Higher-Order Spectra, Equivariant Hodge–Deligne Polynomials, and Macdonald-Type Equations

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Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract We define notions of higher-order spectra of a complex quasi-projective manifold with an action of a finite group G and with a G-equivariant automorphism of finite order, some of their refinements and give Macdonald-type equations for them.

Keywords Group actions • Macdonald-type equations • Orbifold Euler characteristic • Spectrum

Subject Classifications:14L30, 55M35, 57R18

1 Introduction

For a "good" topological space X, say, a union of cells in a finite CW-complex or a quasi-projective complex analytic variety, the Euler characteristic $\chi(X)$, defined as the alternating sum of the dimensions of the cohomology groups with compact support, is an additive invariant. In [17], I.G. Macdonald derived a formula for the Poincaré polynomial of a symmetric product. For the Euler characteristic, this formula gives the following. Let $S^nX = X^n/S_n$ be the nth symmetric power of the

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space X. Then one has [17]

$$1 + \sum_{n=1}^{\infty} \chi(S^n X) t^n = (1-t)^{-\chi(X)}.$$

We can interpret this formula as a formula for an invariant (here the Euler characteristic) expressing the generating series of the values of the invariant for the symmetric powers of a space as a series not depending on the space (here simply $(1-t)^{-1}$) with an exponent which is equal to the value of the invariant for the space itself. We call such an equation a *Macdonald-type equation*. In [12], formulae of this type were considered for some generalizations of the Euler characteristic (with values in certain rings). If the ring of values is not a number ring ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$), to formulate these equations, one needs to use so-called power structures over the rings [10] which can be defined through (pre-)λ-ring structures on them.

Here we consider other generalizations of these formulas. We consider another additive invariant which is finer than the Euler characteristic: the (Hodge) spectrum. The (Hodge) spectrum was first defined in [20, 21] for a germ of a holomorphic function on $(\mathbb{C}^n, 0)$ with an isolated critical point at the origin. It can also be defined for a pair (V, φ), where V is a complex quasi-projective variety and φ is an automorphism of V of finite order: [5]. (The spectrum of a germ of a holomorphic function is essentially the spectrum of its motivic Milnor fibre defined in [5].)

Traditionally the spectrum is defined as a finite collection of rational numbers with integer multiplicities (possibly negative ones) and therefore as an element of the group ring $\mathbb{Z}[\mathbb{Q}]$ of the group $\mathbb{Q}$ of rational numbers. Let $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck group of pairs (V, φ), where V is a quasi-projective variety and φ is an automorphism of V of finite order (with the addition defined by the disjoint union). The group $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ is a ring with the multiplication defined by the Cartesian product of varieties and with the automorphism defined by the diagonal action. The Euler characteristic can be interpreted as a ring homomorphism from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to the ring of integers $\mathbb{Z}$. The spectrum can be regarded as a sort of generalized Euler characteristic. (The spectrum of a pair (V, φ) determines the Euler characteristic of V in a natural way.) Namely, it can be viewed as a group homomorphism from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to the group ring $\mathbb{Z}[\mathbb{Q}]$, but it is not a ring homomorphism. Rational numbers (i.e., elements of the group $\mathbb{Q}$) are in one-to-one correspondence with the elements of the group $(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}$: $r \longleftrightarrow (\{r\}, [r])$, where $\{r\}$ is the fractional part of the rational number r and $[r]$ is its integer part. In this way, the group rings $\mathbb{Z}[\mathbb{Q}]$ and $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$ can be identified as abelian groups. (The isomorphism is not a ring homomorphism!) This permits to consider the spectrum as an element of the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$. Moreover, the corresponding map from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$ is a ring homomorphism.

The group ring $\mathbb{Z}[\mathscr{A}]$ of an abelian group $\mathscr{A}$ has a natural λ-structure. We use this fact to show that the spectrum of a pair (V, φ) as an element of $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$ also satisfies a Macdonald-type equation.

For a quasi-projective variety with an action of a finite group, one has the orbifold Euler characteristic defined in [7, 8] (see also [1, 14]) and the higher-order Euler characteristics defined in [1, 3]. These notions can be extended to some generalizations of the Euler characteristic.

For a complex quasi-projective manifold V with an action of a finite group G and with a G-equivariant automorphism φ of finite order, one can define the notion of an orbifold spectrum as an element of the group ring $\mathbb{Z}[\mathbb{Q}]$: [9]. This spectrum takes into account not only the logarithms of the eigenvalues of the action of the transformation φ on the cohomology groups but also the so-called ages of elements of G at their fixed points (both being rational numbers). Algebraic manipulations with these two summands are different. The first ones behave as elements of $\mathbb{Q}/\mathbb{Z}$, whereas the second ones as elements of $\mathbb{Q}$. This explains why the existence of a Macdonald-type equation for this spectrum is not clear. However, if one considers the "usual" Hodge spectrum as an element of the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$ and applies the orbifold approach to the summand $\mathbb{Z}$ (thus, substituting it by $\mathbb{Q}$), one gets a version of the orbifold spectrum with values in the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}]$ permitting a Macdonald-type equation. This version of the orbifold spectrum determines the one from [9] in a natural way. Moreover, taking into account the weight filtration as well, one can consider a refinement of this notion with values in the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}]$. The last notion is equivalent to the notion of the equivariant orbifold Hodge–Deligne polynomial.

Applying the traditional method to define higher-order Euler characteristics through the orbifold one to the described notions, we define higher-order spectra of a triple (V, G, φ) with a quasi-projective G-manifold V, some of their refinements, and give Macdonald-type equations for them.

2 λ-Structure on the Group Ring of an Abelian Group

Let $\mathscr{A}$ be an abelian group (with the sum as the group operation) and let $\mathbb{Z}[\mathscr{A}]$ be the group ring of $\mathscr{A}$. The elements of $\mathbb{Z}[\mathscr{A}]$ are finite sums of the form $\sum\limits_{a \in \mathscr{A}} k_a\{a\}$ with $k_a \in \mathbb{Z}$. (We use the curly brackets in order to avoid ambiguity when $\mathscr{A}$ is a subgroup of the group $\mathbb{R}$ of real numbers: $\mathbb{Z}$ or $\mathbb{Q}$.) The ring operations on $\mathbb{Z}[\mathscr{A}]$ are defined by $\sum k'_a\{a\} + \sum k''_a\{a\} = \sum (k'_a + k''_a)\{a\}$, $(\sum k'_a\{a\})(\sum k''_a\{a\}) = \sum\limits_{a,b \in \mathscr{A}} (k'_a \cdot k''_b)\{a+b\}$.

Let R be a commutative ring with a unit. A λ-ring structure on R (sometimes called a pre-λ-ring structure; see, e.g., [16]) is an "additive-to-multiplicative" homomorphism $\lambda : R \to 1 + T \cdot R[[T]]$ ($a \mapsto \lambda_a(T)$) such that $\lambda_a(T) = 1 + aT + \ldots$. This means that $\lambda_{a+b}(T) = \lambda_a(T) \cdot \lambda_b(T)$ for $a, b \in R$.

The notion of a λ-ring structure is closely related to the notion of a power structure defined in [10]. Sometimes a power structure has its own good description which permits to use it, e.g., for obtaining formulae for generating series of some invariants. A power structure over a ring R is a map $(1+T{\cdot}R[[T]])\times R \to 1+T{\cdot}R[[T]]$, $(A(T),m) \mapsto (A(T))^m$ ($A(t) = 1 + a_1T + a_2T^2 + \ldots$, $a_i \in R$, $m \in R$) possessing all the basic properties of the exponential function: see [10]. A λ-structure on a ring defines a power structure over it. On the other hand, there are, in general, many λ-structures on a ring corresponding to one power structure over it.

The group ring $\mathbb{Z}[\mathscr{A}]$ of an abelian group $\mathscr{A}$ can be considered as a λ-ring. The λ-ring structure on $\mathbb{Z}[\mathscr{A}]$ is natural and must be well known. However, we have not found its description in the literature. Therefore, we give here a definition of a λ-structure on the ring $\mathbb{Z}[\mathscr{A}]$. (A similar construction was discussed in [11] for the ring of formal "power" series over a semigroup with certain finiteness properties.)

The group ring $\mathbb{Z}[\mathscr{A}]$ can be regarded as the Grothendieck ring of the group semiring $S[\mathscr{A}]$ of maps of finite sets to the group $\mathscr{A}$. Elements of $S[\mathscr{A}]$ are the equivalence classes of the pairs (X,ψ) consisting of a finite set X and a map $\psi : X \to \mathscr{A}$. (Two pairs (X_1,ψ_1) and (X_2,ψ_2) are equivalent if there exists a bijective map $\xi : X_1 \to X_2$ such that $\psi_2 \circ \xi = \psi_1$.) The group ring $\mathbb{Z}[\mathscr{A}]$ is the Grothendieck ring of the semiring $S[\mathscr{A}]$. Elements of the ring $\mathbb{Z}[\mathscr{A}]$ are the equivalence classes of maps of finite virtual sets (i.e., formal differences of sets) to $\mathscr{A}$. For a pair (X,ψ) representing an element a of the semiring $S[\mathscr{A}]$, let its nth symmetric power $S^n(X,\psi)$ be the pair $(S^nX,\psi^{(n)})$ consisting of the nth symmetric power $S^nX = X^n/S_n$ of the set X and of the map $\psi^{(n)} : S^nX \to \mathscr{A}$ defined by $\psi^{(n)}(x_1,\ldots,x_n) = \sum_{i=1}^{n} \psi(x_i)$. One can easily see that the series

$$\lambda_a(T) = 1 + [S^1(X,\psi)]T + [S^2(X,\psi)]T^2 + [S^3(X,\psi)]T^3 + \ldots$$

defines a λ-structure on the ring $\mathbb{Z}[\mathscr{A}]$ (or rather a λ-structure on the semiring $S[\mathscr{A}]$ extendable to a λ-structure on $\mathbb{Z}[\mathscr{A}]$ in a natural way).

The power structure over the ring $\mathbb{Z}[\mathscr{A}]$ corresponding to this λ-structure can be described in the following way. Let $A(T) = 1 + a_1T + a_2T^2 + \ldots$, where $a_i = [(X_i,\psi_i)]$, $m = [(M,\psi)]$ with finite *sets* X_i and M (thus, a_i and m being actually elements of the semiring $S[\mathscr{A}]$). Then

$$(A(T))^m = 1+\sum_{n=1}^{\infty}\left(\sum_{\{n_i\}:\sum in_i=n}\left[\left(((M^{\sum_i n_i}\setminus\Delta)\times\prod_i X_i^{n_i})\,/\prod_i S_{n_i}\,,\psi_{\{n_i\}}\right)\right]\right)\cdot T^n\,,$$

where Δ is the big diagonal in $M^{\sum_i n_i}$ (consisting of $(\sum n_i)$-tuples of points of M with at least two coinciding ones), the group $\prod_i S_{n_i}$ acts on $(M^{\sum_i n_i}\setminus\Delta)\times\prod_i X_i^{n_i}$ by permuting simultaneously the factors in $M^{\sum_i n_i} = \prod_i M^{n_i}$ and in $\prod_i X_i^{n_i}$, and the map $\psi_{\{n_i\}} : ((M^{\sum_i n_i}\setminus\Delta)\times\prod_i X_i^{n_i})/\prod_i S_{n_i} \to \mathscr{A}$ is defined by

$$\psi_{\{n_i\}}(\{y_i^j\},\{x_i^j\}) = \sum_i (i\cdot\psi(y_i^j) + \psi_i(x_i^j))\,,$$

where y_i^j and x_i^j, $j = 1, \ldots, n_i$, are the jth components of the point in M^{n_i} and in $X_i^{n_i}$ respectively (cf. [10, Eq. (1)]); a similar construction for the Grothendieck ring of quasi-projective varieties with maps to an abelian manifold was introduced in [18].

The ring $R[z_1, \ldots, z_n]$ of polynomials in $z_1, \ldots, z_n$ with the coefficients from a λ-ring R carries a natural λ-structure: see, e.g., [16]. The same holds for the ring $R[z_1^{1/m}, \ldots, z_n^{1/m}]$ of fractional power polynomials in $z_1, \ldots, z_n$. In terms of the corresponding power structure, one can write

$$(1-T)^{-\sum_{\underline{k}} a_{\underline{k}} \underline{z}^{\underline{k}}} = \prod_{\underline{k}} \lambda_{a_{\underline{k}}}(\underline{z}^{\underline{k}} T),$$

where $\underline{z} = (z_1, \ldots, z_n)$, $\underline{k} = (k_1, \ldots, k_n)$, $\underline{z}^{\underline{k}} = z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$.

The ring $R(G)$ of representations of a group G is regarded as a λ-ring with the λ-structure defined by

$$\lambda_{[\omega]}(T) = 1 + [\omega]t + [S^2\omega]T^2 + [S^3\omega]T^3 + \ldots,$$

where ω is a representation of G and $S^n\omega$ is its nth symmetric power.

3 The Spectrum and the Equivariant Hodge–Deligne Polynomial

Let V be a complex quasi-projective variety with an automorphism φ of finite order. For a rational α, $0 \leq \alpha < 1$, let $H_\alpha^k(V)$ be the subspace of $H^k(V) = H_c^k(V; \mathbb{C})$ (the cohomology group with compact support) consisting of the eigenvectors of φ_* with the eigenvalue $\mathbf{e}[\alpha] := \exp(2\pi\alpha i)$. The subspace $H_\alpha^k(V)$ carries a natural complex mixed Hodge structure.

Definition 1 (See, e.g., [5]) The *(Hodge) spectrum* $\mathrm{hsp}(V, \varphi)$ of the pair (V, φ) is defined by

$$\mathrm{hsp}(V, \varphi) = \sum_{k,p,q,\alpha} (-1)^k \dim(H_\alpha^k(V))^{p,q} \cdot \{p + \alpha\} \in \mathbb{Z}[\mathbb{Q}].$$

The spectrum $\mathrm{hsp}(V, \varphi)$ can be identified either with the fractional power polynomial (Poincaré polynomial)

$$p_{(V,\varphi)}(t) = \sum_{k,p,q,\alpha} (-1)^k \dim(H_\alpha^k(V))^{p,q} \cdot t^{p+\alpha} \in \mathbb{Z}[t^{1/m}]$$

or with the equivariant Poincaré polynomial

$$\overline{e}_{(V,\varphi)}(t) = \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V))^{p,q}\omega_{\mathbf{e}[\alpha]} \cdot t^p \in R_f(\mathbb{Z})[t]\,,$$

where $R_f(Z)$ is the ring of finite order representations of the cyclic group $\mathbb{Z}$ and $\omega_{\mathbf{e}[\alpha]}$ is the one-dimensional representation of $\mathbb{Z}$ with the character equal to $\mathbf{e}[\alpha]$ at 1. Both rings $\mathbb{Z}[t^{1/m}]$ and $R_f(\mathbb{Z})[t]$ carry natural λ-structures and thus power structures. However, the natural power structure over $\mathbb{Z}[t^{1/m}]$ is not compatible with the multiplication of spaces: the map

$$p_\bullet : K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[t^{1/m}]$$

is not a ring homomorphism. Therefore, a natural Macdonald-type equation for the spectrum is formulated in terms of the equivariant Poincaré polynomial $\overline{e}_{(V,\varphi)}(t)$. Moreover, a stronger statement can be formulated in terms of the equivariant Hodge–Deligne polynomial of the pair (V, φ).

Definition 2 ([6], see also [19]) The *equivariant Hodge–Deligne polynomial* of the pair (V, φ) is

$$e_{(V,\varphi)}(u, v) = \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V))^{p,q}\omega_{\mathbf{e}[\alpha]} \cdot u^p v^q \in R_f(\mathbb{Z})[u, v]\,,$$

One has $\overline{e}_{(V,\varphi)}(t) = e_{(V,\varphi)}(t, 1)$.

Let S^nV be the nth symmetric power of the variety V. The transformation $\varphi : V \to V$ defines a transformation $\varphi^{(n)} : S^nV \to S^nV$ in a natural way.

Theorem 1 *One has*

$$1 + e_{(V,\varphi)}(u, v)T + e_{(S^2V,\varphi^{(2)})}(u, v)T^2 + e_{(S^3V,\varphi^{(3)})}(u, v)T^3 + \ldots = (1-T)^{-e_{(V,\varphi)}(u,v)}\,, \tag{1}$$

where the RHS of (1) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[u, v]$.

The *proof* is essentially contained in [4] where J. Cheah proved an analogue of (1) for the usual (non-equivariant) Hodge–Deligne polynomial. Theorem 1 can be deduced from the arguments of Cheah [4] by taking care of different eigenspaces.

Theorem 1 means that the natural map $e_\bullet$ from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to $R_f(\mathbb{Z})[u, v]$ is a λ-ring homomorphism.

Corollary 1 *One has*

$$1 + \overline{e}_{(V,\varphi)}(t)T + \overline{e}_{(S^2V,\varphi^{(2)})}(t)T^2 + \overline{e}_{(S^3V,\varphi^{(3)})}(t)T^3 + \ldots = (1-T)^{-\overline{e}_{(V,\varphi)}(t)}\,, \tag{2}$$

where the RHS of (2) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[t]$.

4 The Orbifold Spectrum and the Equivariant Orbifold Hodge–Deligne Polynomial

Let X be a topological G-space and G a finite group. Let G_* be the set of conjugacy classes of elements of G. For an element $g \in G$, let $X^{\langle g \rangle} = \{x \in X : gx = x\}$ be the fixed point set of g, and let $C_G(g) = \{h \in G : h^{-1}gh = g\} \subset G$ be the centralizer of g. The group $C_G(g)$ acts on the fixed point set $X^{\langle g \rangle}$. The orbifold Euler characteristic $\chi^{\mathrm{orb}}(X, G)$ can be defined by

$$\chi^{\mathrm{orb}}(X, G) = \sum_{[g] \in G_*} \chi(X^{\langle g \rangle}/C_G(g)) .$$

Refinements of this notion taking into account the mixed Hodge structure on the cohomology groups use the so-called ages of elements of the group as shifts of the corresponding graded components of the mixed Hodge structure (see, e.g., [2, 22]).

Let V be a complex quasi-projective manifold of dimension d with an action of a finite group G and with a G-equivariant automorphism φ of finite order. One can say that the notion of the orbifold spectrum of the triple (V, G, φ) is inspired by the notion of the orbifold Hodge–Deligne polynomial: [2].

Let G_*, $V^{\langle g \rangle}$ and $C_G(g)$ be defined as above. The group $C_G(g)$ acts on the fixed point set $V^{\langle g \rangle}$. Let $\hat{\varphi}$ be the transformation of the quotient $V^{\langle g \rangle}/C_G(g)$ induced by φ. For a point $x \in V^{\langle g \rangle}$, the *age* of g (or *fermion shift number*) is defined in the following way [15, Sect. 2.1], [23, Eq. (3.17)]. The element g acts on the tangent space $T_x V$ as a complex linear operator of finite order. It can be represented by a diagonal matrix with the diagonal entries $\mathbf{e}[\beta_1], \ldots, \mathbf{e}[\beta_d]$, where $0 \le \beta_i < 1$ for $i = 1, \ldots, d$ and $\mathbf{e}[r] := \exp(2\pi i r)$ for a real number r. The *age* of the element g at the point x is defined by $\mathrm{age}_x(g) = \sum_{i=1}^{d} \beta_i \in \mathbb{Q}_{\ge 0}$. For a rational number $\beta \ge 0$, let $V_\beta^{\langle g \rangle}$ be the subspace of the fixed point set $V^{\langle g \rangle}$ consisting of the points x with $\mathrm{age}_x(g) = \beta$. (The subspace $V_\beta^{\langle g \rangle}$ of $V^{\langle g \rangle}$ is a union of connected components of the latter one.)

Definition 3 (cf. [9]) The *orbifold spectrum* of the triple (V, G, φ) is

$$\mathrm{hsp}^{\mathrm{orb}}(V, G, \varphi) = \sum_{[g] \in G_*} \sum_{\beta \in \mathbb{Q}_{\ge 0}} \mathrm{hsp}(V_\beta^{\langle g \rangle}/C_G(g), \hat{\varphi}) \cdot \{\beta\} \in \mathbb{Z}[\mathbb{Q}] .$$

As above the spectrum $\mathrm{hsp}^{\mathrm{orb}}(V, G, \varphi)$ can be identified with the orbifold Poincaré polynomial

$$p^{\mathrm{orb}}_{(V,G,\varphi)}(t) = \sum_{[g] \in G_*} \sum_{\beta \in \mathbb{Q}_{\ge 0}} p_{(V_\beta^{\langle g \rangle}/C_G(g), \hat{\varphi})}(t) \cdot t^\beta \in \mathbb{Z}[t^{1/m}] .$$

It can be regarded as a reduction of the equivariant orbifold Poincaré polynomial

$$\overline{e}^{\text{orb}}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \overline{e}_{(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi})}(t)\cdot t^\beta \in R_f(\mathbb{Z})[t^{1/m}]$$

or of the equivariant orbifold Hodge–Deligne polynomial

$$e^{\text{orb}}_{(V,\varphi)}(u,v) = \sum_{k,p,q,\alpha,[g],\beta} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g)))^{p,q}\omega_{\mathbf{e}[\alpha]}\cdot u^p v^q (uv)^\beta$$

(an element of $R_f(\mathbb{Z})[u,v][(uv)^{1/m}]$).

As it was explained above, the presence of (rational) summands of different nature—elements of the quotient $\mathbb{Q}/\mathbb{Z}$ and elements of $\mathbb{Q}$ itself—leads to the situation when the existence of a Macdonald-type equation for the orbifold spectrum (and for the orbifold Poincaré polynomial) is doubtful. On the other hand, there exist Macdonald-type equations for the equivariant orbifold Poincaré polynomial and for the equivariant orbifold Hodge–Deligne polynomial (see Sect. 5). This inspires the definition of the corresponding version of the orbifold spectrum.

Definition 4 The *orbifold pair spectrum* $\text{hsp}^{\text{orb}}_2(V,G,\varphi)$ of (V,G,φ) is

$$\sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi}))^{p,q}\{(\alpha, p+\beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}]\,.$$

The word *pair* is used, in particular, to distinguish this notion from the one defined in [9]. Moreover, taking into account the weight filtration as well, one gets a certain refinement of this notion.

Definition 5 The *orbifold triple spectrum* $\text{hsp}^{\text{orb}}_3(V,G,\varphi)$ of (V,G,φ) is

$$\sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi}))^{p,q}\{(\alpha, p+\beta, q+\beta)\}$$

(an element of $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}\times\mathbb{Q}]$).

5 Higher-Order Spectrum and Equivariant Hodge–Deligne Polynomial

The notions of the higher-order spectrum of a triple (V,G,φ) and of the higher-order equivariant Hodge–Deligne polynomial of it are inspired by the notions of the higher-order Euler characteristic [1, 3] and of the corresponding higher-order generalized Euler characteristic [12]. For a topological G-space X, the Euler characteristic χ of order k can be defined by

$$\chi^{(k)}(X,G) = \sum_{[g]\in G_*} \chi^{(k-1)}(X^{\langle g\rangle}, C_G(g)),$$

where $\chi^{(0)}(X,G) := \chi(X/G)$ (see the notations in Sect. 4). One can see that $\chi^{(1)}(X,G) := \chi^{orb}(X,G)$. As for the orbifold Euler characteristic (i.e., for the Euler characteristic of order 1), refinements of these notions taking into account the mixed Hodge structure should use the age shift.

Let (V,G,φ), $V_\beta^{\langle g\rangle}$ and $\hat{\varphi}$ be as in Sect. 4 and let $k \geq 1$.

Definition 6 The *spectrum of order k* of the triple (V,G,φ) is

$$\mathrm{hsp}^{(k)}(V,G,\varphi) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g), \varphi)\cdot\{\beta\} \in \mathbb{Z}[\mathbb{Q}],$$

where $\mathrm{hsp}^{(0)}(V,G,\varphi) := \mathrm{hsp}(V/G,\hat{\varphi})$.

The orbifold spectrum is the spectrum of order 1.

Like above the spectrum of order k can be described by the corresponding order k Poincaré polynomial:

$$p^{(k)}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} p^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(t)\cdot t^\beta \in \mathbb{Z}[t^{1/m}],$$

where $p^{(1)}_\bullet(t) := p^{\mathrm{orb}}_\bullet(t)$.

It can be regarded as a reduction of the *equivariant order k Poincaré polynomial*

$$\overline{e}^{(k)}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \overline{e}^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(t)\cdot t^\beta \in R_f(\mathbb{Z})[t^{1/m}]$$

or of the *equivariant order k Hodge–Deligne polynomial*

$$e^{(k)}_{(V,G,\varphi)}(u,v) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} e^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(u,v)(uv)^\beta \in R_f(\mathbb{Z})[u,v][(uv)^{1/m}].$$

The following definition is an analogue of the definition of the orbifold pair and triple spectra in Sect. 4.

Definition 7 The *pair spectrum of order k* of (V,G,φ) is

$$\mathrm{hsp}_2^{(k)}(V,G,\varphi) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}_2^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g),\varphi)\{(0,\beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}].$$

The *triple spectrum of order k* of (V, G, φ) is

$$\begin{aligned}&\mathrm{hsp}_3^{(k)}(V, G, \varphi)\\ &= \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}_3^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g), \varphi)\{(0, \beta, \beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}]\,.\end{aligned}$$

The following statement is a Macdonald-type equation for the equivariant order k Hodge–Deligne polynomial. For $n \geq 1$, the Cartesian power V^n of the manifold V is endowed with the natural action of the wreath product $G_n = G \wr S_n = G^n \rtimes S_n$ generated by the componentwise action of the Cartesian power G^n and by the natural action of the symmetric group S_n (permuting the factors). Also one has the automorphism $\varphi^{(n)}$ of V^n induced by φ. The triple $(V_n, G_n, \varphi^{(n)})$ should be regarded as an analogue of the symmetric power of the triple (V, G, φ).

Example 1 Let $f(z_1, \ldots, z_n)$ be a quasi-homogeneous function with the quasi-weights $q_1, \ldots, q_n$ (and with the quasi-degree 1), and let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group of its symmetries ($f(g\underline{z}) = f(\underline{z})$ for $g \in G$). The Milnor fiber $M_f = \{f = 1\}$ of f is an $(n-1)$-dimensional complex manifold with an action of a group G and with a natural finite-order automorphism φ (the monodromy transformation or the exponential grading operator):

$$\varphi(z_1, \ldots, z_n) = (\mathbf{e}[q_1]z_1, \ldots, \mathbf{e}[q_n]z_n)\,.$$

For $s \geq 1$, let $\mathbb{C}^{ns} = (\mathbb{C}^n)^s$ be the affine space with the coordinates $z_i^{(j)}$, $1 \leq i \leq n$, $1 \leq j \leq s$. The system of equations $f(z_1^{(j)}, \ldots, z_n^{(j)}) = 0$, $j = 1, \ldots, s$, defines a complete intersection in $\mathbb{C}^{ns}$. Its Milnor fiber $M = \{f(z_1^{(j)}, \ldots, z_n^{(j)}) = 1,\ \text{for } j = 1, \ldots, s\}$ is the sth Cartesian power of the Milnor fiber M_f of f and has a natural action of the wreath product G_s. The spectrum of a complete intersection singularity is defined by a choice of a monodromy transformation. A natural monodromy transformation on M is the sth Cartesian power $\varphi^{(s)}$ of the monodromy transformation φ. Thus, the triple $(M, G_s, \varphi^{(s)})$ can be regarded as an analogue of the sth symmetric power of the triple (M_f, G, φ).

Theorem 2 *Let V be a (smooth) quasi-projective G-manifold of dimension d with a G-equivariant automorphism φ of finite order. One has*

$$\begin{aligned}&1 + \sum_{n=1}^{\infty} e^{(k)}_{(V^n, G_n, \varphi^{(n)})}(u, v) \cdot T^n\\ &= \left(\prod_{r_1,\ldots,r_k\geq 1} \left(1 - (uv)^{(r_1r_2\cdots r_k)d/2} \cdot T^{r_1r_2\cdots r_k}\right)^{r_2r_3^2\cdots r_k^{k-1}} \right)^{-e^{(k)}_{(V,G,\varphi)}(u,v)}, \qquad (3)\end{aligned}$$

where the RHS of (3) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[u, v][(uv)^{1/m}]$.

Proof In [13], there were defined equivariant generalized higher-order Euler characteristics of a complex quasi-projective manifold with commuting actions of two finite groups G_O and G_B as elements of the extension $K_0^{G_B}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ of the Grothendieck ring $K_0^{G_B}(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective G_B-varieties ($\mathbb{L}$ is the class of the complex affine line with the trivial action), and there were given Macdonald-type equations for them: [13, Theorem 2]. One can see that these definitions and the Macdonald-type equations can be applied when instead of an action of a finite group G_B, one has a finite order action of the cyclic group $\mathbb{Z}$. The equivariant order k Hodge–Deligne polynomial is the image of the equivariant generalized Euler characteristic of order k under the map $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}] \to R_f(\mathbb{Z})[u, v][(uv)^{1/m}]$. Since this map is a λ-ring homomorphism (Theorem 1), the Macdonald-type equation for the equivariant generalized Euler characteristic of order k implies (3).

Corollary 2 *In the situation described above, one has*

$$1 + \sum_{n=1}^{\infty} \mathrm{hsp}_{\nu}^{(k)}(V^n, G_n, \varphi^{(n)}) \cdot T^n$$

$$= \left(\prod_{r_1,\ldots,r_k \geq 1} \left(1 - a^{(\nu)}_{r_1,\ldots,r_k} T^{r_1 r_2 \cdots r_k}\right)^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-\mathrm{hsp}_{\nu}^{(k)}(V,G,\varphi)},$$

where $\nu = 2, 3$*,*

$$a^{(2)}_{r_1,\ldots,r_k} = \{(0, (r_1 r_2 \cdots r_k)d/2)\},$$

$$a^{(3)}_{r_1,\ldots,r_k} = \{(0, (r_1 r_2 \cdots r_k)d/2, (r_1 r_2 \cdots r_k)d/2)\},$$

and the RHS is understood in terms of the power structures over the group rings $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}]$ *for* $\nu = 2$ *and* $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}]$ *for* $\nu = 3$.

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