcal{E}}$ be the set of free variables of $\mathbf{IRS}_\Omega^{\mathbb{E}}$, an assignment is a map

$$v : VAR_{\mathcal{E}} \longrightarrow E_{\psi_\Omega(\varepsilon_{\Omega+1})}$$

such that $v(a_i^\alpha) \in E_{\alpha+1}$ for all $i < \omega$ and ordinals α. Again an assignment canonically lifts to all $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms by setting

$$\begin{aligned} v(\mathbb{E}_\alpha) &= E_\alpha, \\ v([x \in t \mid F(x, s_1, \dots, s_n)]) &= \{x \in v(t) \mid F(x, v(s_1), \dots, v(s_n))\}. \end{aligned}$$

The difference between here and the case of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ is that for a given term t, it is no longer possible to ascertain the location of $v(t)$ within the E-hierarchy solely by looking at the syntactic structure of t. It is however possible to place an upper bound on that location using the following function

$$\begin{aligned} m(\mathbb{E}_\alpha) &: = \alpha, \\ m(a_i^\alpha) &: = \alpha, \\ m([x \in t \mid F(x, s_1, \ldots, s_n)]) &: = \max(m(t), m(s_1), \ldots, m(s_n)) + 1. \end{aligned}$$

It can be observed that $v(s) \in E_{m(s)+1}$ for any s, however in general $m(s)$ is only an upper bound on a term's position in the E-hierarchy.

Theorem 4.25 (Soundness for $\mathbf{IRS}_\Omega^{\mathbb{E}}$) *Suppose* $\Gamma[s_1, \ldots, s_n]$ *is a finite set of* $\Pi^{\mathcal{E}}$ *formulae with* $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, $\Delta[s_1, \ldots, s_n]$ *a set containing at most one* $\Sigma^{\mathcal{E}}$ *formula and*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}] \quad \textit{for some operator } \mathcal{H} \textit{ and some } \alpha, \rho < \Omega.$$

Then for any assignment v,

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(s_1), \ldots, v(s_n)] \rightarrow \bigvee \Delta[v(s_1), \ldots, v(s_n)],$$

where $\bigwedge \Gamma$ *and* $\bigvee \Delta$ *stand for the conjunction of formulae in* Γ *and the disjunction of formulae in* Δ *respectively, by convention* $\bigwedge \emptyset := \top$ *and* $\bigvee \emptyset := \bot$.

Proof The proof is by induction on α. Note that the derivation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ contains no inferences of the form $(\forall R)_\infty$, $(\exists L)_\infty$ or $(\Sigma^{\mathcal{E}}\text{-}Ref)$ and all cuts have $\Delta_0^{\mathcal{E}}$ cut formulae.

All axioms apart from (E6) and (E7) are clearly sound under the interpretation, the soundness of (E6) and (E7) follows from Lemma 4.2.

Now suppose the last inference was $(\mathcal{E}b\exists R)$, so amongst other premises we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}] \Rightarrow \text{fun}(t, s_i, s_j) \wedge A(t, \bar{s}) \quad \text{for some } \alpha_0 < \alpha.$$

Applying the induction hypothesis yields

$$\begin{aligned} E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \rightarrow & [\text{fun}(v(t), v(s_i), v(s_j)) \wedge A(v(t), \bar{s})] \\ & \text{where } v(\bar{s}) := v(s_1), \ldots, v(s_n). \end{aligned}$$

Suppose $\Gamma[v(\bar{s})]$ holds in $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$, so we have

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \text{fun}(v(t), v(s_i), v(s_j)) \wedge A(v(t), v(\bar{s})).$$

It remains to note that the function space ${}^{v(s_i)}v(s_j)$ is a member of $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$ and thus

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models (\exists x \in {}^{v(s_i)}v(s_j))A(x, v(\bar{s})).$$

as required.

Now suppose the last inference was $(\mathcal{E}b\exists L)_\infty$, thus amongst other premises we have

$$\mathcal{H} \Big|\!\frac{\alpha_0}{\rho}\; \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \wedge A(p, \bar{s}) \Rightarrow \Delta[\bar{s}] \quad \text{for all terms } p \text{ and some } \alpha_0 < \alpha. \tag{20}$$

For the remainder of this case fix an arbitrary valuation v_0. Let $\beta_0 := m(s_i)$, $\beta_1 := m(s_j)$ and $\beta := \max(\beta_0, \beta_1) + 2$. Choose k such that a_k^β does not occur in any of the terms in $\bar{s}$. As a special case of (20) we have

$$\mathcal{H} \Big|\!\frac{\alpha_0}{\rho}\; \Gamma[\bar{s}], \mathrm{fun}(a_k^\beta, s_i, s_j) \wedge A(a_k^\beta, \bar{s}) \Rightarrow \Delta[\bar{s}]\,.$$

Applying the induction hypothesis we get

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \wedge [\mathrm{fun}(v(a_k^\beta), v(s_i), v(s_j)) \wedge A(v(a_k^\beta), v(\bar{s}))] \to \bigvee \Delta[v(\bar{s})] \tag{21}$$

for all valuations v. In particular (21) holds true for all valuations v which coincide with v_0 on $\bar{s}$. By the choice of a_k^β it follows that

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v_0(\bar{s})] \to \bigvee \Delta[v_0(\bar{s})]$$

as required.

All other cases may be treated in a similar manner to those above, using similar reasoning to Theorem 3.25. ☐

Lemma 4.26 *Suppose* ***IKP***$(\mathcal{E}) \vdash \Rightarrow A$ *for some* $\Sigma^{\mathcal{E}}$ *sentence* A*, then there exists an* $n < \omega$*, which we may compute from the derivation, such that*

$$\mathcal{H}_\sigma \Big|\!\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\; \Rightarrow \; A \quad \textit{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Proof Suppose **IKP**$(\mathcal{E}) \vdash \Rightarrow A$, then by Theorem 4.24 we can explicitly calculate some $1 \leq m < \omega$ such that

$$\mathcal{H}_0 \Big|\!\frac{\Omega\cdot\omega^m}{\Omega+m}\; \Rightarrow A\,.$$

Applying partial cut elimination for $\mathbf{IRS}_\Omega^{\mathbb{E}}$ Theorem 4.10 we get

$$\mathcal{H}_0 \Big|\!\frac{\omega_{m-1}(\Omega\cdot\omega^m)}{\Omega+1}\; \Rightarrow A\,.$$

Finally by applying collapsing for $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ Theorem 4.13 we get

$$\mathcal{H}_{\omega_m(\Omega\cdot\omega^m)} \left|\frac{\psi_\Omega(\omega_m(\Omega\cdot\omega^m))}{\psi_\Omega(\omega_m(\Omega\cdot\omega^m))}\right. \Rightarrow A$$

as required. □

Theorem 4.27 *If A is a $\Sigma^{\mathcal{E}}$-sentence and* $\boldsymbol{IKP}(\mathcal{E}) \vdash \Rightarrow A$ *then there is an ordinal term $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$, which we may compute from the derivation, such that*

$$E_\alpha \models A.$$

Proof By Lemma 4.26 we can determine some $m < \omega$ such that

$$\mathcal{H}_\sigma \left|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\right. \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Let $\alpha := \psi_\Omega(\sigma)$. Applying boundedness Lemma 4.12 we get

$$\mathcal{H} \left|\frac{\alpha}{\alpha}\right. \Rightarrow A^{\mathbb{E}_\alpha}.$$

Now Theorem 4.25 yields

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A^{\mathbb{E}_\alpha}.$$

It follows that

$$E_\alpha \models A$$

as required. □

Remark 4.28 Suppose $A \equiv \exists x C(x)$ is a $\Sigma^{\mathcal{E}}$ sentence and $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$. As in the case of $\mathbf{IKP}(\mathcal{P})$, as well as the ordinal term α given by Theorem 4.27, it is possible to compute a specific $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ term s such that $E_\alpha \models C(s)$. Moreover this process can be carried out inside $\mathbf{IKP}(\mathcal{E})$. These results will be verified in [28].

As in the foregoing cases we also have a conservativity result.

Theorem 4.29 $\boldsymbol{IKP}(\mathcal{E}) + \Sigma^{\mathcal{E}}$*-Reflection is conservative over* $\boldsymbol{IKP}(\mathcal{E})$ *for* $\Sigma^{\mathcal{E}}$*-sentences.*

Remark 4.30 An obvious question is whether the conservativity results of Theorems 2.37, 3.29, 4.29 can be lifted to formulae with free variables? This would require ordinal analyses with set parameters. For classical Kripke-Platek set theory this has been carried out by the first author in [8]. The second author thinks that this result can be lifted to the intuitionistic context. However it is likely that this extension requires a fair amount of extra work since the linearity and decidability of the ordinal representation system would have to be sacrificed.

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Machine-Checked Proof-Theory for Propositional Modal Logics

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Abstract We describe how we machine-checked the admissibility of the standard structural rules of weakening, contraction and cut for multiset-based sequent calculi for the unimodal logics S4, S4.3 and K4De, as well as for the bimodal logic S4C recently investigated by Mints. Our proofs for both S4 and S4.3 appear to be new while our proof for S4C is different from that originally presented by Mints, and appears to avoid the complications he encountered. The paper is intended to be an overview of how to machine-check proof theory for readers with a good understanding of proof theory.

1 Introduction

Sequent calculi provide a rigorous basis for meta-theoretic studies of various logics. The central theorem is cut-elimination/admissibility, which states that detours through lemmata can be avoided, since it can help to show many important logical properties like consistency, interpolation, and Beth definability. Cut-free sequent calculi are also used for automated deduction, for nonclassical extensions of logic programming, and for studying the connection between normalising lambda calculi and functional programming. Sequent calculi, and their extensions, therefore play an important role in logic and computation.

Meta-theoretic reasoning about sequent calculi is error-prone because it involves checking many combinatorial cases, with some being very difficult, but many being very similar. Invariably, authors resort to expressions like "the other cases are similar", or "we omit details". The literature contains many examples of meta-theoretic proofs with serious and subtle errors in the original pencil-and-paper proofs. For example, the cut-elimination theorem for the modal "provability logic" GL, where

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$\Box\varphi$ can be read as "φ is provable in Peano Arithmetic", has a long and chequered history which has only recently been resolved [11].

Here, we describe how we formalised cut-elimination for traditional, propositional, multiset-based sequent calculi without explicit structural rules for the propositional modal logics S4, S4.3, K4De and S4C using the interactive proof-assistant Isabelle/HOL. As far as we know, the proofs for S4 and S4.3 are new, and avoid the complexities of previous proofs for these logics. Our results also confirm the recent claim of cut-elimination for S4C due to Mints, although our proof is different, and avoids the complications he encountered in his proofs.

In Sect. 2.1, we briefly describe traditional sequent calculi, discuss the need for multisets, and describe the general form of our main theorems. In Sect. 2.2 we describe the modal logics we study. In Sect. 2.3 we give a brief overview of how interactive proof assistants work. In Sect. 3 we show how we encode formulae, sequents and rules, showing a sequent rule as an example. In Sect. 4 we describe how we encoded the notion of derivability, giving rise to what we call "implicit derivations". In Sect. 4.4 we show how we encoded "explicit derivations" as concrete tree data structures, and the functions used to reason about them. In Sect. 5 we describe how we generalised the forms of our sequent rules to easily capture rule skeletons extended with arbitrary contexts which are essential to make weakening admissible. In Sect. 6 we describe how we encoded the properties of weakening, invertible of some rules, and contraction in Isabelle. In Sect. 7 we describe how we generalised our previous work on explicit derivations to facilitate inductive proof of properties (such as the admissibility of contraction or cut), and in Sect. 8 we describe this further specifically for cut-admissibility. In Sects. 9–12 we describe the cut-admissibility proofs for the specific logics S4, S4.3, K4De and S4C. The remaining sections discuss related work and conclude.

We assume the reader is familiar with basic proof-theory and higher-order logic, but assume that the reader is a novice in interactive proof assistants. Our Isabelle code can be found at http://users.cecs.anu.edu.au/~jeremy/isabelle/2005/seqms/. Some of this work was reported informally in [13] and also, more formally, in [6].

2 Preliminaries

2.1 Sequents Built from Multisets Versus Sets

Proof-theorists typically work with sequents $\Gamma \vdash \Delta$ where Γ and Δ are "collections" of formulae. The "collections" found in the literature increase in complexity from simple sets for classical logic [8], to multisets for linear logic [9], to ordered lists for substructural logics [7], to complex tree structures for display logics [1]. A sequent rule typically has a rule name, a (finite) number of premises, a side-condition and a conclusion. Rules are read top-down as "if all the premises hold then the conclusion holds". A derivation of the judgement $\Gamma \vdash \Delta$ is typically a finite tree of judgements with root $\Gamma \vdash \Delta$ where parents are obtained from children by "applying a rule". We use "derivation" to refer to a proof *within* a calculus, reserving "proof" for a meta-theoretic proof of a theorem *about* the calculus.

Sequent calculi typically contain three structural rules called weakening, contraction and cut. These rules are bad for automated reasoning using backward proof-search since they can be applied at any time. Thus for backward proof-search, we are interested in sequent calculi which do not contain explicit rules for weakening, contraction and cut. The traditional way to design such calculi is to assume that sequents are built out of multisets, omit these rules from the calculus itself, and prove that each of these rules is admissible. That is, for each rule, we have to prove that the conclusion sequent is derivable if each of its premises are derivable. For example, our work does not regard the cut rules shown below as being part of the system:

$$(\text{cut})\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \qquad (\text{cut})\ \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Thus our results will be lemmata of the form: if $\Gamma \vdash A, \Delta$ is (cut-free) derivable and $\Gamma, A \vdash \Delta$ is (cut-free) derivable then $\Gamma \vdash \Delta$ is (cut-free) derivable.

2.2 *Our Modal Logics*

The sequent calculi we study are designed to reason about the meta-theory of the basic modal logics S4, S4.3, K4De (called GTD by Mints) and S4C. Semantically, the first three are mono-modal logics characterised, respectively, by Kripke frames having: reflexive and transitive frames; reflexive, transitive and linear frames; and transitive and dense frames. The logic S4C, called dynamic topological logic, is a bimodal logic where $\Box$ is captured by a reflexive and transitive binary relation $R_\Box$ and where $\circ$ is captured by a serial and discrete linear relation $R_\circ$ with an interaction between them of "confluency":

$$\forall x \forall y \forall z \exists u . R_\Box(x, y) \ \& \ R_\circ(x, z) \Rightarrow R_\circ(y, u) \ \& \ R_\Box(z, u). \tag{1}$$

The Hilbert-calculi for these logics are obtained by extending a traditional Hilbert-calculus for classical propositional logic with the axioms and inference rules as shown below using the naming conventions given in Fig. 1:

Logic	Axioms	Rules
S4	K,$\Box\bot$,4,T	RN$\Box$
S4.3	K,$\Box\bot$,4,T,.3	RN$\Box$
K4De (GTD)	K,$\Box\bot$,4,De	RN$\Box$
S4C	K,$\Box\bot$,K$_\circ$,T,4,C,$\circ\bot$	RN$\Box$, RN$\circ$

Axiom Name	Schema	Rule Name	Schema
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	RN$\Box$	$\varphi/\Box\varphi$
$\Box\bot$	$\Box\bot \leftrightarrow \bot$	RN$\circ$	$\varphi/\circ\varphi$
De	$\Box\Box\varphi \to \Box\varphi$		
T	$\Box\varphi \to \varphi$		
4	$\Box\varphi \to \Box\Box\varphi$		
.3	$\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$		
C	$\circ\Box\varphi \to \Box \circ \varphi$		
$K_\circ$	$\circ(\varphi \to \psi) \leftrightarrow (\circ\varphi \to \circ\psi)$		
$\circ\bot$	$\circ\bot \leftrightarrow \bot$		

Fig. 1 Various axioms and inference rules

The modal logic S4C is designed to capture the basic logic for hybrid systems [4] where Eq. (1) captures the lower semi-continuity of the linear discrete relation with respect to the topological interpretation of the $\Box$-connective.

2.3 Interactive Proof Assistants

Interactive proof-assistants are now a mature technology for "formalising mathematics" [23]. They come in many different flavours as indicated by the names of some of the most popular ones `Mizar`, `HOL`, `Coq`, `LEGO`, `NuPrl`, `NqThm`, `Isabelle`, `λ-Prolog`, `HOL-Light`, `LF`, `ELF`, `Twelf`, with apologies to those whose favourite proof-assistant we have omitted.

Most of the modern proof-assistants are implemented using a modern functional programming language such as ML, which was specifically designed for the implementation of, and interaction with, such proof-assistants.

The lowest levels typically implement a typed λ-calculus with hooks provided to allow the encoding of further logical notions such as equality of terms on top of this base implementation. The base implementation is usually very small, comprising of a few hundred lines of code, so that this code can be scrutinised by experts to ensure its correctness.

Almost all aspects of proof-checking eventually compile down to a type-checking problem using this small core, so that trust rests on strong typing and a well-scrutinised small core of (ML) code.

Most proof-assistants also allow the user to create a proof-transcript which can be cross-checked using other proof-assistants to guarantee correctness.

Figure 2 shows how these logical frameworks typically work. Thus given some goal β and an expression which claims that α is implied by the conjunction of β_1 up to β_n, the Isabelle engine pattern-matches α and β to find a substitution θ such

Fig. 2 Backward chaining in logical frameworks

$$\begin{array}{ll} [\beta_1 \;;\; \beta_2 \;;\; \cdots \;;\; \beta_n] \Longrightarrow \alpha & \beta \\ \\ \theta = match(\beta, \alpha) & \beta_1\theta \;;\; \beta_2\theta \;;\; \cdots \;;\; \beta_n\theta \end{array}$$

that $\alpha\theta = \beta$, and then reduces the original goal β to the n subgoals $\beta_1\theta, \ldots, \beta_n\theta$ (note that n may be 0). We can then repeat this procedure on each $\beta_i\theta$ until all subgoals are proved (which requires that each final step produces no new subgoals, i.e., has $n = 0$). The pattern matching required is usually higher order unification. The important point is that the logical framework keeps track of sub-goals and the current proof state.

The syntax of the "basic propositions" such as α, β is defined via an "object logic", which is a parameter. Different "object logics" can be invoked using the same logical-framework for the task at hand.

The logical properties of ";" such as associativity or commutativity, and properties of the "$\Longrightarrow$" such as classicality or linearity are determined by the "meta-logic", which is usually fixed for the logical framework in question.

For example, the meta-logic of Isabelle [20] is higher-order typed intuitionistic logic with connectives $\Longrightarrow$ (implication), !! ($\forall$), == (equality), and no negation, while the object-logic is classical higher-order logic (HOL) using $\longrightarrow$, `ALL` ($\forall$), =, `EX` ($\exists$), and $\sim$ (not) [10]. Unlike in classical first-order logic, which has terms and formulae, functions and predicates, in Isabelle's meta-logic and in HOL we just have terms (where a formula is a term of type boolean), and functions (where a predicate is a function whose return type is boolean). Further, functions are themselves terms, of a function type, and "higher order" simply means that functions can accept other functions as arguments and can produce functions as results. This allows a uniform treatment of all these entities.

As noted, the meta-logic allows propositions such as $[\beta_1; \beta_2] \Longrightarrow \alpha$, which in fact is the pretty-printer's rendering of $\beta_1 \Longrightarrow (\beta_2 \Longrightarrow \alpha)$. Think of this as meaning "from β_1 and β_2 one may infer α". Since the object-logic (HOL) contains the connectives & and $\longrightarrow$ with their usual classical semantics, we find that $\beta_1 \& \beta_2 \longrightarrow \alpha$ means the same (but in a classical rather than intuitionistic setting) as $\beta_1 \Longrightarrow (\beta_2 \Longrightarrow \alpha)$. But to direct Isabelle to actually use an inference to reduce α to $\beta_1\theta, \ldots, \beta_n\theta$ as explained above, we need the first (meta-logical) form. Thus we shall see two logical syntaxes: $\Longrightarrow$, !! (and ; as explained above) for the Isabelle intuitionistic meta-level, and $\longrightarrow$, `ALL`, &, `EX` and $\sim$ for the classical HOL object-level. Together they are referred to as Isabelle/HOL [26].

3 A Deep Embedding of Formulae, Sequents and Rules

Recall that the meta-logic provides us with a method for backward chaining via expressions of the form (see Fig. 2):

$$[\beta_1 \; ; \cdots \; ; \; \beta_n] \Longrightarrow \alpha.$$

The usual method for obtaining the power for reasoning about sequent derivations is to use the full power of higher-order classical logic (HOL) to build the basic object-level propositions β_i.

Isabelle's incarnation of HOL provides the usual connectives of logic such as conjunction, disjunction, implication, negation and the higher order quantifiers. But it also provides many powerful facilities allowing us to define new types, define functions which accept and return other functions as arguments, and even define infinite sets using inductive definitions [26].

For example, the following HOL expressions would capture the usual inductive definition of the set `even_nat` of even natural numbers by encoding the facts that "zero is even, and if n is even then so is $n + 2$", where `:` stands for set membership $\in$:

```
0 : even_nat
n : even_nat ==> n + 2 : even_nat
```

Most proof-assistants will automatically generate an induction principle from a given inductive definition. For example, Isabelle will automatically generate the usual induction principle which states that we can prove a property P holds of all even naturals if we can show that $P(0)$ holds and we can show that $P(n)$ implies $P(n + 2)$. An implicit assumption which facilitates such induction principles is that the inductive definitions are the *only* way to construct its members. Thus, if m is an even natural, then it is either 0, or is of the form $n + 2$ for some ("smaller") even natural n. Together, they form the base case and the inductive step of an inductive definition that defines the set `even_nat` as the smallest set of terms $0, 0 + 2, 0 + 2 + 2, \ldots$. It is implicit in these definitions that an inference step such as `n` : `even_nat` $\Longrightarrow$ `n` + `2` : `even_nat` may be applied only finitely many times.

We previously said that we shall see two syntaxes: a meta-level intuitionistic logic and an object-level classical HOL syntax. Since we wish to reason about sequent calculi for modal logics, we now need to encode a third logical syntax: namely the syntax of modal sequents.

To encode sequent calculus into HOL we first encode terms for capturing the grammar for recognising formulae as below where comments are enclosed in `(*` and `*)`:

```
datatype formula
       = FC string (formula list)    (* formula connective *)
       | FV string                   (* formula variable   *)
       | PP string                   (* prim prop          *)
```

We use three type constructors `FC`, `FV` and `PP` which encode, respectively, formula connectives, formula variables, and atomic formulae (primitive propositions) as HOL terms. Each of them takes one string argument which is simply the string we want to use for that construction. The formula connective constructor `FC` also accepts a list of formulae, which constitute its subformulae. For example, the term `FC "&&" [FV "A", PP "q"]` encodes $A \wedge q$ where we use "`&&`" as the string for conjunction of classical logic. Since we want to encode modal logics, we require only the classical connectives, plus three unary modalities `FC "Box" [.]` for $\Box$. and `FC "Dia" [.]` for $\Diamond$. and `FC "Circ" [.]` for $\circ$.

Isabelle's HOL allows us to form sets and multisets of objects of an arbitrary type, so the HOL expressions `formula set` and `formula multiset` capture the types of modal formula sets and modal formula multisets.

Using these types we can build a sequent type using a constructor `Sequent`:

```
datatype 'a sequent = Sequent "'a multiset" "'a multiset"
```

Here `'a` is a type variable and the datatype `'a sequent` demands that the constructor `Sequent` is followed by two multisets of items of type `'a`. For example, the datatype `formula sequent` would require our sequents to be constructed out of multisets of formulae (of type `formula`). An alternative infix notation for the constructor `Sequent` is $\vdash$ or `|-`.

We define the type for our sequent rules by the type definition:

```
types 'a psc = "'a list * 'a"       (* single rule *)
```

Such a sequent rule is a pair (ps, c) of a list of items ps (the premises) and a single item c (the conclusion): the items are of some type `'a` which is a parameter. We shall instantiate the type variable `'a` with the type `formula sequent` to obtain sequents built from two multisets of modal formulae.

Note that in common parlance we may say that (ps, c) is a rule meaning that ps and c may be instantiated in any way. Such a "rule" is a schema which can be instantiated to give infinitely many rule instances. When describing the Isabelle implementation we may refer to a specific pair (ps, c) as a "rule", although in the context of logical rules, this could be better described as a specific instance of a rule schema; where we describe our Isabelle theorems involving "sets of rules", these will usually be the infinite sets of instances of a finite set of rule schemata.

Thus, we can use the HOL type-declaration below to declare that rls is a set of sequent rules, where each element of rls is a pair (ps, c) whose first component ps is a list of its premise sequents, and whose second component c is its conclusion sequent:

```
rls :: formula sequent psc set
```

Each sequent consists of two multisets of items of type `formula`, and inductively define the set *rls* by giving a finite collection of rule schemata, each denoting an infinite set of instances, which belong to this set. For example, the traditional rule $(\vdash \wedge)$ for introducing a conjunction into the right hand side of a sequent, as shown below, can be given by the encoding below it where we use the string `&&` to encode $\wedge$, "+" for multiset union, and `{#A#}` to denote a singleton multiset:

$$(\vdash \wedge)\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

$$\texttt{([G} \vdash \texttt{\{\#A\#\} + D, G} \vdash \texttt{\{\#B\#\} + D], G} \vdash \texttt{\{\#A \&\& B\#\} + D)} \in \texttt{rls}$$

When this clause appears in the definition of `rls`, it means that this sequent rule is in `rls` for each possible value of A, B, G, D of the appropriate type.

Having encoded the notions of formulae and sequents into HOL, we are now in a position to encode the notion of derivability and derivations. As we shall explain shortly, the notion of derivability and derivations are subtly different in the following senses:

Derivability we write inductively defined predicates in HOL to capture the set of sequents derivable from a given, possibly empty, set of potential leaf sequents, using a given set of rules defined using the encoding of formulae and sequents described above. The base case will capture that every given leaf is vacuously derivable, and the inductive case will capture that a sequent c is derivable if the rule set contains a rule (ps, c) where each of the premises in ps is itself derivable. We do not construct an actual derivation, as such, but just ensure that there exists a sequence of sequent rule applications which can take us from the given leaf sequents to the given end-sequent. We therefore use the word "implicit" to describe such derivations.

Derivation (trees) we create a new object type called `dertree` which will allow us to encode an explicit tree as a HOL term to represent an actual derivation of the given sequent from the given leaves using the given set of rules. We therefore use the word "explicit" to describe such derivations.

4 Implicit and Explicit Derivations

In Sect. 4.1, we give an inductively defined predicate `derrec` for capturing the set of all recursively derivable sequents. In Sect. 4.2, we describe the principle of induction that is automatically generated by Isabelle/HOL from `derrec` and describe how it can be used to prove an arbitrary property P of such sequents. In Sect. 4.3, we describe our other implicit derivability predicates in less detail. In Sect. 4.4 we describe how we encoded explicit derivation trees. In Sect. 4.5 we describe how we can move to and fro between these two notions.

4.1 Defining Derivability (Implicitly) in Isabelle

We are now in a position to encode the set `derrec` of "recursively derivable sequents" given a set `plvs` of (potential) leaf sequents and a given set `rls` of sequent rules. The set `derrec rls plvs` is defined inductively as shown below (the Isabelle code is precisely as it appears in the Isabelle theory file). It defines simultaneously the predicates `derrec` (that a single sequent is derivable) and `dersrec` (that all sequents in a list are derivable).

Definition 4.1 (*`derrec, dersrec`*) `derrec rls plvs` is the set of endsequents which are derivable from the set `plvs` of potential leaves using the set `rls` of sequent rules.

`dersrec rls plvs` is the set of lists of endsequents which are all derivable from potential leaves `plvs` using sequent rules `rls`:

```
consts   (* these are type declarations *)
  derrec  :: "'a psc set => 'a set => 'a set"
  dersrec :: "'a psc set => 'a set => 'a list set"

inductive "derrec rls plvs" "dersrec rls plvs"
  intrs (* the clauses defining members of these two
           mutually defined inductive sets          *)
    dpI   "eseq : plvs ==> eseq : derrec rls plvs"
   derI   "[| (ps, eseq) : rls ; ps : dersrec rls plvs |]
             ==> eseq : derrec rls plvs"
   dlNil  "[] : dersrec rls plvs"
  dlCons  "[| seq : derrec rls plvs ;
             seqs : dersrec rls plvs |]
             ==> seq # seqs : dersrec rls plvs"
```

We now explain the Isabelle code and why it achieves the meanings for `derrec` and `dersrec` given in the definition. These are two mutually inductively defined sets each of which depends on the other. The type declarations mean that where `plvs` is a set of (potential) leaf sequents and `rls` is a set of "rules", instances of (*premise list, conclusion*) pairs, then `derrec rls plvs` is a set of sequents. A sequent is in `derrec rls plvs` if and only if finite repeated application of the clauses of the definition require it to be, and likewise `dersrec rls plvs`. We now describe the four clauses, each of which is preceded by its name:

`dpI` The base case of the inductive definition of `derrec` captures that each initial sequent `eseq` from `plvs` is itself (vacuously) derivable from the initial leaf set `plvs` using the rules `rls`. The `:` stands for set membership $\in$.

`derI` If `(ps, eseq)` is the list of premises and the conclusion of a rule, and the premise list `ps` satisfies `dersrec rls plvs`, meaning that the premises `ps` are all derivable (see below), then the conclusion `eseq` is derivable.

`dlNil` An empty list of sequents satisfies `dersrec rls plvs`

`dlCons` If `seq` satisfies `derrec rls plvs` and the list `seqs` satisfies `dersrec rls plvs` then the list `seq # seqs` satisfies `dersrec rls plvs`. The symbol `#` denotes appending an item `seq` to the front of a list `seqs` to form a longer list.

Note that the clauses `dlNil` and `dlCons` give us that a list is in `dersrec rls plvs` if all its members are in `derrec rls plvs`; and since these clauses give *all* members of `dersrec rls plvs`, this “if” is in fact “if and only if”.

In fact the actual Isabelle/HOL code is more general, in that the things being derived are of a parametric type `'a` and need not be sequents, but could be formulae or other constructs, and a “rule” merely consists of a list of “premises” and a “conclusion”. We describe it in terms of sequents, here, merely to place it in the context of our cut-admissibility proofs.

4.2 Inductive Proofs via Automated Inductive Principles

We use inductive definitions because correct induction principles are generated automatically by Isabelle from the inductive definition of `derrec`. A heavily simplified version of the induction principle automatically generated for proving an arbitrary property `P` by the definition of the inductive set `derrec` is shown below using meta-level intuitionistic connectives (`==>`, `!!`, `;`) and object-level classical HOL connectives (`ALL`, `-->`, `:`).

```
1 !! x.!! P.
2 [| x : derrec rls plvs ;
3   (ALL c. c : plvs ⟶ P(c)) ;
4   (ALL c. ALL ps. (ps, c) : rls ⟶ (ALL y : (set ps). P(y)) ⟶ P(c))
5 |] ==> P(x)
```

An explanation is:

1 for all sequents x and all properties P
2 if x is derivable from (potential) leaves `plvs` using rules `rls`, and
3 P holds for every sequent c in `plvs`, and
4 for each rule (ps, c), P of each premise in ps implies P of its conclusion c,
5 then P holds of x

We can visualise this induction principle as below where we replace the meta-level `==>` by a horizontal line and replace the meta-level `;` with juxtaposition of premises and replace `:` by set membership $\in$:

$$\frac{x \in derrec\ rls\ plvs \quad \forall c \in plvs.P\ c \quad \forall(ps, c) \in rls.(\forall p\ in\ ps.P\ p) \Rightarrow P\ c}{P\ x}$$

This is an induction principle which we use often in proof-theory: prove that some property holds of the leaves of a derivation, and prove that the property is preserved from the premises to the conclusion of each rule. For example, consider the standard translation from sequents of LK to formulae given by $\tau(A_1, \ldots, A_n \vdash B_1, \ldots, B_m) = A_1 \wedge \cdots \wedge A_n \to B_1 \vee \cdots \vee B_m$. We typically use this translation to argue that all derivable sequents are valid in the semantics of first-order logic. The proof proceeds by showing that the translation of the leaves of a derivation are all valid, and showing that if the translations of the premises are valid then the translations of the conclusion are valid, for every rule. Note that no explicit derivation is created by this induction principle since it uses derivability (implicit derivations).

Thus this induction principle is really a lemma, but our formal encoding of it requires one more definition.

Definition 4.2 For all sets `A` and all unary predicates `P`, the property `Ball A P` holds iff every member `x` of `A` satisfies `P`:

```
Ball_def: "Ball A P == ALL x. x : A --> P x"
```

The following is the formal inductive principle described informally above which is generated by Isabelle/HOL, automatically, using "?" to show arguments that are implicitly universally quantified.

Lemma 4.1 (derrec-induction) *For every sequent* `x`*, every rule set* `rls`*, every list of leaves* `plvs`*, and every property* `P`*, if*

(a) `x` *is derivable from potential leaves* `plvs` *using the rules in* `rls`*, and*
(b) every sequent `c` *in* `plvs` *obeys* `P`*, and*
(c) for every sequent `c` *and premise list* `ps` *if* `(ps, c)` *is a rule in* `rls`*, and each premise in* `ps` *is derivable from potential leaves* `plvs` *using rules in* `rls` *and every premise from* `ps` *obeys* `P` *then* `c` *obeys* `P`

then `x` *obeys* `P`*:*

```
standard drs.inductr:
 "[| ?x : derrec ?rls ?plvs ;
     !!c. c : ?plvs ==> ?P c ;
     !!c ps. [| (ps, c) : ?rls ;
                ps : dersrec ?rls ?plvs ;
                Ball (set ps) ?P |] ==> ?P c
  |] ==> ?P ?x"
```

Proof Isabelle automatically generates an induction principle (not shown) from the definition of `derrec`. Since the definition also involves defining `dersrec` (which expresses that a list of items are all derivable), the automatically generated principle involves a property P_1 of derivable sequents and a property P_2 of lists of derivable sequents. Naturally we choose property P_2 of a list to be that all members of the list satisfy P_1. That instantiation gives us the lemma. ■

Intuitively, Isabelle converts object-level classical implications (⟶) into meta-level intuitionistic implications (`==>`), allowing us to use the lemma itself for subgoaling.

Using these inductive principles we proved the following lemma about derivability using Isabelle, where the question marks indicate free-variables which are implicitly universally quantified:

Lemma 4.2 *If each premise in* `ps` *is derivable from leaves* `plvs` *using rules* `rls`*, and* `eseq` *is derivable from* `ps` *using* `rls`*, then* `eseq` *is derivable from* `plvs` *using* `rls`*:*

$$[|?\mathtt{ps} \subseteq \mathtt{derrec}\ ?\mathtt{rls}\ ?\mathtt{plvs}\ ;\ ?\mathtt{eseq} \in \mathtt{derrec}\ ?\mathtt{rls}\ ?\mathtt{ps}|] \Longrightarrow ?\mathtt{eseq} \in \mathtt{derrec}\ ?\mathtt{rls}\ ?\mathtt{plvs}$$

4.3 Further Implicit Derivability Predicates

We briefly describe the remaining functions we used to describe derivability.

Definition 4.3 (*derivable rule*) For a list of sequents `lvs` and a sequent `eseq`, `(lvs, eseq)` is a *derivable* rule with respect to the rule set `rls` if we can construct an implicit derivation using rules in `rls` whose leaves are *exactly* the sequents `lvs` (in the same order), and whose endsequent is `eseq`.

We formalise this notion using functions `derl` (for the derivable rules) and `dersl` (an auxiliary function).

Definition 4.4 (`derl, dersl`) For a list of sequents `lvs` and a sequent `eseq`, the pair `(lvs, eseq)` is in `derl rls` if it is a derivable rule with respect to `rls`.

For lists of sequents `lvs` and `eseqs`, the pair `(lvs, eseqs)` is in `dersl rls` if there is a sequence of rule instances from `rls` which take us from (exactly) the list of leaf sequents `lvs` to the list of endsequents `eseqs`. We envisage a number of implicit derivations drawn side-by-side, whose endsequents are the members of the list `eseqs`.

```
types 'a psc = "'a list * 'a"      (* single step inference *)
consts (* these are type definitions *)
  derl      :: "'a psc set => 'a psc set"
  dersl     :: "'a psc set => ('a list * 'a list) set"

inductive "derl rls" "dersl rls"
intrs
asmI "([eseq], eseq) : derl rls"
```

```
dtderI "[| (lvs, eseq) : rls ; (lvss, lvs) : dersl rls |]
                    ==> (lvss, eseq) : derl rls"
dtNil "([], []) : dersl rls"
dtCons "[| (lvs, eseq) : derl rls ; (lvss, eseqs) : dersl rls|]
                    ==> (lvs @ lvss, eseq # eseqs) : dersl rls"
```

This formalises the notion of a derivable rule: `derl rls` is the set of derivable rules with respect to `rls`.

Where an inference rule `'a psc` is a list of premises `ps` and a conclusion `c`, a "derived rule" is of the same type. We define `derl rls` to be the set of rules derivable from the rule set `rls`. This, like `derrec`, was defined as an inductive set. So (`lvs`, `eseq`) ∈ `derl` `rls` reflects the shape of an implicit derivation tree: `lvs` is a list of exactly the leaves used, in the correct order, whereas `eseq` ∈ `derrec` `rls` `plvs` holds even if the set of (potential) leaves `plvs` contains extra sequents.

We note that the definition means that $([c], c) \in$ `derl rls`: that is, the "trivial" derived rules are included. To define `derl rls` to exclude the "trivial" derived rules would complicate results such as Theorem 4.1.

The formal Isabelle definitions of `derl` used also the function `dersl`, which represents several implicit derivation trees side-by-side:

(`lvss`, `eseqs`) ∈ `dersl` `rls` when the list `lvss` is the concatenation of their lists of leaves, and `eseqs` is the list of their endsequents.

Theorem 4.1 *With respect to some given set of rules* `rls`*:*

(a) the items derivable from a set `plvs` *of leaves are the items derivable from the set of sequents derivable from* `plvs`*:*

```
derrec_trans_eq:
  "derrec ?rls ?plvs = derrec ?rls (derrec ?rls ?plvs)"
```

(b) derivability (whether defined using `derrec` *or* `derl`*) using the set of derived rules is equivalent to derivability using the original set of rules:*

```
derrec_derl_deriv_eq :
  "derrec (derl ?rls) ?plvs = derrec ?rls ?plvs"
derl_deriv_eq : "derl (derl ?rls) = derl ?rls"
```

Finally, we can define the notion of an admissible rule.

Definition 4.5 (*admissible*, *adm*) A rule `(ps, c)` is *admissible* with respect to a rule set `rls` if, assuming its premises (leaves) `ps` are derivable from the empty set {} of leaves using rules from `rls`, then so is its conclusion (endsequent) `c`:

```
consts  (* this is a type declaration *)
  adm :: "'a psc set => 'a psc set"

inductive "adm rls"
  intrs   (* inductive defn of the set of admissible rules *)
    I  "(ps : dersrec rls {} --> c : derrec rls {})
        ==> (ps, c) : adm rls"
```

Using Definition 4.5 we obtained the following four results, which were surprisingly tricky since `adm` is not monotonic in its argument `rls`, where <= encodes ⊆.

Theorem 4.2 *With respect to some given set of rules* `rls`*:*

(a) every derivable rule is admissible;
(b) the admissible rules are closed under admissibility;
(c) the admissible rules are closed under admissibility after derivability;
(d) the admissible rules are closed under derivability.

```
"derl ?rls <= adm ?rls"            "adm (adm ?rls) = adm ?rls"
"adm (derl ?rls) = adm ?rls"       "derl (adm ?rls) = adm ?rls"
```

4.4 Explicit Derivation Trees: A Deep Embedding of Derivations

The main advantage of the method outlined in the previous section was that there was no concrete representation of a derivation. That is, we relied on the proof-assistant to perform pattern-matching and rule instantiations in an appropriate way, so that all we needed was to capture the idea that derivations began with leaves and ended with a single end-sequent.

When we reason about cut-elimination, often we are required to perform transformations on explicit derivations. We therefore need a representation of such trees inside our encoding. In previous work [6], we described such an encoding using the following datatype:

```
datatype seq dertree = Der seq (seq dertree list)
                     | Unf seq
```

The declaration states that a derivation tree can either be an `Unf`inished (unproved) leaf sequent built using the constructor `Unf`, or it can be a pair `(seq, dts)` consisting of a conclusion sequent `seq` and a list `dts` of (sub-)derivation trees bound together using the constructor `Der`.

Definition 4.6 Given an object `dt` of type `dertree`, `conclDT dt` returns the first argument of the constructors `Der` and `Unf` as the conclusion (endsequent) of `dt`.

For a tree `dt` which is not an `Unf`inished leaf, `nextUp dt` returns the list of trees whose conclusions are the premises of the last rule of `dt`, and `botRule dt` returns the bottom rule (premise list and conclusion) of `dt`.

```
primrec
  conclDT_Der: "conclDT (Der seq dts) = seq"
  conclDT_Unf: "conclDT (Unf seq) = seq"

  nextUp_Der: "nextUp (Der seq dts) = dts"
  botRule_Der: "botRule (Der seq dts) = (map conclDT dts, seq)"
```

Here, `map conclDT dts` applies `conclDT` to each member of the list `dts` of derivation trees and hence returns the premises of the bottom rule.

Our use of `dertee` can be seen as an even deeper embedding of proof-theory into Isabelle/HOL since it utilises the proof-assistant to describe an explicit derivation rather than the implicit existence of such a derivation as encoded by our derivability predicates from the previous section.

4.5 To and Fro Between Explicit and Implicit Derivations

Omitting details now, suppose we define `valid rls dt` to hold when derivation tree `dt` correctly uses rules from `rls` only and has no `Unf`inished leaves: that is, the leaves of `dt` are all instances of the conclusions of rules which have no premises (i.e., such as $\Gamma, A \vdash A, \Delta$). We linked our two approaches for specifying the derivable sequents by proving:

Lemma 4.3 *If derivation tree* `dt` *is valid w.r.t. the rules* `rls` *then its endsequent is (implicitly) derivable from the empty set of leaves using* `rls`*:*

```
valid_derrec:
      "valid ?rls ?dt ==> conclDT ?dt : derrec ?rls {}"
```

Lemma 4.4 *If the end-sequent* `eseq` *is (implicitly) derivable from the empty set* {} *of leaves using rules* `rls` *then there exists an explicit derivation tree* `dt` *which is valid w.r.t.* `rls`*, whose end-sequent is* `eseq`*:*

```
derrec_valid:
        "?eseq : derrec ?rls {}
         ==>  EX dt. valid ?rls dt & conclDT dt = ?eseq"
```

Thus we now know that the implicit derivations captured by our derivability predicate `derrec` can be faithfully captured using the deeper embedding using explicit `dertree` derivation trees. Indeed, the lemmas allow us to move freely between the two embeddings at will to omit or include details as required [6].

5 Subformula Relation, Rule Skeletons and Extensions with Contexts

Our generalised definition of formulae allows a single definition of the immediate (proper) subformula relation, `ipsubfml`, which will not need to be changed when new connectives are added.

Definition 5.1 If a formula `P` is in the set obtained from the list of formulae `Ps` then `P` is a proper subformula of any larger formula `FC conn Ps` created using a formula-connective `conn` and `Ps`:

```
consts  (* this is a type-declaration for function ipsubfml  *)
    ipsubfml :: "(formula * formula) set"
inductive "ipsubfml" (* proper immediate subformula relation *)
    intrs
    ipsI "P : set Ps ==> (P, FC conn Ps) : ipsubfml"
```

For example, `(f, Box f) : ipsubfml` because `Box f` is the abbreviation `Box f == FC "Box" [f]` where `conn` is the string `"Box"` and `Ps` is the formula-list `[f]`.

In Sect. 3 we showed that the traditional $\wedge R$ rule from LK could be encoded as below using a sequent consisting of a pair (Γ, Δ) of multisets of formulae, written $\Gamma \vdash \Delta$, where multiset braces are written as `{#` and `#}` and multiset union is written as `+`:

$$(\,[\,\mathtt{G} \vdash \{\#\mathtt{A}\#\} + \mathtt{D}\,,\ \mathtt{G} \vdash \{\#\mathtt{B}\#\} + \mathtt{D}\,],\ \ \mathtt{G} \vdash \{\#\mathtt{A}\ \&\&\ \mathtt{B}\#\} + \mathtt{D}\ \,) \in \mathtt{rls}$$

The essence of the rule is more succinctly described by the rule skeleton $\mathcal{R}_s$ shown below left. We now describe how we can uniformly extend $\mathcal{R}_s$ with the context $X \vdash Y$ to obtain the extended rule $\mathcal{R}_e$ shown below at right:

$$\mathcal{R}_s = \frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \qquad\qquad \mathcal{R}_e = \frac{X \vdash Y, A \quad X \vdash Y, B}{X \vdash Y, A \wedge B}$$

Definition 5.2 If the sequent `seqXY` is the pair `(X, Y)`, representing the sequent $X \vdash Y$, and the sequent `seqUV` is the pair `(U, V)`, representing the sequent $U \vdash V$, then `extend seqUV seqXY` is the sequent `(X+U, Y+V)`, representing the sequent $X, U \vdash Y, V$ since `seqXY + seqUV` is `(X+U, Y+V)` by the pointwise extension of + to pairs of multisets and the function `pscmap` allows us to modify a rule `(ps, c)` by applying an arbitrary function `f` to each of its components:

```
consts   (* this is a type declaration *)
  extend :: "'a sequent => 'a sequent => 'a sequent"
  extrs  :: "'a sequent psc set => 'a sequent psc set"

defs
  extend_def : "extend seqXY seqUV ==  seqXY + seqUV"
  pscmap_def : "pscmap f (ps, c) = (map f ps, f c)"
```

We can now take a set `rules` of rule skeletons and produce their uniform extension with arbitrary context `flr` (for "formulae left and right") representing $X \vdash Y$.

Definition 5.3 (*`extrs`*) Given a rule set `rules`, the inductively defined set `extrs rules` is the set of rules consisting of all uniform extensions of all rules in `rules`:

```
inductive "extrs rules"
  intrs
    I "psc : rules ==> pscmap (extend flr) psc = epsc
       ==> epsc : extrs rules"
```

For example, we can now use functions `extend` and `pscmap` so that

$$\texttt{extend}\,(X \vdash Y)\ (U \vdash V) = (X + U) \vdash (Y + V)$$
$$\mathcal{R}_e = \texttt{pscmap}\ (\texttt{extend}\,(X \vdash Y))\ \mathcal{R}_s$$

Thus `pscmap` uniformly extends the skeleton provided by $\mathcal{R}_s$ with arbitrary contexts X and Y on respective sides to encode $\mathcal{R}_e$ using multiset addition $+$. So `extrs` S means the set of all such extensions of all rules in the set S.

Then we define `lksss`, the set of rules for Gentzen's LK; we show just a selection. The rules below are the (skeletons of some of the) traditional invertible logical introduction rules from LK (without any context):

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \qquad \frac{\vdash A, B}{\vdash A \vee B} \qquad \frac{B \vdash \quad \vdash A}{A \to B \vdash} \qquad \frac{A, B \vdash}{A \wedge B \vdash} \qquad \frac{A \vdash \quad B \vdash}{A \vee B \vdash} \qquad \frac{A \vdash B}{\vdash A \to B}$$

Using `&&` for $\wedge$, `v` for $\vee$ and `--` for $\neg$, we can encode the logical introduction rules as shown below, to obtain the set `lksir` of LK right introduction rule skeletons, where `{#}` rather than `{##}` is the empty-multiset:

Definition 5.4 (*`lksir`*) `lksir` is the set of right logical introduction rules, in the form without any context and using the form which is invertible, as shown above.

```
inductive "lksir" (* LK right introduction rule skeletons *)
 intrs
  andr
   "([{#} |- {#A#}, {#} |- {#B#}], {#} |- {#A && B#}) : lksir"
  orr   "([{#} |- {#A#} + {#B#}], {#} |- {#A v B#})    : lksir"
  negr  "([{#A#} |- {#}], {#} |- {#--A#})              : lksir"
  impr  "([{#A#} |- {#B#}], {#} |- {#A -> B#})         : lksir"
```

Similar rules `lksil` (not shown) give the skeletons of the traditional invertible rules for left-introduction. By adding the initial sequent "axiom" $A \vdash A$ with an empty list `[]` of premises below we obtain the set of "unextended" rules `lksne` for LK:

Definition 5.5 (`lksne`) `lksne` is the set of rules, not extended by any arbitrary context, without structural rules, for LK.

```
inductive "lksne" (* LK rule skeletons before being extended *)
  intrs
    axiom  "([], {#A#} |- {#A#})        : lksne"
    ilI    "(ps, c) : lksil ==> (ps, c) : lksne"
    irI    "(ps, c) : lksir ==> (ps, c) : lksne"
```

We can now form the full extended set `lksss` of rules for LK, by extending each rule skeleton `psc` from `lksne` by an arbitrary pair (X, Y) of contexts `flr` (for `f`ormulae `l`eft and `r`ight) regarded as a sequent $X \vdash Y$:

Definition 5.6 (*`lksss`*) `lksss` is the set of rules, extended by arbitrary contexts, without structural rules, for LK.

```
inductive "lksss"
  intrs
    extI "psc : lksne ==> pscmap (extend flr) psc : lksss"
```

Now, we can encode the skeleton shown below right of the traditional K-rule shown below left:

$$\frac{\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Delta}\,(K) \qquad\qquad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,(SK)$$

Definition 5.7 (*`SK`*) `SK` is the set of instances of the skeleton of the K rule of modal logic

```
inductive "SK"
  intrs
      I "([X |- {#A#}], mset_map Box X |- {#Box A#}) : SK"
```

Note that X is a multiset, and $\Box X$ is informal notation for applying $\Box$ to each member of X; this is implemented using `mset_map`, used in the encoded `SK` rule. Using a similar notation we write $\Box B^k$ for $(\Box B)^k$, the multiset containing n copies of $\Box B$. Development of `mset_map` and relevant lemmas is in the source files `Multiset_no_le.{thy,ML}`.

By extending the skeletons of the LK rules and extending only the conclusion of the skeleton (SK) of the K rule above, we could obtain an encoding of the traditional sequent calculus for the modal logic K:

```
inductive "lksK"
  intrs
    extI "psc : lksne ==> pscmap (extend flr) psc : lksK"
       K "(ps, c) : SK ==> (ps, extend flr c)      : lksK"
```

Since we actually handle more complex logics, but not K as such, we have not made this a formal definition.

Note that most of these definitions use the Isabelle feature for inductively defined sets, even though many of them are not actually inductive (i.e., recursive). We do this because Isabelle automatically generates useful theorems for them, including rules which help us prove or use an expression such as `rl : lksne`.

6 The Weakening, Inversion and Contraction Properties

We now encode the weakening, inversion and contraction as properties.

Definition 6.1 A set `rls` of rules satisfies the weakening admissibility property if, whenever a sequent $X \vdash Y$ is derivable, any larger sequent $(X \vdash Y) + (U \vdash V) = (X, U \vdash Y, V)$ is derivable:

```
consts (* type for function wk_adm using type variable 'a *)
   wk_adm    :: "'a sequent psc set => bool"

wk_adm_def : "wk_adm rls ==
     ALL XY. XY : derrec rls {} -->
                 (ALL UV. XY + UV : derrec rls {})"
```

Here, the variable `rls` is forced to be a set of sequent rules by the type of `wk_adm`, and thence the variables `XY` and `UV` will be forced to be of type `sequent` by the typing restrictions on the inputs to `derrec`.

Definition 6.2 (*`inv_rl`*) A rule `(ps, c)` is *invertible* with respect to a set `rls` of rules if, whenever `c` is derivable using `rls`, so is every member of `ps`:

```
inv_rl.simps:
  "inv_rl rls (ps, c) =
          (c : derrec rls {} --> ps : dersrec rls {})"
```

Here, the definition of `dersrec` hides a universal quantifier over the members of the list `ps`: see Definition 4.1.

To encode contraction, we utilise an axiomatic type class for sequents, described in more detail elsewhere [6]. Thus we can write $(A \vdash 0) + (A \vdash 0) \leq (X \vdash Y)$ to mean that the multiset X contains at least two copies of A and write $(X \vdash Y) - (A \vdash 0)$ for the sequent obtained by deleting one of these copies from X. Similarly we can write $(0 \vdash A) + (0 \vdash A) \leq (X \vdash Y)$ to mean that the multiset Y contains at least two copies of A and write $(X \vdash Y) - (0 \vdash A)$ for the sequent obtained by deleting one of these copies from Y. More generally, we can write $UV + UV \leq XY$ to assert that the sequent $XY - UV$ can be obtained from XY by contracting the contents of the sequent UV. Thus, if the multiset of all formulae in UV (on both sides) is the singleton multiset `{#A#}` we know that the skeleton of the relevant contraction rule is one of:

$$\frac{A, A \vdash}{A \vdash} \qquad \frac{\vdash A, A}{\vdash A}$$

Definition 6.3 A set `rls` of rules satisfies the contraction admissibility property for the formula `A` if, whenever a derivable sequent $X \vdash Y$ satisfies $(A \vdash 0) + (A \vdash 0) \leq (X \vdash Y)$, the sequent $(X \vdash Y) - (A \vdash 0)$ is derivable, and likewise for $0 \vdash A$.

```
ctr_adm_def : "ctr_adm rls A ==
    ALL UV. ms_of_seq UV = {#A#} -->
        (ALL XY. XY : derrec rls {} --> UV + UV <= XY -->
          XY - UV : derrec rls {})"
```

The first condition `ms_of_seq As = {#?A#}`, asserts that the formulae on both sides of the sequent `As` form the singleton multiset `{#?A#}`, thus capturing that the contraction can happen on either side of the turnstile.

7 Generalising Cut-Admissibility Proofs

We now show how our previous work [6] on multicut admissibility for LK can be formulated to make it as general as possible. We first give details of induction principles and lemmata for "structural" induction over implicit derivations obtained via our derivability predicates and then describe their analogues for explicit derivation trees.

7.1 A General Framework for Reasoning About Implicit Derivations

The initial sequents of our sequent calculi will be allowed to apply to arbitrary formulae, not only atoms, and this excludes the possibility of proving height-preserving invertibility. This, and also the form of our contraction rule, which allows just one contraction per derivation step, prevents us from proving a height-preserving contraction-admissibility result. For proofs of contraction-admissibility, without height-preservation, an induction principle which also involves the size or structure of the relevant formula is required. Furthermore, proving cut-admissibility requires induction on both size of formula and derivation height (or a proxy for it). We therefore require a double induction on height (or proxy) and formula size (as measured by any well-founded subformula relation).

Our first induction principle could be seen as using a lexicographic ordering (n, m) where n is the sub-formula relation and m is the (inverse of the) distance from the end-sequent in the original derivation.

We use a relation `sub` on formulae: it could be any relation on formulae, but we use the (immediate) sub-formula relation. To put our general results in context, we may refer to `sub` as a "sub-formula relation". In general we want `sub` to be well-founded; more generally our theorems will apply to the "well-founded part" of `sub`.

In regard to the height measure, or distance from the original end-sequent, our first induction principle, instead of assuming that a property holds for all derivations of lesser height, merely assumes that it holds for sub-derivations.

Definition 7.1 (*`wfp`*) For a binary relation `sub`, a formula A is in `wfp sub`, the "well-founded part" of `sub`, iff there is not any infinite descending chain $\ldots, A_2, A_1, A$ such that $(A_1, A), (A_2, A_1), \ldots$ are all in `sub`.

Definition 7.2 (*`gen_step`*) For a formula `A`, a property `P`, a subformula relation `sub`, a set of sequents `derivs`, and a particular rule `r = (ps, c)`, where `ps` is a list of premises and `c` is the conclusion of `r`:

`gen_step P A sub derivs r` means
If

(a) forall `A'` such that `(A', A)` $\in$ `sub` and all sequents `s` $\in$ `derivs` the property `P A' s` holds, and
(b) for every premise `p` $\in$ `ps` both `p` $\in$ `derivs` and `P A p` holds, and
(c) `c` $\in$ `derivs`

then `P A c` holds.

```
gen_step_def :
 "gen_step P A sub derivs (ps, c) =
  ( (ALL A'. (A', A) : sub --> Ball derivs (P A'))
     --> (ALL p : set ps. p : derivs & P A p) --> c : derivs
   --> P A c)"
```

In this text, `ALL p : set ps` means $\forall p \in ps$. Typically `derivs` will be the set of sequents derivable using a given set `rls` of rules, and a given set of leaves `plvs`, so `derivs` = `derrec rls plvs`.

Intuitively, given a fixed rule `r = (ps, c)`, a fixed formula `A`, a fixed property `P` and a fixed relation `sub`, Definition 7.2(a) formalises for any derivable sequent `s` that `(A, c)` is "less than" `(A', s)` if `(A', A)` ∈ `sub`. Definition 7.2(b) formalises for any premise `p` from `ps` that `(A, p)` is "less than" `(A, c)` if `p` is a premise of `c` in the rule `r`. Thus, it can be seen as a particular instance of a lexicographic ordering on formula-sequent pairs where (A_1, s_1) is "less than" (A_2, s_2) if $(A_1, A_2) \in$ `sub` or, if $A_1 = A_2$ and s_2 is a premise of c via the particular rule (instance) $r = (ps, c)$.

Alternatively, by Definition 7.2, `gen_step` describes the situation where if a property P is true generally for sub-formulae A', and for the premises of a particular rule then the property holds for the conclusion of that rule.

The main theorem, named `gen_step_lem` and given as Theorem 7.1 below, states that if this step case can be proved for all possible rule instances then `P` holds for all cases.

Theorem 7.1 (*`gen_step_lem`*) *For a formula `A`, a property `P`, a subformula relation `sub`, a sequent `S` and a set of rules `rls`: If*

(a) `A` is in the well-founded part of the subformula relation `sub`, and
(b) for all formulae `A'` and all rules `r` in `rls`, the induction step condition `gen_step P A' sub (derrec rls {}) r` holds, and
(c) sequent `S` is `rls`-derivable

then `P A S` holds.

```
gen_step_lem:
  "[| ?A : wfp ?sub ;
      ALL A'. ALL r : ?rules.
                    gen_step ?P A' ?sub (derrec ?rules {}) r ;
      ?S : derrec ?rules {} |]
    ==> ?P ?A ?S"
```

Proof We combine the principle of well-founded induction, applied to the formula A and the well-founded subfomula relation `sub`, with the induction principle derrec-induction for `derrec` shown as Lemma 4.1, which is provided by Isabelle as a consequence of the inductive definition of `derrec`. ■

7.2 Induction for Two-Premise Subtrees

We now turn to the induction principle used for deriving cut-admissibility, or indeed any property P of two-premise implicit derivations. In the diagram below, to prove $P(cl, cr)$, for example, to prove that a cut between cl and cr is admissible, the induction assumption is that $P(psl_i, cr)$ and $P(cl, psr_j)$ hold for all i and j:

$$\dfrac{\dfrac{psl_1 \ldots psl_n}{cl}\,\rho_l \qquad \dfrac{psr_1 \ldots psr_m}{cr}\,\rho_r}{\mathcal{C}}\,(cut\ ?)$$

A proof of $P(cl, cr)$ using this induction assumption inevitably proceeds according to what the rules ρ_l and ρ_r are, and further, for a cut-formula A, whether it is principal in either or both of ρ_l and ρ_r. But our proofs also use induction on the size of the cut-formula, or, more generally, on some well-founded relation on formulae. So we actually consider a property $P\ A\ (cl, cr)$ where A is the cut-formula, psl are the premises $psl_1 \ldots psl_n$ of rule ρ_l, and cl is its conclusion, and analogously for ρ_r and cr. In proving $P\ A\ (cl, cr)$, in addition to the inductive assumption above, we assume that $P\ A'\ (da, db)$ holds generally for $(A', A) \in$ `sub` and all sequents da and db which are "`rls`-derivable", i.e., derivable from the empty set of leaves using rules from rls. These intuitions give the following definition `gen_step2sr` of a condition which permits one step of the inductive proof:

Definition 7.3 (*`gen_step2sr`*) For a formula `A`, a property `P`, a subformula relation `sub`, a set of rules `rls`, sequent rules `(psl, cl)`, and `(psr, cr)`: `gen_step2sr P A sub rls ((psl, cl), (psr, cr))` means:
If

(a) `P A' (da, db)` holds for all subformulae `A'` of `A` and all `rls`-derivable sequents `da` and `db`, and
(b) for each premise `pa` in `psl`, `pa` is `rls`-derivable and `P A (pa, cr)` holds, and
(c) for each premise `pb` in `psr`, `pb` is `rls`-derivable and `P A (cl, pb)` holds, and
(d) `cl` and `cr` are `rls`-derivable,

then `P A (cl, cr)` holds.

```
gen_step2sr_simp :
"gen_step2sr P A sub rls ((psl, cl), (psr, cr)) =
( (ALL A'. (A', A) : sub -->
   (ALL da:derrec rls {}.
    ALL db:derrec rls {}. P A' (da, db)))
  -->
  (ALL pa:set psl. pa : derrec rls {} & P A (pa, cr)) -->
  (ALL pb:set psr. pb : derrec rls {} & P A (cl, pb)) -->
    cl : derrec rls {} --> cr : derrec rls {}
     --> P A (cl, cr) )"
```

The main theorem `gen_step2sr_lem` below for proving an arbitrary property P states that if the step of the inductive proof goes through in all cases, i.e., for all possible final rule instances $\rho_l = (psl, cl)$ on the left and $\rho_r = (psr, cr)$ on the right, then P holds for all formulae A and sequents cl and cr on the left and right respectively.

Theorem 7.2 (*`gen_step2sr_lem`*) *If `A` is in the well-founded part of the subformula relation; sequents `seql` and `seqr` are `rls`-derivable ; and for all formulae `A'`, and all rules `(psl, cl)` and `(psr, cr)`, our induction step condition `gen_step2sr P A' sub rls ((psl, cl), (psr, cr))` holds, then `P A (seql, seqr)` also holds.*

```
gen_step2sr_lem :
 "[| ?A : wfp ?sub ;
     ?seql : derrec ?rls {} ; ?seqr : derrec ?rls {} ;
     ALL A'. ALL (psl, cl):?rls. ALL (psr, cr):?rls.
           gen_step2sr ?P A' ?sub rls ((psl, cl), (psr, cr)) |]
     ==> ?P ?A (?seql, ?seqr)"
```

Proof As with Lemma 7.1, the proof of this involves combining induction principles available to us. It is more complex than Lemma 7.1 because we had to deal with the well-founded induction on the sub-formula relation and `derrec`-induction (Lemma 4.1) on the *two* implicit derivations which provide the two premises of the cut. ■

This enables us to split up an inductive proof, by showing, separately, that `gen_step2sr` holds for particular cases of the final rules `(psl, cl)` and `(psr, cr)` on each side. In some cases these results apply generally to different calculi.

For example, the inductive step for the case where the cut-formula A is parametric, not principal, on the left is encapsulated in the following theorem where `prop2 car ?erls ?A (?cl, ?cr)`, which is equivalent to `(?cl, ?cr) : car ?erls ?A`, means that the conclusion of a cut on `A` with premises `cl` and `cr` is derivable using rules `erls`. Below, `:#` stands for membership of a multiset, and `~` stands for classical negation, and `wk_adm` refers to weakening admissibility for a system of rules, defined formally in Definition 6.1.

Theorem 7.3 *If weakening is admissible for the rule set `erls`, all extensions of some rule `(ps, U |- V)` are in the rule set `erls`, and the final rule instance `pscl` of the left hand (implicit) subtree is an extension of `(ps, c)` where the cut-formula `A` is not in `V` (meaning that `A` is parametric on the left), then `gen_step2sr (prop2 car ?erls) ?A ?sub ?rls (?pscl, ?pscr)` holds.*

```
lcg_gen_step:
"[| wk_adm ?erls ;
    extrs {(?ps, ?U |- ?V)} <= ?erls ;
    ~ ?A :# ?V ;
    ?pscl = pscmap (extend (?W |- ?Z)) (?ps, ?U |- ?V) |]
==> gen_step2sr (prop2 car ?erls) ?A ?any ?erls (?pscl, ?pscr)"
```

Notice that so far we have dealt with a shallow embedding of derivations; it does not apply to proofs which require derivation trees to be represented explicitly. As noted in Sect. 4.4, the derivability of a sequent is equivalent to the existence of a valid derivation tree for it, and so now we describe the similar approach for explicit derivation trees.

7.3 Induction Principles for Explicit Derivation Trees

Sometimes we need to proceed by induction on (for example) the length of a derivation by which a sequent can be obtained, rather than by the fact of a sequent having been obtained earlier in the same derivation. At other times, we not only need to do induction on height, but we may also have to transform the immediate premises in some way, for example, by utilising the admissibility of weakening or contraction.

We could change our (notion of implicit derivations) derivability predicate `derrec rls plvs` with a third argument `ht`, say, so that `derrec rls plvs ht` captured the set of sequents derivable from the leaves in `plvs` using rules from `rls` with height `ht`. But then it becomes much harder to incorporate the transformations of the immediate premises of an end-sequent using the weakening and contraction lemmata since we have no explicit access to the derivation itself. So to compare (say) the heights of derivations, we must be able to define them and for this we need to look at explicit derivation trees.

We can use explicit derivation trees to perform a proof equivalent to one using Theorem 7.1, by using the following definitions and lemmata.

Definition 7.4 (*`gen_step_tr`*) For all properties `P`, all formulae `B`, all "subformula" relations `sub` and all (explicit) derivation trees `dta`:
`gen_step_tr P B sub dta` means:
if

(a) `P C dtb` holds for all subformulae `C` of `B` and all derivation trees `dtb`, and
(b) `P B dtsub` holds for all the immediate subtrees `dtsub` of `dta`

then `P B dta` holds.

```
gen_step_tr_def:
  "gen_step_tr P B sub dta ==
    (ALL C. (C, B) : sub --> (ALL dtb. P C dtb)) -->
    (ALL dtsub:set (nextUp dta). P B dtsub) --> P B dta"
```

Lemma 7.1 (`gen_step_tr_lem`) *For all properties* `P`*, for all formulae* `A`*, for all relations* `sub`*, for all derivations* `dt`*, if* `A` *is in the well-founded part of* `sub`*, and* `gen_step_tr P B sub dtb` *holds for all formulae* `B` *and all derivations* `dtb`*, then* `P A dt` *holds.*

```
gen_step_tr_lem:
  "[| ?A : wfp ?sub ;
      ALL B dtb. (gen_step_tr ?P B ?sub dtb) |]
      ==> ?P ?A ?dt"
```

Definition 7.5 (`gen_step2_tr`) For all properties `P`, for all formulae `B`, for all "sub-formula" relations `sub`, for all pairs `(dta, dtb)` of derivation trees:
`gen_step2_tr P B sub (dta, dtb)` means:
if

(a) `P C (dtaa, dtbb)` holds for every sub-formula `C` of `B` and all derivation trees `dtaa` and `dtbb`, and
(b) `P B (dtp, dtb)` holds for all immediate subtrees `dtp` of `dta`, and
(c) `P B (dta, dtq)` holds for all immediate subtrees `dtq` of `dtb`

then `P B (dta, dtb)` holds:

```
gen_step2_tr.simps:
 "gen_step2_tr P B sub (dta, dtb) =
  ((ALL C. (C, B):sub --> (ALL dtaa dtbb. P C (dtaa, dtbb)))
  --> (ALL dtp:set (nextUp dta). P B (dtp, dtb))
  --> (ALL dtq:set (nextUp dtb). P B (dta, dtq))
  --> P B (dta, dtb))"
```

Lemma 7.2 (`gen_step2_tr_lem`) *For all properties* `P`*, for all formulae* `A`*, for all relations* `sub`*, for all derivation trees* `dta` *and* `dtb`*, if* `A` *is in the well-founded part of* `sub`*, and* `gen_step2_tr P B sub (dtaa, dtbb)` *holds for all formulae* `B` *and all derivations* `dtaa` *and* `dtbb`*, then* `P A (dta, dtb)` *holds:*

```
gen_step2_tr_lem:
 "[| ?A : wfp ?sub ;
     ALL B dtaa dtbb. gen_step2_tr ?P B ?sub (dtaa, dtbb) |]
     ==> ?P ?A (?dta, ?dtb)"
```

These properties are exact analogues, for explicit derivation trees, of the properties `gen_step` and `gen_step2sr` and Theorems 7.1 and 7.2, with (for example) Lemma 8.2 linking them.

However, the purpose of using explicit derivation trees is to define different induction patterns. For example, we defined an induction pattern which depends on the inductive assumption that the property *P* holds for the given tree on one side, and any smaller tree on the other side.

Definition 7.6 (*`measure`*) For all `a`, all `b`, and all functions `f::'a => nat`, the pair `(a, b)` is in `measure f` iff `f a` < `f b`:

```
measure_eq: "((?a, ?b) : measure ?f) = (?f ?a < ?f ?b)"
```

Definition 7.7 (*`height_step2_tr`*) For all properties `P`, for all formulae `A`, for all subformula relations `sub`, for all pairs `(dta, dtb)` of derivations, *`height_step2_tr P A sub (dta, dtb)`* means:
if

(a) `P B (a, b)` holds for all subformulae `B` of `A` and for all derivation trees `A` and `B`, and
(b) `P A (tp, dtb)` holds for all derivation trees `tp` of smaller height than `dta`, and
(c) `P A (dta, tq)` holds for all derivation trees `tq` of smaller height than `dtb`

then `P A (dta, dtb)` holds.

```
height_step2_tr_def:
"height_step2_tr P A sub (dta, dtb) =
 ((ALL B. (B, A) : sub --> (ALL a b. P B (a, b))) -->
 (ALL dtp. heightDT dtp < heightDT dta --> P A (dtp, dtb)) -->
 (ALL dtq. heightDT dtq < heightDT dtb --> P A (dta, dtq)) -->
 P A (dta, dtb))"
```

In some cases we found that this wasn't enough, and defined a more general pattern, in which the inductive assumption applies where the sum of the heights of the two trees is smaller.

Definition 7.8 (*`sumh_step2_tr`*) For a property `P`, a formula `A`, a subformula relation `sub`, and a pair of derivations `(dta, dtb)`,
`sumh_step2_tr P A sub (dta, dtb)` means:
if

(a) `P B (a, b)` holds for all subformulae `B` of `A` and all derivation trees `a` and `b`, and
(b) for all derivation trees `dtaa` and `dtbb`, we have `heightDT dtaa` + `heightDT dtbb` < `heightDT dta` + `heightDT dtb` implies `P A (dtaa, dtbb)`

then `P A (dta, dtb)` holds

```
sumh_step2_tr_eq:
 "sumh_step2_tr P A sub (dta, dtb) =
   ((ALL B. (B, A) : sub --> (ALL a b. P B (a, b))) -->
   (ALL dtaa dtbb. heightDT dtaa + heightDT dtbb <
      heightDT dta + heightDT dtb --> P A (dtaa, dtbb)) -->
   P A (dta, dtb))"
```

We could of course generalise this by replacing `heightDT` by any natural number function, which may be different for trees on the left and right sides. Indeed it could be further generalised to any well-founded relation on pairs of derivation trees.

Each of these properties is successively weaker since the corresponding inductive assumption is stronger, hence P applies to correspondingly wider classes of derivations: as formalised next.

Lemma 7.3 *For a property* `P`*, a formula* `A`*, a relation* `sub`*, and for a pair* `(dta, dtb)` *of derivations:*

(a) `gen_step2_tr` *implies* `height_step2_tr`
(b) `height_step2_tr` *implies* `sumh_step2_tr`

```
gs2_tr_height:
  "gen_step2_tr ?P ?A ?sub (?dta, ?dtb) ==>
             height_step2_tr ?P ?A ?sub (?dta, ?dtb)"

hs2_sumh:
   "height_step2_tr ?P ?A ?sub (?dta, ?dtb) ==>
             sumh_step2_tr ?P ?A ?sub (?dta, ?dtb)"
```

Accordingly we need the lemma that proving these step results is sufficient for only the weakest of them.

Lemma 7.4 (`sumh_step2_tr_lem`) *For a property* `P` *and a formula* `A` *in the well-founded part of a relation* `sub`*, if* `sumh_step2_tr P A sub (dta, dtb)` *holds for all derivations* `dta` *and* `dtb` *then* `P A (dtaa, dtbb)` *holds for all derivations* `dtaa` *and* `dtbb`*:*

```
sumh_step2_tr_lem:
 "[|  ?A : wfp ?sub ;
      ALL A dta dtb. sumh_step2_tr ?P A ?sub (dta, dtb) |]
  ==> ?P ?A (?dtaa, ?dtbb)"
```

We are now in a position to define the statement of cut-admissibility in Isabelle, and to apply all of these results.

8 Statement of Cut-Admissibility in Isabelle

Definition 8.1 (`cas, car`) For all formulae A, and all pair of sequents:

`car rls` A holds if the sequent obtained by applying the cut rule on formula A to them is derivable: that is, $(X_l \vdash Y_l, X_r \vdash Y_r) \in$ `car rls` A iff $(X_l, (X_r - A) \vdash (Y_l - A), Y_r)$ is `rls`-derivable;

`cas rls` A holds if cut-admissibility on A is available for that pair of sequents: that is, $(X_l \vdash Y_l, X_r \vdash Y_r) \in$ `cas rls` A means that *if* $X_l \vdash Y_l$ and $X_r \vdash Y_r$ are `rls`-derivable, *then* $(X_l \vdash Y_l, X_r \vdash Y_r) \in$ `car rls` A.

```
car_eq:
  "((Xl |- Yl, Xr |- Yr) : car rls A) =
   ((Xl + (Xr - {#A#}) |- Yl - {#A#} + Yr) : derrec rls {})"

cas_eq:
 "((seql, seqr) : cas rls A) =
    (seql : derrec rls {} & seqr : derrec rls {}
      --> (seql, seqr) : car rls A)"
```

When we are talking about proving `cas` or `car` by induction on the (implicit) derivation of the two sequents, that is, we are talking about two sequents which are derivable, then these two concepts become equivalent. This is because the definition of `gen_step2sr` only involves the property of the pair of sequents in the cases where those two sequents are derivable. Recall that `prop2` simply gives an equivalent concept with a different type.

Lemma 8.1 *The induction steps for proving* `cas` *and* `car` *are equivalent:*

```
prop2_def     : "prop2 f rls A seqs == seqs : f rls A"

gs2_cas_eq_car: "gen_step2sr (prop2 cas ?rls) ?A ?sub ?rls =
                 gen_step2sr (prop2 car ?rls) ?A ?sub ?rls"
```

Definition 8.2 (`casdt`) For any set `rls` of rules and any formula `A`, two *valid* (ie. no unproved leaves, and all steps are rules of `rls`) derivation trees `dtl` and `dtr` satisfy `casdt rls A` iff their conclusions satisfy `car`:

```
casdt_eq:
   "((?dtl, ?dtr) : casdt ?rls ?A) =
          (valid ?rls ?dtl & valid ?rls ?dtr
             --> (conclDT ?dtl, conclDT ?dtr) : car ?rls ?A)"
```

Here is the lemma linking the induction step for cut-admissibility in terms of implicit derivability with the corresponding induction step for explicit derivation trees.

Lemma 8.2 (`gs2_tr_casdt_sr`) *Given two derivation trees* `dta` *and* `dtb`*, a cut-formula* `A`*, a sub-formula relation* `sub`*, and a rule set* `rls`*, if the bottom rules of those trees satisfy the step condition* `gen_step2sr` *for cut-admissibility, then the two trees satisfy the step condition* `gen_step2_tr` *for cut-admissibility:*

```
gs2_tr_casdt_sr:
   "gen_step2sr (prop2 cas ?rls) ?A ?ipsubfml ?rls
        (botRule ?dta, botRule ?dtb) ==>
    gen_step2_tr (prop2 casdt ?rls) ?A ?ipsubfml (?dta, ?dtb)"
```

In fact the two concepts are essentially equivalent:

Theorem 8.1 (`gs2_casdt_equiv`) *Given a set of derivation rules* `rls`*, a formula* `A`*, a sub-formula relation* `ipsubfml` *and two bottom rules* `pscl` *and* `pscr`*, then the following are equivalent:*

(a) if `pscl` *and* `pscr` *are in* `rls`*, then they satisfy the step condition* `gen_step2sr` *for cut-admissibility (for implicit derivations)*
(b) all trees `dta` *and* `dtb` *whose bottom rules are* `pscl` *and* `pscr` *respectively, satisfy the step condition* `gen_step2_tr` *for cut-admissibility (for explicit derivations)*

```
gs2_casdt_equiv:
"(?pscl : ?rls -->?pscr : ?rls --> gen_step2sr (prop2 cas ?rls)
    ?A ?ipsubfml ?rls (?pscl, ?pscr)) =
(ALL dta dtb.  botRule dta = ?pscl --> botRule dtb = ?pscr -->
  gen_step2_tr (prop2 casdt ?rls) ?A ?ipsubfml (dta, dtb))"
```

We are now ready to apply our formalisation work to particular calculi.

9 Weakening, Contraction and Cut Admissibility for S4

There exist both pen and paper [19, 25] and a formalised proof [5] of mix-elimination for sequent calculi for S4 containing explicit weakening and contraction rules. As stated previously, explicit structural rules are detrimental for automated reasoning, giving a practical motivation for proving that such rules are admissible. This is our goal.

Troelstra and Schwichtenberg also state cut-elimination for a sequent calculus G3s [25] for S4 that contains no explicit structural rules. Unfortunately, their "proof" only discusses one actual transformation, and in particular overlooks one non-trivial case—when Cut is applied on a formula $\Box A$, with both premises being an instance of the G3s R$\Box$ rule (shown below). In this case, the deduction cannot be transformed by simply permuting the Cut, or introducing a new Cut of smaller rank, on the sequents in the original deduction. Greater detail is given later in this section.

$$\text{R}\Box\ \frac{\Box\Gamma \vdash A, \Diamond\Delta}{\Gamma', \Box\Gamma \vdash \Box A, \Diamond\Delta, \Delta'}$$

Goubault [14] acknowledges the problem posed by absorbing Weakening into the R$\Box$ rule. However, his solutions are given in the context of typed λ-calculi for a minimal version of S4, interpreted as a sequent calculus through a version of the Curry-Howard correspondence. Based on a proposal from [2], Goubault-Larrecq replaces the R$\Box$ rule by a different rule with multiple premises (for subformulae within the principal formula), along with both re-write and garbage collection rules for the λ terms involved. While this solution could possibly be extended to sequent calculi, the creation of new premises and hence branching is detrimental to backward proof search. Our solution presented in this section also has the advantage of being significantly simpler.

Negri [18] proves various admissibility theorems for S4, but the calculus involved is labelled. These labels include elements of the Kripke semantics within the calculus, and so the resulting theorems are thus not entirely syntactical proofs. Furthermore, there are rules in the calculus which deal only with reachability between worlds. While perhaps not as inefficient as the standard structural rules, these rules nevertheless do not operate on logical connectives. In particular to S4, from the perspective of automated reasoning, applying all possible instances of the transitivity rule (shown below) or checking whether the transitivity rule has been saturated can be a very time-consuming process.

$$\text{Transitivity}\ \frac{xRz, xRy, yRz, \Gamma \vdash \Delta}{xRy, yRz, \Gamma \vdash \Delta}$$

R is the accessibility relation. x, y, z are worlds.

9.1 Calculus for S4

The sequent calculus we use for S4 is based on the calculus G3cp [25], with the addition of two modal rules. Note that the initial sequents $\Gamma, \varphi \vdash \varphi, \Delta$ do not require that φ be atomic, and that there are only rules for $\Box$ formulae since $\Diamond\varphi$ is interpreted as $\neg\Box\neg\varphi$. The rules of the calculus are shown in Figs. 3 and 4. Note that the clause `boxI` in the inductive definition for `gs4_rls` applies `extend` only to the conclusion, corresponding to the appearance of the two sets Σ and Δ in the conclusion of the rule $S4\Box$.

Initial Sequents

$$\text{id}\ \frac{}{\Gamma, \varphi \vdash \varphi, \Delta}$$

Classical rules

$$\text{L}\wedge\ \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \qquad \text{R}\vee\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

$$\text{L}\to\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta} \qquad \text{R}\to\ \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}$$

$$\text{L}\neg\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

Modal rules

$$\text{Refl}\ \frac{\Gamma, \varphi, \Box\varphi \vdash \Delta}{\Gamma, \Box\varphi \vdash \Delta} \qquad \text{S4}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi}{\Sigma, \Box\Gamma \vdash \Box\varphi, \Delta}$$

Fig. 3 Sequent calculus GS4 for S4

The Isabelle encoding of the calculus is modular, with the overall calculus, `gs4_rls`, built up from separate declarations of the id rule, the classical rules acting on antecedents and succedents, and the two modal rules.

9.2 Weakening for S4

Intuitively, weakening is admissible for a system of rules if, whenever the conclusion c of a rule (ps, c) is weakened to c', there is a rule with conclusion c' and premises ps' which are (optionally) weakened counterparts of ps.

The following definition seeks to formalise this condition.

Definition 9.1 A set of rules `rls` satisfies `ext_concl` iff: for every list of premises `ps` and conclusion `c` that form a rule (ρ_1 say) in `rls`, and for all possible sequents `UV`, there exists a list of premises `ps'` such that the premises `ps'` and the extended conclusion `c + UV` also form an instance of some rule (ρ_1' say) in `rls` and for every premise `P` from `ps` there is a corresponding premise `p'` in `ps'` such that `p'` is either `P` itself or is an extension of `P`:

```
inductive "lksne" intrs (* skeletons of LK rules *)
  axiom   "([], {#A#} |- {#A#}) : lksne"
    ilI   "psc : lksil ==> psc  : lksne"
    irI   "psc : lksir ==> psc  : lksne"

inductive "lksss" intrs (* extended skeletons for LK *)
   extI   "psc : lksne ==> pscmap (extend flr) psc     : lksss"

inductive "lkrefl" intrs (* refl rule skeleton *)
      I   "([{#A#} + {#Box A#} |- {#}], {#Box A#} |- {#}) : lkrefl"

inductive "lkbox" intrs (* S4 Box rule skeleton *)
      I   "([gamma + mset_map Box gamma |- {#A#}],
                      mset_map Box gamma |- {#Box A#})  : lkbox"

inductive "gs4_rls" intrs
    lksI  "psc : lksss ==> psc : gs4_rls"
   reflI  "psc : lkrefl ==> pscmap (extend flr) psc  : gs4_rls"
          (* Box rule allows extra formulae in conclusion only *)
    boxI  "(prem, conc) : lkbox ==>
                                (prem, extend flr conc) : gs4_rls"
```

Fig. 4 Isabelle rules for GS4

$$\frac{\mathtt{p}_1 \ldots \mathtt{p}_k}{\mathtt{c}} \ (\rho_1) \qquad\qquad \frac{\mathtt{p'}_1 \ldots \mathtt{p'}_k}{\mathtt{c\ +\ UV}} \ (\rho'_1) \qquad\qquad \mathtt{p}_i \le \mathtt{p'}_i$$

In the Isabelle text `(ps, ps') : allrel r` means that `ps` and `ps'` are lists of the same length where each corresponding pair of their members is in `r`. The relation $\leq$ for sequents is defined in terms of $\leq$ for multisets, that is, $X \vdash Y \leq X' \vdash Y'$ means $X \leq X'$ and $Y \leq Y'$.

```
ext_concl_def:
 "ext_concl rls ==
   ALL (ps, c) : rls. ALL UV. EX ps'.
   (ps', c + UV) : rls & (ps, ps') : allrel {(p, p'). p <= p'}"

inductive "allrel r" intrs
    allrel_Nil  "([], []) : allrel r"
    allrel_Cons "[| (ha, hb) : r ; (ta, tb) : allrel r |]
                ==>  (ha # ta, hb # tb) : allrel r"
```

Lemma 9.1 *If rule set* `rls` *obeys* `ext_concl` *then* `rls` *admits weakening:*

```
wk_adm_ext_concl: "ext_concl ?rls ==> wk_adm ?rls"
```

The lemma `wk_adm_ext_concl` is so simple it can be proved directly by the induction principle for `derrec` Lemma 4.1 (without using `gen_step_lem`). Use of lemmas like `gen_step_lem` is really only for the purpose of breaking up the proofs, so that various different cases of `gen_step` (ie various final rules of the derivation) can be put into separate lemmata, some of which may be able to be reused for different calculi.

Lemma 9.2 *The set of rule `gs4_rls` satisfies `ext_concl`.*

```
gs4_ext_concl: "ext_concl gs4_rls"
```

Corollary 9.1 *The rules of S4 satisfy weakening admissibility.*

```
gs4_wk_adm: "wk_adm gs4_rls"
```

9.3 Invertibility and Contraction for S4

We now describe how we captured the traditional proof of invertibility.

Suppose that we are given a calculus consisting of the rule set `drls` and suppose that we want to reason about the derivability predicate `derrec` defined earlier. Let `derivs` be the set `derrec drls` of all sequents that are derivable from the empty set of leaves using the rules of `drls`. Suppose that we wish to prove that every rule in `irls` is invertible w.r.t. `drls` (where `irls` is usually a subset of `drls`).

Omitting details, the function `invs_of irls c` returns the set of sequents obtainable by applying each rule of `irls` to the sequent `c` *backwards* once. That is, a sequent `seq` is in `invs_of irls c` if applying some rule ρ of `irls` to `c` backwards, once, will give `seq` as one of the premises of ρ.

To prove that a rule `(ps, concl)` is invertible w.r.t. `drls` requires us to prove that each sequent `seq` from the list `ps` of premises is in `derivs` if `concl` is in `derivs`. To prove that each rule in a set of rules `irls` is invertible w.r.t. `drls` requires us to prove that the above property holds for each rule `(ps, concl)` from `irls`: that is, `invs_of irls concl <= derivs` where `<=` encodes the subset relation.

Traditional proofs of invertibility proceed by an induction on the structure of a given derivation of a sequent `concl ∈ derivs`. Assuming that the final rule in this derivation is `(ps, concl)`, the induction hypothesis is to assume that the invertibility lemma holds for each premise in `ps`. That is, we assume that every sequent `seq` obtained by applying any rule from `irls` *backwards*, once, to any premise `P` in `ps` is itself in `derivs`:

```
ALL p:set ps. invs_of irls p <= derivs
```

Use of the induction hypothesis stated above can then be encoded in `inv_step` as follows. Let an "`irls`-inverse" of a sequent s be a sequent s' obtained from s by applying any rule from `irls` backwards once.

Definition 9.2 (*`inv_step`*) For a given set `derivs` of derivable sequents, for a rule set `irls`, and for every rule instance `(ps, concl)`, the property: `inv_step derivs irls (ps, concl)` means:

If every premise in `ps` being in `derivs` implies that every "`irls`-invert" of premises in `ps` is in `derivs`,
then every "`irls`-invert" of the conclusion `concl` is in `derivs`.

```
inv_step.simps:
   "inv_step derivs irls (ps, concl) =
        (set ps <= derivs
              --> (ALL p:set ps. invs_of irls p <= derivs)
         --> invs_of irls concl <= derivs)"
```

This is the key result for doing invertibility by stating various cases of the induction step as separate lemmata.

The expression `UNION (set ?ps) (invs_of ?irls)` represents the set X of all sequents obtained by applying some rule from `irls` backwards once to every sequent `P` from a list of sequents `ps` viewed as a set:

$$X := \bigcup_{\texttt{P} \in \texttt{set ps}} (\texttt{invs\_of ?irls p})$$

Then, `(set ?ps Un UNION (set ?ps) (invs_of ?irls))` represents the union of X and the list of sequents `ps` treated as a set, ie `(set ps)` $\cup X$.

The property `inv_stepm` is weaker than `inv_step` but is monotonic in its first argument, which makes reusing lemmata such as `lks_inv_stepm` possible as follows.

Definition 9.3 (*`inv_stepm`*) For all rule sets `drls`, for all rule sets `irls`, for all rules `(ps, concl)`, the expression `inv_stepm drls irls (ps, concl)` means: the `irls`-inverses of `concl` are derivable using `derrec drls` from `(set ps)` and the `irls`-inverses of every `P` $\in$ `set ps`:

```
inv_stepm.simps:
"inv_stepm drls irls (ps, concl) =
  (invs_of irls concl <=
     derrec drls (set ps Un UNION (set ps) (invs_of irls)))"
```

Lemma 9.3 (*`inv_step_mono`*) *`inv_stepm` is monotonic in its first argument:*

```
inv_step_mono:
   "[| inv_stepm ?drlsa ?irls ?psc ; ?drlsa <= ?drlsb |]
     ==> inv_stepm ?drlsb ?irls ?psc"
```

Lemma 9.4 (*`inv_step_m`*) *For every set `drls` of rules and every set `plvs` of sequents, the function `derrec drls plvs` returns the set of sequents derivable from `plvs` using the rules of `drls`. Let us call this set of sequents `derivs`. For every set `drls` of rules used for derivations, for every rule set `irls`, for every rule `psc`, if `inv_stepm drls irls psc` holds then so does `inv_step derivs irls psc` for any set of leaf sequents `plvs`:*

```
inv_step_m:
   "inv_stepm ?drls ?irls ?psc
            ==> inv_step (derrec ?drls ?plvs) ?irls ?psc"
```

Lemma 9.5 (*`gen_inv_by_step`*) *For every rule set `rls` which is used to construct a set `derrec rls` of derivations from the empty set of leaves, for every rule set `irls`, every rule `psc` from `irls` is invertible w.r.t. `rls` if every rule instance `(ps, concl)` from `rls` obeys*
`inv_step (derrec rls ) irls (ps, concl)`:

```
gen_inv_by_step:
   "[| Ball ?rls (inv_step (derrec ?rls {}) ?irls) ;
       ?psc : ?irls |]
     ==> inv_rl ?rls ?psc"
```

Lemma 9.6 *Every instance of the rule Refl, extended with arbitrary contexts, is invertible in the rule set `gs4_rls`:*

```
Ball (extrs lkrefl) (inv_rl gs4_rls)
```

Proof Suppose that $\Gamma, \Box\varphi \vdash \Delta$ is derivable. We can show that the premise $\Gamma, \varphi, \Box \varphi \vdash \Delta$ is derivable by applying weakening, which has already been shown to be admissible in `gs4_rls`. ■

Lemma 9.7 *Every instance of the rule set `lksss` (of classical propositional logic) is invertible in the rule set `gs4_rls`:*

```
Ball lksss (inv_rl gs4_rls)
```

Proof By Lemma 9.5, it suffices to prove `(inv_step (derrec gs4_rls {}) lksss) psc` for every rule `psc` from `gs4_rls`. By Lemma 9.4, it suffices to prove `inv_stepm gs4_rls lksss psc` for every rule `psc` from `gs4_rls`. Here, `lksss == extrs lksne`, the rule set `lksne` extended with arbitrary contexts. We proceed by cases on each rule `psc` *in* `gs4_rls`:

`psc` = Refl. Immediate, the inverse of rule Refl is an instance of weakening.

```
"?psc : extrs lkrefl
  ==> inv_stepm gs4_rls (extrs lksne) ?psc"
```

`psc` is from LK. Where the rule `psc` is a classical rule, we first prove that the set of classical rules is invertible w.r.t. itself:

```
"?psc : extrs lksne ==>
    inv_stepm (extrs lksne) (extrs lksne) ?psc"
```

Since the rules `lksss` are a subset of the rules `gs4_rls`, we can use (the monotonicity) Lemma 9.3 to obtain:

```
"?psc : extrs lksne
  ==> inv_stepm gs4_rls (extrs lksne) ?psc"
```

`psc` = $S4\Box$. When the last rule is $S4\Box$ (with arbitrary contexts in conclusion only to make weakening admissible) we prove a general result. If the rule set `rls` contains exactly one rule `extcs (ps, c)` which is the rule (skeleton) `(ps, c)` with only the conclusion extended by an arbitrary context, and `rl` is any member (instance) of `rls`, then `inv_stepm rls (extrs (ips, ic)) rl` holds for any rule `(ips, ic)` extended with arbitrary contexts if the (skeleton of the) conclusion `ic` and the (skeleton of the) conclusion `c` are disjoint:

```
inv_stepm_disj_cs:
    "[| seq_meet ?c ?ic = 0 ;
        extcs {(?ps, ?c)} = ?rls ;
        ?rl : ?rls |]
      ==> inv_stepm ?rls (extrs {(?ips, ?ic)}) ?rl"
```

In particular, we can put `extcs (?ps, ?c)` to be the rule $S4\Box$ and put `(extrs (?ips, ?ic))` to be any rule from `lksss` since the skeletons of the conclusions of the `lksss` rules contain only the principal formula of the respective rule and none of these is a $\Box$-formula. ■

Theorem 9.1 (*`inv_rl_gs4_refl`* and *`inv_rl_gs4_lks`*) *The Refl (`lkrefl`) rule and all Classical (`lksss`) rules are invertible within `gs4_rls`.*

Proof The theorem is simply the conjunction of Lemmas 9.6 and 9.7. We explain some of the cases in English to highlight the new aspects.

Consider invertibility for the R$\vee$ rule. We proceed by an induction on height, and use the induction principle `gen_inv_by_step` from Lemma 9.5.

Case 1 Axiom If $\Gamma \vdash \varphi \vee \psi, \Delta$ is an axiom, and $\varphi \vee \psi$ is principal, then $\Gamma = \Gamma', \varphi \vee \psi$. The derivation for $\Gamma \vdash \varphi, \psi, \Delta$ is then:

$$\mathrm{L}\vee \frac{\mathrm{id}\,\dfrac{}{\Gamma', \varphi \vdash \varphi, \psi, \Delta} \qquad \mathrm{id}\,\dfrac{}{\Gamma', \psi \vdash \varphi, \psi, \Delta}}{\Gamma', \varphi \vee \psi \vdash \varphi, \psi, \Delta}$$

If $\varphi \vee \psi$ is parametric in (id), then $\Gamma \vdash \Delta$ is (id), and so is $\Gamma \vdash \varphi, \psi, \Delta$.

Case 2 Principal If $\Gamma \vdash \varphi \vee \psi, \Delta$ is not an axiom, but $\varphi \vee \psi$ is principal, then R$\vee$ must have been the last rule applied. Invertibility follows immediately from the premises of the R$\vee$ rule.

Case 3 Parametric If $\Gamma \vdash \varphi \vee \psi, \Delta$ is not an axiom, and $\varphi \vee \psi$ is parametric, then an application of a new instance of that last rule (perhaps using the induction hypothesis) obtains the necessary endsequent. This is because all rules allow arbitrary contexts in their conclusion (and premises when the premises contain context). To illustrate, consider the two cases when the last rule used to originally derive $\Gamma \vdash \varphi \vee \psi, \Delta$ is either the Refl or the S4$\Box$ rule:

- If the last rule was Refl then $\Gamma = \Gamma', \Box A$ and the original derivation is:

$$\mathrm{Refl}\,\frac{\overset{\Pi}{\Gamma', A, \Box A \vdash \Delta, \varphi \vee \psi}}{\Gamma', \Box A \vdash \Delta, \varphi \vee \psi}$$

  Applying the inductive hypothesis on the premises gives a derivation of $\Gamma', A, \Box A$
  $\vdash \Delta, \varphi, \psi$. Applying Refl to this gives the required $\Gamma', \Box A \vdash \varphi, \psi, \Delta$.
- If the last rule was S4$\Box$ then $\Gamma = \Sigma, \Box\Gamma'$ and $\Delta = \Box A, \Delta'$ and the original derivation looks like:

$$\mathrm{S4}\Box\,\frac{\overset{\Pi}{\Gamma', \Box\Gamma' \vdash A}}{\Sigma, \Box\Gamma' \vdash \Box A, \Delta', \varphi \vee \psi}$$

  To derive $\Gamma \vdash \varphi, \psi, \Delta$, simply apply a new instance of S4$\Box$ to the original premise, this time with φ, ψ as the context instead of $\varphi \vee \psi$:

$$\mathrm{S4}\Box\,\frac{\overset{\Pi}{\Gamma', \Box\Gamma' \vdash A}}{\Sigma, \Box\Gamma' \vdash \Box A, \Delta', \varphi, \psi}$$

■

Theorem 9.2 (`gs4_ctr_adm`) *Contraction is admissible for* `gs4_rls`.

```
gs4_ctr_adm: "ctr_adm gs4_rls ?A"
```

Proof The cases for the G3cp and Refl rules are handled in the standard manner as in the literature (see [25] and [17]) using the invertibility results above. The formalisation performs the necessary transformations using a simple instantiation `gen_ctr_adm_step` (not shown) of the induction principle `gen_step_lem` of Theorem 7.1.

When the rule above the contraction is an instance of the S4□ rule, there are two possible cases. Either one or both copies of the contraction-formula exist within the context of the S4□ rule, or both copies are principal.

In the first case, deleting one copy still leaves an instance of the rule. That is, if the contraction-formula is A, with A in the succedent, then the original rule instance is as shown below where either $\Box\varphi = A$ or $A \in \Delta$:

$$\text{S4}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi}{\Sigma, \Box\Gamma \vdash \Box\varphi, A, \Delta}$$

Applying the S4□ rule without introducing the shown second copy of A in the conclusion above gives a proof of $\Sigma, \Box\Gamma \vdash \Box\varphi, \Delta$ as required since an occurrence of A is still in the succedent as $\Box\varphi = A$ or $A \in \Delta$. Similarly, if A is in the context Σ the new S4□ rule instance is then:

$$\text{S4}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi}{\Sigma - A, \Box\Gamma \vdash \Box\varphi, A, \Delta}$$

The harder case occurs when both instances of the contraction-formula A are principal. Due to the nature of the S4□ rule this requires A to occur in the antecedent only, as there cannot be two principal formulae in the succedent. As only boxed formulae are principal, A has form $\Box B$. The original rule instance is thus represented by:

$$\text{S4}\Box\ \frac{B, B, \Box B, \Box B, \Gamma, \Box\Gamma \vdash \varphi}{\Sigma, \Box B, \Box B, \Box\Gamma \vdash \Box\varphi, \Delta}$$

The copies of $\Box B$ and B can be contracted upon, first using the induction hypothesis that the result applies to preceding sequents in the derivation, and then on the rank of the contraction-formula. The S4□ rule can then be applied to give the required conclusion.

$$\text{S4}\Box\ \frac{B, \Box B, \Gamma, \Box\Gamma \vdash \varphi}{\Sigma, \Box B, \Box\Gamma \vdash \Box\varphi, \Delta}$$

In the Isabelle proof, this step is unfortunately rather more tedious. A significant number of proof steps in the formalisation are dedicated to manipulating the ordering of formulae to convince the proof assistant that the S4□ rule can be applied after applying the induction hypotheses, and that the resulting sequent is indeed what is required. ■

9.4 Cut-Admissibility for S4

We first state a lemma used several times in the proof of cut-admissibility.

Lemma 9.8 *Given two (explicit) derivation trees* `dta` *and* `dtb`*, a cut-formula* `A`*, a sub-formula relation* `sub`*, and a rule set* `rls`*, if the bottom rules of those trees satisfy the step condition* `gen_step2sr` *for cut-admissibility, then the two trees satisfy the step condition* `sumh_step2_tr` *for cut-admissibility:*

```
gs2_car_sumhs_tr:
  "gen_step2sr (prop2 car ?rls) ?A ?sub ?rls
               (botRule ?dta, botRule ?dtb)
   ==> sumh_step2_tr (prop2 casdt ?rls) ?A ?sub (?dta, ?dtb)"
```

Proof By combining Lemmas 7.3, 8.2 and 8.1. ■

Theorem 9.3 (`gs4_cas`) *Cut is admissible in the calculus* `gs4_rls`*.*

```
gs4_cas:
  "(?Xl |- mins ?A ?Yl, mins ?A ?Xr |- ?Yr) : cas gs4_rls ?A"
```

Proof Our proof essentially uses a double induction on level and rank, where level measures the sum of the heights of the derivation trees for the left and right premises of the cut, and rank measures the complexity of the cut-formula. It uses Lemma 7.4, in which `?sub` is instantiated to the immediate subformula relation.

The two most difficult cases to consider correspond to when the cut-formula is principal below an application of the S4□ rule on the left, and also principal in either the Refl or the S4□ rule on the right. As these are all modal rules, the Cut in question must be on a boxed formula, □A.

In the former case, the original Cut has form:

$$\text{Cut on } \Box A\ \dfrac{\text{S4}\Box\ \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array}}{\Sigma, \Box\Gamma_L \vdash \Delta_L, \Box A} \qquad \text{Refl}\ \dfrac{\begin{array}{c}\Pi_r\\ A, \Box A, \Gamma_R \vdash \Delta_R\end{array}}{\Box A, \Gamma_R \vdash \Delta_R}}{\Sigma, \Box\Gamma_L, \Gamma_R \vdash \Delta_L, \Delta_R}$$

This is transformed as follows:

$$\text{Refl*}\ \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array} \qquad \dfrac{\text{S4}\Box\ \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array}}{\Sigma, \Box\Gamma_L \vdash \Delta_L, \Box A} \qquad \begin{array}{c}\Pi_r\\ A, \Box A, \Gamma_R \vdash \Delta_R\end{array}}{A, \Sigma, \Box\Gamma_L, \Gamma_R \vdash \Delta_L, \Delta_R}\ \text{Cut on } \Box A}{\Sigma, \Gamma_L, \Box\Gamma_L, \Box\Gamma_L, \Gamma_R \vdash \Delta_L, \Delta_R}\ \text{Cut on } A}{\Sigma, \Gamma_L, \Box\Gamma_L, \Gamma_R \vdash \Delta_L, \Delta_R}\ \text{Contraction-admissibility}}{\Sigma, \Box\Gamma_L, \Gamma_R \vdash \Delta_L, \Delta_R}$$

Here Refl* means multiple uses of Refl, once for each member of Γ_L. Importantly, the new Cut on $\Box A$ has lower level, and the Cut on A is of smaller rank. Thus both can be eliminated by the induction hypotheses.

For the latter case, when S4$\Box$ is principal on both sides, the original Cut has form:

$$\text{Cut on } \Box A \;\; \dfrac{\text{S4}\Box \; \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array}}{\Sigma_L, \Box\Gamma_L \vdash \Delta_L, \Box A} \qquad \text{S4}\Box \; \dfrac{\begin{array}{c}\Pi_r\\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B\end{array}}{\Box A, \Sigma_R, \Box\Gamma_R \vdash \Box B, \Delta_R}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box B, \Delta_L, \Delta_R}$$

The normal process of reducing Cut level would apply Cut on the left cut-sequent and the premise of the right cut-sequent, as follows:

$$\text{Cut on } \Box A \;\; \dfrac{\text{S4}\Box \; \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array}}{\Sigma_L, \Box\Gamma_L \vdash \Delta_L, \Box A} \qquad \begin{array}{c}\Pi_r\\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B\end{array}}{A, \Sigma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash B, \Delta_L}$$

Unfortunately, this results in a deduction where we can no longer recover the $\Box B$ present in the conclusion of the original Cut. The nature of the calculus and the S4$\Box$ rule means that new box formulae cannot be introduced in any succedent which contains some context Δ (or where there are additional formula Σ in the antecedent). As stated earlier, this case is omitted in the cut-elimination theorem of Troesltra and Schwichtenberg [25].

To overcome this issue without introducing the complications and new branching rule in the solution of Goubault [14], we modify the original derivation of the left premise to produce one of equal height upon which we can still apply the induction hypothesis on level. The new application of the S4$\Box$ rule differs from the original by simply not adding any context in the conclusion. Formally, the Σ and Δ of the generic S4$\Box$ rule in Fig. 3 are $\emptyset$ in the new S4$\Box$ instance below:

$$\text{S4}\Box \;\; \dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array} \qquad \text{Cut on } \Box A \;\; \dfrac{\text{S4}\Box\text{ (new)} \; \dfrac{\begin{array}{c}\Pi_l\\ \Gamma_L, \Box\Gamma_L \vdash A\end{array}}{\Box\Gamma_L \vdash \Box A} \qquad \begin{array}{c}\Pi_r\\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B\end{array}}{A, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash B}}{\Gamma_L, \Box\Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash B} \;\text{Cut on } A \\ \hdashline \Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash B \end{array} \;\text{Contraction-admissibility}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box B, \Delta_L, \Delta_R}$$

In the formalised proof, this instance is the only case where the inductive principle of Lemma 7.4 is actually required. As the combined height of the derivations leading to $\Box\Gamma_L \vdash \Box A$ and $A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B$ is lower than the level of the original Cut, the induction hypothesis on level can be applied. In all the other cases Theorem 7.2 would have sufficed. So in fact in all the other cases the property we prove is `gen_step2sr` ... and we use Lemma 9.8 to link it to the required property `sumh_step2_tr` ... where the ellipses indicate arguments to each function as appropriate. ■

10 Weakening, Contraction and Cut Admissibility for S4.3

There exists a syntactic pen and paper proof of cut-admissibility for S4.3 in the literature [22], however the calculus involved contains Weakening and Contraction as explicit rules, and mix-elimination is proved rather than cut. There also exist published semantic proofs of closure under Cut for both sequent and hypersequent calculi for S4.3 [12, 15]. To our knowledge, there are no published papers for S4.3 providing a sequent calculus devoid of structural rules and proving cut-elimination per se.

Labelled calculi [3, 18] are perhaps the closest representatives in the literature. As noted previously, while these calculi do not use Weakening or Contraction, they explicitly include the semantics of the logic in the calculi, along with corresponding operations on world accessibility rather than logical operators, thus they are not purely syntactic.

10.1 Calculus for S4.3

The rules of the sequent calculus for S4.3 are listed in Fig. 5. The calculus is based on the version of Goré [12], but with Weakening absorbed into the modal rules. Note, in the S4.3$\Box$ rule of Fig. 5, that $\vec{\Phi} = \{\varphi_1, \ldots, \varphi_n\}$ and $\vec{\Phi}_{-i} = \{\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n\}$ for $1 \leq i \leq n$.

For backward proof search, the S4.3$\Box$ rule can be thought of as producing a new premise for all boxed formula in its conclusion, each of these formula being unboxed separately in its own premise. Thus the general statement of the rule contains an indeterminate number of premises, one is necessary for each $\varphi_i \in \vec{\Phi}$. For the sake of simplicity and clarity, at times only one of these premises will be shown as a representative for all n premises. That is, the rule will be represented in the following form shown below at left:

$$\text{S4.3}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta} \qquad \text{S4.3}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta}\ \forall\psi.\Box\psi \notin \Sigma \cup \Delta$$

There are two different versions of the S4.3$\Box$ rule: either the context $(\Sigma \cup \Delta)$ can contain any formulae, as shown above left, or they cannot include top-level boxed-formulae, as shown above right. In the latter case, the $\Box\Gamma$ and $\Box\vec{\Phi}$ in the conclusion of the S4.3$\Box$ rule must correspond to exactly all the top-level boxed formulae within that sequent. The two versions of the calculus are in fact equivalent, following a proof of the admissibility of Weakening for the latter, however, for efficient backward proof search, the version above right is preferred as it is invertible and hence does not require backtracking during proof search.

Zero Premise Rule (Axiom)

$$\text{id}\ \frac{}{\Gamma, \varphi \vdash \varphi, \Delta}$$

Classical rules

$$\text{L}\wedge\ \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \qquad \text{R}\vee\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

$$\text{L}\!\to\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta} \qquad \text{R}\!\to\ \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}$$

$$\text{L}\neg\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

Modal rules

$$\text{Refl}\ \frac{\Gamma, \varphi, \Box\varphi \vdash \Delta}{\Gamma, \Box\varphi \vdash \Delta}$$

$$\text{S4.3}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi_1, \Box\vec{\Phi}_{-1} \ \cdots\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i} \ \cdots\ \Gamma, \Box\Gamma \vdash \varphi_n, \Box\vec{\Phi}_{-n}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta}\ \dagger$$

Fig. 5 Sequent calculus for S4.3 where † is $\forall\psi.\Box\psi \notin \Sigma \cup \Delta$

Henceforth, Σ and Δ within the S4.3$\Box$ rule will be restricted from containing the $\Box$ operator at the top-level. In Isabelle, this is implemented by creating a new type of formula, based on the default formula type. HOL's `typedef` allows a concise method of declaring new types as a subset of an existing type, where `~=` stands for inequality:

```
typedef nboxfml =
    "{f::formula. ALL (a::formula). f ~= FC ''Box'' [a]}"
```

The Isabelle formalisation of the overall calculus is based on the calculus for S4 given in Fig. 3. The only change is in the S4.3$\Box$ rule, which requires the mapping function `nboxseq` to create a new premise for each individual boxed formula in the succedent. The code for this is given in Fig. 6.

```
(* Functions to unbox one formula for each premise *)
consts
  ithprem :: "formula multiset => formula list => formula
               => formula sequent"
  nprems  :: "formula multiset => formula list
               => formula sequent list"

(* The boxes in the succedent are treated as a list As.
   "ms_of_list (remove1 Ai As)" is the multiset consisting of
   all elements in "As", with one copy of "Ai" removed. *)
defs
  ithprem_def :
    "ithprem Gamma As Ai ==
        mset_map Box Gamma + Gamma |-
        {#Ai#} + mset_map Box (ms_of_list (remove1 Ai As))"
  nprems_def :
    "nprems Gamma As == map (ithprem Gamma As) As"

consts (* type definitions for functions *)
  gs43_rls :: "formula sequent psc set"
    s43box :: "formula sequent psc set"

(* The S4.3 box rule *)
inductive "s43box"
 intrs
    I "(nprems gamma As, mset_map Box gamma |-
         mset_map Box (ms_of_list As))            : s43box"

(* The S4.3 calculus as an extension of the LK calculus *)
inductive "gs43_rls"
 intrs
      lksI "psc : lksss ==> psc : gs43_rls"
     reflI "psc : lkrefl ==>
                          pscmap (extend flr) psc : gs43_rls"
    (* boxI allows extra formulae in conclusion only,
       and enforces the ‘dagger’ condition of Figure 5 *)
    boxI  : "(p, c) : lkbox ==>
                       (p, extend (nboxseq flr) c) : gs43_rls"
```

Fig. 6 S4.3 calculus as encoded in Isabelle

10.2 Weakening for S4.3

As the S4.3□ rule does not allow arbitrary contexts, weakening-admissibility must be proved by induction, in this case on both height and rank (of the implicit derivation tree, i.e., using Lemma 7.1). To simplify the case for the S4.3□ rule and its multiple premises, we prove weakening-admissibility for the antecedent and succedent separately, and only considering a single formula at a time. The Isabelle encodings for these properties are given below. The induction itself proceeds on the height of the derivation, with a sub-induction on the rank of the formula A being inserted into the conclusion.

Definition 10.1

`wk_adm_single_antec rls` means:
  For any `rls`-derivable sequent S, and any single formulae A,
  if $S \in$ `derrec rls {}` then $S+$`({#A#} |- {#})` $\in$ `derrec rls {}`.
`wk_adm_single_succ rls` means:
  For any `rls`-derivable sequent S, and any single formulae A,
  if $S \in$ `derrec rls {}` then $S+$`({#} |- {#A#})` $\in$ `derrec rls {}`.

Lemma 10.1 (`wk_adm_sides`) *For a set of rules* `rls`*, if* `wk_adm_single_antec` *and* `wk_adm_single_succ` *both hold then so does* `wk_adm`.

Proof By multiset induction, repeatedly applying the results for single formulae. ■

Theorem 10.1 (`gs43_wk_adm`) *Weakening is admissible for the calculus consisting of the set of rule* `gs43_rls`.

Proof In the case of the S4.3□ rule, if A is not boxed, then it is allowed to be contained in the context of the rule's conclusion. The derivability of the original premises, followed by an application of a new S4.3□ rule including A as part of its context, then gives the required sequent. The difficulty arises when A is a boxed formula, say $A = \Box B$. For the sake of clarity, the representation of the original sequent can be split into its boxed and non-boxed components, so the original derivation is:

$$\text{S4.3}\Box \; \frac{\begin{array}{c}\Pi \\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}\end{array}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta}$$

When A is to be added to the antecedent, the induction on height can be used to add $A = \Box B$ to each of the original premises. Following this by an application of the sub-induction on formula rank, allows the addition of B, giving the derivability of $B, \Box B, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}$. An application of the S4.3□ rule then completes the case:

$$\text{S4.3}\Box \; \frac{B, \Box B, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}}{\Box B, \Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta}$$

The final case to consider is that of adding $A = \Box B$ to the succedent. The goal once again is to use the S4.3□ rule to give the desired conclusion. From the original premises $\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}$, the induction hypothesis on height (inserting $\Box B$) gives the derivability of $\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box B$. A different application of the S4.3□ rule, bringing in empty contexts, on the original premises also gives the derivability of $\Box\Gamma \vdash \Box\vec{\Phi}$. Applying the induction on formula rank then shows that $\Box\Gamma \vdash B, \Box\vec{\Phi}$ is derivable.

At this point, the derivability of all necessary premises for a new S4.3□ rule instance has been proven. These are sequents of the form $\Gamma, \Box\Gamma \vdash \varphi'_i, \Box\vec{\Phi}'_{-i}$ where $\vec{\Phi}' = \vec{\Phi}, B$ and φ' is from the multiset $\vec{\Phi} \cup \{B\}$ as appropriate. The final rule application is then:

$$\text{S4.3}\Box\ \frac{\Gamma, \Box\Gamma \vdash \varphi'_i, \Box\vec{\Phi}'_{-i}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Box B, \Delta} \qquad \blacksquare$$

10.3 Invertibility and Contraction for S4.3

As for S4, we prove inversion lemmata for the G3cp and Refl rules within the overall calculus.

Theorem 10.2 (*`inv_rl_gs43_refl`* and *`inv_rl_gs43_lks`*) *Refl (`lkrefl`) and all Classical rules (`lksss`) are invertible within the calculus `gs43_rls`.*

Proof Since the inverse of the Refl rule is an instance of weakening, which we have shown is admissible, the only notable case occurs for the G3cp rules, where the last rule applied in the original derivation is S4.3□. The proof uses the induction principle of Lemma 9.5.

If the original derivation is as shown below left then proving invertibility for G3cp requires showing the derivability of all premises after applying a G3cp rule backwards from the endsequent of the S4.3□ rule. The classical rules do not operate on boxed formulae, so this rule can only modify Σ or Δ upwards into Σ' and Δ' respectively as shown below right:

$$\text{S4.3}\Box\ \frac{\overset{\Pi}{\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta} \qquad \text{G3cp rule}\ \frac{\Sigma', \Box\Gamma \vdash \Box\vec{\Phi}, \Delta'}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta}$$

Clarifying again, invertibility of the G3cp rule requires deriving $\Sigma', \Box\Gamma \vdash \Box\vec{\Phi}, \Delta'$. The usual tactic would apply another instance of the S4.3□ rule to the original premises, but bringing in a different context. However, this does not admit a proof if there are boxed formula in Σ' or Δ'. For example, if the G3cp rule is L∧ and the principal formula is $A \wedge \Box B$ then Σ' contains a boxed formula, $\Box B$, which cannot be introduced within the (box-free) context of a new S4.3□ rule application.

To accommodate this case, the premises of the modal rule are used to derive the conclusion without any context. Then weakening-admissibility is used to bring the remaining formulae in the premise of the G3cp rule:

$$\text{Weakening-admissibility}\ \dfrac{\text{S4.3}\Box\ \dfrac{\begin{array}{c}\Pi\\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}\end{array}}{\Box\Gamma \vdash \Box\vec{\Phi}}}{\Sigma', \Box\Gamma \vdash \Box\vec{\Phi}, \Delta'}$$

■

For S4, proving invertibility is sufficient to lead to a contraction admissibility proof. However, using invertibility alone does not allow an obvious transformation when dealing with the S4.3□ rule. In order to prove contraction-admissibility, we first require the following lemma:

Lemma 10.2 (`gs43_refl`) *The rule R-refl is admissible in* `gs43_rls`.

$$\textit{R-refl}\ \dfrac{\Gamma \vdash \Delta, \Box A}{\Gamma \vdash \Delta, A}$$

The corresponding statement of the lemma in Isabelle (not shown) states that if a sequent `seq` *is derivable in* `gs43_rls` *and the sequent is equivalent to* $X \vdash Y, \Box A$ *for any* X *and* Y*, then* $X \vdash Y, A$ *is also derivable.*

Proof By an induction on the structure of the (implicit) derivation tree, using the `derrec`-induction principle, Lemma 4.1. The analysis is on the last rule applied in deriving $\Gamma \vdash \Delta, \Box A$.

Case 1 The last rule applied was id. If $\Box A$ is parametric then $\Gamma \vdash \Delta$ is an axiom, and the conclusion will be also. If $\Box A$ is principal, then $\Gamma = \{\Box A\} \cup \Gamma'$ and the following transformation is applied:

$$\text{Refl}\ \dfrac{\text{id}\ \dfrac{}{A, \Box A, \Gamma' \vdash \Delta, A}}{\Box A, \Gamma' \vdash \Delta, A}$$

Case 2 The last rule applied was from G3cp. No rules in G3cp operate on a boxed formula, so $\Box A$ must be parametric. The induction hypothesis on height is thus applicable to the premise of the G3cp rule. Applying the original G3cp on the resulting sequent gives the desired conclusion.

Case 3 The last rule applied was Refl. As in Case 2, $\Box A$ must be parametric, as Refl only operates on boxed formula in the antecedent.

Case 4 The last rule applied was S4.3□. Then one premise of the original deduction un-boxes $\Box A$. Using Refl for each member of Γ' (denoted by Refl*) followed by weakening admissibility on this premise is enough to produce the conclusion. For clarity, here we express $\Gamma = \Sigma \cup \Box\Gamma'$ and $\Delta = \Box\vec{\Phi} \cup \Delta'$. The original derivation is:

$$\dfrac{\begin{matrix}\Pi_1 \\ \Gamma', \Box\Gamma' \vdash \Box\vec{\Phi}, A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \Gamma', \Box\Gamma' \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A\end{matrix}}{\Sigma, \Box\Gamma' \vdash \Box\vec{\Phi}, \Delta', \Box A}$$

This is transformed into:

$$\begin{array}{rc} & \Pi_1 \\ & \Gamma', \Box\Gamma' \vdash \Box\vec{\Phi}, A \\ \cline{2-2} \text{Refl}^* & \Box\Gamma' \vdash \Box\vec{\Phi}, A \\ \hdashline \text{Weakening-admissibility} & \Sigma, \Box\Gamma' \vdash \Box\vec{\Phi}, \Delta', A \end{array}$$

$\blacksquare$

Theorem 10.3 (`gs43_ctr_adm`) *Contraction is admissible in* `gs43_rls`.

Proof We use the induction principle Lemma 7.1, for implicit derivation trees. If the last rule used in the derivation was the S4.3□ rule, there are two cases to consider. The case where the contraction-formula is parametric is handled by simply re-applying another instance of the S4.3□ rule as in the S4 case. Similarly, when the contraction-formula is principal in the antecedent, then the proof proceeds as for S4. Specifically, one copy of $\Box A$ from $\Sigma, \Box A, \Box A, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta$ must be removed. The original derivation is:

$$\begin{array}{rc} & \Pi \\ & A, A, \Box A, \Box A, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i} \\ \cline{2-2} \text{S4.3}\Box & \Sigma, \Box A, \Box A, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta \end{array}$$

By contracting twice using first the induction hypothesis on height, then the induction hypothesis on rank, on all premises followed by an application of the S4.3□ rule, the desired endsequent is obtained:

$$\begin{array}{rc} & \Pi \\ & A, A, \Box A, \Box A, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i} \\ \hdashline \text{IH on height} & A, A, \Box A, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i} \\ \hdashline \text{IH on rank} & A, \Box A, \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i} \\ \cline{2-2} \text{S4.3}\Box & \Sigma, \Box A, \Box\Gamma \vdash \Box\vec{\Phi}, \Delta \end{array}$$

When the contraction-formula is principal in the succedent, there are two possible premises to consider. Either a premise "un-boxes" one of the contraction-formulae,[1] or it leaves both boxed. The original deduction is:

---

[1] Technically, there are two syntactically identical premises which individually un-box one of the two copies of $\Box A$.

$$\frac{\begin{array}{c}\Pi_1\\ \Gamma, \Box\Gamma \vdash \Box\vec{\Phi}, A, \Box A\end{array} \qquad \begin{array}{c}\Pi_2\\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A, \Box A\end{array}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Box A, \Box A, \Delta}$$

In the latter case, the induction hypothesis can be directly applied, removing one copy of the boxed formulae:

$$\text{IH on height}\ \frac{\begin{array}{c}\Pi_2\\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A, \Box A\end{array}}{\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A}$$

In the former case, we use Lemma 10.2 to produce the following:

$$\text{S4.3}\Box\ \frac{\text{IH on rank}\ \dfrac{\text{R-refl}\ \dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Box\Gamma \vdash \Box\vec{\Phi}, A, \Box A\end{array}}{\Gamma, \Box\Gamma \vdash \Box\vec{\Phi}, A, A}}{\Gamma, \Box\Gamma \vdash \Box\vec{\Phi}, A} \qquad \text{IH on height}\ \dfrac{\begin{array}{c}\Pi_2\\ \Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A, \Box A\end{array}}{\Gamma, \Box\Gamma \vdash \varphi_i, \Box\vec{\Phi}_{-i}, \Box A}}{\Sigma, \Box\Gamma \vdash \Box\vec{\Phi}, \Box A, \Delta}$$

■

10.4 Cut-Admissibility for S4.3

Theorem 10.4 (`gs43_cas`) *Cut is admissible in the calculus* `gs43_rls`.

Proof As with Theorem 9.3, we use the induction principle of Lemma 7.4, involving induction on the sums of heights of two explicit trees, although for the majority of cases the simpler principle Theorem 7.2 would suffice. So again, in those cases, we prove `gen_step2sr` ... and we use Lemma 9.8 to link it to the required property `sumh_step2_tr` ..., where the ellipses indicate arguments to each function as appropriate.

When S4.3□ leads to the left cut-sequent, and the Refl rule is used on the right, the transformation mimics the corresponding case for S4. However, for the case where S4.3□ is principal on both sides we require a new transformation. For clarity, the premises above the S4.3□ rule on the left are given as two cases, depending on whether the cut-formula is un-boxed or not. The boxed formula in the succedents of the premises are also distinguished by the superscripts L and R for left and right cut premises respectively. Explicitly, these are $\vec{\Phi}^L = \{\varphi_1^L, \ldots, \varphi_i^L, \ldots, \varphi_n^L\}$ and $\vec{\Phi}^R = \{\psi_1^R, \ldots, \psi_k^R, \ldots, \psi_m^R\}$. The original cut thus has the form:

$$
\text{Cut on } \Box A \;\dfrac{\begin{array}{c} \text{S4.3}\Box\;\dfrac{\begin{array}{c}\Pi_L^a\\ \Gamma_L, \Box\Gamma_L \vdash \varphi_i^L, \Box\vec{\Phi}^L_{-i}, \Box A\end{array} \qquad \begin{array}{c}\Pi_L^b\\ \Gamma_L, \Box\Gamma_L \vdash A, \Box\vec{\Phi}^L\end{array}}{\underline{\Sigma_L, \Box\Gamma_L \vdash \Box\vec{\Phi}^L, \Delta_L, \Box A}} \\ \vdots \\ \vdots \end{array} \qquad \begin{array}{c}\Pi_R\\ \text{S4.3}\Box\;\dfrac{A, \Box A, \Gamma_R, \Box\Gamma_R \vdash \psi_k^R, \Box\vec{\Phi}^R_{-k}}{\Box A, \Sigma_R, \Box\Gamma_R \vdash \Box\vec{\Phi}^R, \Delta_R}\end{array}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \Box\vec{\Phi}^R, \Delta_L, \Delta_R}
$$

To remove this cut, the derivation is transformed into one where the principal rule (S4.3□) is applied last to produce the desired endsequent. The problem is then proving that the premises of the following S4.3□ rule application are derivable. This in itself requires two different transformations of the original derivation, depending on the two forms that the premises can take; either the un-boxed formula in the succedent originated from the left cut premise, that is from $\Box\vec{\Phi}^L$, or from the right, within $\Box\vec{\Phi}^R$. These cases are named $\mathcal{P}_L$ and $\mathcal{P}_R$ respectively. The final S4.3□ rule used in our new transformation is then:

$$
\text{S4.3}\Box\;\dfrac{\begin{array}{c}\mathcal{P}_L\\ \underline{\Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \varphi_i^L, \Box\vec{\Phi}^L_{-i}, \Box\vec{\Phi}^R} \\ \vdots \end{array} \qquad \begin{array}{c}\mathcal{P}_R\\ \Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \psi_k^R, \Box\vec{\Phi}^R_{-k}\end{array}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \Box\vec{\Phi}^R, \Delta_L, \Delta_R}
$$

For both transformations, the same idea behind the principal S4□ rule case is used. We first derive the original cut-sequents but without their original contexts. These new sequents will be called $\mathcal{D}_L$ and $\mathcal{D}_R$ respectively, that is, $\mathcal{D}_L = \Box\Gamma_L \vdash \Box\vec{\Phi}^L, \Box A$ and $\mathcal{D}_R = \Box A, \Box\Gamma_R \vdash \Box\vec{\Phi}^R$. These are derived using the derivations in the original cut, but applying new instances of the S4.3□ rule. Importantly, the derivations of the new sequents $\mathcal{D}_L$ and $\mathcal{D}_R$ have the same height as the original cut-sequents. This is the case where the induction principle of Lemma 7.4 is required.

$$
\text{S4.3}\Box\;\dfrac{\begin{array}{c}\Pi_L^a\\ \Gamma_L, \Box\Gamma_L \vdash \varphi_i^L, \Box\vec{\Phi}^L_{-i}, \Box A\end{array} \qquad \begin{array}{c}\Pi_L^b\\ \Gamma_L, \Box\Gamma_L \vdash A, \Box\vec{\Phi}^L\end{array}}{\mathcal{D}_L = \Box\Gamma_L \vdash \Box\vec{\Phi}^L, \Box A}
$$

$$
\text{S4.3}\Box\;\dfrac{\begin{array}{c}\Pi_R\\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash \psi_k^R, \Box\vec{\Phi}^R_{-k}\end{array}}{\mathcal{D}_R = \Box A, \Box\Gamma_R \vdash \Box\vec{\Phi}^R}
$$

Having introduced all the necessary notation and pre-requisites, the first actual case to consider is deriving $\mathcal{P}_L$. The induction on level allows $\mathcal{D}_R$ to be cut, on cut-formula $\Box A$, with all of the sequents given by the derivation Π_L^a above the original left S4.3□ rule. The transformation performs n cuts, for all premises corresponding to the formulae in $\vec{\Phi}^L$. The results of this cut then match exactly with $\mathcal{P}_L$ after using the admissibility of Weakening to introduce the formulae of Γ_R in the antecedent.

$$\dfrac{\dfrac{\begin{matrix}\Pi^a_L\\ \Gamma_L, \Box\Gamma_L \vdash \varphi^L_i, \Box\vec{\Phi}^L_{-i}, \Box A\end{matrix} \qquad \begin{matrix}\mathcal{D}_R\\ \Box A, \Box\Gamma_R \vdash \Box\vec{\Phi}^R\end{matrix}}{\Gamma_L, \Box\Gamma_L, \Box\Gamma_R \vdash \varphi^L_i, \Box\vec{\Phi}^L_{-i}, \Box\vec{\Phi}^R}\;\text{Cut on }\Box A}{\mathcal{P}_L = \Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \varphi^L_i, \Box\vec{\Phi}^L_{-i}, \Box\vec{\Phi}^R}\;\text{Weakening-admissibility}$$

To derive the sequents in $\mathcal{P}_R$, the induction hypothesis on level is used to cut $\mathcal{D}_L$ with all of the premises above the right S4.3$\Box$ in the original cut, with cut-formula $\Box A$. The induction on formula rank on A is then used to cut the sequent resulting from Π^b_L with all these new sequents. Finally, contraction-admissibility allows the removal of the extra copies of $\Box\Gamma$ and $\Box\vec{\Phi}^L$, and concludes the case.

$$\dfrac{\dfrac{\begin{matrix}\Pi^b_L\\ \Gamma_L, \Box\Gamma_L \vdash A, \Box\vec{\Phi}^L\\ \hline \vdots\\ \vdots\\ \vdots\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_L\\ \Box\Gamma_L \vdash \Box\vec{\Phi}^L, \Box A\end{matrix} \qquad \begin{matrix}\Pi_R\\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash \psi^R_k, \Box\vec{\Phi}^R_{-k}\end{matrix}}{\Box\Gamma_L, A, \Gamma_R, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \psi^R_k, \Box\vec{\Phi}^R_{-k}}\;\text{Cut on }\Box A}{\Gamma_L, \Box\Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \Box\vec{\Phi}^L, \psi^R_k, \Box\vec{\Phi}^R_{-k}}\;\text{Cut on }A}{\mathcal{P}_R = \Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \Box\vec{\Phi}^L, \psi^R_k, \Box\vec{\Phi}^R_{-k}}\;\text{Contraction-admissibility}$$

To conclude, the transformations above derive $\mathcal{P}_L$ and $\mathcal{P}_R$ while reducing cut-level or cut-rank. These are the premises of an instance of the S4.3$\Box$ rule which results in the conclusion of the original cut. This completes the cut-admissibility proof. ■

11 Weakening, Contraction and Cut Admissibility for GTD

We now describe Isabelle proofs of cut admissibility for a sequent calculus for the logic GTD described in [16]. Axiomatically, GTD is K with the additional axiom $\Box A \Leftrightarrow \Box\Box A$. The sequent inference rules involving $\Box$, allowing arbitrary context in the conclusion so as to make weakening admissible, are shown below:

$$\frac{\Box\Gamma, \Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Delta}(\vdash \Box) \qquad\qquad \frac{\Box\Gamma, \Gamma \vdash \Box A}{\Sigma, \Box\Gamma \vdash \Box A, \Delta}(\Box \vdash)$$

The skeletons of the above two rules are encoded as `GTD` shown below by factoring out the form of A as either B or as $\Box B$:

```
inductive "GTD"
 intrs
    I  "A = B | A = Box B ==>
                ([mset_map Box X + X |- {#A#}],
                  mset_map Box X     |- {#Box B#}) : GTD"
```

11.1 Calculus for GTD

We now look at proving cut admissibility for a version of GTD without structural rules, where the box rules have their conclusions (only) extended with an arbitrary context, which permits weakening to be admissible.

We define the rules of the sequent calculus as follows. The rules used for classical logic (before extending them with a context) form the set `lksne` where the rule sets `idrls`, `lksil` and `lksir` are the axioms and the left and right logical introduction rules: see Fig. 3.

Definition 11.1 (*lkssx*) Given, a rule set `xrls`, every rule of `xrls` is in the rule set `lkssx xrls`, and every rule `psc` in rule set `lknse` gives a rule in `lkssx xrls` obtained by uniformly extending both the premise and conclusion of `psc` with an arbitrary context (sequent) `flr`:

```
inductive "lkssx xrls"
  intrs
      x   "psc : xrls ==> psc : lkssx xrls"
    extI  "psc : lksne ==> pscmap (extend flr) psc : lkssx xrls"
```

Definition 11.2 (*extcs*) Given a rule set `rules`, the rule set `extcs rules` is obtained by extending only the conclusion `c` of each rule `(ps, c)` in `rules` by an arbitrary context (sequent) `flr` (while leaving the premises unchanged):

```
inductive "extcs rules"
  intrs
     I   "(ps, c) : rules ==> (ps, extend flr c) : extcs rules"
```

The rule set `lkssx (extcs GTD)` for GTD is obtained by extending only the conclusion of the rule `GTD` and by extending every rule of `lknse`.

11.2 Weakening-Admissibility for GTD

First we prove weakening admissibility, using a lemma which allows us to apply Lemma 9.1.

Lemma 11.1 *For any rule sets* `rls` *and* `rlsa`

(a) `extrs rlsa` ∪ `extcs rls` *satisfies* `ext_concl`
(b) `extrs rlsa` ∪ `extcs rls` *satisfies weakening admissibility*

```
extrs_cs_ext_concl: "ext_concl (extrs ?rlsa Un extcs ?rls)"
wk_adm_extrs_cs: "wk_adm (extrs ?rlsa Un extcs ?rls)"
```

Proof The first is easy. The second follows using Lemma 9.1. ■

Corollary 11.1 *GTD satisfies weakening admissibility.*

```
wk_adm_lkssx_cs: "wk_adm (lkssx (extcs ?xrls))"
```

Proof Since the rule set `lkssx (extcs GTD)` for GTD is also equal to `extrs lksne` ∪ `extcs GTD`, the result follows from Lemma 11.1. ■

11.3 Inversion and Contraction-Admissibility for GTD

For contraction admissibility, first we need to prove invertibility of the classical logical rules. The general method for doing so was described in Sect. 9.3.

Recall the predicate `inv_stepm`, which is used in an inductive proof of invertibility. Its three arguments are:

`drls` first, the set of *derivation rules* with respect to which the invertibility (a case of admissibility) is defined,
`irls` second, the set of rules whose invertibility is being considered (the *inversion rules*)
`(ps, c)` third, the *final rule* of a derivation—since we are talking about proving the invertibility result by induction on the derivation, the inductive hypothesis is that the invertibility result applies to the premises `ps` of this final rule.

By Lemma 9.3, `inv_stepm` (although not `inv_step`) is monotonic in the derivation rules argument. For its second argument the following holds.

Lemma 11.2 *For a given set* `drls` *of derivation rules and a given final rule* `psc`*, if* `inv_stepm` *applies for inversion rule sets* `irlsa` *and* `irlsb`*, then it applies for* `irlsa` ∪ `irlsb`*.*

```
inv_stepm_Un:
"[| inv_stepm ?drls ?irlsa ?psc ;
    inv_stepm ?drls ?irlsb ?psc |]
   ==> inv_stepm ?drls (?irlsa Un ?irlsb) ?psc"
```

So far as the third argument is concerned, the requirement to prove a rule is invertible is simply that `inv_stepm` ... applies for all cases of the third argument (see Lemmas 9.4 and 9.5): thus the lemmata we use are expressed to apply to single cases of the third argument.

We now describe the lemmata used as building-blocks for the required invertibility result.

Lemma 11.3 *(a)* `inv_stepm` *... applies where the derivation rules and the set of rules to be inverted are the classical logical rules* `extrs lksne`*, and the final rule is any one of those rules*

```
lks_inv_stepm:
   "?psc : extrs lksne ==>
     inv_stepm (extrs lksne) (extrs lksne) ?psc"
```

(b) where the set of inversion rules is the set of extensions of a single skeleton whose conclusion is `ic`*, and the set of derivation rules is the set of extensions of a single skeleton rule whose conclusion is* `c`*, and these skeleton conclusions* `ic` *and* `c` *are disjoint (i.e., have no formula in common on the same side of the turnstile), and the final rule is one of those derivation rules, then* `inv_stepm`*...applies*

```
inv_stepm_disj:
  "[| seq_meet ?c ?ic = 0 ;
      extrs {(?ps, ?c)} = ?rls ; ?rl : ?rls |]
     ==> inv_stepm ?rls (extrs {(?ips, ?ic)}) ?rl"
```

(c) as for (b), except that the set of derivation rules is the set of extensions in the conclusion only (using `extcs`*) of the single skeleton*

```
inv_stepm_disj_cs:
"[| seq_meet ?c ?ic = 0 ;
    extcs {(?ps, ?c)} = ?rls ; ?rl : ?rls |]
   ==> inv_stepm ?rls (extrs {(?ips, ?ic)}) ?rl"
```

(d) where the set of inversion rules and the set of derivation rules are each the set of extensions of a single skeleton rule whose conclusion has a single formula, and if those two skeletons' conclusions are equal then the two skeletons are equal, then `inv_stepm`*... applies*

```
inv_stepm_scrls:
"[| extrs {?srl} = ?rls ; ?rl : ?rls ;
    ?srl : scrls ; ?irl : scrls ;
    snd ?srl = snd ?irl --> ?srl = ?irl |]
  ==> inv_stepm ?rls (extrs {?irl}) ?rl"
```

Parts (b) and (c) (`inv_stepm_disj` and `inv_stepm_disj_cs`) are for the case where the principal formula of the rule to be inverted is in the context of the conclusion of the last rule of the derivation: the first premise gives us that the formula to be inverted is not the principal formula of the rule, though it is expressed in a way which is relevant to a case where the rules in question have more than just one principal formula.

Part (d) (`inv_stepm_scrls`), whose proof uses part (b), uses the fact that for each formula involved there are unique introduction rules for the left and right sides of $\vdash$, so an inversion step is either parametric or gives us the premise(s) of the last rule applied.

Lemma 11.4 *Every rule of* `lksss` *is invertible in the calculus for GTD.*

```
gtdns_inv_rl: "Ball (extrs lksne) (inv_rl (lkssx (extcs GTD)))"
```

Proof This uses Lemmas 9.4, 9.5 and 11.3. ■

Then, to prove contraction admissibility, we follow an approach very similar to Sect. 9.3. For the rules ($\Box \vdash$) and ($\vdash \Box$), the proof for the cases where either of these is the final rule is just the same as for the S4$\Box$ rule in Sect. 9.3.

Lemma 11.5 *Contraction is admissible in GTD.*

```
gtdns_ctr_adm: "ctr_adm (lkssx (extcs GTD)) ?A"
```

11.4 Cut-Admissibility for GTD

Now, for cut admissibility, the difficult cases are where the last rule on both sides is one of the two box rules ($\vdash \Box$) and ($\Box \vdash$):

$$\frac{\Box\Gamma, \Gamma \vdash B}{\Sigma, \Box\Gamma \vdash \Box B, \Delta}(\vdash \Box) \qquad\qquad \frac{\Box\Gamma, \Gamma \vdash \Box B}{\Sigma, \Box\Gamma \vdash \Box B, \Delta}(\Box \vdash)$$

Since the proof is effectively the same whichever of these two rules is on the right, we define a unary function `s4g` . such that

- `s4g` $(\lambda B.\ \{B, \Box B\})$ is all instances of either $(\vdash \Box)$ or $(\Box \vdash)$,
- `s4g` $(\lambda B.\ \{B\})$ is all instances of $(\vdash \Box)$, and
- `s4g` $(\lambda B.\ \{\Box B\})$ is all instances of $(\Box \vdash)$

where the function `prs B` encapsulates the choices of B and/or $\Box B$, as required and where `s4g prs` below encodes only the skeletons of the rules above: see the definition of GTD at the start of Sect. 11. Formally,

Definition 11.3 (*s4g*) `s4g prs` is the set of instances of the following rule where $A \in \mathtt{prs} B$:

$$\frac{\Box\Gamma, \Gamma \vdash A}{\Box\Gamma \vdash \Box B}$$

```
inductive "s4g prs"
  intrs I "A : prs B ==>
                     ([mset_map Box X + X |- {#A#}],
                       mset_map Box X |- {#Box B#}) : s4g prs"
```

The case of the $(\vdash \Box)$ rule on the left is dealt with in Sect. 9.4: depending on whether we have the rule $(\vdash \Box)$ or $(\Box \vdash)$ on the right, we may need to change B to $\Box B$ in the diagrams there.

For the case where we have the $(\Box \vdash)$ rule on the left, the original derivation is as in the following diagram, where B' is B or $\Box B$.

$$\text{Cut on } \Box A\ \dfrac{\Box\vdash \dfrac{\begin{matrix}\Pi_l \\ \Gamma_L, \Box\Gamma_L \vdash \Box A\end{matrix}}{\Sigma_L, \Box\Gamma_L \vdash \Delta_L, \Box A} \qquad \Box\vdash \text{ or } \vdash\Box\ \dfrac{\begin{matrix}\Pi_r \\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B'\end{matrix}}{\Box A, \Sigma_R, \Box\Gamma_R \vdash \Box B, \Delta_R}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box B, \Delta_L, \Delta_R}$$

As in Sect. 9.4, we modify the original derivation of a premise, in this case the right premise, by simply not adding any context in the conclusion. This produces a derivation of equal height upon which we can still apply the induction hypothesis on level. Formally, the Σ and Δ of the generic box rule $(\vdash \Box)$ or $(\Box \vdash)$ are $\emptyset$ in the new instance below:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_l \\ \Gamma_L, \Box\Gamma_L \vdash \Box A\end{matrix} \qquad \dfrac{\begin{matrix}\Pi_r \\ A, \Box A, \Gamma_R, \Box\Gamma_R \vdash B'\end{matrix}}{\Box A, \Box\Gamma_R \vdash \Box B}\ \Box\vdash \text{ or } \vdash\Box \text{ (new)}}{\Gamma_L, \Box\Gamma_L, \Box\Gamma_R \vdash \Box B}\ \text{Cut on } \Box A}{\Gamma_L, \Box\Gamma_L, \Gamma_R, \Box\Gamma_R \vdash \Box B}\ \text{Weakening-admissibility}}{\Sigma_L, \Sigma_R, \Box\Gamma_L, \Box\Gamma_R \vdash \Box B, \Delta_L, \Delta_R}\ \Box\vdash$$

For the cut-elimination proof we also use results for the parametric cases, that is, where the cut-formula appears in the context of the last rule on either side above the cut. This includes cases where that rule is in `extrs` ... (where the rule has a context which appears in premises and conclusion) and where that rule is in `extcs` … (where the rule has a context which appears only in the conclusion).

The following lemma is used for the common situation of a cut which is parametric with respect to the last rule of the left-hand derivation.

Lemma 11.6 (`lcg_gen_step`) *Consider a set* `erls` *of derivation rules, for which weakening is admissible, and which contains all extensions of a skeleton rule ρ with premises* `ps` *and conclusion $U \vdash V$. Consider two derivations of which the final rule of the left side is an extension of ρ. Then for a cut-formula A which is not contained in V, and any subformula relation* `sub`*, the inductive step condition* `gen_step2_sr` *… holds for the admissibility of a cut on A.*

```
lcg_gen_step:
  "[| wk_adm ?erls ;
      extrs {(?ps, ?U |- ?V)} <= ?erls ;
      ~ ?A :# ?V ;
      ?pscl = pscmap (extend (?W |- ?Z)) (?ps, ?U |- ?V) |]
    ==>
    gen_step2sr (prop2 car ?erls) ?A ?sub ?erls (?pscl, ?pscr)"
```

A similar lemma `lcg_gen_steps_extcs` holds for the case where only extensions in the conclusion of ρ are contained in `erls`.

```
lcg_gen_steps_extcs:
   "[| wk_adm ?rls ;
       extcs {(?ps, ?c)} <= ?rls ; ~ ?A :# succ ?c |]
     ==> gen_step2sr (prop2 car ?rls) ?A ?sub ?rls
                     ((?ps, extend ?flr ?c), ?psr, ?cr)"
```

Finally we need to deal with the cases of matching instances of the usual logical introduction rules. Here we use a general result giving requirements for certain cases of the final rules on either side of a putative cut to satisfy the step condition for cut-admissibility.

It uses a property `c8_ercas_prop`, which encodes the property that a cut which is principal (i.e., the cut formula is introduced by a logical introduction rule in the final step) on both sides is reducible to cuts on sub-formulae. It is loosely defined as follows:

Definition 11.4 (*`c8_ercas_prop`*) Given a set of derivation rules `prls`, a cut-formula A, a subformula relation `psubfml`, and a set of skeleton rules (typically logical introduction rules) `rls`,
`c8_ercas_prop psubfml prls A rls` means:
assuming that we have cut-admissibility for cut-formulae which are smaller than A according to `psubfml`, where two derivations have as their final sequents $X_l \vdash A, Y_l$ and $X_r, A \vdash Y_r$, and on both sides the final rule introduces A using logical introduction rules in `rls`, then $X_l, X_r \vdash Y_l, Y_r$ is derivable, that is, the cut on A is admissible.

Of course, whether `c8_ercas_prop` holds depends on the specific set of logical rules. Beyond that, however, the following lemma is quite general.

Lemma 11.7 *Given a set of derivation rules* `drls`*, a cut-formula A, and a subformula relation* `psubfml`*, if*

- `drls` *satisfy weakening admissibility*
- *there is a set* `rls` *of skeleton rules all of whose extensions are contained in* `drls`
- *all rules in* `rls`*, other than axiom rules* $B \vdash B$*, have a single formula in their conclusion*
- *the axiom rules are also in* `drls`
- `drls` *and* `rls` *satisfy* `c8_ercas_prop`
- *the final rules of two derivations are extensions of rules in* `rls`

then the step condition `gen_step2sr` *for cut-admissibility for the two derivations is satisfied.*

```
gs2sr_alle:
  "[| wk_adm ?drls ;
      c8_ercas_prop ?psubfml ?drls ?A ?rls ;
      ?rls <= iscrls ;
      idrls <= ?drls ;
      extrs ?rls <= ?drls ;
      (?psa, ?ca) : extrs ?rls ;
      (?psb, ?cb) : extrs ?rls |]
    ==> gen_step2sr (prop2 car ?drls) ?A ?psubfml ?drls
                    ((?psa, ?ca), ?psb, ?cb)"
```

We apply this result to the logic GTD using first another general result.

Lemma 11.8 (*`gen_lksne_c8`*) *If a set of derivation rules* `drls` *satisfies weakening admissibility and contraction admissibility, and contains the extensions of the logical introduction rule skeletons* `lksne` *then the condition* `c8_ercas_prop` *is satisfied (for the usual immediate proper subformula relation and for any cut-formula).*

```
gen_lksne_c8:
   "[| ALL A'. ctr_adm ?drls A' ;
       wk_adm ?drls ; extrs lksne <= ?drls |]
     ==> c8_ercas_prop ipsubfml ?drls ?A lksne"
```

Corollary 11.2 [`gtdns_lksne_c8`] *GTD satisfies* `c8_ercas_prop` *in relation to the logical introduction rule skeletons* `lksne`*.*

```
gtdns_lksne_c8:
     "c8_ercas_prop ipsubfml (lkssx (extcs GTD)) ?A lksne"
```

Finally we get the cut admissibility result. Here, `mins` A M means multiset M with one additional copy of A inserted.

Theorem 11.1 (`gtdns_casdt, gtdns_cas`) *GTD satisfies cut-admissibility.*

```
gtdns_casdt: "(?dt, ?dta) : casdt (lkssx (extcs GTD)) ?A"

  gtdns_cas: "(?Xl |- mins ?A ?Yl, mins ?A ?Xr |- ?Yr) :
                              cas (lkssx (extcs GTD)) ?A"
```

12 Weakening, Contraction and Cut Admissibility for Dynamic Topological Logic S4C

We now describe Isabelle proofs of the cut admissibility of the logic S4C described by Mints [16]. This system has two “modal” operators, $\Box$ and $\circ$. The $S4$-axioms hold for $\Box$, $\circ$ commutes with the boolean operators, and the following are given:

$$\circ(A \to B) \leftrightarrow (\circ A \to \circ B)$$
$$\circ\bot \leftrightarrow \bot$$
$$\circ\Box A \to \Box \circ A$$

The following sequent rules are given for S4C by Mints [16]

$$\frac{\circ^k A, \Gamma \vdash \Delta, \circ^k B}{\Gamma \vdash \Delta, \circ^k (A \to B)}(\vdash\to) \qquad \frac{\Gamma \vdash \Delta, \circ^k A \quad \circ^k B, \Gamma \vdash \Delta}{\circ^k (A \to B), \Gamma \vdash \Delta}(\to\vdash)$$

$$\frac{\circ^k A, \Gamma \vdash \Delta}{\circ^k \Box A, \Gamma \vdash \Delta}(\Box \vdash) \qquad \frac{\Gamma \vdash \Delta}{\circ\Gamma \vdash \circ\Delta}(\circ) \qquad \frac{\mathcal{B} \vdash A}{\mathcal{B} \vdash \Box A}(\vdash \Box)$$

In the $(\vdash \Box)$ rule, $\mathcal{B}$ must consist of “$\Box$-formulae”, that is, formulae of the form $\circ^k \Box A$.

As Mints omits the other logical operators, we include, for them, the usual logical introduction rules with the principal and side formulae preceded by $\circ^k$ just as with the $(\vdash\to)$ and $(\to\vdash)$ rules shown above.

Our version of the calculus contains no explicit structural rules, so we prove invertibility of the logical rules and contraction admissibility. The presence of the ($\circ$) rule makes the proof more complicated and is handled similarly to our handling of contraction in proving cut admissibility for GTD.

As we have no structural rules, we use a presentation of the system which

- allows an arbitrary context to be added to the conclusion (only) of the ($\vdash \Box$) and ($\circ$) rules
- uses a version of the ($\Box \vdash$) rule which includes the principal formula in the premise

$$\frac{\Gamma \vdash \Delta}{\Sigma, \circ\Gamma \vdash \circ\Delta, \Pi}(\circ) \qquad \frac{\mathcal{B} \vdash A}{\Gamma, \mathcal{B} \vdash \Box A, \Delta}(\vdash \Box) \qquad \frac{\circ^k \Box A, \circ^k A, \Gamma \vdash \Delta}{\circ^k \Box A, \Gamma \vdash \Delta}(\Box \vdash)$$

12.1 Calculus for S4C

We now describe how we encoded the sequent calculus. First we define the rules which can be extended by an arbitrary context in their premises and conclusion. Without the context, these rules form the set `s4cnsne`.

Applying `nkmap` k to a rule applies $\circ^k$ to each formula appearing in that rule, and `funpow` f x means applying f to x, n times, i.e., $f^n(x)$.

```
inductive "s4cnsne"
  intrs
       id  "psc : idrls ==> psc        : s4cnsne"
  circ_il  "rl : lksil ==> nkmap k rl  : s4cnsne"
  circ_ir  "rl : lksir ==> nkmap k rl  : s4cnsne"
   circ_T  "rl : lkrefl ==> nkmap k rl : s4cnsne"

inductive "lkrefl"
  intrs
  I  "([{#A#} + {#Box A#} |- {#}], {#Box A#} |- {#}) : lkrefl"

defs
  nkmap_def : "nkmap k == pscmap (seqmap (funpow Circ k))"

inductive "s4cns"
 intrs
   extI "rl : s4cnsne ==> pscmap (extend (U |- V)) rl : s4cns"
 extcsI "(ps, c) : circ Un s4cbox ==>
                              (ps, extend (U |- V) c) : s4cns"
```

```
inductive "circ"
  intrs
      I "([seq], seqmap Circ seq) : circ"

inductive "s4cbox"
  intrs
     boxI "M : msboxfmls ==> ([M |- {#A#}],
                              M |- {#Box A#}) : s4cbox"

inductive "msboxfmls"
  intrs
     I "ALL f. f :# M --> f : boxfmls ==> M : msboxfmls"

inductive "boxfmls"
  intrs
     I "funpow Circ k (Box B) : boxfmls"
```

We first prove the admissibility of weakening and contraction.

12.2 Weakening for S4C

Weakening admissibility was straightforward using Lemma 9.1.

12.3 Inversion and Contraction-Admissibility for S4C

Invertibility of the logical introduction rules was dealt with using multiple lemmata showing various cases of `inv_stepm`, as described in Sect. 9.3: as noted there, a proof of invertibility can be split up into

- the invertibility of various different rules
- cases of what the last rule in the derivation, from whose conclusion we wish to apply one of the inverted rules

As in Sect. 9.3, we make significant use of Lemma 9.3.

We then prove contraction admissibility. This uses predicates and results which are essentially Definition 7.2 and Lemma 7.1, but instantiated to apply to the property of contraction admissibility, giving the property `ctr_adm_step` and a lemma `gen_ctr_adm_step`.

We now look at proving `ctr_adm_step` for each possible case for the last rule of a derivation.

Lemma 12.1 *If*

- *rule set* `lrls` *consists of rules which are the identity (axiom) rules* $A \vdash A$*, or are rules with a single formula in their conclusion,*
- *all rules in* `lrls` *have the "subformula" property (which here means that for every premise* other than a premise which contains the conclusion, *every formula in that premise is a subformula of a formula in the conclusion*
- *the rule set* `drls` *(derivation rules) contains the extensions of* `lrls`
- *in regard to the derivation rules* `drls`*, the inverses of extensions of* `lrls` *are admissible*
- *rule* (ps, c) *is an extension of a rule of* `lrls`

then the contraction admissibility step `ctr_adm_step` *holds for the final rule* (ps, c) *and the derivation rule set* `drls`*.*

So the conclusion of this lemma means: assuming that

- contraction on formulae A' smaller than A is admissible, and
- contraction on A is admissible in the sequents ps

then contraction on A in sequent c is admissible.

```
gen_ctr_adm_step_inv:
   "[| ?epsc : extrs ?lrls ;
       ?lrls <= iscrls ;
       extrs ?lrls <= ?drls ;
       Ball ?lrls (subfml_cp_prop ?sub) ;
       Ball (extrs ?lrls) (inv_rl ?drls) |]
     ==> ctr_adm_step ?sub (derrec ?drls {}) ?epsc ?A"

subfml_cp_prop.simps:
   "subfml_cp_prop sub (ps, c) =
      (ALL p:set ps. c <= p
       | (ALL fp. ms_mem fp p -->
          (EX fc. ms_mem fc c & (fp, fc) : ub)))"
```

Then the other cases of `ctr_adm_step` were proved separately:

Lemma 12.2 *In S4C, for derivations with final rules* $(\vdash \Box)$ *and* $(\circ)$ *(extended in their conclusions), the inductive contraction admissibility step* `ctr_adm_step` *holds.*

```
ctr_adm_step_s4cbox_r:
  "[| (?ps, ?c) : extcs s4cbox ; extcs s4cbox <= ?drls |]
     ==> ctr_adm_step ?sub (derrec ?drls {}) (?ps, ?c) ?A"
```

```
ctr_adm_step_circ_r:
  "[| (?ps, ?c) : extcs circ ; extcs circ <= ?drls |]
     ==> ctr_adm_step ipsubfml (derrec ?drls {}) (?ps, ?c) ?A"
```

Consequently, we get contraction admissibilty. The only case not covered above is for the reflexivity rule ($\Box \vdash$), in its form where the principal formula is copied to the premise. This is required for contraction admissibility, which becomes simple with the rule in this form.

Lemma 12.3 (`s4cns_ctr_adm`) *Contraction is admissible in GTD.*

```
s4cns_ctr_adm: "ctr_adm s4cns ?A"
```

12.4 Cut-Admissibility for S4C

To prove cut admissibility for a sequent calculus containing an explicit contraction rule, two methods are

- to prove mix-elimination directly, where the property proved by induction on the derivation is that any instance of the mix rule is admissible; in effect this was done in [6] for the more complex logic GLS,
- in respect of the derivations on either side of the cut, to look up the derivation skipping over consecutive instances of contraction on the cut-formula, and consider the various cases of the next rule on either side above those contractions.

We do something similar to the second approach here, but we look up the derivations on either side to find the last rule before a consecutive sequence of ($\circ$) rules. For this we use the theorem `top_circ_ns`. In some cases we also need the fact that if the bottom rule is not ($\circ$), then the tree asserted to exist is actually the original one. The function `forget` exists simply to prevent automatic case splitting of its argument: logically it does nothing.

Lemma 12.4 (`top_circ_ns`) *Given a valid (explicit) derivation tree* `dt`*, then there is a valid (explicit) tree* `dtn` *and an integer k such that*

- *the bottom rule of* `dtn` *is not* ($\circ$)*,*
- *the conclusions c and c' of* `dt` *and* `dtn` *are related by $c = \circ^k c'$*
- *height of* `dt` *= height of* `dtn` *$+k$*
- `dt` *and* `dtn` *iff $k = 0$ iff the bottom rule of* `dt` *is not* ($\circ$)

```
top_circ_ns:
   "valid ?rls ?dt
    ==> EX dtn k.
           botRule dtn ~: extcs circ & valid ?rls dtn
         & seqmap (funpow Circ k) (conclDT dtn) <= conclDT ?dt
         & heightDT ?dt = heightDT dtn + k
```

```
          & forget ((k = 0) = (botRule ?dt ~: extcs circ)
          & (k = 0) = (dtn = ?dt))"

  forget_def: "forget f == f"
```

But one easy case is where the last rule on *both* sides is the $(\circ)$ rule: then we can apply cut (on a smaller formula) to the premises of the $(\circ)$ rules, and then apply the $(\circ)$ rule. So when we look at the $(\circ)$ rules on both sides immediately preceding the cut, we need only bother about the case where the number of those $(\circ)$ rules is zero on one side.

First, the case where both rules are the $(\vdash \Box)$ rule. The fact that the conclusions of both the $(\circ)$ rule and the $(\vdash \Box)$ rule may be extended by an arbitrary context complicates matters. Consider the following diagram of a number of $(\circ)$ rules followed by the $(\vdash \Box)$ rule.

$$\cfrac{\cfrac{\mathcal{M} \vdash A}{\Gamma, \mathcal{M} \vdash \Box A, \Delta}\;(\vdash \Box)}{\Gamma', \circ^k\Gamma, \circ^k\mathcal{M} \vdash \circ^k\Box A, \circ^k\Delta, \Delta'}\;(\circ^*)$$

In this case we can instead construct the following derivation tree, which is of the same height.

$$\cfrac{\cfrac{\mathcal{M} \vdash A}{\mathcal{M} \vdash \Box A}\;(\vdash \Box)}{\circ^k\mathcal{M} \vdash \circ^k\Box A}\;(\circ^*)$$

Thus we can use, in proving an inductive step, the fact that $\circ^k\mathcal{M} \vdash \circ^k\Box A$ is derivable, and with a derivation of the same height as that of $\Gamma', \circ^k\Gamma, \circ^k\mathcal{M} \vdash \circ^k\Box A, \circ^k\Delta, \Delta'$. This will be used in our proofs without further comment.

Now, where the cut-formula is within $\circ^k\Delta, \Delta'$ (where this is the derivation tree on the left of a desired cut), or within $\Gamma', \circ^k\Gamma$ (where this tree is on the right), the cut is admissible because we can start from the derivable sequent $\mathcal{M} \vdash A$ and apply $(\vdash \Box)$ and $\circ$ rule without any extra formulae in the conclusions, as discussed above. In this case we just use weakening admissibility to obtain the result of the cut.

These situations are covered by Lemma 12.5 (`s4cns_cs_param_l`”) below and the symmetric result `s4cns_cs_param_r`”.

Lemma 12.5 *Let the left premise subtree of a desired cut be* `dt`*, with* `dtn` *and* k *as in Lemma 12.4, let the bottom rule of* `dtn` *be an extension (of the conclusion) of a rule in* `s4cns` *whose conclusion is* $cl \vdash cr$*, and let* C *not be in* $\circ^k cr$*. Then the inductive step* `sumh_step2_tr` *for proving cut-admissibility with cut-formula* C *holds (where* `list` *and* `lista` *are names automatically generated by Isabelle for the lists of premises of final rules).*

```
s4cns_cs_param_l'':
   "[| (?ps, ?cl |- ?cr) : s4cns ; valid s4cns ?dtn ;
       botRule ?dtn : extcs {(?ps, ?cl |- ?cr)} ;
       count (mset_map (funpow Circ ?k) ?cr) ?C = 0 |]
     ==> sumh_step2_tr (prop2 casdt s4cns) ?C ?sub
                        (Der (seqmap (funpow Circ ?k)
                              (conclDT ?dtn) + ?flr) ?list,
                         Der ?dtr ?lista)"
```

A similar pair of results, discussed later (see Lemma 12.6), covers the case where the rule above the $(\circ)$ rules is a skeleton rule which is extended by an arbitrary context in its conclusion *and* its premises.

Now we can assume that the cut-formula is within the principal part of the rule before the $(\circ)$ rules (noting that for the $(\vdash \Box)$ rule the "principal part" means the entire $\mathcal{M} \vdash \Box A$). Then there must be zero $(\circ)$ rules on the right side: because if there are zero $\circ$ rules on the left, then the cut-formula must be $\Box A$, whence there would also be zero $(\circ)$ rules on the right.

In the diagrams, *(cut ?)* represents the instance of the cut rule which we are aiming to show is admissible.

$$\dfrac{\dfrac{\dfrac{\mathcal{M} \vdash A}{\Gamma, \mathcal{M} \vdash \Box A, \Delta}\,(\vdash \Box)}{\Gamma', \circ^k\Gamma, \circ^k\mathcal{M} \vdash \circ^k\Box A, \circ^k\Delta, \Delta'}\,(\circ^*) \qquad \dfrac{\circ^k\Box A, \mathcal{M}' \vdash B}{\Gamma'', \circ^k\Box A, \mathcal{M}' \vdash \Box B, \Delta''}\,(\vdash \Box)}{\Gamma', \circ^k\Gamma, \Gamma'', \circ^k\mathcal{M}, \mathcal{M}' \vdash \Box B, \circ^k\Delta, \Delta', \Delta''}\,(\mathit{cut}\ ?)$$

Here we do the cut, by induction, before the $(\vdash \Box)$ rule on the right, using a derivation similar to that on the left, but without any context, then we apply the $(\vdash \Box)$ rule, introducing the required context.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{M} \vdash A}{\mathcal{M} \vdash \Box A}\,(\vdash \Box)}{\circ^k\mathcal{M} \vdash \circ^k\Box A}\,(\circ^*) \qquad \circ^k\Box A, \mathcal{M}' \vdash B}{\circ^k\mathcal{M}, \mathcal{M}' \vdash B}\,(\mathit{inductive\ cut})}{\Gamma', \circ^k\Gamma, \Gamma'', \circ^k\mathcal{M}, \mathcal{M}' \vdash \Box B, \circ^k\Delta, \Delta', \Delta''}\,(\vdash \Box)$$

For the other cases, we first consider the "parametric" cases, where the last rule above the $(\circ)$ rules is an extension ρ' of a rule ρ in `s4cnsne`, and the principal formula of ρ is not the "de-circled" cut-formula A. Recall that `s4cnsne` consists of the axiom, logical introduction rules and the $(\Box \vdash)$ rule, as skeletons (i.e., not extended with context), but with $\circ^k$ applied to their formulae.

$$\dfrac{\dfrac{\dfrac{X' \vdash Y', A}{X \vdash Y, A}\,(\rho')}{W, \circ^k X \vdash \circ^k Y, \circ^k A, Z}\,(\circ^*) \qquad \circ^k A^m, U \vdash V}{\mathrm{W}, \circ^k X, U \vdash \circ^k Y, Z, V}\,(\mathit{cut}\ ?)$$

Here we must apply the $\circ$ rule the requisite number of times to the premise(s) of ρ', then apply (using the inductive hypothesis) cut on $\circ^k A$ to each of them, and finally apply ρ'' which we get by applying $\circ^k$ to ρ, and then extending it appropriately.

This uses the result that if a rule is in `s4cnsne` then so is the result of applying $\circ^k$ to all formulae in its premises and conclusion.

```
s4cnsne_nkmap: "?r : s4cnsne ==> nkmap ?k ?r : s4cnsne"
```

$$\dfrac{\dfrac{X' \vdash Y', A}{\mathrm{W}, \circ^k X' \vdash \circ^k Y', \circ^k A, Z}(\circ^*) \qquad \circ^k A^m, U \vdash V}{\dfrac{\mathrm{W}, \circ^k X', U \vdash \circ^k Y', Z, V}{\mathrm{W}, \circ^k X, U \vdash \circ^k Y, Z, V}(\rho'')}(\textit{inductive cut})$$

Lemma 12.6 (`s4cns_param_l'`) and the symmetric result `s4cns_param_r'` cover this case.

Lemma 12.6 *Let the left premise subtree of a desired cut be* `dt`*, with* `dtn` *and k as in Lemma 12.4, let the bottom rule of* `dtn` *be an extension of a rule in* `s4cnsne` *whose conclusion is cl ⊢ cr, and let C not be in* $\circ^k cr$*. Then the inductive step* `sumh_step2_tr` *for proving cut-admissibility with cut-formula C holds.*

```
s4cns_param_l':
   "[| (?ps, ?cl |- ?cr) : s4cnsne ;
       botRule ?dtn : extrs {(?ps, ?cl |- ?cr)} ;
       valid s4cns ?dtn ;
      count (mset_map (funpow Circ ?k) ?cr) ?C = 0 ;
      Suc (heightDTs ?list) = heightDT ?dtn + ?k |]
     ==> sumh_step2_tr (prop2 casdt s4cns) ?C ?sub
                       (Der (seqmap (funpow Circ ?k)
                            (conclDT ?dtn) + ?flr) ?list,
                        Der ?dtr ?lista)"
```

It is similar for the parametric case on the right. The axiom rule is trivial in all cases.

For the $(\vdash \Box)$ rule on the left, where the rule on the right is an extension of rule ρ whose principal formula is the "de-circled" cut-formula, the only case remaining is where ρ is $(\Box \vdash)$.

$$\dfrac{\dfrac{\dfrac{\mathcal{M} \vdash A}{X, \mathcal{M} \vdash \Box A, Y}(\vdash \Box)}{X', \circ^k X, \circ^k \mathcal{M} \vdash \circ^k \Box A, \circ^k Y, Y'}(\circ^*) \qquad \dfrac{\dfrac{\circ^{k''} A, U \vdash V}{\circ^{k''} \Box A, U \vdash V}(\Box \vdash)}{\circ^{k'+k''} \Box A, \circ^{k'} U, U' \vdash \circ^{k'} V, V'}(\circ^*)}{X', \circ^k X, \circ^k \mathcal{M}, \circ^{k'} U, U' \vdash \circ^k Y, Y', \circ^{k'} V, V'}(\textit{cut?})$$

Here $k' + k'' = k$, but since we also have that $k = 0$ or $k' = 0$, this means that $k' = 0$ and $k'' = k$. The following diagram omits a final use of the admissibility of weakening.

$$\begin{prooftree}
\AxiomC{$\mathcal{M} \vdash A$}
\RightLabel{$(\circ^*)$}
\UnaryInfC{$\circ^k \mathcal{M} \vdash \circ^k A$}
\AxiomC{$\mathcal{M} \vdash A$}
\RightLabel{$(\vdash \Box)$}
\UnaryInfC{$\mathcal{M} \vdash \Box A$}
\RightLabel{$(\circ^*)$}
\UnaryInfC{$\circ^k \mathcal{M} \vdash \circ^k \Box A$}
\AxiomC{$\circ^k \Box A, \circ^k A, U \vdash V$}
\dashedLine
\RightLabel{(\textit{inductive cut})}
\BinaryInfC{$\circ^k \mathcal{M}, \circ^k A, U \vdash V$}
\dashedLine
\RightLabel{(\textit{inductive cut})}
\BinaryInfC{$\circ^k \mathcal{M}, \circ^k \mathcal{M}, U \vdash V$}
\RightLabel{(\textit{ctr})}
\UnaryInfC{$\circ^k \mathcal{M}, U \vdash V$}
\end{prooftree}$$

Next we look at the case of the $(\vdash \Box)$ rule on the right, but for this, since the cut-formula must be a $\Box$-formula, all cases have already been dealt with.

Finally, there is the case where the last rules (above the final sequence of $\circ$-rules) on both sides are extensions of rules in `s4cnsne`. Most of these cases have been covered, i.e., the axiom rules, and the "parametric" cases, where the "de-circled" cut-formula is not the principal formula of the rule.

So there remain the cases where the rules on either side are the logical introduction rules. For these, the proofs are essentially the same as for other logics generally, except that we need to allow for a number of circles. Conceptually it is easiest to imagine that in each case the final $\circ$ rules are moved upwards to precede the final logical introduction rules, although we didn't actually prove it this way.

We proved that the usual logical introduction rules, with $\circ^k$ applied to principal and side formulae (as used in S4C), satisfy `c8_ercas_prop` (Definition 11.4). Recall that this means that assuming cut admissibility on smaller formulae, us have cut admissibility of a more complex formula where the last rule on either side is a logical introduction rule.

Lemma 12.7 [*`s4cns_c8_ercas`*] *S4C satisfies `c8_ercas_prop` in relation to the logical introduction rule skeletons `lksil` cup `lksir`, with $\circ^k$ applied to all formulae.*

```
s4cns_c8_ercas: "c8_ercas_prop (circrel ipsubfml) s4cns ?A
    (nkmap ?k ` (lksil Un lksir))"
```

The following diagrams show an example. We let $t = k + k' = l + l'$. In this case we do not make use of the fact that either k or l must be zero.

$$\begin{prooftree}
\AxiomC{$X \vdash \circ^{k'} A, Y$}
\AxiomC{$X \vdash \circ^{k'} B, Y$}
\RightLabel{$(\vdash \wedge)$}
\BinaryInfC{$X \vdash \circ^{k'} (A \wedge B), Y$}
\RightLabel{$(\circ^*)$}
\UnaryInfC{$\circ^k X \vdash \circ^t (A \wedge B), \circ^k Y$}
\AxiomC{$U, \circ^{l'} A, \circ^{l'} B \vdash V$}
\RightLabel{$(\wedge \vdash)$}
\UnaryInfC{$U, \circ^{l'} (A \wedge B) \vdash V$}
\RightLabel{$(\circ^*)$}
\UnaryInfC{$\circ^l U, \circ^t (A \wedge B) \vdash \circ^l V$}
\dashedLine
\RightLabel{(\textit{cut ?})}
\BinaryInfC{$\circ^k X, \circ^l U \vdash \circ^k Y, \circ^l V$}
\end{prooftree}$$

The diagram above is simplified by not including the extra context which may be introduced in the conclusion of the $(\circ)$ rules. This is replaced by

$$
\cfrac{
\begin{array}{c}
\cfrac{X \vdash \circ^{k'} A, Y}{\circ^k X \vdash \circ^t A, \circ^k Y}\,(\circ^*)\\ \hline \vdots\\ \vdots\\ \vdots
\end{array}
\qquad
\cfrac{\cfrac{X \vdash \circ^{k'} B, Y}{\circ^k X \vdash \circ^t B, \circ^k Y}\,(\circ^*) \qquad \cfrac{U, \circ^{l'} A, \circ^{l'} B \vdash V}{\circ^l U, \circ^t A, \circ^t B \vdash \circ^l V}\,(\circ^*)}{\circ^k X, \circ^l U, \circ^t A \vdash \circ^k Y, \circ^l V}\,(\textit{inductive cut})
}{
\cfrac{\circ^k X, \circ^k X, \circ^l U \vdash \circ^k Y, \circ^k Y, \circ^l V}{\circ^k X, \circ^l U \vdash \circ^k Y, \circ^l V}\,(\textit{contraction})
}\,(\textit{inductive cut})
$$

Again, we can use weakening admissibility to get the extra context which was introduced by the $(\circ)$ rules, but omitted from the first diagram.

Finally we combine these results to get the cut admissibility result, in terms of explicit derivation trees, and then in terms of derivability.

Theorem 12.1 (*`s4cns_casdt, s4cns_cas`*) *S4C satisfies cut-admissibility.*

```
s4cns_casdt: "(?dta, ?dtb) : casdt s4cns ?A"
  s4cns_cas: "(?cl, ?cr) : cas s4cns ?A"
```

12.5 Comparing Our Proofs and the Proofs of Mints

The slides for the presentation of Mints [16] contains a very abbreviated treatment of cut-admissibility for S4C. We attempted to follow the proof shown there, but were unable to. The slides state a lemma ("Substitution Lemma"), that the following rule is admissible

$$\frac{\mathcal{B} \vdash \Box C \quad \Box C, \Gamma \vdash \Delta}{\mathcal{B}, \Gamma \vdash \Delta}$$

As a lemma it is undoubtedly correct (it is a particular case of cut admissibility). However, as part of the proof of cut-admissibility we were unable to prove it as it stands—it appears to need (at least) an assumption that cuts on C are admissible.

13 Related Work

We may compare this approach with that of Pfenning [21]. Pfenning uses the propositions-as-types paradigm, where a type represents (partially) a sequent. More precisely, for intuitionistic logic, a type `hyp A -> hyp B -> conc C` represents a sequent containing `A` and `B` in its antecedent, and `C` in its succedent. For classical logic, `neg A -> neg B -> pos C -> pos D -> #` represents a sequent containing `A` and `B` in its antecedent, and `C` and `D` in its succedent. A term of a given type represents a derivation of the corresponding sequent.

Pfenning's proof of cut-admissibility proceeds by a triple induction, using structural induction on the formula and the two terms representing the derivations. It therefore most closely resembles our proofs involving explicit derivations, as described in Sect. 7.3.

However in Sect. 7.3 we go on to measure properties (such as the height) of an explicit derivation. It seems as though Pfenning's approach does not allow the possibility of doing that.

Tews [24] describes the use of Coq to prove cut-elimination for propositional multi-modal logics. In Coq, types are identified with terms, and each term has a type: a type has the type **Type**. A proposition is a type whose inhabitants are its proofs, so $A \to B$ means both the type of proofs of the proposition $A \to B$ and the type of functions which take proofs of A to proofs of B. Since types can depend on terms, this gives a dependently typed system, which can provide a way of capturing side-conditions in the type system. For example, the type counted_list A n is the type of lists of items of type A and whose length is n.

Tews uses a (single) list of formulae as a sequent, where formulae which would appear on the other side of a two-sided sequent are negated. He proves that for the rule sets he uses, for any reordering s' of the conclusion s of a rule, there is a corresponding rule whose conclusion is s', and, assuming sets of rules and hypotheses closed under reordering, that provability is also closed under reordering. He defines an (object-logic) proof as the type proof, similar to our definition of the type `dertree`, but the type definition also incorporates the requirement that each "node" of the tree must be in the given set of rules. This is an example of a dependent type, where the type proof depends on the term rules.

He proves cut-elimination both semantically (by proving soundness and cut-free completeness) and syntactically (where the proof implements a cut-elimination procedure). Thus his work includes extensive formalisation of the semantics of the logics. His proofs use the modal ranks of formulae, and involve formalising substitution, which we did not find necessary, and in some cases require proving depth-preserving admissibility of rules.

14 Further Work and Conclusion

We have proved cut-admissibility for several different sequent calculi, ranging from the well-known logics S4 and S4.3 to GTD and S4C described recently in [16]. In other work not described here we also proved cut-admissibility for GTD, for a calculus containing explicit contraction and weakening rules, in both the ways described at the start of Sect. 12.4.

We have shown how the proofs can be split up into components some of which were expressed in lemmata which can be reused in similar proofs for other calculi. This was of significant value, as was the use of the type classes described in [6]. It remains to generalise our framework so that these results follow simply by instantiating these general concepts.

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Intuitionistic Decision Procedures Since Gentzen

Roy Dyckhoff

1 Introduction

Gentzen solved the decision problem for intuitionistic propositional logic in his doctoral thesis [31]; this paper reviews some of the subsequent progress. Solutions to the problem are of importance both for general philosophical reasons and because of their use in implementations of proof assistants (such as *Coq* [4], widely used in software verification) based on intuitionistic logic. Our focus is on calculi and procedures that can be understood in relation to traditional proof theory.

We tend (despite their importance) to avoid implementation issues, e.g. the use of AVL trees rather than lists [73], structure sharing techniques [44], binary decision diagrams [33], caching and dependency directed backjumping [34] and prefix unification [72], in favour of relatively simple calculi where questions such as cut admissibility can be raised and, ideally by syntactic methods, answered. We ignore the first-order case, for which see Schütte [63], Franzén et al. [60] and Otten [55]. For implementations see Otten's ILTP website [56]. We also have our own implementations of several of the calculi mentioned here, using our own Prolog software YAPE ("Yet Another Proof Engine") [15] allowing sequent calculus rules to be coded clearly and proofs to be displayed using LaTeX, either as trees or linearly, as illustrated in Sect. 8.7. We are particularly interested in questions of

1. **termination** (hence decidability),
2. **bicompleteness** (extractability of models from failed proof searches),
3. **determinism** (avoidance of backtracking),
4. **simplicity** (allows easier reasoning about systems).

We include a short discussion of labelled calculi; concerning termination therein, we refer to some recent literature.

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2 Gentzen's Calculus, LJ

Gentzen [31] solved the decision problem for **Int** with a calculus **LJ**, in which the antecedent of each sequent is a list of formulae and the succedent is either empty or a single formula. Since lists rather than sets are used, and the "operational" rules act only on the first element of the list, rules of *Exchange*, *Contraction* and *Thinning* (hereinafter called *Weakening*) are required. *Initial sequents* are of the form $A \Longrightarrow A$, where A is a formula. Let C indicate either a formula or an empty succedent. If A is a formula and Γ a list, then A, Γ is the list with head A and tail Γ; and similarly Γ, Δ is the list obtained by concatenating lists Γ and Δ. The important rules (for intuitionistic implication) are

$$\frac{\Gamma \Longrightarrow A \quad B, \Delta \Longrightarrow C}{A \to B, \Gamma, \Delta \Longrightarrow C}\ L\to \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \to B}\ R\to$$

Gentzen took negation as a primitive notion, with rules as follows:

$$\frac{\Gamma \Longrightarrow A}{\neg A, \Gamma \Longrightarrow}\ L\neg \qquad \frac{A, \Gamma \Longrightarrow}{\Gamma \Longrightarrow \neg A}\ R\neg$$

which have the virtue of being illustrations of the succedent being empty.

But in practice we will take negation as a defined notion, using $\neg A =_{def} A \to \bot$, and allowing also $\bot, \Gamma \Longrightarrow C$ as an initial sequent (and now we can replace empty succedents by $\bot$). For completeness we show also his rules for conjunction and disjunction:

$$\frac{A_i, \Gamma \Longrightarrow C}{A_0 \wedge A_1, \Gamma \Longrightarrow C}\ L\wedge_i \qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Longrightarrow C \quad B, \Gamma \Longrightarrow C}{A \vee B, \Gamma \Longrightarrow C}\ L\vee \qquad \frac{\Gamma \Longrightarrow A_i}{\Gamma \Longrightarrow A_0 \vee A_1}\ R\vee_i$$

Gentzen's approach is not (as one may find suggested in the literature) a root-first approach, but to see what sequents (from the finite range of possibilities) are initial, what can be inferred from them, and so on. It may thus be regarded as an early instance of Maslov's "Inverse Method" [48]. Defining a sequent to be *reduced* if its antecedent contains no more than three occurrences of any formula, and after showing that a derivation of a reduced sequent can be modified into one where all the sequents are reduced, one can see an obvious finiteness argument exploiting the subformula property. Kosta Došen observed in [12] that Gentzen's "three occurrences" can be reduced to "two occurrences". (B subsumes $A \to B$, so, in the rule $L\to$, we may need a copy of $A \to B$ in Γ but we don't need one in Δ. In other words, $B, \Delta \Longrightarrow C$ and $B, A \to B, \Delta \Longrightarrow C$ are inter-derivable.)

2.1 Calculi of Ono, Ketonen and Kleene, Troelstra's G3i

Gentzen's **LJ** is not a calculus suitable for solving the decision problem root-first—note especially the "context-splitting" nature of $L\!\to$. Ono [54], Ketonen [42] and Kleene [43] observed in various ways that it was better to incorporate structural rules (like *Exchange*, *Weakening* and *Contraction*) into the notation and/or into the operational rules: for example, in Kleene's treatment one has

$$\frac{A\to B, \Gamma \Longrightarrow A \quad B, A\to B, \Gamma \Longrightarrow C}{A\to B, \Gamma \Longrightarrow C}\ L\!\to \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A\to B}\ R\!\to$$

with the convention that two sequents are "cognate" (and thus are interchangeable) iff exactly the same formulae appear in the antecedents (regardless of number and order) and they have the same succedent. Thus, using modern terminology found in [68] but not in [43], A, Γ is now the multiset sum of the multisets A and Γ. Care needs to be taken with rules if sets are used, since A, Γ as a pattern can match both with Γ containing A and with Γ not containing A.

Note that $A\to B$ can be omitted from the second premiss of $L\!\to$ (as before, since it is subsumed by B), but not from the first, lest completeness be lost.

This incorporation of *Weakening* into the rules allows a "root-first" approach, i.e. the root-first construction of a tree with the sequent to be decided at the root and expansion of the tree by choice of a principal formula and generation of the appropriate rule instance and thus of part of the next level of the tree. Typically (provided the search can be shown to terminate) such an expansion is done depth-first rather than breadth-first.

Troelstra's book (with Schwichtenberg) [68] gives a good treatment of this transformation of Gentzen's calculi by incorporation of the structural rules, including acknowledgment of the inspiration provided by Dragalin [13].

2.2 Maehara's Calculus, m-G3i

Maehara [47] introduced[1] an important variant of Kleene's calculus: succedents can now be arbitrary (finite) collections Δ of formulae rather than just empty or singular. The rules for implication and disjunction are then

$$\frac{A\to B, \Gamma \Longrightarrow A, \Delta \quad B, A\to B, \Gamma \Longrightarrow \Delta}{A\to B, \Gamma \Longrightarrow \Delta}\ L\!\to \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A\to B, \Delta}\ R\!\to$$

---

[1] A referee points out that this paper gave a classical provability interpretation of intuitionistic logic and an early syntactic proof of its faithfulness, a result conjectured by Gödel (and proved semantically by McKinsey and Tarski), as could usefully have been mentioned in [20].

$$\dfrac{A, A\vee B, \Gamma \Longrightarrow \Delta \quad B, A\vee B, \Gamma \Longrightarrow \Delta}{A\vee B, \Gamma \Longrightarrow \Delta}\,L\vee \qquad \dfrac{\Gamma \Longrightarrow A, B, A\vee B, \Delta}{\Gamma \Longrightarrow A\vee B, \Delta}\,R\vee$$

which have the virtues (1) that $L\to$ and $R\vee$ are invertible and hence (2) that all the non-determinism in root-first search pertains to the $R\to$ rule and the choice of its principal formula $A\to B$ in the succedent for analysis. (We consider $R\neg$ as a special case of $R\to$.) Without loss of completeness one may remove the repetition of the principal formula $A \vee B$ from the premisses of $L\vee$ and $R\vee$.

Proofs in this system can be much smaller than those in the single-succedent calculus: see Egly and Schmitt [23].

Approximately this calculus is used as a basis in tableau theorem proving; one advantage is that counter-models can be extracted from failed searches. (Note that the rule $R\vee$ is classical here but not in **G3i**.) In other words, the calculus is *bicomplete*.

The same calculus (presented as a tableau calculus) appears, attributed to Beth [5], in Fitting's thesis [27] and in his book [28]. It can also be found in Curry's book [11]. As noted by von Plato [58], a similar system was developed by Gentzen, and can be found in his Nachlass.

Fitting's notion of "tableau" is a finite sequence of configurations, each obtained from its predecessor by applying a rule to one of the problems (i.e. sequents) therein; each *configuration* is a finite collection of problems (each of which has to be solved for the configuration to be *closed*). A problem is *solved* when it is an initial sequent. The tableau is *closed* (and the proof complete) when one (without loss of generality, the last) of its configurations is closed. Some rule applications replace a problem by just one problem while others replace it by two; in each case the new configuration is just the old configuration in which the old problem has been replaced by the new ones. There is no need to apply rules to solved problems or to closed configurations.

Using sequent notation rather than signed formulae, and (as in proof theory rather than tableau practice) writing the proof from bottom to top, and (for clarity) marking principal formulae with a box), here is a closed tableau proving the sequent $\Longrightarrow (p\to q\to r)\to(p\to q)\to p\to r$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p\to q\to r, p\to q, p \Longrightarrow p, r;\quad p\to q\to r, p\to q, p, q \Longrightarrow p, r;\quad p\to q\to r, p\to q, p, q, q\to r \Longrightarrow q, r;\quad p\to q\to r, p\to q, p, q, q\to r, r \Longrightarrow r}}{p\to q\to r, p\to q, p \Longrightarrow p, r;\quad p\to q\to r, p\to q, p, q \Longrightarrow p, r;\quad p\to q\to r, p\to q, p, q, \boxed{q\to r} \Longrightarrow r}\,L\to}{p\to q\to r, p\to q, p \Longrightarrow p, r;\quad \boxed{p\to q\to r}, p\to q, p, q \Longrightarrow r}\,L\to}{p\to q\to r, \boxed{p\to q}, p \Longrightarrow r}\,L\to}{p\to q\to r, p\to q \Longrightarrow \boxed{p\to r}}\,R\to}{p\to q\to r \Longrightarrow \boxed{(p\to q)\to p\to r}}\,R\to}{\Longrightarrow \boxed{(p\to q\to r)\to(p\to q)\to p\to r}}\,R\to$$

As can be seen, each problem in the "last" (i.e. uppermost) configuration is solved; hence, that configuration is closed. Conjunctive branching is handled at each $L\to$ step by adding an extra problem, thus hiding the tree structure familiar to us from

Gentzen's work; but, backtracking (because of the rule $R\!\to$) is not made explicit. The repetition (as one moves upwards) of closed problems is official, but inessential. Backtracking could be added by allowing branching—one branch should then be required to be closed; only then is it feasible to extract counter-models from failed searches. [In this proof there are no backtracking possibilities.]

Termination is assured by the subformula property: some form of loop-checking is required.

An interesting variation is the calculus **GHPC** of Dragalin [13]; by omitting Δ in the first premiss, without loss of completeness, this has a non-invertible $L\!\to$ rule, incorporating a form of focusing useful in the proof theory of the multi-succedent **m-G4ip**.

3 Vorob'ev's Calculus, G4ip

Vorob'ev introduced in [70, 71] an important calculus now known as **G4ip**. Others (Hudelmaier [39–41], Dyckhoff [14]) rediscovered (and refined) the same calculus some 40 years later. See also Lincoln et al. [45]. The goal is to avoid use of loop-checking (messy to reason about, tricky to implement). The formula $\neg\neg(p \vee \neg p)$ easily illustrates the need for loop-checking in Fitting's calculus. The key idea is to replace, in the single succedent calculus[2] **G3ip**, the left rule for implication $L\!\to$ by four rules, according to the form of the implication's antecedent A, exploiting the intuitionistic equivalences

1. $\bot \to B \equiv \top$,
2. $P \wedge (P \to B) \equiv P \wedge B$,
3. $(C \wedge D) \to B \equiv C \to (D \to B)$,
4. $(C \vee D) \to B \equiv (C \to B) \wedge (D \to B)$,
5. $C \wedge ((C \to D) \to B) \equiv C \wedge (D \to B)$.

to reduce the formula's complexity (in a carefully measured sense) and a bit of proof theory to show completeness. The effect is that root-first depth-first proof search terminates, i.e. root-first application of inference rules decreases the sequent's "size" rather than allowing it to oscillate up and down without termination. A measure of "size" (due to Hudelmaier) can be found in [68]; another can be found in [14].

The rules for left-implication are thus as follows (with P atomic):

$$\frac{\Gamma, P, B \Longrightarrow E}{\Gamma, P, P \to B \Longrightarrow E}\ L0\!\to \qquad \frac{\Gamma, C \to (D \to B) \Longrightarrow E}{\Gamma, (C \wedge D) \to B \Longrightarrow E}\ L\wedge\!\to$$

$$\frac{\Gamma, C \to B, D \to B \Longrightarrow E}{\Gamma, (C \vee D) \to B \Longrightarrow E}\ L\vee\!\to \qquad \frac{\Gamma, C, D \to B \Longrightarrow D \quad \Gamma, B \Longrightarrow E}{\Gamma, (C \to D) \to B \Longrightarrow E}\ L\!\to\!\to$$

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[2] The final "p" in the name indicates "propositional".

of which each but the last is invertible. There is no need for the case $\bot \to B$ to be included, since this formula is equivalent to $\top$. In each case the size of each premiss is less than that of the conclusion.

Formulae $P \to B$ are called *atomic implications*; those of the form $(C \to D) \to B$ are called *nested implications*. Proving completeness of the resulting system is a challenging exercise.

3.1 Hudelmaier's Refinements of Vorob'ev's Calculus

First appearance of Hudelmaier's rediscovery of Vorob'ev's work is in [39], i.e. in 1988 and then in his thesis, published in 1992 as [40].

The novelty (apart from some different proof methods) w.r.t. **G4ip** is in [41], ensuring proofs are of linear rather than exponential depth, by use of **fresh** proposition variables P in the cases ($L{\vee}{\to}$ and $L{\to}{\to}$) where a **compound** subformula (B, resp. D) from the conclusion is duplicated into a premiss. So these rules are replaced by the following:

$$\frac{\Gamma, C \to P, D \to P, P \to B \Longrightarrow E}{\Gamma, (C \vee D) \to B \Longrightarrow E}\ L{\vee}{\to}' \qquad \frac{\Gamma, C, D \to P, P \to B \Longrightarrow P \quad \Gamma, B \Longrightarrow E}{\Gamma, (C \to D) \to B \Longrightarrow E}\ L{\to}{\to}'$$

This allows one to show the decision problem to be in $O(n\ log\ n)$-SPACE. (In 1977 Ladner showed **S4**, and hence also **Int**, to be in PSPACE; in 1979 Statman [65] showed **Int** to be P-SPACE-hard.)

More precisely, Hudelmaier showed the value of $L{\vee}'$ but had difficulty[3] with $L{\to}{\to}'$, so he adopted a more complicated rule; the difficulty was overcome by Fiorino [3, 26]. The proof of soundness is easy: given proofs of the premisses of $L{\to}{\to}'$, substitute D for P in the first, cut with a proof of $D \to D$, and then use the sound rule $L{\to}{\to}$ to infer the desired conclusion. As for completeness, note that the rule

$$\frac{C, D \to P, P \to B, \Gamma \Longrightarrow P}{C, (C \to D) \to B, \Gamma \Longrightarrow D}$$

is invertible[4] in **G3ip** (or any other standard calculus for **Int**), and thus, in a proof by induction on sequent size of completeness of **G4ip**, a sequent matching the conclusion may be reduced to the corresponding premiss.

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[3] He commented "Unfortunately, this method does not work for the second problem.".

[4] As established using a cut of the conclusion with $D, D \to P \Longrightarrow P$ and a cut with $C, D \to P, P \to B \Longrightarrow (C \to D) \to B$.

3.2 Dyckhoff's Refinements of Vorob'ev's Calculus

Novelty[5] (apart from different proof methods) of [14] is to have (in addition to the single succedent calculus **G4ip**) a multi-succedent calculus **m-G4ip**. ([14] called them **LJT** and **LJT***.) This is closer to tableau methods used in implementations and allows easy extraction of a counter-model from a failed proof search [57] (joint work with Pinto). For the multi-succedent version, use Maehara's rule $R\to$ and replace (in each of the four special left rules for implication) each succedent formula E by Δ. (But the first premiss of $L\to\to$ should have, as in Dragalin's calculus **GHPC**, just one formula D in its succedent, lest the rule be unsound.) This can be combined with Hudelmaier's depth-reduction techniques. Various refinements of the multi-succedent version have been developed and implemented by a group in Milan (Avellone, Ferrari, Fiorentino, Fiorino, Miglioli†, Moscato and Ornaghi); one of the most recent papers is [24]. Their proof methods are almost entirely semantic.

3.3 Proof Theory of Vorob'ev's Calculus

Vorob'ev's proof of completeness of the calculus rests on a lemma now seen as the completeness of a single-succedent focused calculus **LJQ'**: see Dyckhoff and Lengrand [17] for details, and its extension to a multi-succedent focused calculus **LJQ*** (a variant of a calculus in Herbelin's thesis [36]). Root-first proof search in **LJQ'** occasionally focuses on the succedent and analyses it until either it is atomic or the rule $R\to$ is used; in particular, the $L\to$ rule requires (in the first premiss, but not in the second) a focus on the succedent. The completeness of this approach is a useful fact, exploited not just in Vorobev's [70, 71] but also in Hudelmaier's [41] (in which it is mentioned as "folklore"). The same trick is applicable elsewhere, e.g. in guarded logic and labelled calculi.

Dyckhoff and Negri [18] give a direct proof of completeness (w.r.t. an axiomatic presentation, via *Cut*-admissibility, rather than w.r.t. semantics), showing that *Contraction* is admissible in **G4ip** and hence (with explicit cut reduction steps) that *Cut* is admissible. This approach generalises to the multi-succedent case, and even shows the completeness of a first-order version (without, alas, the depth-boundedness ...). Dyckhoff, Kesner and Lengrand [16] show (for the implicational fragment **G4ip**$^\to$ only) how to make the cut reduction system strongly normalising.

4 Weich's Thesis

Weich [73, 74] made several excellent contributions: verified constructive completeness proofs, in *MINLOG* and in *Coq*, from which *Scheme* or *OCaml* programs may be extracted; pruning of the search by use of counter-models generated earlier in

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[5] As realised by the author after too long a delay.

the search (“an improvement both astonishing and significant”); a “conditional normal form” for formulae, obtained by pre-processing: essentially, $A{\to}B$ where A is a conjunction of atoms and B is one of $\bot$, P, $Q \vee R$, $(Q{\to}R){\to}\bot$, $(Q{\to}R){\to}S$. This reduces some of the run-time expansions that are otherwise repeated in different branches of the search. (P, Q, R, S indicate atoms.)

5 Easy Optimisations

Once the succedent is empty (or just $\bot$), one can revert to classical logic. Search can be pruned if a new subproblem (arising from choice of instance of non-invertible rule) isn’t solvable classically.

“Simplification”: once an atom p is added to the antecedent, all formulae in the sequent are simplified by putting $p = \top$ and reducing (e.g. with $\top \wedge A \equiv A$).

The same works if a negated atom $\neg p$ is added to the antecedent; the sequent is simplified by replacing p throughout by $\bot$ (and simplifying accordingly, e.g. with $\bot \vee A \equiv A$).

When a problem is analysed into two subproblems, and the first is solved, one may use [73] information from it in the second; e.g. the rules (one for multi-succedent; two for single-succedent calculi).

$$\frac{A, \Gamma \Longrightarrow \Delta \quad B, \Gamma \Longrightarrow A, \Delta}{A \vee B, \Gamma \Longrightarrow \Delta}\; L\vee' \qquad \frac{A, \Gamma \Longrightarrow G \quad B, \Gamma \Longrightarrow A \vee G}{A \vee B, \Gamma \Longrightarrow G}\; L\vee^*$$

$$\frac{A, \Gamma \Longrightarrow G \quad B, A \to G, \Gamma \Longrightarrow G}{A \vee B, \Gamma \Longrightarrow G}\; L\vee''$$

Several other easy optimisations are to be found in Franzén’s [29], Ferrari et al’s [24] and Weich’s [73].

6 Goal-Directed Pruning

We say that the atom P occurs *strictly positively* in $\bot$; and in P; and in $A \wedge B$ iff in one of A and B; in $A \vee B$ iff in both A and B; and in $A \to B$ iff in B. In brief, P *sp-occurs* in the formula. The following is based on a result in [67, p. 69].

Theorem 1 *If* $\Gamma \Longrightarrow P$ *in* **G4ip**, *then there is some formula in* Γ *in which* P *sp-occurs.*

Proof By induction on the derivation and case analysis:

1. The last step has $\bot$ (resp. P) principal; then $\bot \in \Gamma$ (resp. $P \in \Gamma$) and P sp-occurs therein.
2. The last step has $A \wedge B$ principal and derivable premiss $A, B, \Gamma' \Longrightarrow P$; by the induction hypothesis we can find a suitable formula either in Γ' or in $\{A, B\}$. In the latter case, P sp-occurs in $A \wedge B$.
3. The last step has $A \vee B$ principal and derivable premisses $A, \Gamma' \Longrightarrow P$ and $B, \Gamma' \Longrightarrow P$; by the induction hypothesis, we can find a suitable formula either in Γ' or in both A and B. In the latter case, P sp-occurs in both A and B (and hence in $A \vee B$).
4. The remaining cases are similar. □

Thus, the sequent

$$(p \to s) \to t, (c \to p) \to b \Longrightarrow p$$

cannot be reduced (but would be reduced if we had $p = t$). Reduction of a sequent to two new sequents using (root-first) the rule $L{\to}{\to}$ generates premisses of which the first may not be derivable even if the conclusion is derivable; the Theorem can be used here to prune the search space. One can see this as a weak form of "goal-directedness".

One would like to strengthen this to the claim that the formula given by the theorem can be taken as principal, thus (in general) allowing many possible choices of principal formula to be ignored. A counterexample is given by the sequent $q \vee r, q \to p, r \to p \Longrightarrow p$. A counter-example with only atomic and nested implications in the antecedent is $(a \to a) \to b, b \to (q \vee r), q \to p, r \to p \Longrightarrow p$; the only derivation of this in **G4ip** ends with $(a \to a) \to b$ principal—but p does not sp-occur therein.

Weich [73] presents and justifies a much stronger form of goal-directedness.

7 Mints' Classification

Mints [51] gave a convenient classification of subclasses of **Int**, and their complexity. Let $|S|$ be the formula equivalent of a sequent S. By introduction of new variables (following Skolem 1920 and Wajsberg 1938), one can in linear time replace a formula A by a sequent S_A so that A is provable iff $|S_A|$ is provable, where the succedent of S_A is atomic and the antecedent consists of formulae that (with P, Q, R atomic) are one of

(0) **negated atoms** $\neg P$,
(1) **atoms** P,
(2) **implications** $P \to Q$,
(3) **binary implications** $P \to (Q \to R)$,
(4) **nested implications** $(P \to Q) \to R$,

(5) **implied disjunctions** $P \to (Q \vee R)$,
(6) **negative implications** $P \to (\neg Q)$,
(7) **converse negative implications** $\neg Q \to P$.

Thus, it suffices to consider only sequents where the antecedent X consists of formulae of these eight types (and the succedent is atomic).

"Simplification" allows us to remove all formulae of type 0 or 1. For consistency with Mints' paper we avoid this step.

According to the types of formulae used in X, one has complexity results: if all formulae of X are of type 2, 3 or 4 we talk of the class [2,3,4], and similarly for other classes.

One then has that

- The class [2,3,4] (and any superclass) is PSPACE-complete,
- the class [1,2,5,6] is NP-complete (and any superclass is NP-hard),
- the class [0,1,2,3,6] (and any subclass) is in LIN,
- the class [0,1,2,4,5,7] (and any subclass) is in P,
- ….

Note that the class [0,1,2,3,5,6] is the zero-order case of "coherent logic", recently the subject of theoretical study and automation [6].

From the perspective of **G4ip**, the difficulty of proof search is dealing with "nested implications", i.e. formulae of type (4) and their variant (7). So the surprise is that (provided we exclude formulae of type (3) and their variant (6)) while allowing formulae of type (4) and their variant (7), the decision problem is in P. This is achieved using a resolution method [50], a variant of the familiar "forward chaining" method that disposes linearly of [0,1,2,3,6]. Tammet [66] implemented this method. But the verdict [56] by the ILTP website authors is "Prover seems to be incorrect".

8 Ensuring the Subformula Property

G4ip lacks the subformula property, and has been criticised by some for this failing, apparently on philosophical grounds.

Despite a strong feeling that it doesn't matter (because it is still analytic in a weak but adequate sense), we consider henceforth some further approaches that ensure that proofs have the subformula property:

1. Underwood's calculus.
2. Intercalation calculus of Sieg and Cittadini.
3. Implication-locking (Franzén).
4. Loop-checking (two approaches).
5. The calculus **LJPm*** of Mints.
6. The calculi **IG**^{r} and **SIC** of Corsi and Tassi.
7. The calculus **LSJ** of Ferrari, Fiorentino and Fiorino.
8. The calculus **GLJ** of RD (unpublished).

8.1 Underwood's Calculus

Underwood [69] gave a constructive completeness proof for a calculus presented rather in terms of Kripke semantics than proof theory. As reconstructed by Weich [73], this is as follows, with antecedents and succedents regarded as sets: rules for conjunction and disjunction are rather standard, with provisos about not being used if (used root-first) they fail to add a new formula to one of the sets. Rules for implication are thus:

$$\frac{A{\to}B, \Gamma \Longrightarrow A, \Delta \quad B, A{\to}B, \Gamma \Longrightarrow \Delta}{A{\to}B, \Gamma \Longrightarrow \Delta}\; L{\to}\;\; (A \notin \Delta;\; B \notin \Gamma)$$

$$\frac{A, \Gamma \Longrightarrow B, A{\to}B, \Delta}{A, \Gamma \Longrightarrow A{\to}B, \Delta}\; \mathit{RSimp}\; (B \notin \Delta) \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A{\to}B, \Delta}\; R{\to}\;\; (A \notin \Gamma)$$

Branches are bounded in length by the square of the number of the end-sequent's subformulae, hence termination without loop-checking. This is the basis for the extraction of an algorithm by Caldwell [8]. (An early version was in use by Constable in NuPRL about 1991.)

8.2 Intercalation Calculus of Sieg and Cittadini

Sieg and Cittadini [64], building on earlier work for classical logic by Byrnes, present a system (the "intercalation calculus") geared towards use in a pedagogical system *AProS*, in use at Carnegie Mellon University and elsewhere; for such a system, lack of the subformula property would be confusing. Thanks to this property, the search space is finite; a loop-checker is required. *Questions* (i.e. sequents) are of the form $\alpha;\ \beta ?G$, where G is the "goal formula" and α and β are sets of formulae, the former being assumptions and the latter the formulae derived from assumptions. As examples of the rules, we present those for implication in more traditional notation:

$$\frac{\alpha;\ \beta \Longrightarrow A \quad \alpha;\ \beta, B \Longrightarrow G}{\alpha;\ \beta \Longrightarrow G}\; L{\to} \quad \mathit{where}\ A{\to}B \in \alpha \cup \beta,\ B \notin \alpha \cup \beta,\ A \neq G$$

$$\frac{A, \alpha;\ \beta \Longrightarrow B}{\alpha;\ \beta \Longrightarrow A{\to}B}\; R{\to}$$

The objective here, of course, is the finding of normal natural deduction proofs rather than being an efficient decision procedure. Herbelin's calculus **LJT** [36] would be another approach to this objective, allowing (in principle) the discovery or enumeration of all such proofs.

8.3 Implication-Locking (Franzén's approach)

Franzén [29, 60] uses the notion of *covering*: Γ *covers* A if

- $A \in \Gamma$, or
- $A \equiv B \wedge C$ and Γ covers **both** B and C, or
- $A \equiv B \vee C$ and Γ covers **one of** B and C, or
- $A \equiv B \to C$ and Γ covers C.

The rule $R\!\to$ is then specialised to the two cases: the usual one (a *transfer instance*) if Γ does not cover the antecedent A of the principal formula, and the special one (infer $\Gamma \Longrightarrow A \to B$ from $\Gamma \Longrightarrow B$ when Γ covers A) (similar, respectively, to Underwood's $R\!\to$ and *RSimp*). There is then the restriction that, on each branch, every two instances of $L\!\to$ must be separated by a transfer instance of $R\!\to$. In other words, implications are "locked" until "released" by a transfer. This is enough to ensure termination.

8.4 Loop-Checking (The Bern Approach)

For simplicity, we ignore disjunction and absurdity. We may therefore restrict $L\!\to$ to cases where the succedent formula is an atom. Left rules are *cumulative*, i.e. the principal formula is duplicated to the premiss. So a loop can only occur during a phase when nothing new is added to the antecedent, and in the succedent a formula appears and later (i.e. higher up the proof branch) appears again. Without loss of generality, one can restrict to the case where this formula is an atom. Sequents now contain an extra component, the *history* H (the set of such atoms; quite simple). Then, if as one moves root-first one uses the left rule for implication with conclusion having atomic succedent P, this use is *blocked* if already $P \in H$, but otherwise P is added to the history. If a new formula is added to the antecedent, the history at the premiss is emptied. See [37] (by Heuerding et al. 1996) for details.

8.5 Loop-Checking (The St Andrews Approach)

Howe [38] presented a variation of the Bern approach. Sequents again contain an extra component, the *history*. Loops are found earlier at the cost of some extra data storage. In some cases this dramatically cuts the search time, but in general makes it slightly slower.

8.6 System LJpm* of Mints

Mints' inference rules [52] operate on *tableaux*, i.e. lists $\mathcal{T}$ of multi-succedent sequents (the *components* of $\mathcal{T}$). We use ";" for the "append" operation on lists, where [52] uses a "$\star$"; and, for emphasis, we parenthesise components. A tableau is *initial* iff **one** of its components is an initial sequent. A *proof* is a tree, each leaf of which is an initial tableau. Use of (Mints') tableaux rather than just of sequents avoids backtracking at the meta-level: all the inference rules are invertible.

$$\frac{\mathcal{T};\ (A, A \vee B, \Gamma \Longrightarrow \Delta);\ \mathcal{T}' \qquad \mathcal{T};\ (B, A \vee B, \Gamma \Longrightarrow \Delta);\ \mathcal{T}'}{\mathcal{T};\ (A \vee B, \Gamma \Longrightarrow \Delta);\ \mathcal{T}'}\ L\vee$$

$$\frac{\mathcal{T};\ (\Gamma \Longrightarrow \Delta, A \vee B, A, B);\ \mathcal{T}'}{\mathcal{T};\ (\Gamma \Longrightarrow \Delta, A \vee B);\ \mathcal{T}'}\ R\vee$$

Conjunctive branching (as in $L\vee$) replaces one tableau by two, whereas disjunctive branching (**not yet illustrated**) adds components to a tableau. [Not the same usage as Fitting.] Here are the rules for implication:

$$\frac{\mathcal{T};\ (A \to B, \Gamma \Longrightarrow \Delta, A);\ \mathcal{T}' \qquad \mathcal{T}; (B, A \to B, \Gamma \Longrightarrow \Delta); \mathcal{T}'}{\mathcal{T};\ (A \to B, \Gamma \Longrightarrow \Delta);\ \mathcal{T}'}\ L\to$$

$$\frac{\mathcal{T};\ \ (\Gamma \Longrightarrow \Delta, A \to B);\ \ (A, \Gamma \Longrightarrow B);\ \ \mathcal{T}'}{\mathcal{T};\ (\Gamma \Longrightarrow \Delta, A \to B);\ \mathcal{T}'}\ R\to$$

in which note the conjunctive branching in the first (i.e. the rule has two premisses) and the disjunctive branching (by addition of the new component $(A, \Gamma \Longrightarrow B)$ to the tableau) in the second. Note that the principal formula is always, except in $R\to$, duplicated into the premisses. $\Gamma \Longrightarrow \Delta$ *subsumes* $\Gamma' \Longrightarrow \Delta'$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ (as sets of formulae); then one forbids any tableau extension step if some new sequent subsumes some component of some tableau lower down the tree, i.e. loops must be detected (might be costly) and avoided. A finiteness argument then shows that this ensures termination.

8.7 System IG^r of Corsi and Tassi

We present (in our own 2-D style) the system **IG**^{r} of Corsi and Tassi [10] (implicational part: the other parts present no difficulties). Its main features are (a) that it is depth-bounded (b) that it has the subformula property and (c) bicompleteness. *Initial sequents* are, as usual, those with an atom on both left and right.) The superfix r stands for a regularity condition, enforced by the use of $\mathcal{B}$ and $\mathcal{H}$.

$\mathcal{B}$ is for the *blocked* formulae, i.e. the set of all not yet unblocked formulae on the path from here to root that have been principal for $L\neg$ or $L\to$. Until they are unblocked, they cannot be principal for these rules again. Blocking is similar to Franzén's notion of locking. $\mathcal{B}$ is *cleared* (i.e. set to {}) whenever (as one proceeds up such a path) there is a use of $R\to$ or $R\neg$.

$\mathcal{H}$ is for a *History*, i.e. the set of all formulae on the path from here to root that are principal for $R\neg$ or $R\to$. $\mathcal{H}$ is never cleared.

Implicit in the following is that we can write $\neg A, \mathcal{B}$ iff $\neg A \notin \mathcal{B}$, etc. Where we cannot write it, the rule cannot be used.

$$\dfrac{\neg A, \Gamma \xRightarrow[\mathcal{H}]{\neg A, \mathcal{B}} A, \Delta}{\neg A, \Gamma \xRightarrow[\mathcal{H}]{\mathcal{B}} \Delta}\, L\neg \qquad \dfrac{A \to B, \Gamma \xRightarrow[\mathcal{H}]{A\to B, \mathcal{B}} A, \Delta \qquad B, \Gamma \xRightarrow[\mathcal{H}]{A\to B, \mathcal{B}} \Delta}{A \to B, \Gamma \xRightarrow[\mathcal{H}]{\mathcal{B}} \Delta}\, L\to$$

$$\dfrac{A, \Gamma \xRightarrow[\neg A, \mathcal{H}]{\{\}}}{\Gamma \xRightarrow[\mathcal{H}]{\mathcal{B}} \neg A, \Delta}\, R\neg \qquad \dfrac{A, \Gamma \xRightarrow[A\to B, \mathcal{H}]{\{\}} B}{\Gamma \xRightarrow[\mathcal{H}]{\mathcal{B}} A \to B, \Delta}\, R\to$$

$$\dfrac{\Gamma \xRightarrow[\neg A, \mathcal{H}]{\mathcal{B}} \Delta}{\Gamma \xRightarrow[\neg A, \mathcal{H}]{\mathcal{B}} \neg A, \Delta}\, AF\neg \qquad \dfrac{\Gamma \xRightarrow[A\to B, \mathcal{H}]{\mathcal{B}} B, \Delta}{\Gamma \xRightarrow[A\to B, \mathcal{H}]{\mathcal{B}} A \to B, \Delta}\, AF\to$$

The rules $AF\neg$ and $AF\to$ implement what the authors call an *A Fortiori* condition, seen most clearly in the second of these two rules.

Here is a proof in Gentzen's tree-style of the sequent corresponding to the formula that is the type of the *S* combinator:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, p\to q, p\to q\to r \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{p\to q} p, r}\, Ax \quad \dfrac{\dfrac{}{q, p, p\to q\to r \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{p\to q\to r, p\to q} p, r}\, Ax \quad \dfrac{\dfrac{}{q\to r, q, p \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{q\to r, p\to q\to r, p\to q} q, r}\, Ax \quad \dfrac{}{r, q, p \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{q\to r, p\to q\to r, p\to q} r}\, Ax}{q\to r, q, p \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{p\to q\to r, p\to q} r}\, L\to}{q, p, p\to q\to r \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{p\to q} r}\, L\to}{p, p\to q, p\to q\to r \xRightarrow[p\to r,(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{\{\}} r}\, L\to}{p\to q, p\to q\to r \xRightarrow[(p\to q)\to p\to r,(p\to q\to r)\to(p\to q)\to p\to r]{\{\}} p\to r}\, R\to}{p\to q\to r \xRightarrow[(p\to q\to r)\to(p\to q)\to p\to r]{\{\}} (p\to q)\to p\to r}\, R\to}{\xRightarrow[\{\}]{\{\}} (p\to q\to r)\to(p\to q)\to p\to r}\, R\to$$

with shape

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1} \qquad \dfrac{\dfrac{}{2} \qquad \dfrac{\dfrac{}{3} \quad \dfrac{}{4}}{5}\, L\to}{6}\, L\to}{7}\, L\to}{\dfrac{8}{9}\, R\to}\, R\to}{10}\, R\to$$

A proof in the **G4ip** calculus looks (in tree form) much the same but with fewer formulae at each node and without branching, using the rule $L0\!\!\to$ rather than $L\!\!\to$. (Here is the $\mathbf{IG}^r$ proof again in linear style, using a one-dimensional layout $\mathcal{B}; \mathcal{H} : \Gamma \Longrightarrow \Delta$ for each sequent)

$$[p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [p, p \to q, p \to q \to r] \Longrightarrow [p, r] \tag{1}$$

$$[p \to q \to r, p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [q, p, p \to q \to r] \Longrightarrow [p, r] \tag{2}$$

$$[q \to r, p \to q \to r, p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [q \to r, q, p] \Longrightarrow [q, r] \tag{3}$$

$$[q \to r, p \to q \to r, p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [r, q, p] \Longrightarrow [r] \tag{4}$$

$$[p \to q \to r, p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [q \to r, q, p] \Longrightarrow [r] \quad \text{By}L\to \ \text{from } 3, 4 \tag{5}$$

$$[p \to q]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [q, p, p \to q \to r] \Longrightarrow [r] \quad \text{By}L\to \ \text{from } 2, 5 \tag{6}$$

$$[]; [p \to r, (p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [p, p \to q, p \to q \to r] \Longrightarrow [r] \quad \text{By}L\to \ \text{from } 1, 6 \tag{7}$$

$$[]; [(p \to q) \to p \to r, (p \to q \to r) \to (p \to q) \to p \to r] : [p \to q, p \to q \to r] \Longrightarrow [p \to r] \quad \text{By}R\to \ \text{from } 7 \tag{8}$$

$$[]; [(p \to q \to r) \to (p \to q) \to p \to r] : [p \to q \to r] \Longrightarrow [(p \to q) \to p \to r] \quad \text{By}R\to \ \text{from } 8 \tag{9}$$

$$[]; [] : [] \Longrightarrow [(p \to q \to r) \to (p \to q) \to p \to r] \quad \text{By}R\to \ \text{from } 9 \tag{10}$$

The same calculus was rediscovered in 2013–2014 by Goré et al., renamed “IntHistGC” and implemented with several optimisations such as caching and backjumping [34].

8.8 *The Calculus* SIC *of Corsi and Tassi*

SIC is a variant of the system $\mathbf{IG}^r$ in the same paper [10]; the essential difference is that backtracking (because of disjunctive branching) is incorporated into the calculus, and thus each node of the tree is a stack of ordinary sequents rather than just one such sequent. This is very similar to Mints’ notion of tableau. Sequents are pushed onto the stack to indicate all the alternative possibilities (according to the different implicational succedent formulae); as they are tried and found unsolvable, they are popped, and failure occurs when the stack is empty. The goal is thus achieved (as reflected in the paper’s title, “Intuitionistic logic freed of all metarules”) that all use of “global metarules” is thus replaced by use of “local metarules”, incorporated into the rules of the calculus.

8.9 *The Calculus* LSJ *of Ferrari, Fiorentini and Fiorino*

Sequents are [25] of the form $\Gamma \overset{\Theta}{\Rightarrow} \Delta$, the components Γ, Θ, Δ being sets (of formulae) rather than multisets. Let us use $<$ for $\leq$ without equality.

The semantics (using only finite models) of such a sequent is that $(K, V, w) \Vdash \Gamma \overset{\Theta}{\Rightarrow} \Delta$ iff, whenever both

1. for every $H \in \Theta$ and $w' \in K$ with $w < w'$, one has $(K, V, w') \Vdash H$,
2. for every $G \in \Gamma$, one has $(K, V, w) \Vdash G$,

then for some $D \in \Delta$ one has $(K, V, w) \Vdash D$. Negation is defined as usual. We omit the rules for disjunction, dual to those for conjunction. The rules are

$$\frac{}{\bot, \Gamma \overset{\Theta}{\Rightarrow} \Delta}\ L\bot \qquad \frac{}{A, \Gamma \overset{\Theta}{\Rightarrow} \Delta, A}\ Id$$

$$\frac{A, B, \Gamma \overset{\Theta}{\Rightarrow} \Delta}{A \wedge B, \Gamma \overset{\Theta}{\Rightarrow} \Delta}\ L\wedge \qquad \frac{\Gamma \overset{\Theta}{\Rightarrow} \Delta, A \quad \Gamma \overset{\Theta}{\Rightarrow} \Delta, B}{\Gamma \overset{\Theta}{\Rightarrow} \Delta, A \wedge B}\ R\wedge$$

$$\frac{B, \Gamma \overset{\Theta}{\Rightarrow} \Delta \quad \Gamma \overset{B,\Theta}{\Longrightarrow} \Delta, A \quad \Theta, \Gamma \overset{B}{\Rightarrow} A}{A \to B, \Gamma \overset{\Theta}{\Rightarrow} \Delta}\ L\to \qquad \frac{A, \Gamma \overset{\Theta}{\Rightarrow} \Delta, B \quad A, \Theta, \Gamma \overset{\{\}}{\Rightarrow} B}{\Gamma \overset{\Theta}{\Rightarrow} \Delta, A \to B}\ R\to$$

A syntactic proof of cut-admissibility for this calculus seems difficult; a semantic proof is in [25]. Using our own implementation of **LSJ**, with Prolog cuts to prune the search space wherever seemed appropriate, the first (indeed, only) proof we found of the formula that is the type of the *S* combinator is 87 lines long. It is possible that, with differently placed cuts in the implementation, a shorter proof would be found. An associated calculus, building on the approach of [57], gives bicompleteness.

8.10 The Calculus **GLJ**

We recall Sambin and Valentini's system **GLS'** from [61] for the classical provability logic **GL** (implicational and modal part: the other parts are standard). Antecedents and succedents are sets; so Γ, A and Γ, Δ stand for the unions $\{A\} \cup \Gamma$ and $\Gamma \cup \Delta$ (and similarly for A, Γ). When we see $A \supset B, \Gamma$ in the conclusion of a schematic rule, it is implicit that $A \supset B \notin \Gamma$; similarly with Δ rather than Γ, and for other connectives. Actual proofs may for clarity be written with apparent repetitions. All rules (except *RR*) are invertible—[61] calls this *doubly sound*. *Initial sequents* are, as is almost usual, those with a formula common to both left and right (or with $\bot$ on the left). The rules (in each of which Γ and Δ are disjoint, and with Π and Σ disjoint sets of atoms) are:

$$\frac{\Gamma \Longrightarrow \Delta, A \quad B, \Gamma \Longrightarrow \Delta}{A \supset B, \Gamma \Longrightarrow \Delta}\ L\supset \frac{A, \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \supset B}\ R\supset \frac{\Gamma, \Box\Gamma, \Box D \Longrightarrow D}{\Pi, \Box\Gamma \Longrightarrow \Box\Delta, \Box D, \Sigma}\ RR$$

The rule *RR* has the property that if the conclusion is valid then, for some choice of a boxed formula in its succedent, the corresponding premiss is valid. ([61] also calls this *doubly sound*.)

Root-first proof search in **GLS'** terminates. The argument (from [61]) is as follows. First, used root-first, every rule other than *RR* reduces the number of connectives. Second, as we proceed up a branch, the set of boxed formulae in the antecedent occasionally expands but never shrinks: thus, if a sequent $\Pi, \Box\Gamma \Longrightarrow \Box\Delta, \Sigma$ is a conclusion of *RR*, the antecedent of every sequent above it will contain a formula $\Box D$, with $D \in \Delta$, and the antecedent of every sequent at or below it cannot contain such a formula (since search is required to stop at initial sequents). So all the sequents in a branch are different. By the subformula property their number is finite, so search along any branch terminates.

Consider the standard embedding $\cdot^{\boxdot}$ of **Int** into **GL**, in which notice the distinctions between *classical* and *intuitionistic* implication, $A \supset B$ and $A \to B$, and between *classical* and *intuitionistic* negation, $\sim A$ and $\neg A$:

$$\begin{array}{rcl rcl}
\bot^{\boxdot} & := & \bot & P^{\boxdot} & := & P \wedge \Box P \\
(A \wedge B)^{\boxdot} & := & A^{\boxdot} \wedge B^{\boxdot} & (A \vee B)^{\boxdot} & := & A^{\boxdot} \vee B^{\boxdot} \\
(\neg A)^{\boxdot} & := & \sim A^{\boxdot} \wedge \Box(\sim A^{\boxdot}) & (A \to B)^{\boxdot} & := & (A^{\boxdot} \supset B^{\boxdot}) \wedge \Box(A^{\boxdot} \supset B^{\boxdot})
\end{array}$$

The interpretations of the intuitionistic implication rules are then

$$\frac{\Gamma^{\boxdot}, (A \to B)^{\boxdot} \Longrightarrow A^{\boxdot}, \Delta^{\boxdot} \quad \Gamma^{\boxdot}, B^{\boxdot} \Longrightarrow \Delta^{\boxdot}}{\Gamma^{\boxdot}, (A \to B)^{\boxdot} \Longrightarrow \Delta^{\boxdot}}\ L\!\to^{\boxdot} \qquad \frac{\Gamma^{\boxdot}, A^{\boxdot} \Longrightarrow B^{\boxdot}}{\Gamma^{\boxdot} \Longrightarrow (A \to B)^{\boxdot}, \Delta^{\boxdot}}\ R\!\to^{\boxdot}$$

and these need to be justified as sound rules in **GLS’** (in which *Weakening* is known to be admissible). This is routine.

We now specialise the rules of **GLS’** and show them in the language of **Int**, giving us a novel calculus **GLJ**. In the following, P ranges over atoms, Π and Σ are sets of atoms; Θ and Ψ are sets of either atoms, classical negations or classical mplications, implicitly treated as boxed; Γ and Δ are arbitrary sets of formulae—and all are just formulae of **Int**, apart from the classical negations and implications (in Θ and Ψ).

Sequents are now of the form $\Pi;\Theta;\Gamma \Longrightarrow \Delta;\Psi;\Sigma$. Provability of a formula A will match the derivability of the sequent $[]; []; [] \Longrightarrow A; []; []$. We use the classical notation $A \supset B$ for implications after being moved by rule $L\!\to$ from Γ to Θ (or by rule $R\!\to$ from Δ to Ψ), so that, when moved back to Γ by variants of RR, they are correctly analysed (by $L\supset$).

$$\frac{}{\boxed{P}, \Pi;\Theta;\Gamma \Longrightarrow \Delta;\Psi;\Sigma, \boxed{P}}\ Ax1 \qquad \frac{}{\Pi;\boxed{A}, \Theta;\Gamma \Longrightarrow \Delta;\Psi, \boxed{A};\Sigma}\ Ax2$$

$$\frac{}{\Pi;\Theta;\boxed{\bot}, \Gamma \Longrightarrow \Delta;\Psi;\Sigma}\ L\bot \qquad \frac{\Pi;\Theta;\Gamma \Longrightarrow \Delta;\Psi;\Sigma}{\Pi;\Theta;\Gamma \Longrightarrow \Delta, \boxed{\bot};\Psi;\Sigma}\ R\bot$$

$$\frac{P, \Pi;P, \Theta;\Gamma \Longrightarrow \Delta;\Psi;\Sigma}{\Pi;\Theta;\boxed{P}, \Gamma \Longrightarrow \Delta;\Psi;\Sigma}\ LAt$$

$$\frac{\Pi;\Theta;\Gamma \Longrightarrow \Delta;\Psi;P, \Sigma \quad \Pi;\Theta;\Gamma \Longrightarrow \Delta;P, \Psi;\Sigma}{\Pi;\Theta;\Gamma \Longrightarrow \boxed{P}, \Delta;\Psi;\Sigma}\ RAt$$

$$\frac{\Pi;\Theta;A, B, \Gamma \Longrightarrow \Delta;\Psi;\Sigma}{\Pi;\Theta;\boxed{A \wedge B}, \Gamma \Longrightarrow \Delta;\Psi;\Sigma}\ L\wedge$$

$$\frac{\Pi;\Theta;\Gamma \Longrightarrow \Delta, A;\Psi;\Sigma \quad \Pi;\Theta;\Gamma \Longrightarrow \Delta, B;\Psi;\Sigma}{\Pi;\Theta;\Gamma \Longrightarrow \Delta, \boxed{A \wedge B};\Psi;\Sigma}\ R\wedge$$

Rules for $\vee$ are dual to those for $\wedge$, so need not be shown here.

$$\frac{\Pi; A \supset B, \Theta; \Gamma \Longrightarrow \Delta, A; \Psi; \Sigma \quad \Pi; A \supset B, \Theta; B, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}{\Pi; \Theta; \boxed{A \to B}, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}\ L{\to}$$

$$\frac{\Pi; \Theta; \Gamma \Longrightarrow \Delta, A; \Psi; \Sigma \quad \Pi; \Theta; B, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}{\Pi; \Theta; \boxed{A \supset B}, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}\ L{\supset}$$

$$\frac{\Pi; \Theta; A, \Gamma \Longrightarrow \Delta, B; \Psi; \Sigma \quad \Pi; \Theta; \Gamma \Longrightarrow \Delta; \Psi, A \supset B; \Sigma}{\Pi; \Theta; \Gamma \Longrightarrow \Delta, \boxed{A \to B}; \Psi; \Sigma}\ R{\to}$$

$$\frac{\Pi; {\sim}A, \Theta; \Gamma \Longrightarrow \Delta, A; \Psi; \Sigma}{\Pi; \Theta; \boxed{\neg A}, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}\ L{\neg} \qquad \frac{\Pi; \Theta; \Gamma \Longrightarrow \Delta, A; \Psi; \Sigma}{\Pi; \Theta; \boxed{{\sim}A}, \Gamma \Longrightarrow \Delta; \Psi; \Sigma}\ L{\sim}$$

$$\frac{\Pi; \Theta; A, \Gamma \Longrightarrow \Delta; \Psi; \Sigma \quad \Pi; \Theta; \Gamma \Longrightarrow \Delta; \Psi, {\sim}A; \Sigma}{\Pi; \Theta; \Gamma \Longrightarrow \Delta, \boxed{\neg A}; \Psi; \Sigma}\ R{\neg}$$

Finally, we have rules corresponding to the *RR* rule of **GLS'**. There are three of these, since the formulae in Ψ can be either atoms, classical implications or classical negations. These are the only rules that are not invertible.

$$\frac{[]; A \supset B, \Theta; A, \Theta \Longrightarrow B; []; []}{\Pi; \Theta; [] \Longrightarrow []; \Psi, \boxed{A \supset B}; \Sigma}\ RR{\supset} \frac{[]; {\sim}A, \Theta; A, \Theta \Longrightarrow []; []; []}{\Pi; \Theta; [] \Longrightarrow []; \Psi, \boxed{{\sim}A}; \Sigma}\ RR{\sim} \frac{\Pi; P, \Theta; \Theta \Longrightarrow []; []; P}{\Pi; \Theta; [] \Longrightarrow []; \Psi, \boxed{P}; \Sigma}\ RRAt$$

In these rules $RR{\supset}$, $RR{\sim}$ and $RRAt$, it is required that Π and Σ are disjoint and Θ is disjoint from (respectively) $\Psi, A \supset B$, from Ψ, ${\sim}A$ and from Ψ, P.

This calculus **GLJ** doesn't quite have the subformula property: for example, $L{\to}$ and $R{\to}$ turn intuitionistic implications into classical implications. To obtain it, we can either decree that $A \supset B$ is a subformula of $A \to B$ and that ${\sim}A$ is a subformula of $\neg A$, or adjust the calculus slightly (at the expense of some extra search). But it does have the termination property, by an extension of the argument for **GLS'** above. Countermodel construction from failed searches seems to be routine. But, the first proof found (of the type of the *S* combinator) has 5,185 lines. **GLJ** is thus presented as an example, which could have been written down at any time after 1982, of a complete calculus with an easy termination argument—but not as efficient as **G4ip** or **m-G4ip**.

9 Labelled Calculi

Many authors (Castellini, Catach, Fitting, Gabbay, Kanger, Maslov, Negri, Russo, Schmidt, Simpson, Tishkovsky, Vigano, …) have exploited labels (aka "prefixes") in sequent calculi (or tableau calculi), one motivation being to make the inference rules invertible (and another being to allow uniform development of analytic calculi from frame conditions rather than from axioms). Some have criticised this as a lack of syntactic purity, i.e. as the presence of "semantic pollution"; others defend it as allowing calculi for otherwise unmanageable logics. Read [59] mounts a strong defence. Goré has a useful survey [32] in the context of modal logics. Using labelled tableaux, Schmidt and Tishkovsky have implemented a generic tableau calculus generator [62], geared rather towards description logics; this can generate a JAVA-based prover, or could be combined with a tableau-based theorem prover such as LOTREC [46] or the Tableaux Work Bench [1].

For **Int**, and using sequent calculus notation rather than tableaux, one statement of the method is by Dyckhoff and Negri [20]. This covers a wide range of intermediate logics—all those where the first-order frame conditions in Kripke semantics can be presented as geometric (aka coherent) implications, i.e. (in fact) all that can be presented semantically using first-order formulae, since every first-order theory has a coherent conservative extension [21, 22]. This approach solves the problem of backtracking; but termination is a problem, with various approaches, including the "unrestricted blocking" rule of [62] and another method in [30, 53].

9.1 Calculus G3i

The calculus just mentioned (by Dyckhoff and Negri [20]) is as follows (rules for $\vee$ are omitted, being dual to those for $\wedge$):

$$\frac{}{\boxed{x : \bot}, \Gamma \Longrightarrow \Delta}\,L\bot \qquad \frac{}{\boxed{x \leq y, x : P}, \Gamma \Longrightarrow \Delta, \boxed{y : P}}\,Ax$$

$$\frac{x : A, x : B, \Gamma \Longrightarrow \Delta}{\boxed{x : A \wedge B}, \Gamma \Longrightarrow \Delta}\,L\wedge \qquad \frac{\Gamma \Longrightarrow \Delta, x : A \quad \Gamma \Longrightarrow \Delta, x : B}{\Gamma \Longrightarrow \Delta, \boxed{x : A \wedge B}}\,R\wedge$$

$$\frac{x \leq y, x : A \rightarrow B, \Gamma \Longrightarrow \Delta, y : A \quad x \leq y, x : A \rightarrow B, y : B, \Gamma \Longrightarrow \Delta}{\boxed{x \leq y, x : A \rightarrow B}, \Gamma \Longrightarrow \Delta}\,L\rightarrow$$

$$\frac{x \leq y, y : A, \Gamma \Longrightarrow \Delta, y : B}{\Gamma \Longrightarrow \Delta, \boxed{x : A \rightarrow B}}\,R\rightarrow$$

$$\frac{x \leq x, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta}\,Ref \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Longrightarrow \Delta}{\boxed{x \leq y, y \leq z}, \Gamma \Longrightarrow \Delta}\,Trans$$

with y fresh in $R\rightarrow$, i.e. not occurring in the conclusion.

Derivations can be restricted to those in which the label x used in the *Ref* rule already occurs in the conclusion.

This calculus does not terminate (e.g. on Peirce's formula).

Negri [53] shows how to add a loop-checking mechanism to ensure termination and build finite counter-models directly from a failed proof search. The effect on complexity isn't clear; the loop-checking is expensive. Further details of related methods are in Garg et al. [30]; there is also related work by Schmidt et al. [62] and by Antonsen and Waaler [2].

10 Focused Calculi

Naive implementations of the calculi mentioned above spend a great deal of time looking along lists to find a formula of a certain form.

A better approach is to take the next formula and either analyse it (i.e. generate appropriate subproblems) or put it aside in a suitable place for later use. For example, atomic formulae can be examined (to see if the branch closes) or (if that fails) put into a list of atoms; and succedent conjunctions can be put aside until all non-branching rules have been dealt with. This can be regarded as a naive form of *focusing*. So can, to some extent, the calculus **G4ip**, with its connections to the focused calculus **LJQ**.

But several authors, notably McLaughlin and Pfenning [49], have more logic-based approaches. For lack of space, we omit their presentation.

11 Challenges and Open Problems

1. Find a simple calculus for **Int** that (a) has the termination property (ideally, with linear depth) and (b) avoids backtracking through rules, but without implementing the usual meta-level "list of disjunctive goals to be tried one after another". This can be done for classical logic and for Gödel-Dummett logic [19]. Is there a fundamental complexity result (yet to be discovered) that forbids this? Note that linear temporal logic is PSPACE-hard but has a terminating calculus [7] with all rules invertible. (Termination here depends on a form of history mechanism.) Is there a combination of the **G4ip** ideas and labelling that solves this problem?
2. Find, develop and simplify uniform methods for ensuring termination in labelled calculi.
3. Find syntactic (i.e. non-semantic) methods for proving cut admissibility for calculi with sequents with several components, e.g. **LSJ** and **GLJ**.
4. Is there a calculus that combines the good features of **G4ip** (where it is the nested implications—formulae of type (4) and their variant (7)—that are problematic) and Mints' resolution method (where these are less of a problem: his class [0,1,2,4,5,7] is in P). Or do we get, not the good, but the bad features of both?
5. Develop more proofs of correctness and completeness using proof assistants like *NuPRL*, *Coq* and *Agda*, extending work of Underwood [69], Caldwell [8], Weich [73, 74] and allowing extraction of verified software in (e.g.) Haskell, Scheme or

OCaml. There is some recent work (unpublished) by Larchey-Wendling on **LSJ** (and on **G4ip**) in this direction.

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The Operational Perspective: Three Routes

Solomon Feferman

For Gerhard Jäger, in honor of his 60th birthday.

Let me begin with a few personal words of appreciation, since Gerhard Jäger is one of my most valued friends and long time collaborators. It's my pleasure to add my tribute to him for his outstanding achievements and leadership over the years, and most of all for having such a wonderful open spirit and being such a fine person.

I first met Gerhard at the 1978 logic colloquium meeting in Mons, Belgium. He was attending that with Wolfram Pohlers and Wilfried Buchholz, with both of whom I had long enjoyed a stimulating working relationship on theories of iterated inductive definitions. From our casual conversations there, it was clear that Gerhard was already someone with great promise in proof theory. But things really took off between us a year later when we both visited Oxford University for the academic year 1979–1980. Gerhard had just finished his doctoral dissertation with Kurt Schütte and Wolfram Pohlers. I remember that we did a lot of walking and talking together, though I had to walk twice as fast to keep up with him. We talked a lot about proof theory and in particular about my explicit mathematics program that I had introduced in 1975 and had expanded on in my Mons lectures; Gerhard was quick to take up all my questions and to deal with them effectively. Since then, as hardly needs saying, he became a leader in the development of the proof theory of systems of explicit mathematics and related systems in the applicative/operational framework (among many contributions to a number of other areas), and he went on to establish in Bern a world center for studies in these subjects.

It was also through Gerhard that I was able to come in useful contact with a number of his students and members of his group, and most particularly with Thomas Strahm, who then became a second very important collaborator of mine, both on explicit mathematics and the unfolding program, of which I'll say something below. Since

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some time now, Gerhard and Thomas have been working with me on a book on the foundations of explicit mathematics, and in the last few years we have made great progress on that with the assistance of my former student Ulrik Buchholtz, for which I'm very grateful.[1] Finally on the personal front I want to add that my wife, Anita, and I have had the pleasure over the years of visiting Gerhard and his family, first in Zürich, and then in Bern, and we want to thank him and his wife Corinna for their generous, ever-ready welcome and hospitality.

This article consists of four sections, beginning in Sect. 1 with an explanation of the general features of the operational perspective. That is then illustrated in the remaining three sections by the explicit mathematics program, operational set theory, and the unfolding program, resp. The material of this article is by no means exhaustive of the work carried out under the operational perspective; instead, it concentrates on those areas with which I have been personally involved and that I thus know best, but references with a wider scope are given where relevant. I have two readers in mind: the general reader with a background in logic on the one hand and the expert in applicative theories on the other. For the former I have emphasized the aims of the work and filled in its background. For the latter, I have added new points toward the end of each of Sects. 2–4 that I hope will be worthy of attention. In particular, Sect. 2 has material on the development of constructive and predicative mathematics in systems of explicit mathematics, Sect. 3 deals with problems that arose in my development of OST and sketches Gerhard Jäger's solution to them, and Sect. 4 concludes with new conjectures on the unfolding of systems of operational set theory.

1 The Operational Perspective

Operations are ubiquitous in mathematics but not adequately accounted for in current global (or universal) foundational schemes. In particular, the only operations that have a direct explanation in set theory are those represented by functions *qua* many-one relations, so cannot explain operations such as union and power set that are supposed to be applicable to arbitrary sets. The attempt of Church [14, 15] to provide a foundation of mathematics in purely operational terms that would be an alternative to set theory was shown to be inconsistent, and later efforts at similar programs such as that of Fitch [18] have had only very limited success. In any case, one should not expect a "one size fits all" theory of operations; witness the great conceptual variety of computational, algebraic, analytic and logical operations among others. Nevertheless, there is a core theory of operations that can readily be adapted to a number of local purposes by suitable expansions in each case. This has the following features:

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[1] Let me also take this opportunity to thank Thomas Strahm and Thomas Studer for organizing the December 2014 meeting in honor of Gerhard Jäger, and for arranging for me to participate via Skype since I was unable to attend in person.

(i) Operations are in general allowed to be partial. For example, the operations of division in algebra and integration and differentiation in analysis are not everywhere defined.
(ii) Operations may be applied to operations. For example, one has the operation of composition of two operations, the operation of n-times iteration of a given operation, the "do…until…" operation of indefinite iteration, etc.
(iii) In consequence of (ii), a generally adaptable theory of operations is type-free.
(iv) Extensionality is not assumed for operations. For example, the theory should allow the indices of partial recursive functions to appear as one model.
(v) The language of the theory is at least as expressive as the untyped lambda-calculus and the untyped combinatory calculus.
(vi) Though logical operations of various kinds on propositions and predicates may appear in particular applications, first-order classical or intuitionistic predicate logic is taken as given.

These features form the general operational perspective.

In accordance with (i)–(iii), in any particular expansion of the basic theory, we will want some way of introducing *application terms* $s, t, u, \ldots$, generated from variables and constants by closure under application, written $s(t)$ or st, and then to express that a term t is defined, in symbols $t\downarrow$. In the original operational approach that I took in my article [23] on explicit mathematics, the basic relations included a three place relation $\mathrm{App}(x, y, z)$, informally read as expressing that the operation x applied to y has the value z. Then application (pseudo-)terms were introduced contextually, first in (pseudo-)formulas $t \simeq z$ expressing that t is defined and has the value z: for t a variable or constant, this is simply taken to be ordinary equality, while for t of the form $t_1 t_2$, this is taken to be $\exists x, y(t_1 \simeq x \wedge t_2 \simeq y \wedge \mathrm{App}(x, y, z))$. Finally, $t\downarrow$ is defined to be $\exists z(t \simeq z)$ and $t_1 \simeq t_2$ is defined to be $\forall z(t_1 \simeq z \leftrightarrow t_2 \simeq z)$. An elegant alternative to this approach was provided by Beeson in 1981 (cf. [5, Chap. VI.1]) under the rubric, the Logic of Partial Terms (LPT). In LPT, terms are now first class citizens and the expressions $t\downarrow$ are taken to be basic formulas governed by a few simple axioms and rules, among which we have suitable restrictions on universal and existential instantiation. In this system, $t_1 \simeq t_2$ is defined by the formula $t_1\downarrow \vee\, t_2\downarrow \rightarrow t_1 = t_2$. Most of the work after 1981 on the systems within the operational perspective has taken LPT as basic, but we will see in Sect. 3 below that there may still be cases where the original approach is advantageous. Note that LPT contains the "strictness" axioms that if a relation $\mathrm{R}(t_1, \ldots, t_n)$ holds then $t_i\downarrow$ holds for each i; in particular, that is the case for the equality relation. By $t_1 t_2, \ldots, t_n$ we mean the result of successive application by association to the left.

The minimal theory we use has two basic constants **k** and **s** (corresponding to Curry's combinators **K** and **S**) with axioms: (i) $\mathbf{k}xy = y$, and (ii) $\mathbf{s}xy\downarrow\ \rightarrow \mathbf{s}xyz \simeq xz(yz)$. These serve to imply that with any term $t(x, \ldots)$ we may associate a term $\lambda x.t(x, \ldots)$ in which the variable x is not free, and is such that $(\lambda x.t(x, \ldots))y \simeq t(y, \ldots)$. Moreover we can construct a universal recursor or fixed point operator **r** (sometimes denoted **rec**), i.e. a term for which $\mathbf{r}f\downarrow \wedge\ \mathbf{r}fx \simeq f(\mathbf{r}f)x$ is provable. In all the applications of the operational perspective below, axioms (i) and (ii) are further supplemented by suitable axioms for pairing and projection operations **p**, $\mathbf{p}_0$, and $\mathbf{p}_1$, and definition by cases **d**.[2] For the purposes below, let's call these *the basic operational axioms*, whether with respect to the App formulation or that in LPT.

2 Explicit Mathematics

My work with the operational approach began with the explicit mathematics program in [23]. Here is what led me to that. In the years following the characterization [19, 76] of predicative analysis in terms of an autonomous progression of ramified analytic systems whose limit is at the ordinal Γ_0, I had explored various ways to simplify conceptually the formal treatment of predicativity via unramified systems.[3] Moreover, that would be important to see which parts of mathematical practice could be accounted for on predicative grounds going beyond [85]. Independently of that work, in [21] I had made use of extensions of Gödel's functional ("Dialectica") interpretation to determine the proof-theoretical strength of various subsystems of analysis by the adjunction of the unbounded minimum operator as well as the Suslin-Kleene operator. The two pursuits came closer together in the article "Theories of finite type related to mathematical practice" [24] for the *Handbook of Mathematical Logic*. As with Gödel's interpretation, that made use of the functional finite type structure over the natural numbers.

Meanwhile, Errett Bishop's novel informal approach to constructive analysis [7] had made a big impression on me and I was interested in seeing what kind of more or less direct axiomatic foundation could be given for it that would explain how it managed to look so much like classical analysis in practice while admitting a constructive interpretation. Closer inspection showed that this depended on dealing with all kinds of objects (numbers, functions, sets, etc.) needed for analysis as if they are given by explicit presentations, each kind with an appropriate "equality" relation, and that operations on them are conceived to lead from and to such presentations preserving the given equality relations. In other words, the objects are conceived of as given *intensionally*, while a classical reading is obtained by instead working *extensionally* with the equivalence classes with respect to the given equality relations. Another aspect of Bishop's work that was more specific to its success was his systematic use of witnessing data as part of what constitutes a given object, such as modulus of convergence for a real number and modulus of (uniform) continuity for a function of real numbers. Finally, his development did not require restriction to intuitionistic logic (though Bishop himself abjured the Law of Excluded Middle).

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[2] There are several possible formulations of the definition by cases operator. In the one originally taken in [23], sometimes called definition by cases on V, this takes the form $\mathbf{d}xyuv = (x \text{ if } u = v, \text{else } y)$. However, when added to the axioms for **k** and **s**, extensionality is inconsistent for operations. More restrictive versions have subsequently been used, mainly definition by cases on the natural numbers, allowing both extensionality and totality of operations; cf. [59].

[3] First steps in that direction had already been made in [19] via the system IR. For subsequent explorations cf. [20, 22, 26].

Stripped to its core, the ontology of Bishop's work is given by a universe of objects, each conceived to be given explicitly, among which are operations and classes (*qua* classifications). This led to my initial formulation of a system T_0 of explicit mathematics in [23] in which that approach to constructive mathematics could be directly formalized. In addition, I introduced a second system T_1, obtained by the adjunction of the unbounded minimum operator so as to include a foundation of predicative mathematics. The theory T_0 was formulated in a single sorted language with basic relations =, App, Cl, and η where $\mathrm{Cl}(x)$ expresses that x is a class(ification) and $y\ \eta\ x$ expresses that y has the property given by x when $\mathrm{Cl}(x)$ holds. Variables A, $B, C, \ldots, X, Y, Z$, are introduced to range over the objects satisfying Cl, and $y \in X$ is also written for $y\ \eta\ x$ where $x = X$. The basic logic of T_0 is the classical first-order predicate calculus.[4] The axioms of T_0 include the basic operational axioms, and the remaining axioms are operationally given class existence axioms. For example, we have an operation **prod** which takes any pair X, Y of classes to produce their cartesian product, $X \times Y$ and another operation **exp** which takes X, Y to the cartesian power Y^X, also written $X \to Y$. The formation of such classes is governed by an *Elementary Comprehension Axiom* scheme (ECA) that tells which properties determine classes in a uniform way from given classes. These are given by formulas φ in which classes may be used as parameters to the right of the membership relation and in which we do not quantify over classes, and the uniformity is provided by operations $\mathbf{c}_\varphi$ applied to the parameters of φ.[5] But to form general products we need further notions and an additional axiom. Given a class I, by an I-termed sequence of classes is meant an operation f with domain I such that for each $i \in I$ the value of $f(i)$ is a class X_i; one wishes to use this to define $\Pi X_i[i \in I]$. It turns out that in combination with ECA a more basic operation is that of forming the join (or disjoint sum) $\sum X_i[i \in I]$ whose members are all pairs (i, y) such that $y \in X_i$; an additional *Join axiom* (J) is needed to assure existence of the join as given by an operation $\mathbf{j}(I, f)$. Finally, we have an operation $\mathbf{i}(A, R)$ and associated axiom (IG) for *Inductive Generation* which produces the class of objects accessible under the relation R (a class of ordered pairs) hereditarily within the class A. In particular, IG may be used to produce the class $\mathbf{N}$ of natural numbers, then the class $\mathbf{O}$ of countable tree ordinals, and so on.

In later expositions of systems of explicit mathematics, the language of LPT was used instead of the App relation for the operational basis, and the natural numbers $\mathbf{N}$ were taken to be a basic class for which several forms of the principle of induction were distinguished for proof theoretic purposes, as will be explained below. Also, in the approach to the formalization of Explicit Mathematics due to Jäger [49], it turned out to be more convenient to treat classes extensionally but each with many possible representations within the universe $\mathbf{V}$ of individuals.[6] Membership has its usual meaning, but a new basic relation is needed, namely that an object x

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[4] In [25] I also examined T_0 within intuitionistic logic.

[5] The scheme ECA can be finitely axiomatized by adding constants for the identity relation, the first-order logical operations for negation, conjunction, existential quantification, and inverse image of a class under an operation.

[6] There is a difference in terminology, though: Jäger used 'types' for our classes.

names or *represents* the class X, written $\Re(x, X)$. In these terms, for example, one has operations **prod** and **exp** such that whenever $\Re(x, X)$ and $\Re(y, Y)$ hold then $\Re(\mathbf{prod}(x, y), X \times Y)$ and $\Re(\mathbf{exp}(x, y), X \to Y)$ hold.

The only difference of T_1 from T_0 lies in the adjunction of a numerical choice operator $\boldsymbol{\mu}$ as a basic constant, together with the axiom:

$$(\mu) \quad f \in (\mathbf{N} \to \mathbf{N}) \to \mu f \in \mathbf{N} \wedge [\exists x (fx = 0) \to f(\mu f) = 0],$$

from which the unbounded[7] least number operator can be defined. This is equivalent to assumption of the operator $\mathbf{E}_0$ for quantification over **N**. Later on a third system T_2 was introduced by adjoining a constant for the Suslin-Kleene operator $\mathbf{E}_1$ for choosing a descending sequences g from a non-well-founded tree in the natural numbers represented by an operation f. Models of the basic operational axioms of T_0 are provided in the natural numbers by taking $\mathrm{App}(x, y, z)$ to be the relation $\{x\}(y) \simeq z$; thus the extensions of the total operations are just the recursive functions. Similarly, a model of the operational part of T_1 is given by indices of partial $\prod_1^1$ functions, so in this case the extensions of the total operations are just the hyperarithmetic functions. Finally, in the case of T_2, one uses indices of the functions partial recursive in the E_1 functional. In general, given any model $(A, App, \ldots)$ of the operational axioms with or without these special operators, one obtains a model of the class construction axioms by a transfinite inductive definition of names of classes with suitable codes for the operations on classes.[8]

The proof-theoretical study of subsystems of T_0 began in [23] and was continued in [25]. Since then the proof theory of subsystems of the T_i (either given directly or by interpretation) has greatly proliferated and has become the dominant part of research in explicit mathematics, continuing until this day. The paper Jäger et al. [55] provides a useful survey of a considerable part of such work that begins with a relatively weak theory BON (Basic Theory of Operations and Numbers). That adds to the applicative language the constants **N, 0**, and **sc** as well as a constant $\mathbf{r}_\mathrm{N}$ for primitive recursion on **N**. Over the basic operational theory, BON has the usual axioms for 0 and successor; for primitive recursion, we have an axiom which asserts that for arbitrary f and g, total on **N** and $\mathbf{N}^3$(each to **N**), resp., the operation $h = \mathbf{r}_\mathrm{N} fg$ is total on $\mathbf{N}^2$ to **N**, and satisfies $hx\mathbf{0} = fx$ and $hx(\mathbf{sc}(y)) = gxy(hxy)$. Several forms of induction are considered over BON; the full scheme, called *formula induction*, ($\mathrm{F\text{-}I_N}$) is of the usual form for each formula $\varphi(x)$ in the language, namely $\varphi(\mathbf{0}) \wedge (\forall x \in \mathbf{N})(\varphi(x) \to \varphi(\mathbf{sc}(x)) \to (\forall x \in \mathbf{N})\varphi(x)$. The single special case of this for $\varphi(x)$ of the form $fx = \mathbf{0}$ (where f is a variable) is called *operation induction* ($\mathrm{O\text{-}I_N}$), and when f is further assumed to be total from **N** to $\{\mathbf{0}, \mathbf{1}\}$ that is called *set induction* ($\mathrm{S\text{-}I_N}$). Finally, the case for $\varphi(x)$ of the form $fx \in \mathbf{N}$ is called **N**-*induction* ($\mathrm{N\text{-}I_N}$). The paper Jäger et al. [55] summarizes

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[7] In certain subsystems of T_1 with restricted induction we need to add to the (μ) axiom that if $\mu f \in \mathbf{N}$ then $f \in (\mathbf{N} \to \mathbf{N})$.

[8] Parts of T_0 relate to Aczel's Frege structures and Martin-Löf's constructive theory of types; cf. for example, Beeson [5], Chaps. XI and XVII. But neither of these approaches goes on to the adjunction of non-constructive functional operators like μ (or E_0) and E_1.

the proof-theoretical strength of all combinations of these with BON and then with BON plus the axioms for μ and $\mathbf{E}_0$.[9] For reasons to be seen in a moment, let me single out only two of their theorems:

(i) BON + (F-$\mathrm{I_N}$) ≡ PA, and
(ii) BON(μ) + (S-$\mathrm{I_N}$) ≡ PA,

where ≡ is the relation of proof-theoretical equivalence; we also have conservation of the l.h.s. over the r.h.s in each of these results. (BON(μ) is BON plus the (μ) axiom.) The result (i) is part of the folklore of the subject, and (ii) was established in [37].

Let us now look at these systems within the language of T_0 and T_1, resp. There it is natural to also consider *class induction* (C-$\mathrm{I_N}$), i.e. the case of the induction scheme where $\varphi(x)$ is of the form $x \in X$. Under the Elementary Comprehension Axiom scheme (ECA), that implies (F-$\mathrm{I_N}$) for the formulas of the language of BON. Moreover, under the assumption of the (μ) axioms we can alternatively use set induction to obtain all those instances. Put in these terms it turns out that we have the following from Feferman and Jäger [38]:

(iii) BON + ECA + (C-$\mathrm{I_N}$) ≡ PA, and
(iv) BON(μ) + ECA + (S-$\mathrm{I_N}$) ≡ PA.

These results are of significance with respect to the question: what parts of mathematics are accounted for in different parts of the explicit mathematics systems? The results (iii) and (iv) are relevant to constructive and predicative mathematics, resp., as follows.

A careful examination of Bishop and Bridges [8]—a reworking and expansion of Bishop [7]—shows that all its work in constructive analysis can be formalized in the theory BON + ECA + (C-$\mathrm{I_N}$), hence requires no more principles for its justification than given by Peano Arithmetic. The typical choice of notions and style of argument is presented in [25, pp. 176ff]. Closer inspection shows that much of Bishop and Bridges [8] can already be carried out in BON + ECA + (S-$\mathrm{I_N}$), which is equivalent in strength to PRA. Although the theory of measure and integration presented in Bishop [7] made use of Borel sets, and thus of the countable join of classes and the countable tree ordinals, Bishop and Bridges [8] substituted for that an approach to measure and integration that does not require J or IG at all.

Turning now to predicative mathematics, it is easily seen that all the redevelopment of 19th century analysis on those grounds as sketched in [85] can be carried out in the system BON(μ) + ECA + (S-$\mathrm{I_N}$) of (iv) above. The natural question to be raised is how much of modern analysis can be carried out in that system. In that respect we can make use of extensive detailed notes that I prepared in the period 1977–1981 but never published at the time; a scanned copy of those notes with an up-to-date introduction is now available in [35]. That work supports my conjecture [28, 30] that all scientifically applicable mathematics can be formalized in a system conservative over PA, namely

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[9] For the proof theory of systems of explicit mathematics with $\mathbf{E}_1$ see Jäger and Strahm [60] and Jäger and Probst [58].

BON(μ) + ECA + (S-I_N). To carry this out in the case of 19th c. analysis, systematic use is made of Cauchy completeness rather than the impredicative l.u.b. principle, and sequential compactness is used in place of the Heine-Borel theorem. Then for 20th c. analysis, Lebesgue measurable sets and functions are introduced directly via the Daniell approach without first going through the impredicative operation of outer measure; the existence of non-measurable sets cannot be proved in the system. Moving on to functional analysis, again the "positive" theory can be developed, at least for separable Banach and Hilbert spaces, and can be applied to various L_p spaces as principal examples. Among the general results that are obtained are usable forms of the Riesz Representation Theorem, the Hahn-Banach Theorem, the Uniform Boundedness Theorem, and the Open Mapping Theorem. The notes conclude with the spectral theory for compact self-adjoint operators on a separable Hilbert space.

This of course invites comparison with the work of Simpson [77] that examines various parts of mathematics from the standpoint of the Reverse Mathematics program initiated by Harvey Friedman. That centers on five subsystems of second order arithmetic: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1 - CA_0$. Each of these beyond the first is given by a single second-order axiom scheme in addition to the induction axiom for N in the form (C-I_N). In contrast to our work, which permits the free representation of practice in the full variable finite type structure over N, all mathematical notions considered by Simpson are represented in the second-order language by means of considerable coding. The main aim of the Reverse Mathematics program is to show that for a substantial part of practice, if a given mathematical theorem follows from a suitable one of the five axioms above then it is equivalent to it, i.e. the implication can be reversed. For comparison with our work, much of predicative analysis falls under these kinds of results obtained for WKL_0 and ACA_0, of proof-theoretical strength PRA and PA respectively. Thus, on the one hand Simpson's results are more informative than ours, since the strength of various individual theorems of analysis is sharply determined. On the other hand, the exposition for the work in WKL_0 and ACA_0 is not easily read as a systematic development of predicative analysis, as it is in our notes. Still, the Simpson book is recommended as a rich resource of other interesting results that could be incorporated into our approach through explicit mathematics.

Of course, predicative analysis—as measured by the explication of Feferman [19] and Schütte [76]—in principle goes far beyond what can be reduced to PA. First of all, there are a number of interesting subsystems of second-order arithmetic that are of the same proof-theoretical strength as the union of the ramified analytic systems up to Γ_0. Among these we have the system $\Sigma^1_1 - DC + BR$, where BR is the Bar Rule; the proof-theoretical equivalence in this case was first established in Feferman [26] and later (as a special case of a more general statement) in Feferman and Jäger [36]. In the latter publication, another system of this type is formulated as the autonomous iteration of the Π^0_1 comprehension axiom. Finally, Friedman et al. [41] showed by model-theoretic methods that the system ATR_0 is also of the same strength as full predicative analysis.[10] Since that may be given by a single axiom over ACA_0 (cf., ibid. p. 204), it follows that results in analysis and other parts of mathematics that are provably equivalent to (that axiom of) ATR_0 are impredicative. Simpson [78, 79] gives a number of examples of theorems from descriptive set theory that are equivalent to ATR_0, such as that every uncountable closed (or analytic) set contains a perfect subset.

By contrast to these systems of second order arithmetic, in [23] I conjectured that a certain subsystem ${T_1}^{(N)}$ of T_1 is equivalent in strength to predicative analysis; in the notation here, that system may be written as BON(μ) + ECA + (F-I_N) + J. However, Glass and Strahm [43] showed that ${T_1}^{(N)}$ is proof-theoretically equivalent to the iteration of ${\Pi^0}_1 - CA$ through all $\alpha < \varphi\, \varepsilon_0 0$, hence still far below Γ_0. It is an open question whether there is a natural subsystem of T_1 of strength full predicative analysis. Marzetta and Strahm [65] give a partial answer to this question by the employment of axioms for universes.[11]

Finally, let us turn to the evaluation of the proof-theoretical strength of the full systems T_0 and T_1 of explicit mathematics. In unpublished notes from 1976, I showed how to interpret T_0 in $\Delta^1_2 - CA + BI$ (cf. [26, p. 218]). Then Jäger and Pohlers [57] determined an upper bound for the proof-theoretic ordinal—call it κ—of the latter system, and Jäger [46] gave a proof in T_0 of transfinite induction on α for each $\alpha < \kappa$, thus closing the circle.[12] One of the main results of Glass and Strahm [43] is that proof-theoretically, T_1 is no stronger than T_0. In a personal communication, Dieter Probst has sketched arguments to show that also T_2 is no stronger than T_0, but natural variants of E_1 lead to stronger systems.[13]

NB. Currently, work is well advanced on a book being coauthored with Gerhard Jäger and Thomas Strahm with the assistance of Ulrik Buchholtz in which much of the foundations of explicit mathematics will be exposited in a systematic way. In the meantime, Buchholtz has set up an online bibliography of explicit mathematics and closely related topics at http://www.iam.unibe.ch/~til/em_bibliography/ that can be searched chronologically or by author and by title. The plans are to maintain this independently of the publication of the book. At the time of writing it consists of 127 items; readers are encouraged to let us know if there are further items that should be added.

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[10] Then Jäger [47] and Avigad [3] showed that they are of the same proof-theoretical strength, by proof-theoretical methods, the first via theories of iterated admissible sets without foundation and the second via fixed point theories.

[11] In Sect. 4 below I conjecture that the unfolding of a suitable subsystem of T_0 is equivalent in strength to predicative analysis.

[12] Recently, Sato [74] has shown how to establish the reduction of $\Delta^1_2 - CA + BI$ to T_0 without going through the ordinal notation system for κ.

[13] Another interesting group of questions concerns the strength over T_0 (or its restricted version $T_0\restriction$) of the principle MID that I introduced in [27]. That expresses that if f is any monotone operation from classes to classes then f has a least fixed point. Takahashi [80] showed that T_0 + MID is interpretable in $\Pi^1_2 - CA + BI$, and then Rathjen [69] showed that it is much stronger than T_0. Next, exact strength of $T_0\restriction$ + MID was determined by Glass et al. [42]. A series of further results by Rathjen for the strength of T_0 + MID and T_0 + UMID, where UMID is a natural uniform version of the principle, are surveyed in the paper Rathjen [70].

3 Operational Set Theory

I introduced operational set theory in notes [33], eventually published in detail in [34].[14] This is an applicative based reformulation and extension of some systems of classical set theory ranging in strength from KP to ZFC and beyond. For the system of strength ZFC, this goes back in spirit to the theory of sets and functions due to von Neumann [84], but Beeson [6] is a direct predecessor as an operational theory. In addition to extending the range of this approach through intermediate systems down to KP, my work differs from both of these in its primary concern, namely to state various large cardinal notions in general applicative terms and, among other things, use those to explain admissible analogues in the literature. Significant further work on the strength of various systems of operational set theory has been carried out by Jäger [50–53] and Jäger and Zumbrunnen [61].[15] The last of these is of particular significance for the following, since it shows that one of the main conjectures of [34] is wrong and that it (and related other conjectures) need to be modified in order to obtain the intended consequences; Gerhard Jäger has suggested two ways to do that that will be described later in this section.

The system OST allows us to explain in uniform operational terms the informal idea from Zermelo [86] that any definite property of elements of a set determines the subset of that set separated by the given property. Namely, represent the truth values "truth" and "falsity" by 1 and 0, resp., and let $\mathrm{B} = \{0, 1\}$. In the applicative extension of the usual set-theoretical language, write $f : a \to b$ for $\forall x(x \in a \to fx \in b)$, and $f : a \to \mathrm{V}$ for $\forall x(x \in a \to fx{\downarrow})$. Then definite properties of subsets of a set a may be identified with operations $f : a \to \mathrm{B}$, and the uniform separation principle is given by an operation S such that for each such a and f, $\mathrm{S}(f, a)$ is defined and exists as a set, $\{x : x \in a \wedge fx = 1\}$. Furthermore, it allows us to explain in uniform operational terms the idea from von Neumann [84] that if a is a set and f is an operation from a into the universe of sets then the range of f on a is a set. Namely, we have an operation R such that for each $f : a \to \mathrm{V}$, $\mathrm{R}(f, a)$ is defined and exists as a set, $\{y : \exists x(x \in a \wedge y = fx)\}$. Finally, OST allows us to express a uniform form of global choice by means of an operation C such that for each f, $\exists x(fx = 1)$ then $\mathrm{C}f$ is defined and $f(\mathrm{C}f) = 1$.

In a little more detail, here is a description of the theory OST essentially as presented in [34]. Its language is an expansion of the language of usual set theory by the atomic applicative formulas, together with a number of constants to be specified along with their axioms. The basic logic is LPT, the logic of partial terms. The axioms of OST fall into five groups. Group 1 axioms are those for **k** and **s**. Group 2 axioms consist of extensionality and the existence of the empty set **0**, unordered pair, union

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[14] To explain some anomalies of the dates of subsequent work on this subject, it should be noted that my 2009 paper was submitted to the journal *Information and Control* in December 2006 and in revised form in April 2008. In the meantime, Jäger [50] had appeared and so I could refer to it in that revised version.

[15] Further important work is contained in Zumbrunnen [87] and Sato and Zumbrunnen [75]. Constructive operational theories of sets have been treated by Cantini and Crosilla [12, 13] and Cantini [11].

and ω; we take $\mathbf{1} = \{\mathbf{0}\}$ and $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\}$. Group 3 axioms are for the logical operations **el**, **cnj**, **neg** and $\mathbf{uni}_\mathrm{b}$ (bounded universal quantification), with the obvious intended axioms, the last of which is that if $f : a \to \mathbf{B}$ then $\mathbf{uni}_\mathrm{b}(f, a) \in \mathrm{B}$ and $\mathbf{uni}_\mathrm{b}(f, a) = \mathbf{1} \leftrightarrow (\forall x \in a)(fx = \mathbf{1})$. The Group 4 axioms are for **S** (Separation), **R** (Replacement) and **C** (Choice), as described above. Finally, Group 5 is the scheme of set induction for *all* formulas in the language of OST; when that is restricted to the formula $x \in a$ we write OST$\restriction$ for the system.

From the Group 3 (logical) axioms it is shown that one can associate with each Δ_0 formula $\varphi(\underline{x})$ in the language of ordinary set theory (where $\underline{x} = x_1, \ldots, x_n$) a closed term t_φ that is defined and maps $\mathbf{V}^n$ into $\mathbf{B}$, and which satisfies $\forall \underline{x}(t_\varphi(\underline{x}) = 1 \leftrightarrow \varphi(\underline{x}))$. Thus OST satisfies the separation axiom for bounded formulas. Then using Replacement and Choice, one obtains the Δ_0-Collection axiom. Hence KP (here taken to include the axiom of infinity) is contained in OST. This leads us to the $\geq$ direction of the following.

(i) OST $\equiv$ KP.

My proof in [34] for the $\leq$ direction went via an interpretation of OST in KP + V= L, beginning with an interpretation of the operations as those given by codes for the partial functions that are Σ_1 definable in parameters. An alternative proof of that bound was given by Jäger [50] using his theories of numbers and ordinals for transfinite inductive definitions.

To obtain systems of strength full set theory and beyond, one adds constants **uni** for unbounded universal quantification with axiom (Uni) like that for $\mathbf{uni}_\mathrm{b}$, and **P** for the power set operation, the latter with axiom (Pow) which states that for each x, $\mathbf{P}x$ is defined and $\forall y(y \in \mathbf{P}x \leftrightarrow y \subseteq x)$.[16] Then we have:

(ii) OST$\restriction$ + (Pow) + (Uni) $\equiv$ ZFC.

The proof that ZFC is contained in the left side is easy, using that every formula is now represented by a definite operation. The proof that OST$\restriction$ + (Pow) + (Uni) can be reduced to ZFC + (V = L) was given by Jäger [50]. On the other hand, as shown in Jäger [51] and Jäger and Krähenbuhl [56], the unrestricted system OST + (Pow) + (Uni) is of the same strength as NBG extended by a suitable form of $\Sigma^1_1 - \mathrm{AC}$.

Next, in view of (i) and (ii) it is natural to ask what the strength is of OST + (Pow). Of course we have that this contains KP + (Pow), where that is formulated with an additional constant **P** as above. The problem concerns the other direction. In this case Jäger [50] showed that

(iii) OST + (Pow) is interpretable in KP + (Pow) + V = L.

But the latter theory is much stronger than KP + (Pow) as shown by Rathjen [71],[17] so one can't use (iii) to determine the strength of OST + (Pow). Nevertheless, Rathjen [72] has been able to establish the following there, using novel means:

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[16] Rathjen [71] uses (Pow(**P**)) for our formulation in the language of KP as well, in order to distinguish it from the usual power set axiom formulated without the additional constant. It would have been better to do that in [34], but not having done so I here follow the notation from there.

[17] An earlier such result for the system with a restricted form of set induction is due to Mathias [66].

(iv) OST + (Pow) ≡ KP + (Pow).

Let's turn now to the problematic conjectures of [34]. These concern the natural formulation in operational terms of an ordinal κ being regular, inaccessible, and Mahlo, resp., as well as a notion of being 2-regular due to Aczel and Richter [2] that is equivalent to being Π^1_1-indescribable (cf. [73, pp. 329–331]). Using lower case Greek letters to range over ordinals, the first of these is defined in the language of OST by

$\text{Reg}(\kappa) := (\kappa > 0) \wedge \forall \alpha, f\ [\alpha < \kappa \wedge (f : \alpha \to \kappa) \to \exists \beta < \kappa\ (f : \alpha \to \beta)]$.
Then being inaccessible is defined by

$\text{Inacc}(\kappa) := \text{Reg}(\kappa) \wedge \forall \alpha < \kappa\ \exists \beta < \kappa\ [\text{Reg}(\beta) \wedge \alpha < \beta]$.
The statements of regularity and inaccessibility of the class **Ω** of ordinals are defined analogously by:

$\text{Reg} := \forall \alpha, f\ [(f : \alpha \to \mathbf{\Omega}) \to \exists \beta (f : \alpha \to \beta)]$.

$\text{Inacc} := \text{Reg} \wedge \forall \alpha\, \exists\, \beta\ [\text{Reg}(\beta) \wedge \alpha < \beta]$.

Let Fun(a) be the usual set-theoretical formula expressing that the set a is a function, i.e. a many-one binary relation; for x in dom(a), $a(x)$ is the unique y with $\langle x, y\rangle \in a$. Then among the immediate consequences of the OST axioms are, first, that there is a closed term **op** such that for each set a, $\mathbf{op}a\downarrow$ and if Fun(a) and $f = \mathbf{op}a$ then for each $x \in \text{dom}(a), fx = a(x)$ and, second, there is a closed term **fun** such that for each f, a, if $f : a \to \mathbf{V}$ then $\mathbf{fun}(f, a)\downarrow$ and if $c = \mathbf{fun}(f, a)$ then Fun(c) and for each $x \in \text{dom}(c)$, $c(x) = fx$. Thus the above notions and statements of regularity and inaccessibility can be read as usual in the ordinary language of set theory. That led me mistakenly to assert in Theorem 10 of [34] that OST + (Inacc) is interpretable in KPi + V = L, and to conjecture that OST + (Inacc) is equivalent in strength to KPi.[18] This has been proved to be wrong by [61], who show that OST + (Inacc) is equivalent in strength to the extension KPS of KP by the statement Inacc when read in ordinary set-theoretical terms, denoted SLim for "strong limit axiom." KPS proves that for any κ that satisfies Reg(κ), L_κ is a standard model of ZFC without the power set axiom; hence KPS is much stronger than second-order arithmetic.

My mistake was that the notion of regularity here—while natural in the context of ordinary set theory—does not correspond to that used in KP viewed as a theory for admissible sets. Namely, as presented in Jäger [48], that is given by the additional predicate Ad(x) expressing that x is admissible, with the appropriate axioms. Then KPi asserts the unboundedness of the admissibles, in the sense that $\forall x \exists y[\text{Ad}(y) \wedge x \in y]$. So the question arises as to whether there is a natural extension of OST that is equivalent in strength to KPi. My first thought was that there should be some notion of universe, Uni(x), formulated in the language of OST, that is analogous to the notion of admissibility, Ad(x), in the language of KP, such that when we extend OST by the statement $\forall x \exists y[\text{Uni}(y) \wedge x \in y]$ we obtain a system of the same strength as KPi. In e-mail exchanges with Gerhard Jäger early in the summer of 2014, I made several proposals for the definition of Uni(x) to do just that, but each proved to be defective. One of these proposals was to say that u is a universe if it

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[18] However, I did say that I had not checked the details. In fact, I hadn't thought them through at all.

is a transitive set that contains all the constants of OST, is closed under application, satisfies the basic set-theoretic axioms of OST and the axioms for **S**, **R** and **C** under the hypotheses suitably relativized to u. But Jäger pointed out that the system OST + $\forall x \exists u[\mathrm{Uni}(u) \wedge x \in u]$ is still stronger than KPi, by the results of his paper [53].

In going over this situation, Jäger noted that the applicative structure must also be relativized in explaining the notion of a universe in the language of OST. This first led him to make the following suggestion. Namely, one returns to the formulation of the applicative basis in terms of the ternary App relation, rather than the logic of partial terms. Then a universe is defined to be a pair $\langle u, a\rangle$ such that (i) u is a transitive set with $a \subseteq u^3$, (ii) whenever $(f, x, y) \in a$ then $\mathrm{App}(f, x, y)$, (iii) u contains ω and all the constants of OST, and (iv) all the axioms of OST hold when relativized to u provided that the App relation is replaced by the set a. Jäger outlined a proof that OST + $\forall x \exists u, a[\mathrm{Uni}(\langle u, a\rangle) \wedge x \in u]$ is proof-theoretically equivalent to KPi. More recently, in work in progress Jäger [54], he has proposed another way of modifying the notion of regularity (and thence inaccessibility) so as to stay within the language and logic of partial terms while relativizing it to a universe in the preceding sense. In the new approach one adds a predicate $\mathrm{Reg}(u, a)$ to the language satisfying certain axioms similar to (i)–(iv) and in addition an assumption of the linear ordering of those pairs $\langle u, a\rangle$ for which $\mathrm{Reg}(u, a)$ holds. Then in place of the above condition Inacc on the class of ordinals one can consider the statement Lim-Reg (abbreviated LR), which asserts that $\forall x \exists u,a[\mathrm{Reg}(u, a) \wedge x \in u]$. The main result of Jäger [54] is that OST + LR is proof-theoretically equivalent to KPi. The advantage of his second approach is that one can re-express further large cardinal notions such as Mahlo, etc., much as before. Assuming this is successful, we can look forward to a reexamination of my original aim to use OST as a vehicle to restate various large cardinal notions in applicative terms in order to explain the existing admissible analogues that are in the literature.

4 The Unfolding Program

In Sect. 2 I spoke of my work on unramified systems of predicative strength[19] as being one precursor to the development of explicit mathematics. That mainly had to do with the potential use of such systems as a means to determine which parts of classical analysis could be justified on predicative grounds. But one of the articles indicated, [26], was concerned with more basic conceptual aims, namely those advanced by Kreisel [63] who suggested the study of principles of proof and definition that "we recognize as valid once we have understood (or, as one sometimes says, 'accepted') certain given concepts." The two main examples Kreisel gave of this were finitism and predicativity, and in both cases, he advanced for that purpose the use of some form of autonomous transfinite progressions embodying a "high degree of self-reflection." My aim in [26] was to show in the case of predicativity how that might be generated

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[19] Cf. Footnote 2.

instead by "a direct finite rather than transfinite reflective process, and without alternative use of the well-foundedness notion in the axioms." The motivation was that if one is to model actual reflective thought then one should not invoke the transfinite in any way. But not long after that work I realized that what is implicit in accepting certain basic principles and concepts can be explained more generally in terms of a notion of reflective closure of schematic systems, where schemata are considered to be open-ended using symbols for free predicate variables P, as in the scheme for induction on the natural numbers. One crucial engine in the process of reflection is the employment of the substitution rule A(P)/A(B), where B(x) is a formula that one has come to recognize as meaningful in the course of reflection, and where by A(B) is meant the result of substituting B(t) for each occurrence P(t) in A(P). I described the notion of reflective closure and its application to the characterization of predicativity in a lecture for a meeting in 1979 on the work of Kurt Gödel, but was only to publish that work in the article, "Reflecting on incompleteness" [29]. For the technical apparatus I introduced there an axiomatization[20] of the semantic theory of truth of Kripke [64] in which the truth predicate may consistently be applied to statements within which it appears by treating truth and falsity as partial predicates.

Though the axiomatic theory of truth employed in [29] proved to be of independent interest, as an engine for the explanation of reflective closure it still had an air of artificiality about it. I was thus led to reconsidering the entire matter in [31] in which the notion of unfolding of open-ended schematic systems was introduced in close to its present form by means of a basic operational framework. As formulated there, given a schematic system S, the question is: which operations and predicates—and which principles concerning them ought to be accepted if one has accepted S? And under the heading of operations one should consider both operations on the domain D_S of individuals of S and operations on the domain Π of predicates; both domains are included in an overarching domain V. For the underlying general theory of operations applicable to arbitrary members of V, in [31] I made use of a type 2 theory of partial functions and (monotone) partial functionals, generated by explicit definition and least fixed point recursion, and that is what was followed in the paper Feferman and Strahm [39] for the unfolding of a schematic system NFA of non-finitist arithmetic. Later, in order to simplify various matters in the treatment of finitist arithmetic, the work on NFA was reformulated in Feferman and Strahm [40] so as to use instead the basic operational language and axioms on V essentially as described at the end of Sect. 1 above, and that is what has been followed in subsequent work on unfolding.

Here are a few details for the unfolding of NFA, which in many ways is paradigmatic. The axioms of NFA itself are simply the usual ones for 0, sc and pd together with the induction scheme given as $\mathrm{P}(0) \wedge \forall x[\mathrm{P}(x) \rightarrow \mathrm{P}(\mathrm{sc}(x))] \rightarrow \forall x(\mathrm{P}(x))$ where P is a free predicate variable. The language of the unfolding of NFA adds a number of constants, the predicate symbol $\mathrm{N}(x)$, the predicate symbol $\Pi(x)$, and the relation $y \in x$ for x such that $\Pi(x)$. The axioms of the unfolding U(NFA) consist of the following five groups: (I) The axioms of NFA relativized to N. (II) The partial combinatory axioms, with pairing, projections and definition by

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[20] Since referred to as KF in the literature.

cases. (III) An axiom for the characteristic function of equality on N. (IV) Axioms for various constants in the domain Π of predicates, namely for the natural numbers, equality, and the free predicate variable P, and for the logical operations ¬, ∧, and ∀. (V) An axiom for the join of a sequence of predicates, given by $\mathbf{j}(f)$ when $f : \mathrm{N} \rightarrow \Pi$. The full unfolding U(NFA) is then obtained by applying the substitution rule A(P)/A(B) where B is an arbitrary formula of the unfolding language. A natural subsystem of this called the *operational unfolding* of NFA and denoted U_0(NFA) is obtained by restricting to axiom groups (I)–(III) with the formulas B in the substitution rule restricted accordingly. In U_0(NFA) one successively constructs terms $t(x)$ intended to represent each primitive recursive function, by means of the recursion operator and definition by cases. Applying the substitution rule it is then shown by induction on the formula $t(x)\downarrow$ that each such term defines a total operation on the natural numbers. Thus the language of PA may be interpreted in that of U_0(NFA) and so by application of the substitution rule once more, we have PA itself included in that system. Moving on to U(NFA), the domain of predicates is expanded considerably by use of the join operation. Once one establishes that a primitive recursive ordering $\prec$ satisfies the schematic transfinite induction principle TI($\prec$, P) with the free predicate variable P, one may apply the substitution rule to carry out proofs by induction on $\prec$ with respect to arbitrary formulas. In particular, one may establish existence of a predicate corresponding to the hyperarithmetical hierarchy along such an ordering, relative to any given predicate p in Π. Then by means of the usual arguments, if one has established in U(NFA) the schematic principle of transfinite induction along a standard ordering for an ordinal α, one can establish the same for $\varphi\alpha 0$, hence the same for each ordinal less than Γ_0. Thus U(NFA) contains the union of the ramified analytic systems up to Γ_0. The main results of Feferman and Strahm [39] are that U_0(NFA) is proof-theoretically equivalent to PA and is conservative over it, and U(NFA) is proof-theoretically equivalent to the union of the ramified analytic systems up to Γ_0 and is conservative over it. In other words, U(NFA) is proof-theoretically equivalent to predicative analysis. In addition we showed that the intermediate system U_1(NFA) without the join axiom (V) is proof-theoretically equivalent to the union of the ramified systems of finite level.

The unfolding of finitist arithmetic was later taken up in Feferman and Strahm [40]; two open-ended schematic systems of finitist arithmetic are treated there, denoted FA and FA + BR, resp. The basic operations on individuals are the same as in NFA together with the characteristic function of equality, while those on predicates are given by ⊥, ∧, ∨, and ∃. Reasoning now is applied to sequents $\Gamma \rightarrow \mathrm{A}$, and the basic assumptions are the usual ones for 0, sc and pd, and the induction rule in the form: from $\Gamma \rightarrow \mathrm{P}(0)$ and $\Gamma, \mathrm{P}(x) \rightarrow \mathrm{P}(\mathrm{sc}(x))$, infer $\Gamma \rightarrow \mathrm{P}(x)$, with P a free predicate variable. Now the substitution rule is applied to sequent inference rules of the form $\Sigma_1, \Sigma_2, \ldots, \Sigma_n \Rightarrow \Sigma$; we may substitute for P throughout by a formula B to obtain a new such rule. The first main result of Feferman and Strahm [40] is that all three unfoldings of FA are equivalent in strength to PRA. That is in accord with the informal analysis of finitism by Tait [81]. On the other hand, Kreisel [62, pp. 169–172], had sketched an analysis of finitism in terms of a certain autonomous progression and alternatively "for a more attractive formulation" without progressions but with the use of the Bar Rule, BR, that is equivalent to PA. The rule BR allows one to infer from the no-descending sequence property NDS(f, $\prec$) for a decidable ordering $\prec$, where f is a free function variable, the principle of transfinite induction on the ordering TI($\prec$, P), with the free predicate variable P. The second main result of Feferman and Strahm [40] is that all three unfoldings of FA + BR are equivalent in strength to PA, thus in accord with Kreisel's analysis of finitism.

Extending the unfolding program to still weaker theories, Eberhard and Strahm [16, 17] have dealt with three unfolding notions for a basic system FEA of feasible arithmetic. Besides the operational unfolding U_0(FEA) and (full) predicate unfolding U(FEA), they introduced a more general truth unfolding system U_T(FEA) obtained by adding a truth predicate for the language of the predicate unfolding.[21] Their main result is that the provably total functions of binary words for all three systems are exactly those computable in polynomial time.

The most recent result in the unfolding program is due to Buchholtz [9] who determined the proof-theoretic ordinal of $\mathrm{U(ID_1)}$, where the usual system of one arithmetical inductive definition $\mathrm{ID_1}$ is recast in open-ended schematic form. That is taken to expand NFA and for each arithmetical A(P, x) in which P has only positive occurrences one assumes the following principles for the predicate constant P_A associated with A: (i) $\forall x(\mathrm{A}(\mathrm{P}_A, x) \to \mathrm{P}_A(x))$ and (ii) $\forall x(\mathrm{A}(\mathrm{P}, x) \to \mathrm{P}(x)) \to \forall x(\mathrm{P}_A(x) \to \mathrm{P}(x))$, with P the free predicate variable. The axioms of $\mathrm{U(ID_1)}$ are similar to those of U(NFA), except that for Axiom (V), the join operation more generally takes an operation f from a predicate p to predicates $fx = q_x$ to the disjoint sum $\mathbf{j}(f)$ of the q_x's over the x's in p. The main result of Buchholtz [9] is that $|\mathrm{U(ID_1)}| = \psi(\Gamma_{\Omega+1})$ (= $\psi_\Omega(\Gamma_{\Omega+1})$). This invites comparison with the famous result of Howard (cf. [1]) according to which $|\mathrm{ID_1}| = \psi(\varepsilon_{\Omega+1})$, previously denoted $\varphi\, \varepsilon_{\Omega+1} 0$.[22] In addition, Buchholtz et al. [10] show that a number of proof-theoretic results for systems of strength Γ_0 have direct analogues for suitable systems of strength $\psi(\Gamma_{\Omega+1})$. Finally, [9, p. 48] presents very plausible conjectures concerning the unfolding of schematic theories of iterated inductive definitions generalizing the results for $\mathrm{ID_1}$.

Readers may already have guessed that the unfolding of NFA and $\mathrm{ID_1}$ can be recast in terms of systems S of explicit mathematics. For that purpose it is simplest to return to the original syntax of those systems and use Cl(x) in place of $\Pi(x)$. Note that with the variables X, Y, Z,… taken to range over Cl, every second-order formula over the applicative structure is expressible as a formula of the language. The substitution rule now takes the form $\varphi(X)/\varphi(\{x : \psi(x)\}\}$ where ψ is an arbitrary formula of the language, and where in the conclusion of the rule each instance of the form $t \in X$ that occurs in φ is replaced by $\psi(t)$. In place of the operations on

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[21] This follows the proposed formulation of U(NFA) via a truth predicate in Feferman [31, p. 14].

[22] Ulrik Buchholtz originally thought that $\psi(\Gamma_{\Omega+1})$ is the same as the ordinal H(1) of Bachmann [4]. This seemed to be supported by Aczel [1] who wrote (p. 36) that H(1) may have proof theoretical significance related to those of the ordinals ε, Γ_0 and $\varphi\, \varepsilon_{\Omega+1} 0$. And Miller [67, p. 451] had conjectured that "H(1) [is] the proof-theoretic ordinal of $\mathrm{ID_1}$* which is related to $\mathrm{ID_1}$ as predicative analysis $\mathrm{ID_0}$* is to first-order arithmetic $\mathrm{ID_0}$." However, Wilfried Buchholz recently found that the above representation of H(1) in terms of the ψ function is incorrect. This suggests one should revisit the bases of Aczel's and Miller's conjectures.

predicates in Π we now use the corresponding operations on classes.[23] Thus, in place of U(NFA) we would consider the system U * (S) generated by the substitution rule from the system S = BON + ECA + J + (C-$\mathrm{I_N}$), where the class induction axiom on N takes the place of the induction scheme of NFA. So it is reasonable to conjecture that U * (S) in this case is of the same strength as predicative analysis.[24] Similarly, we may obtain an analogue of U(ID_1) by making use of the Inductive Generation Axiom IG of T_0. Recall that IG takes the form that we have an operation $\mathbf{i}(A, R)$ that is defined for all classes A and R, and whose value is a class I that satisfies the closure condition

$$\mathrm{Clos}(A,R, I) := \forall x \in A[(\forall y((y, x) \in R \to y \in I) \to x \in I]$$

together with the minimality condition

$$\mathrm{Min}(A, R, \varphi) := \mathrm{Clos}(A, R, \varphi) \to (\forall x \in I)\, \varphi(x),$$

where φ is an arbitrary formula. In its place the schematic form IG$\restriction$ restricts the minimality condition to formulas $\varphi(x)$ of the form $x \in X$, i.e.

$$\mathrm{Clos}(A, R, X) \to I \subseteq X.$$

Let us denote by IG(O) the instance of IG used to generate the class of countable tree ordinals and by IG(O)$\restriction$ the same with the restricted minimality condition. Then with S as above, the system S + IG(O)$\restriction$ is analogous to ID_1, so we may expect that U * (S + IG(O)$\restriction$) is equivalent in strength to U(ID_1) and so its proof theoretic ordinal would be equal to $\psi(\Gamma_\Omega + 1)$. But now we can also form the system S + IG$\restriction$ and it is natural to ask what the strength is of its unfolding U * (S + IG$\restriction$). This would seem to encompass autonomously iterated systems ID (cf. [68, p. 332]).

One of the motivations for [31] was to give substance to the idea of Gödel [44] that consideration of axioms for the existence of inaccessible cardinals and the hierarchies of Mahlo cardinals more generally “show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far.” (Cf. [45, p. 182]). My idea was that this could be spelled out by the unfolding of a suitable schematic system of set theory, but the details in [31] were rather sketchy. These now can be spelled out as follows, using the language of OST as a point of departure. For unfoldings, we could either take the U(·) approach by adding the predicate $\Pi(x)$ or the variant U*(·) approach by adding the predicate $\mathrm{Cl}(x)$ as in systems of explicit mathematics. For simplicity I shall follow the latter here. Take S-OST to be the schematic version of OST, which is obtained by replacing the set induction axiom scheme by its class version $\forall x[\forall y(y \in x \to y \in X) \to x \in X] \to \forall x(x \in X)$. Then we can consider U * (S-OST ± Pow ± Uni), where (Pow) is formulated as in Sect. 3 above with a symbol **P** for the power set operation, and (Uni)

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[23] Alternatively, one can of course work with the formulation of explicit mathematics in terms of the representation relation $\mathfrak{R}(x, X)$.

[24] This would provide another answer to the question of finding a system of explicit mathematics of the same strength as predicative analysis.

is the axiom for unbounded universal quantification with **uni** as the corresponding basic operation. In particular, we would be interested in characterizing the three systems, U * (S-OST), U * (S-OST + Pow) and, finally U * (OST + Pow + Uni).

Now with OST $\equiv$ KP $\equiv$ ID_1, one may think of S-OST as analogous to schematic ID_1, so that I conjecture that U * (S-OST) $\equiv$ U(ID_1). Secondly, Rathjen [72] has studied KP + AC + Pow($\wp$) using relativized ordinal analysis methods, and shown that this system proves the existence of the cumulative hierarchy of V_α's for all $\alpha < \psi(\varepsilon_{\Omega+1})$ and moreover that that is best possible. Thus I conjecture that U * (S-OST + Pow) proves the existence of the cumulative hierarchy of V_α's for all $\alpha < \psi(\Gamma_{\Omega+1})$ and that that is best possible. Finally, I expect that the analogous results for the unfolding of S-OST + Pow + Uni would make use of the ordinal notation system up to $\psi(\Gamma_{\mathrm{ORD}+1})$ in a suitable sense. Thus the unfolding of the system S-OST + (Pow) + (Uni) would be equivalent in strength to the extension of ZFC by a certain range of small large cardinals. It would then be another question to see how far that goes in terms of the standard classifications of such.

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Some Remarks on the Proof-Theory and the Semantics of Infinitary Modal Logic

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Abstract We investigate the (multiagent) infinitary version $\mathbf{K}_{\omega_1}$ of the propositional modal logic $\mathbf{K}$, in which conjunctions and disjunctions over countably infinite sets of formulas are allowed. It is known that the *natural* infinitary extension $\mathbf{LK}^{\Box}_{\omega_1}$ (here presented as a *Tait-style* calculus, $\mathbf{TK}^{\sharp}_{\omega_1}$) of the standard sequent calculus $\mathbf{LK}^{\Box}_{p}$ for $\mathbf{K}$ is *incomplete* with respect to Kripke semantics. It is also known that in order to axiomatize $\mathbf{K}_{\omega_1}$ one has to add to $\mathbf{LK}^{\Box}_{\omega_1}$ new initial sequents corresponding to the infinitary propositional counterpart BF_{ω_1} of the Barcan Formula. We introduce a generalization of standard Kripke semantics, and prove that $\mathbf{TK}^{\sharp}_{\omega_1}$ is sound and complete with respect to it. By the same proof strategy, we show that the stronger system $\mathbf{TK}_{\omega_1}$, allowing *countably infinite* sequents, axiomatizes $\mathbf{K}_{\omega_1}$, although it provably does not admit cut-elimination.

Keywords Modal logic · Infinitary logic · Kripke semantics · Tait-style calculi · Cut-elimination

1 Introduction

Let $\mathbf{LK}_p$ be the propositional fragment of Gentzen's sequent calculus for classical logic. As is well-known (more or less since the mid 1950's, see e.g. [11]), the sequent calculus $\mathbf{LK}^{\Box}_{p}$ obtained by adding to $\mathbf{LK}_p$ the inference rule

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ \texttt{K} \quad (\text{where } \Box\Gamma := \{\Box\psi \mid \psi \in \Gamma\})$$

provides an adequate axiomatization of the minimal normal propositional modal logic $\mathbf{K}$, semantically defined as the set of all modal formulas which are valid in every Kripke frame. Furthermore, this axiomatization satisfies the subformula property,

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since it is not difficult to verify that $\mathbf{LK}^{\Box}_{p}$ admits cut-elimination. Of course, the same does hold for the corresponding multiagent system based on an arbitrarily fixed set A of agents—below, we will identify without limitations A with the set of natural numbers ω.

Let us now move on to consider the *infinitary* (multiagent) modal version $\mathbf{K}_{\omega_1}$ of $\mathbf{K}$ (the symbol 'ω_1' denotes the first uncountable ordinal). In the language of $\mathbf{K}_{\omega_1}$, featuring the infinitary connectives $\bigvee$ (*countable* disjunction) and $\bigwedge$ (*countable* conjunction) in place of the corresponding finitary connectives $\vee$ and $\wedge$, many interesting modal operators which encode infinitary formulas within a finitary framework (typically, fixed point operators) become directly *definable*—for instance that of *common knowledge*, C:

$$\mathsf{C}\varphi := \bigwedge\{\mathsf{E}^n\varphi \mid n \geq 1\}$$

where $\mathsf{E}\varphi$ (*everyone knows, that* φ) is defined as

$$\mathsf{E}\varphi := \bigwedge\{\Box_i\varphi \mid i < \omega\}$$

and

$$\mathsf{E}^n\varphi := \overbrace{\mathsf{E}\ldots\mathsf{E}}^{n}\varphi.$$

It is however a known fact (see e.g. [12, 15, 17]) that the *natural* infinitary extension $\mathbf{LK}^{\Box}_{\omega_1}$ of the sequent calculus $\mathbf{LK}^{\Box}_{p}$, which is simply obtained by replacing the rules for $\vee$ and $\wedge$ with their infinitary counterparts

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \bigvee\Phi}\ (\varphi \in \Phi) \qquad \frac{\ldots\ \varphi, \Gamma \Rightarrow \Delta\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\bigvee\Phi, \Gamma \Rightarrow \Delta}$$

and

$$\frac{\ldots\ \Gamma \Rightarrow \Delta, \varphi\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma \Rightarrow \Delta, \bigwedge\Phi} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\bigwedge\Phi, \Gamma \Rightarrow \Delta}\ (\varphi \in \Phi)$$

is not Kripke complete. In particular, the schema

$$\bigwedge\Box\Phi \to \Box\bigwedge\Phi \qquad (BF_{\omega_1})$$

that is the infinitary propositional counterpart of the famous *Barcan Formula* of quantified modal logic,[1] is trivially *valid in all Kripke frames*, but *is not derivable* in $\mathbf{LK}^{\Box}_{\omega_1}$.

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[1] Alternatively, BF_{ω_1} may be seen as an infinitary version of the $\mathbf{K}$-tautology $\Box\varphi\wedge\Box\psi \to \Box(\varphi\wedge\psi)$.

Notice that, as a consequence, the basic modal logic of common knowledge **KC** (see e.g. [3]) cannot be embedded in $\mathbf{LK}^{\Box}_{\omega_1}$, since BF_{ω_1} (together with its *converse* CBF_{ω_1}, which is instead derivable in $\mathbf{LK}^{\Box}_{\omega_1}$), is essentially needed in order to derive the *fixed point axiom* $\mathsf{C}\varphi \leftrightarrow \mathsf{E}\varphi \wedge \mathsf{EC}\varphi$ of **KC**:

$$\begin{tikzcd}
\mathsf{C}\varphi \arrow[r, leftrightarrow] & \mathsf{E}^1\varphi \wedge \mathsf{E}^2\varphi \wedge \mathsf{E}^3\varphi \wedge \dots \arrow[d, leftrightarrow] & \\
& \mathsf{E}\varphi \wedge (\mathsf{E}(\mathsf{E}^1\varphi) \wedge \mathsf{E}(\mathsf{E}^2\varphi) \wedge \dots) \arrow[d, shift right=2, "BF_{\omega_1}"] & \\
& \mathsf{E}\varphi \wedge \mathsf{E}(\mathsf{E}^1\varphi \wedge \mathsf{E}^2\varphi \wedge \dots) \arrow[u, shift right=2, "CBF_{\omega_1}"'] \arrow[r, leftrightarrow] & \mathsf{E}\varphi \wedge \mathsf{EC}\varphi
\end{tikzcd}$$

Indeed, BF_{ω_1} plays a key role in the axiomatization of $\mathbf{K}_{\omega_1}$. It has been proved by Tanaka ([15]; see also [17]) that the sequent calculus $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$—that is, $\mathbf{LK}^{\Box}_{\omega_1}$ plus all instances of $\Rightarrow BF_{\omega_1}$ as further initial sequents—axiomatizes $\mathbf{K}_{\omega_1}$. A complete Hilbert-style calculus ($\mathbf{KL}_{\omega_1}$) axiomatizing $\mathbf{K}_{\omega_1}$ and featuring BF_{ω_1} as an axiom had been earlier provided by Radev in [12].

An alternative route is followed in the present paper. We work with *Tait-style* (i.e. one-sided) sequent calculi—but this is of course not essential—and we hide BF_{ω_1}, so to speak, in the syntax. More precisely, after some preliminaries on the (multiagent) modal infinitary language in *negation normal form* we adopt (Sect. 2), we consider (Sect. 3) the two calculi $\mathbf{TK}^{\sharp}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$: the essential difference between them is that *sequents* are *finite* (sets of formulas) in $\mathbf{TK}^{\sharp}_{\omega_1}$, whereas they can also be *countably infinite* in $\mathbf{TK}_{\omega_1}$. It is a known fact that this difference does not matter as far as non-modal infinitary logic is concerned; it does however matter when modal operators are added: while $\mathbf{TK}^{\sharp}_{\omega_1}$ is indeed nothing but the one-sided version of $\mathbf{LK}^{\Box}_{\omega_1}$, the calculus $\mathbf{TK}_{\omega_1}$ turns out to be equivalent to $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$.

As we said, $\mathbf{LK}^{\Box}_{\omega_1}$ (our $\mathbf{TK}^{\sharp}_{\omega_1}$) is *incomplete* with respect to Kripke semantics. Yet it is a natural calculus to consider. In Sect. 4 we introduce a *generalized* Kripke semantics (*standard* Kripke semantics being but a limit case of the generalized one) and show that, on the one side, $\mathbf{TK}^{\sharp}_{\omega_1}$ is valid with respect to the generalized semantics while, on the other side, BF_{ω_1} admits a very simple *generalized* countermodel.

In Sect. 5 we prove a *completeness* theorem for $\mathbf{TK}^{\sharp}_{\omega_1}$ with respect to the *generalized* Kripke semantics, thus providing an adequate relational semantics for this system. Actually, one and the same proof strategy—a suitable adaptation of the familiar canonical model construction—gives also, as a bonus, a smooth completeness proof for $\mathbf{TK}_{\omega_1}$ with respect to *standard* Kripke semantics, which is alternative to the completeness proofs given in [15, 17] for $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$, and in [12] for $\mathbf{KL}_{\omega_1}$.

The question concerning *cut-free axiomatizability* of $\mathbf{K}_{\omega_1}$ is discussed in the concluding Sect. 6. In particular, we show by suitable counterexamples that $\mathbf{TK}_{\omega_1}$, as well as some natural variants of this calculus, *do not* admit cut-elimination. As far as we know, only one cut-free axiomatization of $\mathbf{K}_{\omega_1}$ is presently available, Tanaka's calculus $\mathbf{TLM}_{\omega_1}$ ([16]), whose sequents are *trees of standard sequents*. The cut-elimination theorem for $\mathbf{TLM}_{\omega_1}$ is however proved only semantically.

2 The Infinitary Modal Language $\mathcal{L}^{\Box}_{\omega_1}$

The alphabet of our infinitary multiagent propositional language $\mathcal{L}^{\Box}_{\omega_1}$ comprises the symbols:

- $p_0, \widetilde{p}_0, p_1, \widetilde{p}_1, p_2, \widetilde{p}_2 \ldots$: denumerably many propositional atoms and negated propositional atoms (*literals*);
- $\bigwedge, \bigvee, \Box_i, \widetilde{\Box}_i$ $(i < \omega)$: logical operators.[2]

We denote by Lit (Lit^+) the set of all (positive) literals.

The *formulas* of the language $\mathcal{L}^{\Box}_{\omega_1}$ are generated starting from the literals by applying as usual the modal operators $\Box_i$ and $\widetilde{\Box}_i$ $(i < \omega)$, and by forming *countable* disjunctions ($\bigvee$) and conjunctions ($\bigwedge$).

More precisely, the set FM of all $\mathcal{L}^{\Box}_{\omega_1}$*-formulas* is the least fixed point of the monotone operator F such that

$$\begin{aligned} F(X) = \mathsf{Lit} &\cup \{\Box_i x \mid x \in X, i < \omega\} \cup \{\widetilde{\Box}_i x \mid x \in X, i < \omega\} \\ &\cup \{\textstyle\bigwedge Z \mid Z \subseteq X, |Z| \le \omega\} \cup \{\textstyle\bigvee Z \mid Z \subseteq X, |Z| \le \omega\}. \end{aligned}$$

Equivalently,

$$\mathsf{FM} = \bigcup_{\alpha<\omega_1} \mathsf{FM}^{\alpha}$$

where FM^{α} $(\alpha < \omega_1)$ is defined by transfinite induction as follows:

(i) $\mathsf{FM}^0 = \mathsf{Lit}$,
(ii) $\mathsf{FM}^{\beta+1} = F(\mathsf{FM}^{\beta})$,
(iii) $\mathsf{FM}^{\lambda} = \bigcup_{\beta<\lambda} \mathsf{FM}^{\beta}$ (where λ is a limit ordinal).

The *rank* of a formula φ is the least ordinal $\beta < \omega_1$ such that $\varphi \in \mathsf{FM}^{\beta}$. The notion of *subformula* of a formula is defined as usual; notice that $\mathsf{sbf}(\varphi)$, the set of all subformulas of φ, is always countable.

Notational conventions 2.1 Henceforth, the lower-case Greek letters φ, ψ, χ, possibly with indices, are used as metavariables for formulas. Capital Greek letters $\Gamma, \Delta, \Phi, \Psi, \ldots$ will range over *countable subsets* of FM, while capital Roman letters $C, D, \ldots$ will range over *finite subsets* of FM.

Notice that the connective '$\neg$' is not contained in the alphabet of $\mathcal{L}^{\Box}_{\omega_1}$; officially, $\mathcal{L}^{\Box}_{\omega_1}$-formulas are always in *negation normal form* (*NNF*). It is however convenient to introduce *negation* in the metalanguage. The map

$$\varphi \in \mathsf{FM} \ \mapsto \ \neg\varphi \in \mathsf{FM}$$

---

[2] There is no special reason behind our choice of taking $\widetilde{\Box}_i$ (where $\Box\varphi \equiv \neg\widetilde{\Box}_i\varphi$) as a primitive operator in place of the more familiar $\Diamond_i$ (where $\Box_i\varphi \equiv \neg\Diamond_i\neg\varphi$).

is defined inductively in the natural way:

(i) $\neg p_k := \widetilde{p}_k, \quad \neg\widetilde{p}_k := p_k,$
(ii) $\neg\Box_i\psi := \widetilde{\Box}_i\psi, \quad \neg\widetilde{\Box}_i\psi := \Box_i\psi,$
(iii) $\neg\bigwedge\Psi := \bigvee\{\neg\psi \mid \psi \in \Psi\}, \quad \neg\bigvee\Psi := \bigwedge\{\neg\psi \mid \psi \in \Psi\}.$

Thus $\neg\neg\varphi$ and φ are syntactically identical (notation: $\neg\neg\varphi \equiv \varphi$). Observe that φ need not be a subformula of $\neg\varphi$.

Notational conventions 2.2 • $\Phi, \psi := \Phi \cup \{\psi\}; \quad \Phi, \Psi := \Phi \cup \Psi; \quad \dots;$
- $\neg\Phi := \{\neg\varphi \mid \varphi \in \Phi\};$
- $\varphi \wedge \psi := \bigwedge\{\varphi, \psi\}; \quad \varphi \vee \psi := \bigvee\{\varphi, \psi\};$
- $\varphi \to \psi := \neg\varphi \vee \psi;$
- $\top := \bigwedge\emptyset; \quad \bot := \bigvee\emptyset;$
- $\Box_i\Phi := \{\Box_i\varphi \mid \varphi \in \Phi\}; \quad \widetilde{\Box}_i\Phi := \{\widetilde{\Box}_i\varphi \mid \varphi \in \Phi\}.$

Note that $\neg\top \equiv \bot$ and $\neg\bot \equiv \top$.

We conclude the present section by introducing one further notion, which however will not be used until Sect. 5.

Definition 2.3 (*Environment*) Let Γ be a subset of FM. The *environment* of Γ, in symbols $\mathcal{E}[\Gamma]$, is the *least* subset of FM satisfying the following conditions:

(i) $\Gamma \subseteq \mathcal{E}[\Gamma]$;
(ii) $\mathcal{E}[\Gamma]$ is closed under subformulas and under negation;
(iii) $\mathcal{E}[\Gamma]$ is closed under *finite* conjunctions and disjunctions, as well as under $\Box_i$ and $\widetilde{\Box}_i$ $(i < \omega)$;
(iv) for each formula $\psi \in \mathcal{E}[\Gamma]$ and $\bigwedge\Phi \in \mathcal{E}[\Gamma]$, and for every $i < \omega$, the formula $\Box_i \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}$ belongs to $\mathcal{E}[\Gamma]$.

Lemma 2.4 *For every* countable *set* Γ *of formulas, its environment* $\mathcal{E}[\Gamma]$ *is* countable.

Proof Straightforward. $\Box$

3 Tait-Style Infinitary Modal Calculi

As anticipated in the introductory section, we work with *Tait-style* sequent calculi. Along with the *NNF*-syntax of our language $\mathcal{L}^{\Box}_{\omega_1}$, this format allows a considerable economy and elegance in the presentation of the proof systems under investigation.

The sequents to be derived are therefore not the usual *two-sided* sequents, but rather *one-sided* sequents, that is *sets of formulas*, having a *disjuctive* reading in the intended interpretation.[3]

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[3] In a classical environment, there is an obvious correspondence between two-sided and one-sided sequents. '$\Gamma \Rightarrow \Delta$' $\rightsquigarrow$ '$\neg\Gamma, \Delta$', and 'Γ' $\rightsquigarrow$ '$\neg\Gamma_1 \Rightarrow \Gamma_2$' for each partition (Γ_1, Γ_2) of Γ.

$$\frac{}{p, \neg p}\,\texttt{ID}\ \ (p \in \mathsf{Lit}^+) \qquad \frac{C}{C, D}\,\texttt{W} \qquad \frac{C, \varphi \qquad D, \neg\varphi}{C, D}\,\texttt{CUT}$$

$$\frac{C, \varphi}{C, \bigvee \Phi}\,\texttt{OR}\ \ (\varphi \in \Phi) \qquad \frac{\dots\ C, \varphi\ \dots\ (\mathit{all}\ \varphi \in \Phi)}{C, \bigwedge \Phi}\,\texttt{AND}$$

Fig. 1 $\mathbf{T}^{\sharp}_{\omega_1}$

$$\frac{}{p, \neg p}\,\texttt{ID}\ \ (p \in \mathsf{Lit}^+) \qquad \frac{\Gamma}{\Gamma, \Delta}\,\texttt{W} \qquad \frac{\Gamma, \varphi \qquad \Delta, \neg\varphi}{\Gamma, \Delta}\,\texttt{CUT}$$

$$\frac{\Gamma, \varphi}{\Gamma, \bigvee \Phi}\,\texttt{OR}\ \ (\varphi \in \Phi) \qquad \frac{\dots\ \Gamma, \varphi\ \dots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma, \bigwedge \Phi}\,\texttt{AND}$$

Fig. 2 $\mathbf{T}^{*}_{\omega_1}$

A choice concerning the *cardinality of sequents* must however be taken right from the start: shall we confine to *finite* sequents only? Or shall we allow countably *infinite* sequents too?

As far as infinitary classical *truth-functional* logic is concerned, it makes no essential difference which of the two alternatives we adopt (see e.g. [4]). Indeed, let us start by considering three Tait-style *non-modal* propositional calculi $\mathbf{T}^{\sharp}_{\omega_1}$, $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$, whose inference rules[4] are shown in Figs. 1, 2 and 3.

$\mathbf{T}^{\sharp}_{\omega_1}$ derives *finite* sets of formulas, whereas both $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ derive *countable*, possibly infinite sets of formulas (recall the Notational conventions 2.1). Modulo this difference, $\mathbf{T}^{\sharp}_{\omega_1}$ and $\mathbf{T}^{*}_{\omega_1}$ have the "same" inference rules; on the other side, $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ differ only in the $\bigvee$-introduction rule.[5] Notice also that the generalized cut-rule with countably many premises

---

[4] Where formulas are of course intended to belong to the $\Box_i$- and $\widetilde{\Box}_i$-less fragment $\mathcal{L}_{\omega_1}$ of $\mathcal{L}^{\Box}_{\omega_1}$. **Caution**: in all the calculi under consideration in this paper sequents are *sets*, not *multisets*. This means that *contraction* is hidden in the logical inference rules: the principal formula of an inference may occur as a side formula in the premise(s); e.g.

$$\frac{C, \varphi \vee \psi, \psi}{C, \varphi \vee \psi} \quad \text{and} \quad \frac{\Gamma, \Phi, \bigvee \Phi}{\Gamma, \bigvee \Phi}$$

are instances of `OR`, respectively `OR`$^+$.

[5] Of course, because of the presence of the rule `W`, the rule `OR` is a derived rule in $\mathbf{T}_{\omega_1}$.

$$\boxed{\mathbf{T}_{\omega_1} = (\mathbf{T}^{*}_{\omega_1} - \mathtt{OR}) + \frac{\Gamma, \Phi}{\Gamma, \bigvee \Phi}\ \mathtt{OR}^{+}}$$

Fig. 3 $\mathbf{T}_{\omega_1}$

$$\frac{\Gamma, \Phi \qquad \ldots\ \neg\varphi, \Delta\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma, \Delta}\ \mathtt{CUT}^{+}$$

is admissible (as a derived rule) in $\mathbf{T}_{\omega_1}$; actually $\mathbf{T}_{\omega_1}$ is equivalent to the calculus obtained from $\mathbf{T}^{*}_{\omega_1}$ by replacing the rules $\mathtt{OR}^{+}$ and $\mathtt{CUT}$ with the rules $\mathtt{OR}$ and $\mathtt{CUT}^{+}$, respectively.

It is immediately verified that

(1) for an arbitrary formula φ, the sequents

$$\varphi, \neg\varphi \qquad \text{and} \qquad \top$$

are cut-free derivable in any of these calculi, the latter through a vacuous application of the $\mathtt{AND}$ rule;

(2) for every C and Γ,

$$\mathbf{T}^{\sharp}_{\omega_1} \vdash C \ \Leftrightarrow\ \mathbf{T}^{*}_{\omega_1} \vdash C \quad \text{and} \quad \mathbf{T}^{*}_{\omega_1} \vdash \Gamma \ \Rightarrow\ \mathbf{T}_{\omega_1} \vdash \Gamma.$$

In fact, also the second arrow in (2) above can be reversed and, in a sense to be specified, the three calculi are *equivalent*. Let us write

- '$\models_{\omega_1} \Gamma$' to mean that $\bigvee \Gamma$ is truth-functionally valid: for every valuation $\mathsf{v} : \mathsf{Lit}^{+} \longrightarrow \{0, 1\}$ there is some $\varphi \in \Gamma$ such that $\mathsf{v}(\varphi) = 1$ (v being extended from positive literals to arbitrary $\mathcal{L}_{\omega_1}$-formulas in the natural way);
- '$\vdash_0 \Gamma$' to mean that Γ is *cut-free derivable*.

Then the relations between the three calculi, their soundness and semantic completeness, as well as the cut-elimination property for $\mathbf{T}^{\sharp}_{\omega_1}$, can be summarized as follows on the basis of known results.

Proposition 3.1 *Let C be a finite set of $\mathcal{L}_{\omega_1}$-formulas, and $\{\Gamma_1, \ldots, \Gamma_n\}$ be a finite, possibly empty set of countable sets of $\mathcal{L}_{\omega_1}$-formulas. Then the following are equivalent:*

(1) $\mathbf{T}^{\sharp}_{\omega_1} \vdash_0 \bigvee \Gamma_1, \ldots, \bigvee \Gamma_n, C$;
(2) $\mathbf{T}^{\sharp}_{\omega_1} \vdash \bigvee \Gamma_1, \ldots, \bigvee \Gamma_n, C$;
(3) $\mathbf{T}^{*}_{\omega_1} \vdash \Gamma_1, \ldots, \Gamma_n, C$;
(4) $\mathbf{T}_{\omega_1} \vdash \Gamma_1, \ldots, \Gamma_n, C$;
(5) $\models_{\omega_1} \Gamma_1, \ldots, \Gamma_n, C$.

Fig. 4 $\mathbf{TK}^{\sharp}_{\omega_1}$, $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$

$$\mathbf{TK}^{\sharp}_{\omega_1} := \mathbf{T}^{\sharp}_{\omega_1} + \frac{\neg C, \varphi}{\neg\Box_i C, \Box_i \varphi}\,\mathtt{K}^{\sharp}_i \quad (i < \omega)$$

$$\mathbf{TK}^{*}_{\omega_1} := \mathbf{T}^{*}_{\omega_1} + \frac{\neg \Gamma, \varphi}{\neg\Box_i \Gamma, \Box_i \varphi}\,\mathtt{K}_i \quad (i < \omega)$$

$$\mathbf{TK}_{\omega_1} := \mathbf{T}_{\omega_1} + \frac{\neg \Gamma, \varphi}{\neg\Box_i \Gamma, \Box_i \varphi}\,\mathtt{K}_i \quad (i < \omega)$$

Proof (1) ⇒ (2), (3) ⇒ (4), (4) ⇒ (5): trivial.
(2) ⇒ (3): straightforward, because $\mathbf{T}^{*}_{\omega_1} \vdash \neg \bigvee \Gamma, \Gamma$.
(5) ⇒ (1): completeness and cut-elimination for $\mathbf{T}^{\sharp}_{\omega_1}$ are proved in [14]. □

Notice that the syntactic proof of cut-elimination for $\mathbf{T}^{\sharp}_{\omega_1}$ given in [14] can be easily adapted to $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ (see e.g. [7]; cp. also [8] and [4]).[6]

Let us now extend the above calculi with modal inference rules, with the aim of capturing—by means of an adequate infinitary Tait-style calculus—the infinitary modal propositional logic determined by the class of all Kripke frames, i.e. the infinitary version $\mathbf{K}_{\omega_1}$ of the (multiagent) modal system **K**. Taking into account the need for a preliminary choice concerning the cardinality of sequents, we shall consider on the basis of the previous investigation the three candidates $\mathbf{TK}^{\sharp}_{\omega_1}$, $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ shown in Fig. 4.

The present scenario turns out to be radically different from the previous (non-modal) one: now the alternative "finite vs countable sequents" does matter! Indeed, we have:

(a) $\mathbf{TK}^{\sharp}_{\omega_1}$ admits cut-elimination, but is *incomplete* (though obviously sound) with respect to Kripke semantics;
(b) $\mathbf{TK}^{*}_{\omega_1}$ too admits cut-elimination and is *incomplete* (although obviously sound) with respect to Kripke semantics; but it is not equivalent to $\mathbf{TK}^{\sharp}_{\omega_1}$ in the sense in which $\mathbf{T}^{\sharp}_{\omega_1}$ and $\mathbf{T}^{*}_{\omega_1}$ are equivalent according to Proposition 3.1;
(c) $\mathbf{TK}_{\omega_1}$ is instead *sound and complete* with respect to Kripke semantics; yet it provably does not admit cut-elimination.

---

[6] The cut-elimination property of (the two-sided version of) $\mathbf{T}^{\sharp}_{\omega_1}$ is exploited in [6] to prove cut-elimination for linear-time temporal logic **LTL**, by faithfully embedding **LTL** into infinitary non-modal propositional logic.

As to point **(a)**, a syntactic proof of cut-elimination for $\mathbf{TK}^{\sharp}_{\omega_1}$ can be obtained by adapting Tait's proof of the cut-elimination theorem for $\mathbf{T}^{\sharp}_{\omega_1}$ ([14], see also [16])—the same does hold for $\mathbf{TK}^{*}_{\omega_1}$. The semantical incompleteness of $\mathbf{TK}^{\sharp}_{\omega_1}$ follows in turn as a consequence of cut-elimination, because of the following easily verifiable facts:

Fact 3.2 The schema ("*Barcan Formula*")

$$\begin{gathered}\bigwedge \Box_i \Phi \rightarrow \Box_i \bigwedge \Phi, \\ \text{or, as a finite sequent,} \quad \neg \bigwedge \Box_i \Phi,\ \Box_i \bigwedge \Phi\end{gathered} \tag{BF_{ω_1}}$$

is valid in every Kripke model (see the next section).

Fact 3.3 The instance

$$\neg \bigwedge \{\Box_i p_n \mid n < \omega\}, \Box_i \bigwedge \{p_n \mid n < \omega\}$$

of BF_{ω_1} has no *cut-free* derivation in $\mathbf{TK}^{\sharp}_{\omega_1}$.

Notice that BF_{ω_1} is (cut-free) derivable in $\mathbf{TK}_{\omega_1}$ as follows:

$$\dfrac{\dfrac{\dfrac{\cdots \left\{ \dfrac{}{\neg\Phi, \varphi}\,{\scriptstyle \mathtt{ID,\ W}} \right\} \cdots \quad (\varphi \in \Phi)}{\neg\Phi, \bigwedge \Phi}\,{\scriptstyle \mathtt{AND}}}{\neg\Box_i \Phi, \Box_i \bigwedge \Phi}\,{\scriptstyle \mathtt{K}_i}}{\neg \bigwedge \Box_i \Phi, \Box_i \bigwedge \Phi}\,{\scriptstyle \mathtt{OR}^{+}}$$

On the other side, the "*converse Barcan Formula*"

$$\Box_i \bigwedge \Phi \rightarrow \bigwedge \Box_i \Phi$$

is derivable already in $\mathbf{TK}^{\sharp}_{\omega_1}$:

$$\dfrac{\cdots \left\{ \dfrac{\dfrac{\neg\varphi, \varphi}{\neg \bigwedge \Phi, \varphi}\,{\scriptstyle \mathtt{OR}}}{\neg\Box_i \bigwedge \Phi, \Box_i \varphi}\,{\scriptstyle \mathtt{K}_i} \right\} \cdots \quad (\varphi \in \Phi)}{\neg\Box_i \bigwedge \Phi, \bigwedge \Box_i \Phi}\,{\scriptstyle \mathtt{AND}}$$

As to point **(b)**, it is immediate to verify that $\mathbf{TK}^{\sharp}_{\omega_1}$ and $\mathbf{TK}^{*}_{\omega_1}$ derive the same *finite* sequents,

$$\mathbf{TK}^{\sharp}_{\omega_1} \vdash C \;\Leftrightarrow\; \mathbf{TK}^{*}_{\omega_1} \vdash C \qquad \text{for every } C, \tag{3.1}$$

like the corresponding non-modal calculi. Hence BF_{ω_1} is not derivable in $\mathbf{TK}^{*}_{\omega_1}$ as well. On the other side, contrary to Proposition 3.1,

$$\mathbf{TK}^{*}_{\omega_1} \vdash \Gamma \;\not\Rightarrow\; \mathbf{TK}^{\sharp}_{\omega_1} \vdash \bigvee \Gamma. \tag{3.2}$$

For instance, let $\Psi := \{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2 \ldots , \Box_i \bigwedge_{n<\omega} p_n\}$. Then $\mathbf{TK}^{*}_{\omega_1} \vdash \Psi$

$$\dfrac{\cdots\left\{\dfrac{}{\{\neg p_n\}_{n<\omega}, p_m}\,\texttt{ID, W}\right\}\cdots\ (m<\omega)}{\dfrac{\{\neg p_n\}_{n<\omega}, \bigwedge_{n<\omega} p_n}{\{\neg\Box_i p_n\}_{n<\omega}, \Box_i \bigwedge_{n<\omega} p_n}\,\texttt{K}_i}\,\texttt{AND}$$

but, as a semantic argument in the next section (Fact 4.6) will show,[7] $\mathbf{TK}^{\sharp}_{\omega_1} \nvdash \bigvee \Psi$.

As to the two claims made in point **(c)**, these will be addressed in Sect. 5, where the semantic completeness of $\mathbf{TK}_{\omega_1}$ is proved, and in Sect. 6, where the question concerning the non eliminability of the cut rule in this calculus is discussed. Notice that merely on the basis of what has been established so far, in particular (3.1) and (3.2) above, we already know that $\mathbf{TK}^{*}_{\omega_1}$ *is not closed* under the inference rule $\texttt{OR}^{+}$ characteristic of $\mathbf{TK}_{\omega_1}$.

We conclude this section by stating a simple equivalence result, by which the key role played in the present context by the Barcan Formula BF_{ω_1} is made fully evident.

Let $\mathbf{TKB}^{\sharp}_{\omega_1}$ and $\mathbf{TKB}^{*}_{\omega_1}$ be the calculi obtained from $\mathbf{TK}^{\sharp}_{\omega_1}$, resp. $\mathbf{TK}^{*}_{\omega_1}$, by adding all the instances of BF_{ω_1} as new initial sequents. Then:

Proposition 3.4 *For every countable set* Γ *of formulas, the following are equivalent:*

(1) $\mathbf{TK}_{\omega_1} \vdash \Gamma$;
(2) $\mathbf{TKB}^{*}_{\omega_1} \vdash \Gamma$;
(3) $\mathbf{TKB}^{\sharp}_{\omega_1} \vdash \bigvee \Gamma$.

Proof (2) ⇒ (1): obvious, since $\mathbf{TK}_{\omega_1} \vdash BF_{\omega_1}$.
(3) ⇒ (2): it is sufficient to observe that the inversion of $\texttt{OR}^{+}$ is admissible (as a derived rule) in $\mathbf{TK}^{*}_{\omega_1}$:

$$\dfrac{\Gamma, \bigvee\Phi \qquad\qquad \dfrac{\cdots\left\{\dfrac{}{\neg\varphi, \Phi}\,\texttt{ID, W}\right\}\cdots\ (\varphi\in\Phi)}{\neg\bigvee\Phi, \Phi}\,\texttt{AND}}{\Gamma, \Phi}\,\texttt{CUT}$$

---

[7] A simple syntactic argument is also at hand, by using Fact 3.3 and (see below) point (ii) in the proof of Proposition 3.4.

(1) ⇒ (3): first of all, observe that for every Γ, Δ, C :

(i) $\mathbf{TK}^{\sharp}_{\omega_1} \vdash \neg \bigvee \Gamma, \bigvee(\Gamma \cup \Delta)$;
(ii) $\mathbf{TK}^{\sharp}_{\omega_1} \vdash \neg \bigvee(\Gamma \cup C), \bigvee \Gamma, C$;
(iii) $\mathbf{TK}^{\sharp}_{\omega_1} \vdash \neg \bigvee(\Gamma \cup \Delta), \bigvee \Gamma, \bigvee \Delta$.

The easy verification is left to the reader.
Next, given a $\mathbf{TK}_{\omega_1}$-derivation $\mathfrak{D} \vdash \Gamma$, we produce a $\mathbf{TKB}^{\sharp}_{\omega_1}$-derivation $\mathfrak{D}' \vdash \bigvee \Gamma$ arguing by transfinite induction on the height of $\mathtt{h}(\mathfrak{D}) < \omega_1$ of $\mathfrak{D}$[8] and taking cases according to the final inference R of $\mathfrak{D}$. The case $R = \mathtt{ID}$ is trivial; the cases $R = \mathtt{W}$, $R = \mathtt{CUT}$ are easily dealt with using the I.H. together with (i) and (ii) above. Let us spell out the details only for the cases $R = \mathtt{OR}^+$ ($R = \mathtt{AND}$ is similar) and $R = \mathtt{K}_i$.
[$R = \mathtt{OR}^+$]: then, taking into account the possibility that the principal formula is a side formula in the premise, $\mathfrak{D}$ has the form:

$$\dfrac{\begin{matrix}\vdots \\ \Delta, \bigvee \Phi, \Phi\end{matrix}}{\Delta, \bigvee \Phi}\;\mathtt{OR}^+$$

We obtain $\mathfrak{D}' \vdash \bigvee(\Delta \cup \{\bigvee \Phi\})$ in $\mathbf{TKB}^{\sharp}_{\omega_1}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \textbf{I.H.} \\ \bigvee(\Delta \cup \Phi \cup \{\bigvee \Phi\})\end{matrix}}{\bigvee(\Delta \cup \Phi), \bigvee \Phi}\;\mathtt{CUT}\text{ with (ii)}}{\bigvee \Delta, \bigvee \Phi}\;\mathtt{CUT}\text{ with (iii)}}{\bigvee(\Delta \cup \{\bigvee \Phi\}), \bigvee \Phi}\;\mathtt{CUT}\text{ with (i)}}{\bigvee(\Delta \cup \{\bigvee \Phi\})}\;\mathtt{OR}$$

[$R = \mathtt{K}_i$]: then $\mathfrak{D}$ has the form:

$$\dfrac{\begin{matrix}\vdots \\ \neg\Delta, \varphi\end{matrix}}{\neg\Box_i \Delta, \Box_i \varphi}\;\mathtt{K}_i$$

---

[8] The height of a derivation $\mathfrak{D}$ in $\mathbf{TK}_{\omega_1}$ is defined in the standard way: $\mathtt{h}(\mathfrak{D}) = 0$ if $\mathfrak{D}$ is an axiom ($\mathtt{ID}$), and $\mathtt{h}(\mathfrak{D}) = \sup\{\mathtt{h}(\mathfrak{D}_i) + 1 \mid i \in I\}$ if $\mathfrak{D}$ results from derivations $\{\mathfrak{D}_i\}_{i \in I}$ by an application of an $|I|$-premises inference rule $\mathtt{W}$, $\mathtt{OR}^+$, $\mathtt{CUT}$ ($|I| \leq 2$), $\mathtt{AND}$ ($|I| \leq \omega$). Of course $\mathtt{h}(\mathfrak{D})$ can exceed finite ordinals due to the presence of the rule $\mathtt{AND}$, possibly having denumerably many premises.

We obtain $\mathfrak{D}' \vdash \bigvee(\neg\Box_i\Delta \cup \{\Box_i\varphi\})$ in $\mathbf{TKB}^{\sharp}_{\omega_1}$ as follows:

$$\cfrac{\cfrac{\cfrac{\overset{BF_{\omega_1}}{\bigvee\neg\Box_i\Delta,\Box_i\bigwedge\Delta} \qquad \cfrac{\cfrac{\overset{\vdots\ \textbf{I.H.}}{\bigvee(\neg\Delta\cup\{\varphi\})}}{\neg\bigwedge\Delta,\varphi}\ \text{CUT with (ii)}}{\neg\Box_i\bigwedge\Delta,\Box_i\varphi}\ \mathrm{K}_i^{\sharp}}{\bigvee\neg\Box_i\Delta,\Box_i\varphi}\ \text{CUT}}{\bigvee(\neg\Box_i\Delta\cup\{\Box_i\varphi\}),\Box_i\varphi}\ \text{CUT with (i)}}{\bigvee(\neg\Box_i\Delta\cup\{\Box_i\varphi\})}\ \text{OR}$$

□

4 Generalized (and Standard) Kripke Frames

As we saw, $\mathbf{TK}^{\sharp}_{\omega_1}$ is *incomplete* with respect to Kripke semantics. In order to provide this calculus with a (possibly natural) *adequate* semantics, we introduce here a generalization of standard Kripke models.

A *generalized* (multiagent) Kripke frame is a pair

$$\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}\rangle$$

where W is a nonempty set and, for each $i < \omega$, $\mathfrak{R}_i$ is a *nonempty family* of binary relations over W which is *downward directed* with respect to inclusion, i.e. it satisfies:

$$(\forall R \in \mathfrak{R}_i)(\forall S \in \mathfrak{R}_i)(\exists T \in \mathfrak{R}_i)(T \subseteq R \cap S) \quad (i < \omega). \tag{DD}$$

Standard Kripke frames are of course a special case of generalized Kripke frames, namely the case in which the family $\mathfrak{R}_i$ reduces to a singleton $\{R_i\}$—singletons trivially satisfy the condition (DD)—for each $i < \omega$. Intuitively, one may think of a generalized frame as a "well-behaved" system of approximations of a standard frame. The relations in $\mathfrak{R}_i$ are the available approximations of the "real" accessibility relation (not necessarily contained in $\mathfrak{R}_i$) for agent i—$R \in \mathfrak{R}_i$ being a *finer* approximation than $S \in \mathfrak{R}_i$ just in case that $R \subseteq S$. The condition (DD) then amounts to a *no-conflict* request, saying that any two approximations have a common refinement in $\mathfrak{R}_i$—see also Remark 4.4 below.[9]

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[9] Kripke-style semantics based on generalized Kripke frames *not* satisfying the condition (DD) have been considered in the literature (under diverse names, like *multi-relational* semantics, or *multiplex* semantics), see e.g. [2, 5, 13], in order to model various types of non-normal modal logics, in particular deontic systems allowing conflicts of obligation (in which case the relations $R \in \mathfrak{R}_i$ are seen as distinct, possibly conflicting normative standards for agent i).

When dealing with standard frames and models (see below) we will keep to the familiar way of presentation, by identifying $\{R_i\}$ with R_i. In other words, we write $\mathsf{S} = \langle W, \{R_i\}_{i<\omega} \rangle$ instead of $\mathsf{S} = \langle W, \{\{R_i\}\}_{i<\omega} \rangle$, when S is standard.

A valuation over a generalized frame G is, as usual, a map

$$\mathsf{v} : \mathsf{Lit}^+ \longrightarrow 2^W.$$

Finally, a *generalized Kripke model* is a triple

$$\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v} \rangle$$

where $\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega} \rangle$ is a generalized Kripke frame and v is a valuation over G.

Given a generalized model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v} \rangle$, an element w of $W_{\mathcal{M}} = W$ and a formula $\varphi \in \mathsf{FM}$, the relation

$$\mathcal{M}, w \vDash_g \varphi$$

is defined inductively as follows:

(i) $\mathcal{M}, w \vDash_g p_k$ iff $w \in \mathsf{v}(p_k)$;
(ii) $\mathcal{M}, w \vDash_g \widetilde{p}_k$ iff $w \notin \mathsf{v}(p_k)$;
(iii) $\mathcal{M}, w \vDash_g \bigwedge \Psi$ iff $\mathcal{M}, w \vDash_g \psi$ for each $\psi \in \Psi$;
(iv) $\mathcal{M}, w \vDash_g \bigvee \Psi$ iff $\mathcal{M}, w \vDash_g \psi$ for some $\psi \in \Psi$;
(v) $\mathcal{M}, w \vDash_g \Box_i \psi$ iff there exists a relation $R \in \mathfrak{R}_i$ such that, for every $u \in W$, wRu implies $\mathcal{M}, u \vDash_g \psi$ $(i < \omega)$;
(vi) $\mathcal{M}, w \vDash_g \widetilde{\Box}_i \psi$ iff for each $R \in \mathfrak{R}_i$ there is a state $u \in W$ such that wRu and $\mathcal{M}, u \nvDash_g \psi$ $(i < \omega)$

where '$\mathcal{M}, w \nvDash_g \varphi$' is short for '*not*$(\mathcal{M}, w \vDash_g \varphi)$'.

Notice that, for every formula φ, we have

$$\mathcal{M}, w \vDash_g \neg\varphi \quad \text{iff} \quad \mathcal{M}, w \nvDash_g \varphi$$

as expected.

Truth of a formula φ in a generalized model $\mathcal{M}$, in symbols

$$\mathcal{M} \models_g \varphi$$

as well as *generalized universal validity* of a formula, in symbols

$$\models_g \varphi$$

are defined as usual:

- $\mathcal{M} \models_g \varphi$ iff $\mathcal{M}, w \vDash_g \varphi$ for all $w \in W_{\mathcal{M}}$;
- $\models_g \varphi$ iff $\mathcal{M} \models_g \varphi$ for all generalized models $\mathcal{M}$.

All these notions are extended to arbitrary countable sets of formulas Γ according to the following

Notational conventions 4.1 • $\mathcal{M}, w \vDash_g \Gamma$ iff $\mathcal{M}, w \vDash_g \varphi$ *for some* $\varphi \in \Gamma$;
- $\mathcal{M} \models_g \Gamma$ iff $\mathcal{M}, w \vDash_g \Gamma$ for all $w \in W_{\mathcal{M}}$;
- $\models_g \Gamma$ iff $\mathcal{M} \models_g \Gamma$ for all generalized models $\mathcal{M}$.

Caution: note the *disjunctive reading* of $\mathcal{M}, w \vDash_g \Gamma$!

Standard models coincide with *generalized models based on standard frames* $\mathsf{S} = \langle W, \{R_i\}_{i<\omega}\rangle$. In fact, as it is immediately seen, in this case the two *non-standard* clauses (v) and (vi) of the above inductive definition become the usual:

(v)$_{st}$ $\mathcal{M}, w \vDash \Box_i\psi$ iff wR_iu implies $\mathcal{M}, u \vDash \psi$ for every $u \in W$ $(i < \omega)$;
(vi)$_{st}$ $\mathcal{M}, w \vDash \widetilde{\Box}_i\psi$ iff there exists a state $u \in W$ s.t. wR_iu and $\mathcal{M}, u \nvDash \psi$ $(i < \omega)$

and so the satisfaction relation '$\mathcal{M}, w \vDash_g \varphi$' boils down to the familiar one '$\mathcal{M}, w \vDash \varphi$'.

Henceforth, when dealing with *standard* models $\mathcal{M}$, we will drop the index g from '$\mathcal{M}, w \vDash_g \varphi$' and from all the related notions, including those of Convention 4.1.

Remark 4.2 It is easy to verify that every generalized Kripke model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{V}\rangle$ satisfying the condition that the relation $R_i^* = \bigcap \mathfrak{R}_i$ belongs to $\mathfrak{R}_i$ $(i < \omega)$, is equivalent to the *standard* model

$$\mathcal{M}^* = \langle W, \{R_i^*\}_{i<\omega}, \mathsf{V}\rangle$$

in the sense that for every $w \in W$ and every formula φ of $\mathcal{L}^{\Box}_{\omega_1}$:

$$\mathcal{M}, w \models_g \varphi \quad \text{iff} \quad \mathcal{M}^*, w \models \varphi.$$

In view of the downward directedness condition (DD) it then follows that a generalized Kripke model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{V}\rangle$ in which every family $\mathfrak{R}_i$ is *finite* (even more so, one in which the set of states W is *finite*) is nothing but a standard Kripke model in disguise.

Trivially, the three calculi $\mathbf{TK}^{\sharp}_{\omega_1}$, $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ are valid with respect to the standard Kripke semantics. Yet, after having generalized the notions of Kripke frame, Kripke model and universal validity in the way described above, we can easily realize that the soundness of the necessitation rule and of the characteristic *schema K*:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

of the minimal normal modal system **K** *is not lost*. As a consequence, we have that the infinitary system $\mathbf{TK}^{\sharp}_{\omega_1}$ is in fact *valid* with respect to the *generalized* semantics.

Theorem 4.3 (**g-*Validity for*** $\mathbf{TK}^{\sharp}_{\omega_1}$) *For every finite set* C *of* $\mathcal{L}^{\Box}_{\omega_1}$*-formulas:*

$$\vdash^{\sharp} C \quad \Rightarrow \quad \models_g C.$$

Proof It is sufficient to check that the modal rules $\mathtt{K}^{\sharp}_i$ of $\mathbf{TK}^{\sharp}_{\omega_1}$ preserve truth in any generalized model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{V}\rangle$; that is, for every $i < \omega$:

$$\mathcal{M} \models_g \{\neg\psi_1, \dots, \neg\psi_n, \varphi\} \;\Rightarrow\; \mathcal{M} \models_g \{\neg\Box_i\psi_1, \dots, \neg\Box_i\psi_n, \Box_i\varphi\}.$$

So, assume that $\mathcal{M} \models_g \{\neg\psi_1, \dots \neg\psi_n, \varphi\}$ and suppose, towards a contradiction, that $\mathcal{M} \not\models_g \{\neg\Box_i\psi_1, \dots, \neg\Box_i\psi_n, \Box_i\varphi\}$; then

$$\mathcal{M}, w \vDash_g \Box_i\psi_1, \;\dots\; , \mathcal{M}, w \vDash_g \Box_i\psi_n \tag{4.1}$$

and

$$\mathcal{M}, w \nvDash_g \Box_i\varphi \tag{4.2}$$

for some $w \in W$.

By (4.1), there exist relations $R_1, \dots, R_n \in \mathfrak{R}_i$ such that

$$\text{for } 1 \le k \le n: \quad (\forall u \in W)\,(wR_ku \Rightarrow \mathcal{M}, u \vDash_g \psi_k). \tag{4.3}$$

Applying the downward directedness condition (DD) to $\mathfrak{R}_i$, let $T \in \mathfrak{R}_i$ be such that $T \subseteq R_1 \cap \dots \cap R_n$. Then, by (4.3):

$$\text{for } 1 \le k \le n: \quad (\forall u \in W)\,(wTu \Rightarrow \mathcal{M}, u \vDash_g \psi_k). \tag{4.4}$$

On the other side, it follows by (4.2) and $T \in \mathfrak{R}_i$ that there exists a state $t \in W$ such that

$$wTt \text{ and } \mathcal{M}, t \nvDash_g \varphi. \tag{4.5}$$

Hence, by (4.4) and (4.5):

$$\mathcal{M}, t \nvDash_g \{\neg\psi_1, \dots, \neg\psi_n, \varphi\} \tag{4.6}$$

in contrast with our assumption $\mathcal{M} \models_g \{\neg\psi_1, \dots \neg\psi_1, \varphi\}$. □

Remark 4.4 It is easy to see that the instances of the modal rule $\mathtt{K}^{\sharp}_{\mathtt{i}}$ in which the set C is either empty, or a singleton, are sound *independently* of the condition (DD). In other words, (DD) is *not needed* in order to show that the necessitation rule RN (from φ to infer $\Box_i\varphi$) and the regularity rule RR (from $\varphi \to \psi$ to infer $\Box_i\varphi \to \Box_i\psi$) are truth-preserving in any generalized model. (CC) plays instead an essential role when the *finite* set C contains at least two formulas or, equivalently, in order to show that the schema $\Box_i \varphi \wedge \Box_i \psi \rightarrow \Box_i(\varphi \wedge \psi)$ (the finitary version of BF_{ω_1}) is valid—indeed, by dropping (DD) a generalized countermodel to this formula can be easily constructed.[10]

Now, we show that the Barcan Formula BF_{ω_1} is not universally valid with respect to the *generalized* Kripke semantics.

Fact 4.5 **(g-countermodel for BF_{ω_1})** $\not\models_g BF_{\omega_1}$.

Proof Let us consider the generalized Kripke model

$$\mathcal{C} = \langle \mathbb{N}^*, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$$

where:

- $\mathbb{N}^* := \mathbb{N} \cup \{*\}$ (the natural numbers plus a new state $*$);
- for every $i < \omega$, $\mathfrak{R}_i = \mathfrak{R} := \{R_X \mid X \in cof(\mathbb{N})\}$, where:
  - $cof(\mathbb{N})$ is the set of all *cofinite* subsets of $\mathbb{N}$,
  - $R_X := \{\langle *, n\rangle \mid n \in X\}$;
- $\mathsf{v}(p_n) := \mathbb{N} \setminus \{n\}$, for each $p_n \in \mathsf{Lit}^+$.

The so defined family $\mathfrak{R}$ of relations satisfies the condition (DD) since on the one side cofinite subsets of $\mathbb{N}$ are closed under finite intersection, and on the other side, by definition

$$R_X \cap R_Y = R_{X\cap Y} \in \mathfrak{R} \quad (X, Y \in cof(\mathbb{N})).$$

Now, for $i < \omega$ we have, by construction, that for each $n \geq 0$:

$$(\forall w \in \mathbb{N}^*)(* R_{\mathbb{N}\setminus\{n\}} w \;\Rightarrow\; \mathcal{C}, w \models_g p_n) \tag{4.7}$$

whence

$$\mathcal{C}, * \models_g \Box_i p_n. \tag{4.8}$$

Thus

$$\mathcal{C}, * \models_g \bigwedge\{\Box_i p_n \mid n < \omega\}. \tag{4.9}$$

On the other side, for every $X \in cof(\mathbb{N})$ and every $k \in X$:

$$* R_X k \quad and \quad \mathcal{C}, k \not\models_g p_k. \tag{4.10}$$

---

[10] In the usual Hilbert-style axiomatization of **K** (classical tautologies, axiom-schema K and inference rules `MP`, `RN`) one can equivalently replace K with the axiom schema $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ together with the inference rule `RR`. Multi-relational semantics in which the condition (DD) is not requested (see fn. 9) are thus typically used to model subsystems of **K** in which `RN` and `RR` are accepted, while $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ is rejected.

Hence

$$\mathcal{C}, * \nvDash_g \Box_i \bigwedge\{p_n \mid n < \omega\}. \tag{4.11}$$

We conclude from (4.9) and (4.11) that

$$\mathcal{C} \nvDash_g \bigwedge\{\Box_i p_n \mid n < \omega\} \to \Box_i \bigwedge\{p_n \mid n < \omega\}$$

the latter formula being an instance of BF_{ω_1}. □

Hence, Theorem 4.3 and Fact 4.5 supplement the underivability proof of BF_{ω_1} in $\mathbf{TK}^{\sharp}_{\omega_1}$, mentioned in the previous section, with a semantical argument. We can also use the above generalized Kripke model $\mathcal{C}$, together with Theorem 4.3, to see that the formula

$$\bigvee\{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2, \dots, \Box_i \bigwedge_{n<\omega} p_n\}$$

is not derivable in $\mathbf{TK}^{\sharp}_{\omega_1}$, as claimed without proof in Sect. 3. Indeed, by (4.8) and (4.11), we have

Fact 4.6 $\mathcal{C}, * \nvDash_g \bigvee\{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2, \dots, \Box_i \bigwedge_{n<\omega} p_n\}$.

In conclusion, observe that the results mentioned in point **(b)** of Sect. 3 show that $\mathbf{TK}^{*}_{\omega_1}$ *is not* sound with respect to the generalized Kripke semantics. Of course, both $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ are sound with respect to the narrower class of all the generalized Kripke frames $\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}\rangle$ satisfying the *countable* downward directedness condition

$$(\forall\mathfrak{S} \subseteq \mathfrak{R}_i)(|\mathfrak{S}| \leq \omega \to (\exists T \in \mathfrak{R}_i)(\forall S \in \mathfrak{S})(T \subseteq S)). \tag{DD$_{\omega_1}$}$$

However, *only* $\mathbf{TK}_{\omega_1}$ is also *complete* with respect to this special class of generalized frames. Thus the problem of finding an adequate Kripke-style semantic characterization of $\mathbf{TK}^{*}_{\omega_1}$ remains open.

5 Completeness Theorems for $\mathbf{TK}^{\sharp}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$

We are going to prove, via a *canonical model* technique, a completeness theorem for $\mathbf{TK}^{\sharp}_{\omega_1}$ with respect to the *generalized* Kripke semantics, and a completeness theorem for $\mathbf{TK}_{\omega_1}$ with respect to the *standard* Kripke semantics. As the reader will see, our proofs of the two results run closely parallel and, in fact, eventually diverge only in one very specific key point.

We start with the common part of the two proofs. The notion of *environment* $\mathcal{E}[\Gamma]$ of a set Γ of formulas (see Sect. 2) is employed here for the first time. In the following, we will often make a tacit use of the closure properties of environments, as well as of Lemma 2.4, saying that the environment of a *countable* set of formulas is, in turn, *countable*.

By convenience, let us denote by $\mathbf{J}$ an arbitrarily fixed element of $\{\mathbf{TK}^{\sharp}_{\omega_1}, \mathbf{TK}_{\omega_1}\}$. Recall that the rule `OR` is available also in $\mathbf{TK}_{\omega_1}$ as a derived rule.

Definition 5.1 Let Γ be a countable set of formulas:

(i) $\mathbf{J} \rhd \Gamma \ := \ \mathbf{J} \vdash C$ for some *finite* subset C of Γ;
(ii) Γ is $\mathbf{J}$-*consistent* iff $\mathbf{J} \not\rhd \neg\Gamma$;
(iii) Γ is $\mathbf{J}$-*saturated* iff:

(a) Γ is $\mathbf{J}$-*consistent*,
(b) $\Gamma \cup \neg\Gamma = \mathcal{E}[\Gamma]$,
(c) for all Φ such that $\bigwedge \Phi \in \mathcal{E}[\Gamma] : \Phi \subseteq \Gamma \Rightarrow \bigwedge \Phi \in \Gamma$;

(iv) Γ is $\mathbf{J}$-*uniform* iff for every Φ such that $\bigwedge \Phi \in \mathcal{E}[\Gamma]$:
if $\mathbf{J} \rhd \neg\Gamma, \varphi$ for each $\varphi \in \Phi$, then $\mathbf{J} \rhd \neg\Gamma, \bigwedge \Phi$.

Lemma 5.2 (***Saturated sets***) *Any* $\mathbf{J}$*-saturated set* Γ *satisfies:*

(1) For all $\varphi \in \mathcal{E}[\Gamma]$*, either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*, and not both.*
(2) For all $\varphi \in \mathcal{E}[\Gamma]$*, if* $\mathbf{J} \rhd \neg\Gamma, \varphi$ *then* $\varphi \in \Gamma$.
(3) For all Φ *such that* $\bigwedge \Phi \in \mathcal{E}[\Gamma]$*:* $\Phi \subseteq \Gamma$ *iff* $\bigwedge \Phi \in \Gamma$.
(4) For all Φ *such that* $\bigvee \Phi \in \mathcal{E}[\Gamma]$*:* $\Phi \cap \Gamma \neq \emptyset$ *iff* $\bigvee \Phi \in \Gamma$.

Proof
(1): immediate, by (a) and (b) of Definition 5.1. (iii).
(2): suppose $\mathbf{J} \vdash \neg C, \varphi$ for some $C \subseteq \Gamma$. If $\varphi \notin \Gamma$ then, by (1), $\neg\varphi \in \Gamma$ and so $D := C \cup \{\neg\varphi\}$ is finite subset of Γ such that $\mathbf{J} \vdash \neg D$, against the $\mathbf{J}$-consistency of Γ.
(3): if Φ is such that $\bigwedge \Phi \in \mathcal{E}[\Gamma]$, then also $\Phi \subseteq \mathcal{E}[\Gamma]$ by Definition 2.3. Suppose now that $\bigwedge \Phi \in \Gamma$, and let $\varphi \in \Phi$: as $\mathbf{J} \vdash \neg \bigwedge \Phi, \varphi$ trivially, we have that $\mathbf{J} \rhd \neg\Gamma, \varphi$ and so that $\varphi \in \Gamma$ by (2). Hence $\Phi \subseteq \Gamma$. The converse direction is given by (c) of Definition 5.1.(iii).
(4): by (1) and (3), $\bigvee \Phi \in \Gamma$ iff $\bigwedge \neg\Phi \notin \Gamma$ iff $\neg\Phi \not\subseteq \Gamma$ iff $\Phi \cap \Gamma \neq \emptyset$. ☐

Fact 5.3 (**Distributivity of** $\vee$ over $\bigwedge$)

$$\mathbf{J} \vdash_{(0)} \psi \vee \bigwedge \Phi \ \leftrightarrow\ \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}.$$

Proof

$$\dfrac{\cdots\left\{\dfrac{\dfrac{\neg\psi, \psi, \varphi \qquad\quad \neg\varphi, \psi, \varphi}{\neg(\psi \vee \varphi), \psi, \varphi}\,\text{AND}}{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi, \varphi}\,\text{OR}\right\}\cdots \quad (\varphi \in \Phi)}{\begin{array}{c}\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi, \bigwedge \Phi \\ \hline\hline \neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi \vee \bigwedge \Phi\end{array}\ \text{OR}}\,\text{AND}$$

$$\dfrac{\cdots\left\{\begin{array}{c}\dfrac{\neg\psi,\psi,\varphi \qquad \dfrac{\neg\varphi,\psi,\varphi}{\neg\bigwedge\Phi,\psi,\varphi}\,\texttt{OR}}{\neg(\psi\vee\bigwedge\Phi),\psi,\varphi}\,\texttt{AND}\\ \hline\hline \neg(\psi\vee\bigwedge\Phi),\psi\vee\varphi\end{array}\texttt{OR}\right\}\cdots\ (\varphi\in\Phi)}{\neg(\psi\vee\bigwedge\Phi),\bigwedge\{\psi\vee\varphi\mid\varphi\in\Phi\}}\,\texttt{AND}$$

□

Notational convention 5.4 For $i < \omega$:

$$\sqrt[i]{\Gamma} := \{\varphi \mid \Box_i\varphi \in \Gamma\}.$$

Lemma 5.5 *For every countable set* Γ *of formulas:*

(1) If Γ *is* **J***-consistent then, for all* $\psi \in \mathcal{E}[\Gamma]$, *either* $\Gamma \cup \{\psi\}$ *or* $\Gamma \cup \{\neg\psi\}$ *is* **J***-consistent.*
(2) If Γ *is* **J***-saturated and* $\Box_i\psi \in \mathcal{E}[\Gamma] \setminus \Gamma$ *then* $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ *is* **J***-consistent.*
(3) If Γ *is finite, then* Γ *is* **J***-uniform.*
(4) If Γ *is* **J***-uniform then, for every* $\psi \in \mathcal{E}[\Gamma]$, $\Gamma \cup \{\psi\}$ *is* **J***-uniform.*

Proof (1) is immediate by the CUT rule, and (3) is trivial.
(2): under the assumptions suppose that $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is not **J**-consistent. Then, for some finite set C such that $\Box_i C \subseteq \Gamma$, we have $\mathbf{J} \vdash \neg C, \psi$ and so, by an application of the rule $\mathrm{K}^\sharp$, $\mathbf{J} \vdash \neg\Box_i C, \Box_i\psi$. Thus $\mathbf{J} \rhd \neg\Gamma, \Box_i\psi$, and it now follows by (2) of Lemma 5.2 that $\Box_i\psi \in \Gamma$, against the assumption.
(4): assume that Γ is **J**-uniform, and suppose $\mathbf{J} \rhd \neg\Gamma, \neg\psi, \varphi$ for all $\varphi \in \Phi$. Then $\mathbf{J} \rhd \neg\Gamma, \bigwedge\{\neg\psi\vee\varphi \mid \varphi \in \Phi\}$ by the **J**-uniformity of Γ, plus the closure properties of environments $\mathcal{E}[\Gamma]$. The conclusion follows by CUT using the sequent $\neg\bigwedge\{\neg\psi\vee\varphi \mid \varphi \in \Phi\}, \neg\psi, \bigwedge\Phi$ which is derivable in **J** by Fact 5.3. □

Lemma 5.6 (Saturation) *Every* **J**-consistent and **J**-uniform *countable set of formulas* Γ *can be extended to a countable set* $\Gamma^* \supseteq \Gamma$ *which is* **J***-saturated.*

Proof Assume that the countable set Γ is both **J**-consistent and **J**-uniform. First of all, being Γ countable, we know (Lemma 2.4) that its environment $\mathcal{E}[\Gamma]$ is countable; so let $\chi_0, \chi_1, \chi_2 \ldots$ be an arbitrarily fixed enumeration of $\mathcal{E}[\Gamma]$.
Define now inductively, for $n \geq 0$, a set $\Gamma_n \subseteq \mathcal{E}[\Gamma]$ provably satisfying:

(1) Γ_n is **J**-consistent and **J**-uniform;
(2) $\Gamma \subseteq \Gamma_n \subseteq \Gamma_{n+1}$.

Basis:

- $\Gamma_0 := \Gamma$

Step $\Gamma_k \rightsquigarrow \Gamma_{k+1}$:

Supposing that $\Gamma_0, \ldots, \Gamma_k$ have been defined in a way that (1) and (2) are satisfied, consider the formula χ_k. Since Γ_k is **J**-consistent, at least one of the sets $\Gamma_k \cup \{\chi_k\}$ and $\Gamma_k \cup \{\neg\chi_k\}$ must be **J**-consistent by (1) of Lemma 5.5.

—If $\Gamma_k \cup \{\chi_k\}$ is **J**-consistent, we set:

- $\Gamma_{k+1} := \Gamma_k \cup \{\chi_k\}$.

— If $\Gamma_k \cup \{\chi_k\}$ is not **J**-consistent, and so $\Gamma_k \cup \{\neg\chi_k\}$ is **J**-consistent, we set:

- $\Gamma_{k+1} := \Gamma_k \cup \{\neg\chi_k\}$, in case χ_k is not of the form $\bigwedge \Phi$ for some Φ;
- $\Gamma_{k+1} := \Gamma_k \cup \{\neg\chi_k, \neg\varphi\}$, in case χ_k is of the form $\bigwedge \Phi$, where $\varphi \in \Phi$ is chosen in a way such that $\Gamma_k \cup \left\{\neg \bigwedge \Phi, \neg\varphi\right\}$ is **J**-consistent.

Observe that in the latter case such a formula $\varphi \in \Phi$ always exists. Indeed, otherwise we would have

$$\text{for every } \varphi \in \Phi : \ \mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}), \varphi \tag{5.1}$$

and in turn, since $\Gamma_k \cup \{\neg \bigwedge \Phi\}$ is **J**-uniform by (4) of Lemma 5.5 and the fact that Γ_k is **J**-uniform:

$$\begin{aligned} &\mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}), \bigwedge \Phi, \\ &\text{i.e. } \mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}) \end{aligned} \tag{5.2}$$

against the **J**-consistency of $\Gamma_k \cup \{\neg\chi_k\}$.

Clearly, (1) and (2) are preserved in the step $\Gamma_k \rightsquigarrow \Gamma_{k+1}$. Finally, set:

$$\Gamma^* := \bigcup_{n \geq 0} \Gamma_n .$$

Now $\Gamma^* \supseteq \Gamma$ and Γ^* is easily seen to be saturated. Indeed, it is **J**-consistent by (1) and (2), and $\Gamma^* \cup \neg\Gamma^* = \mathcal{E}[\Gamma] = \mathcal{E}[\Gamma^*]$ by construction. As to the last condition, suppose $\bigwedge \Phi = \chi_n \notin \Gamma^*$. Then, by the construction, $\Gamma_{n+1} = \Gamma_n \cup \{\neg \bigwedge \Phi, \neg\varphi\}$ for some $\varphi \in \Phi$; so $\neg\varphi \in \Gamma^*$ and, by **J**-consistency of Γ^*, $\varphi \notin \Gamma^*$. Hence $\Phi \nsubseteq \Gamma^*$. □

The following Lemma, which makes an essential use of the Barcan Formula, *does hold only* for the calculus $\mathbf{TK}_{\omega_1}$.

Lemma 5.7 *Let* Γ *be a* $\mathbf{TK}_{\omega_1}$*-saturated set of formulas, and let* $\psi \in \mathcal{E}[\Gamma]$ *be such that* $\Box_i \psi \notin \Gamma$. *Then* $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ *is* $\mathbf{TK}_{\omega_1}$*-uniform.*

Proof Under the assumptions, suppose that for each $\varphi \in \Phi$, $\mathbf{TK}_{\omega_1} \rhd \neg(\sqrt[i]{\Gamma} \cup \{\neg\psi\}), \varphi$; that is

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\sqrt[i]{\Gamma}, \psi, \varphi. \tag{5.3}$$

Then, by applying `OR` twice, followed by $\mathtt{K}_i$, we have:

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\Box_i\sqrt[i]{\Gamma}, \Box_i(\psi \vee \varphi) \tag{5.4}$$

and, by taking into account that $\Box_i \sqrt[i]{\Gamma} \subseteq \Gamma$:

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\Gamma, \Box_i(\psi \vee \varphi). \tag{5.5}$$

The set Γ is $\mathbf{TK}_{\omega_1}$-saturated by assumption, and by Definition 2.3 the formulas $\Box_i(\psi \vee \varphi)$ (for $\varphi \in \Phi$) and $\bigwedge\{\Box_i(\psi \vee \varphi) \mid \varphi \in \Phi\}$ all belong to $\mathcal{E}[\Gamma]$. Thus (5.5), together with (2) and (3) of Lemma 5.2, yields:

$$\bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\} \in \Gamma. \tag{5.6}$$

On the other side, we have

$$\mathbf{TK}_{\omega_1} \vdash \neg \bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi) \tag{5.7}$$

which is shown as follows by making use of the fact that BF_{ω_1} is derivable in $\mathbf{TK}_{\omega_1}$ (see Sect. 3):

$$\dfrac{\dfrac{\dfrac{}{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi \vee \bigwedge \Phi}\ \text{Fact 5.3}}{\neg\Box_i \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi)}\ \text{K}}{\neg \bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi)}\ \text{CUT with } BF_{\omega_1}$$

Now (5.6) and (5.7) yield

$$\mathbf{TK}_{\omega_1} \rhd \neg\Gamma, \Box_i(\psi \vee \bigwedge \Phi). \tag{5.8}$$

It follows by (2) of Lemma 5.2 that $\Box_i(\psi \vee \bigwedge \Phi) \in \Gamma$, and so that $\psi \vee \bigwedge \Phi \in \sqrt[i]{\Gamma}$. Therefore $\mathbf{TK}_{\omega_1} \rhd \neg\sqrt[i]{\Gamma}, \psi \vee \bigwedge \Phi$ and finally

$$\mathbf{TK}_{\omega_1} \rhd \neg(\sqrt[i]{\Gamma} \cup \{\neg\psi\}), \bigwedge \Phi$$

using a CUT with $\mathbf{TK}^{(\sharp)}_{\omega_1} \vdash \neg(\chi_1 \vee \chi_2), \chi_1, \chi_2$. $\Box$

We are now ready to define the *canonical models* for $\mathbf{TK}_{\omega_1}$ and $\mathbf{TK}^{\sharp}_{\omega_1}$. Let

- $SAT_{\omega_1} := \{\Gamma \mid \Gamma \subseteq \mathsf{FM}, |\Gamma| \leq \omega, \Gamma \text{ is } \mathbf{TK}_{\omega_1}\text{-saturated}\}$;
- $SAT^{\sharp}_{\omega_1} := \{\Gamma \mid \Gamma \subseteq \mathsf{FM}, |\Gamma| \leq \omega, \Gamma \text{ is } \mathbf{TK}^{\sharp}_{\omega_1}\text{-saturated}\}$.

Definition 5.8 ($\mathbf{TK}_{\omega_1}$*- and* $\mathbf{TK}^{\sharp}_{\omega_1}$*-universal model*)

(1) $\mathcal{U}_{\omega_1}$ is the *standard* Kripke model

$$\mathcal{U}_{\omega_1} = \langle SAT_{\omega_1}, \{R_i\}_{i<\omega}, \mathsf{V}\rangle$$

where

- $\Gamma R_i \Delta :\Leftrightarrow \sqrt[i]{\Gamma} \subseteq \Delta \quad (\Gamma, \Delta \in SAT_{\omega_1}, i < \omega)$;
- $\mathsf{v}(p) := \{\Gamma \in SAT_{\omega_1} \mid p \in \Gamma\}$ for every $p \in \mathsf{Lit}^+$.

(2) $\mathcal{U}^{\sharp}_{\omega_1}$ is the *generalized* Kripke model

$$\mathcal{U}^{\sharp}_{\omega_1} = \langle SAT^{\sharp}_{\omega_1}, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$$

where

- for $i < \omega$, $\mathfrak{R}_i := \{R_i^C \mid C \subseteq_{\text{fin}} \mathsf{FM}\}$, with $\Gamma R_i^C \Delta :\Leftrightarrow \sqrt[i]{\Gamma} \cap C \subseteq \Delta$ $(\Gamma, \Delta \in SAT^{\sharp}_{\omega_1}, i < \omega)$
- $\mathsf{v}(p) := \{\Gamma \in SAT^{\sharp}_{\omega_1} \mid p \in \Gamma\}$ for every $p \in \mathsf{Lit}^+$.

Note that, trivially,

$$R_i^{C\cup D} \subseteq R_i^C \cap R_i^D \;\text{ and }\; R_i^{C\cup D} \in \mathfrak{R}_i,$$

so that $\langle SAT^{\sharp}_{\omega_1}, \{\mathfrak{R}_i\}_{i<\omega}\rangle$ satisfies the characteristic condition (DD) of a generalized frame.

Theorem 5.9

(1) For every $\Gamma \in SAT_{\omega_1}$, for every formula $\varphi \in \mathcal{E}[\Gamma]$:

$$\mathcal{U}_{\omega_1}, \Gamma \vDash \varphi \;\Leftrightarrow\; \varphi \in \Gamma.$$

(2) For every $\Gamma \in SAT^{\sharp}_{\omega_1}$, for every formula $\varphi \in \mathcal{E}[\Gamma]$:

$$\mathcal{U}^{\sharp}_{\omega_1}, \Gamma \vDash_g \varphi \;\Leftrightarrow\; \varphi \in \Gamma.$$

Proof For both (1) and (2) we argue by (transfinite) induction on (the rank of) φ. The cases $\varphi \in \mathsf{Lit}$ and $\varphi \equiv \bigwedge \Phi$, $\bigvee \Phi$ are immediate by Definition 5.9 and Lemma 5.2, (3)–(4). The case $\varphi \equiv \Box_i \psi$ requires instead separate arguments for $\mathcal{U}_{\omega_1}$ and $\mathcal{U}^{\sharp}_{\omega_1}$, see below. Finally, the case $\varphi \equiv \widetilde{\Box}_i \psi$ easily reduces to the previous one by (1) of Lemma 5.2.
$(\mathcal{U}_{\omega_1}) : \varphi \equiv \Box_i \psi$.
[$\Leftarrow$]: immediate by the definition of R_i and the induction hypothesis.
[$\Rightarrow$]: suppose that $\Box_i \psi \notin \Gamma$. Then $\Theta := \sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is $\mathbf{TK}_{\omega_1}$-consistent by (2) of Lemma 5.5 as well as $\mathbf{TK}_{\omega_1}$-uniform by Lemma 5.7. Applying Lemma 5.6 to Θ we find a $\Delta \in SAT_{\omega_1}$ such that $\Gamma R_i \Delta$ and $\psi \notin \Delta$. The conclusion $\mathcal{U}_{\omega_1}, \Gamma \nvDash \Box_i \psi$ follows by applying the induction hypothesis.
$(\mathcal{U}^{\sharp}_{\omega_1}) : \varphi \equiv \Box_i \psi$.
[$\Leftarrow$]: suppose $\Box_i \psi \in \Gamma$, and let $C := \{\psi\}$. Then $R_i^C \in \mathfrak{R}_i$ and, for every $\Delta \in SAT^{\sharp}_{\omega_1}$ such that $\Gamma R_i^C \Delta$ we obviously have $\psi \in \Delta$, and so also $\mathcal{U}^{\sharp}_{\omega_1}, \Delta \vDash_g \psi$ by the induction hypothesis. Hence $\mathcal{U}^{\sharp}_{\omega_1}, \Gamma \vDash_g \Box_i \psi$.

[$\Rightarrow$]: suppose that $\Box_i\psi \notin \Gamma$. To conclude $\mathcal{U}^\sharp_{\omega_1}, \Gamma \nvDash_g \Box_i\psi$ it is sufficient to find, for every finite set C of formulas, a set $\Delta \in SAT^\sharp_{\omega_1}$ such that $\Gamma R^C_i \Delta$ and $\mathcal{U}^\sharp_{\omega_1}, \Delta \nvDash_g \psi$.

This is done as follows: given C, let $D := (\sqrt[i]{\Gamma} \cap C) \cup \{\neg\psi\}$. D is $\mathbf{TK}^\sharp_{\omega_1}$-consistent, for otherwise by the rule $\mathtt{K}^\sharp_i$ we would have $\mathbf{TK}^\sharp_{\omega_1} \rhd \neg\Gamma, \Box_i\psi$ and so by (2) of Lemma 5.2 $\Box_i\psi \in \Gamma$ against the assumption. On the other side, D is finite and so is also $\mathbf{TK}^\sharp_{\omega_1}$-uniform by (3) of Lemma 5.5. It now follows by Lemma 5.6 that there exists a set $\Delta \in SAT^\sharp_{\omega_1}$ such that $D \subseteq \Delta$, and so $\Gamma R^C_i \Delta$, as well as $\neg\psi \in \Delta$, $\psi \notin \Delta$, and finally $\mathcal{U}^\sharp_{\omega_1}, \Delta \nvDash_g \psi$ by the induction hypothesis. ☐

Corollary 5.10 *For every finite set* $C \subseteq \mathsf{FM}$*:*

$$\mathbf{TK}^\sharp_{\omega_1} \nvdash C \quad \Rightarrow \quad \mathcal{U}^\sharp_{\omega_1} \nvDash_g C.$$

Hence $\mathbf{TK}^\sharp_{\omega_1}$ *is sound and complete with respect to* generalized *Kripke semantics.*

Proof If $\mathbf{TK}^\sharp_{\omega_1} \nvdash C$ then $\neg C$ is clearly $\mathbf{TK}^\sharp_{\omega_1}$-consistent; being a finite set, $\neg C$ is also $\mathbf{TK}^\sharp_{\omega_1}$-uniform by (3) of Lemma 5.5. Then by Lemma 5.6 there exists a set $\Theta \in SAT^\sharp_{\omega_1}$ such that $\neg C \subseteq \Theta$, and so also $C \cap \Theta = \emptyset$. It follows by (2) of Theorem 5.9 that $\mathcal{U}^\sharp_{\omega_1}, \Theta \nvDash_g C$. ☐

Corollary 5.11 *For every countable set* $\Gamma \subseteq \mathsf{FM}$*:*

$$\mathbf{TK}_{\omega_1} \nvdash \Gamma \quad \Rightarrow \quad \mathcal{U}_{\omega_1} \nvDash \Gamma.$$

Hence $\mathbf{TK}_{\omega_1}$ *is sound and complete with respect to* standard *Kripke semantics.*

Proof If $\mathbf{TK}_{\omega_1} \nvdash \Gamma$ then also $\mathbf{TK}_{\omega_1} \nvdash \bigvee\Gamma$. So we can argue as above taking $C = \{\bigvee\Gamma\}$ and using (1) of Theorem 5.9 to conclude that $\mathcal{U}_{\omega_1}, \Theta \nvDash \bigvee\Gamma$, hence also $\mathcal{U}_{\omega_1}, \Theta \nvDash \Gamma$, for some $\Theta \in SAT_{\omega_1}$ containing $\bigvee\Gamma$. ☐

6 $\mathbf{TK}_{\omega_1}$ does not admit cut-elimination

As we anticipated, the `CUT` rule cannot be eliminated from $\mathbf{TK}_{\omega_1}$. This will be now demonstrated by exhibiting a suitable example of a valid sequent which is not cut-free derivable in the calculus under investigation.

Below, $\{q^k_n \mid k, n \geq 0\}$ is a set of pairwise distinct positive literals. As previously done, we write '$\vdash_0$' to denote *cut-free* derivability.

Fact 6.1 For every $m \geq 0$ and every $\Phi \subseteq \{q^k_n \mid n \geq 0, k \leq m\}$, if

$$\Theta \subseteq \Delta_\Phi := \neg\Box_i\Phi, \Big\{\bigvee_n \neg\Box_i q^k_n \mid k \geq 0\Big\}, \Box_i \bigwedge_k q^k_k$$

then $\mathbf{TK}_{\omega_1} \nvdash_0 \Theta$.

Proof We argue by transfinite induction on the height $\mathtt{h}(\mathfrak{D})$ of $\mathbf{TK}_{\omega_1}$-derivations $\mathfrak{D}$.

Let $\tau < \omega_1$. Assume (I.H.) that for no set Θ satisfying the hypotheses there is a $\mathbf{TK}_{\omega_1}$-derivation $\mathfrak{D} \vdash_0 \Theta$ with $\mathtt{h}(\mathfrak{D}) < \tau$.

Suppose, by way of contradiction, that for some $m \geq 0$, some $\Phi \subseteq \{q_n^k \mid n \geq 0, k \leq m\}$ and some $\Lambda \subseteq \Delta_\Phi$ there is a cut-free derivation $\mathfrak{D}$ of Λ with $\mathtt{h}(\Lambda) = \tau$. Let $\mathtt{R}$ be the final inference of $\mathfrak{D}$. Clearly $\mathtt{R}$ must be one of $\mathtt{W}$, $\mathtt{OR}^+$, $\mathtt{K}_i$.

If $\mathtt{R} = \mathtt{W}$ we are immediately in contradiction with the I.H.

If $\mathtt{R} = \mathtt{OR}^+$, let $\bigvee_n \neg\Box_i q_n^j$ (for some $j \geq 0$) be the principal formula of the inference and $\mathfrak{D}'$ be the subderivation of the premise Λ'. Then $\Lambda' \subseteq \Delta_{\{q_n^k \mid n \geq 0,\, k \leq r\}}$, where $r = \max(m, j)$. Since $\mathtt{h}(\mathfrak{D}') < \tau$, we are in contradiction with the I.H. again.

Finally, if $\mathtt{R} = \mathtt{K}_i$, it is easily seen that from the subderivation of the premise we would also get a derivation

$$\mathfrak{D}' \vdash \neg\Phi', \bigwedge_k q_k^k$$

for some $\Phi' \subseteq \Phi$. But this is clearly impossible by the boundedness condition on Φ and the soundness of $\mathbf{TK}_{\omega_1}$. □

Proposition 6.2 *The calculus* $\mathbf{TK}_{\omega_1}$ *does not admit cut-elimination. For instance, the sequent*

$$\Delta \;:=\; \Big\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\Big\}, \Box_i \bigwedge_k q_k^k$$

is derivable, but not cut-free derivable*, in* $\mathbf{TK}_{\omega_1}$.

Proof $\mathbf{TK}_{\omega_1} \nvdash_0 \Delta$ by Fact 6.1, since $\Delta \equiv \Delta_\Phi$ with $\Phi = \emptyset$.

On the other side, Δ can be derived as follows by making use of a $\mathtt{CUT}$ with an appropriate instance of BF_{ω_1} (which we know being derivable in $\mathbf{TK}_{\omega_1}$):

$$\dfrac{\dfrac{\cdots\left\{\dfrac{\dfrac{\neg q_k^k, q_k^k}{\neg\Box_i q_k^k, \Box_i q_k^k}\,\mathtt{K}_i}{\bigvee_n \neg\Box_i q_n^k, \Box_i q_k^k}\,\mathtt{OR}\right\}\cdots\,(k\geq 0)}{\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\}, \bigwedge_k \Box_i q_k^k}\,\mathtt{W}\,,\,\mathtt{AND} \qquad \begin{array}{c}\vdots\\ \neg\bigwedge_k \Box_i q_k^k, \Box_i \bigwedge_k q_k^k\end{array}}{\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\}, \Box_i \bigwedge_k q_k^k}\,\mathtt{CUT}$$

□

We conclude with a further negative result, showing how a *seemingly natural* way out of the difficulty emerging from Proposition 6.2 is in turn doomed to failure.

Let us consider the calculus $\mathbf{TK}^\circ_{\omega_1}$ obtained from $\mathbf{TK}_{\omega_1}$ by replacing the rule $\mathtt{OR}^+$ with the stronger (and clearly sound) rule

$$\frac{\Gamma, \{\Phi_m\}_{m\in I}}{\Gamma, \{\bigvee \Phi_m\}_{m\in I}}\ \mathtt{OR}^{\circ}\ \ (I \text{ countable})$$

by means of which *countably many* disjunctions can be *simultaneously* introduced.

Indeed, the sequent Δ of Fact 6.2 becomes *cut-free derivable* in $\mathbf{TK}^{\circ}_{\omega_1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\ldots \neg q^k_k, q^k_k \ldots\ \ (k \geq 0)}{\{\neg q^k_k \mid k \geq 0\}, \bigwedge_k q^k_k}\ \mathtt{W,\ AND}}{\{\neg \Box_i q^k_k \mid k \geq 0\}, \Box_i \bigwedge_k q^k_k}\ \mathtt{K}_i}{\{\neg \Box_i q^k_n \mid k \geq 0, n \geq 0\}, \Box_i \bigwedge_k q^k_k}\ \mathtt{W}}{\{\bigvee_n \neg \Box_i q^k_n \mid k \geq 0\}, \Box_i \bigwedge_k q^k_k}\ \mathtt{OR}^{\circ}$$

But unfortunately, a new counterexample to cut-elimination comes out! For $n \geq 0$, let

$$\varphi_n := \begin{cases} \neg \Box_i p_0, & \text{if } n = 0; \\ \bigwedge \{\Box_i p_0, \ldots, \Box_i p_k, \neg \Box_i p_{k+1}\}, & \text{if } n = k+1. \end{cases}$$

Fact 6.3 For every $X \subseteq_{\text{fin}} \omega$, if

$$\Theta \subseteq \Gamma_X := \{\varphi_n\}_{n \geq 0}, \{\neg \Box_i p_m\}_{m \in X}, \Box_i \bigwedge_n p_n$$

then $\mathbf{TK}^{\circ}_{\omega_1} \nvdash_0 \Theta$.

Proof As in the proof of Fact 6.2 we argue by transfinite induction on the height of $\mathbf{TK}^{\circ}_{\omega_1}$-derivations.

Let $\tau < \omega_1$. Assume (I.H.) that for no set Θ satisfying the hypotheses there is a $\mathbf{TK}^{\circ}_{\omega_1}$-derivation $\mathfrak{D} \vdash_0 \Theta$ with $\mathtt{h}(\mathfrak{D}) < \tau$.
Suppose, by way of contradiction, that for some set $X \subseteq_{\text{fin}} \omega$ and some $\Lambda \subseteq \Gamma_X$ there is a cut-free derivation $\mathfrak{D}$ of Λ with $\mathtt{h}(\Lambda) = \tau$. Let $\mathtt{R}$ be the final inference of $\mathfrak{D}$. Necessarily $\mathtt{R}$ is be one of $\mathtt{W}$, $\mathtt{AND}$, $\mathtt{K}_i$.

If $\mathtt{R} = \mathtt{W}$ we are immediately in contradiction with the I.H.

If $\mathtt{R} = \mathtt{AND}$, let $\varphi_{j+1} = \bigwedge\{\Box_i p_0, \ldots, \Box_i p_j, \neg \Box_i p_{j+1}\}$ (for some $j \geq 0$) be the principal formula of the inference, and let $\mathfrak{D}'$ be the subderivation of the $j+1$-th premise Λ' (the one having $\neg \Box_i p_{j+1}$ as secondary formula) of this inference. Then $\Lambda' \subseteq \Gamma_{X \cup \{j+1\}}$, and since $\mathtt{h}(\mathfrak{D}') < \tau$ we are in contradiction with the I.H. again.

If $\mathtt{R} = \mathtt{K}_i$, then there would be a finite set $Y = X \cup \{0\} \subseteq \omega$ and a derivation

$$\mathfrak{D}' \vdash \{\neg p_k\}_{k \in Y}, \bigwedge_n p_n$$

which is clearly impossible by the finiteness of Y and the soundness of $\mathbf{TK}^{\circ}_{\omega_1}$. $\Box$

Proposition 6.4 *The calculus* $\mathbf{TK}^{\circ}_{\omega_1}$ *does not admit cut-elimination. For instance, the sequent*

$$\Gamma \ :=\ \{\varphi_n\}_{n\geq 0}, \Box_i \bigwedge_n p_n$$

is derivable, but not cut-free derivable, *in* $\mathbf{TK}^{\circ}_{\omega_1}$.

Proof $\mathbf{TK}^{\circ}_{\omega_1} \nvdash_0 \Gamma$ by Fact 6.3, since $\Gamma \equiv \Gamma_X$ with $X = \emptyset$.

On the other side, Γ can be derived (in fact, already in $\mathbf{TK}_{\omega_1}$) by making use of `CUT`. First of all, we verify:

$$\mathbf{TK}_{\omega_1} \vdash_0 \varphi_0, \dots, \varphi_m, \Box_i p_m \quad \text{for each } m \geq 0. \tag{6.1}$$

This is easily proved by induction on m:

- $m = 0$:

$$\dfrac{\neg p_0, p_0}{\neg\Box_i p_0, \Box_i p_0}\,\mathrm{K}_i$$

- $m = k + 1$:

$$\dfrac{\overset{\text{I.H.}}{\varphi_0, \Box_i p_0} \quad \cdots \quad \overset{\text{I.H.}}{\varphi_0, \dots, \varphi_k, \Box_i p_k} \quad \dfrac{\neg p_{k+1}, p_{k+1}}{\neg\Box_i p_{k+1}, \Box_i p_{k+1}}\,\mathrm{K}_i}{\varphi_0, \dots, \varphi_k, \bigwedge\{\Box_i p_0, \dots, \Box_i p_k, \neg\Box_i p_{k+1}\}, \Box_i p_{k+1}}\,\texttt{W, AND}$$

Next, using BF_{ω_1}, we obtain the following derivation of Γ in $\mathbf{TK}_{\omega_1}$:

$$\dfrac{\dfrac{\cdots\left\{\begin{matrix}(6.1)\\ \varphi_0, \dots, \varphi_m, \Box_i p_m\end{matrix}\right\}\cdots\ (m\geq 0)}{\{\varphi_n\}_{n\geq 0}, \bigwedge_n \Box_i p_n}\,\texttt{W , AND} \qquad \begin{matrix}\vdots\\ \neg\bigwedge_n \Box_i p_n, \Box_i \bigwedge_n p_n\end{matrix}}{\{\varphi_n\}_{n\geq 0}, \Box_i \bigwedge_n p_n}\,\texttt{CUT}$$

□

7 Concluding Remarks

The counterexample to cut-elimination given in Proposition 6.4 seems to involve infinitary conjunction in an essential way. It is then natural to ask whether a strengthening of the `AND` rule might be of some use. In analogy to the strengthening $\texttt{OR}^{\circ}$ of the disjunction rule $\texttt{OR}^{+}$ discussed in the previous section, we may look for a rule allowing the simultaneous introduction of countably many conjunctions in the conclusion. One possible candidate we experimented with is the following rule (featuring uncountably many premises):

$$\frac{\ldots\ \Gamma, \Psi\ \ldots\ (\mathit{all}\ \Psi \lhd \{\Phi_i \mid i \in I\})}{\Gamma, \{\bigwedge \Phi_i \mid i \in I\}}\ \texttt{AND}^{\circ}$$

where $\{\Phi_i \mid i \in I\}$ is a countable family of countable sets of formulas, and $\Psi \lhd \{\Phi_i \mid i \in I\}$ is short for $(\forall i \in I)(\Psi \cap \Phi_i \neq \emptyset)$.

Bad news again! Let $\mathbf{TK}^{\circ\circ}_{\omega_1}$ be the calculus obtained from $\mathbf{TK}^{\circ}_{\omega_1}$ by replacing the rule `AND` with the above rule `AND`°. It is easy to check that the sequent Γ of Proposition 6.4, providing a counterexample to cut-elimination for $\mathbf{TK}^{\circ}_{\omega_1}$, becomes cut-free derivable in $\mathbf{TK}^{\circ\circ}_{\omega_1}$. Unfortunately, a new (rather involved) counterexample can be provided, showing that even $\mathbf{TK}^{\circ\circ}_{\omega_1}$ does not admit cut-elimination.

The ensuing impression is that further attempts to look at other variants of $\mathbf{TK}_{\omega_1}$, in order to find a "conventional" sequent-style calculus for $\mathbf{K}_{\omega_1}$ allowing (syntactical, if possible) cut-elimination, are likely destined to bring to a deadlock. On the other side, we think that further investigations on the proof theory of infinitary modal logic within the framework of "non conventional" sequent-style calculi (deep sequent systems [1], labeled sequent systems [9, 10], and other proof systems which make explicit use of semantic parameters in the syntax), as well as a deeper exploration of the relationship between infinitary modal logic and modal fixed point logics, are worth to be pursued.

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From Subsystems of Analysis to Subsystems of Set Theory

Wolfram Pohlers

Dedicated to Gerhard Jäger on the occasion of his 60th birthday

1 Introduction

It is a pleasure for me to contribute to a volume to honor Gerhard Jäger's work in proof theory on the occasion of his 60th birthday.

I had the advantage to accompany his first steps into proof theory. A period during which the emphasis of a certain part of proof theory changed from studies of subsystems of Analysis to the study of subsystems of set theory. A change whose details were nearly exclusively worked out by Gerhard Jäger.

To honor this aspect of Gerhard Jäger's contribution to proof theory I am going to try to give a non technical and very personally biased account of how we got from subsystems of Analysis to subsystems of set theory. This is, however, only one aspect of Gerhard's work. But it is the aspect to which I have the closest bonds.

I want to express my warmest thanks to Wilfried Buchholz who improved the text by a series of helpful remarks and corrections.

2 Ordinal Analysis for Predicative Systems

Anyone who knows me will guess that the "certain part" of proof theory I am talking about is *ordinal analysis*. To distinguish ordinal analysis from Analysis in the sense

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of second order number theory I will always capitalize Analysis if I mean Analysis in the sense of second order number theory.

To stress the necessities that brought us to change to subsystems of set theory I will put some emphasis on the time before the change.

2.1 Ordinal Analysis

To make clear what I am talking about let us resume some of the basic facts of ordinal analysis. It means the computation of the proof theoretic ordinal of a mathematical theory.

The proof theoretic ordinal of a theory T is commonly defined as the supremum of the order–types of elementarily definable order relations whose well–foundedness is provable within the theory T. But ordinal analysis is in fact much more than just knowing the proof theoretic ordinal of a theory T. I claim that you know nearly everything about the meta–mathematics of a mathematical theory once you have an ordinal analysis of it. I will, however, not deepen this claim here. Later I will mention an example.

Determining the proof theoretic ordinal of a theory T of course requires that we can talk about well–foundedness in the language of T. Since well–foundedness in an arithmetical language is a genuine Π^1_1–notion this needs a second order language. The situation is, however, not so bad since we can express second order Π^1_1–statements in a first order logic with free second order variables.

There is a method that goes back to Gerhard Gentzen how such an information can be achieved. I want to describe this method in more modern terms. These more modern terms are based on a strengthening of the ω–completeness theorem due to Henkin [16] and Orey [26] which says that ω–logic is complete for Π^1_1–sentences in the standard structure $\mathbb{N}$.

The strengthening I mean is that there is even a cut–free semi–formal proof system for ω–logic that is complete for Π^1_1 sentences in the following sense.

2.1 Theorem *A* Π^1_1*–sentence* $(\forall X)\phi(X)$ *is true in the standard structure* $\mathbb{N}$ *iff there is an ordinal* $\alpha < \omega_1^{ck}$ *such that* $\left|\frac{\alpha}{0}\right. \phi\,(X)$.

I promised not to become too technical. Therefore I try to explain the notions in the theorem in a non–technical way. Clearly ω_1^{ck} denotes the Church–Kleene ordinal, the first ordinal that cannot be represented by a recursive well–ordering on the natural numbers.

The ω–rule is a rule with infinitary, i.e., ω many premises saying that from $\psi(n)$ for every $n \in \mathbb{N}$ you can infer $((\forall x)\psi(x)$. A proof tree in ω–logic is therefore an ω–branching well–founded infinitary tree whose depth is canonically measured by an ordinal. The notion $\left|\frac{\alpha}{0}\right. \phi$ then denotes that there is a cut–free derivation for ϕ in ω–logic whose depth is bounded by α.

A cut–free derivation in ω–logic is of course not uniquely determined by its end formula but there is a certain border for the complexities of the possible cut–free derivations which cannot be undercut. We may thus define the *truth complexity* $\mathrm{tc}((\forall X)\phi(X))$ *of a* Π^1_1*–sentence* as the least ordinal α for which we have a cut-free derivation $\vdash^{\alpha}_{0} \phi(X)$ in ω–logic.[1]

The main theorem that relates truth complexities and proof theoretic ordinals is the Boundedness Theorem which goes back to Gentzen's paper [15]. We state it in an improved form which is due to Arnold Beckmann [2]. For the theorem let

$$(\forall X)\Pi_{\prec}(X) \;:\Leftrightarrow (\forall X)[(\forall x)[(\forall y \prec x) y \in X \rightarrow x \in X] \rightarrow (\forall x)[x \in X]]$$

denote the formula that expresses the well–foundedness of an elementarily definable relation $\prec$.

2.2 Theorem (Boundedness) *Assume that* $\prec$ *is a transitive well–founded binary relation on the natural numbers that is elementarily definable and* $\vdash^{\alpha}_{0} \Pi_{\prec}(X)$ *. Then its order–type is less than or equal to* α*.*

For well–orderings $\prec$ *of limit type its order–type and the truth complexity of* $(\forall X)\Pi_{\prec}(X)$ *coincide.*[2]

Ordinal analyses for "predicative" arithmetical theories follow, in principle, all the same pattern.

- Unravel formal proofs into proofs within ω–logic.
- Eliminate the cuts.
- Use the Boundedness Theorem to obtain an upper bound for the proof theoretic ordinal.[3]

Clearly there is ample room for numerous refinements that are needed to obtain ordinal analyses for skew intermediate axiom systems but in general this pattern remains retained.

The classical result is Gentzen's analysis of Peano arithmetic. Following the above pattern the upper bound for the proof theoretic ordinal is obtained by first observing that every formal derivation of a formula ϕ in Peano Arithmetic can be unraveled into an infinitary derivation $\vdash^{\omega\cdot 2}_{n} \phi$ with $n < \omega$ in ω–logic with cut. Here $\vdash^{\alpha}_{\rho} \phi$ denotes an ω–derivation of length $\leq \alpha$ whose cut formulas have all complexities below ρ. The main step is Gentzen's cut elimination theorem.

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[1] In case that ϕ is a sentence not containing free variables, the calculus $\vdash^{\alpha}_{0} \phi$ is just a verification of ϕ in the structure $\mathbb{N}$. Details about semi–formal systems in general and the verification calculus $\vdash^{\alpha}_{0} \phi$ together with a proof of Theorem 2.1 can be found in [34].

[2] This formulation is quite different from Gentzen's approach who did not use infinitary logic. However, the idea of the proof is the same. In our formulation Gentzen's result says that ω^{α} is an upper bound for the order–type of $\prec$. A result that sufficed to obtain the proof theoretic ordinal for Peano Arithmetic. Already Schütte improved Gentzen's result in showing that the order–type of $\prec$ is $\leq \omega \cdot \alpha$ (cf. [38] Theorem 23.1).

[3] I concentrate on the computation of upper bounds, since only there *meta–mathematical* methods are needed. Lower bounds are obtained by exhausting the *mathematical* power of an axiom system.

2.3 Theorem (Gentzen) $\underset{\rho+1}{\overset{\alpha}{|\!\!\!-\!\!-}}\ \phi$ implies $\underset{\rho}{\overset{\omega^\alpha}{|\!\!\!-\!\!-}}\ \phi$,

which states that the cut–degree of a derivation in ω–logic can be decreased by 1 for the price of raising the length of the derivation by one ω–power.[4] By iterated use of cut elimination we obtain the first fixed–point of the function $\alpha \mapsto \omega^\alpha$, i.e., $\varepsilon_0 := \min \{\xi \,|\, \omega^\xi = \xi\}$, as an upper bound for the proof theoretic ordinal of Peano Arithmetic.

2.2 Ramified Analysis

Gentzen's work was a first attempt towards a—at least partial—solution of the second problem in Hilbert's 1900 list of mathematical problems, the consistency of Analysis.[5] Analysis is formalizable within second order number theory. The difference to Peano Arithmetic is the fact that we now also allow quantifiers over sets of natural numbers, represented by second order variables, and reinforce the logic by the *comprehension scheme*

$$(\exists X)[(\forall y)[y \in X \leftrightarrow F(y)]] \tag{CA}$$

in which $F(y)$ is an arbitrary formula of Analysis, i.e., second order arithmetic, which must not contain the second order variable X freely.

For Analysis, however, the pattern which I sketched in the previous section does not longer work. The reason for this failure is the comprehension scheme. Since $F(y)$ is an arbitrary formula of Analysis it may itself contain arbitrary nestings of second order quantifiers which causes us to run into circles. To handle this "impredicativity" in the comprehension scheme is the true challenge. Due to this impredicativity it is unclear if there is an infinitary rule that allows us to deal with second order quantifiers in a similar way as the ω–rule allows us to handle first order quantifiers.

A first step to meet this challenge is *Ramified Analysis* in which the comprehension scheme is restricted to *ramified comprehension.*

To explain the ramification we declare quantifiers ranging over natural numbers as quantifiers of stage 0. The stage of a formula of Ramified Analysis is the maximum of the stages of the quantifiers occurring in it. So every formula in the language of first order Peano Arithmetic obtains stage 0.

If $\phi(x)$ is a formula of stage α we call $\{x \,|\, \phi(x)\}$ a *comprehension–term* of stage $\alpha + 1$. Second order quantifiers $(\mathsf{Q}X^\alpha)\phi(X)$ for $\mathsf{Q} \in \{\forall, \exists\}$ are supposed to range

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[4] Again the formulation deviates considerably from Gentzen's original theorem. He did not use infinitary derivations but obtained the result by a complicated assignment of ordinals to the nodes in a finite derivation.

[5] In Sect. 3.3 I will briefly comment on Hilbert's Programme.

over all comprehension–terms of stages less than α.

By ramification we avoid circularities and obtain an infinitary rule similar to the ω–rule saying that from $\phi(S)$ for all comprehension–terms S of stages less than α we can conclude $(\forall X^\alpha)\phi(X)$.

Augmenting ω–logic with this $(\forall X^\alpha)$—and the obvious dual $(\exists X^\alpha)$—rule we obtain a semi–formal system which enjoys cut elimination in the following form. Observe that the formulas in Ramified Analysis have now transfinite complexities.

2.4 Theorem *Any semi–formal derivation* $\left|\frac{\alpha}{\beta+\omega^\rho}\right.\phi$ *of Ramified Analysis reduces to* $\left|\frac{\varphi_\rho(\alpha)}{\beta}\right.\phi$.

Here $\varphi_\rho(\alpha)$ denotes the Veblen function which is defined such that $\varphi_0(\alpha) := \omega^\alpha$ and φ_ρ enumerates the common fixed–points of the functions φ_ξ for $\xi < \rho$. Hence $\varphi_1(0) = \varepsilon_0$.

The pattern to obtain upper bounds still works using the infinitary systems of Ramified Analysis. The more tricky part is now the unraveling of formal derivations in Analysis. It has turned out that this is possible up to systems in which the comprehension scheme is restricted to Δ^1_1–formulas.[6]

Since there are no circularities in Ramified Analysis all systems that can be embedded into Ramified Analysis are regarded to be *predicative* which means that all apparent circularities in such systems can be resolved.

It follows from Theorem 2.4 that Ramified Analysis is governed by the Veblen function φ. Ordinals that are closed under the Veblen function φ, viewed as a binary function, are *strongly critical*. Therefore every theory that can be embedded into Ramified Analysis has a proof theoretic ordinal less than or equal to the first strongly critical ordinal Γ_0. A result that is independently due to Kurt Schütte [37] and Sol Feferman [11] who have moreover shown that every ordinal less than Γ_0 can be predicatively justified. Predicatively justified is here understood in a very technical (i.e., non–philosophical) sense. Let RA_α denote the part of Ramified Analysis that only contains formulas and comprehension–terms of stages less than α and only allows infinitary derivations of lengths less than α. Roughly speaking an ordinal α is predicatively justified if it can be represented by a well–order for which not only the definition is non–circular but also the proof of its well–foundedness can be embedded into RA_α.

In this technical sense the Schütte–Feferman ordinal Γ_0 is the exact bound for predicativity and it has become common to call axiom systems with proof theoretical ordinals less than or equal to Γ_0 predicative.

However, the methods of predicative proof theory are not restricted to systems with ordinals less than or equal to Γ_0 as Gerhard Jäger and his school have shown in their project of metapredicativity. So I would like to draw a (technical) bound between predicative and impredicative systems there, where the methods of predicative proof theory fail.

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[6] There are variations of the comprehension scheme, e.g., choice schemata, and even stronger systems such as the system ATR_0 of arithmetical transfinite recursion (cf. [39]) which can also be embedded into Ramified Analysis but this is inessential for our story.

3 Ordinal Analyses for Impredicative Axiom Systems

Having learned many facts about predicative proof theory in Schütte's lectures and seminars, my interest turned to impredicative axiom systems. The most famous analysis of an impredicative axiom system which existed at that time was that by Gaisi Takeuti [41] for second order number theory with the Π^1_1–comprehension scheme and Bar induction. Yet it was not genuinely an ordinal analysis but rather a consistency proof in the style of Gentzen. In my dissertation I analyzed Takeuti's proof and converted it into an ordinal analysis in terms of an ordinal notation system Σ developed by Schütte.[7] Although I was able to master the technique I did, at that time, not really understand what was going on in Takeuti's reduction procedure. Only much later that became clear by studies of Wilfried Buchholz (cf. [7]).

3.1 ν–fold Iterated Inductive Definitions

However, Takeuti's techniques turned out to be very useful in confirming the long conjectured proof theoretic ordinals of axiom systems for iterated inductive definitions.

In abstract terms an inductive definition on the set of natural numbers is a monotone operator $\Gamma : \mathrm{Pow}(\mathbb{N}) \longrightarrow \mathrm{Pow}(\mathbb{N})$. Such an operator possesses a least fixed–point $I_\Gamma = \bigcap \{S \,|\, \Gamma(S) \subseteq S\}$.

A subset of the natural numbers is *inductively definable* if it is a slice of a fixed–point of an elementarily definable inductive definition.

To obtain an axiomatization of inductive definitions we introduce a set constant I_ϕ for every arithmetical formula $\phi(X, x)$ in which the second order variable X must only occur positively. This ensures that the operator

$$\Gamma_\phi(S) := \{n \,|\, \mathbb{N} \models \ \phi(S, n)\}$$

is monotone and thus an inductive definition on the natural numbers. We therefore commonly talk about *positively definable inductive definitions.* The intended interpretation for the constant I_ϕ is the least fixed–point of the inductive definition Γ_ϕ. This can be axiomatized in a first order way by the closure axiom

$$(\forall x)[\phi(I_\phi, x) \to x \in I_\phi]$$

and the induction scheme

$$(\forall y)[\phi(\psi, y) \to \psi(y)] \to (\forall x)[x \in I_\phi \to \psi(x)].$$

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[7] Actually the system in my dissertation was weaker than Takeuti's. The ordinal of the full system was not available in Σ.

We obtain iterations by liberalizing the condition that the inductive definition has to be elementarily definable but allow previously introduced set constants for fixed–points in its definition. This can be iterated along any definable well–ordering. The details (cf. [9, 12]) are not important here.

For an elementarily definable inductive definition Γ_ϕ we obtain its fixed–point as a comprehension–term $\{x \mid (\forall X)[(\forall y)[\phi(X, y) \to y \in X] \to x \in X]\}$, i.e., by Π^1_1–comprehension. Consequently the first ordinal analyses of the theories ID_ν for ν–fold iterated inductive definitions were obtained by embedding these theories into systems of iterated Π^1_1–comprehensions which then could be handled by Takeuti's technique (cf. [27, 28]).

Although this yielded a correct computation of the upper bounds for the proof theoretic ordinals of the theories ID_ν the method was, due to the complicated reduction procedure à la Takeuti, completely opaque. It was Sol Feferman's constant nagging for a more perspicuous method that kept us (if I may speak also in the name of Wilfried Buchholz) working on the problem. Wilfried Buchholz succeeded in developing his Ω–rules which, however, did not completely satisfy myself for reasons which I will discuss below.

3.2 Buchholz' Ω–Rule

For the following part we need a rough idea of Buchholz' Ω–rule. This rule is closely related to the hyperjump operation. One possibility to present a hyperjump is the transition from a *constructive number class* to the *next constructive transfinite number class*. The transition is given by the rule

$$(\forall x)[x \in \mathcal{O}_n \Rightarrow \{e\}(x) \in \mathcal{O}_{n+1}] \;\Rightarrow\; 3^{n+1} \cdot 5^e \in \mathcal{O}_{n+1}.$$

Inspired by a paper [19] by Howard on "a system of abstract constructive ordinals" Buchholz invented a rule that mimicked the construction of higher "number classes". To explain the rule we use the language with the constants I_ϕ for fixed points and replace the closure axiom by a rule

$$\vdash^{\alpha}_{\rho} \chi \to \phi(I_\phi, n) \;\Rightarrow\; \vdash^{\beta}_{\rho} \chi \to n \in I_\phi\,.$$

We[8] introduce the "first derivation class"[9] as the class of cut–free derivations in ω–arithmetic with only positive occurrences of fixed–point constants and the "second derivation class" which comprises derivations also with cuts and—more important—also arbitrary lengths and negative occurrences of fixed–point constants. There is a

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[8] When formulating such rules I tacitly assume $\alpha < \beta$. I mention the side formula χ just to be not too severely cheating. A rigid definition is preferably done in the framework of some form of sequent calculus.

[9] This is my naming to emphasize the relationship to the hyperjump operation.

class of constructive "derivation operations", which we do not want to explain further, and we can formulate the rule as follows.

(Ω–rule) *If there is a derivation operation F that converts any derivation $\frac{\alpha}{0}\!\!\!\vert\;\; \chi \to n \in I_\phi$ in the first derivation class into a derivation $\vert\!\frac{F(\alpha)}{\rho}\; \chi \to \psi$ then we can infer $\vert\!\frac{F(\Omega)}{\rho}\; n \in I_\phi \to \psi$.*

The similarity to the hyperjump operation should now be obvious. The first derivation class plays the role of Kleene's $\mathcal{O}$ and the "constructive derivation operations" adopt the role of the recursive functions.

Clearly this rule can be extended to Ω_ν–rules which mimic ν–fold iterated applications of the hyperjump and are thus apt for the handling of ν–fold iterations of inductive definitions (cf. [5]).

The Ω rule is an infinitary rule with Ω many premises. The corresponding derivations are infinitary Ω–branching well–founded trees. It is easy to see that the formal theory of inductive definitions is easily embedded into this system. The closure axiom is immediate from the closure rule.

To obtain also the induction scheme we start with a derivation $\vert\!\frac{\alpha}{0}\; n \in I_\phi$ in the first derivation class and transform it into a derivation

$$\vert\!\frac{F(\alpha)}{\rho}\; (\forall y)[\phi(\psi, y) \to \psi(y)] \to \psi(n)$$

by replacing all occurrences of $k \in I_\phi$ by $\psi(k)$ and adding the premise $(\forall y)[\phi(\psi, y) \to \psi(y)]$. The only rule that can be violated by this operation is the closure rule. In this case, however, we obtain the premise $\vert\!\frac{F(\alpha_0)}{\rho}\; (\forall y)[\phi(\psi, y) \to \psi(y)] \to \phi(\psi, n)$ from which we logically infer

$$\vert\!\frac{F(\alpha)}{\rho}\; (\forall y)[\phi(\psi, y) \to \psi(y)] \to \psi(n)\,.$$

Using the Ω–rule followed by an ω–rule we finally obtain

$$\vert\!\frac{F(\Omega)+1}{\rho}\; (\forall y)[\phi(\psi, y) \to \psi(y)] \to (\forall x)[x \in I_\phi \to \psi(x)]\,.$$

The calculi with Ω_ν–rules allow cut elimination. Therefore any derivation of a formula with only positive occurrences of constants for non–iterated fixed–points can be converted into a derivation in the "first derivation class", which, in principle, are verifications. Therefore a boundedness theorem holds true for derivations of the first "derivation class"—the situation is, however, different from that in predicative theories. We will come back to that in a later section. Nevertheless the suprema of the lengths of the eventually obtained derivations in the "first derivation class" yield upper bounds for the proof theoretic ordinals of the theories for iterated inductive definitions.

3.3 A Remark on Hilbert's Programme

With Buchholz' Ω_ν–rules we had a perspicuous way to determine the upper bounds for iterated inductive definitions. At least much more perspicuous than Takeuti's reduction procedure. Still I myself was not completely satisfied. To explain why, I have to give a brief avowal of my personal motivation for doing ordinal analysis.

My starting point is a certain aspect of Hilbert's Programme. Though I believe that—due to Gödel's second incompleteness theorem—Hilbert's programme failed in so far, that elementary consistency proofs of Analysis are impossible, I nevertheless think that there is another important aspect of Hilbert's programme: The elimination of *"ideal objects"*.

As I see it it is not completely clear what Hilbert understood by "ideal objects" in general. However, there are pretty concrete hints what he meant by *"real statements"* in contrast to ideal ones. Such a hint can be found in his 1927 talk given in Hamburg. Here is my translation of the passage. *"The physicist requires for a theory that its theorems can be formally derived from the laws of nature and its hypotheses alone without referring to outside perceptions. Only certain combinations and conclusions of physical laws are checkable by experiments—this is also true for my proof theory in which only real statements are verifiable."*[10]

But what are the mathematical analogs of experimentally checkable statements? Of course we cannot make experiments in mathematics but we can compute. A good analog for an experimentally checkable statement is therefore a statement whose instances are verifiable by computations, i.e., Π^0_1–statements. That of course does not mean that we can *prove* Π^0_1–statements by computations but that we can check their instances. This situation is comparable to that in physics where we also cannot "prove" the consequences of a theory experimentally but can check *instances* of its predictions.

The analog situation for checking the instances of Π^0_2–consequences of a mathematical theory T could therefore consist in finding a function F_T that helps us to *design "experiments"* for the theory T. That means that whenever we have a Π^0_2–sentence $(\forall x)(\exists y)R(x, y)$ there is a number k such that $F_T(k + m)$ fixes for all m the frame for finitely many "experiments" in which we can check the instances $(\exists y)R(m, y)$ primitive recursively.[11]

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[10] Der Physiker verlangt gerade von einer Theorie, dass ohne die Heranziehung von anderweitigen Bedingungen aus den Naturgesetzen oder Hypothesen die besonderen Sätze allein durch Schlüsse, also auf Grund eines reinen Formelspiels abgeleitet werden. Nur gewisse Kombinationen und Folgerungen der physikalischen Gesetze können durch Experimente kontrolliert werden—so wie in meiner Beweistheorie nur die realen Aussagen unmittelbar einer Verifikation fähig sind. (Cited from [18]).

[11] Here it is necessary to allow additional number parameters in Π^0_2–sentences. A more elaborated discussion on the interaction of Hilbert's programme and ordinal analysis will appear in [33].

3.4 $\boldsymbol{\Pi^0_2}$*–Analysis*

Pursuing Hilbert's programme for the elimination of ideal objects in this strict sense means that the procedure for the elimination of "ideal objects" in a T–proof of a Π^0_2–sentence should provide us with a function F_T that designs an experiment for T. Clearly the function F_T has to be definable and computable without reference to ideal means, which excludes functional interpretations with functionals of higher types. Higher types are to be considered as ideal—although such functional interpretations are of interest(s) in their own.

By "ideal methods" I understand axioms or rules that axiomatize ideal objects, e.g., sets obtained by comprehensions, ordinals obtained by reflections, etc. *Let us call the program of elimination of ideal objects in T–proofs of* Π^0_2*–sentences the* Π^0_2*–analysis of the theory T.*

This is an ambitious aim and it was not at all clear from the beginning how far this program could be realized. From our present knowledge we have to admit, that the elimination of ideal methods costs the price of long transfinite recursions in the definition of the functions F_T. This is, at least at a first glance, against the spirit of Hilbert's programme who claimed that *"operating the infinite can only be secured in the finite"*.[12] Of course one could argue that the infinities needed are countable ordinals that can be represented by elementarily—at least primitive recursively—definable and decidable order relations on the natural numbers. Therefore one could regard these infinities as only slight extensions of the finite, if we understand "the finite" as natural numbers in their natural ordering. So we could extend Kronecker's aphorism *"the natural numbers are made by God all other is man–made"* to *"the (constructive countable) ordinals are made by God all other is man–made"*. Whereas I think that the question in how far the ordinals are "God–made" and how these ordinals can be represented is one of the deepest problems in foundational research even outside ordinal analysis or even proof theory. But this is a discussion I will address elsewhere.

Π^0_2–analysis is to be seen in contrast to ordinal analysis which we may also call Π^1_1–analysis due to its closeness to the computation of truth complexities of Π^1_1–sentences. Nevertheless, Π^1_1–analysis can be considered as a first and less ambitious step towards Π^0_2–analysis. This is because Π^1_1–statements correspond to Σ_1–statements over $\mathsf{L}_{\omega_1^{ck}}$ with parameters—a fact that in another context will become

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[12] Das Operieren mit dem Unendlichen kann nur durch das Endliche gesichert werden [17].

important in a moment. Π^1_1–statements can thus be viewed as abstractions of Σ^0_1–statements with parameters in so far, that computability, i.e., ω–computability, is generalized to ω_1^{ck}–computability. However, ω_1^{ck}–computability is proof–theoretically much easier to handle since there are many ordinals with good closure properties below ω_1^{ck}—among them the proof theoretic ordinals of axiomatic theories—which is not true for ω.

Admittedly one of the more practical reasons why we then first concentrated on Π^1_1–analyses was the work of Gentzen (e.g., in his 1943 paper [15]) and his descendants which showed that elimination procedures for proofs of Π^1_1–sentences are possible whereas there were hardly examples for the feasibility of Π^0_2–analyses.

Against this background the reasons why I was not completely satisfied by the calculus with Ω_ν–rules were twofold.[13]

First, the derivations in the "first derivation class" still contain the "impredicative" closure rule, which may be considered as an "ideal method" in the axiomatization of the least fixed–point. Therefore the reduction to derivations in the "first derivation class" does not yet include a complete elimination of "ideal methods".

Secondly, more severely, the methods in the reduction procedure also included "ideal methods". As mentioned before, the Ω-rule and its iterations correspond to iterations of hyperjumps and are therefore not directly formalizable in an elementary basis theory (such as Primitive Recursive Arithmetic PRA) plus a certain amount of transfinite induction along simple predicates. It did therefore not really match the aims I had in mind.

Therefore I tried to find an alternative method which was closer to the proven techniques of predicative proof theory. The starting point for the development of this alternative approach was the Boundedness Theorem for inductive definitions which I mentioned at the end of Sect. 3.2.

3.5 A Brief Résumé of Inductive Definitions

To formulate the Boundedness Theorem for inductive definitions we briefly resume some basic facts about inductive definitions.

The fixed–point of an inductive definition Γ as the least Γ–closed subset of the natural numbers is a simple example of an impredicative definition. A definition that recurs to an entity—the collection of Γ–closed sets—of which it is itself a member. On the other hand one has the impression that viewing fixed–points from this angle is the wrong aspect. It suppresses the aspect of the inductive generation of the fixed–point from below.

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[13] This, of course, does not mean that these rules are dubious. Contrarily I think that the power of these ingenious rules, especially in respect to second order systems, should be more extensively studied.

When defining terms, formulas and other basic logical notions inductively, we always assume to develop these notions from below by iterating the definition steps. In all these examples the iteration becomes stationary after at most ω–many steps. For arbitrary inductive definitions, however, i.e., arbitrary monotone operators, it may happen that we have to iterate the operator along arbitrarily large segments of the countable ordinals in order to reach a fixed–point.

To emphasize the idea of iterating an inductive definition we define the iteration–stage of an inductive definition Γ by transfinite recursion. We put

$$I_\Gamma^\alpha := \Gamma(I_\Gamma^{<\alpha}) \text{ with } I_\Gamma^{<\alpha} := \bigcup_{\xi<\alpha} I_\Gamma^\xi.$$

Then it is obvious by cardinality reasons that there is a countable ordinal σ such that $I_\Gamma^\sigma = I_\Gamma^{<\sigma}$. The least such ordinal is the *closure ordinal* $||\Gamma||$ of the inductive definition Γ. If σ is the closure ordinal of an inductive definition Γ we easily obtain that I_Γ^σ is *the* fixed–point of Γ, i.e., that $I_\Gamma^\sigma = I_\Gamma$. If we denote by $|n|_\Gamma := \min\{\xi \,|\, n \in I_\Gamma^\xi\}$ the stage of an element $n \in I_\Gamma$ we obtain $||\Gamma|| = \sup\{|n|_\Gamma + 1 \,|\, n \in I_\Gamma\}$.

According to [25][14] let $\kappa^{\mathfrak{S}}$ be the supremum of all closure ordinals of inductive definitions $\Gamma : \mathrm{Pow}(S) \longrightarrow \mathrm{Pow}(S)$ that are elementarily definable in a structure $\mathfrak{S}$. Then we obtain $\kappa^{\mathbb{N}} = \omega_1^{ck}$. In analogy to this ordinal we define for an axiom system T that axiomatizes inductive definitions above $\mathfrak{S}$ the ordinal

$$\kappa^T := \sup\{|n|_\Gamma + 1 \,|\, T \vdash n \in I_\Gamma \text{ and } \Gamma \text{ is elementarily definable in } \mathfrak{S}\}.$$

Denoting the axiom system for ν–fold iterated inductive definitions above $\mathbb{N}$ by ID_ν and its proof theoretic ordinal by $|\mathsf{ID}_\nu|$ it is not very difficult to show that we then obtain

$$\kappa^{\mathsf{ID}_\nu} = |\mathsf{ID}_\nu|. \tag{1}$$

Determining the proof theoretic ordinal of these theories therefore means to determine the supremum of the stages of elements which provably belong to a fixed–point of an elementarily definable inductive definition. Observe that the definition of the ordinal κ^T is not a second order definition and thus only uses sentences (without free second order variables). This observation is of some importance as we will see in a moment.

3.6 Infinitary Logic for Inductive Definitions

The Boundedness Theorem mentioned in Sect. 3.2 can now be stated in the following form.

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[14] This is probably the right place to mention that Moschovakis' book [25] was of central importance for the proof theoretic study of inductive definition.

3.1 Theorem *Let* Γ *be an elementarily definable positive inductive definition. Then* $\left|\frac{\alpha}{0}\right. n \in I_\Gamma$ *implies* $|n|_\Gamma \leq \alpha$.

Together with (1) we get an upper bound for the proof theoretic ordinals for theories for inductive definitions from Theorem 3.1.

Although the statement of Theorem 3.1 is very close to that of Theorem 2.2 its proof is much simpler. It bases on the simple fact that in less than α many steps you only can say something about $I_\Gamma^{<\alpha}$.

To come back to the starting point of my alternative approach we observe that using the Boundedness Theorem in the form of Theorem 3.1 the impredicativity of the closure rules

$$\left|\frac{\alpha}{0}\right. \psi \to \phi(I_\phi, x) \Rightarrow \left|\frac{\beta}{0}\right. \psi \to x \in I_\phi$$

as they occur in derivations of the first derivation class can be resolved by replacing I_ϕ by its stages $I_\phi^{<\alpha}$ according to Theorem 3.1. Since $\alpha < \beta$ it then becomes the semantically correct implication

$$\psi \to \phi(I_\phi^{<\alpha}, x) \;\Rightarrow\; \psi \to x \in I_\phi^{<\beta}. \tag{2}$$

An implication which is at least locally predicative. The obvious idea was then to define $n \in I_\phi^{<\alpha}$ by an infinite disjunction $\bigvee_{\xi<\alpha} \phi(I_\phi^{<\xi}, n)$ of length α and use a canonical semi–formal system for infinitary logic together with this local predicativity in the ordinal analysis of iterated inductive definitions.

3.7 Semantical Cut–Elimination

However, there is a serious obstacle to this obvious idea. In the calculus with Ω_ν rules the definition of the derivation operations together with the elimination procedure secure that the lengths of derivations in the "first derivation class" are always countable (even constructively countable) ordinals. The ordinals α and β in Eq. (2) are thus countable (even constructively countable). Since proof theoretic ordinals are always ordinals less than ω_1^{ck} (and similarly $\kappa^{\mathbb{N}} = \omega_1^{ck}$, hence $\kappa^T < \omega_1^{ck}$) this is essential for obtaining useful upper bounds. This can, in general, not be secured if we use just infinitary logic.

Here we have to mention a peculiarity of semi–formal systems which only derive sentences. Since every sentence possesses a definite truth value in the intended standard structure (which is the structure of natural numbers in our case) such systems allow for semantical cut–elimination. Say that $\left|\frac{\alpha}{\rho}\right.$ is a *semi–formal system for a structure* $\mathfrak{S}$ iff $\mathfrak{S} \models \phi$ holds true if and only if there are ordinals α and ρ such that $\left|\frac{\alpha}{\rho}\right. \phi$.

3.2 Theorem (Semantical Cut–elimination) *Let* $\overset{\alpha}{\underset{\rho}{\vdash}}$ *be a semi–formal system for a structure* $\mathfrak{S}$ *that only derives sentences. Then* $\overset{\alpha}{\underset{\rho}{\vdash}} \phi$ *already implies* $\overset{\alpha}{\underset{0}{\vdash}} \phi$.

A rigid proof of the theorem by induction on α needs a rigid definition of the rules of the semi–formal system which we do not want to give here (since we promised not to become too technical).[15] To sketch the proof we show

$$\overset{\alpha}{\underset{\rho}{\vdash}} \psi \to \phi \textit{ and } \mathfrak{S} \models \psi \text{ implies} \overset{\alpha}{\underset{0}{\vdash}} \phi \tag{3}$$

by induction on α. The key point in the proof is of course the application of a cut

$$\overset{\alpha_0}{\underset{\rho}{\vdash}} \psi \to \psi_1 \,, \quad \overset{\alpha_0}{\underset{\rho}{\vdash}} \psi_1 \to \phi \quad \Rightarrow \overset{\alpha}{\underset{\rho}{\vdash}} \psi \to \phi \,.$$

From $\mathfrak{S} \models \psi$ we get by the correctness of the semi–formal system $\mathfrak{S} \models \psi_1$ and therefrom $\overset{\alpha_0}{\underset{0}{\vdash}} \phi$ by the induction hypothesis. Clearly (3) entails the claim. ☐

The point in the Semantical Cut–elimination Theorem is that it not only claims the eliminability of cuts but also gives a bound for the length of the cut free derivation, i.e., the verification.[16]

3.8 Local Predicativity

We have seen in Eq. (1) that we do not need a second order formula and therefore no free second order variables to express the proof theoretic ordinal for inductive definitions. Therefore one could expect that Semantical Cut–elimination along with the Boundedness Theorem 3.1 make ordinal analysis for inductive definitions trivial. That this is not the case is the need for "ideal elements" in the definition of the semi–formal system.

In order to express the completed fixed–point we have to introduce an "ideal ordinal" Ω whose defining axiom is

$$(\forall x)[\phi(I_\phi^{<\Omega}, x) \to x \in I_\phi^{<\Omega}].$$

Possible interpretations for Ω are therefore ω_1, the first uncountable cardinal or ω_1^{ck}, as $\phi(X, x)$ is supposed to be an X–positive formula in the first order language of arithmetic. But also other, smaller, interpretations for Ω are possible (cf. [32], Sect. 9.7).

Clearly this ideal ordinal[17] Ω also appears as measure for the derivation lengths of semi–formal derivations. A derivation $\overset{\alpha}{\underset{\rho}{\vdash}} n \in I_\phi^{<\Omega} \to \psi$ will in general require an $\alpha \geq \Omega$. Semantical Cut–elimination and the Boundedness Theorem 3.1 will thus in

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[15] For a rigid proof cf. [32].

[16] A bound that in general is useless for proof theoretic studies.

[17] In former publications I sometimes talked about "virtual ordinals".

general produce ordinals above Ω, too big as to serve for a useful upper bound for the proof theoretic ordinal. The salvation for this problem is the "elimination of the ideal element Ω".

As just indicated derivation lengths above Ω are unavoidable in general. We can only expect derivation lengths below Ω for sentences without negative occurrences of $I_\phi^{<\Omega}$. To realize this expectation we are confronted with a new feature that is characteristic for impredicative proof theory: *the need for a collapsing procedure that collapses derivations of sentences with only positive occurrences of* $I_\phi^{<\Omega}$ *into derivations of lengths below* Ω. Since ordinals are transitive and thus not collapsible this can only be realized by measuring the derivation lengths with ordinals taken from a subset of all ordinals that incorporates gaps large enough for the necessary collapsing.[18]

This was originally obtained by taking ordinals from a notation system with sufficiently large gaps above Ω and a transitive segment below Ω. The notation system allowed for the definition of a collapsing function, say ψ_Ω, that collapsed ordinals above Ω in the notation system into ordinals of the segment below Ω. The outcome was a collapsing theorem in the following form.

3.3 Theorem (Collapsing Theorem) *Let* ψ *be a sentence in the language of inductive definitions that only contains positive occurrences of* $I_\phi^{<\Omega}$ *and assume* $\vdash^{\alpha}_{\Omega+1} (\forall x)[\phi(I_\phi^{<\Omega}, x) \to x \in I_\phi^{<\Omega})] \ \to \ \psi$ *. Then we obtain* $\vdash^{\psi_\Omega(\omega^\alpha)}_{\Omega} \psi$ *.*[19]

Having in mind the Boundedness Theorem 3.1 we can replace all occurrences of $I_\phi^{<\Omega}$ in ψ by $I_\phi^{<\psi_\Omega(\omega^\alpha)}$. The Collapsing Theorem thus provides actually an elimination of the ideal ordinal Ω.

This is the basic idea of local predicativity (cf. [29]) although narrated in more modern terms. With some teething troubles this idea turned out to work also for iterated inductive definitions although the definition of the collapsing functions at that time were still pretty clumsy (cf. [31]). It culminated in the contribution to SLN 897.

4 Towards Set Theory

Of course it was tempting to try to transfer this technique to ramified analysis in order to obtain direct analyses for subsystems of classical Analysis. The naive idea to introduce an "ideal" ordinal Ω and to extend the stages of ramified Analysis beyond Ω turned out to be much too naive. It was even not clear how to formulate the "defining axiom" for Ω in terms of ramified analysis.

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[18] To find the subsets of the class of ordinals with the adequate gaps is actually the most challenging task in the ordinal analysis of strong axiom systems.

[19] A proof of this theorem (in a more modern form working already with operator controlled derivations which will be mentioned below) is in [32] Lemma 9.4.5.

The crucial hint was given by an adaption of the Buchholz Ω–rule to subsystems of (unramified) Analysis. Using the language of second order number theory the rule can be put in the following form (cf. [10] for details).

(Ω–rule) *If there is a derivation operation* F *that converts every derivation* $\left|\frac{\alpha}{0}\right. \chi \to (\forall X)\phi(X)$ *in the first derivation class into a derivation* $\left|\frac{F(\alpha)}{\rho}\right. \chi \to \psi$ *then we can infer* $\left|\frac{F(\Omega)}{\rho}\right. (\forall X)\phi(X) \to \psi$.

The Ω–rule is the only rule with Ω many premises in the Buchholz calculus and its main formula is the negative occurrence of the formula $(\forall X)\phi(X)$, i.e., a Σ^1_1–formula. This was at first glance unexpected. According to all experience infinitary rules served to derive universal quantifiers. The conjecture then was that the reason for this change of behavior is the hyperarithmetical quantifier theorem which states that every Σ^1_1–sentence corresponds to a Π^1_1–sentence relativized to hyperarithmetical sets.

This suggested the idea to modify ramified analysis in such a way that universal quantifiers of stage Ω vary over hyperarithmetical sets and to axiomatize Ω by a form of the recursion theoretic boundedness theorem.[20] The attempts to realize this idea, however, met a lot of awkward difficulties. Difficulties that appeared not to be insurmountable but needed so much coding machinery that the whole project became practically unmanageable (at least for me) and thus ceased to be fun.

On the other hand we knew that the hyperarithmetical sets are the intersection of the constructible sets in $L_{\omega_1^{ck}}$ with the powerset of the natural numbers. The idea was therefore: *why not work directly with constructible sets and avoid the awkward coding*. Searching the literature for examples we found an article by Sol Feferman [13] in which he treated "predicatively reducible systems of set theory" and a paper by Harvey Friedman [14] treating "set theoretic foundations for constructive analysis". These papers, however, reduced set theoretic axiom systems to known subsystems of Analysis. What we wanted to do was the converse way.

4.1 Ramified Set Theory

The aim was therefore to replace ramified analysis by "ramified set theory". Because of lack of examples everything had to be done from scratch. Designing a language for "ramified set theory" is not too difficult. One can more or less directly use the language of the constructible hierarchy with its stages L_α as additional constants. Therefore there are ground terms L_α of stage α for all ordinals α and composed terms $\{x \in \mathsf{L}_\alpha \,|\, \phi^{\mathsf{L}_\alpha}(x, a_1, \ldots, a_n)\}$ of stage α where ϕ is a formula in the language of set theory and $a_1, \ldots, a_n$ are previously defined terms of ramified set theory of stages less than α. There is also a canonical decoration for the language of ramified

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[20] Which says in its simplest form that for every Σ^1_1–definable set M of well-orderings there is an ordinal $\xi < \omega_1^{ck}$ that bounds the order–types of the well–orderings in M. A fact which actually is a consequence of the Boundedness Theorem 2.2. Cf. [2].

set theory (in the sense of [34]) which leads to a canonical semi–formal system for ramified set theory.

At that time not everything was as clear and smooth as we see it today and there were many difficulties to overcome. Fortunately at that time there was a very eager and interested student in Munich, Gerhard Jäger, with whom we could discuss the situation intensively and who was looking for a diploma thesis. I encouraged Schütte—since I had not yet passed my "habilitation" I was not allowed to supervise diploma students myself—to let Gerhard Jäger work on this problem. This was very ambitious for a diploma thesis because there were hardly patterns for doing proof theory directly in the constructible hierarchy. As a matter of course, Gerhard mastered the problem, starting with a still predicative system. This led to an excellent diploma thesis partly published as *"Beweistheorie von KPN"* [21]. Here I should perhaps also mention the big influence of Barwise's book [1] on *"Admissible sets and structures"* on our discussions. An influence similar to that of Moschovaki's book *"Elementary induction on abstract structures"* [25] on our work on inductive definitions.

Gerhard received his diploma degree with distinction and continued to work in this direction. In his dissertation *"Die konstruktible Hierarchie als Hilfsmittel zur beweistheoretischen Untersuchung von Teilsystemen der Analysis"* [20] he extended this work further and it culminated in his Habilitationsschrift *"Theories for admissible sets. A unifying approach"* [23] published by Bibliopolis in 1986. He so laid the fundament for all further research in this direction.

Already in 1982 we had a joint paper [24] published in the "Sitzungsberichte der Bayerischen Akademie der Wissenschaften", unfortunately in German, in which we gave an ordinal analysis of Δ^1_2–comprehension with the (classical) bar induction via an ordinal analysis of the impredicative theory KPi, a set theory that axiomatizes an admissible universe which is also the union of admissible universes.

I mention this result since it may serve as an example for my previous remark that *"you know nearly everything about the meta–mathematics of a theory once you have an ordinal analysis of it"*. Having analyzed KPi and thus also knowing the proof theoretic ordinal of Δ^1_2–comprehension, Gerhard Jäger succeeded in proving the open conjecture that Feferman's theory T_0 for explicit mathematics is equivalent to Δ^1_2 comprehension with bar induction (cf. [22]). A claim which then seemed to be impenetrable by other means.[21] He solved it by giving a well–ordering proof for the ordinal notation system obtained by the analysis of KPi within the theory T_0. There are of course also other examples of *"knowing nearly everything"*, mostly connected with Π^0_2 analyses and associated combinatorial principles, which I cannot go into further.

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[21] During the preparation of this article I received a preprint by Kentaro Sato "*A new model construction by making a detour via intutionistic theories. II: Interpretability lower bounds for Feferman's explicit mathematics T_0*" in which he establishes the equivalence without use of ordinal analysis.

4.2 More Recent Developments

Since then many advances took place. Wilfried Buchholz [6] introduced operator controlled derivations as a simplification of local predicativity. As a matter of fact this is much more than just a simplification. Local predicativity fails for theories stronger than Π_2–reflection. In his analysis of Π_3–reflection Michael Rathjen [35] introduced a new technique based on thinning operations on the ordinals. A technique which led to analyses of theories up to the strength of Σ_1–Separation, a theory that is equivalent to Π^1_2–comprehension [36]. Operator controlled derivations play an important role in these analyses.

There is also progress in extending the elimination procedures to proofs of Π^0_2–statements—which were my original aim.

The basis therefor was laid by Andreas Weiermann. There are two papers, one joint with Adam Cichon and Wilfried Buchholz [8] in which they developed a new approach to subrecursive hierarchies which is essential for such analyses, the other, joint with Benjamin Blankertz [4], in which they used such hierarchies to obtain Π^0_2–analyses. Benjamin Blankertz later developed the technical details in a very general setting in his dissertation [3]. Jan Carl Stegert [40] in his dissertation simplified Blankertz' work and extended it to axiom systems for reflection and stability.

I myself am busy to collect all these results in a monograph about the proof theory of stability. This is work in progress.

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Restricting Initial Sequents: The Trade-Offs Between Identity, Contraction and Cut

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Abstract In logical sequent calculi, initial sequents expressing the axiom of identity can be required to be atomic, without affecting the deductive strength of the system. When extending the logical system with right- and left-introduction rules for atomic formulas, one can analogously require that initial sequents be restricted to "uratoms", which are undefined (not even vacuously defined) atoms. Depending on the definitional clauses for atoms, the resulting system is possibly weaker than the unrestricted one. This weaker system may however be preferable to the unrestricted system, as it enjoys cut elimination and blocks unwanted derivations arising from non-wellfounded definitions, for example in the context of paradoxes.

1 Introduction

In standard sequent calculi of first-order logic, initial sequents

$$A \vdash A,$$

which express the axiom of identity, can be restricted to atomic A. For non-atomic A, the sequent $A \vdash A$ can then be derived using the right- and left-introduction rules for the logical constants occurring in A. This way of presenting sequent calculi is quite common and has certain technical advantages, such as in the area of automated theorem proving, and also in proof theory itself. For example, arguments establishing

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height-preserving admissibility (see [12]) or arguments concerning rank-preserving admissibility as in Lemma 2 below depend on it. There is even a philosophical rationale behind this procedure: If there are *specific* rules to generate a complex proposition on the right or left side of the turnstile, these specific rules should be used. The statement $A \vdash A$, which is completely *unspecific* as to the structure of A, should only be made when no specific way of introducing A is available, that is, when A is atomic.

This situation changes, when we extend our logical system with rules for atomic formulas ("atoms"). If right- and left-introduction rules for atoms are available, then these atoms are still atomic in the logical sense, that is, they do not contain logical constants, but they are no longer atoms in the semantical sense as they have a specific meaning given by these rules. This is the case in the theory of definitional reflection (see [6–8, 14]). There one extends the logical framework with right-introduction rules for atoms based on the clauses of a definition. This definition has the form of an extended logic program allowing for logically complex formulas in bodies of clauses. The rule of definitional reflection complements these rules with a left-introduction rule based on a kind of inversion principle. However, the points discussed here apply to any extension of logical systems that provides right- and left-introduction rules for atoms (see, for example, [1, 11]).

2 The Formal System of Intuitionistic Logic with Definitional Reflection

We consider intuitionistic propositional logic, which is sufficient to make our point. Let upper case Latin letters denote formulas in this language, let lower case Roman letters denote atoms (which in our simplified framework are propositional letters), and let upper case Greek letters denote finite multisets of formulas. We suppose that a *definition* $\mathbb{D}$ is given, which consists of finitely many clauses of the form

$$a \Leftarrow A$$

Such a clause is called a *(defining) clause for a*. We furthermore assume that with every $\mathbb{D}$ its domain $dom(\mathbb{D})$ is associated, which is a set of atoms containing those atoms for which there is a definitional clause in $\mathbb{D}$, but possibly further atoms. The elements in $dom(\mathbb{D})$ are called the atoms *defined* by $\mathbb{D}$. We allow for atoms defined by $\mathbb{D}$ without there being a clause for them. These elements of $dom(\mathbb{D})$ are called *vacuously defined*. Atoms that are not defined by $\mathbb{D}$ and thus do not belong to $dom(\mathbb{D})$ are called *uratoms*. If $a \in dom(\mathbb{D})$, let $\mathbb{D}(a)$ be the set of *defining conditions* of a, that is, the set $\{A_1, \ldots, A_n\}$ if the clauses for a in $\mathbb{D}$ are as follows:

$$\begin{cases} a \Leftarrow A_1 \\ \quad \vdots \\ a \Leftarrow A_n \end{cases}$$

If a is vacuously defined, then $\mathbb{D}(a)$ is empty. If a is an uratom, then $\mathbb{D}(a)$ is undefined. (In other words, if a is not defined by $\mathbb{D}$, then $\mathbb{D}(a)$ is undefined in the metalogical sense.)

Our system of intuitionistic logic with definitional reflection over the definition $\mathbb{D}$, called $\mathbf{LI}(\mathbb{D})$, has the following rules of inference, where the antecedent of a sequent is understood as a multiset of formulas.

$$(I)\ \frac{}{\Gamma, a \vdash a}$$

$$(\top)\ \frac{}{\Gamma \vdash \top} \qquad\qquad (\bot)\ \frac{}{\Gamma, \bot \vdash A}$$

$$(\vdash\wedge)\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\wedge B} \qquad\qquad (\wedge\vdash)\ \frac{\Gamma, A, B \vdash C}{\Gamma, A\wedge B \vdash C}$$

$$(\vdash\vee)\ \frac{\Gamma \vdash A}{\Gamma \vdash A\vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A\vee B} \qquad (\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A\vee B \vdash C}$$

$$(\vdash\rightarrow)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A\rightarrow B} \qquad\qquad (\rightarrow\vdash)\ \frac{\Gamma, A\rightarrow B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A\rightarrow B \vdash C}$$

$$(\vdash\mathbb{D})\ \frac{\Gamma \vdash C}{\Gamma \vdash a}\ C \in \mathbb{D}(a) \qquad\qquad (\mathbb{D}\vdash)\ \frac{\{\Gamma, a, C \vdash A : C \in \mathbb{D}(a)\}}{\Gamma, a \vdash A} \qquad a \in dom(\mathbb{D})$$

Without the definitional rules in the last line, this is a standard variant of the intuitionistic propositional sequent calculus which we call **LI**. The last line contains the rules of definitional closure ($\vdash\mathbb{D}$) and definitional reflection ($\mathbb{D}\vdash$), which for any atom a which is defined by $\mathbb{D}$, delivers right- and left-introduction rules. The right-introduction rule says that a can be inferred from each defining condition of a. The left-introduction rule says that everything that can be inferred from each defining condition of a can be inferred from a itself. If the clauses defining a are viewed as its inductive definition, the left-introduction rule for a expresses the extremal clause for this inductive definition: "Nothing else defines a". For further discussion see [8, 14]. Note that the rules ($\vdash\mathbb{D}$) and ($\mathbb{D}\vdash$) only apply to those atoms a which are defined by $\mathbb{D}$ (i.e., $a \in dom(\mathbb{D})$) and not to uratoms. In view of ($\bot$), we can disregard vacuously defined atoms, if we identify a vacuously defined atom a with an atom a defined by the clause $a \Leftarrow \bot$.

For **LI** it is well-known that the structural rules of thinning, contraction and cut

$$(Thin)\ \frac{\Gamma\vdash C}{\Gamma, A\vdash C} \qquad (Contr)\ \frac{\Gamma, A, A\vdash C}{\Gamma, A\vdash C} \qquad (Cut)\ \frac{\Gamma\vdash A \quad \Gamma, A\vdash C}{\Gamma\vdash C}$$

are admissible (see, for example, [2, 12]). As thinning is admissible, the version of cut with separated contexts

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}$$

is admissible, too.

The admissibility of thinning and contraction extends from **LI** to **LI**($\mathbb{D}$). For thinning this is obvious, for contraction this is due to the fact that in ($\mathbb{D}\vdash$) the atom a is repeated in the premisses, which means that we have already built an implicit contraction into ($\mathbb{D}\vdash$). In fact, this implicit contraction is not even needed. Without loss of deductive power, we can replace ($\mathbb{D}\vdash$) with its contraction-free variant $(\mathbb{D}\vdash)^{\mathrm{cf}}$:

$$(\mathbb{D}\vdash)^{\mathrm{cf}} \ \frac{\{\Gamma, C \vdash A : C \in \mathbb{D}(a)\}}{\Gamma, a \vdash A}$$

This can be seen as follows. Consider the system $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$, which results from **LI**($\mathbb{D}$) by replacing ($\mathbb{D}\vdash$) with $(\mathbb{D}\vdash)^{\mathrm{cf}}$. For a derivation $\mathcal{D}$ in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$, the $\mathbb{D}$-rank $r_{\mathbb{D}}(\mathcal{D})$ is the maximum number of applications of $(\mathbb{D}\vdash)^{\mathrm{cf}}$, where the maximum is taken over all branches of $\mathcal{D}$. More precisely,

$$\begin{aligned} r_{\mathbb{D}}(\mathcal{D}) &= 0, \ \text{if } \mathcal{D} \text{ is}(I), (\top) \text{ or } (\bot), \\ r_{\mathbb{D}}(\mathcal{D}) &= r_{\mathbb{D}}(\mathcal{D}_1), \text{if } \mathcal{D} \text{ is of the form } \frac{\mathcal{D}_1}{\Gamma \vdash A}, \text{and the last step} \\ &\ \text{is different from } (\mathbb{D}\vdash)^{\mathrm{cf}}, \\ r_{\mathbb{D}}(\mathcal{D}) &= max\{r_{\mathbb{D}}(\mathcal{D}_1), r_{\mathbb{D}}(\mathcal{D}_2)\}, \text{if } \mathcal{D} \text{ is of the form } \frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\Gamma \vdash A}, \text{and the last step} \\ &\ \text{is different from } (\mathbb{D}\vdash)^{\mathrm{cf}}, \\ r_{\mathbb{D}}(\mathcal{D}) &= max_{C \in \mathbb{D}(a)}\{r_{\mathbb{D}}(\mathcal{D}_C)\} + 1, \text{if } \mathcal{D} \text{ is of the form} \frac{\{\mathcal{D}_C : C \in \mathbb{D}(a)\}}{\Gamma, a \vdash A}(\mathbb{D}\vdash)^{\mathrm{cf}}. \end{aligned}$$

Then we can show the following.

Lemma 1 (Invertibility lemma) *If $\mathcal{D}$ is a derivation of $\Gamma, a \vdash A$ in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$, then for each $C \in \mathbb{D}(a)$ we can find a derivation $\mathcal{D}_C$ of $\Gamma, C \vdash A$ such that $r_{\mathbb{D}}(\mathcal{D}_C) \leq r_{\mathbb{D}}(\mathcal{D})$.*

Proof The only non-trivial case obtains when $\mathcal{D}$ is an initial sequent $\Gamma, a \vdash a$. Here $r_{\mathbb{D}}(\mathcal{D}) = 0$. We use that for any C we can find a derivation $\mathcal{D}'_C$ of $\Gamma, C \vdash C$ in **LI**. Since $(\mathbb{D}\vdash)^{\mathrm{cf}}$ is not used in $\mathcal{D}'_C$, we know that $r_{\mathbb{D}}(\mathcal{D}'_C) = 0$. Then, for $C \in dom(\mathbb{D})$, let $\mathcal{D}_C$ be

$$\frac{\begin{array}{c}\mathcal{D}'_C \\ \Gamma, C \vdash C\end{array}}{\Gamma, C \vdash a}(\vdash \mathbb{D})$$

Since $r_{\mathbb{D}}(\mathcal{D}'_C) = 0$, we know that $r_{\mathbb{D}}(\mathcal{D}_C) = 0$. □

Now it is easy to show that contraction is admissible in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$. More precisely, we can show the following.

Lemma 2 *If $\mathcal{D}$ is a derivation of $\Gamma, A, A \vdash C$ in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$, then we can find a derivation $\mathcal{D}'$ of $\Gamma, A \vdash C$ in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$such that $r_{\mathbb{D}}(\mathcal{D}') \leq r_{\mathbb{D}}(\mathcal{D})$.*

Proof by induction on the triple $\langle r_{\mathbb{D}}(\mathcal{D}), deg(A), h(\mathcal{D})\rangle$, where $deg(A)$ is the logical complexity of A and $h(\mathcal{D})$ is the height of $\mathcal{D}$ (that is, the length of its longest branch). As an example we present the case in which $(\mathbb{D}\vdash)^{\mathrm{cf}}$ is applied in the last step and the atom a introduced by $(\mathbb{D}\vdash)^{\mathrm{cf}}$ is the contraction formula:

$$\mathcal{D}: \quad \frac{\left\{\begin{array}{c}\mathcal{D}_C \\ \Gamma, a, C \vdash A\end{array} \ : \ C \in \mathbb{D}(a)\right\}}{\Gamma, a, a \vdash A}(\mathbb{D}\vdash)^{\mathrm{cf}}$$

We assume that a is not vacuously defined–otherwise the case is trivial. Obviously, $r_{\mathbb{D}}(\mathcal{D}_C) < r_{\mathbb{D}}(\mathcal{D})$ for every $C \in \mathbb{D}(a)$. Applying the invertibility lemma (Lemma 1) to the premiss derivations $\mathcal{D}_C$ we obtain derivations

$$\left\{\begin{array}{c}\mathcal{D}'_C \\ \Gamma, C, C \vdash A\end{array} \ : \ C \in \mathbb{D}(a)\right\}$$

such that $r_{\mathbb{D}}(\mathcal{D}'_C) < r_{\mathbb{D}}(\mathcal{D})$ for every C. Therefore, by induction hypothesis, we obtain derivations

$$\left\{\begin{array}{c}\mathcal{D}''_C \\ \Gamma, C \vdash A\end{array} \ : \ C \in \mathbb{D}(a)\right\}$$

such that $r_{\mathbb{D}}(\mathcal{D}''_C) \leq r_{\mathbb{D}}(\mathcal{D}'_C) < r_{\mathbb{D}}(\mathcal{D})$ for every C. From those we obtain a derivation

$$\mathcal{D}': \quad \frac{\left\{\begin{array}{c}\mathcal{D}''_C \\ \Gamma, C \vdash A\end{array} \ : \ C \in \mathbb{D}(a)\right\}}{\Gamma, a \vdash A}$$

such that $r_{\mathbb{D}}(\mathcal{D}') \leq r_{\mathbb{D}}(\mathcal{D})$. □

3 The Failure of Cut in LI($\mathbb{D}$)

The system **LI**($\mathbb{D}$) does not enjoy the admissibility of cut. Consider the following definition:

$$\mathbb{D}_r \ \{ r \Leftarrow \neg r$$

(with $\neg r$ abbreviating $r \to \bot$.) Using the right- and left-introduction rules for r and the rules for implication, this leads to derivations of both $r \vdash \bot$ and of $\vdash r$:

$$\dfrac{\dfrac{\dfrac{}{r, r\to\bot\vdash r}\,(I) \qquad \dfrac{}{r, \bot \vdash r}\,(I)}{r \to \bot, r \vdash \bot}\,(\to\vdash)}{r \vdash \bot}\,(\mathbb{D}_r\vdash) \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{r, r \to \bot \vdash r}\,(I) \qquad \dfrac{}{r, \bot \vdash r}\,(I)}{r \to \bot, r \vdash \bot}\,(\to\vdash)}{r \vdash \bot}\,(\mathbb{D}_r\vdash)}{\vdash \neg r}\,(\vdash\to)}{\vdash r}\,(\vdash\mathbb{D}_r) \tag{1}$$

If cut were admissible, $\vdash \bot$ would be derivable, which is not the case as there is no right-introduction rule for $\bot$. Note that we impose no restriction on the form of definitional clauses, in particular no well-foundedness restriction. The definition $\mathbb{D}_r$ may be considered a propositional short form of Russell's paradox, obtained from the following clause for comprehension:

$$t \in \{x : A(x)\} \Leftarrow A(t)$$

by instantiating $A(x)$ with $x \notin x$ and t with $\{x : x \notin x\}$, and then abbreviating $\{x : x \notin x\} \in \{x : x \notin x\}$ by r.

In view of the fact that $(\mathbb{D}\vdash)$ is not stronger than its contraction-free variant $(\mathbb{D}\vdash)^{\mathrm{cf}}$, instead of (1) we may consider the following pair of derivations in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{r, r\to\bot\vdash r}\,(I) \quad \dfrac{}{r, \bot\vdash\bot}\,(I)}{r\to\bot, r\vdash\bot}\,(\to\vdash)}{r\to\bot\vdash r\to\bot}\,(\vdash\to)}{r\to\bot\vdash r}\,(\vdash\mathbb{D}_r) \qquad \dfrac{}{\bot\vdash\bot}\,(I)}{r \to \bot \vdash \bot}\,(\to\vdash)}{r \vdash \bot}\,(\mathbb{D}_r\vdash)^{\mathrm{cf}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{r, r\to\bot\vdash r}\,(I) \quad \dfrac{}{r, \bot\vdash\bot}\,(I)}{r\to\bot, r\vdash\bot}\,(\to\vdash)}{r\to\bot\vdash r\to\bot}\,(\vdash\to)}{r\to\bot\vdash r}\,(\vdash\mathbb{D}_r) \qquad \dfrac{}{\bot\vdash\bot}\,(I)}{r \to \bot \vdash \bot}\,(\to\vdash)}{r \vdash \bot}\,(\mathbb{D}_r\vdash)^{\mathrm{cf}}}{\vdash r}\,(\vdash\mathbb{D}_r) \tag{2}$$

This shows that it is not a particular form of contraction of atoms that needs to be used in the derivation of the paradox, but rather contraction for implicational formulas, which is the essential form of contraction in $\mathbf{LI}(\mathbb{D})$.

4 The Trade-Off Between Contraction and Cut

We have shown by an example that in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$, which by means of the rules $(\wedge\vdash)$ and $(\to\vdash)$ implicitly contains contraction, and in which therefore the explicit contraction rule is admissible (Lemma 2), the rule of cut is not admissible. If we consider a contraction-free variant, in which $(\wedge\vdash)$ and $(\to\vdash)$ are replaced with

$$\frac{\Gamma, A\vdash C}{\Gamma, A\wedge B\vdash C} \quad \frac{\Gamma, B\vdash C}{\Gamma, A\wedge B\vdash C} \quad \text{and} \quad \frac{\Gamma\vdash A \quad \Delta, B\vdash C}{\Gamma, \Delta, A\to B\vdash C}$$

respectively, then cut can be eliminated. This is shown in detail in [13]. Therefore we obtain a trade-off between contraction and cut, when we add definitional rules to the logical system.

Result 1 *If the logical system contains implicit or explicit contraction, then the admission of cut makes the system inconsistent. If it contains neither implicit nor explicit contraction, then cut is admissible.*

This corresponds to the observation dating back to Fitch [5] that removing the rule of contraction may be used as a strategy to cope with the paradoxes. This result can even be refined. The form of contraction used in counterexample (1) is contraction of an atom defined by $\mathbb{D}_r$ with an atom of the same shape used in an initial sequent. In [15, 16] it was claimed that prohibiting this specific sort of contraction would be a more specific way of keeping cut admissible in the system with definitional reflection than abolishing contraction altogether. This claim is not invalidated by the fact that in $\mathbf{LI}^{\mathrm{cf}}(\mathbb{D})$ no contraction of atoms is needed, as example (2) shows. In fact, it carries over to the present situation *mutatis mutandis*. The contraction of $r \to \bot$, which is implicit in the lowermost application of $(\to\vdash)$, is a contraction of an occurrence of $r \to \bot$, in which r stems from an initial sequent, with an occurrence of $r \to \bot$, in which r is a result of $(\vdash\mathbb{D}_r)$. This means that there is still an identification of occurrences of atoms which are generated by different (structural vs. meaning-giving) rules, though in example (2) it is not definitional reflection $(\mathbb{D}_r\vdash)^{\mathrm{cf}}$ but the introduction of the atom r on the right side according to $(\vdash\mathbb{D}_r)$, which is involved. Identifying such critical forms of contraction can, for example, be achieved by attaching labels to formulas that indicate when definitional rules are applied (see [3, 4]). A further elaboration of this topic, which will result in a more finegrained specification of the rules of contraction, is beyond the scope of this paper.

5 Restricting Initial Sequents: The Admissibility of Cut

In initial sequents $\Gamma, a \vdash a$ of $\mathbf{LI}(\mathbb{D})$, a can be an arbitrary atom. According to what was said in the introduction, we now restrict the atom a in initial sequents to uratoms by replacing (I) with the following rule:

$$(I)^{\circ}\quad \frac{}{\Gamma, a \vdash a}\quad a \notin dom(\mathbb{D})$$

As the rule of definitional reflection we use the contraction-free rule $(\mathbb{D}\vdash)^{\mathrm{cf}}$. The resulting system, with $(I)^{\circ}$ instead of (I) and $(\mathbb{D}\vdash)^{\mathrm{cf}}$ instead of $(\mathbb{D}\vdash)$, is called $\mathbf{LI}^{\circ}(\mathbb{D})$. Lemmas 1 and 2 continue to hold for $\mathbf{LI}^{\circ}(\mathbb{D})$. The proof of Lemma 1 is now trivial, as initial sequents involving a defined atom a can no longer occur. Due to the restriction on initial sequents, Lemma 2 is easier to prove. Therefore in $\mathbf{LI}^\circ(\mathbb{D})$ contraction is admissible. The counterexample (2) to cut no longer works, as it uses initial sequents for r. These initial sequents are not available in $\mathbf{LI}^\circ(\mathbb{D}_r)$, because r is defined in $\mathbb{D}_r$, and is thus not an uratom.

In fact, in the system $\mathbf{LI}^\circ(\mathbb{D})$ we can eliminate cuts. To demonstrate this, we use as an induction measure the $\mathbb{D}$-*weight* of a formula occurrence in a derivation. Unlike the $\mathbb{D}$-rank as used in Lemmas 1 and 2, the $\mathbb{D}$-weight is not a measure of a derivation $\mathcal{D}$, but a measure of a formula occurrence at a certain place in (a sequent in) a derivation $\mathcal{D}$. In the following, upper indices distinguish occurrences of formulas. For example, C^1 and C^2 denote different occurrences of the formula C. It is always assumed that a formula occurrence below an inference line corresponds to or results from a particular formula occurrence (or from particular formula occurrences) above the line. For example, in an application of $(\vee\vdash)$ of the form

$$\frac{A_1^1, \ldots, A_n^1, A^1 \vdash C^1 \quad A_1^2, \ldots, A_n^2, B^1 \vdash C^2}{A_1^3, \ldots, A_n^3, (A \vee B)^1 \vdash C^3}$$

it is assumed that, for all i $(1 \leq i \leq n)$, the occurrence A_i^3 corresponds to the occurrences A_i^1 and A_i^2, the occurrence C^3 corresponds to the occurrences C^1 and C^2, and the occurrences of A and B as immediate subformulas of $(A \vee B)^1$ correspond to the occurences A^1 and B^1, respectively.

Then the $\mathbb{D}$-weight $w_{\mathbb{D}}(C^1)$ of a formula occurrence C^1 in a given derivation is defined by induction on the construction of the derivation.

Each formula occurrence in $(I)^\circ$, $(\top)$ or $(\bot)$ has $\mathbb{D}$-weight 0.

If the last step is

$$(\vdash\to)\ \frac{A_1^1, \ldots, A_n^1, A^1 \vdash B^1}{A_1^2, \ldots, A_n^2 \vdash (A \to B)^1}$$

then $w_{\mathbb{D}}(A_i^2) = w_{\mathbb{D}}(A_i^1)$ for all i $(1 \leq i \leq n)$, and $w_{\mathbb{D}}((A \to B)^1) = max\{w_{\mathbb{D}}(A^1), w_{\mathbb{D}}(B^1)\}$. For $(\wedge\vdash)$ and $(\vdash\vee)$ the $\mathbb{D}$-weight is defined in the same way.

If the last step is

$$(\vee\vdash)\ \frac{A_1^1, \ldots, A_n^1, A^1 \vdash C^1 \quad A_1^2, \ldots, A_n^2, B^1 \vdash C^2}{A_1^3, \ldots, A_n^3, (A \vee B)^1 \vdash C^3}$$

then $w_{\mathbb{D}}(A_i^3) = max\{w_{\mathbb{D}}(A_i^1), w_{\mathbb{D}}(A_i^2)\}$ for all i $(1 \leq i \leq n)$, $w_{\mathbb{D}}(C^3) = max\{w_{\mathbb{D}}(C^1), w_{\mathbb{D}}(C^2)\}$ and $w_{\mathbb{D}}((A \vee B)^1) = max\{w_{\mathbb{D}}(A^1), w_{\mathbb{D}}(B^1)\}$, and analogously for $(\vdash\wedge)$.

If the last step is

$$(\to\vdash)\ \frac{A_1^1, \ldots, A_n^1, (A \to B)^1 \vdash A^1 \quad A_1^2, \ldots, A_n^2, B^1 \vdash C^1}{A_1^3, \ldots, A_n^3, (A \to B)^2 \vdash C^2}$$

then $w_{\mathbb{D}}(A_i^3) = max\{w_{\mathbb{D}}(A_i^1), w_{\mathbb{D}}(A_i^2)\}$ for all i $(1 \leq i \leq n)$, $w_{\mathbb{D}}(C^2) = w_{\mathbb{D}}(C^1)$, and $w_{\mathbb{D}}((A \to B)^2) = max\{w_{\mathbb{D}}((A \to B)^1), w_{\mathbb{D}}(A^1), w_{\mathbb{D}}(B^1)\}$.

If we consider derivations with cut and if the last step is

$$(Cut)\ \frac{A_1^1, \ldots, A_n^1 \vdash A^1 \quad A_1^2, \ldots, A_n^2, A^2 \vdash C^1}{A_1^3, \ldots, A_n^3 \vdash C^2}$$

then $w_{\mathbb{D}}(A_i^3) = max\{w_{\mathbb{D}}(A_i^1), w_{\mathbb{D}}(A_i^2)\}$ for all i $(1 \leq i \leq n)$, and $w_{\mathbb{D}}(C^2) = w_{\mathbb{D}}(C^1)$.

If the last step is a $\mathbb{D}$-rule, then $w_{\mathbb{D}}(a)$ is increased by 1. More precisely, if this step is

$$(\vdash \mathbb{D})\ \frac{A_1^1, \ldots, A_n^1 \vdash C^1}{A_1^2, \ldots, A_n^2 \vdash a^1}\ C \in \mathbb{D}(a)$$

then $w_{\mathbb{D}}(A_i^2) = w_{\mathbb{D}}(A_i^1)$ for all i $(1 \leq i \leq n)$, and $w_{\mathbb{D}}(a^1) = w_{\mathbb{D}}(C^1) + 1$. If it is

$$(\mathbb{D}\vdash)^{\mathrm{cf}}\ \frac{\{A_1^i, \ldots, A_n^i, C_i^1 \vdash A^i : 1 \leq i \leq k\}}{A_1^{k+1}, \ldots, A_n^{k+1}, a^1 \vdash A^{k+1}}\ \text{where } \mathbb{D}(a) = \{C_1, \ldots, C_k\},$$

then $w_{\mathbb{D}}(A_j^{k+1}) = max_{1 \leq i \leq k}\{w_{\mathbb{D}}(A_j^i)\}$ for all j $(1 \leq j \leq n)$, and $w_{\mathbb{D}}(a^1) = max_{1 \leq i \leq k}$ $\{w_{\mathbb{D}}(C_i^1)\} + 1$. If $\mathbb{D}(a) = \emptyset$, and thus $k = 0$, then $w_{\mathbb{D}}(A_j^1) = w_{\mathbb{D}}(A^1) = 0$ for all j $(1 \leq j \leq n)$ and $w_{\mathbb{D}}(a^1) = 1$.

It is important to notice that the weight increases not only at the application of $(\mathbb{D}\vdash)^{\mathrm{cf}}$, but also at the application of $(\vdash \mathbb{D})$. However, the crucial point is that only the formula a introduced by $(\vdash \mathbb{D})$ or $(\mathbb{D}\vdash)^{\mathrm{cf}}$ is affected by the increase of weight, not the parametric context formulas. Putting it another way, we may consider the $\mathbb{D}$-rank to be a measure of a sequent within a derivation, namely the $\mathbb{D}$-rank of the subderivation with this sequent as its end-sequent. In contradistinction to that, the $\mathbb{D}$-weight is a measure of a formula occurrence within a sequent, namely the $\mathbb{D}$-weight of this formula occurrence within the subderivation that has this sequent as its end-sequent.

Now cut can be eliminated even though contraction is admissible. More precisely, we can show *weight-preserving cut elimination* in the following sense.

Theorem 1 *Consider* $\mathbf{LI}^{\circ}(\mathbb{D})$ *extended with the cut rule* (Cut)*. Suppose a derivation* $\mathcal{D}$ *in this system is given that ends with an application*

$$\frac{A_1^1, \ldots, A_n^1 \vdash A^1 \quad A_1^2, \ldots, A_n^2, A^2 \vdash C^1}{A_1^3, \ldots, A_n^3 \vdash C^2}$$

of cut such that the derivations of its premisses are cut-free. Then we can construct a cut-free derivation of $A_1^4, \ldots, A_n^4 \vdash C^3$ *such that* $w_{\mathbb{D}}(A_i^4) \leq w_{\mathbb{D}}(A_i^3)$ *for all* i $(1 \leq i \leq n)$ *and* $w_{\mathbb{D}}(C^3) \leq w_{\mathbb{D}}(C^2)$.

Proof by induction on the triple $\langle w_{\mathbb{D}}(A), deg(A), h(\mathcal{D})\rangle$, where $w_{\mathbb{D}}(A) = max(w_{\mathbb{D}}(A^1), w_{\mathbb{D}}(A^2))$, and, as before, $deg(A)$ is the logical complexity of A and $h(\mathcal{D})$ is the height of $\mathcal{D}$. The value $w_{\mathbb{D}}(A)$ is also called the *weight of the cut formula*. The reduction steps are as usual in cut elimination. We just indicate where the $\mathbb{D}$-rules are involved, and where the definition of *weight* comes into play. The main reduction of the $\mathbb{D}$-rules reduces

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash B\end{array}}{\Gamma\vdash a}\,(\vdash\mathbb{D}) \qquad \dfrac{\left\{\begin{array}{c}\mathcal{D}_C\\ \Gamma, C\vdash A\end{array} \;:\; C\in\mathbb{D}(a)\right\}}{\Gamma, a\vdash A}\,(\mathbb{D}\vdash)^{\mathrm{cf}}}{\Gamma\vdash A} \qquad B\in\mathbb{D}(a)$$

to

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash B\end{array} \quad \begin{array}{c}\mathcal{D}_B\\ \Gamma, B\vdash A\end{array}}{\Gamma\vdash A}$$

by reducing a cut with cut-formula a to a cut with cut-formula B of lower weight.

An example of a permutation of cut with an application of a $\mathbb{D}$-rule is the reduction of

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, A\vdash C\end{array}}{\Gamma, A\vdash a}\,(\vdash\mathbb{D})}{\Gamma\vdash a} \quad C\in\mathbb{D}(a)$$

to

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A\vdash C\end{array}}{\Gamma\vdash C}}{\Gamma\vdash a}\,(\vdash\mathbb{D})$$

Here it is crucial that even though the weight of the cut formula A might not decrease, the weight of a is not increased, as it is solely dependent on the weight of C, which is untouched by the transformation. A measure such as the rank $r_{\mathbb{D}}$ that applies to sequents in a proof rather than formula occurrences in a sequent could not deliver this behaviour.

The restriction of initial sequents to uratoms becomes significant, when we consider the situation

$$\frac{\Gamma\vdash a^1 \quad \Gamma, a^2\vdash a^3}{\Gamma\vdash a^4}$$

which is reduced to

$$\Gamma \vdash a^1$$

As the right premiss of the cut is an initial sequent, $w_{\mathbb{D}}(a^2) = w_{\mathbb{D}}(a^3) = w_{\mathbb{D}}(a^4) = 0$. According to our restriction on initial sequents, a is an uratom. It is easy to see that the weight of any occurrence of an uratom is 0, which means that $w_{\mathbb{D}}(a^1) = 0$. Thus $w_{\mathbb{D}}(a^1) = w_{\mathbb{D}}(a^4) = 0$. The latter equation would not necessarily hold if we admitted initial sequents with defined atoms. If a is not an uratom, then it is possible that $w_{\mathbb{D}}(a^1) > 0$ and thus $w_{\mathbb{D}}(a^1) > w_{\mathbb{D}}(a^4)$, contrary to what is claimed in the theorem. □

The admissibility of contraction and cut in **LI**°($\mathbb{D}$) implies that the unrestricted rule (I) is not derivable[1] in **LI**°($\mathbb{D}$). For if it were derivable, then using the definition $\mathbb{D}_r$ and applying cut to the derivations (2), we could derive $\vdash \bot$.

6 The Trade-Off Between Identity and Contraction/Cut

We have shown that by restricting initial sequents to the case where a is an uratom, we obtain a system in which both contraction and cut are admissible. If we do not restrict initial sequents, contraction is still admissible, but cut ceases to be admissible. This means that, in a sense, we have traded identity against contraction and cut.[2]

Result 2 *If identity as expressed by initial sequents is restricted to uratoms, then the rules of contraction and cut are admissible. If identity is admitted for any atom, whether defined or not, then the admission of cut makes the system inconsistent.*

It should be noted that, in the presence of unrestricted identity, a restriction on cut is an option which should not be excluded. A way of restricting cut to cases where it continues to be admissible, consists, for example, in using certain term assignments and corresponding provisos for the application of cut, as sketched in [17]. Here we do not want to enter a philosophical discussion on which one of unrestricted identity, unrestricted contraction or unrestricted cut is the preferential rule, but just point to the trade-offs between these principles in the presence of definitional reflection.

As the counterexample given in (1) or (2) is from the domain of paradoxes, this shows that restrictions on any of the principles of identity, contraction and cut can block paradoxes (see [16]).

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[1] As the rule (I) is an axiom, derivability and admissibility mean the same.

[2] In the system with unseparated contexts chosen here, unrestricted identity plus cut implies contraction, so that cut and contraction cannot be separated. This was pointed out to the author by Roy Dyckhoff.

7 Restricted Initial Sequents in Logic Programming

Within logic, and in particular within the discussion of the paradoxes, the restriction of initial sequents, and thus of identity, has never been properly considered, in contrast, for example, to the issue of contraction, which is a strong topic in this debate. However, in the realm of logic programming, this issue has always been present. In fact, the idea of definitional reflection has been developed in close parallel with related issues in logic programming (see [8]). The restriction of initial sequents considered here was first proposed by Kreuger [10]. He motivated it by considerations concerning the operational interpretation of definitional reflection in the programming language GCLA, which implemented definitional reflection. Instead of formally specifying a domain $dom(\mathbb{D})$, he adds the clause $a \Leftarrow a$ to $\mathbb{D}$ to trivialize the application of $(\mathbb{D}\vdash)$ for every a which in our terminology is an uratom.

In their proof-theoretic framework for logic programming, Jäger and Stärk [9] use a classical one-sided Schütte-Tait-style sequent calculus, which contains rules for the evaluation of atoms that correspond to our rules of definitional closure and reflection. They develop a three-valued semantics for this system and explicitly consider identity-free derivations to be the proof-theoretic approach most faithful to this semantics. For identity-free derivations they prove a cut elimination theorem by translating proofs into a system with ramified $\mathbb{D}$-rules, for which the cut elimination proof is completely standard, and then retranslate cut-free proofs. This translation and retranslation crucially depends on the fact that identity in the unrestricted form (I) is lacking. This method can easily be carried over to the situation considered here. Thus Jäger and Stärk implicitly point to the trade-off between unrestricted identity and the availability of cut, which was the main topic of this paper.

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Higman's Lemma and Its Computational Content

Helmut Schwichtenberg, Monika Seisenberger and Franziskus Wiesnet

Dedicated to Gerhard Jäger on the occasion of his 60th birthday

Abstract Higman's Lemma is a fascinating result in infinite combinatorics, with manyfold applications in logic and computer science. It has been proven several times using different formulations and methods. The aim of this paper is to look at Higman's Lemma from a computational and comparative point of view. We give a proof of Higman's Lemma that uses the same combinatorial idea as Nash-Williams' indirect proof using the so-called minimal bad sequence argument, but which is constructive. For the case of a two letter alphabet such a proof was given by Coquand. Using more flexible structures, we present a proof that works for an arbitrary well-quasiordered alphabet. We report on a formalization of this proof in the proof assistant Minlog, and discuss machine extracted terms (in an extension of Gödel's system T) expressing its computational content.

Keywords Higman's Lemma · Inductive definitions · Minimal bad sequence · Program extraction

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1 Introduction

Without exaggeration it can be said that Higman's Lemma [15] is one of the most often proven theorems in Mathematical Logic and Theoretical Computer Science. The fascination of this theorem is due to the fact that it has various formulations and is of interest in different areas such Proof theory, Constructive Mathematics, Reverse Mathematics, and Term rewriting, as we will briefly discuss further below.

Nash-Williams [19] gave a very concise classical proof using the so-called minimal bad sequence argument. In the following we briefly recall well-quasiorderings and sketch Nash-Williams' proof.

Definition A binary relation $\preceq$ on a set A is a well-quasiorder (wqo) if (i) it is transitive and (ii) every infinite sequence in A is "good", i.e., $\forall_{(a_i)_{i<\omega}} \exists_{i,j} (i < j \wedge a_i \preceq a_j)$.

Let A^* denote the set of finite sequences ("words") with elements in A. We call a word $[a_1, \dots, a_n]$ embeddable ($\preceq^*$) in $[b_1, \dots, b_m]$ if there exists a strictly increasing map $f : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $a_i \preceq b_{f(i)}$ for all $i \in \{1, \dots n\}$.

Now Higman's Lemma says

If $(A, \preceq)$ is a well-quasiorder, then so is $(A^*, \preceq^*)$.

Nash-Williams' proof proceeds as follows. That a bad sequence of words, i.e., a sequence that is not good, is impossible is basically a consequence of two facts: (a) for each bad sequence exists a bad sequence with is smaller in a lexicographical sense, and (b), if there exists a bad sequence, then exists also a minimal bad sequence with respect to this lexicographical order. We give the proof in more detail:

(1) In order to show "wqo $(A, \preceq)$ implies wqo $(A^*, \preceq^*)$" assume for contradiction that there is a bad sequence of words in A^*.
(2) Among all infinite bad sequences of words we choose (using classical dependent choice) a minimal bad sequence, i.e., a sequence $(w_i)_{i<\omega}$, such that, for all n, $w_0, \dots, w_n$ starts an infinite bad sequence, but $w_0, \dots, w_{n-1}, v$, where v is a proper initial segment of w_n, does not.
(3) Since for all i $w_i \neq []$, let $w_i = a_i * v_i$. By Ramsey's theorem and the fact that our alphabet A is a well-quasiorder, there exists an infinite subsequence $a_{\kappa_0} \preceq a_{\kappa_1} \preceq \cdots$ of the sequence $(a_i)_{i<\omega}$. This also determines a corresponding sequence $w_0, \dots, w_{\kappa_0 - 1}, v_{\kappa_0}, v_{\kappa_1}, \dots$.
(4) The sequence $w_0, \dots, w_{\kappa_0 - 1}, v_{\kappa_0}, v_{\kappa_1}, \dots$ must be bad (otherwise also $(w_i)_{i<\omega}$ would be good), but this contradicts the minimality in (2).

The computational content of Nash-Williams' proof was first investigated by Murthy [17], by applying Friedman's A-translation in the interactive theorem prover NuPRL to the classical proof. Murthy represented functions as relations to eliminate choice, and used second order classical logic. However, due to the size of the translated proof and program, the resulting program could only be run on trivial input. In [29], the second author formalized Nash-Williams' proof in the proof assistant Minlog, by applying a refined version of the A-translation and, contrary to Murthy, not eliminating the axiom of classical dependent choice, but rather adding computational content to it using bar recursion. This resulted in a considerable smaller extracted program, but still with infeasible run-times due to the eager evaluation strategy of Minlog's term language. Reasonable results have been only obtained recently thanks to a Minlog extension which translates extracted terms to Haskell. Other formalizations of the classical proof include Herbelin's formalization of Murthy's A-translated proof in Coq [14] and Sternagel's formalization of Nash-Williams' proof in Isabelle [30] which also provides a proof of Kruskal's theorem. However, [30] does not include the extraction of a program. Recently, Powell [21] applied Gödel's Dialectica Interpretation to this proof. The interpretation yields a program, but no formalization has been provided so far.

In this paper, we aim at a constructive proof (without choice) which has the same underlying construction as Nash-Williams' proof but allows us to directly read off the program. For a $\{0, 1\}$-alphabet such a proof was given by Coquand and Fridlender [6, 7]. Here we provide a proof and a formalization for full Higman's Lemma, and also discuss how this proof is related to other constructive proofs. The paper is organized as follows: We give a constructive reformulation of Nash-Williams' proof in Sect. 2 and comment on its formalization in Sect. 3. In Sect. 4 we spell out the computational content of some of the proofs. Each time it comes in the form of a term (in an extension T^+ of Gödels T) machine extracted from a formalization of the respective proof. We give an overview on existing formalizations of the Coquand/Fridlender proof at the end of Sect. 2 and add a comparison with other constructive proofs in Sect. 5.

2 A Constructive Reformulation of Nash-Williams' Proof

The objective of this section is to present a constructive proof of Higman's Lemma that uses the same combinatorial idea as Nash-Williams' classical proof and generalizes the proof by Coquand and Fridlender. Such a proof (without formalization) has been given in [28]. However, if one is interested in the computational content one has to reformulate this proof and to change it at various places to make the computational content visible (see also the remark at the end of this section). We use an inductive characterization of a binary relation satisfying condition (ii) in the definition of a well-quasiorder; such relations have been called "almost full" in [33]. Our characterization is via a "bar" predicate which comes in two variants, one for the alphabet and one for words, see below for a definition. Thus, the statement we are going to prove is

$$\mathrm{BarA}_{\preceq}[\,] \to \mathrm{BarW}_{\preceq}[\,].$$

Throughout the whole paper we assume $\preceq$ to be a binary relation on a set A which is decidable in the sense that it is given by a binary total function into the booleans; transitivity will not be needed. It suffices to let A be the set of natural numbers. Most of our notions will depend on the $\preceq$-relation. However, we usually suppress this dependence, since $\preceq$ will be kept fixed most of the time.

Notation We use

$a, b, \dots$	for letters, i.e., elements of a A,
$as, bs, \dots$	for finite sequences of letters, i.e., elements of A^*,
$v, w, \dots$	for words, i.e., elements of A^*,
$vs, ws, \dots$	for finite sequences of words, i.e., elements of A^{**}.

Definition (*Higman embedding, inductive*) The embedding relation $\preceq^*$ on A^* is defined inductively by the following axioms (written as rules):

$$\frac{}{[\,] \preceq^* [\,]} \qquad \frac{v \ \preceq^* \ w}{v \ \preceq^* \ a{*}w} \qquad \frac{a \preceq b \quad v \ \preceq^* \ w}{a{*}v \ \preceq^* \ b{*}w}$$

where $*$ denotes the cons operation on lists.

Definition GoodA *as* expresses that a finite sequence *as* of letters is good; note that finite sequences grow to the left, i.e., a finite sequence is *good* if there are two elements such that the one to the left is larger than or equal to w.r.t. $\preceq$ to the one on the right. A sequence is called *bad* if it is not good. Furthermore, we use

$$\begin{aligned}\mathrm{Ge}_{\exists}(a, as) &:= \exists_{i<|as|} a \succeq (as)_i,\\ \mathrm{Ge}_{\forall}(a, as) &:= \forall_{i<|as|} a \succeq (as)_i,\\ \mathrm{Ge}_{\exists\forall}(a, ws) &:= \exists_{i<|ws|} \forall_{j<|(ws)_i|} a \succeq (ws)_{i,j}.\end{aligned}$$

A finite sequence $as = [a_{n-1}, .., a_0]$ is *decreasing* if $a_j \succeq a_i$ whenever $j \geq i$. Further, BSeq *as* determines the "first" bad subsequence occurring in *as*:

$$\begin{aligned}\mathrm{BSeq}\,[\,] &:= [\,],\\ \mathrm{BSeq}\,(a{*}as) &:= \begin{cases} a{*}\mathrm{BSeq}\,as & \text{if } \neg\mathrm{Ge}_{\exists}(a, as),\\ \mathrm{BSeq}\,as & \text{otherwise.}\end{cases}\end{aligned}$$

Definition We inductively define a set BarA $\subseteq A^*$ by the following rules:

$$\frac{\mathrm{GoodA}\ as}{\mathrm{BarA}\ as} \qquad \frac{\forall_a\,\mathrm{BarA}\ a{*}as}{\mathrm{BarA}\ as}.$$

BarW *ws* is defined similarly, using the corresponding GoodW *ws*. However, since GoodW is a predicate on words, it refers to the embedding relation $\preceq^*$ on A^* rather than $\preceq$ directly.

As in the end we are interested in getting a program that for any sequence of words yields witnesses that this sequence is good we also prove the following.

Proposition (BarWToGoodInit) BarW[ ] *implies that every infinite sequence of words has a good initial segment.*

Proof Let f be a variable of type `nat=>list nat`. We show, more generally,

$$\forall_{ws,f,n}(\text{BarW } ws \to \text{Rev}(\bar{f}n) = ws \to \exists_m \text{GoodW}(\text{Rev}(\bar{f}m)))$$

by induction on BarW. The proposition then follows with $ws = []$.

1. GoodW ws. Assume that there are an infinite sequence f and a number n such that $\text{Rev}(\bar{f}n) = ws$ (i.e., $[f(n-1), \ldots, f0] = ws$). Since ws is good, we can take m to be n.
2. Using the induction hypothesis

$$\forall_{w,f,n}(\text{Rev}(\bar{f}n) = w{*}ws \to \exists_m \text{GoodW}(\text{Rev}(\bar{f}m)))$$

with fn, f and $n+1$, we only have to prove $\text{Rev}(\bar{f}(n+1)) = fn{*}ws$, which follows from $\text{Rev}(\bar{f}n) = ws$. □

Note that the reverse direction expresses a form of bar induction. However, for the proof below the present direction suffices.

In the following we want to first highlight the idea behind the constructive proof. This is best done by showing how the steps (1)–(4) in the proof of Nash-Williams given in the introduction are dealt with in the inductive proof.

(1) Prove inductively "BarA [] → BarW[]".

(2) The minimality argument will be replaced by structural induction on words.

(3) Given a sequence $ws = [w_n, \ldots, w_0]$ s.t. $w_i = a_i{*}v_i$, we are interested in all decreasing subsequences $[a_{\kappa_l}, \ldots, a_{\kappa_0}]$ of maximal length and their corresponding sequences $v_{\kappa_l}, \ldots, v_{\kappa_0}, w_{\kappa_0 - 1}, \ldots, w_0$. The sequences $[a_{\kappa_l}, \ldots, a_{\kappa_0}]$ form a forest. In the proof these sequences will be computed by the procedure Forest which takes ws as input and yields a forest labeled by pairs in $A^{**} \times A^*$. In the produced forest the right-hand components of each node form such a descending subsequence $[a_{\kappa_i}, \ldots, a_{\kappa_0}]$ and the corresponding left-hand component consists of the sequence $[v_{\kappa_i}, \ldots, v_{\kappa_0}, w_{\kappa_0 - 1}, \ldots, w_0]$. If we extend the sequence ws to the left by a word $a{*}v$, then in the existing forest either new nodes, possibly at several places, are inserted, or a new singleton tree with root node $\langle v{*}ws, [a]\rangle$ is added. Now the informal idea of the inductive proof is: if in Forest ws new nodes cannot be inserted infinitely often (without ending up with a good left-hand component in a node) and if also new trees cannot be added infinitely often, then ws can not be extended badly infinitely often. Formally, this will be captured by the statement:

$$\forall_{ws}(\text{BarW}(\text{BSeq}(\text{Heads } ws)) \to \text{BarF}(\text{Forest } ws) \to \text{BarW } ws).$$

(4) The first part of item (4) corresponds to GoodWForestToGoodW.

Definition For a finite sequence ws of words let Heads ws denote the finite sequence consisting of the starting letters of the non-empty words. We call a finite sequence ws of words admissible (Adm ws) if each word in ws is non-empty.

Notation We use t for elements in $T(A^{**} \times A^{*})$, i.e., trees labeled by pairs in $A^{**} \times A^{*}$, and ts, ss for elements in $(T(A^{**} \times A^{*}))^{*}$, i.e., forests. The tree with root $\langle ws, as\rangle$ and list of subtrees ts is written $\langle ws, as\rangle ts$. We use the destructors Left and Right for pairs and the destructors Root and Subtrees for trees. For better readability we set:

$$\begin{aligned}
&\text{Newtree}\,\langle ws, as\rangle && := \langle ws, as\rangle[\,],\\
&\text{Roots}\,[t_{n-1}, \dots, t_0] && := [\text{Root}\,t_{n-1}, \dots, \text{Root}\,t_0],\\
&\text{Lefts}\,[\langle vs_{n-1}, as_{n-1}\rangle, \dots, \langle vs_0, as_0\rangle] && := [vs_{n-1}, \dots, vs_0],\\
&\text{Rights}\,[\langle vs_{n-1}, as_{n-1}\rangle, \dots, \langle vs_0, as_0\rangle] && := [as_{n-1}, \dots, as_0].
\end{aligned}$$

Definition Let $ws \in A^{**}$ be a sequence of words. Then Forest $ws \in (T(A^{**} \times A^{*}))^{*}$ is recursively defined by

$$\begin{aligned}
&\text{Forest}\,[\,] := [\,],\\
&\text{Forest}\,[\,]{*}ws := \text{Forest}\,ws,\\
&\text{Forest}\,(a{*}v){*}ws :=\\
&\begin{cases}
\text{InsertF}(\text{Forest}\,ws, v, a) & \text{if } \text{Ge}_{\exists}(a, \text{BSeq}(\text{Heads}\,ws)),\\
\text{Newtree}\,\langle v{*}ws, [a]\rangle{*}(\text{Forest}\,ws) & \text{otherwise}
\end{cases}
\end{aligned}$$

where

$$\text{InsertF}(ts, v, a) := \text{map}\left(\lambda_t \begin{bmatrix} \text{if } \text{Ge}_{\forall}(a, \text{Right}\,(\text{Root}\,t))\\ \quad \text{InsertT}(t, v, a)\\ \quad t \end{bmatrix}\right) ts$$

and

$$\begin{aligned}
&\text{InsertT}(\langle vs, as'\rangle ts, v, a) :=\\
&\begin{cases}
\langle vs, as'\rangle \text{InsertF}(ts, v, a) & \text{if } \text{Ge}_{\exists\forall}(a, \text{Rights}(\text{Roots}\,ts)),\\
\langle vs, as'\rangle(\text{Newtree}\,\langle v{*}vs, a{*}as'\rangle{*}ts) & \text{otherwise.}
\end{cases}
\end{aligned}$$

Example Take as (almost full) relation the natural numbers with $\leq$. For better readability we use underlining rather than parentheses to indicate a list. We will only use one-digit numbers, hence every digit stands for a natural number. Then Forest $[\underline{28}, \underline{421}, \underline{69}, \underline{35}]$ is

$$([\underline{8}, \underline{421}, \underline{69}, \underline{35}], \underline{2}) \qquad \dfrac{([\underline{21}, \underline{5}], \underline{43}) \qquad ([\underline{9}, \underline{5}], \underline{63})}{([\underline{5}], \underline{3})}$$

and Forest $[\underline{52}, \underline{28}, \underline{421}, \underline{69}, \underline{35}]$ is

$$\dfrac{([\underline{2}, \underline{8}, \underline{421}, \underline{69}, \underline{35}], \underline{52})}{([\underline{8}, \underline{421}, \underline{69}, \underline{35}], \underline{2})} \qquad \dfrac{\dfrac{([\underline{2}, \underline{21}, \underline{5}], \underline{543})}{([\underline{21}, \underline{5}], \underline{43})} \qquad ([\underline{9}, \underline{5}], \underline{63})}{([\underline{5}], \underline{3})}$$

If we "project" each node to its right-hand-side we obtain

$$\underline{2} \qquad \dfrac{\underline{43} \qquad \underline{63}}{\underline{3}} \qquad \text{and} \qquad \dfrac{\underline{52}}{\underline{2}} \qquad \dfrac{\dfrac{\underline{543}}{\underline{43}} \qquad \underline{63}}{\underline{3}}$$

The leaves of e.g. the final tree are the maximal decreasing subsequences of the heads [52463] of $[\underline{52}, \underline{28}, \underline{421}, \underline{69}, \underline{35}]$. Recall that the left-hand-side of each leaf consists of the sequence $[v_{\kappa_i}, \dots, v_{\kappa_0}, w_{\kappa_0 - 1}, \dots, w_0]$, and its right-hand-side is the maximal descending subsequence $[a_{\kappa_i}, \dots, a_{\kappa_0}]$ of $[a_n, \dots, a_0]$. In the example the leaf $([\underline{2}, \underline{8}, \underline{421}, \underline{69}, \underline{35},], \underline{52},)$ has exactly this form: $\underline{52}$ is a maximal descending subsequence of $\underline{52463}$, and we have $[\underline{52}, \underline{28}, \underline{421}, \underline{69}, \underline{35}] = [(5 * \underline{2}), (2 * \underline{8}), \underline{421}, \underline{69}, \underline{35}]$.

Definition Let $t \in T(A^{**} \times A^*)$. Then t is a *tree with a good leaf* (GLT t) if there is a leaf with a good left side. We inductively define the predicate $\mathrm{BarF} \subseteq (T(A^{**} \times A^*))^*$ by the rules

$$\dfrac{\mathrm{GLT}(ts)_i}{\mathrm{BarF}\, ts} \qquad \dfrac{\forall_{a,v}(\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts)) \to \mathrm{BarF}(\mathrm{InsertF}(ts, v, a)))}{\mathrm{BarF}\, ts}$$

Lemma (GoodWProjForestToGoodW, BSeqHeadsEqRhtsRootsForest)

(a) $\forall_{ws,i}(i < \mathrm{Lh}\, ws \to \mathrm{GLT}(\mathrm{Forest}\ ws)_i \to \mathrm{GoodW}\ ws)$.
(b) $\forall_{ws}(\mathrm{Adm}\, ws \to \mathrm{BSeq}(\mathrm{Heads}\, ws) = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\ (\mathrm{Forest}\ ws))))$.

Proof Both parts follow from the construction of Forest; the proof of (a) is rather laborious and involves a number of auxiliary notions. However, since we are mainly interested in computational content and this lemma has none, we do not give details. □

Lemma (BarFNil, BarFAppd)

(a) BarF[ ].
(b) $\forall_{t,ts}(\mathrm{BarF}[t] \to \mathrm{BarF}\, ts \to \mathrm{BarF}\, t{*}ts)$.

Proof (a) BarF[ ] follows from the second rule of the definition of BarF, using ex-falso-quodlibet.

(b) This assertion holds since InsertF is defined by a map operation. In more detail, using # for the concatenation of two lists, we prove

$$\forall_{ts}(\mathrm{BarF}\, ts \to \forall_{ss}(\mathrm{BarF}\, ss \to \mathrm{BarF}\, ts\#ss))$$

by induction on BarFts. The base case is straightforward as $\mathrm{GLT}(ts)_i$ implies $\mathrm{GLT}(ts\#ss)_i$. In the step case we have

$$\begin{aligned}\mathrm{ih}_1\colon \forall_{v,a}(&\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts)) \to\\ &\forall_{ss}(\mathrm{BarF}\, ss \to \mathrm{BarF}(\mathrm{InsertF}(ts, v, a)\# ss)))\end{aligned}$$

and need to prove $\forall_{ss}(\mathrm{BarF}\, ss \to \mathrm{BarF}\, ts\# ss)$. Fix $ss \in A^*$ and use induction on $\mathrm{BarF}\, ss$. The base case again is easy since $\mathrm{GLT}(ss)_i$ implies that there is j such that $\mathrm{GLT}(ts\# ss)_j$. In the step case we have

$$\mathrm{ih}_2\colon \forall_{v,a}(\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ss)) \to \mathrm{BarF}\, ts\#\mathrm{InsertF}(ss, v, a))$$

as well as its “strengthening”

$$\mathrm{ih}_{2a}\colon \forall_{v,a}(\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ss)) \to \mathrm{BarF}(\mathrm{InsertF}(ss, v, a))).$$

To show $\mathrm{BarF}\, ts\# ss$, assume v, a with $\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts\# ss))$ and show BarF $(\mathrm{InsertF}(ts\# ss, v, a))$.

Case 1. $\neg\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts))$, i.e., new nodes are only added to ss; ts remains unchanged. Then $\mathrm{BarF}\, ts\# ss'$ follows by ih_2.

Case 2. $\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts)))$. First assume that new nodes are added to both ts and ss, i.e., $\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ss))$. In this case we use with v, a and $\mathrm{InsertF}(ss, v, i)$. We still need to show $\mathrm{BarF}(\mathrm{InsertF}(ss, v, a))$, which holds because of ih_{2a}.

Now assume $\neg\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ss))$, i.e., new nodes are only added to ts. In this case we apply ih_1 with v, a and ss where we use ih_{2a} and the definition of BarF to obtain $\mathrm{BarF}\, ss$. □

The next lemma tells us that a forest consisting of only one tree, in which we continue to insert new nodes by InsertF operations, eventually becomes good.

Lemma (BarFNew) *Assume* BarA [ ]. *Then*

$$\forall_{ws_0}(\mathrm{BarW}\ ws_0 \to \forall_{as_0}\mathrm{BarF}[\mathrm{Newtree}\ \langle ws_0, as_0\rangle]).$$

Proof $\mathrm{Ind}_1(\mathrm{BarW})$. 1.1. GoodW ws_0. Then $\mathrm{GLT}(\mathrm{Newtree}\ \langle ws_0, as_0\rangle)$, i.e., BarF $[\mathrm{Newtree}\ \langle ws_0, as_0\rangle]$. 1.2. Assume

$$\mathrm{ih}_1\colon \forall_{w,as}\mathrm{BarF}[\mathrm{Newtree}\ \langle w{*}ws_0, as\rangle].$$

Let $as_0 \in A$. Instead of proving $\mathrm{BarF}[\mathrm{Newtree}\ \langle ws_0, as_0\rangle]$ we show more generally that this assertion holds for all t with $\mathrm{Root}\, t = \langle ws_0, as_0\rangle$ and (a) $\mathrm{Subtrees}\, t$ in BarF, and (b) $\mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}(\mathrm{Subtrees}\, t)))$ in BarA. We do this by main induction on (b) and side induction on (a), i.e., we prove

$$\begin{aligned}&\forall_{as}(\mathrm{BarA}\, as \to \neg\mathrm{GoodA}\, as \to\\ &\forall_{ts}(\mathrm{BarF}\, ts \to as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\, ts)) \to \mathrm{BarF}[\langle ws_0, as_0\rangle ts])).\end{aligned}$$

$\mathrm{Ind}_2(\mathrm{BarA})$. 2.1. GoodA *as*. Then the conclusion follows immediately by ex-falso-quodlibet with the premise $\neg\mathrm{GoodA}\,as$. 2.2. BarA *as* is obtained by the second rule. We assume *as* and

$$\mathrm{ih}_2 : \forall_{a,ts}(\mathrm{BarF}\,ts \to a{*}as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts)) \to \mathrm{BarF}[\langle ws_0, as_0\rangle ts]),$$

and have to show

$$\forall_{ts}(\mathrm{BarF}\,ts \to as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts)) \to \mathrm{BarF}[\langle ws_0, as_0\rangle ts]).$$

$\mathrm{Ind}_3(\mathrm{BarF})$. 3.1. Fix $(ts)_i$ such that $\mathrm{GLT}(ts)_i$. By the first clause of BarF, for any t such that $\mathrm{Subtrees}\,t = ts$, $\mathrm{GLT}(ts)_i$ implies $\mathrm{BarF}[t]$. 3.2. Fix ts with $as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts))$ and assume the induction hypothesis

$$\begin{aligned}\mathrm{ih}_3 : \forall_{v,a}(&\mathrm{Ge}_\exists(a, \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts))) \to\\ &as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}(\mathrm{InsertF}(ts, v, a)))) \to\\ &\mathrm{BarF}[\langle ws_0, as_0\rangle \mathrm{InsertF}(ts, v, a)])\end{aligned}$$

together with its strengthening

$$\mathrm{ih}_{3a} : \forall_{v,a}(\mathrm{Ge}_\exists(a, \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts))) \to \mathrm{BarF}(\mathrm{InsertF}(ts, v, a))).$$

To show $\mathrm{BarF}[\langle ws_0, as_0\rangle ts]$ we use the second clause, i.e., prove

$$\forall_{v,a}(\mathrm{Ge}_\exists(a, \mathrm{Head}[as_0]) \to \mathrm{BarF}(\mathrm{InsertF}([\langle ws_0, as_0\rangle ts], v, a))).$$

We fix v and a with $\mathrm{Ge}_\exists(a, as_0)$ and prove the statement by case distinction on how a relates to *as*, i.e., whether nodes in the existing subtrees *ts* need to be inserted, or whether a new subtree has to be added.

Case 1. $\mathrm{Ge}_\exists(a, as)$. In this case we have

$$as = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\,ts)) = \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}(\mathrm{InsertF}(ts, v, a))))$$

and by applying ih_3 we obtain $\mathrm{BarF}[\langle ws_0, as_0\rangle \mathrm{InsertF}(ts, v, a)]$.

Case 2. $\neg\mathrm{Ge}_\exists(a, as)$. In this case we need to show

$$\mathrm{BarF}[\langle ws_0, as_0\rangle \mathrm{Newtree}\ \langle w{*}ws_0, a{*}as_0\rangle{*}ts]$$

which can be obtained by applying ih_2 to a and $\mathrm{Newtree}\ \langle w{*}ws_0, a{*}as_0\rangle{*}ts$ provided we can show

$$\mathrm{BarF}(\mathrm{Newtree}\ \langle w{*}ws_0, a{*}as_0\rangle{*}ts).$$

This follows from $\mathrm{BarF}\,ts$ and $\mathrm{BarF}[\mathrm{Newtree}\ \langle w{*}ws_0, a{*}as_0\rangle]$ via BarFAppd. The former holds by ih_{3a}, the latter follows by ih_1.

Now, the proof of the general assertion is completed. Since BarA [ ] by assumption and BarF[ ] by BarFNil, we may in the assertion put $as = [\,]$ and $ts = [\,]$ and end up with BarW $ws \to$ BarF[Newtree $\langle ws_0, as_0 \rangle$]. □

Theorem (Higman) BarA [ ] → BarW[ ].

Proof Assume $\mathrm{BarA}_{[\,]}$. We show more generally

$$\begin{aligned}
&\forall_{as}(\mathrm{BarA}\; as \to \\
&\ \forall_{ts}(\mathrm{BarF} ts \to \\
&\forall_{ws}(\mathrm{Adm}\, ws \to \mathrm{BSeq}(\mathrm{Heads}\, ws) = as \to \mathrm{Forest}\ ws = ts \to \mathrm{BarW}\ ws))).
\end{aligned}$$

$\mathrm{Ind}_1(\mathrm{BarA})$. 1.1. GoodA as. Then, the result follows by ex-falso-quodlibet since for any ws, BSeq(Heads ws) is bad.

1.2. Let $as \in A^*$ and assume

$$\begin{aligned}
\mathrm{ih}_1 : {} & \forall_{a,ts}(\mathrm{BarF} ts \to \\
& \quad \forall_{ws}(\mathrm{Adm}\, ws \to \mathrm{BSeq}(\mathrm{Heads}\, ws) = a{*}as \to \mathrm{Forest}\ ws{=}ts \to \mathrm{BarW}\ ws)).
\end{aligned}$$

$\mathrm{Ind}_2(\mathrm{BarF})$. 2.1. $\mathrm{GLT}\,(ts)_i$. Then, by GoodWProjForestToGoodW, for any ws such that Forest $ws = ts$ we obtain GoodW ws and hence BarW ws. 2.2. Fix ts and assume

$$\begin{aligned}
\mathrm{ih}_2 : {} & \forall_{v,a}(\mathrm{Ge}_{\exists}(a, \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\, ts))) \to \\
& \quad \forall_{ws}(\mathrm{Adm}\, ws \to \mathrm{BSeq}(\mathrm{Heads}\, ws) = as \to \mathrm{Forest}\ ws{=}\mathrm{InsertF}(ts, v, a) \to \\
& \qquad \mathrm{BarW}\ ws))
\end{aligned}$$

as well as the strengthening of the induction hypothesis

$$\mathrm{ih}_{2a} : \forall_{v,a}(\mathrm{Ge}_{\exists}(a, \mathrm{Heads}(\mathrm{Rights}(\mathrm{Roots}\, ts))) \to \mathrm{BarF}(\mathrm{InsertF}(ts, v, a))).$$

Assume that we have ws such that BSeq(Heads ws) $= as$ and Forest $ws = ts$. In order to prove BarW ws, we fix a word w and show BarW $w{*}ws$ by induction on the structure of w:

$\mathrm{Ind}_3(w)$. 3.1. BarW$[\,]{*}ws$ holds since the empty word is embeddable in any word. 3.2. Assume that we have a word of form $a{*}w$. We show BarW$(a{*}w){*}ws$ by case analysis on whether or not $\mathrm{Ge}_{\exists}(a, as)$.

Case 1. $\mathrm{Ge}_{\exists}(a, as)$.

In this case, we have

$$\begin{aligned}
\mathrm{BSeq}(\mathrm{Heads}((a{*}w){*}ws)) &= as, \\
\mathrm{Forest}((a{*}w){*}ws) &= \mathrm{InsertF}(ts, w, a).
\end{aligned}$$

By BSeqHeadsEqRhtsRootsForest and the definition of $\mathrm{Forest}\,(a{*}w){*}ws$, we know that at least one node has been inserted into Forest ws. In this situation, we may apply ih_2 (to $\mathrm{InsertF}\,(ts, w, a)$ and $(a{*}w){*}ws$) and conclude $\mathrm{BarW}\,(a{*}w){*}ws$.

Case 2. $\neg\mathrm{Ge}_{\exists}(a, as)$. Then we have

$$\begin{aligned}\mathrm{BSeq}\,(\mathrm{Heads}((a{*}w){*}ws)) &= a{*}as,\\ \mathrm{Forest}\,(a{*}w){*}ws) &= \mathrm{Newtree}\,\langle w{*}ws, [a]\rangle{*}ts.\end{aligned}$$

By ih_{2a} and ih_3, we have $\mathrm{BarF}ts$ and $\mathrm{BarW}\,w{*}ws$. Hence, by BarFNew applied to $w{*}ws$ and $[a]$, we obtain $\mathrm{BarF}[\mathrm{Newtree}\,\langle w{*}ws, [a]\rangle]$. By BarFAppd we may conclude

$$\mathrm{BarF}[\mathrm{Newtree}\,\langle w{*}ws, [a]\rangle{*}ts].$$

Now we are able to apply ih_1 (to a, $\mathrm{Newtree}\,\langle w{*}ws, [a]\rangle{*}ts$ and $(a{*}w){*}ws$) and end up with $\mathrm{BarW}\,(a{*}w){*}ws$. This completes the proof of the general assertion.

Now, by putting $as = [\,]$, $ts = [\,]$ and $ws = [\,]$ and the fact that $\mathrm{BarF}[\,]$ always holds (by BarFNil) we obtain $\mathrm{BarA}\,[\,] \to \mathrm{BarW}[\,]$. □

Remark In order to make the computational content behind the inductive proof visible, it is essential to use a "positive" formulation of a well-quasiorder, that is, a definition using two rules, as was pointed out, e.g., in [10]. Having a proof of BarW ws implies that the proof yields the information whether BarW ws was obtained by the first rule or by the second. In the first case the result can be read off, in the second we continue with looking at a proof of BarW $w * ws$. If we used a definition consisting of only one rule, i.e., an $\mathrm{acc}_{\preceq}$-notion as in [28], BarW ws would correspond to

$$\neg\exists_i (i < \mathrm{Lh}\, ws \to (ws)_i \text{ embeds into } ws) \to \mathrm{BarW}\, w{*}ws$$

where the test whether or not the premise holds results in a brute-force search; it is not given by the proof itself.

In the next section we discuss a formalization of this proof. For the special case of $\{0,1\}$ there are formalizations in Agda (Fridlender), Minlog (Seisenberger), Isabelle (Berghofer, [3]) and Coq (Berghofer). The formalization of the general case is much more elaborate. Such a formalization has been given in Coq, by Delobel.[1] However, its computational content has not been extracted and investigated. It would suffer from the point made in the previous remark, and its usage of the Set/Prop distinction (see footnote 2) in Coq. Here we want to demonstrate how to get hold of the computational content of a (non-trivial) proof by means of an extracted term, and that this term clearly represents the computationally relevant aspects of the underlying proof.

---

[1] http://coq.inria.fr/V8.2pl1/contribs/HigmanS.html.

3 Formalization

Why should we formalize the rather clear proof given in the previous section? There is of course the obvious reason that we want to be sure that it is correct. However, in addition we might want to get hold of its computational content. We will present this content in the form of an "extracted term", in (an extension T^+ of) Gödel's T. This term can be applied to another term representing an infinite sequence of words, and then evaluated (i.e., normalized). The normal form is a numeral determining a good finite initial segment of the input sequence.

When formalizing we of course need a theory (or formal system) where this is done. Now what features of such a theory are essential for our task? First of all, we have to get clear about (i) what "computational content" is, and (ii) where it arises. We use the Kleene-Kreisel concept of modified realizability for the former. In fact, we will have a formula expressing "the term t is a realizer for the formula A" inside our formal system. For the latter, we take it that computational content *only* arises from inductive predicates; prime examples are the Bar predicates introduced in the previous section. But then a particular aspect becomes prominent: we need "non computational" (n.c.) universal quantification [1] written $\forall^{\mathrm{nc}}$ to correctly express the type of a computational problem of the form

$$\forall^{\mathrm{nc}}_{as}(\mathrm{BarA}\ as \to A).$$

Its intended computational content is a function f mapping a witness that as is in BarA into a realizer of A. It is important that f does *not* get as as an argument.[2]

On the more technical side, we use TCF [26], a form of HA^{ω} extended by inductively defined predicates and n.c. logical connectives. TCF has the (Scott-Ershov) partial continuous functions as its intended model.

It is also mandatory to use a proof assistant to help with the task of formalization. We use Minlog[3] [2], which is designed to support these features.

Space does not permit to present the full formalization[4] of the constructive proof above of Higman's Lemma. We restrict ourselves to comment on some essential aspects.

Most important are of course the basic definitions of the data structures (free algebras) and predicates involved. Their formal definitions are very close to the informal ones above and do not need to be spelled out. However, already at this level computational content crops up: an inductive predicate may or may not have computational content. Examples for the former are the Bar predicates, and for the latter the GoodA predicate. It is convenient to define the GoodA predicate inductively,

---

[2] A similar phenomenon is addressed in Coq [5] by the so-called Set/Prop distinction. However, enriching the logic by n.c. universal quantification (and similarly n.c. implication) seems to be more flexible.

[3] See http://www.minlog-system.de.

[4] See http://www.git/minlog/examples/bar/higman.scm.

but—since it is decidable—we can also view it as a (primitive) recursive boolean valued function.

The first point in the proof above where we have to be careful with n.c. quantification is the inductive definition of BarA, with clauses

$$\begin{aligned}&\texttt{InitBarA}\colon \forall^{\mathrm{nc}}_{\preceq,as}(\mathrm{GoodA}_{\preceq}as \to \mathrm{BarA}_{\preceq}as),\\ &\texttt{GenBarA}\colon \quad \forall^{\mathrm{nc}}_{\preceq,as}(\forall_a \mathrm{BarA}_{\preceq}a * as \to \mathrm{BarA}_{\preceq}as).\end{aligned}$$

The (free) algebra of witnesses for this inductive predicate is called `treeA`. In the clause `GenBarA` the generation tree of $\mathrm{BarA}_{\preceq}as$ should have infinitely many predecessors indexed by a, hence we need $\forall_a$. However, the outside quantifier is $\forall^{\mathrm{nc}}_{\preceq,as}$, since we do not want to let the argument as be involved in the computational content of $\mathrm{BarA}_{\preceq}as$. Hence `treeA` has constructors

$$\begin{aligned}&\texttt{CInitBarA}\colon \texttt{treeA},\\ &\texttt{CGenBarA}\colon \quad (\texttt{nat} \Rightarrow \texttt{treeA}) \Rightarrow \texttt{treeA}.\end{aligned}$$

A similar (but slightly more involved) comment applies to the inductive definition of BarF. For readability we omit the dependency on $\preceq$ here. The clauses are

$$\texttt{InitBarF}\colon \forall^{\mathrm{nc}}_{ts,i}(i < \mathrm{Lh}\, ts \to \mathrm{GLT}\ (ts)_i \to \mathrm{BarF} ts),$$

and `GenBarF`:

$$\begin{aligned}\forall^{\mathrm{nc}}_{ts}(\forall_{tas,a,v}(&tas = \mathrm{ProjF}\, ts \to \mathrm{Ge}_{\exists\forall}(a, \mathrm{Roots}\, tas) \to\\ &\mathrm{BarF}(\mathrm{InsertF}(ts, v, a)) \to\\ &\mathrm{BarF} ts).\end{aligned}$$

We need the concept of the "A-projection" of a tree t, where each rhs of a label in t is projected out. Here only the A-projection of *ts* (but *not ts*) is used computationally. More precisely, the predecessors of BarF*ts* are all $\mathrm{InsertF}(ts, v, a)$ for v, a with $\mathrm{Ge}_{\exists\forall}(a, \mathrm{Rights}(\mathrm{Roots}\, ts))$. To decide the latter, we need (computationally) Rights(Roots *ts*), i.e., the A-projection of *ts*.

The (free) algebra of witnesses for the inductive predicate BarF is called `treeF`; its constructors are

$$\begin{aligned}&\texttt{CInitBarF}\colon \texttt{treeF},\\ &\texttt{CGenBarF}\colon \quad (\texttt{list lntree nat} \Rightarrow \texttt{nat} \Rightarrow \texttt{list nat} \Rightarrow \texttt{treeF}) \Rightarrow \texttt{treeF}.\end{aligned}$$

4 Extraction

We now spell out the computational content of some of the proofs above. Each time it comes in the form of a term (in T^+) machine extracted from a formalization of the respective proof.

When reading the extracted terms please note that lambda abstraction is displayed via square brackets; so `[n]n+m` means $\lambda_n n + m$. Our notation $\langle ws, as\rangle ts$ for the tree with root $\langle ws, as\rangle$ and list of subtrees ts is displayed as `(ws pair as)%ts`. Also types are implicit in variable names; for example, `n`, `a` both range over natural numbers. One can also use the display string for a type as a variable name of this type; for example, `treeW` is a name for a variable of type `treeW`.

4.1 *BarWToGoodInit*

We use typed variable names

```
f:    nat=>list nat
gw:   nat=>list nat
hwfa: list nat=>(nat=>list nat)=>nat=>nat
```

The term extracted from the proof of the proposition BarWToGoodInit is

```
[treeW]
 Rec treeW=>(nat=>list nat)=>nat=>nat)treeW([f,a]a)
 ([gw,hwfa,f,a]hwfa(f a)f(Succ a))
```

It takes some effort to understand such an extracted term. The recursion operator on `treeW` with value type `alpha` has type

```
treeW=>alpha=>((list nat=>treeW)=>(list nat=>alpha)=>alpha)
     =>alpha
```

Let `Leaf: treeW` and `Branch: (list nat=>treeW)=>treeW` be the constructors of `treeW`. Then $\Phi :=$ `(Rec treeW=>alpha)` is given by the recursion equations

$$\begin{aligned} \Phi(\texttt{Leaf}) &:= G, \\ \Phi(\texttt{Branch}(g)) &:= H(g, \lambda_v \Phi(g(v))). \end{aligned}$$

Here the value type `alpha` is `(nat=>list nat)=>nat=>nat`, and

$$\begin{aligned} G &:= \lambda_{f,a} a, \\ H(\texttt{gw}, \texttt{hwfa}) &:= \lambda_{f,a} \texttt{hwfa}(f(a), f, a+1). \end{aligned}$$

4.2 BarFNil, BarFAppd

For BarFNil we have the simple extracted term

```
CGenBarF([tas,a,v]CInitBarF)
```

For BarFAppd we use the variable names

```
g:    list lntree nat=>nat=>list nat=>treeF
htat: list lntree nat=>nat=>list nat=>treeF=>nat=>treeF
hat:  list lntree nat=>nat=>list nat=>nat=>treeF
```

Then the extracted term is

```
[wqo,treeF]
(Rec treeF=>treeF=>nat=>treeF)treeF([treeF0,a]CInitBarF)
([g,htat,treeF0]
 (Rec treeF=>nat=>treeF)treeF0([a]CInitBarF)
 ([g0,hat,a]
   CGenBarF
   ([tas,a0,v]
     [if (LargerARExAll wqo a0 Roots((Lh tas--a)init tas))
       (htat((Lh tas--a)init tas)a0 v
       [if (LargerARExAll wqo a0 Roots((Lh tas--a)rest tas))
         (g0((Lh tas--a)rest tas)a0 v)
         (CGenBarF g0)]
       a)
       (hat((Lh tas--a)rest tas)a0 v a)]))))
```

The recursion operator on `treeF` with value type `alpha` has type

```
treeF=>alpha=>
     ((list lntree nat=>nat=>list nat=>treeF)=>
      (list lntree nat=>nat=>list nat=>alpha)=>alpha)=>alpha
```

`LargerARExAl wqo a ws` means $\exists_{i<|ws|}\forall_{j<|(ws)_i|} a \succeq (ws)_{i,j}$. `treeF` has constructors `CInitBarF: treeF` and `CGenBarF: (list nat=>treeF)=>` texttttreeF. Then $\Phi :=$ `(Rec treeF=>alpha)` is given by the recursion equations

$$\begin{aligned}\Phi(\texttt{CInitBarF}) &:= G,\\ \Phi(\texttt{CGenBarF}(g)) &:= H(g, \lambda_v \Phi(g(v))).\end{aligned}$$

The value type of the first recursion is `treeF=>nat=>treeF`, and

$$\begin{aligned} G &:= \lambda_{\mathtt{treeF},a}\mathtt{CInitBarF}, \\ H(g,\mathtt{htat}) &:= \lambda_{\mathtt{treeF0}}K(g,\mathtt{htat},\mathtt{treeF0}) \end{aligned}$$

with $K(g,\mathtt{htat},\mathtt{treeF0})$ given by

```
(Rec treeF=>nat=>treeF)treeF0([a]CInitBarF)
([g0,hat,a]
  CGenBarF
  ([tas,a0,v]
    [if (LargerARExAll wqo a0 Roots((Lh tas--a)init tas))
      (htat((Lh tas--a)init tas)a0 v
      [if (LargerARExAll wqo a0 Roots((Lh tas--a)rest tas))
        (g0((Lh tas--a)rest tas)a0 v)
        (CGenBarF g0)]
      a)
      (hat((Lh tas--a)rest tas)a0 v a)]))
```

The inner recursion is on `treeF` again, with value type `nat=>treeF`, and

$$\begin{aligned} G_1 &:= \lambda_a\mathtt{CInitBarF}, \\ H_1(\mathtt{g0},\mathtt{hat}) &:= \lambda_a\mathtt{CGenBarF}\dots \end{aligned}$$

4.3 BarFNew

With the variable names

```
gw: list nat=>treeW  hw:   list nat=>list nat=>treeF
ga: nat=>treeA       hatt: nat=>treeF=>treeF
```

we extract

```
[wqo,treeA,treeW]
 (Rec treeW=>list nat=>treeF)treeW([v]CInitBarF)
 ([gw,hw,v]
   (Rec treeA=>treeF=>treeF)treeA([treeF]CInitBarF)
   ([ga,hatt,treeF]
     (Rec treeF=>treeF)treeF CInitBarF
     ([g,g0]
       CGenBarF
       ([tas,a,v0]
         [if (LargerARExAll wqo a Roots Subtrees Head tas)
           (g0 Subtrees Head tas a v0)
           (hatt a
```

```
             (cBarFAppd wqo(hw v0(a::v))
                        (CGenBarF g)Lh Subtrees Head tas))]])))
     (CGenBarF([tas,a,v0]CInitBarF)))
```

This time we have three nested recursions: an outer one on `treeW` with value type `list nat=>treeF`, then on `treeA` with value type `treeF=>treeF`, and innermost on `treeF` with value type `treeF`. This corresponds to the three elimination axioms used in the proof. Notice that the computational content `cBarFAppd` of the theorem `BarFAppd` appears as a constant inside the term.

4.4 Higman

```
[wqo,treeA]
 (Rec treeA=>treeF=>list list nat=>list lntree list nat=>treeW)
 treeA
 ([treeF,ws,tas]CInitBarW)
 ([ga,ha,treeF]
   (Rec treeF=>list list nat=>list lntree list nat=>treeW)treeF
   ([ws,tas]CInitBarW)
   ([g,h,ws,tas]
    CGenBarW
    ([v](Rec list nat=>treeW)v(CGenBarW([v0]CInitBarW))
     ([a,v0,treeW]
      [if (LargerAR wqo a(BSeq wqo Heads ws))
  (h tas a v0((a::v0)::ws)(InsertAF wqo tas a))
  (ha a (cBarFAppd wqo(cBarFNew wqo treeA treeW a:)
   (CGenBarF g)Lh tas)
      ((a::v0)::ws)
      ((a: %(Nil lntree list nat))::tas))]]))))
 (CGenBarF([tas,a,v]CInitBarF))
 (Nil list nat)
 (Nil lntree list nat)
```

4.5 Experiments

To run the extracted terms we need to "animate" the theorems involved. This means that the constant denoting their computational content (e.g., `cBarFAppd` for the theorem `BarFAppd`) unfolds into the term extracted from the proof of the theorem. Then for an arbitrary infinite sequence extending e.g. the example in Sect. 2 we obtain the expected good initial segment.

In more detail, we first have to animate the computationally relevant propositions in the proof given above of Higman's Lemma. Then we need to prove and animate lemmas relating to the particular relation `NatLe`:

`BarNatLeAppdOne`: $\forall_{i,m,as}(i + \mathrm{Lh}(as) = m + 1 \to \mathrm{BarA}_{\leq}(as\#[m]))$,
`BarANilNatLe`: $\mathrm{BarA}_{\leq}[]$,
`HigmanNatLe`: $\mathrm{BarW}_{\leq}[]$.

Using these we can prove the final proposition

$$\texttt{GoodWInitNatLe:}\ \forall_f \exists_n \mathrm{GoodW}_{\leq}(\mathrm{Rev}(\bar{f}n)).$$

Let `neterm` be the result of normalizing the term extracted from this proof. Next we provide an infinite sequence (extending the example in Sect. 2), e.g. in the form of a program constant:

```
(add-program-constant "Seq" (py "nat=>list nat"))
(add-computation-rules
"Seq 0" "5::2:"
"Seq 1" "2::8:"
"Seq 2" "4::2::1:"
"Seq 3" "6::9:"
"Seq 4" "3::5:"
"Seq(Succ(Succ(Succ(Succ(Succ n)))))" "0:")
```

Finally we run the our normalized extracted term by evaluating

```
(pp (nt (mk-term-in-app-form neterm (pt "Seq"))))
```

(Here `nt` means "normalize term" and `pt` means "parse term"). The result is 4, the length of a good initial segment of our infinite sequence.

5 Related Work: Other Proofs of Higman's Lemma

As mentioned at the beginning of Sect. 2, our constructive proof of Higman's Lemma does not need transitivity; it works for arbitrary almost full relations. However, in the following discussion we disregard this fine point and assume that the underlying relation is a well-quasiorder. This will make it easier to compare different proofs in the literature.

There are quite a number of constructive proofs of Higman's Lemma, thus the natural question arises: are they all different? The number of proofs is due to the fact that researchers from different areas, algebra, proof theory, constructive mathematics, term rewriting, to name a few, became interested in Higman's Lemma. In addition, there are various formulations of a well-quasiorder which include different proof principles. These are for instance proofs using ordinal notation systems and transfinite induction as used in [24, 25] or inductively defined predicates and structural induction as used in [9, 18, 23]. Below we argue that these proofs are the same from a computational point of view.

The proof theoretic strength of Higman's Lemma is that of Peano Arithmetic, i.e. ϵ_0, as was shown in [12] using the constructive proof of [25]. Speaking in terms of Reverse Mathematics, Higman's Lemma can be proven in the theory ACA_0. In term rewriting theory, Higman's Lemma and its generalization to trees, Kruskal's Theorem, are used to prove termination of string rewriting systems and term rewriting systems respectively. The orders whose termination is covered by these two theorems are called simplification orders. They form an important class since the criterion of being a simplification order can be checked syntactically. A constructive proof, e.g., as given in [4], moreover yields a bound for the longest possible bad sequence. In the case of Higman's Lemma the reduction length, expressed in terms of the Hardy hierarchy, H_α, assuming a finite alphabet A, is as follows. If we have a bad sequence $(t_i)_{i<n}$, fulfilling the condition $|t_i| \leq |t_0| + k \times i$, where k is a constant and $|t|$ denotes the size of t, then the length n of the sequence is bound by $\Phi(|t_0|)$ where Φ is an elementary function in $H_{\omega^{\omega^{|A|}}}$ [4, 31]. This bound is optimal since there are term rewriting systems which "reach" these bounds [32].

5.1 Equivalent Formulations of a Well-Quasiorder

We define the maximal ordertype of a well-quasiorder $(A, \preceq)$ as the supremum of the ordertypes of all extensions of $(A, \preceq)$ to a linear order. Equivalently, in a more constructive manner, the maximal ordertype can be defined by the height of the tree of all bad sequences ($\mathrm{Bad}_\preceq$) with elements in A. A reification of a quasi order $(A, \preceq)$ into a wellordering $(\sigma, <)$ is a map

$$r : \mathrm{Bad}_\preceq \to \sigma,$$

such that for all $a{*}as \in \mathrm{Bad}_\preceq$ we have $r(a{*}as) < r(as)$. On the set $\mathrm{Bad}_\preceq$ of bad sequences in A we define a relation $\ll_A$ by $as' \ll_A as$ iff $as' = a{*}as$ for some $a \in A$. The accessible part of the relation $\ll_A \subseteq \mathrm{Bad}_\preceq \times \mathrm{Bad}_\preceq$ is inductively given by the rule

$$\frac{\forall_{as'}(as' \ll_A as \to \mathrm{acc}_{\ll_A} as')}{\mathrm{acc}_{\ll_A} as}$$

It is obvious that the following are equivalent for a quasiorder $(A, \preceq)$:

(i) $(A, \preceq)$ is a well-quasiorder (i.e., $\mathrm{Wqo}(A, \preceq)$).
(ii) $(A, \preceq)$ has a maximal ordertype.
(iii) There is a reification of $(A, \preceq)$ into a wellorder.
(iv) $(\mathrm{Bad}_{\preceq}, \ll_A)$ is wellfounded, i.e., $\mathrm{acc}_{\ll_A}[\,]$.
(v) $\mathrm{BarA}_{\preceq}[]$.

5.2 A Generic Proof of Higman's Lemma

In the following we sketch a generic proof of

$$\mathrm{Wqo}(A, \preceq) \to \mathrm{Wqo}(A^*, \preceq^*).$$

which differs from the proof presented in the earlier sections. We start by choosing a characterization of a well-quasiorder, either using ordinal notations ((ii) or (iii)) or inductive definitions ((iv) or (v)). (Note that in the latter case we need to generalize the statement; for instance, in (iv) we prove more generally $\mathrm{acc}_{\ll_A} as \to \mathrm{acc}_{\ll_{(A_{as})^*}} []$ and use this proof with $as = []$, and in the proof below instead of $A_{[a]}$ we use everywhere A_{a*as}, etc.). Here we define A_{as} as the set of all elements that extend as badly, i.e. $\forall_i as_i \not\preceq a$. Similarly, we define $A_{[a]}$ to be the set of all elements b such that $a \not\preceq_A b$. Assume that for our choice of characterization we are able to prove (with the obivous extension of $\preceq$ to $\cup$ and $\times$):

$$\begin{aligned}
(a)&\quad \forall a\, \mathrm{Wqo}(A_{[a]}, \preceq) \to \mathrm{Wqo}(A, \preceq),\\
(b)&\quad A \subseteq B \to \mathrm{Wqo}(A, \preceq) \to \mathrm{Wqo}(B, \preceq),\\
(c)&\quad \mathrm{Wqo}(A) \wedge \mathrm{Wqo}(B) \to \mathrm{Wqo}(A \cup B),\\
(d)&\quad \mathrm{Wqo}(A) \wedge \mathrm{Wqo}(B) \to \mathrm{Wqo}(A \times B).
\end{aligned}$$

Assume $\mathrm{Wqo}(\preceq)$. We proceed to prove Higman's Lemma by using either structural induction or transfinite induction, depending on our choice. From the induction hypothesis we get

$$\mathrm{Wqo}(A_{[a]}) \to \mathrm{Wqo}({A_{[a]}}^*). \tag{1}$$

By (a) it suffices to prove $\forall_v \mathrm{Wqo}(A^*_{[v]})$. Let $v = [a_1, \dots, a_n]$. The main combinatorial idea is now contained in the following statement

$$(A)^*_{[[a_1,\dots,a_n]]} \subseteq \bigcup\{(A_{[a_1]})^* \times A \times (A_{[a_2]})^* \times \cdots \times A \times (A_{[a_l]})^* \mid l < n\} \tag{2}$$

which holds by a simple combinatorial argument. Using (b) we are done once we have shown

$$\mathrm{Wqo}(\bigcup\{(A_{[a_1]})^* \times A \times (A_{[a_2]})^* \times \cdots \times A \times (A_{[a_l]})^* \mid l < n\}).$$

But this follows immediately from (c), (d) and (1).

Remark Instantiated versions of this proof, using characterizations (ii), (iii), (iv) or (v) of a well-quasiorder, can be found in the following articles: (ii) is used by de Jongh and Parikh [8] and Schmidt [24]. (iii) is used in the proof by Schütte and Simpson [25] (and Hasegawa [13]) (and is the characterization which is most promising in terms of generalizations beyond Kruskal's Theorem). (iv) has been used by Fridlender [9], using an acc notation. His proof is a reformulation of the proof by Richman and Stolzenberg [23]. To a less formal extent this characterization is also used in [18], where also structural induction and a similar construction describing the space to which a sequence can be extended badly are used. Characterization (v): the proof in [9] can be easily reformulated using (v). Fridlender [10] gives a variant where he does not need the decidability of $\preceq_A$. His proof is a type theoretic version of an intuitionistic proof by Veldman, later published in [34].

Finally, the proof of [18] forms the basis of the formalization and proof of Higman's Lemma in [16], in ACL2. Their work however starts with a program solving the problem, and then proving its properties rather than extracting the program from the proof.

Remark Higman's Lemma extends naturally to Kruskal's Theorem, the corresponding statement for trees. Constructive proofs of Kruskal's Theorem have been given by Schmidt [24] using characterization (ii), by Rathjen and Weiermann [22] and Hasegawa [13] using (iii), and in [27] using (iv). Finally, also Goubault-Larrecq's proof [11] which generalizes the proof in [18] falls under this category.

It remains to compare how the computational content behind this generic proof of Higman Lemma is related to the constructive proof given in this paper. Although we have not yet formalized the proof above, it is quite obvious that the construction, in particular Eq. (2) differs from the construction in our proof, and therefore would result in a different algorithm.

6 Conclusion and Further Work

We presented and formalized a constructive proof of Higman's Lemma that contains the same combinatorial idea as Nash-Williams' indirect proof, and extracted and discussed its inherent program in detail. We also argued that a number of constructive proofs of Higman's Lemma are based on a combinatorial idea different from ours. It is still open to make that claim formal, i.e. to formalize the proof presented in the previous section, and compare the resulting program with our extracted program.

Similarly, there are a number of formalizations of Nash-Williams' classical proof as mentioned in the introduction. It would be worthwhile to confirm that they, in principle, lead to the same algorithm, which also corresponds to the algorithm in our extracted program.

Equally interesting is the question which of the discussed proofs are most suitable for applications such as termination of string- and term rewriting systems, see e.g. [11, 30, 35] for recent discussions on applications to termination proofs. A particularly promising application has been given in [20]. It will be worth checking how our alternative proof of Higman's Lemma and its extracted program can be utilized with regard to these applications or further generalizations.

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How to Reason Coinductively Informally

Anton Setzer

Dedicated to Gerhard Jäger on occasion of his 60th Birthday

Abstract We start by giving an overview of the theory of indexed inductively and coinductively defined sets. We consider the theory of strictly positive indexed inductive definitions in a set theoretic setting. We show the equivalence between the definition as an indexed initial algebra, the definition via an induction principle, and the set theoretic definition of indexed inductive definitions. We review as well the equivalence of unique iteration, unique primitive recursion, and induction. Then we review the theory of indexed coinductively defined sets or final coalgebras. We construct indexed coinductively defined sets set theoretically, and show the equivalence between the category theoretic definition, the principle of unique coiteration, of unique corecursion, and of iteration together with bisimulation as equality. Bisimulation will be defined as an indexed coinductively defined set. Therefore proofs of bisimulation can be carried out corecursively. This fact can be considered together with bisimulation implying equality as the coinduction principle for the underlying coinductively defined set. Finally we introduce various schemata for reasoning about coinductively defined sets in an informal way: the schemata of corecursion, of indexed corecursion, of coinduction, and of corecursion for coinductively defined relations. This allows to reason about coinductively defined sets similarly as one does when reasoning about inductively defined sets using schemata of induction. We obtain the notion of a coinduction hypothesis, which is the dual of an induction hypothesis.

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1 Introduction

When reasoning about inductive defined sets such as the natural numbers, we are used to argue informally while referring to the induction hypothesis. When for instance showing $\forall x, y, z \in \mathbb{N}.(x + y) + z = x + (y + z)$, we do not define first a relation $R(z) \Leftrightarrow \forall x, y.(x + y) + z = x + (y + z)$ and then argue that R is closed under 0 and successor S. Instead one proves $(x + y) + 0 = x + (y + 0)$ and proves $(x + y) + \mathrm{S}(z) = x + (y + \mathrm{S}(z))$ by using the induction hypothesis $(x + y) + z = x + (y + z)$.

Although these two versions are obviously equivalent, the version using the induction hypothesis is much more lightweight, and easier to teach to students.

When referring to coinductively defined sets, i.e. final coalgebras, we are currently usually following principles which are similar to referring to the closure of the relation R under 0, S in inductive definitions. For instance when showing that two elements of a labelled transition system are bisimilar, one defines a relation on pairs of states of the transition system and shows that it is a bisimulation relation.

In this article we will discuss how to argue about coinductively defined sets in a similar way as we argue about inductive sets. This is made easier by following the approach in [1, 2, 27, 28] of introducing final coalgebras by their elimination rules rather than their introduction rules. For instance, instead of defining the set of streams of natural numbers as a set closed under $\mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream}$ (and allowing infinite sequences of cons applications), we define Stream as a set such that we have $\mathrm{head} : \mathrm{Stream} \to \mathbb{N}$, and $\mathrm{tail} : \mathrm{Stream} \to \mathrm{Stream}$. This makes it easier to describe what the correct use of the corecursion hypothesis is: we can define $s : A \to \mathrm{Stream}$ by defining $\mathrm{head}(s(a)) \in \mathbb{N}$ and $\mathrm{tail}(s(a)) \in \mathrm{Stream}$. For defining tail we can use the corecursion hypothesis, i.e. define $\mathrm{tail}(s(a)) = s(a')$ for some a' (depending on a).

Coinduction is the dual of induction. In Sect. 3 we will review the well-known fact that the principle of induction is equivalent to the fact that there is only one solution for the equations defining a function by the principle of iteration. Therefore the principle of induction is just one way of expressing the fact that the principle of iteration has a unique solution. Dually, coinduction is a principle expressing that the principle of coiteration has a unique solution. In Sect. 8, Theorem 8.7 we will show that this principle is equivalent to the fact that bisimulation on coalgebras implies equality. Bisimulation can be defined coinductively. Therefore we can give proofs of bisimilarity by corecursion. Therefore coinduction can be considered as the principle that we can give proofs of equality by corecursion over the coinductive definition of bisimulation. The coinduction hypothesis is essentially the corecursion hypothesis in defining elements of the bisimilarity corecursively.

We hope that such schemata will make arguing about coinductively defined sets easier and less technical than it is at the moment.

We will in this article often use “*coinductively defined set*” for final coalgebra. The reason is that we want to use a terminology which suggests the use of corecursion and coinduction principles like those developed in this article, and which makes it clear that coinductively defined sets are the dual of inductively defined sets.

Content of this article We will start by introducing some notations in Sect. 2, where we will transfer notations from dependent type theory into set theory. Then we review in Sect. 3 the theory of indexed inductive definitions, and prove the equivalence between the category theoretic definition and the definition by induction. We use here restricted indexed inductive definitions as introduced in Peter Dybjer's and the author's articles [11, 13]. In Sect. 4 the notions of iteration and primitive recursion and their equivalence, if uniqueness is added, are reviewed. The main purpose of Sects. 3 and 4 is to motivate analogous definitions for coinductively defined sets, and make clear how they are obtained by dualising the concepts related to inductively defined sets. Our set theoretic definition of inductive definitions is based on defining its elements as terms, which are well-founded (in most standard examples therefore finitary) objects, and which can be represented in set theory in a straightforward way. Defining the elements of coinductively defined sets is more complicated, since the naïve interpretation using constructors would result in non-well-founded sets [3], whereas in ZF set theory all sets are well-founded. In Sect. 5 we give one way of introducing elements of coinductive sets set theoretically. Our construction is defined in such a way that it reflects the fact that coinductively defined sets are formed by giving their elimination rules or observations. In Sect. 6 we introduce the notions of coiteration, corecursion, and show the equivalence of those principles. In Sect. 7 we discuss a more convenient way of introducing elements of coinductively defined sets corecursively without having for each index to define a function. In Sect. 8 we introduce bisimulation, a principle of coinduction, and show that this principle of coinduction is equivalent to unique coiteration/corecursion. Finally in Sect. 9 we introduce various schemata for reasoning about coinductively defined sets informally. The schemata we introduce are corecursion, indexed corecursion, coinduction, and coinduction for bisimulation relations. We finish with a conclusion in Sect. 10.

We want to note that most of the material in Sects. 3 and 4 is well known in the theory of initial algebras and inductively defined sets. The purpose of those sections is to give an overview over the theory of indexed inductive definitions, so that it is easier to see in later sections how coinductively defined sets are the dual of inductively defined sets. Sect. 5 is the adaption of a well known categorical construction to the indexed case. We hope the fact that it is rather concrete and reflects the fact that coinductively defined sets are formed by their elimination rules or observations helps to get a better understanding of coinductively defined sets. The main contribution of this article are in Sects. 6–9, where the last section demonstrates, how to reason informally about coinductively defined sets.

We will work in this article set theoretically. The main reason for this is that the goal of this article is that ordinary mathematicians, who not necessarily work in type theory, should be able to use the schemata introduced in this article for reasoning about coalgebras. We believe that the reasoning principles can be transferred to extensional type theory, although further work is needed in order to make sure that all principles type-check. A transfer to intensional type theory, and therefore proof assistants such as Agda, would require further modifications. The main problem is that in order to obtain decidable type checking in intensional type theory one needs to replace final coalgebras by weakly final coalgebras. So coinduction can only be used to prove that elements are bisimilar rather than equal.

Related Work The equivalence between induction principles and category theoretic definition of initial algebras is well known, in case of inductive-recursive definitions it has for instance been worked out in [12], although the equivalence of inductive definitions has been known much longer. The reduction of indexed inductive-definitions to Petersson-Synek Trees has been developed in container theory, see esp. [14, 19] but as well [6, 21]. There are various set theoretic models of final coalgebras, examples are de Bruin [8], Barr [7] or Aczel [4]. The equivalence between final coalgebras and bisimulation as equality and iteration is well known in the theory of coalgebras, see for instance the articles and textbooks by Rutten and Sangiori [24–26] (the theory is much older). The notion of bisimulation of processes was initially defined by Park [22] and Milner [20] as a greatest fixed point, and therefore as a coinductively defined relation. Dybjer has defined a set theoretic interpretation of type theory in [9] and with the author in [10].

In our previous article [27] we introduced coalgebras into type theory by giving formation-, elimination-, introduction- and equality-rules. There we argued, that coalgebras are formed by giving their elimination rules, and that the introduction rules and equality rules are derived. We didn't explore the principle of coinduction in that article. The current article elaborates on this, however not in the context of type theory but in a general set theoretic setting. The difficulty is that in intensional type theory we obtain only weakly final coalgebras.

2 Notations

In the following, we will work mainly set theoretically, using for simplicity the theory of Zermelo Fraenkel set theory with the axiom of choice. Since our inspiration comes from Martin-Löf type theory, we will simulate basic constructions in type theory in set theory.

We will work in this article in the set theoretical model of type theory, as introduced for instance in Sect. 6 of [10]. In this model inductively defined sets are modelled as sets of terms, introduced by constructors, and function types are modelled as set theoretic functions. Since the idea of this article is to work directly in set theory, we will identify inductively defined sets with the least set introduced by constructors, and function types with the set theoretic function set.

Assumption 2.1 (a) We assume a finite set of constructor symbols $\mathrm{C}_1, \ldots, \mathrm{C}_n$ together with an arity $\mathrm{arity}(\mathrm{C}_i) \in \mathbb{N}$ associated with each of them.

(b) We assume a Gödel number $\ulcorner \mathrm{C}_i \urcorner \in \mathbb{N}$ associated with each C_i such that $\ulcorner \mathrm{C}_i \urcorner \neq \ulcorner \mathrm{C}_j \urcorner$ for $i \neq j$.

(c) We assume some standard encoding of sequences of sets $a_1, \ldots, a_n$ as a set $\langle a_1, \ldots, a_n \rangle$, including the case $n = 0$. We assume this is done in such a way that there are functions which obtain from a code $\langle a_1, \ldots, a_n \rangle$ its length n and the ith element a_i.

Definition 2.2 (a) Let Set be the collection of sets.
(b) We will in the following use set theoretic notation for function application, i.e. we will write $f(a)$ for the application of f to a.
(c) If C is an n-ary constructor we define

$$\begin{aligned} &\mathrm{C} : \mathrm{Set}^n \to \mathrm{Set}, \\ &\mathrm{C}(t_1, \ldots, t_n) := \langle \ulcorner \mathrm{C} \urcorner, t_1, \ldots, t_n \rangle. \end{aligned}$$

Definition 2.3 (a) Let $A \in \mathrm{Set}$ and $B(x) \in \mathrm{Set}$ depending on $x \in A$. We define the dependent function set as

$$(a \in A) \to B(a) := \{f \in A \to \bigcup_{a \in A} B(a) \mid \forall a \in A. f(a) \in B(a)\}$$

and the dependent product as

$$(a \in A) \times B(a) := \{\langle a, b \rangle \mid a \in A, b \in B(a)\}.$$

Let π_0 and π_1 be the first and second projections, i.e. $\pi_0(\langle a, b \rangle) = a$, $\pi_1(\langle a, b \rangle) = b$.

(b) $$\begin{aligned} &(x_1 \in A_1) \to (x_2 \in A_2) \to \cdots \to (x_n \in A_n) \to B \\ &\quad := (x_1 \in A_1) \to ((x_2 \in A_2) \to (\cdots \to ((x_n \in A_n) \to B) \cdots)). \end{aligned}$$

(c) $$\begin{aligned} &(x_1 \in A_1) \times (x_2 \in A_2) \times \cdots \times (x_n \in A_n) \\ &\quad := (x_1 \in A_1) \times ((x_2 \in A_2) \times (\cdots \times (x_n \in A_n) \cdots)). \end{aligned}$$

(d) For $A, B \in \mathrm{Set}$ let $A + B := \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$, where inl, inr are unary constructors.
(e) $\times$ binds stronger than $+$ and $+$ binds stronger than $\to$.
(f) Let $\mathbf{1} := \{*\}$ where $*$ is a 0-ary constructor.
(g) Let for a relation $R(x_1, \ldots, x_n)$

$$\widehat{R}(x_1, \ldots, x_n) := \begin{cases} \mathbf{1} & \text{if } R(x_1, \ldots, x_n), \\ \emptyset & \text{Otherwise.} \end{cases}$$

When writing an argument of a function as being an element of a relation, we write $R(x_1, \ldots, x_n)$ instead of $\widehat{R}(x_1, \ldots, x_n)$. For instance
$(n \in \mathbb{N}) \to (n > 0) \to \cdots$ means more precisely $(n \in \mathbb{N}) \to (n \widehat{>} 0) \to \cdots$.

(h) When having functions $f : (x \in A) \to (y \in B(x)) \to C(x, y)$ we write $f(x, y)$ for $f(x)(y)$, similarly for functions with more arguments.

(i) When referring to a function $f : (x \in A) \to (y \in B(x)) \to C(x, y)$ in a diagram we sometimes need its uncurried form $\widehat{\mathrm{f}} : (x \in A) \times (y \in B(x)) \to C(x, y)$. In order to reduce notational overhead we will usually write f instead of $\widehat{\mathrm{f}}$.

(j) When defining $f{:}(x \in (A \times B)) \to (c \in C(x)) \to D(x, c)$ we write $f(a, b, c)$ instead of $f(\langle a, c\rangle, c)$, similarly for longer products or functions with more arguments.

Definition 2.4 (a) For $\mathrm{I} \in \mathrm{Set}$ let $\mathrm{Set}^{\mathrm{I}}$ be the category of I-indexed sets with objects $A \in \mathrm{I} \to \mathrm{Set}$ and morphisms $f : A \to B$ being set theoretic functions $f : (i \in \mathrm{I}) \to A(i) \to B(i)$.

(b) For $A, B \in \mathrm{Set}^{\mathrm{I}}$, Let $A +_{\mathrm{Set}^{\mathrm{I}}} B := \lambda i.A(i) + B(i), A \times_{\mathrm{Set}^{\mathrm{I}}} B = \lambda i.A(i) \times B(i)$. Furthermore, let

$$\mathrm{inl}_{\mathrm{Set}^{\mathrm{I}}} := \lambda i, x.\mathrm{inl}(x) : A \to A + B$$

similarly for inr, π_0, π_1.

(c) For $X \subseteq \mathrm{Set}^{\mathrm{I}}$ let

$$\begin{aligned} \textstyle\bigcap_{\mathrm{Set}^{\mathrm{I}}} X &:= \lambda i. \textstyle\bigcap\{y(i) \mid y \in X\}, \\ \textstyle\bigcup_{\mathrm{Set}^{\mathrm{I}}} X &:= \lambda i. \textstyle\bigcup\{y(i) \mid y \in X\}. \end{aligned}$$

(d) For $X, Y \in \mathrm{Set}^{\mathrm{I}}$ let $X \subseteq_{\mathrm{Set}^{\mathrm{I}}} Y :\Leftrightarrow \forall i \in \mathrm{I}.X(i) \subseteq Y(i)$.

(e) We will usually omit the index $\mathrm{Set}^{\mathrm{I}}$ in the notations introduced above.

3 Initial Algebras and Inductively Defined Sets

We consider in the following the theory of simultaneous inductive definitions of sets $\mathrm{D}(i)$ for $i \in \mathrm{I}$. We fix $\mathrm{I} \in \mathrm{Set}$.

In [11, 13] Dybjer and the author introduced indexed inductive-recursive definitions. We defined an indexed inductively defined set $\mathrm{U} : \mathrm{I} \to \mathrm{Set}$ while simultaneously recursively defining a function $\mathrm{T} : (i \in \mathrm{I}) \to \mathrm{U}(i) \to E[i]$ for some type $E[i]$. $\mathrm{U}(i)$ was a universe of codes for elements of a type, and $\mathrm{T}(i, u)$ was the type corresponding to code u. The special case of indexed inductively defined sets (more precisely strictly positive indexed inductively defined sets) is obtained by taking $E[i] = \mathbf{1}$. Therefore T is equal to $\lambda i, x.*$. T becomes trivial and can be omitted. We call the set defined inductively in the following D instead of U and omit in the following T.

In [11, 13] we considered two versions of indexed inductive(-recursive) definitions, restricted and generalised ones. Generalised inductive definitions have constructors of the form

$$\begin{aligned}&\mathrm{C} : (x_1 \in A_1) \to (x_2 \in A_2(x_1)) \to \cdots \to (x_n \in A_n(x_1, \ldots, x_{n-1}))\\&\quad \to \mathrm{D}(\mathrm{i}(x_1, \ldots, x_n))\end{aligned}$$

whereas in restricted ones the index of the result of C is given by the first argument, so

$$\begin{aligned}&\mathrm{C} : (i \in \mathrm{I}) \to (x_1 \in A_1(i)) \to (x_2 \in A_2(i, x_1)) \to \cdots \to (x_n \in A_n(i, x_1, \ldots, x_{n-1}))\\&\quad \to \mathrm{D}(i).\end{aligned}$$

Restricted indexed inductive definitions allow decidable case distinction on elements of the set D defined inductively: an element of $\mathrm{D}(i)$ must be of the form $\mathrm{C}(i, x_1, \ldots, x_n)$ for one of the constructors of D. In case of general indexed inductive definitions we can in general not decide whether $\mathrm{C}(x_1, \ldots, x_n)$ forms an element of $\mathrm{D}(i)$, since we can in general not decide whether $\mathrm{i}(x_1, \ldots, x_n) = \mathrm{i}$.

We consider in the following only restricted indexed inductive definitions, since indexed inductive definitions are here mainly treated in order to motivate coinductively defined sets later, for which restricted ones are the natural choice.

Strictly positive restricted indexed inductive definitions are the least sets closed under constructors like C as before. In a notation borrowed from the type theoretic theorem prover Agda we write for the fact that Tree is this least set:

$$\begin{array}{lll}\multicolumn{3}{l}{\mathrm{data\ D} : \mathrm{I} \to \mathrm{Set\ where}}\\ \quad\mathrm{C} & : & (i \in \mathrm{I}) \to (x_1 \in A_1(i)) \to \cdots \to (x_n \in A_n(i, x_1, \ldots, x_{n-1})) \to \mathrm{D}(i)\\ \quad\mathrm{C}' & : & (i \in \mathrm{I}) \to (y_1 \in A'_1(i)) \to \cdots \to (y_m \in A'_m(i, y_1, \ldots, y_{m-1})) \to \mathrm{D}(i)\\ \quad\cdots & & \end{array}$$

Strict positivity means that $A_k(i, \vec{x})$ are either sets which were defined before $\mathrm{D}(i)$ was introduced (non-inductive arguments), or are of the form $(b \in B(i, \vec{x})) \to \mathrm{D}(j(i, \vec{x}, b))$ (inductive arguments). Since we do not know anything about $\mathrm{D}(i)$, later arguments cannot depend on previous inductive arguments.[1]

Therefore we obtain an equivalent inductive definition by moving all inductive arguments to the end. Now we can replace all non-inductive arguments by one single one by forming a product (and letting the later arguments depend on the projections). The inductive arguments $((b \in B_1(i, \vec{x})) \to \mathrm{D}(j_1(i, \vec{x}, b))) \to \cdots \to ((b \in B_k(i, \vec{x})) \to \mathrm{D}(j_k(i, \vec{x}, b)) \to$ can be replaced by $((b \in (B_1(i, \vec{x}) \times \cdots \times B_k(i, \vec{x}))) \to \mathrm{D}(j'(i, \vec{x}, b))$ for some suitable j' (in the special case where there is no inductive argument, we obtain an inductive argument $\emptyset \to \mathrm{D}$). Therefore an inductive definition can be replaced by one having constructors of the form

$$\mathrm{C}_k : (i \in \mathrm{I}) \to (a \in A_k(i)) \to ((b \in B_k(i, a)) \to \mathrm{D}(j(i, a, b))) \to \mathrm{D}(i).$$

---

[1] This holds only in indexed inductive-definitions; in indexed inductive-recursive definitions arguments can depend on T applied to previous inductive arguments.

Assuming we have constructors $\mathrm{C}_0, \ldots, \mathrm{C}_{n-1}$ we can replace all constructors by one single one of type

$$
\begin{aligned}
\mathrm{C} : (i \in \mathrm{I})& \\
&\rightarrow (k \in \{0, \ldots, n-1\}) \\
&\rightarrow (a \in A_k(i)) \\
&\rightarrow ((b \in B_k(i, a)) \rightarrow \mathrm{D}(j(i, a, b))) \\
&\rightarrow \mathrm{D}(i)
\end{aligned}
$$

which after merging the two non-inductive arguments into one becomes

$$\mathrm{C} : (i \in \mathrm{I}) \rightarrow (a \in \mathrm{A}(i)) \rightarrow ((b \in \mathrm{B}(i, a)) \rightarrow \mathrm{D}(\mathrm{j}(i, a, b))) \rightarrow \mathrm{D}(i).$$

This is the Petersson-Synek Tree [23], which is an indexed version of Martin-Löf's W-type. The Petersson-Synek trees subsume all strictly positive inductive definitions. They are initial algebras of indexed containers in the theory of containers, see [6, 21]. In [14, 19] a formal proof that initial algebras of indexed containers and therefore Petersson-Synek trees subsume all indexed inductive definitions is given. We write in the following Tree instead of D and tree for the constructor C. Let us fix in the following A, B, j:

Assumption 3.1 (a) In the following assume

$$
\begin{aligned}
\mathrm{I} &\in \mathrm{Set}, \\
\mathrm{A} &: \mathrm{I} \rightarrow \mathrm{Set}, \\
\mathrm{B} &: (i \in \mathrm{I}) \rightarrow \mathrm{A}(i) \rightarrow \mathrm{Set}, \\
\mathrm{j} &: (i \in \mathrm{I}) \rightarrow (a \in \mathrm{A}(i)) \rightarrow \mathrm{B}(i, a) \rightarrow \mathrm{I}.
\end{aligned}
$$

(b) Let tree be a constructor of arity 3.

In the above we have

$$
\begin{aligned}
\mathrm{tree} : (i \in \mathrm{I}) \rightarrow (a \in \mathrm{A}(i)) \rightarrow ((b \in \mathrm{B}(i, a)) \rightarrow \mathrm{Tree}(\mathrm{j}(i, a, b))) \\
\rightarrow \mathrm{Tree}(i).
\end{aligned}
$$

In the data-notation introduced above we denote this by:

$$
\begin{aligned}
&\mathrm{data\ Tree} : \mathrm{I} \rightarrow \mathrm{Set\ where} \\
&\quad \mathrm{tree} \quad : \ (i \in \mathrm{I}) \rightarrow (a \in \mathrm{A}(i)) \rightarrow ((b \in \mathrm{B}(i, a)) \rightarrow \mathrm{Tree}(\mathrm{j}(i, a, b))) \\
&\qquad\qquad \rightarrow \mathrm{Tree}(i).
\end{aligned}
$$

We will now repeat the well-known argument, that the categorical definition of inductive definitions is equivalent to the induction principle. The dual of this argument will then be used to determine the equivalence between the categorical definition of coalgebras and the corresponding coinduction principle.

Definition 3.2 (a) Let the functor $\mathrm{F} : \mathrm{Set}^{\mathrm{I}} \to \mathrm{Set}^{\mathrm{I}}$ be given by

$$\begin{array}{l}\mathrm{F}(X, i) := (a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to X(\mathrm{j}(i, a, b))) \\ \text{and for } f : X \to Y \\ \mathrm{F}(f) : \mathrm{F}(X) \to \mathrm{F}(Y), \\ \mathrm{F}(f, i, \langle a, g \rangle) := \langle a, \lambda b. f(\mathrm{j}(i, a, b), g(b)) \rangle.\end{array}$$

(b) An F-algebra, where F is as above, is a pair (X, f) such that $X \in \mathrm{Set}^{\mathrm{I}}$ and $f : \mathrm{F}(X) \to X$.

(c) The categorical definition[2] of Tree is that (Tree, tree) is an initial F-algebra,[3] which means:

- (Tree, tree) is an F-algebra.
- For any other F-algebra (X, f) there exists a unique $g : \mathrm{Tree} \to X$ s.t. the following diagram commutes

$$\begin{tikzcd}\mathrm{F}(\mathrm{Tree}) \arrow[r, "\mathrm{tree}"] \arrow[d, "\mathrm{F}(g)"'] & \mathrm{Tree} \arrow[d, "\exists! g"] \\ \mathrm{F}(X) \arrow[r, "f"] & X\end{tikzcd}$$

We call g the unique F-algebra homomorphism into (X, f).

(d) The inductive definition of Tree is given by[4]

- (Tree, tree) is an F-algebra,
- for any formula $\varphi(i, x)$ depending on $i \in \mathrm{I}$ and $x \in \mathrm{Tree}(i)$ we have that if

$$\begin{array}{l}\forall i \in \mathrm{I}. \forall a \in \mathrm{A}(i). \forall f \in (b \in \mathrm{B}(i, a)) \to \mathrm{Tree}(\mathrm{j}(i, a, b)). \\ \quad (\forall b \in \mathrm{B}(i, a). \varphi(\mathrm{j}(i, a, b), f(b))) \\ \quad \to \varphi(i, \mathrm{tree}(i, a, f))\end{array} \qquad (\mathrm{Prog}(\varphi))$$

then

$$\forall i \in \mathrm{I}. \forall x \in \mathrm{Tree}(i). \varphi(i, x).$$

We call the assumption $\mathrm{Prog}(\varphi)$ that "φ is progressive".

---

[2] Note that we deviate from standard category theory in so far as we fix the function tree: tree is the curried version of the constructor, which we introduced before. In standard category theory both the set Tree and the function tree can be arbitrary, and therefore the initial algebra is only unique up to isomorphism. Note as well that above we had the convention that we identify tree with its uncurried form $\widehat{\mathrm{tree}}$. Without this convention one would say that (Tree, $\widehat{\mathrm{tree}}$) is an F-algebra.

[3] Here F is as above, i.e. strictly positive.

[4] Again tree is the curried version of the constructor defined before.

(e) The set theoretic definition of Tree is given by

$$\mathrm{Tree} = [\![\,\mathrm{Tree}\,]\!]$$

where $$[\![\,\mathrm{Tree}\,]\!] := \bigcap\{X \in \mathrm{Set}^{\mathrm{I}} \mid (X, \mathrm{tree}) \text{ is an F-algebra}\}.$$

Lemma 3.3 $[\![\,\mathrm{Tree}\,]\!]$ *is a set.*

Proof We repeat the standard argument. Define by induction on the ordinals F^{α}, $\mathrm{F}^{<\alpha} \in \mathrm{Set}^{\mathrm{I}}$,

$$\begin{array}{lcl} \mathrm{F}^{\alpha}(i) & := & \{\mathrm{tree}(i, a, f) \mid \langle a, f\rangle \in \mathrm{F}(\mathrm{F}^{<\alpha}, i)\}, \\ \mathrm{F}^{<\alpha} & := & \bigcup_{\beta<\alpha}\mathrm{F}^{\beta}. \end{array}$$

F is monotone, and therefore $\mathrm{F}^{\alpha} \subseteq \mathrm{F}^{\beta}$ for $\alpha < \beta$. Let κ be a regular infinite cardinal, $\kappa > \mathrm{card}(B(i, a))$ for $i \in \mathrm{I}$ and $a \in \mathrm{A}(i)$ (where $\mathrm{card}(x)$ is the cardinality of x).

We show that $(\mathrm{F}^{<\kappa}, \mathrm{tree})$ is an F-algebra. Assume $\langle a, f\rangle \in \mathrm{F}(\mathrm{F}^{<\kappa}, i)$. Then $a \in \mathrm{A}(i)$, $f \in (b \in \mathrm{B}(i, a)) \to \mathrm{F}^{<\kappa}(\mathrm{j}(i, a, b))$. Therefore, for $b \in \mathrm{B}(i, a)$ there exist $\beta < \kappa$ s.t. $f(b) \in \mathrm{F}^{\beta}(\mathrm{j}(i, a, b))$. By the regularity of κ and $\kappa > \mathrm{card}(\mathrm{B}(i, a))$ there exists a $\gamma < \kappa$ s.t. for all $b \in \mathrm{B}(i, a)$ we have $f(b) \in \mathrm{F}^{\gamma}(\mathrm{j}(i, a, b))$. Therefore $\mathrm{tree}(i, a, f) \in \mathrm{F}^{\gamma+1}(i) \subseteq \mathrm{F}^{<\kappa}(i)$.

It follows $[\![\,\mathrm{Tree}\,]\!] \subseteq \mathrm{F}^{<\kappa}$ which is a set.

In fact $[\![\,\mathrm{Tree}\,]\!] = \mathrm{F}^{<\kappa}$, since one can show by induction on α that for any F-algebra (X, tree) we have $\mathrm{F}^{\alpha} \subseteq X$, and therefore $\mathrm{F}^{<\kappa} \subseteq X$, so $(\mathrm{F}^{\kappa}, \mathrm{tree})$ is the initial algebra.

The following theorem is well known. We show it since it provides the key idea for the coinduction principle introduced later.

Theorem 3.4 *The following is equivalent:*

(a) The categorical definition of Tree.
(b) The inductive definition of Tree.
(c) The set theoretic definition of Tree.

Proof (a) implies (b): Let $\varphi(i, x)$ be progressive. Define $E \in \mathrm{Set}^{\mathrm{I}}$, $E(i) := \{x \in \mathrm{Tree}(i) \mid \varphi(i, x)\}$. By progressivity of φ we obtain $\mathrm{tree} : \mathrm{F}(E) \to E$, therefore (E, tree) is an F-algebra. Let $h := \lambda i.x.x : E \to \mathrm{Tree}$ be the embedding function, g the unique F-algebra homomorphism $E \to \mathrm{Tree}$, and consider

$$\begin{tikzcd}
\mathrm{F}(\mathrm{Tree}) \arrow[r, "\mathrm{tree}"] \arrow[d, "\mathrm{F}(g)"'] & \mathrm{Tree} \arrow[d, "\exists g"] \\
\mathrm{F}(E) \arrow[r, "\mathrm{tree}"] \arrow[d, "\mathrm{F}(h)"'] & E \arrow[d, "h"] \\
\mathrm{F}(\mathrm{Tree}) \arrow[r, "\mathrm{tree}"] & \mathrm{Tree}
\end{tikzcd}$$

The upper diagram commutes by definition of g. The lower diagram obviously commutes. $h \circ g : \mathrm{Tree} \to \mathrm{Tree}$ and the identity function $\mathrm{id} : \mathrm{Tree} \to \mathrm{Tree}$ are two functions which make the outer diagram commute. By uniqueness of this function we get that $h \circ g = \mathrm{id}$, i.e. $\forall i \in \mathrm{I}.\forall x \in \mathrm{Tree}(i).g(i, x) = x$, and therefore $\forall i \in \mathrm{I}.\forall x \in \mathrm{Tree}(i).x \in E(i)$, $\forall i \in \mathrm{I}.\forall x \in \mathrm{Tree}(i).\varphi(i, x)$.

Proof of (b) implies (a): Let (X, f) be an F-homomorphism. The existence of a unique g follows as for the recursion theorem in set theory: One first defines for $i \in \mathrm{I}$ and $t \in \mathrm{Tree}(i)$ $\mathrm{TC}(i, t)$ as the least set such that,

- if $t = \mathrm{tree}(i, a, g)$, $b \in \mathrm{B}(i, a)$ then $\langle \mathrm{j}(i, a, b), g(b)\rangle \in \mathrm{TC}(i, t)$,
- if $\langle i', \mathrm{tree}(i', a, g)\rangle \in \mathrm{TC}(i, t)$ and $b \in \mathrm{B}(i', a)$ then $\langle \mathrm{j}(i', a, b), g(b)\rangle \in \mathrm{TC}(i, t)$.

So $\mathrm{Tree}(i, t)$ contains all proper subtrees of t and contains for every tree its subtrees.

Then it follows that we have course of value induction on Tree, i.e. if φ is course of value progressive, written $\mathrm{Prog}_{\mathrm{coursevalue}}(\varphi)$, i.e.

$$\forall i \in \mathrm{I}.\forall t \in \mathrm{Tree}(i).(\forall \langle i', t'\rangle \in \mathrm{TC}(i, t).\varphi(i', t')) \to \varphi(i, t)$$

then $\forall i \in \mathrm{I}.\forall t \in \mathrm{Tree}(i).\varphi(i, t)$. When showing this one shows first by induction on $i \in \mathrm{I}$, $t \in \mathrm{Tree}(i)$ that $\forall i \in \mathrm{I}.\forall t \in \mathrm{Tree}(i).\forall \langle i', y\rangle \in \mathrm{TC}(i, t).\varphi(i', y)$, which implies $\forall i \in \mathrm{I}.\forall t \in \mathrm{Tree}(i).\varphi(i, t)$. Let $\mathrm{TC}'(i, t) \in \mathrm{Set}^{\mathrm{I}}$, $\mathrm{TC}'(i, t, i') = \{t' \mid \langle i', t'\rangle \in \mathrm{TC}(i, t)\}$. Then one shows by course of value induction that for every $i \in \mathrm{I}$, $t \in \mathrm{Tree}(i)$ there exists a unique function $g : \mathrm{TC}'(i, t) \to X$ which fulfils the condition of the iteration principle given by the categorical diagram, restricted to $\mathrm{TC}'(i, t)$. We now obtain a function $g : \mathrm{Tree} \to X$ fulfilling the same equations, and show easily its uniqueness.

Proof of (c) implies (b): Assume φ is progressive. Define E as in the direction "(a) implies (b)". (E, tree) is an F-algebra, therefore $\mathrm{Tree} \subseteq E$.

Proof of (b) implies (c): We show first by induction on Tree that $\forall i \in \mathrm{I}.\forall x \in \mathrm{Tree}(i).x \in [\![\mathrm{Tree}]\!](i)$, therefore $\mathrm{Tree} \subseteq [\![\mathrm{Tree}]\!]$. Furthermore, (b) implies that (Tree, tree) is an F-algebra, and therefore $[\![\mathrm{Tree}]\!] \subseteq \mathrm{Tree}$.

4 Iteration, Recursion, Induction

In Sect. 6 we will introduce the principles of coiteration and corecursion. In order to see that these principles are the dual of iteration and primitive recursion, we repeat in this section the definition of those principles as well as the principle of type theoretic induction. We will give as well the (well-known) proof that the principles of being an initial F-algebra, of unique iteration, of unique primitive recursion, and of type theoretic induction are equivalent, which will as well be dualised in Sect. 6.

Definition 4.1 Assume Tree : I → Set and tree : F(Tree) → Tree.[5]

(a) By "(Tree, tree) satisfies the principle of unique iteration" we mean the following: Assume

$$\begin{aligned} &X : \mathrm{I} \to \mathrm{Set}, \\ &f : (i \in \mathrm{I}) \to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i,a)) \to X(\mathrm{j}(i,a,b)))) \\ &\qquad\qquad\quad \to X(i). \end{aligned}$$

Then there exists a unique $g : \mathrm{Tree} \to X$ such that

$$g(i, \mathrm{tree}(i, a, h)) = f(i, \langle a, \lambda b.g(\mathrm{j}(i, a, b), h(b))\rangle).$$

(b) By "(Tree, tree) satisfies the principle of unique primitive recursion" we mean the following: Assume

$$\begin{aligned} &X : \mathrm{I} \to \mathrm{Set}, \\ &f : (i \in \mathrm{I}) \to ((a \in \mathrm{A}(i)) \times \\ &\qquad\qquad\quad ((b \in \mathrm{B}(i,a)) \to (\mathrm{Tree}(\mathrm{j}(i,a,b)) \times X(\mathrm{j}(i,a,b))))) \\ &\qquad\qquad \to X(i). \end{aligned}$$

Then there exists a unique $g : \mathrm{Tree} \to X$ such that

$$g(i, \mathrm{tree}(i, a, h)) = f(i, \langle a, \lambda b.\langle h(b), g(\mathrm{j}(i, a, b), h(b))\rangle\rangle).$$

(c) By "(Tree, tree) satisfies the principle of unique type theoretic induction" we mean the following: Assume

$$\begin{aligned} &X : (i \in \mathrm{I}) \to \mathrm{Tree}(i) \to \mathrm{Set}, \\ &f : (i \in \mathrm{I}) \to ((a \in \mathrm{A}(i)) \times \\ &\qquad\qquad (h : (b \in \mathrm{B}(i,a)) \to ((t \in \mathrm{Tree}(\mathrm{j}(i,a,b))) \times X(\mathrm{j}(i,a,b),t)))) \\ &\qquad\qquad \to X(i, \mathrm{tree}(i, a, \pi_0 \circ h)). \end{aligned}$$

Then there exists a unique $g : (i \in \mathrm{I}) \to (t \in \mathrm{Tree}(i)) \to X(i,t)$ such that

$$g(i, \mathrm{tree}(i, a, h)) = f(i, \langle a, \lambda b.\langle h(b), g(\mathrm{j}(i, a, b), h(b))\rangle\rangle).$$

(d) By "(Tree, tree) satisfies the principle of iteration, primitive recursion or type theoretic induction" we mean that it satisfies the corresponding principle as above, but omitting the condition that g is unique.

Theorem 4.2 *Assume* Tree *is a set s.t.* tree : F(Tree) → Tree. *The following are equivalent*

---

[5] Note that in contrast to other sections, tree can be an arbitrary function of this type, and Tree is assumed just to be an element of $\mathrm{Set}^{\mathrm{I}}$.

(a) (Tree, tree) *is an initial* F*-algebra.*
(b) (Tree, tree) *satisfies the principle of unique iteration.*
(c) (Tree, tree) *satisfies the principle of unique primitive recursion.*
(d) (Tree, tree) *satisfies the principle of type theoretic induction.*
(e) (Tree, tree) *satisfies the principle of unique type theoretic induction.*

Proof (a) and (b) are equivalent since the principle of iteration is nothing but the commutativity of the diagram spelt out.

(a) implies (e): Define for X, f as in the definition of type-theoretic induction

$$\begin{aligned}
&X' : \mathrm{I} \to \mathrm{Set},\\
&X'(i) = (t \in \mathrm{Tree}(i)) \times X(i,t),\\[1em]
&h : \mathrm{F}(X') \to X',\\
&h(i, \langle a, k\rangle) \quad = \langle \mathrm{tree}(i, a, \pi_0 \circ k), f(i, \langle a, k\rangle)\rangle.
\end{aligned}$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{F}(\mathrm{Tree}) \arrow[r, "\mathrm{tree}"] \arrow[d, "\mathrm{F}(g')"] \arrow[dd, bend right, "{\mathrm{F}(\pi_0 \circ g'), \mathrm{F}(\mathrm{id})}"'] & \mathrm{Tree} \arrow[d, "\exists! g'"'] \arrow[dd, bend left, "{\pi_0 \circ g', \mathrm{id}}"] \\
\mathrm{F}(X') \arrow[r, "h"] \arrow[d, "\mathrm{F}(\pi_0)"] & X' \arrow[d, "\pi_0"'] \\
\mathrm{F}(\mathrm{Tree}) \arrow[r, "\mathrm{tree}"] & \mathrm{Tree}
\end{tikzcd}$$

There exists a unique g' such that the upper part of the diagram commutes. The lower part of the diagram commutes trivially. Both $\pi_0 \circ g'$ and id : Tree $\to$ Tree make the outer diagram commute. By uniqueness we get $\pi_0 \circ g' = \mathrm{id}$. Therefore $g'(i, t) = \langle t, g(i, t)\rangle$ for some $g : (i : \mathrm{I}) \to \mathrm{Tree}(i) \to X'(i, t)$, and we see immediately that g satisfies the equations for type-theoretic induction.

Assume g_0 is another solution for the equations for type theoretic induction in the theorem. Let $g_0' : \mathrm{Tree} \to X'$, $g_0'(i, t) = \langle t, g_0(i, t)\rangle$. Then the upper diagram above with g' replaced by g_0' commutes as well. By uniqueness of g' it follows $g_0' = g'$ and therefore $g_0 = g$.

Obviously, (e) implies (d).

(d) implies (c). We immediately obtain (d) implies the principle of primitive recursion, since it is a special case of type theoretic induction. We get as well unique primitive recursion: Assume X, f as in the definition of unique primitive recursion, and let $g, g' : \mathrm{Tree} \to X$ be two solutions for the primitive recursion equation. Let

$$\begin{aligned}
&X' : (i \in I) \to \mathrm{Tree}(i) \to \mathrm{Set},\\
&X'(i, t) = g(i, t) \mathrel{\widehat{=}} g'(i, t).
\end{aligned}$$

Let f' be of the type of the function underlying the principle of type-theoretic induction w.r.t. X', $f'(i, \langle a, h\rangle) = * \in X'(i, t)$, where $t := \mathrm{tree}(i, a, \pi_0 \circ h)$. $f'(i, \langle a, h\rangle) \in X'(i, t)$, since for $b \in \mathrm{B}(i, a)$ we have, with $j' := \mathrm{j}(i, a, b)$, that $\pi_1(h(b)) \in X'(j', \pi_0(h(b)))$, therefore $g(j', \pi_0(h(b))) = g'(j', \pi_0(h(b)))$, and therefore $g(i, t) = g'(i, t)$. Let g'' be defined by the principle of induction w.r.t. X' and f'. Then we have $g'' : (i \in \mathrm{I}) \to (t \in \mathrm{Tree}(i)) \to X'(i, t)$, and therefore for $i \in \mathrm{I}, t \in \mathrm{Tree}(i)$ we have $g(i, t) = g'(i, t)$.

(c) implies (b) since iteration is a special case of primitive recursion.

When dualising inductive definitions, we will not obtain a direct dual of type theoretic induction. We obtain only duals of iteration and recursion. So when dualising the current theorem, we need to omit (d) and (e) and therefore dualise a proof that (a) implies (c). But a proof that (a) implies (c) is essentially the same as the proof that (a) implies (e), where one omits the dependencies of X' on $\mathrm{Tree}(i)$.

5 Modelling Coinductive Sets in Set Theory

In case of inductive definitions it was easy to model inductively defined sets set-theoretically, since we could simply model the well-founded trees set theoretically. When defining coinductively defined sets (or final coalgebras) we obtain non-well-founded trees. If we define the elements as terms introduced by the constructor tree as used before (which was fixed function), then the coinductively defined set would need to be defined as a non-well-founded sets [3]. This can be overcome by introducing coinductively defined sets by their eliminators, and in the following we will give one concrete way of defining them. We can then define a constructor for introducing elements, this constructor is not the function tree defined before. We note that there are many different ways known for defining non-well-founded trees in set theory, our approach here is inspired by Aczel [4]. It can be considered as an indexed explicit version of the standard limit construction of coalgebras. This construction is a category theoretic construction, it is essentially the ω-limit of F^n. One of the earliest versions of such a construction seems to be [5], which is an extension of [3].

One advantage of this concrete representation of coinductively defined sets over other more abstract constructions is that because it is very concrete it is easy to have a feeling of what the elements of the coinductively defined sets are. As one can see the elements of coalgebras are descriptions of the result of applying the eliminators to them (several times in case of the eliminator E_2 which returns an element of the coalgebra). So this construction follows the slogan "an element of a coalgebra is determined by the result of applying the eliminators to it".

Assume I, A, B, j, F as before.

Definition 5.1 • An F-coalgebra (X, f) is given by $X \in \mathrm{Set}^{\mathrm{I}}$ and $f : X \to \mathrm{F}(X)$.

- An F-coalgebra (X, f) is a final F-coalgebra if for any F-coalgebra (Y, g) there exists a unique $h : Y \to X$ s.t.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "\exists ! h"'] & \mathrm{F}(Y) \arrow[d, "\mathrm{F}(h)"] \\
X \arrow[r, "f"] & \mathrm{F}(X)
\end{tikzcd}$$

We will in the following construct a final F-coalgebra $([\![\,\mathrm{Tree}^{\infty}\,]\!], \mathrm{E})$. So we have

$$\begin{aligned}
\mathrm{E} : (i \in \mathrm{I}) &\to [\![\,\mathrm{Tree}^{\infty}\,]\!](i) \\
&\to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](\mathrm{j}(i, a, b)))).
\end{aligned}$$

We can replace the eliminator (or case distinction) E by two eliminators

$$\begin{aligned}
\mathrm{E}_1 \quad &: \quad (i \in \mathrm{I}) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](i) \to \mathrm{A}(i), \\
\mathrm{E}_2 \quad &: \quad (i \in \mathrm{I}) \to (t \in [\![\,\mathrm{Tree}^{\infty}\,]\!](i)) \to (b \in \mathrm{B}(i, \mathrm{E}_1(i, t))) \\
&\qquad \to [\![\,\mathrm{Tree}^{\infty}\,]\!](\mathrm{j}(i, \mathrm{E}_1(i, t), b)).
\end{aligned}$$

E_1 returns the label of the tree and E_2 its subtrees. Since $\mathrm{E}_1, \mathrm{E}_2$ are the two components of E we will in the following freely switch between E and $\mathrm{E}_1, \mathrm{E}_2$.

We summarise that $[\![\,\mathrm{Tree}^{\infty}\,]\!]$ is an F-coalgebra as follows:

$$\begin{aligned}
[\![\,\mathrm{Tree}^{\infty}\,]\!] \quad &: \quad \mathrm{I} \to \mathrm{Set}, \\
\mathrm{E}_1 \quad &: \quad (i \in \mathrm{I}) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](i) \to \mathrm{A}(i), \\
\mathrm{E}_2 \quad &: \quad (i \in \mathrm{I}) \to (b \in \mathrm{B}(i, \mathrm{E}_1(i, t))) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](\mathrm{j}(i, \mathrm{E}_1(i, t), b)).
\end{aligned}$$

The idea for defining $[\![\,\mathrm{Tree}^{\infty}\,]\!] : \mathrm{I} \to \mathrm{Set}$ and E_i as follows: An element of $[\![\,\mathrm{Tree}^{\infty}\,]\!]$ is anything which, when applying E_1 and E_2 to it, returns meaningful results. When applying E_1 we obtain an element of $A(i)$, which we can observe directly. When applying E_2 (with an argument in $B(i, a)$) we obtain an element of $[\![\,\mathrm{Tree}^{\infty}\,]\!](j)$ for some j, which we cannot observe directly. However we can continue applying E_2 several times and then E_1 to obtain an observable result. The observations we have are therefore that we apply several times E_2 to it and then E_1 to it and obtain an element of $A(i)$ for some i.

This means that the observations from an element of $[\![\,\mathrm{Tree}^{\infty}\,]\!](i)$ are if we set $i_0 = i$, an element $a_0 \in \mathrm{A}(i_0)$ which would be the result of applying E_1; we can then continue by choosing an arbitrary $b_0 \in \mathrm{B}(i_0, a_0)$, have now a new index $i_1 = \mathrm{j}(i_0, a_0, b_0)$. For this index we could apply E_1 to it and obtain an element $a_1 \in \mathrm{A}(i_1)$, or apply for an arbitrary $b_1 \in \mathrm{B}(i_0, a_0)$, an element corresponding to index $i_2 = \mathrm{j}(i_0, a_0, b_0)$ and so on.

The observations are therefore a set of sequences $\langle i_0, a_0, b_0, i_1, a_1, b_1, \ldots, i_n, a_n \rangle$, where $i_0 = i, a_k \in \mathrm{A}(i_k), b_k \in \mathrm{B}(i_k, a_k)$ and $i_{k+1} = \mathrm{j}(i_k, a_k, b_k)$. Here, b_k can be chosen freely, whereas a_k is defined uniquely depending on previous occurrences of $b_{k'}$. An element of $[\![\,\mathrm{Tree}^{\infty}\,]\!]$ is determined by those observations and therefore identified with those observations. This gives rise to the following definition of $[\![\,\mathrm{Tree}^{\infty}\,]\!]$ as a set of such sequences:

Definition 5.2 (a) Let for $i \in \mathrm{I}$

$$\begin{aligned}\mathrm{Seq}_{[\![\,\mathrm{Tree}^{\infty}\,]\!]}(i) := \{\langle i_0, a_0, b_0, i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid\ & \\ & n \geq 0, i_0 = i, \\ & (\forall k \in \{0, \ldots, n-1\}.b_k \in \mathrm{B}(i_k, a_k) \wedge i_{k+1} = \mathrm{j}(i_k, a_k, b_k)), \\ & \forall k \in \{0, \ldots, n\}.a_k \in \mathrm{A}(i_k)\}.\end{aligned}$$

(b) Let $[\![\,\mathrm{Tree}^{\infty}\,]\!](i)$ be the set of $t \subseteq \mathrm{Seq}_{[\![\,\mathrm{Tree}^{\infty}\,]\!]}(i)$ such that the following holds:

- $\langle i_0, a_0, b_0, \ldots, i_{n+1}, a_{n+1}\rangle \in t \to \langle i_0, a_0, b_0, \ldots, i_n, a_n\rangle \in t$,
- $\exists! a.\langle i, a\rangle \in t$,
- $\langle i_0, a_0, b_0, \ldots, i_n, a_n\rangle \in t \wedge b_n \in \mathrm{B}(i_n, a_n) \wedge i_{n+1} = \mathrm{j}(i_n, a_n, b_n)$
  $\to \exists! a_{n+1}.\langle i_0, a_0, b_0, \ldots, i_n, a_n, b_n, i_{n+1}, a_{n+1}\rangle \in t$.

(c) Define

$$\begin{aligned}&\mathrm{E}_1 : (i \in \mathrm{I}) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](i) \to \mathrm{A}(i), \\ &\mathrm{E}_1(i, t) := a \qquad \text{if } \langle i, a\rangle \in t.\end{aligned}$$

(d) Define

$$\begin{aligned}&\mathrm{E}_2 : ((i \in \mathrm{I}) \to (t \in [\![\,\mathrm{Tree}^{\infty}\,]\!](i)) \to (b \in \mathrm{B}(i, \mathrm{E}_1(i, t))) \\ &\qquad \to [\![\,\mathrm{Tree}^{\infty}\,]\!](\mathrm{j}(i, \mathrm{E}_1(i, t), b)), \\ &\mathrm{E}_2(i, t, b) := \{\langle i_1, a_1, b_1, \ldots, i_{n+1}, a_{n+1}\rangle \\ &\qquad\qquad \mid \langle i, \mathrm{E}_1(i, t), b, i_1, a_1, b_1, \ldots, i_{n+1}, a_{n+1}\rangle \in t\}.\end{aligned}$$

(e) Define

$$\begin{aligned}&\mathrm{E} : (i \in \mathrm{I}) \to (t \in [\![\,\mathrm{Tree}^{\infty}\,]\!](i)) \\ &\quad \to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](\mathrm{j}(i, a, b)))), \\ &\mathrm{E}(i, t) = \langle \mathrm{E}_1(i, t), \lambda b \in \mathrm{B}(i, \mathrm{E}_1(i, t)).\mathrm{E}_2(i, t, b)\rangle.\end{aligned}$$

Lemma 5.3 $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}$ *in the previous definition are well defined.*

Proof Straightforward by definition.

Lemma 5.4 *Assume*

$$\begin{array}{lll} G & : & \mathrm{I} \to \mathrm{Set}, \\ \widehat{a} & : & (i \in \mathrm{I}) \to G(i) \to \mathrm{A}(i), \\ \widehat{g} & : & (i \in \mathrm{I}) \to (g \in G(i)) \to (b \in \mathrm{B}(i, \widehat{a}(i, g))) \to G(\mathrm{j}(i, \widehat{a}(i, g), b)). \end{array}$$

Then there exists a unique $f : (i \in \mathrm{I}) \to G(i) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](i)$ *such that for all* $i \in \mathrm{I}$, $g \in G(i)$, $b \in \mathrm{B}(i, \widehat{a}(i, g))$ *we have*

$$\begin{array}{lll} \mathrm{E}_1(i, f(i, g)) & = & \widehat{a}(i, g), \\ \mathrm{E}_2(i, f(i, g), b) & = & f(\mathrm{j}(i, \mathrm{E}_1(i, f(i, g)), b), \widehat{g}(i, g, b)). \end{array}$$

Proof Define for $i \in \mathrm{I}$, $g \in G(i)$,

$$\begin{array}{rcl}
f(i,g) & = & \{\langle i_0, a_0, b_0, i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid \\
& & \quad \langle i_0, g_0, a_0, b_0, i_1, a_1, g_1, b_1, \ldots, i_n, g_n, a_n\rangle \in Y(i,g)\} \quad \text{where} \\
Y(i,g) & = & \{\langle i_0, g_0, a_0, b_0, i_1, a_1, g_1, b_1, \ldots, i_n, g_n, a_n\rangle \mid \\
& & \qquad g_0 = g, i_0 = i \\
& & \qquad (\forall j \in \{0, \ldots, n\}. a_j = \widehat{a}(i_j, g_j)), \\
& & \qquad \forall j \in \{0, \ldots, n-1\}. b_j \in B(i_j, a_j) \wedge i_{j+1} = \mathrm{j}(i_j, a_j, b_j) \\
& & \qquad\qquad\qquad\qquad \wedge g_{j+1} = \widehat{g}(i_j, g_j, b_j)\}.
\end{array}$$

One easily sees that $f(i,g) \in [\![\, \mathrm{Tree}^{\infty} \,]\!](i)$.
$\langle i, g, \widehat{a}(i,g)\rangle \in Y(i,g)$, therefore $\langle i, \widehat{a}(i,g)\rangle \in f(i,g)$, therefore
$\mathrm{E}_1(i, f(i,g)) = \widehat{a}(i,g)$.
Furthermore,

$$\begin{array}{rcl}
\mathrm{E}_2(i, f(i,g), b) & = & \{\langle i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid \\
& & \quad \langle i, \mathrm{E}_1(i, f(i,g)), b, i_1, a_1, b_1, \ldots, i_n, a_n\rangle \in f(i,g)\} \\
& = & \{\langle i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid \\
& & \quad \langle i, g, \mathrm{E}_1(i, f(i,g)), b, i_1, g_1, a_1, b_1, \ldots, i_n, g_n, a_n\rangle \in Y(i,g)\} \\
& = & f(\mathrm{j}(i, \mathrm{E}_1(i, f(i,g)), b), \widehat{g}(i,g,b)).
\end{array}$$

Therefore f fulfils the required equations.
Assume now

$$\begin{array}{lcl}
f' : (i \in \mathrm{I}) \to G(i) \to [\![\, \mathrm{Tree}^{\infty} \,]\!](i) & & \text{s.t.} \\
\mathrm{E}_1(i, f'(i,g)) & = & \widehat{a}(i,g), \\
\mathrm{E}_2(i, f'(i,g), b) & = & f'(\mathrm{j}(i, \mathrm{E}_1(i, f'(i,g)), b), \widehat{g}(i,g,b)).
\end{array}$$

Then

$$\langle i', a'\rangle \in f'(i,g) \quad \Leftrightarrow \quad i' = i \wedge a' = \widehat{a}(i,g) \Leftrightarrow \langle i', a'\rangle \in f(i,g).$$

Therefore sequences of length 2 in $f'(i,g)$ and $f(i,g)$ coincide. Furthermore,

$$\mathrm{E}_2(i, f'(i,g), b) \quad = \quad f'(\mathrm{j}(i, \widehat{a}(i,g), b), \widehat{g}(i,g,b)).$$

Therefore

$$\begin{array}{rcl}
\mathrm{E}_2(i, f'(i,g), b) & = & \{\langle i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid \\
& & \quad \langle i, \widehat{a}(i,g), b, i_1, a_1, b_1, \ldots, i_n, a_n\rangle \in f'(i,g)\} \\
& = & f'(\mathrm{j}(i, \widehat{a}(i,g), b), \widehat{g}(i,g,b))
\end{array}$$

which is the same equation as fulfilled by $f(i,g)$. This equation reduces sequences in $f'(i,g)$ of length >2 to sequences of shorter length in some $f'(i', g')$ for some

i', g', similarly for f. Together with the statement about sequences of length 2 above it follows by induction on length(σ)

$$\forall \sigma.\forall i, g.\sigma \in f'(i, g) \Leftrightarrow \sigma \in f(i, g)$$

therefore $\forall i, g.f(i, g) = f'(i, g)$, $f = f'$.

Main Theorem 5.5 ($[\![\mathrm{Tree}^{\infty}]\!]$, E) *is a final* F*-coalgebra.*

Proof ($[\![\mathrm{Tree}^{\infty}]\!]$, E) is an F-coalgebra. Assume (G, g) is an F-coalgebra. Let $g(i, x) = \langle \widehat{a}(i, x), \widehat{g}(i, x) \rangle$. Lemma 5.4 implies that there exists a unique $f : G \to [\![\mathrm{Tree}^{\infty}]\!]$ s.t. the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\exists ! f"'] & \mathrm{F}(G) \arrow[d, "\mathrm{F}(f)"] \\
{[\![\mathrm{Tree}^{\infty}]\!]} \arrow[r, "\mathrm{tree}"] & \mathrm{F}([\![\mathrm{Tree}^{\infty}]\!])
\end{tikzcd}$$

Above we used the notation with keyword data for denoting an inductively defined set as given by its constructors. A similar notation expressing that $[\![\mathrm{Tree}^{\infty}]\!]$ is a final coalgebra with eliminators E_1, E_2, would be:

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{coalg}\ [\![\mathrm{Tree}^{\infty}]\!] : \mathrm{I} \to \mathrm{Set}\ \mathrm{where}} \\
\mathrm{E}_1 & : & (i \in \mathrm{I}) \to [\![\mathrm{Tree}^{\infty}]\!](i) \to \mathrm{A}(i), \\
\mathrm{E}_2 & : & (i \in \mathrm{I}) \to (b \in \mathrm{B}(i, \mathrm{E}_1(i, t))) \to [\![\mathrm{Tree}^{\infty}]\!](\mathrm{j}(i, \mathrm{E}_1(i, t), b)).
\end{array}$$

6 Coiteration and Corecursion

We will in this section introduce the dual of iteration and primitive recursion, namely coiteration and corecursion. We do not know how to directly formulate the dual of type theoretic induction (or dependent primitive recursion), since one cannot directly invert the arrow in a dependent function type. In Sect. 8 we will introduce a principle of coinduction, which can be considered as the dual of induction, although it is not its direct dual.

We show as well that the principles of being a final F-coalgebra, of unique coiteration, and of unique corecursion are equivalent. The definitions and the proof in this section are the exact dual of Sect. 4 (omitting type theoretic induction). Note that the dual of the product × is the disjoint union +. In the principle of primitive recursion we can make use of both the inductive argument and the recursion hypothesis, corresponding to the product (×). In the principle of corecursion we can either return a given element from Tree^{∞} or recursively call the function in question, which is a call to the corecursion hypothesis, corresponding to the disjoint union (+).

Definition 6.1 Assume Tree^{∞} is a set, $\mathrm{E} : \mathrm{Tree}^{\infty} \to \mathrm{F}(\mathrm{Tree}^{\infty})$, and let $\mathrm{E}_1, \mathrm{E}_2$ be the two components of E as defined before.

(a) By "(Tree^{∞}, E) satisfies the principle of unique coiteration" we mean the following: Assume

$$\begin{array}{lcl} X & : & \mathrm{I} \to \mathrm{Set}, \\ \widehat{a} & : & (i \in \mathrm{I}) \to X(i) \to \mathrm{A}(i), \\ \widehat{x} & : & (i \in \mathrm{I}) \to (x \in X(i)) \to (b \in \mathrm{B}(i, \widehat{a}(i, x))) \\ & & \to X(\mathrm{j}(i, \widehat{a}(i, x), b)). \end{array}$$

Then there exists a unique $f : X \to \mathrm{Tree}^{\infty}$ such that

$$\begin{array}{lcl} \mathrm{E}_1(i, f(i, x)) & = & \widehat{\mathrm{a}}(i, x), \\ \mathrm{E}_2(i, f(i, x), b) & = & f(\mathrm{j}(i, \mathrm{E}_1(i, f(i, x)), b), \widehat{\mathrm{x}}(i, x, b)). \end{array}$$

(b) By "(Tree^{∞}, E) satisfies the principle of unique corecursion" we mean the following: Assume

$$\begin{array}{lcl} \widehat{a} & : & (i \in \mathrm{I}) \to X(i) \to \mathrm{A}(i), \\ \widehat{x} & : & (i \in \mathrm{I}) \to (x \in X(i)) \to (b \in \mathrm{B}(i, \widehat{a}(i, x))) \\ & & \to X(\mathrm{j}(i, \widehat{a}(i, x), b)) + \mathrm{Tree}^{\infty}(\mathrm{j}(i, \widehat{a}(i, x), b)). \end{array}$$

Then there exists a unique $f : X \to \mathrm{Tree}^{\infty}$ such that

$$\begin{array}{lcl} \mathrm{E}_1(i, f(i, x)) & = & \widehat{\mathrm{a}}(i, x), \\ \mathrm{E}_2(i, f(i, x), b) & = & \begin{cases} f(\mathrm{j}(i, \mathrm{E}_1(i, f(i, x)), b), x') & \text{if } \widehat{\mathrm{x}}(i, x, b) = \mathrm{inl}(x'), \\ x' & \text{if } \widehat{\mathrm{x}}(i, x, b) = \mathrm{inr}(x'). \end{cases} \end{array}$$

(c) By "(Tree^{∞}, E) satisfies the principle of corecursion or coiteration" we mean that it fills the corresponding principle as above, but omitting the condition that f is unique.

Lemma 6.2 *Assume* Tree^{∞} *is a set,* $\mathrm{E} : \mathrm{Tree}^{\infty} \to \mathrm{F}(\mathrm{Tree}^{\infty})$*, and let* $\mathrm{E}_1, \mathrm{E}_2$ *be the two components of* E *as defined before.*

The following are equivalent

(a) (Tree^{∞}, E) *is a final* F*-coalgebra.*
(b) (Tree^{∞}, E) *satisfies the principle of unique coiteration.*
(c) (Tree^{∞}, E) *satisfies the principle of unique corecursion.*

Proof (a) and (b) are equivalent since $\widehat{a}, \widehat{x}$ are the two components of a morphism $g : X \to \mathrm{F}(X)$, so the unique existence of f as in (b) is equivalent to the unique existence of f in the diagram for defining final coalgebras.

Obviously, (c) implies (b), since coiteration is a special case of corecursion.

(a) implies (c): Define for $X, \widehat{a}, \widehat{x}$ as in the definition of corecursion

$$\begin{aligned}
&X' : \mathrm{I} \to \mathrm{Set},\\
&X'(i) = X(i) + \mathrm{Tree}^\infty(i),\\[1em]
&h : (i \in \mathrm{I}) \to X'(i) \to \mathrm{F}(X', i),\\
&h(i, \mathrm{inl}(x)) \quad = \quad \langle \widehat{a}(i, x), \lambda b.\widehat{x}(i, x, b)\rangle,\\
&h(i, \mathrm{inr}(t)) \quad = \quad \langle \mathrm{E}_0(i, t), \lambda b.\mathrm{inr}(\mathrm{E}_1(i, t, b))\rangle.
\end{aligned}$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{Tree}^\infty \arrow[r, "\mathrm{E}"] \arrow[d, "\mathrm{inr}"] \arrow[dd, bend right=60, "g' \circ \mathrm{inr}, \mathrm{id}"'] & \mathrm{F}(\mathrm{Tree}^\infty) \arrow[d, "\mathrm{F}(\mathrm{inr})"'] \arrow[dd, bend left=60, "\mathrm{F}(g' \circ \mathrm{inr}), \mathrm{F}(\mathrm{id})"] \\
X' \arrow[r, "h"] \arrow[d, "\exists ! g'"] & \mathrm{F}(X') \arrow[d, "\mathrm{F}(g')"'] \\
\mathrm{Tree}^\infty \arrow[r, "\mathrm{E}"] & \mathrm{F}(\mathrm{Tree}^\infty)
\end{tikzcd}$$

There exists a unique g' such that the lower part of the diagram commutes. The upper part of the diagram commutes trivially. Both $g' \circ \mathrm{inr}$ and $\mathrm{id} : \mathrm{Tree}^\infty \to \mathrm{Tree}^\infty$ make the outer diagram commute. By uniqueness we get $g' \circ \mathrm{inr} = \mathrm{id}$. Let $g := g' \circ \mathrm{inl} : X \to \mathrm{Tree}^\infty$. By $g'(\mathrm{inr}(x)) = x$ we have that g satisfies the desired equation. Assume g_0 is another solution for the corecursion equation in the lemma. Let $g'_0 : X' \to \mathrm{Tree}^\infty$, $g'_0(i, \mathrm{inl}(x)) = g_0(x)$, $g'_0(i, \mathrm{inr}(x)) = x$. Then the lower diagram above with g' replaced by g'_0 commutes as well. By uniqueness of g' follows $g'_0 = g'$ and therefore $g_0 = g$.

7 Indexed Corecursion

When defining elements of coinductively defined sets, we often want to define for some $X \in \mathrm{Set}$ and $\widehat{i} : X \to \mathrm{I}$ a function $f : (x \in X) \to \mathrm{Tree}^\infty(\widehat{i}(x))$ corecursively. This can be reduced to corecursion as follows:

Lemma 7.1 *Let* $(\mathrm{Tree}^\infty, \mathrm{E})$ *be a final* F*-coalgebra, where* F *is as before. Assume*

$$\begin{aligned}
&X \in \mathrm{Set},\\
&\widehat{i} \quad : \quad X \to \mathrm{I},\\
&\widehat{a} \quad : \quad (x \in X) \to \mathrm{A}(\widehat{i}(x)),\\
&\widehat{x} \quad : \quad (x \in X) \to (b \in \mathrm{B}(\widehat{i}(x), \widehat{a}(x)))\\
&\qquad\qquad \to \{x \in X \mid \widehat{i}(x) = \mathrm{j}(\widehat{i}(x), \widehat{a}(x), b)\} + \mathrm{Tree}^\infty(\mathrm{j}(\widehat{i}(x), \widehat{a}(x), b))).
\end{aligned}$$

Then there exists a unique $f : (x \in X) \to \mathrm{Tree}^{\infty}(\widehat{i}(x))$*, such that*

$$\begin{aligned}
&\mathrm{E}_1(\widehat{i}(x), f(x)) = \widehat{a}(x),\\
&\mathrm{E}_2(\widehat{i}(x), f(x), b) = \begin{cases} f(y) & \textit{if}\ \widehat{x}(x, b) = \mathrm{inl}(y),\\ t & \textit{if}\ \widehat{x}(x, b) = \mathrm{inr}(t).\end{cases}
\end{aligned}$$

Proof Let $Y : \mathrm{I} \to \mathrm{Set}$, $Y(i) := \{x \in X \mid \widehat{i}(x) = i\}$. f satisfying the equations as stated in the lemma is equivalent to the function

$$\begin{aligned}
&f' : (i \in \mathrm{I}) \to Y(i) \to \mathrm{Tree}^{\infty}(i),\\
&f'(i, x) = f(x).
\end{aligned}$$

satisfying the equations.

$$\begin{aligned}
&\mathrm{E}_1(i, f'(i, x)) = \widehat{a}(x),\\
&\mathrm{E}_2(i, f'(i, x), b) = \begin{cases} f'(\mathrm{j}(i, \widehat{a}(x), b), y) & \text{if}\ \widehat{x}(x, b) = \mathrm{inl}(y),\\ t & \text{if}\ \widehat{x}(x, b) = \mathrm{inr}(t).\end{cases}
\end{aligned}$$

By existence and uniqueness of f' satisfying those equations follows existence and uniqueness of f.

8 Bisimulation and Coinduction

Definition 8.1 Assume Tree^{∞} is a set, $\mathrm{E} : \mathrm{Tree}^{\infty} \to \mathrm{F}(\mathrm{Tree}^{\infty})$, and let $\mathrm{E}_1, \mathrm{E}_2$ be the two components of E as defined before.

(a) Let for $i \in \mathrm{I}$, $t, t' \in \mathrm{Tree}^{\infty}(i)$

$$\mathrm{I}^{\mathrm{Bisim}} := (i \in \mathrm{I}) \times \mathrm{Tree}^{\infty}(i) \times \mathrm{Tree}^{\infty}(i),$$

$$\begin{aligned}
&\mathrm{A}^{\mathrm{Bisim}} : \mathrm{I}^{\mathrm{Bisim}} \to \mathrm{Set},\\
&\mathrm{A}^{\mathrm{Bisim}}(i, t, t') := (\mathrm{E}_0(i, t) = \mathrm{E}_0(i, t'))\quad (\text{more precisely } (\mathrm{E}_0(i, t) \mathrel{\widehat{=}} \mathrm{E}_0(i, t'))),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{B}^{\mathrm{Bisim}} : (i \in \mathrm{I}^{\mathrm{Bisim}}) \to \mathrm{A}^{\mathrm{Bisim}}(i) \to \mathrm{Set},\\
&\mathrm{B}^{\mathrm{Bisim}}(i, t, t', a) := \mathrm{B}(i, a),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{j}^{\mathrm{Bisim}} : (i \in \mathrm{I}^{\mathrm{Bisim}}) \to (a \in \mathrm{A}^{\mathrm{Bisim}}(i)) \to (b \in \mathrm{B}^{\mathrm{Bisim}}(i, a)) \to \mathrm{I}^{\mathrm{Bisim}},\\
&\mathrm{j}^{\mathrm{Bisim}}(i, t, t', *, b) = \langle \mathrm{j}(i, \mathrm{E}_0(i, t), b), \mathrm{E}_1(i, t, b), \mathrm{E}_1(i, t', b)\rangle,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{F}^{\mathrm{Bisim}} : \mathrm{Set}^{\mathrm{I}^{\mathrm{Bisim}}} \to \mathrm{Set}^{\mathrm{I}^{\mathrm{Bisim}}},\\
&\mathrm{F}^{\mathrm{Bisim}}(X, i) = (a \in \mathrm{A}^{\mathrm{Bisim}}(i)) \times ((b \in \mathrm{B}^{\mathrm{Bisim}}(i, a)) \to X(\mathrm{j}^{\mathrm{Bisim}}(i, a, b)).
\end{aligned}$$

We note that, if $(\mathrm{Bisim}, \mathrm{E}^{\mathrm{Bisim}})$ is an $\mathrm{F}^{\mathrm{Bisim}}$-coalgebra, and $\mathrm{E}_0^{\mathrm{Bisim}}$, $\mathrm{E}_1^{\mathrm{Bisim}}$ are the two components of $\mathrm{E}^{\mathrm{Bisim}}$, then

$$\begin{array}{l}\mathrm{E}_0^{\mathrm{Bisim}}(i,t,t') \in (\mathrm{E}_0(i,t) = \mathrm{E}_0(i,t')) \\ \text{i.e. the existence of } \mathrm{E}_0^{\mathrm{Bisim}}(i,t,t') \text{ is equivalent to } \mathrm{E}_0(i,t) = \mathrm{E}_0(i,t') \\ \text{and for } a \in \mathrm{A}^{\mathrm{Bisim}}(i,t,t'), b \in \mathrm{B}^{\mathrm{Bisim}}(i,t,t',a) = \mathrm{B}(i,a) \\ \mathrm{E}_1^{\mathrm{Bisim}}(i,t,t',b) \in \mathrm{Bisim}(\mathrm{j}(i,\mathrm{E}_0(i,t),b), \mathrm{E}_1(i,t,b), \mathrm{E}_1(i,t',b)).\end{array}$$

Definition 8.2 For $X \in \mathrm{Set}$ which is considered as a relation we will in formulae write X instead of $(\exists x.x \in X)$

Lemma 8.3 *Assume the axiom of choice. Assume* $X : \mathrm{I}^{\mathrm{Bisim}} \to \mathrm{Set}$.

There exists a g *s.t.* (X, g) *is an* $\mathrm{F}^{\mathrm{Bisim}}$*-coalgebra iff*

$$\begin{array}{l}\forall i,t,t'.X(i,t,t') \\ \qquad \to \mathrm{E}_0(i,t) = \mathrm{E}_0(i,t') \\ \qquad\quad \wedge \forall b \in \mathrm{B}(i,\mathrm{E}_0(i,t)).X(\mathrm{j}(i,\mathrm{E}_0(i,t),b), \mathrm{E}_1(i,t,b), \mathrm{E}_1(i,t',b)).\end{array}$$

Proof "⇒" is obvious. For "⇐" define $g : X \to \mathrm{F}^{\mathrm{Bisim}}(X)$, $g(i,t,t',x) = \langle *, h\rangle$ where $h(b) =$ some $y \in X(\mathrm{j}(i,\mathrm{E}_0(i,t),b), \mathrm{E}_1(i,t,b), \mathrm{E}_1(i,t',b))$.

Induction is a proof principle which is equivalent to the principle that an F-algebra is an initial F-algebra, or, as we have seen, the principle of unique iteration or unique primitive recursion. Dually coinduction should be a proof principle which is equivalent to the principle that an F-coalgebra is a final F-coalgebra, or equivalently, that it satisfies the principle of unique coiteration or the principle of unique corecursion.

We will see below that the principle of being a final F-coalgebra is equivalent to the fact that bisimulation implies equality. The latter is a proof principle. As it stands it does not seem to be of the same character as the principle of induction as a proof principle. However, bisimulation is a coalgebra, and proofs of bisimulation can therefore be carried out corecursively, and that will give rise to the dual of an induction hypothesis, namely a coinduction hypothesis. This way we obtain proof principle which we believe is of similar character as induction. We will elaborate this in Sect. 9.3 where we will introduce schemata for coinduction.

We therefore call the fact that bisimulation implies equality the principle of coinduction:

Definition 8.4 Let $(\mathrm{Tree}^{\infty}, \mathrm{E})$ be an F-coalgebra.

By "$(\mathrm{Tree}^{\infty}, \mathrm{E})$ satisfies the principle of coinduction" we mean that it satisfies the principle of corecursion and for the final $\mathrm{F}^{\mathrm{Bisim}}$-coalgebra $(\mathrm{Bisim}, \mathrm{E}')$ we have

$$\forall i,t,t'.\mathrm{Bisim}(i,t,t') \to t = t.$$

Remark 8.5 Note that since proofs by bisimulation can be carried out by corecursion on $\mathrm{Bisim}(i,t,t')$ the principle of coinduction becomes a proper proof principle.

Lemma 8.6 *Let* $(\mathrm{Tree}^\infty, \mathrm{E})$ *be an* F*-coalgebra. The following is equivalent*

(i) $(\mathrm{Tree}^\infty, \mathrm{E})$ *is a final* F*-coalgebra.*
(ii) $(\mathrm{Tree}^\infty, \mathrm{E})$ *satisfies the principle of corecursion and for any* $\mathrm{F}^{\mathrm{Bisim}}$*-coalgebra* (X, h) *we have*
$\forall i, t, t'.X(i, t, t') \to t = t'$.
(iii) $(\mathrm{Tree}^\infty, \mathrm{E})$ *satisfies the principle of coinduction.*

Proof By Lemma 6.2, in (i)–(iii) the principle of corecursion is satisfied.

(i) implies (ii): Assume (X, h) is a $\mathrm{F}^{\mathrm{Bisim}}$-coalgebra.

Let

$$G : \mathrm{I} \to \mathrm{Set},$$
$$G(i) := \{\langle t, t' \rangle \in \mathrm{Tree}^\infty \times \mathrm{Tree}^\infty \mid X(i, t, t')\}.$$

Define

$$g : G \to \mathrm{F}(G),$$
$$g(i, \langle t, t' \rangle) = \langle \mathrm{E}_1(i, t), \lambda b.\langle \mathrm{E}_1(i, t, b), \mathrm{E}_1(i, t', b)\rangle\rangle\}.$$

Consider

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\exists!h"'] & \mathrm{F}(G) \arrow[d, "\mathrm{F}(h)"] \\
\mathrm{Tree}^\infty \arrow[r, "\mathrm{E}"] & \mathrm{F}(\mathrm{Tree}^\infty)
\end{tikzcd}$$

There exists a unique h which makes this diagram commute. Both the first and second projection (lifted to $\mathrm{Set}^{\mathrm{I}}$) make this diagram commute. By uniqueness follows they are equal and therefore the assertion follows.

(ii) implies (iii) is obvious since by the previous section there exist such a coalgebra.

(iii) implies (ii) since for any $\mathrm{F}^{\mathrm{Bisim}}$-coalgebra (X, h) we obtain a function $f : X \to \mathrm{Bisim}$. Therefore that $X(i, t, t')$ is inhabited implies that $\mathrm{Bisim}(i, t, t')$ is inhabited.

(ii) implies (i): Let (X, g) be an F-coalgebra and assume h, h' are two solutions which make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, shift right=2, "h"'] \arrow[d, shift left=2, "h'"] & \mathrm{F}(X) \arrow[d, shift right=2, "\mathrm{F}(h)"'] \arrow[d, shift left=2, "\mathrm{F}(h')"] \\
\mathrm{Tree}^\infty \arrow[r, "\mathrm{E}"] & \mathrm{F}(\mathrm{Tree}^\infty)
\end{tikzcd}$$

Let $g(i, x) = \langle \widehat{a}(i, x), \lambda b.\widehat{x}(i, x, b)\rangle$.

Let

$$\begin{aligned}&H : (i \in \mathrm{I}) \to \mathrm{Tree}^{\infty}(i) \to \mathrm{Tree}^{\infty}(i) \to \mathrm{Set},\\ &H(i, t, t') = \{x \in X(i) \mid t = h(i, x), t' = h'(i, x)\}.\end{aligned}$$

It follows for $i \in \mathrm{I}$, $x \in X(i)$ and therefore $x \in H(i, h(i, x), h'(i, x))$ that

$$\mathrm{E}_0(i, h(i, x)) = \pi_0(\mathrm{F}(h)(i, g(i, x))) = \pi_0(g(i, x)) = \mathrm{E}_0(i, h'(i, x))$$

and for $b \in \mathrm{B}(i, \mathrm{E}_0(i, h(i, x)))$

$$\begin{aligned}\mathrm{E}_1(i, h(i, x), b) &= \pi_1(\mathrm{F}(h)(i, g(i, x)))(b)\\ &= h(\mathrm{j}(i, \pi_0(g(i, x)), b), \pi_1(g(i, x))(b))\\ &= h(\mathrm{j}(\cdots), \widehat{x}(i, x, b)),\\ \mathrm{E}_1(i, h'(i, x), b) &= h'(\mathrm{j}(\cdots), \widehat{x}(i, x, b)),\\ \widehat{\mathrm{x}}(i, x, b) &\in H(\mathrm{j}(\cdots), \mathrm{E}_1(i, h(i, x), b), \mathrm{E}_1(i, h'(i, x), b)),\\ \langle *, \lambda b.\widehat{\mathrm{x}}(i, x, b)\rangle &\in \mathrm{F}^{\mathrm{Bisim}}(H)(i, x, b),\\ \lambda i, x.\langle *, \lambda b.\widehat{\mathrm{x}}(i, x, b)\rangle &\in H \to \mathrm{F}^{\mathrm{Bisim}}(H).\end{aligned}$$

Therefore (H, h) is an $\mathrm{F}^{\mathrm{Bisim}}$-coalgebra, $H(i, t, t')$ inhabited implies $t = t'$, and therefore $\forall x \in X(i).h(i, x) = h'(i, x)$.

Main Theorem 8.7 *Assume* Tree^{∞} *is a set,* $\mathrm{E} : \mathrm{Tree}^{\infty} \to \mathrm{F}(\mathrm{Tree}^{\infty})$*, and let* $\mathrm{E}_1, \mathrm{E}_2$ *be the two components of* E *as defined before. The following are equivalent*

(a) $(\mathrm{Tree}^{\infty}, \mathrm{E})$ *is a final* F*-coalgebra.*
(b) $(\mathrm{Tree}^{\infty}, \mathrm{E})$ *satisfies the principle of unique coiteration.*
(c) $(\mathrm{Tree}^{\infty}, \mathrm{E})$ *satisfies the principle of unique corecursion.*
(d) $(\mathrm{Tree}^{\infty}, \mathrm{E})$ *satisfies the principle of coinduction.*

Proof By Lemmata 6.2 and 8.6.

9 Schemata for Corecursive Definitions and Coinductive Proofs

9.1 Schema for Corecursion

By Lemma 6.2 we can introduce elements of the coinductively defined set (final F-coalgebra) ($[\![\, \mathrm{Tree}^{\infty} \,]\!]$, E) as follows:

Assume $A : \mathrm{I} \to \mathrm{Set}$, $[\![\,\mathrm{Tree}^{\infty}\,]\!]$, E_1, E_2 as before. We can define a function

$$f : (i \in \mathrm{I}) \to X(i) \to [\![\,\mathrm{Tree}^{\infty}\,]\!](i)$$

corecursively by defining for $i \in \mathrm{I}$, $x \in X(i)$

- a value $a' := \mathrm{E}_1(i, f(i, x)) \in \mathrm{A}(i)$
- and for $b \in \mathrm{B}(i, a)$ a value $\mathrm{E}_2(i, f(i, x), b) \in [\![\,\mathrm{Tree}^{\infty}\,]\!](i', b)$ where $i' := \mathrm{j}(i, a', b)$ and we can define $\mathrm{E}_2(i, f(i, x), b)$
  - as an element of $[\![\,\mathrm{Tree}^{\infty}\,]\!](i')$ defined before
  - or corecursively define $\mathrm{E}_2(i, f(i, x), b) = f(i', x')$ for some $x' \in X(i')$. Here, $f(i', x')$ will be called the corecursion hypothesis.

As a simple example we consider Streams. Streams are the final F-coalgebra on Set with $\mathrm{F}(X) = \mathbb{N} \times X$. So we have $\mathrm{I} = \mathbf{1}$, $\mathrm{A}(*) = \mathbb{N}$, $\mathrm{B}(*, x) = \mathbf{1}$. Omitting the arguments in $\mathbf{1}$ we obtain $\mathrm{F}(X)$ as above. Let (Stream, E) be the final F-coalgebra, and let head, tail be the two components of E. Then we get

$$\begin{array}{lcl} \mathrm{head} & : & \mathrm{Stream} \to \mathbb{N}, \\ \mathrm{tail} & : & \mathrm{Stream} \to \mathrm{Stream}. \end{array}$$

The above schema is instantiated as follows:

Let $A \in \mathrm{Set}$. We can define

$$f : A \to \mathrm{Stream}$$

corecursively by defining for $a \in A$

- $\mathrm{head}(f(a)) \in \mathbb{N}$ and
- $\mathrm{tail}(f(a)) \in \mathrm{Stream}$, where for defining $\mathrm{tail}(f(a))$ we can
  - either return an element of Stream defined before or
  - corecursively define $\mathrm{tail}(f(a)) = f(a')$ for some $a' \in A$. Here, $f(a')$ will be called the corecursion hypothesis.

So we can for instance define by corecursion

$$\begin{array}{lcl} s \in \mathrm{Stream} & & \text{s.t.} \\ \mathrm{head}(s) & = & 0, \\ \mathrm{tail}(s) & = & s. \end{array}$$

(Here, $A = \mathbf{1}$, and we omit the argument in A.) Or we define

$$\begin{array}{lcl} s' : \mathbb{N} \to \mathrm{Stream} & & \text{s.t.} \\ \mathrm{head}(s'(n)) & = & 0, \\ \mathrm{tail}(s'(n)) & = & s'(n+1). \end{array}$$

or define

$$\begin{array}{lcl} \mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream} & & \text{s.t.} \\ \mathrm{head}(\mathrm{cons}(n, s)) & = & n, \\ \mathrm{tail}(\mathrm{cons}(n, s)) & = & s. \end{array}$$

(Here, $A = \mathbb{N} \times \mathrm{Stream}$, and we curried the function.)

9.2 Schema for Corecursively Defined Indexed Functions

By Lemma 7.1 we have the following schema:

Assume $X \in \mathrm{Set}$, $\widehat{j} : X \to \mathrm{I}$.
We can define

$$f : (x \in X) \to [\![\, \mathrm{Tree}^{\infty} \,]\!](\widehat{i}(x))$$

corecursively by determining for $x \in X$ with $i := \widehat{j}(x)$,

- $a := \mathrm{E}_1(i, f(x)) \in \mathrm{A}(i)$
- and for $b \in \mathrm{B}(i, a)$ with $i' := \mathrm{j}(i, a, b)$ the value $\mathrm{E}_2(i, f(x), b) \in [\![\, \mathrm{Tree}^{\infty} \,]\!](i')$ where we can define $\mathrm{E}_2(i, f(x), b)$ as
  - a previously defined value of $[\![\, \mathrm{Tree}^{\infty} \,]\!](i')$
  - or corecursively define $\mathrm{E}_2(i, f(x), b) = f(x')$ for some x' such that $\widehat{i}(x') = i'$. $f(x')$ will be called the corecursion hypothesis.

As an example consider the coinductively defined set of stacks of a certain height, $\mathrm{Stack} : \mathbb{N} \to \mathrm{Set}$ with destructors

$$\begin{array}{lcl} \mathrm{top} & : & (n \in \mathbb{N}) \to (n > 0) \to \mathrm{Stack}(n) \to \mathbb{N}, \\ \mathrm{pop} & : & (n \in \mathbb{N}) \to (n > 0) \to \mathrm{Stack}(n) \to \mathrm{Stack}(n-1). \end{array}$$

We can define $\mathrm{empty} : \mathrm{Stack}(0)$, where we do not need to define anything since $(0 \widehat{>} 0) = \emptyset$. Furthermore, we can define

$$\begin{array}{l} \mathrm{push} : (n, m \in \mathbb{N}) \to \mathrm{Stack}(n) \to \mathrm{Stack}(n+1) \qquad \text{s.t.} \\ \mathrm{top}(n+1, *, \mathrm{push}(n, m, s)) \; = m, \\ \mathrm{pop}(n+1, *, \mathrm{push}(n, m, s)) = s. \end{array}$$

More complicated examples of indexed coinductively defined sets are state-dependent interactive programs, see [15–18], or bisimulation relations as defined below.

9.3 Schema for Coinduction

When proving that elements of a coinductively defined set are bisimilar one usually defines certain elements which should be shown to be bisimilar simultaneously. This amounts to having

$$\begin{array}{lcl} J & : & \mathrm{Set}, \\ \widehat{i} & : & J \to \mathrm{I}, \\ x_0, x_1 & : & (j \in J) \to \llbracket \mathrm{Tree}^{\infty} \rrbracket(\widehat{i}(j)) \end{array}$$

and showing $\forall j \in J.x_0(j) = x_1(j)$ by proving $\forall j \in J.\mathrm{Bisim}(\widehat{i}(j), x_0(j), x_1(j))$.

Using the same method as in the previous subsection, and using the fact that $b \in \mathrm{Bisim}(i, x, x')$ implies equality, we can show this statement by coinduction by showing the following:

$$\begin{array}{l} \forall j \in J.\mathrm{E}_0(\widehat{i}(j), x_0(j)) = \mathrm{E}_0(\widehat{i}(j), x_1(j)) \wedge \\ \qquad\qquad \forall b \in \mathrm{B}(i, \mathrm{E}_0(\widehat{i}(j), x_0(j))).\mathrm{E}_1(\widehat{i}(j), x_0(j), b) = \mathrm{E}_1(\widehat{i}(j), x_1(j), b) \vee \\ \qquad\qquad\qquad\qquad\qquad\qquad \exists j'.\widehat{i}(j') = \mathrm{j}(\widehat{i}(j), \mathrm{E}_0(\widehat{i}(j), x_0(j), b)) \wedge \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad x_0(j') = \mathrm{E}_0(\widehat{i}(j), x_0(j), b) \wedge \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad x_1(j') = \mathrm{E}_0(\widehat{i}(j), x_1(j), b). \end{array}$$

This means that we have the following principle of coinductive proofs:

Assume

$$\begin{array}{lcl} J & : & \mathrm{Set}, \\ \widehat{i} & : & J \to \mathrm{I}, \\ x_0, x_1 & : & (j \in J) \to \llbracket \mathrm{Tree}^{\infty} \rrbracket(\widehat{i}(j)). \end{array}$$

We can show $\forall j \in J.x_0(j) = x_0(j')$ coinductively by showing

- $\mathrm{E}_0(\widehat{i}(j), x_0(j))$ and $\mathrm{E}_0(\widehat{i}(j), x_1(j))$ are equal
- and for all b that
  $\mathrm{E}_1(\widehat{i}(j), x_0(j), b)$ and $\mathrm{E}_1(\widehat{i}(j), x_0(j), b)$ are equal,
  where we can use either the fact that
  - this was shown before,
  - or we can use the coinduction-hypothesis, which means using the fact $\mathrm{E}_1(\widehat{i}(j), x_0(j), b) = x_0(j')$ and $\mathrm{E}_1(\widehat{i}(j), x_1(j), b) = x_1(j')$ for some $j' \in J$.

Examples of proofs by coinduction in the example of streams with s, s', cons are as follows:

- We show $\forall n \in \mathbb{N}. s = s'(n)$ by coinduction: For using the schema above we have $J = \mathbb{N}$, $x_0(j) = s$, $x_1(j) = s'(n)$. The argument is as follows:
  We show $\forall n \in \mathbb{N}. s = s'(n)$. Assume $n \in \mathbb{N}$. $\mathrm{head}(s) = \mathrm{head}(s'(n))$ and $\mathrm{tail}(s) = s = s'(n+1) = \mathrm{tail}(s'(n))$, where $s = s'(n+1)$ follows by the coinduction hypothesis.
- We show $\mathrm{cons}(0, s) = s$ by coinduction:
  $\mathrm{head}(\mathrm{cons}(0, s)) = 0 = \mathrm{head}(s)$ and $\mathrm{tail}(\mathrm{cons}(0, s)) = s = \mathrm{tail}(s)$, where we did not use the coinduction hypothesis.

9.4 Schema for Coinductively Defined Relations

The previous example can be generalised to arbitrary coinductively defined sets relating elements of an indexed set. A typical example would be the bisimulation relation on a labelled transition system, which we consider below. Assume $\mathrm{I} \in \mathrm{Set}$, $D : \mathrm{I} \to \mathrm{Set}$ (not necessarily a coinductively defined set). Let

$$\mathrm{I}^+ := (i \in \mathrm{I}) \times D(i) \times D(i).$$

Assume

$$\begin{array}{lll} A & : & (i \in \mathrm{I}) \to D(i) \to D(i) \to \mathrm{Set}, \\ B & : & (i \in \mathrm{I}) \to (d \in D(i)) \to (d' \in D(i)) \to A(i, d, d') \to \mathrm{Set}, \\ j & : & (i \in \mathrm{I}) \to (d \in D(i)) \to (d' \in D(i)) \to (a \in A(i, d, d')) \\ & & \to B(i, d, d', a) \to \mathrm{I}, \\ d_0, d_1 & : & (i \in \mathrm{I}) \to (d \in D(i)) \to (d' \in D(i)) \to (a \in A(i, d, d')) \\ & & \to B(i, d, d', a) \to D(j(i, d, d', a)). \end{array}$$

Define

$$\begin{array}{l} \mathrm{F} : \mathrm{Set}^{\mathrm{I}^+} \to \mathrm{Set}^{\mathrm{I}^+}, \\ \mathrm{F}(X, i, d, d') = (a \in A(i, d, d')) \\ \qquad\qquad \times ((b \in B(i, d, d', a)) \\ \qquad\qquad\quad \to X(j(i, d, d', a, b), d_0(i, d, d', a, b), d_1(i, d, d', a, b))). \end{array}$$

Let $(\widehat{\mathrm{B}}, \mathrm{E})$ be the final F-coalgebra, $\mathrm{E}_1, \mathrm{E}_2$ be the two components of E.
Assume

$$\begin{array}{lll} \widehat{J} & : & \mathrm{Set}, \\ \widehat{j} & : & \widehat{J} \to \mathrm{I}, \\ \widehat{d_0}, \widehat{d_1} & : & (j \in \widehat{J}) \to D(\widehat{j}(j)). \end{array}$$

A schema of corecursion (which may be called, if $\widehat{\mathrm{B}}$ is a bisimulation relation and therefore an equality like relation as a coinduction principle) is as follows:

In the above situation we can define a function

$$\widehat{\mathrm{b}} : (j \in J) \to \widehat{\mathrm{B}}(\widehat{j}(j), \widehat{d_0}(j), \widehat{d_1}(j))$$

coinductively by determining for $j \in J$

- an element $\widehat{\mathrm{a}}(j) \in A(\widehat{j}(j), \widehat{d_0}(j), \widehat{d_1}(j))$,
- and for $b \in B(\widehat{j}(j), \widehat{d_0}(j), \widehat{d_1}(j), \widehat{\mathrm{a}}(j))$
  with $i' := j(\widehat{\mathrm{j}}(j), \widehat{d_0}(j), \widehat{d_1}(j), \widehat{\mathrm{a}}(j), b)$,
  $d_i' := d_i(\widehat{\mathrm{j}}(j), \widehat{d_0}(j), \widehat{d_1}(j), \widehat{\mathrm{a}}(j), b)$,
  an element $\widehat{\mathrm{b}}' \in \widehat{\mathrm{B}}(i', d_0', d_1')$,
  where for defining $\widehat{\mathrm{b}}'$ we can use
  - an existing element of $\widehat{\mathrm{B}}(i', d_0', d_1')$
  - or corecursively define $\widehat{\mathrm{b}}' = \widehat{\mathrm{b}}(j')$ for some j'
    such that $\widehat{j}(j') = i', \widehat{d_0}(j') = d_0', \widehat{d_1}(j') = d_1'$.
    $\widehat{\mathrm{b}}(j')$ will be called the corecursion-hypothesis.

As an example we consider bisimulation for a labelled transition system. A labelled transition system is given by set of states S, a set of labels L a relation $\longrightarrow \subseteq \mathrm{S} \times \mathrm{L} \times \mathrm{S}$ where we write $s \xrightarrow{l} s'$ for $\langle s, l, s' \rangle \in \longrightarrow$. Bisimulation $\mathrm{Bisim}(s, s')$ in a transition system can be given by the coalgebraically defined relation $\mathrm{Bisim}(s, s')$ for the eliminators

$$\begin{aligned}
\mathrm{E}_1 \quad : \quad & (s, s' \in \mathrm{S}) \to \mathrm{Bisim}(s, s') \to (l \in \mathrm{L}) \to (s_0 \in \{s_0 \in \mathrm{S} \mid s \xrightarrow{l} s_0\}) \\
& \to ((s_0' \in \{s_0' \in \mathrm{S} \mid s' \xrightarrow{l} s_0'\}) \times \mathrm{Bisim}(s_0, s_0')), \\
\mathrm{E}_2 \quad : \quad & (s, s' \in \mathrm{S}) \to \mathrm{Bisim}(s, s') \to (l \in \mathrm{L}) \to (s_0' \in \{s_0' \in \mathrm{S} \mid s' \xrightarrow{l} s_0'\}) \\
& \to ((s_0 \in \{s_0 \in \mathrm{S} \mid s \xrightarrow{l} s_0\}) \times \mathrm{Bisim}(s_0, s_0')).
\end{aligned}$$

The existence of E_1 and E_2 is equivalent to

$$\begin{aligned}
& \forall s, s' \in \mathrm{S}.\mathrm{Bisim}(s, s') \to \forall l \in \mathrm{L}.\forall s_0 \in \mathrm{S}.(s \xrightarrow{l} s_0) \\
& \qquad \to \exists s_0' \in \mathrm{S}.s' \xrightarrow{l} s_0' \wedge \mathrm{Bisim}(s_0, s_0'), \\
& \forall s, s' \in \mathrm{S}.\mathrm{Bisim}(s, s') \to \forall l \in \mathrm{L}.\forall s_0' \in \mathrm{S}.(s' \xrightarrow{l} s_0') \\
& \qquad \to \exists s_0 \in \mathrm{S}.s \xrightarrow{l} s_0 \wedge \mathrm{Bisim}(s_0, s_0').
\end{aligned}$$

Note that the type of E_1 is equivalent to

$$
\begin{aligned}
\mathrm{E}_1' : {} & (s, s' \in \mathrm{S}) \to \mathrm{Bisim}(s, s') \\
& \to (s_0' \in (l \in \mathrm{L}) \to \{s_0 \in \{s_0 \in \mathrm{S} \mid s \xrightarrow{l} s_0\} \to \mathrm{S}) \\
& \qquad \times ((\langle l, s_0 \rangle \in ((l \in \mathrm{L}) \times \{s_0 \in \mathrm{S} \mid s \xrightarrow{l} s_0\})) \to \mathrm{Bisim}(s_0, s_0'(l, s_0)))
\end{aligned}
$$

similarly for E_2 and both constructors can be unified into one. Therefore this relation is an instance of a strictly positive indexed coinductively defined set as defined in this article.

A proof of bisimulation by corecursion can be done by using the following schema: Let $I \in \mathrm{Set}$, $s, s' : I \to \mathrm{S}$.

We can prove $\forall i \in \mathrm{I}.\mathrm{Bisim}(s(i), s'(i))$ coinductively by defining for any $i \in \mathrm{I}$

- for any $l \in \mathrm{L}$, $s_0 \in \mathrm{S}$ s.t. $s(i) \xrightarrow{l} s_0$ an $s_0' \in \mathrm{S}$ s.t.
  - $s'(i) \xrightarrow{l} s_0'$
  - and s.t. $\mathrm{Bisim}(s_0, s_0')$ where one can for prove the latter by invoking the Coinduction Hypothesis $\mathrm{Bisim}(s(i'), s'(i'))$ for some i' such that $s(i') = s_0$, $s'(i') = s_0'$.
- for any $l \in \mathrm{L}$, $s_0' \in \mathrm{S}$ s.t. $s'(i) \xrightarrow{l} s_0'$ an $s_0 \in \mathrm{S}$ s.t.
  - $s(i) \xrightarrow{l} s_0$
  - and s.t. $\mathrm{Bisim}(s_0, s_0')$ where one can prove the latter by invoking the Coinduction Hypothesis $\mathrm{Bisim}(s(i'), s'(i'))$ for some i' such that $s(i') = s_0$, $s'(i') = s_0'$.

As an example consider $\mathrm{S} = \{*\} \cup \mathbb{N}$, $\mathrm{L} = \mathrm{tick}$ with transitions $* \xrightarrow{\mathrm{tick}} *$ and $n \xrightarrow{\mathrm{tick}} (n+1)$. We show $\forall n \in \mathbb{N}.\mathrm{Bisim}(*, n)$ by coinduction on Bisim. Assume $n \in \mathbb{N}$.

Assume $* \xrightarrow{l} s$. Then $l = \mathrm{tick}$, $s = *$, $n \xrightarrow{\mathrm{tick}} (n+1)$ and by co-IH $\mathrm{Bisim}(*, n+1)$.

Assume $n \xrightarrow{l} s$. Then $l = \mathrm{tick}$, $s = n+1$, $* \xrightarrow{\mathrm{tick}} *$ and by co-IH $\mathrm{Bisim}(*, n+1)$.

10 Conclusion

We have investigated indexed inductive and coinductively defined sets and shown that induction is equivalent to the initial algebra definition and coinduction, corecursion and coinduction are equivalent. We have developed schemata for defining informally elements of coinductively defined sets corecursively and for proving equality of elements by coinduction. We have seen that examples how to actually carry out such definitions and proofs informally.

We belief that carrying out such arguments about coinductively defined sets by corecursion and coinduction informally while referring to the coinduction and corecursion hypothesis makes it more intuitive to carry out such arguments. We hope that it will in the future become as natural to carry out such arguments as it has become natural to define functions into inductively defined sets by primitive recursion and to prove properties by induction in an intuitive way.

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Pointwise Transfinite Induction and a Miniaturized Predicativity

Stanley S. Wainer

For Gerhard Jäger at his 60th.

Abstract The basis of this work is Leivant's [6] theory of ramified induction over N, which has elementary recursive strength. It has been redeveloped and extended in various ways by many people; for example, Spoors and Wainer [13] built a hierarchy of ramified theories whose strengths correspond to the levels of the Grzegorczyk hierarchy. Here, a further extension of this hierarchy is developed, in terms of a predicatively generated infinitary calculus with stratifications of numerical inputs up to and including level ω. The autonomous ordinals are those below Γ_0, but they are generated according to a weak (though quite natural) notion of transfinite induction whose computational strength is "slow" rather than "fast" growing. It turns out that the provably computable functions are now those elementary recursive in the Ackermann function (i.e. Grzegorczyk's ωth level). All this is closely analogous to recent works of Jäger and Probst [5] and Ranzi and Strahm [9] on iterated stratified inductive definitions, but their theories have full, complete induction as basis, whereas ours have only a weak, ramified form of numerical induction at bottom.

1 Introduction

The theory EA(I;O) of Ostrin and Wainer [8] is a stripped-down version of the ramified intrinsic theories of Leivant [6], designed to incorporate the "normal/safe" variable discipline of Bellantoni and Cook [1], which has its origins in an earlier analysis of primitive recursion given by Simmons [12]. A two-sorted analogue of

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Peano arithmetic is developed, with a weaker, “pointwise” or “predicative” induction scheme:

$$A(0) \wedge \forall a(A(a) \to A(a+1)) \to A(x)$$

where a is a safe variable and x is normal. We prefer to call them “output” variables and “input” variables respectively; hence the I;O notation. Input variables are not (at this stage) quantified, so they act as uninterpreted constants. The usual proof theoretic methods apply just as for PA (e.g. embedding and cut elimination in an infinitary arithmetic with ω-rule). But now, because the inductions are only “up to x”, which cannot be substituted for, the natural bounding functions are supplied by the “slow growing” hierarchy rather than the “fast growing” one. Since the slow growing functions below ε_0 are the exponential polynomials, and those below ω^ω are just polynomials, it follows (as Leivant had already previously shown, by different methods) that the provably computable functions of EA(I;O) are the elementary functions (Grzegorczyk’s $\mathcal{E}^3$) and those provably computable in its Σ_1-inductive fragment are the sub-elementary $\mathcal{E}^2$ functions, i.e. those Turing–machine computable in linear space. (By shifting to a binary, rather than our unary, representation of numbers, one sees that the Σ_1-inductive fragment then characterizes polytime.)

Though quite simple in its formulation, EA(I;O) is not very “user friendly”, as it does not permit quantification over inputs x, y, z, and therefore one cannot even show straightforwardly that the provably computable functions—as functions on inputs—are closed under composition. Of course it is true, and Wirz [15] supplies a variety of delicate proof theoretic analyses enabling the derivation of such results, but they also serve to highlight the awkwardness of the logic of EA(I;O). One way to rectify this, as in Spoors and Wainer [13], is to extend the theory conservatively to a new theory EA(I;O)$^+$ allowing quantification over inputs and incorporating also a certain “Σ_1 Reflection Rule” previously employed by Cantini [3]: from $\exists a A(\vec{x}, a)$ derive $\exists y A(\vec{x}, y)$ provided all the free parameters $\vec{x}$ are inputs. The induction however, continues to apply only to formulas of the base theory EA(I;O). One then sees that EA(I;O)$^+$ forms just the first level of a ramified hierarchy of input/output theories, whose provably computable functions coincide, level-by-level, with the Grzegorczyk hierarchy. We first briefly review the main content of Spoors and Wainer [13].

1.1 Input-Output Arithmetic EA(I;O)$^+$

EA(I;O) has the language of arithmetic, with quantified “output” (or “safe”) variables $a, b, c, \ldots$ and unquantified “input” (or “normal”) variables $x, y, z, \ldots$. For convenience other terms and defining axioms are added, for a pairing function $\pi(a, b)$ $(:= 1/2(a+b)(a+b+1)+a+1)$ with inverses π_0, π_1, from which sequence numbers can be constructed using $\pi(s, a)$ to append a to s, and deconstructed by functions $(s)_i$ extracting the ith component. All of these initial functions are subelementary, in fact quadratically bounded. The induction axioms are:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow A(t)$$

where $t = t(\vec{x})$ is a *term on inputs only*, controlling induction-length. Note that if $A(a)$ is progressive then so is $\forall b \leq a.A(b) \equiv \forall b(b \leq a \rightarrow A(b))$, and so a more revealing instance of induction is

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow \forall b \leq t.A(b)\,.$$

In other words, EA(I;O) is essentially a theory of bounded induction, the (implicit) bounds being terms $t(\vec{x})$ dependent on inputs $\vec{x}$ which cannot be universally quantified and then later re-instantiated, as they can be in PA. Call this "input" or "predicative" induction. Note however that there is no restriction on the quantifier complexity of the formula A.

Definition 1.1 *A numerical function* $f : N^k \rightarrow N$ *is "provably computable" or "provably recursive" in EA(I;O) if there is a* Σ_1 *formula* $C_f(\vec{x}, a)$ *(i.e. a bounded formula prefixed by unbounded existential quantifiers) such that* $f(\vec{n}) = m$ *if and only if* $C_f(\vec{n}, m)$ *is true, and EA(I;O)* $\vdash \exists! a C_f(\vec{x}, a)$*, i.e.* f *is provably total on inputs. We shall occasionally use the shorthand* $f(\vec{x})\downarrow$ *for the formula* $\exists a C_f(\vec{x}, a)$.

Theorem 1.2 (Leivant, Ostrin-Wainer) *The provably computable functions of EA(I;O) are exactly the Csillag-Kalmar elementary functions.*

Note 1.3 By carefully restricting the witnessing terms in the existential introduction rule and in induction, to the so-called "basic" ones (i.e. those built out of unary term constructors only: successor, predecessor, π_i) one may also characterize the sub-elementary functions as those provably computable in the Σ_1 inductive fragment. Then increasing induction complexity in EA(I;O) characterizes the successive levels of the Ritchie-Schwichtenberg hierarchy between sub-elementary and elementary.

One sees immediately the deficiencies in the logic of EA(I;O) if one tries to show, simply and directly, that the provably computable functions are closed under composition. For suppose one has proved that $f(x)\downarrow$ and $g(x)\downarrow$, i.e. $\exists a C_g(x, a)$. Then one needs first to reflect the value a of $g(x)$ as an input—*the guiding principle here, is that values a computable from inputs only may themselves be used as inputs.* Thus one obtains $\exists y C_g(x, y)$ and by generalizing over inputs, $\forall y(f(y)\downarrow)$. Then by logic,

$$\exists y C_g(x, y),\ \forall y(f(y)\downarrow) \vdash \exists a, b(C_g(x, b) \wedge C_f(b, a))$$

so by two cuts one immediately derives $\exists a, b(C_g(x, b) \wedge C_f(b, a))$ which is the desired $\exists a C_{f \circ g}(x, a)$.

We therefore now extend EA(I;O) to the new theory EA(I;O)$^+$ by adding Input-quantifier rules (in Tait style):

$$\frac{\Gamma,\ A(t)}{\Gamma,\ \exists x A(x)} \qquad \frac{\Gamma,\ A(y)}{\Gamma,\ \forall x A(x)}$$

(provided t contains only input variables and y is not free in Γ) and also the Σ_1 Reflection rule:

$$\frac{\Gamma(\vec{x}),\ \exists\vec{a}A(\vec{x},\vec{a})}{\Gamma(\vec{x}),\ \exists\vec{y}A(\vec{x},\vec{y})}$$

where $\Gamma(\vec{x}),\ \exists\vec{a}A(\vec{x},\vec{a})$ is a set of Σ_1 formulas all of whose free variables are inputs.

The induction in EA(I;O)$^+$ is the same as that of EA(I;O), it only applies to formulas without input quantifiers. This extension of EA(I;O) is conservative, in that no new provably recursive functions are produced. Upper bounds on provable recursiveness in EA(I;O)$^+$, and its various extensions below, will be obtained by the usual proof theoretic process—embedding into a suitable infinitary system which admits cut elimination.

1.2 EA(I$_2$; I$_1$,O)$^+$, EA(I$_k$; ...I$_2$, I$_1$,O)$^+$ Etc

We now introduce a new level of input variables, and a new tier of inductions over EA(I; O)$^+$ formulas. Henceforth denote I by I$_1$, and call x, y, z the I$_1$ variables. Then the I$_2$ variables are new variables, denoted u, v, w. Add to EA(I$_1$; O)$^+$ these new I$_2$ variables and the new induction principle:

$$A(0) \wedge \forall x(A(x) \to A(x+1)) \to A(t)$$

where A is an EA(I$_1$; O)$^+$-formula, possibly with free I$_2$ parameters, and $t = t(\vec{u})$ is a term containing only I$_2$ variables. This theory is denoted EA(I$_2$; I$_1$, O). Its extension EA(I$_2$; I$_1$, O)$^+$ is obtained by further adding I$_2$-quantifier rules and a Σ_1 reflection rule at level 2:

$$\frac{\Gamma(\vec{u}),\ \exists\vec{x}A(\vec{u},\vec{x})}{\Gamma(\vec{u}),\ \exists\vec{v}A(\vec{u},\vec{v})}$$

where $\Gamma(\vec{u}),\ \exists\vec{x}A(\vec{u},\vec{x})$ is a set of Σ_1 formulas all of whose free variables are I$_2$ inputs.

Definition 1.4 A function f is provably computable in EA(I$_2$; I$_1$, O)$^+$ if, on level-2 inputs $\vec{u}$, it has a Σ_1 defining formula C_f for which $f(\vec{u})\!\downarrow\ \equiv\ \exists a C_f(\vec{u}, a)$ is provable.

Note 1.5 Every function provably computable in EA(I$_1$; O)$^+$ is provably computable in EA(I$_2$; I$_1$, O), for by trivial applications of the level-2 induction principle above, one proves $\exists x(u = x)$ (that is I$_2$ is contained in I$_1$). Hence if $\exists a C_f(\vec{x}, a)$ is provable in EA(I$_1$;O)$^+$ then so is $\exists a C_f(\vec{u}, a)$ in EA(I$_2$; I$_1$, O).

Lemma 1.6 *The functions of Grzegorczyk's $\mathcal{E}^4$ are all provably computable in EA(I$_2$; I$_1$, O)$^+$.*

This is because $2^a\!\downarrow\ \to\ 2^{a+1}\!\downarrow$ is provable in EA(I$_1$; O), and therefore so is $2^x\!\downarrow$ by I$_1$ induction. Then by Σ_1 reflection, $\exists y(2^x = y)$ is EA(I$_1$; O)$^+$-provable and then so is

$\forall x \exists y(2^x = y)$. Now define the superexponential function 2_a by $2_0 = 1$ and $2_{a+1} = 2^{2_a}$. Then $(2_a = x) \to \exists y(2_{a+1} = y)$ and hence $\exists y(2_a = y) \to \exists y(2_{a+1} = y)$ are provable in EA(I_1; O)$^+$. So by I_2 induction in EA(I_2; I_1, O) one obtains $\exists y(2_u = y)$, and then $\forall u \exists v(2_u = v)$ becomes provable in EA(I_2; I_1, O)$^+$. This enables finite iterations of the superexponential, and these provide computational resource-bounds for all $\mathcal{E}^4$ functions.

Adding a new tier of inductions to EA(I_2; I_1, O)$^+$ produces EA(I_3; I_2, I_1, O) and then new I_3 quantifier and reflection rules generate EA(I_3; I_2, I_1, O)$^+$. The provably recursive functions are now those of $\mathcal{E}^5$, allowing one further level of primitive recursion. Repeating this process produces a hierarchy of finitistic theories exhausting primitive recursive arithmetic. The aim here is to investigate the effect of introducing a new input level I_ω "on the top". In order to explore its full potential, a predicatively generated infinitary logic EA(I_ω; …I_2, I_1, O)$^\infty$, with an autonomy condition based on correspondingly (very) weak principles of transfinite induction, seems to be needed.

2 Weak, Pointwise Transfinite Induction

For us, ordinals α, β, γ,..., will be countable "tree ordinals", meaning that each limit λ comes equipped with a uniquely determined fundamental sequence $\{\lambda_i\}_{i\in N}$. We shall only be concerned, here, with relatively small initial segments of the recursive ordinals (in particular Γ_0) so coding them up as ordinal notations will not be a problem, and we regard that as having been done. Thus we'll continue to use lower-case greek letters as variables for ordinals, although they will actually denote their numerical codes. The various functions and relations—e.g. determining whether a number is an ordinal notation, a limit, or a successor, and (if it's a limit) computing the ith member of its fundamental sequence, etcetera—will all be at least primitive recursive, and almost always elementary. See e.g. Feferman [4] for a detailed treatment.

A basic version of transfinite induction up to α is

$$A(0) \wedge \forall\beta(A(\beta) \to A(\beta+1)) \wedge \forall\lambda(\forall i A(\lambda_i) \to A(\lambda)) \to A(\alpha) \ .$$

Weak, pointwise-at-x transfinite induction up to α is the following principle:

$$A(0) \wedge \forall\beta(A(\beta) \to A(\beta+1)) \wedge \forall\lambda(A(\lambda_x) \to A(\lambda)) \to A(\alpha)$$

where x is a numerical input variable. We denote this principle $PTI(x, \alpha, A)$ or just $PTI(x, \alpha)$ when the formula A is understood.

Using this, we can immediately prove, with only a small amount of basic coding apparatus, that the pointwise-at-x descending sequence from α exists. That is

$$\exists s D(s, x, \alpha)$$

where $D(s, x, \alpha)$ is the bounded formula saying that s is the sequence number of ordinal notations such that $(s)_0 = 0$ and $(s)_{lh(s)-1} = \alpha$ and for each $i < lh(s) - 1$ either $(s)_{i+1}$ is a limit λ, in which case $(s)_i = \lambda_x$, or $(s)_{i+1}$ is a successor $\beta + 1$, in which case $(s)_i = \beta$.

Thus $\exists s D(s, x, \alpha)$ expresses the pointwise-at-x well-foundedness of α, and we often abbreviate it as $PWF(x, \alpha)$. The contrast between this Σ^0_1 notion and full Π^1_1 well-foundedness is stark, but even here there are interesting analogies to be drawn with classical proof theory. Whereas the natural subrecursive hierarchies of proof-theoretic bounding functions are "fast" growing in the classical case, they are "slow" growing in the pointwise case. For detailed comparisons between the two, see Schwichtenberg and Wainer [11], Weiermann [14]. Schmerl [10] was the first to formulate such weak, pointwise induction schemes in the context of Peano Arithmetic.

Definition 2.1 The functions L_x and G_x are defined as follows:
$L_x(\alpha) = a$ iff $\exists s(D(s, x, \alpha) \wedge a = lh(s) - 1)$,
$G_x(\alpha) = a$ iff $\exists s(D(s, x, \alpha) \wedge a = \#(s))$
where $\#(s)$ is the number of successor ordinals in the descending sequence s.

Lemma 2.2 *L_x and G_x satisfy the following recursive definitions:*
$L_x(0) = 0, \quad L_x(\beta + 1) = L_x(\beta) + 1, \quad L_x(\lambda) = L_x(\lambda_x) + 1.$
$G_x(0) = 0, \quad G_x(\beta + 1) = G_x(\beta) + 1, \quad G_x(\lambda) = G_x(\lambda_x).$

These functions, being given "pointwise-at-x", are alternative versions of the slow growing hierarchy, and they are both provably defined as immediate consequences of pointwise well-foundedness. They each have their uses, though L_x will be the one used mostly here. For natural systems of ordinal notations L_x and G_x both preserve addition, and in fact they collapse the arithmetic of tree ordinals homomorphically onto ordinary arithmetic, as will be seen in Sect. 4. One usually has $G_x(\alpha) \leq L_x(\alpha) \leq G_{x+1}(\alpha)$.

Of course, even to call $PTI(x, \alpha)$ a *transfinite* induction principle requires a stretch of the imagination, because it is really just a collection of finitary inductions indexed by x and uniformized by α. The following lemma brings this out more clearly.

Lemma 2.3 *In any arithmetical theory containing the basic coding apparatus, $PTI(x, \alpha)$ is provably equivalent to Numerical Induction up to $L_x(\alpha)$.*

Precisely, given any formula $F(a)$, let $A(\alpha) \equiv \forall a \leq L_x(\alpha).F(a)$ where $\forall a \leq L_x(\alpha).F(a)$ stands for $\exists b(L_x(\alpha) = b \wedge \forall a \leq b.F(a))$. Then one may prove

$$PTI(x, \alpha, A) \to (\ F(0) \wedge \forall b(F(b) \to F(b + 1)) \to \forall a \leq L_x(\alpha).F(a)\ ) .$$

Conversely, given any formula $A(\beta)$ let $F(b) \equiv \forall \beta \preceq_x \alpha(L_x(\beta) = b \to A(\beta))$ where $\beta \preceq_x \alpha$ means $\exists s(D(s, x, \alpha) \wedge \exists i < lh(s)((s)_i = \beta))$. Then

$$(\ F(0) \wedge \forall b(F(b) \to F(b + 1)) \to \forall a \leq L_x(\alpha).F(a)\ ) \to PTI(x, \alpha, A) .$$

Proof For the first part, it is only necessary to show that the progressiveness of F implies $A(0)$ and $\forall \beta(A(\beta) \to A(\beta+1))$ and $A(\lambda_x) \to A(\lambda)$ for limits λ. But $F(0)$ implies $\forall a \leq L_x(0).F(a) \equiv A(0)$. And if $\forall b(F(b) \to F(b+1))$ then $\forall \beta(\forall a \leq L_x(\beta).F(a) \to \forall a \leq L_x(\beta+1).F(a))$ and so $\forall \beta(A(\beta) \to A(\beta+1))$. The limit case $A(\lambda_x) \to A(\lambda)$ follows similarly. Therefore $PTI(x, \alpha, A)$ gives $A(\alpha) \equiv \forall a \leq L_x(\alpha).F(a)$.

For the converse, assume A is progressive, i.e. $A(0)$ and $\forall \beta(A(\beta) \to A(\beta+1))$ and $A(\lambda_x) \to A(\lambda)$ for limits λ. Then one easily proves $F(0)$ and for any b, $F(b) \to F(b+1)$. For assume $F(b)$. Then if $\beta \preceq_x \alpha$ and $L_x(\beta) = b+1$, β is either a successor or a limit and its immediate predecessor in the $\preceq_x$-sequence, call it δ, satisfies $L_x(\delta) = b$. Therefore $A(\delta)$ holds and, by the progressiveness of A one immediately gets $A(\beta)$. Hence $F(b+1)$, and so by numerical induction up to $L_x(\alpha)$ we then have $F(L_x(\alpha))$ and therefore $A(\alpha)$. This proves $PTI(x, \alpha, A)$.

Note 2.4 Since $PWF(x, \alpha)$ implies both $L_x(\alpha)$ and $G_x(\alpha)$ are defined, and clearly $G_x(\alpha) \leq L_x(\alpha)$, it immediately follows that $PTI(x, \alpha)$ implies numerical induction up to $G_x(\alpha)$. The converse in this case, however, seems to require also a numerical induction up to $L_x(\alpha)$. In this respect G_x is a little weaker than L_x.

3 The Infinitary System EA($\mathbf{I}_\omega, \ldots \mathbf{I}_2, \mathbf{I}_1, \mathbf{O}$)$^\infty$

The infinitary system derives Tait-style sequents with numerical input declarations:

$$n_\omega{:}I_\omega, \ldots n_j{:}I_j, \ldots n_1{:}I_1, n_0{:}I_0 \vdash^\alpha \Gamma \quad \text{abbreviated } \vec{n}{:}\vec{I} \vdash^\alpha \Gamma\,.$$

Γ is a finite set of closed formulas in the language of arithmetic augmented by elementary term constructors for coding sequences and ordinal notations, and the addition of two new "input predicates": a unary one $I_\omega(..)$ and a binary one written $I_j(n)$, together with their complements $\bar{I}_\omega(..)$ and $\bar{I}_j(n)$. The "level" of Γ is the greatest j such that $I_j(..)$ or $\bar{I}_j(..)$ occurs. It is said to be of "finite level" if I_ω does not occur. Level $O = I_0$ is to be thought of as the domain of all possible "output" values. An important convention will be that a declaration $n_j{:}I_j$ where $n_j = 0$ will often be suppressed (i.e. assumed and not explicitly stated). Of the declared inputs, only finitely-many will be non-zero. An obvious principle is that $n{:}I_j \vdash^\alpha A$ means $\vdash^\alpha I_j(n) \to A$ but the declarations $n_j{:}I_j$ will be kept separate from formulas. The ordinal bounds α on derivations will be notations below Γ_0 but it is not necessary to be too specific about that at this stage. The I_j predicates now distinguish the different levels of input.

The guiding principles are: (Refinement) these become more and more "rarified" or "refined" versions of N as the level j increases; (Stratification) inductions on formulas of finite level are controlled/bounded by inputs declared at a higher level; and (Computation) values computed only from inputs of level $\geq j$ may be regarded as residing at level j. (The Top Level) I_ω is to be thought of as the diagonal intersection: $I_\omega(n) \Rightarrow I_j(n)$ for all $j \le n$. Note that in such infinitary systems, Σ_1 reflection becomes a derived rule (see the comment in Sect. 3.5 below, or Lemma 7 of Spoors and Wainer [13]).

3.1 Logic Rules

Ordinal assignment. To ensure appropriate levels of stratification, the ordinal bounds β on the premises of the logic rules below, bear the following relationship to the ordinal bounds α assigned to the conclusions: (i) $\beta \prec_k \alpha$ where $k{:}I_i$ is a declared input at a level higher than the levels of all formulas in the premises and conclusion; (ii) otherwise, if $I_\omega(..)$ occurs in a formula, then α is a successor of β. (Recall that $\beta \prec_k \alpha$ means that β occurs in the k-descending chain from α: $0 \prec 1 \prec \cdots \prec \gamma \prec \gamma + 1 \prec \cdots \prec \alpha_{k,k} \prec \alpha_k \prec \alpha$ and so $L_k(\beta) < L_k(\alpha)$).

The Axioms are $\vec{n}{:}\vec{I} \vdash^\alpha \Gamma$ where the set Γ contains a true atom (e.g. an equation or inequation between closed terms, or $s \ne s'$, $t \ne t'$, $\bar{I}_s(t)$, $I_{s'}(t')$).

The Cut rule, with cut formula C, is

$$\frac{\vec{n}{:}\vec{I} \vdash^{\beta_0} \Gamma, \neg C \qquad \vec{n}{:}\vec{I} \vdash^{\beta_1} \Gamma, C}{\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma}$$

The $\exists$-rules are:

$$\frac{\vec{n}{:}\vec{I} \vdash^{\beta_0}_{k,C} I_j(m) \qquad \vec{n}{:}\vec{I} \vdash^{\beta_1} A(m), \Gamma}{\vec{n}{:}\vec{I} \vdash^{\alpha} \exists x(I_j(x) \wedge A(x)), \Gamma}$$

where, in addition, $\beta_0 + 1 \preceq_k \beta_1$. Here the left-hand premise "computes" witness m according to the computation rules given below.

The $\forall$-rules are versions of the ω-rule:

$$\frac{n_\omega{:}I_\omega, \ldots \max(n_j, m){:}I_j, \ldots n_0{:}I_0 \vdash^{\beta} A(m), \Gamma \ \text{ for every m in N}}{n_\omega{:}I_\omega, \ldots n_j{:}I_j, \ldots n_0{:}I_0 \vdash^{\alpha} \forall x(I_j(x) \to A(x)), \Gamma}$$

but note that here, the ordinal bound β on the premises does not vary with m. This helps to keep the theory "weak".

The $\vee$, $\wedge$ rules are unsurprising and we don't list them.

The final logic rule allows interaction with computation in the form:

$$\frac{n_\omega{:}I_\omega, \ldots n_j{:}I_j \vdash^{\beta_0}_{k,C} I_j(m) \qquad n_\omega{:}I_\omega, \ldots m{:}I_j, \ldots n_0{:}I_0 \vdash^{\beta_1} \Gamma}{n_\omega{:}I_\omega, \ldots n_j{:}I_j, \ldots n_0{:}I_0 \vdash^{\alpha} \Gamma}$$

3.2 Computation Rules

Here, the relation $\vdash^{\alpha}_{k,C}$ denotes computation relative to $\prec_k$. In each rule, the premise(s) have ordinal bounds $\beta \prec_k \alpha$. The notation simply makes explicit the fact that k is the maximum of all declared inputs at levels higher than the value being computed.

The Computational Axioms are $\vec{n}{:}\vec{I} \vdash^{\alpha}_{k,C} I_{j'}(\ell)$ provided (i) $n{:}I_j$ occurs in the declaration and $\ell \leq n+1$ and $j' \leq j$; or (ii) $n{:}I_\omega$ occurs in the declaration and $\ell \leq n+1$ and $j' \leq n$. Thus $\vdash^{\alpha} I_j(n) \to I_{j'}(\ell)$, so I_j is progressive and is contained in $I_{j'}$ when $j' < j$.

The Lifting Rule, from $I_{j'}$ to I_j when $j' < j$, is:

$$\frac{n_\omega{:}I_\omega, \ldots n_j{:}I_j \vdash^{\beta}_{k',C} I_{j'}(m)}{n_\omega{:}I_\omega, \ldots n_j{:}I_j \vdash^{\alpha}_{k,C} I_j(m)}$$

where $k' = \max(k, n_j)$, recalling that, in the declaration, the blank after $n_j{:}I_j$ means zeros.

When $j = \omega$ the Lifting Rule is, with $k = n$,

$$\frac{n{:}I_\omega \vdash^{\beta}_{k,C} I_j(m) \quad \text{with} \quad j \leq n}{n{:}I_\omega \vdash^{\alpha}_{C} I_\omega(m)}$$

The Computation Rules (call-by-value) are:

$$\frac{n_\omega{:}I_\omega, \ldots n_j{:}I_j \vdash^{\beta_0}_{k,C} I_j(m) \qquad n_\omega{:}I_\omega, \ldots m{:}I_j \vdash^{\beta_1}_{k,C} I_j(\ell)}{n_\omega{:}I_\omega, \ldots n_j{:}I_j \vdash^{\alpha}_{k,C} I_j(\ell)}$$

3.3 Alternative Ordinal Assignment

Alternatively, in each rule above one could simply take $\alpha = \max(\beta_0, \beta_1) + 1$ or $= \beta + 1$. But then, in order to make use of the $\forall$-rule, which requires a uniform bound on all premises, one needs to add an Accumulation Rule, as in Buchholz [2]: from $\vec{n}{:}\vec{I} \vdash^{\beta} \Gamma$ derive $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma$ provided $\beta \prec_k \alpha$ where $k : I_i$ is declared and $i >$ the level of any input predicate I_j occurring in Γ. (It is also a suitably modified version of Mints' Repetition Rule [7].) One easily sees that "predicative" or "ramified" induction follows straight away, whichever method of ordinal assignment is used, since by m cuts on A, $m : I_i \vdash^{m} A(0) \wedge \forall a(A(a) \to A(a+1)) \to A(m)$ and so, if $i >$ the level of A, the accumulation/repetition rule gives $m : I_i \vdash^{\omega} A(0) \wedge \forall a(A(a) \to A(a+1)) \to A(m)$ because $m \prec_m \omega$ for every m. Then an application of $(\forall)$ gives $\vdash^{\omega+1} A(0) \wedge \forall a(A(a) \to A(a+1)) \to \forall x(I_i(x) \to A(x))$.

3.4 Basic Lemmas

Lemma 3.1 *If* $\vec{n}{:}\vec{I} \vdash^{\alpha}_{k,C} I_j(m)$ *by the computation rules alone, then*

$$m \leq f_j^{2^{L_k(\alpha)}}(\max(\vec{n})) \leq f_{j+1}(\max(\vec{n}))$$

where $f = f(L(\alpha))$ *is a functional version of Ackermann:* $f_0(n) = n + 1$, $f_{j+1}(n) = f_j^{2^{L_n(\alpha)}}(n)$ *and* $f_\omega(n) = f_{n+1}(n)$.

Proof If $\vec{n}{:}\vec{I} \vdash^{\alpha}_{k,C} I_j(m)$ comes about by a computational axiom then $m \leq \max \vec{n} + 1 \leq f_0(\max \vec{n})$ and the result follows from the majorisation properties of f (this applies also when $j = \omega$). If it arises by Lifting from $\vec{n}{:}\vec{I} \vdash^{\beta}_{k',C} I_{j'}(m)$ where $j' < j$, then inductively on $j - 1$ one may assume that $m \leq f_j(\max(\vec{n}))$ and therefore $m \leq f_j^{2^{L_k(\alpha)}}(\max(\vec{n}))$ since f_j is a positive, strictly increasing function. When $j = \omega$ the lifting will be from a premise $n{:}I_\omega \vdash^{\beta}_{k,C} I_n(m)$ and so $m \leq f_{n+1}(\max(L_k(\beta), n)) \leq f_\omega(\max(L_k(\alpha), n)$. If the given derivation comes about by the Computation Rule from premises $n_\omega{:}I_\omega, \ldots n{:}I_j \vdash^{\beta_0}_{k,C} I_j(m')$ and $n_\omega{:}I_\omega, \ldots m'{:}I_j \vdash^{\beta_1}_{k,C} I_j(m)$ then $m \leq f_j^{2^{L_k(\beta_1)}}(\max(\vec{n}, m'))$ by the induction hypothesis, and also $m' \leq f_j^{2^{L_k(\beta_0)}}(\max(\vec{n}))$. Composing, $m \leq f_j^{2^{L_k(\beta_0)}+2^{L_k(\beta_1)}}(\max(\vec{n}))$ and the result follows because $2^{L_k(\alpha)} \geq 2^{L_k(\beta_0)} + 2^{L_k(\beta_1)}$. This applies similarly when $j = \omega$.

Lemma 3.2 (Cut elimination) *(i) Suppose* $\vec{n}{:}\vec{I} \vdash^{\gamma} \Gamma, \neg C$ *and* $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma, C$*, both with cut-rank (maximum size of cut formulas)* $\leq r$*. Suppose also that C is either an atom, or a disjunction* $D_0 \vee D_1$ *or of existential form* $\exists x(I_j(x) \wedge D(x))$ *with D of size r (the "size" of input predicates is defined to be zero). Then* $\vec{n}{:}\vec{I} \vdash^{\gamma+\alpha} \Gamma$ *again with cut-rank r. This applies also when* $j = \omega$.

(ii) Hence if $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma$ *with cut-rank* $r + 1$ *then* $\vec{n}{:}\vec{I} \vdash^{\omega^\alpha} \Gamma$ *with cut-rank* $\leq r$ *and (repeating this)* $\vec{n}{:}\vec{I} \vdash^{\alpha^*} \Gamma$ *with cut-rank* 0*, where* $\alpha^* = \exp_\omega^{r+1}(\alpha)$.

Proof (i) The proof is fairly standard, by induction on α. We only do the case $C \equiv \exists x(I_j(x) \wedge D(x))$. First, suppose $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma, C$ comes about by the $\exists$ rule on C, with premises $\vec{n}{:}\vec{I} \vdash^{\beta_0}_{k,C} I_j(m)$ and $\vec{n}{:}\vec{I} \vdash^{\beta_1} \Gamma, D(m), C$. Then by the induction hypothesis (and a weakening of Γ to $\Gamma, D(m)$) one obtains a rank-r derivation of $\vec{n}{:}\vec{I} \vdash^{\gamma+\beta_1} \Gamma, D(m)$. But $\forall$-inversion applied to the other assumption gives $n_\omega{:}I_\omega, \ldots \max(n_j, m){:}I_j, \ldots \vdash^{\gamma} \Gamma, \neg D(m)$. Combining this with the first (computational) premise of the $\exists$-rule (with ordinal bound raised to $\gamma + \beta_0$) gives $\vec{n}{:}\vec{I} \vdash^{\gamma+\beta_1} \Gamma, \neg D(m)$ since $\beta_0 \prec_k \beta_1$ and hence $\gamma + \beta_0 \prec_k \gamma + \beta_1$. Now a Cut of rank r on the formula $D(m)$ gives $\vec{n}{:}\vec{I} \vdash^{\gamma+\alpha} \Gamma$. A similar argument works when I_j is I_ω.

Second, suppose $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma, C$ arises from any other rule. Then C plays no active role in that rule, and occurs only as a side formula. Applying the induction hypothesis to each premise then removes the C and adds γ to the ordinal bound. Then re-application of that final rule gives the desired result since, as noted above, if $\beta \prec_k \alpha$ then $\gamma + \beta \prec_k \gamma + \alpha$.

(ii) Proceed by induction on α. First suppose $\vec{n}{:}\vec{I} \vdash^{\alpha} \Gamma$ arises by a cut of rank $r+1$ with cut formula C. If β_0 and β_1 are the ordinal bounds on the two premises, then applying the induction hypothesis to each premise reduces their cut-ranks to r and increases their ordinal bounds respectively to ω^{β_0} and ω^{β_1}. Say, without loss of generality, that $\beta_0 \preceq_k \beta_1$. Then the sub-derivation with bound ω^{β_0} can be brought up to a bound of ω^{β_1}. By part (i) above, a rank-r derivation of Γ is therefore obtained with tree-ordinal bound $\omega^{\beta_1} + \omega^{\beta_1}$, and since $\beta_1 \prec_k \alpha$ the ordinal assignment allows this to be raised to ω^{α} as required.

Second, if any other rule is the one last used in deriving Γ, suppose the premises have ordinal bounds β_0 and β_1 (the single-premise cases are similar). Then applying the induction hypothesis to the premises, one again reduces their cut-ranks to r, and increases their ordinal bounds to ω^{β_0} and ω^{β_1}. But since $\beta_0, \beta_1 \prec_k \alpha$ it follows that $\omega^{\beta_0}, \omega^{\beta_1} \prec_k \omega^{\alpha}$ and so this final rule may be re-applied to get a rank-r derivation of Γ with ordinal bound ω^{α}.

3.5 *Note on Σ_1 Reflection*

The ($\exists$) and "lifting" rules combine to derive the following version of Σ_1-reflection:

Suppose one has a cut-free derivation of $k : I_{i+1} \vdash^{\alpha} \Gamma$ *where* Γ *is a set of* Σ_1 *formulas of level* $< i$. *Then* $k : I_{i+1} \vdash^{2\cdot\alpha} \Gamma'$ *where* Γ' *results from* Γ *by lifting (some or all) existential quantifiers to level* i.

The proof is by induction on α. Briefly, suppose the premises of the last $\exists$-rule are $k : I_{i+1} \vdash^{\beta_0} I_j(m)$ and $k : I_{i+1} \vdash^{\beta_1} \Gamma, B(m)$ where Γ contains the formula $\exists x(I_j(x) \wedge B(x))$. Then by the induction hypothesis, $k : I_{i+1} \vdash^{2\cdot\beta_1} \Gamma', B'(m)$, and by lifting, $k : I_{i+1} \vdash^{2\cdot\beta_0+1} I_i(m)$. Since $\beta_0 + 1 \preceq_k \beta_1 \prec_k \alpha$ we have $2\cdot\beta_0 + 2 \preceq_k 2\cdot\beta_1 \prec_k 2\cdot\alpha$. Therefore by reapplying the $\exists$-rule and accumulation, $k : I_{i+1} \vdash^{2\cdot\alpha} \Gamma', \exists x(I_i(x) \wedge B'(x))$ as required.

4 Autonomous EA(I_ω, ...I_2, I_1, O)$^\infty$

The autonomous part of EA(I_ω, ...I_2, I_1, O)$^\infty$ is that part which is allowed only to use ordinal bounds generated "predicatively" according to the following rules: ω is allowed, and α is allowed provided there is a derivation $k{:}I_i; \vdash^{\gamma} PWF(k, \alpha)$ for some i and some γ (independent of k) whose set-theoretic ordinal rank $\|\gamma\|$ is less than $\|\alpha\|$.

It is not difficult to check that ω is enough to "get started", for the coding apparatus assumed to be built into the system will enable first, $\omega + m$ to be generated for any finite m, and then $k{:}I_1; \max(\beta, s){:}O \vdash^{\omega} \neg D(s, k, \beta), D(s', k, \beta + \omega)$ where $k{:}I_1; \max(\beta, s){:}O \vdash_C^{\omega+1} O(s')$. Hence by the $\exists$-rule,

$$k{:}I_1;\ \max(\beta, s){:}O \vdash^{\omega+2} \neg D(s, k, \beta), \exists s D(s, k, \beta + \omega)$$

and by the ∀-rule,

$$k{:}I_1;\ \beta{:}O \vdash^{\omega+3} \forall s \neg D(s, k, \beta), \exists s D(s, k, \beta + \omega)$$

which is

$$k{:}I_1;\ \beta{:}O \vdash^{\omega+5} PWF(k, \beta) \to PWF(k, \beta + \omega).$$

A final ∀-rule then gives $k{:}I_1;\ \vdash^{\omega+6} \forall\beta(PWF(k, \beta) \to PWF(k, \beta + \omega))$. Thus from ω one may autonomously generate multiples of ω, and then exponents of them: $2^{\omega\cdot 2}, 2^{\omega\cdot 3}, \ldots$.

It becomes clear from this example, that different levels of pointwise well-foundedness are going to arise, as more and more iterated inductions are needed to derive it. So we define:

$$PWF^{I_j}(k, \alpha) \ \equiv\ \exists s(I_j(s) \wedge D(s, k, \alpha))\ .$$

4.1 *Ordinal Notations Below* Γ_0

Following Feferman [4], with slight modifications, a fundamental sequence $\{\lambda_x\}$ is assigned to each $\lambda = \varphi_\alpha(\beta)$ in the Veblen hierarchy of normal functions (where φ_α enumerates the common fixed points of all φ_γ with $\gamma < \alpha$). Write α_x to denote either the xth member of the fundamental sequence to α if it is a limit, or $\alpha - 1$ if α is a successor.

Definition 4.1 • If $\lambda = \varphi_0(\beta) = \omega^\beta$ with $\beta > 0$ then $\lambda_x = \omega^{\beta_x} \cdot (x + 1)$.
- If $\lambda = \varphi_\alpha(\beta)$ with $\alpha > 0$ and $\beta = 0$ then $\lambda_x = \varphi^x_{\alpha_x}(1)$.
- If $\lambda = \varphi_\alpha(\beta)$ with $\alpha > 0$ and β a successor then $\lambda_x = \varphi^x_{\alpha_x}(\varphi_\alpha(\beta - 1) + 1)$.
- If $\lambda = \varphi_\alpha(\beta)$ with $\alpha > 0$ and β a limit then $\lambda_x = \varphi_\alpha(\beta_x)$

where $\varphi^x_{\alpha_x}$ is the x-times iterate of φ_{α_x}.

By repeated application of φ and addition, one builds up finite terms representing all the ordinals below Γ_0. In particular $\Gamma_0 = \sup \gamma_x$ where $\gamma_0 = \omega$ and $\gamma_{x+1} = \varphi_{\gamma_x}(\gamma_x)$. These terms can be coded, and the fundamental sequences computed, by elementary, or at worst low-level primitive recursive, functions. So they can be proved to exist in EA(I, O) or EA(I_2, I_1, O).

Theorem 4.2 *Assume* $PWF^{I_\omega}(k, \alpha)$ *is autonomously derivable from* $\max(k, \alpha){:}I_\omega$ *with ordinal bound independent of* k*. Then so is :*

$$\forall\beta(PWF^{I_a}(k, \beta) \to PWF^{I_a}(k, \varphi_\alpha(\beta)))$$

where $a = L_k(\alpha) + 1$.

Proof In the background will be the declaration $\max(k,\alpha){:}I_\omega$ which will remain suppressed for reasons of legibility. The main component of the required derivation is computational. Let $s(\beta)$ denote the sequence number s making $D(s,k,\beta)$ true. We show

$$s(\beta) : I_a \vdash^{\gamma_a}_{k,C} I_a(s(\varphi_\alpha(\beta)))$$

by induction on a, for a suitable γ_a.

When $a = 1$ we have $\alpha = 0$ and $\varphi_\alpha(\beta) = \omega^\beta$. In this case there will be an exponential function which computes $s(\omega^\beta)$ from $s(\beta)$ so that $s(\beta) : I_a \vdash^{\gamma_1}_{k,C} I_a(s(\omega^\beta))$ with $\gamma_1 \preceq_k \omega^2$.

For the induction step, assume $a = L_k(\alpha) + 1 \geq 2$ and for every β,

$$s(\beta) : I_{a-1} \vdash^{\gamma_{a-1}}_{k,C} I_{a-1}(s(\varphi_{\alpha_k}(\beta)))\,.$$

The "call-by-value" computation rule enables us to apply this repeatedly k times, starting with $\beta = 1$, to obtain $\vdash^{\gamma_{a-1}+k-1}_{k,C} I_{a-1}(s(\varphi^k_{\alpha_k}(1)))$. Then, since $\varphi^k_{\alpha_k}(1))$ is the kth member of the assigned fundamental sequence to $\varphi_\alpha(0)$, one further step gives $\vdash^{\gamma_{a-1}+k}_{k,C} I_{a-1}(s(\varphi_\alpha(0)))$. Now since k is declared at level ω, and $k \prec_k \omega$, an accumulation gives $\vdash^{\gamma_{a-1}+\omega}_{k,C} I_{a-1}(s(\varphi_\alpha(0)))$.

This process may now be applied again, but starting with $\beta = \varphi_\alpha(0) + 1$. Since $\varphi^k_{\alpha_k}(\varphi_\alpha(0) + 1)$ is the kth member of the fundamental sequence to $\varphi_\alpha(1)$, one obtains $\vdash^{\gamma_{a-1}+\omega\cdot 2}_{k,C} I_{a-1}(s(\varphi_\alpha(1)))$.

Continuing in this way, but with an arbitrary $s(\beta)$ now declared at level I_a, one sees that after $\ell = L_k(\beta)$ iterations the end-result is

$$s(\beta) : I_a \vdash^{\gamma_{a-1}+\omega\cdot(\ell+1)}_{k,C} I_{a-1}(s(\varphi_\alpha(\beta)))$$

after which an accumulation (as $s(\beta) \geq \ell + 1$) gives

$$s(\beta) : I_a \vdash^{\gamma_{a-1}+\omega\cdot\omega}_{k,C} I_{a-1}(s(\varphi_\alpha(\beta)))$$

and then a lifting gives the desired

$$s(\beta) : I_a \vdash^{\gamma_a}_{k,C} I_a(s(\varphi_\alpha(\beta)))$$

where $\gamma_a = \gamma_{a-1} + \omega^2 + 1$. This completes the induction.

Now from this one may derive the following, where s is an arbitrary number and d depends only on the "size" of the formula D:

$$\max(\beta, s) : I_a \vdash^{\gamma_a+d} \neg D(s,k,\beta),\, PWF^{I_a}(k, \varphi_\alpha(\beta))\,.$$

This is because: Either s is not the correct sequence number for $D(s, k, \beta)$, in which case $\neg D(s, k, \beta)$ may easily be derived along with the side formula $PWF^{I_a}(k, \varphi_\alpha(\beta))$. Or else $D(s, k, \beta)$ is true, in which case $s = s(\beta)$ and hence, letting $s_0 = s(\varphi_\alpha(\beta))$, we have $s{:}I_a \vdash^{\gamma_a}_{k,C} I_a(s_0)$ and $s_0{:}I_a \vdash^{\gamma_a+d-2} \neg D(s, k, \beta)$, $D(s_0, k, \varphi_\alpha(\beta))$. Therefore an $(\exists)$ rule with computed witness s_0 again yields $\max(\beta, s){:}I_a \vdash^{\gamma_a+d} \neg D(s, k, \beta), PWF^{I_a}(k, \varphi_\alpha(\beta))$.

A $(\forall)$ rule gives $\beta{:}I_a \vdash^{\gamma_a+d+1} \forall s(I_a(s) \to \neg D(s, k, \beta)), PWF^{I_a}(k, \varphi_\alpha(\beta))$ which immediately yields $\beta{:}I_a \vdash^{\gamma_a+d+2} PWF^{I_a}(k, \beta) \to PWF^{I_a}(k, \varphi_\alpha(\beta))$ and hence $\vdash^{\gamma_a+d+3} \forall\beta(PWF^{I_a}(k, \beta) \to PWF^{I_a}(k, \varphi_\alpha(\beta)))$.

Corollary 4.3 *If $PWF^{I_\omega}(k, \alpha)$ and $PWF^{I_\omega}(k, \beta)$ are autonomously derivable, then so is $PWF^{I_\omega}(k, \varphi_\alpha(\beta))$.*

Proof Suppressing the ordinal bounds, a cut-free derivation of $\max(k, \alpha){:}I_\omega \vdash PWF^{I_\omega}(k, \alpha)$ provides an s_0 such that $\max(k, \alpha){:}I_\omega \vdash_{k,C} I_\omega(s_0)$ where $lh(s_0) = L_k(\alpha) + 1 = a$. Similarly, a cut-free derivation of $\max(k, \beta){:}I_\omega \vdash PWF^{I_\omega}(k, \beta)$ entails computation of s_1 such that $\max(k, \beta){:}I_\omega \vdash_{k,C} I_\omega(s_1)$ and $\vdash D(s_1, k, \beta)$. Then if $s = \max(s_0, s_1)$ we have $\max(k, \alpha, \beta){:}I_\omega \vdash_{k,C} I_\omega(s)$. But a computational axiom yields $s{:}I_\omega \vdash_{k,C} I_a(s_1)$ since $a, s_1 \leq s$, so $\max(k, \alpha, \beta){:}I_\omega \vdash PWF^{I_a}(k, \beta)$. Hence by the theorem a derivation of $PWF^{I_a}(k, \varphi_\alpha(\beta))$ is obtained and one may assume that to be cut-free also. Since the numerical bounds on the (non-zero) inputs are all declared at level I_ω, the existential witnesses may be lifted from I_a to I_ω, thus giving a derivation of $PWF^{I_\omega}(k, \varphi_\alpha(\beta))$.

4.2 Computing $L_x(\varphi_\alpha(\beta))$ and $G_x(\varphi_\alpha(\beta))$

Definition 4.4 Another version of Ackermann: for each fixed $x \in \mathrm{N}$ define $f_a(x, b) = f_a(b)$ where

$$f_0(b) = (x+1)^b, \quad f_{a+1}(b) = (f_a^x \circ succ)^{b+1}(0)\ .$$

Now recall the recursive definitions of L_x and G_x in Sect. 2.

Lemma 4.5 *For each fixed x and all α, β,*

$$L_x(\varphi_\alpha(\beta)) \ \leq\ f_{L_x(\alpha)}(x+1, L_x(\beta)) \quad \textit{and} \quad f_{G_x(\alpha)}(x, G_x(\beta)) \ \leq\ G_x(\varphi_\alpha(\beta))\ .$$

Proof Both parts are similar, so we only do the first. Assuming x fixed throughout, denote $f_a(x+1, b)$ simply as $f_a(b)$. Then proceeding by induction on α and, within that, by induction on β, one has (recalling that α_x also denotes $\alpha - 1$ when α is a successor):

(i) If $\alpha = 0$ then $L_x(\varphi_\alpha(\beta)) = L_x(\omega^\beta) = L_x(\omega^{\beta_x} \cdot (x+1)) + 1$. But, as is easily shown, L_x preserves addition, and therefore $L_x(\varphi_\alpha(\beta)) = L_x(\omega^{\beta_x}) \cdot (x+1) + 1$.

By the induction on β, this in turn is less than or equal to $f_0(L_x(\beta_x)) \cdot (x+2) = f_0(L_x(\beta_x)+1) = f_0(L_x(\beta))$.

(ii) If $\alpha > 0$ and $\beta = 0$ then $L_x(\varphi_\alpha(\beta)) = L_x(\varphi^x_{\alpha_x}(1)) + 1$ and by the induction hypothesis on α_x this is less than or equal to $f^x_{L_x(\alpha_x)}(1) + 1 \leq f^{x+1}_{L_x(\alpha_x)}(1)$ and this is $f_{L_x(\alpha_x)+1}(0) = f_{L_x(\alpha)}(0)$.

(iii) If $\alpha > 0$ and β is a successor, the computation is as in (ii) but with 0 replaced by $\varphi_\alpha(\beta - 1)$. One has $L_x(\varphi_\alpha(\beta)) = L_x(\varphi^x_{\alpha_x}(\varphi_\alpha(\beta-1)+1)) + 1$. By the induction hypothesis and since $f_a(b) > b$, this is $\leq f^{x+1}_{L_x(\alpha_x)}(L_x(\varphi_\alpha(\beta-1)+1))$ and this is $\leq f^{x+1}_{L_x(\alpha)-1} \circ succ(f_{L_x(\alpha)}(L_x(\beta)-1)) = f_{L_x(\alpha)}(L_x(\beta))$.

(iv) If $\alpha > 0$ and β is a limit then $L_x(\varphi_\alpha(\beta)) = L_x(\varphi_\alpha(\beta_x) + 1$. By the induction on β_x, this is $\leq f_{L_x(\alpha)}(L_x(\beta)-1) + 1 \leq f_{L_x(\alpha)}(L_x(\beta))$ as required.

Lemma 4.6 *For ordinal notations $\delta < \Gamma_0$ built up from ω by addition and $\alpha, \beta \mapsto \varphi_\alpha(\beta)$ one has*

$$L_x(\delta) \;\leq\; G_{x+1}(\delta) \;\leq L_{x+1}(\delta)$$

and for each fixed δ these functions, as functions of x, are bounded by finite compositions of the Ackermann function.

Proof First, as previously stated, it is easy to check that both L_x and G_x preserve addition. Further, suppose δ is built up from α and β by $\delta = \varphi_\alpha(\beta)$. Then we may inductively assume the result for both α and β and so from the above lemma we immediately obtain (using basic majorisation properties of f):

$$L_x(\delta) \;\leq\; f_{L_x(\alpha)}(x+1, L_x(\beta)) \;\leq\; f_{G_{x+1}(\alpha)}(x+1, G_{x+1}(\beta)) \;\leq G_{x+1}(\delta)\;.$$

That $G_x(\delta) \leq L_x(\delta)$ for all x is immediate from their definitions.

Now, recalling the version of Ackermann used in Lemma 3.1 (any other version would do, since they are all elementarily reducible to one another) it is easy to see that $f_1(x \cdot (b+1)) \geq 2^{x\cdot(b+1)} \geq (x+1)^b$ and for $a > 0$, $f_{a+1}((x+1)\cdot(b+1)) = f_a^{2^{(x+1)\cdot(b+1)}}((x+1)\cdot(b+1)) \geq (f_a^{x+1} \circ succ)^{b+1}((x+1)\cdot(b+1))$. Comparing this with $f_a(x+1, b)$ of the lemma above, one sees that if $a(x)$ and $b(x)$ are finite compositions of Ackermann which bound $L_x(\alpha)$ and $L_x(\beta)$ respectively, then $f_{a(x)+1}((x+1)\cdot(b(x)+1))$ is another finite composition of Ackermann bounding $L_x(\varphi_\alpha(\beta))$.

4.3 The Strength of EA(I_ω, ...I_2, I_1, O)$^\infty$

Theorem 4.7 *If $PWF^{I_\omega}(k, \alpha)$ is autonomously derivable then so is Weak Transfinite Induction $PTI(k, \alpha, A)$ for any formula A of finite level.*

Proof Assume $\max(k, \alpha)$ is declared at level I_ω. From a cut-free derivation $\vdash^\gamma PWF^{I_\omega}(k, \alpha)$ one can read off (a bound on) the value $a = L_k(\alpha)$ and a derivation of $I_\omega(a)$. By a sequence of cuts on the formula F, assumed to be of "size" r, one easily derives $\vdash^{r+a} \neg F(0), \exists x \leq a.(F(x) \wedge \neg F(x+1)), \forall x \leq a.F(x)$ and then, since the formula is of finite level and a is computed at level I_ω, the ordinal bound may be lifted to $\max(\gamma, r) + \omega$. This proves numerical induction up to $L_k(\alpha)$:

$$\max(k, \alpha){:}I_\omega \vdash^{\max(\gamma,r)+\omega} F(0) \wedge \forall x(F(x) \to F(x+1) \to \forall x \leq L_k(\alpha).F(x)$$

and by Lemma 2.3 this, with an appropriate F, in turn yields $PTI(k, \alpha, A)$.

Theorem 4.8 *The provably computable functions of Aut.EA(I_ω, ...I_1, O)$^\infty$ are exactly those elementary recursive in the Ackermann function.*

Proof Suppose f is provably computable and let C_f be a Σ_1 formula such that for every k, $f(k) = \ell$ iff $C_f(k, \ell)$ holds and $k{:}I_\omega \vdash^\delta \exists z C_f(k, z)$ with δ autonomously generated. We may assume that this derivation is cut-free, and so correct witnesses w for the existential quantifiers in $\exists z C_f(k, z)$ are derived along the way by $k{:}I_\omega \vdash^\delta_{k,C} I_\omega(w)$. By the last Lemma and 3.1 these witnesses are bounded by finite compositions of Ackermann, because if $L(\delta)$ is bounded by a finite Ackermann term, so is the functional version $f(L(\delta))$ of 3.1 which bounds w. Therefore $f(k) =$ the least ℓ such that $C_f(k, \ell)$ is elementary recursive in the Ackermann function.

Conversely, if f is elementary in the Ackermann, it is computable within time, or space, bounded by some finite composition of Ackermann, and therefore by a finite composition of the function $f_{a(x)}(x, b(x))$ in Definition 4.4. But then by Lemma 4.5 this in turn is bounded (as a function of x) by $L_x(\varphi_\alpha(\beta))$ where $a(x) \leq L_x(\alpha)$ and $b(x) \leq L_x(\beta)$. Since, by the last proof, numerical induction up to $L_x(\varphi_\alpha(\beta))$ is available in Autonomous EA(I_ω, ...I_2, I_1, O)$^\infty$, and the progressiveness of the formula expressing the step-by-step computability of f is easily verified, it follows that f is provably defined.

Theorem 4.9 Γ_0 *is the supremum of the autonomous ordinals.*

Proof $\Gamma_0 = \sup \gamma_i$ where $\gamma_0 = \omega$ and $\gamma_{i+1} = \varphi_{\gamma_i}(\gamma_i)$. As shown above, each γ_i is autonomous. However if Γ_0 itself were autonomous then $G_x(\Gamma_0)$ would be bounded by a finite composition of the Ackermann function. But it is not, for $G_x(\Gamma_0) = G_x(\gamma_x)$ and by induction one has $G_x(\gamma_x) \geq f_\omega(x)^x(x)$ where $f_\omega(x)$ is the function $a \mapsto f_a(x, a)$ from Definition 4.4. This has an additional unbounded x-times iterate in its computation, so cannot be bounded by any finite composition of Ackermann, no matter which version of Ackermann one chooses.

Remark 4.10 The finite stages of the hierarchy of input-output theories all embed naturally into the various levels of EA(I_ω, ...I_2, I_1, O)$^\infty$ and one may similarly read off upper bounds on their provable ordinals (i.e. those for which pointwise well-foundedness is provable). Bounds for EA(I_1, O), EA(I_2, I_1, O) etc. are $\varphi_1(0)$, $\varphi_2(0)$ etc., and the bound for the union of these (which amounts to PRA) is the first primitive recursively closed ordinal $\varphi_\omega(0)$.

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