Wolfram Decker · Gerhard Pfister
Mathias Schulze *Editors*

Singularities and Computer Algebra

Festschrift for Gert-Martin Greuel on the Occasion of his 70th Birthday

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Editors
Wolfram Decker
Department of Mathematics
TU Kaiserslautern
Kaiserslautern, Germany

Gerhard Pfister
Department of Mathematics
TU Kaiserslautern
Kaiserslautern, Germany

Mathias Schulze
Department of Mathematics
TU Kaiserslautern
Kaiserslautern, Germany

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Dedicated to Gert-Martin Greuel
on the occasion of his seventieth birthday

Preface

The present proceedings volume arose from a conference on “Singularities and Computer Algebra,” which was held at the Pfalz-Akademie Lambrecht in June 2015 in honor of Gert-Martin Greuel’s 70th birthday . It was attended by roughly 50 participants from Germany, the Netherlands, Switzerland, France, the United Kingdom, Spain, Hungary, Ukraine, Israel, Mexico, Vietnam, Russia and the United States. Most of the participants listed below were influenced by Greuel’s work on singularities and their computational aspects over the last 40 years. Among them, there were colleagues and friends from the early years in Göttingen and Bonn, but also Gert-Martin Greuel’s past and present diploma and Ph.D. students at Kaiserslautern. In particular, each of the invited speakers listed below has collaborated with Greuel in one way or another. These collaborations involved a wide range of topics in singularity theory such as topological and algebraic aspects, classification problems, deformation theory and resolution of singularities. Accordingly, the articles in this volume present a diverse portrait of singularity theory and its evolution over the past few decades.

Greuel’s contributions to mathematics touch on a broad range of topics in singularity theory. His publications list includes more than 100 articles, the importance of which is demonstrated by more than 1000 citations on MathSciNet. With Gerhard Pfister and Hans Schönemann, he developed the computer algebra system SINGULAR, which has since become the computational tool of choice for many singularity theorists. For many years Greuel organized the conference series “Singularities” at the mathematical research institute in Oberwolfach, which was a driving force for the further development of the area. In subsequent years his commitment as director further strengthened the international role of Oberwolfach as one of the most prominent and influential locations of its kind. In order to share his fascination with mathematics with the general public, Greuel created the touring mathematical exhibition IMAGINARY.[1] In Fig. 1 we include a picture of the

[1] See www.imaginary.org.

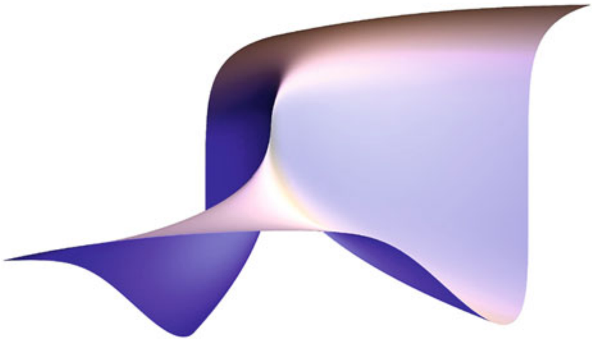

Fig. 1 The E_8 singularity visualized using SURFER

E_8 singularity that is part of the exhibition. The underlying visualization software ("SURFER") is addressed in the article by Stefan Klaus.

We would like to thank everyone who contributed to the success of the conference and to the proceedings. Special thanks go to Petra Bäsell and Cornelia Rottner.

Kaiserslautern, Germany — Wolfram Decker
Kaiserslautern, Germany — Gerhard Pfister
Kaiserslautern, Germany — Mathias Schulze
August 2016

Invited Lectures

Norbert A'Campo
"A real analytic cell complex for the braid group"

Wilfried Bruns
"Normal lattice polytopes"

Igor Burban
"Torsion free sheaves on degenerate elliptic curves and the classical Yang–Baxter equation"

Antonio Campillo
"Polarity maps, singular subschemes, and applications"

Yuriy Drozd
"Minors and categorical resolutions"

Wolfgang Ebeling
"Orbifold zeta functions for dual invertible polynomials"

Anne Frühbis-Krüger
"Vanishing topology of Cohen–Macaulay codimension 2 3-fold"

Xavier Gómez-Mont
"Multiplication by f in the Jacobian algebra as bindings in the spectrum of a hypersurface with an isolated singularity"

Sabir Gusein-Zade
"Higher order Euler characteristics, their generalizations and generating series"

Helmut Hamm
“Divisor class groups of affine complete intersections”

Claus Hertling
“Marked singularities, their moduli spaces, and atlases of Stokes data”

Dmitry Kerner
“Recombination formulas for the spectrum of plane curve singularities”

Stephan Klaus
“Möbius strips, knots, pentagons, polyhedra and the SURFER software”

Viktor Kulikov
“Rational plane quartics and $K3$ surfaces”

Lê Dũng Tráng
“Lipschitz regularity of complex analytic spaces”

Viktor Levandovskyy
“Computational D-module theory and singularities”

András Némethi
“Old and new regarding the Seiberg–Witten invariant conjecture”

Ngô Việt Trung
“Depth and regularity of powers of sums of ideals”

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“Milnor number, unfolding and modality of isolated hypersurface singularities in positive characteristic”

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“Primary decomposition and paralleliziation”

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“Remarks on the Lê–Greuel formula for the Milnor number”

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“Enumeration of real algebraic curves: Local aspects”

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“Hypersurfaces with 1-dimensional singularities”

Joseph Steenbrink
“Resolutions of cubical varieties”

Jan Stevens
“Deforming non-normal isolated surface singularities”

Günther Trautmann
“Algebraic bubbling for vector bundles on surfaces”

Duco van Straten
“Singular aspects of the scientific work of Gert-Martin Greuel”

Jonathan Wahl
“Equisingular moduli of rational surface singularities”

List of Participants

Norbert A’Campo (Universität Basel)
Klaus Altmann (FU Berlin)
Enrique Artal
Mohamed Barakat (TU Kaiserslautern)
Janko Böhm (TU Kaiserslautern)
Wilfried Bruns (Universität Osnabrück)
Igor Burban (Universität zu Köln)
Antonio Campillo (Universidad de Valladolid)
Wolfram Decker
Yuriy Drozd (National Academy of Sciences of Ukraine)
Wolfgang Ebeling (Leibniz Universität Hannover)
Christian Eder (TU Kaiserslautern)
Claus Fieker (TU Kaiserslautern)
Anne Frühbis-Krüger (Leibniz Universität Hannover)
Xavier Gómez-Mont (CIMAT)
Gert-Martin Greuel (TU Kaiserslautern)
Sabir Gusein-Zade (Moscow State University)
Helmut Hamm (Universität Münster)
Claus Hertling (Universität Mannheim)
Dmitry Kerner (Ben-Gurion University of the Negev)
Stephan Klaus (Mathematisches Forschungsinstitut Oberwolfach)
Viktor Kulikov (Steklov Mathematical Institute)
Herbert Kurke (Humboldt-Universität zu Berlin)
Lê Dũng Tráng (Université Aix Marseille and UFC, Fortaleza)
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Gunter Malle (TU Kaiserslautern)
Hannah Markwig (Universität des Saarlandes)
Thomas Markwig (TU Kaiserslautern)
Oleksandr Motsak (TU Kaiserslautern)
András Némethi (A. Rényi Institute of Mathematics)
Ngô Việt Trung (Institute of Mathematics, VAST)
Hong Duc Nguyen
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Joseph Steenbrink (Radboud Universiteit Nijmegen)
Andreas Steenpaß (TU Kaiserslautern)
Jan Stevens (University of Gothenburg)
Günther Trautmann (TU Kaiserslautern)
Duco van Straten (Universität Mainz)
Jonathan Wahl (University of North Carolina)
C.T.C. Wall (University of Liverpool)

Contents

On Some Conjectures About Free and Nearly Free Divisors

Enrique Artal Bartolo, Leire Gorrochategui, Ignacio Luengo, and Alejandro Melle-Hernández

Abstract In this paper we provide infinite families of non-rational irreducible free divisors or nearly free divisors in the complex projective plane. Moreover, their corresponding local singularities can have an arbitrary number of branches. All these examples contradict some of the conjectures proposed by Dimca and Sticlaru. Our examples say nothing about the most remarkable conjecture by A. Dimca and G. Sticlaru, which predicts that every rational cuspidal plane curve is either free or nearly free.

Keywords Free divisors • Nearly free curves

Subject Classifications:14A05, 14R15

1 Introduction

The notion of free divisor was introduced by Saito [21] in the study of discriminants of versal unfoldings of germs of isolated hypersurface singularities. Since then many interesting and unexpected applications to singularity theory and algebraic geometry have been appearing. In this paper, we are mainly focused on complex projective

E. Artal Bartolo
IUMA, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, c/ Pedro Cerbuna 12, 50009 Zaragoza, Spain
e-mail: artal@unizar.es

L. Gorrochategui
Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain
e-mail: leire.gg@gmail.com

I. Luengo • A. Melle-Hernández (✉)
ICMAT (CSIC-UAM-UC3M-UCM), Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense, Plaza de las Ciencias 3, 28040 Madrid, Spain
e-mail: iluengo@mat.ucm.es; amelle@mat.ucm.es

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plane curves, and we adapt the corresponding notions and results to this setup. The results contained in this paper have needed a lot of computations in order to get the correct statements. All of them have been done using the computer algebra system `Singular` [9] through `Sagemath` [24]. We thank `Singular`'s team for such a great mathematical tool and especially to Gert-Martin, to whom we dedicate this paper, for his dedication to `Singular` development.

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d in the polynomial ring, let $C \subset \mathbb{P}^2$ be defined by $f = 0$. Assume that C is reduced. We denote by J_f the Jacobian ideal of f, which is the homogeneous ideal in S spanned by f_x, f_y, f_z. We denote by $M(f) = S/J_f$ the corresponding graded ring, called the Jacobian (or Milnor) algebra of f.

Let I_f be the saturation of the ideal J_f with respect to the maximal ideal (x, y, z) in S and let $N(f) = I_f/J_f$ be the corresponding graded quotient. Recall that the curve $C : f = 0$ is called a *free divisor* if $N((f) = I_f/J_f = 0$; see, e.g., [23].

Dimca and Sticlaru introduced in [13] the notion of nearly free divisor which is a slight modification of the notion of free divisor. The curve C is called *nearly free divisor* if $N(f) \neq 0$ and $\dim_{\mathbb{C}} N(f)_k \leq 1$ for any k.

The main results in [12, 13] and many series of examples motivate the following conjecture.

Conjecture 1.1 ([13])

(i) Any rational cuspidal curve C in the plane is either free or nearly free.
(ii) An irreducible plane curve C which is either free or nearly free is rational.

In [13], the authors provide some interesting results supporting the statement of Conjecture 1.1(i); in particular, Conjecture 1.1(i) holds for rational cuspidal curves of even degree [13, Theorem 4.1]. They need a topological assumption on the cusps which is not fulfilled in general when the degree is odd; see [13, Theorem 4.1].

They proved also that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ or for those curves whose degree is a prime power; see [13, Corollary 4.2] and the discussion in [4].

Using the classification given in [15] of unicuspidal rational curve with a unique Puiseux pair, Dimca and Sticlaru proved in [13, Corollary 4.5] that all of them are either free divisor or nearly free divisor, except the curves of odd degree in one of the cases of the classification.

As for Conjecture 1.1(ii), note that reducible nearly free curves may have irreducible components which are not rational; see [13, Example 2.8]: a smooth cubic with three tangents at aligned inflection points is nearly free (note that the condition of alignment can be removed, at least in some examples computed using `Singular` [9]). For free curves, examples can be found using [29, Theorem 2.7], e.g., $(x^3 - y^3)(y^3 - z^3)(x^3 - z^3)(ax^3 + by^3 + cz^3)$ for generic $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$. The conjectures in [29] give some candidate examples of smaller degree; it is possible to prove that $(y^2z-x^3)(y^2z-x^3-z^3) = 0$ is free (also computed with `Singular` [9]). Dimca and Sticlaru also proposed the following conjecture.

Conjecture 1.2 ([13])

(i) Any free irreducible plane curve C has only singularities with at most two branches.
(ii) Any nearly free irreducible plane curve C has only singularities with at most three branches.

In this paper we give some examples of irreducible free divisors and nearly free divisors in the complex projective plane which are not rational curves giving counterexamples to Conjecture 1.1(ii). Using these counterexamples we have found some examples of irreducible free divisors whose two singular points have any odd number of branches, giving counterexamples to Conjecture 1.2(i). Furthermore, some irreducible nearly free divisors with just one singular point which has four branches giving counterexamples to Conjecture 1.2(ii) are provided too.

Section 2 is devoted to collect well-known results in the theory of free divisors and nearly free divisors mainly from their original papers of Dimca and Sticlaru in [12, 13]. Also a characterization for being nearly free reduced plane curve from Dimca in [10] is recalled. This characterization is similar to the characterization of being free given by du Plessis and Wall in [20].

From Sect. 3.2, it can be deduced that for every odd integer $k \geq 1$, the irreducible plane curve C_{5k} of degree $d = 5k$ defined by

$$C_{5k} : f_{5k} := (y^k z^k - x^{2k})^2 y^k - x^{5k} = 0$$

satisfies:

1. Its geometric genus is $g(C_{5k}) = \frac{(k-1)(k-2)}{2}$;
2. its singular set consists of two points and the number of branches of C_{5k} at each of them is exactly k,
3. C_{5k} is a free divisor, see Theorem 3.9.

This is a counterexample to both the free divisor part of Conjectures 1.1(ii) and 1.2(i).

From Sect. 3.3 it can also be deduced that for any odd integer $k \geq 1$, the irreducible plane curve C_{4k} of degree $d = 4k$ defined by

$$C_{4k} : f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0$$

satisfies:

1. Its geometric genus is $g(C_{4k}) = \frac{(k-1)(k-2)}{2}$;
2. its singular set consists of two points and the number of branches of C_{4k} at each of them is exactly k,
3. C_{4k} is a nearly free divisor, see Theorem 3.11.

This is a counterexample to both the nearly free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(ii) too.

In the families studied above, the number of singular points of the curves is exactly two. In Sect. 3.4, we are looking for curves giving a counterexample to the nearly free divisor part of Conjecture 1.1(ii) with unbounded genus and number of singularities. In particular, for every odd integer $k \geq 1$, the irreducible curve C_{2k} of degree $d = 2k$ defined by

$$C_{2k}: \; f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0$$

satisfies:

1. Its geometric genus is $g(C_{2k}) = \frac{(k-1)(k-2)}{2}$;
2. its singular set $\operatorname{Sing}(C_{2k})$ consists of exactly $3k$ singular points, each of them of type $\mathbb{A}_{k-1}$,
3. C_{2k} is a nearly free divisor, see Theorem 3.12.

This is a counterexample to both the nearly free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(ii) too.

One of the main tools to find such examples is the use of Kummer covers. A Kummer cover is a map $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by

$$\pi_k([x : y : z]) := [x^k : y^k : z^k].$$

Since Kummer covers are finite Galois unramified covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\operatorname{Gal}(\pi_k) \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$, Kummer covers are a very useful, using them one can construct complicated algebraic curves starting from simple ones. We meanly refer to [3, §5] for a systematic study of Kummer covers.

In particular, these families of examples $\{C_{5k}\}$ (which are free divisors), $\{C_{4k}\}$, and $\{C_{2k}\}$ (which are nearly free divisors) are constructed as the pullback under the Kummer cover π_k of the corresponding rational cuspidal curves: the quintic C_5 which is a free divisor, and the corresponding nearly free divisors defined by either, the cuartic C_4, or by the conic C_2.

In the last section, Sect. 4, an irreducible curve C_{49} of degree 49 satisfying

1. its genus is $g(C_{49}) = 0$,
2. its singular set consists of just one singular point which has four branches,
3. C_{49} is a nearly free divisor.

These examples can be constructed as a general element of the unique pencil associated to any rational unicuspidal plane curve; see [8].

2 Free and Nearly Free Plane Curves After Dimca and Sticlaru

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d in the polynomial ring. Let C be the plane curve in $\mathbb{P}^2$ defined by $f = 0$ and assume that C is reduced. We have denoted by J_f the Jacobian ideal of f, which is the homogeneous ideal in S spanned by f_x, f_y, f_z. Let $M(f) = S/J_f$ be the corresponding graded ring, called the Jacobian (or Milnor) algebra of f.

The minimal degree of a Jacobian relation for f is the integer $\mathrm{mdr}(f)$ defined to be the smallest integer $m \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0, \quad (a, b, c) \in S_m^3 \setminus (0, 0, 0). \tag{1}$$

When $\mathrm{mdr}(f) = 0$, then C is a union of lines passing through one point, a situation easy to analyze. We assume from now on that $\mathrm{mdr}(f) \geq 1$.

2.1 Free Plane Curves

We have denoted by I_f the saturation of the ideal J_f with respect to the maximal ideal (x, y, z) in S. Let $N(f) = I_f/J_f$ be the corresponding homogeneous quotient ring.

Consider the graded S-submodule

$$\mathrm{AR}(f) = \{(a, b, c) \in S^3 \mid af_x + bf_y + cf_z = 0\} \subset S^3$$

of *all relations* involving the partial derivatives of f, and denote by $\mathrm{AR}(f)_m$ its homogeneous part of degree m.

Notation 2.1 We set the following: $\mathrm{ar}(f)_k = \dim \mathrm{AR}(f)_k$, $m(f)_k = \dim M(f)_k$, and $n(f)_k = \dim N(f)_k$ for any integer k.

We use the definition of freeness given by Dimca in [10].

Definition 2.2 The curve $C : f = 0$ is a *free divisor* if the following equivalent conditions hold.

1. $N(f) = 0$, i.e. the Jacobian ideal is saturated.
2. The minimal resolution of the Milnor algebra $M(f)$ has the following form

$$0 \to S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \to S^3(-d + 1) \xrightarrow{(f_x, f_y, f_z)} S$$

for some positive integers d_1, d_2.

3. The graded S-module $\mathrm{AR}(f)$ is free of rank 2, i.e. there is an isomorphism

$$\mathrm{AR}(f) = S(-d_1) \oplus S(-d_2)$$

for some positive integers d_1, d_2.

When C is a free divisor, the integers $d_1 \leq d_2$ are called the *exponents* of C. They satisfy the relations

$$d_1 + d_2 = d - 1 \text{ and } \tau(C) = (d-1)^2 - d_1 d_2, \tag{2}$$

where $\tau(C)$ is the *total Tjurina number* of C; see, for instance, [11, 12]. Using deformation results in [23], Sticlaru [25] defines a curve $C \subset \mathbb{P}^2$ to be *projectively rigid* if $(I_f)_d = (J_f)_d$. In particular, if C is free, then it is projectively rigid.

Remark 2.3 This notion of *projectively rigid* differs from the classical one; see, e.g., [16], where a curve is projectively rigid if its equisingular moduli space is discrete. Note that four lines passing through a point define a free divisor, but its equisingular moduli space is defined by the cross ratio.

2.2 Nearly Free Plane Curves

Dimca and Sticlaru introduced a more subtle notion for a divisor to be nearly free; see [13].

Definition 2.4 ([13]) The curve $C : f = 0$ is a *nearly free divisor* if the following equivalent conditions hold.

1. $N(f) \neq 0$ and $n(f)_k \leq 1$ for any k.
2. The Milnor algebra $M(f)$ has a minimal resolution of the form

$$0 \to S(-d-d_2) \to S(-d-d_1+1) \oplus S^2(-d-d_2+1) \to S^3(-d+1) \xrightarrow{(f_0, f_1, f_2)} S \tag{3}$$

   for some integers $1 \leq d_1 \leq d_2$, called the *exponents of* C.
3. There are three syzygies ρ_1, ρ_2, ρ_3 of degrees $d_1, d_2 = d_3 = d - d_1$ which form a minimal system of generators for the first-syzygy module $\mathrm{AR}(f)$.

   If $C : f = 0$ is nearly free, then the exponents $d_1 \leq d_2$ satisfy

$$d_1 + d_2 = d \text{ and } \tau(C) = (d-1)^2 - d_1(d_2 - 1) - 1; \tag{4}$$

see [13]. For both a free and a nearly free curve $C : f = 0$, it is clear that $\mathrm{mdr}(f) = d_1$.

Remark 2.5 In [13] it is shown that to construct a resolution (3) for a given polynomial f, the following conditions must be satisfied:

(i) the integer $b := d_2 - d + 2$,
(ii) three syzygies $r_i = (a_i, b_i, c_i) \in S^3_{d_i}$, $i = 1, 2, 3$, for (f_x, f_y, f_z), i.e.

$$a_i f_x + b_i f_y + c_i f_z = 0,$$

necessary to construct the morphism

$$\bigoplus_{i=1}^{3} S(-d_i - (d-1)) \to S^3(-d+1), \quad (u_1, u_2, u_3) \mapsto u_1 r_1 + u_2 r_2 + u_3 r_3,$$

(iii) one relation $R = (v_1, v_2, v_3) \in \bigoplus_{i=1}^{3} S(-d_i - (d-1))_{b+2(d-1)}$ among r_1, r_2, r_3, i.e. $v_1 r_1 + v_2 r_2 + v_3 r_3 = 0$, necessary to construct the morphism

$$S(-b - 2(d-1)) \to \bigoplus_{i=1,3} S(-d_i - (d-1))$$

by the formula $w \mapsto wR$. Note that $v_i \in S_{b-d_i+d-1}$.

Corollary 2.6 ([13]) *Let $C : f = 0$ be a nearly free curve of degree d with exponents (d_1, d_2). Then $N(f)_k \neq 0$ for $d + d_1 - 3 \leq k \leq d + d_2 - 3$ and $N(f)_k = 0$ otherwise. The curve C is projectively rigid if and only if $d_1 \geq 4$.*

2.3 Characterization of Free and Nearly Free Reduced Plane Curves

Just recently Dimca provides in [10] the following characterization of free and nearly free reduced plane curve C of degree d. For a positive integer r, the following integer is defined:

$$\tau(r)_{\max} := (d-1)(d-r-1) + r^2.$$

Theorem 2.7 ([10]) *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d defined by $f = 0$, and let $r :=$ mdr(f).*

(1) *If $r < \frac{d}{2}$, then $\tau(C) = \tau(r)_{\max}$ if and only if $C : f = 0$ is a free divisor.*
(2) *If $r \leq \frac{d}{2}$, then $\tau(C) = \tau(r)_{\max} - 1$ if and only if C is a nearly free divisor.*

As it is recalled in [10], Theorem 2.7(1) is Corollary of [20, Theorem 3.2] by du Plessis and Wall.

3 High-Genus Curves Which Are Free or Nearly Free Divisors

3.1 Transformations of Curves by Kummer Covers

A Kummer cover is a map $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_k([x : y : z]) := [x^k : y^k : z^k]$. Kummer covers are a very useful tool in order to construct complicated algebraic curves starting from simple ones. Since Kummer covers are finite Galois unramified covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\mathrm{Gal}(\pi_k) \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$, topological properties of the new curves can be obtained, for instance, Alexander polynomial, fundamental group, characteristic varieties, and so on (see [1–3, 5, 6, 17, 18, 28] for papers using these techniques).

Example 3.1 In [28], Uludağ constructs new examples of Zariski pairs using former ones and Kummer covers. He also uses the same techniques to construct infinite families of curves with finite non-abelian fundamental groups.

Example 3.2 In [5, 17], the Kummer covers allow to construct curves with *many cusps* and extremal properties for their Alexander invariants. These ideas are pushed further in [6] where the authors find Zariski triples of curves of degree 12 with 32 ordinary cusps (distinguished by their Alexander polynomial). Within the same ideas, Niels Lindner [18] constructed an example of a cuspidal curve C' of degree 12 with 30 cusps and Alexander polynomial $t^2 - t + 1$. For this, he started with a sextic C_0 with six cusps, admitting a toric decomposition. He pulled back C_0 under a Kummer map $\pi_2 : \mathbb{P}^2 \to \mathbb{P}^2$ ramified above three inflectional tangents of C_0. Since the sextic is of torus type, then the same holds for the pullback. Lindner showed that the Mordell-Weil lattice has rank 2 and that the Mordell-Weil group contains $A_2(2)$.

A systematic study of Kummer covers of projective plane curves has been done by J.I. Cogolludo, J. Ortigas, and the first named author in [3, §5]. Some of their results are collected below.

Let C be a (reduced) projective curve of degree d of equation $F_d(x, y, z) = 0$ and let $\bar{C}_k$ be its transform by a Kummer cover π_k, $k \geq 1$. Note that $\bar{C}_k$ is a projective curve of degree dk of equation $F_d(x^k, y^k, z^k) = 0$.

Definition 3.3 ([3]) We define $P \in \mathbb{P}^2$ such that $P := [x_0 : y_0 : z_0]$. We say that P is a point of *type* $(\mathbb{C}^*)^2$ (or simply of *type* 2) if $x_0y_0z_0 \neq 0$. If $x_0 = 0$ but $y_0z_0 \neq 0$, the point is said to be of *type* $\mathbb{C}^*_x$ (types $\mathbb{C}^*_y$ and $\mathbb{C}^*_z$ are defined accordingly). Such points will also be referred to as *type* 1 points. The corresponding line (either $L_X := \{X = 0\}$, $L_Y := \{Y = 0\}$, or $L_Z := \{Z = 0\}$) the type-1 point lies on will be referred to as its *axis*. The remaining points $P_x := [1 : 0 : 0]$, $P_y := [0 : 1 : 0]$, and $P_z := [0 : 0 : 1]$ will be called *vertices* (or type 0 points), and their axes are the two lines (either L_X, L_Y, or L_Z) they lie on.

Remark 3.4 ([3]) Note that a point of type ℓ, $\ell = 0, 1, 2$ in $\mathbb{P}^2$ has exactly k^ℓ preimages under π_k. It is also clear that the local types of $\bar{C}_k$ at any two points on the same fiber are analytically equivalent. The singularities of $\bar{C}_k$ are described in the following proposition.

Proposition 3.5 ([3]) *Let $P \in \mathbb{P}^2$ be a point of type ℓ and $Q \in \pi_k^{-1}(P)$. The following conditions hold:*

(1) *If $\ell = 2$, then (C, P) and $(\bar{C}_k, Q)$ are analytically isomorphic.*
(2) *If $\ell = 1$, then $(\bar{C}_k, Q)$ is a singular point of type 1 if and only if $m > 1$, where $m := (C \cdot \bar{L})_P$ and $\bar{L}$ is the axis of P.*
(3) *If $\ell = 0$, then $(\bar{C}_k, Q)$ is a singular point.*

Remark 3.6 Using Proposition 3.5 (1), if $\mathrm{Sing}(C) \subset \{xyz = 0\}$, then $\mathrm{Sing}(\bar{C}_k) \subset \{xyz = 0\}$.

Example 3.7 ([3]) In some cases, we can be more explicit about the singularity type of $(\bar{C}_k, Q)$. If P is of type 1, (C, P) is smooth, and $m := (C \cdot \bar{L})_P$, then $(\bar{C}_k, Q)$ has the same topological type as $u_0^k - v_0^m = 0$. In particular, if $m = 2$, then $(\bar{C}_k, Q)$ is of type $\mathbb{A}_{k-1}$.

In order to better describe singular points of type 0 and of type 1 of $\bar{C}_k$, we will introduce the following notation. Let $P \in \mathbb{P}^2$ be a point of type $\ell = 0, 1$ and $Q \in \pi_k^{-1}(P)$ a singular point of $\bar{C}_k$. Denote by μ_P (resp. μ_Q) the Milnor number of C at P (resp. $\bar{C}_k$ at Q). Since $\ell = 0, 1$, then P and Q belong to either exactly one or two axes. If P and Q belong to an axis $\bar{L}$, then $m_P^{\bar{L}} := (C \cdot \bar{L})_P$ (analogous notation for Q). More specific details about singular points of types 0 and 1 can be described as follows.

Proposition 3.8 ([3]) *Under the above conditions and notation, the following conditions hold:*

(1) *For $\ell = 1$, P belongs to a unique axis $\bar{L}$ and*

   a. *$\mu_Q = k\mu_P + (m_P^{\bar{L}} - 1)(k - 1)$,*
   b. *and, if (C, P) is locally irreducible and $r := \gcd(k, m_P^{\bar{L}})$, then (C, Q) has r irreducible components which are analytically isomorphic to each other.*

(2) *For $\ell = 0$, P belongs to exactly two axes $\bar{L}_1$ and $\bar{L}_2$*

   a. *$\mu_Q = k^2(\mu_P - 1) + k(k - 1)(m_P^{\bar{L}_1} + m_P^{\bar{L}_2}) + 1$ (There is a typo in the printed formula in [3]: $k - k^2$ must be added).*
   b. *and, if (C, P) is locally irreducible and $r := \gcd(k, m_P^{\bar{L}_1}, m_P^{\bar{L}_2})$, then (C, Q) has kr irreducible components which are analytically isomorphic to each other.*

3.2 Irreducible Free Curves with Many Branches and High Genus

Let us consider the quintic curve C_5 (see Fig. 1), defined by $f_5 := (yz-x^2)^2y-x^5 = 0$. It has two singular points, $p_1 = [0 : 1 : 0]$ of type $\mathbb{A}_4$ and $p_2 = [0 : 0 : 1]$ of type $\mathbb{E}_8$. Therefore, it is a rational and cuspidal plane curve. This curve is free; see [12, Theorem 4.6]. Let us consider the Kummer cover $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_k([x : y : z]) := [x^k : y^k : z^k]$ and its Kummer transform C_{5k}, defined by $f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k} = 0$.

Theorem 3.9 *For any $k \geq 1$, the curve C_{5k} of degree $d = 5k$ defined by*

$$C_{5k} : f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k} = 0, \tag{5}$$

verifies the following properties:

(1) $\operatorname{Sing}(C_{5k}) = \{p_1, p_2\}$. *The number of branches of C_{5k} at p_2 is k, and at p_1, it equals k (if k is odd) or $2k$ (if k is even).*
(2) *C_{5k} is a free divisor with exponents $d_1 = 2k$, $d_2 = 3k - 1$ and $\tau(C_{5k}) =$* $19k^2 - 8k + 1$.
(3) *C_{5k} has two irreducible components of genus $\frac{(k-2)^2}{4}$ if k is even and irreducible of genus $\frac{(k-1)(k-2)}{2}$ otherwise.*

Proof Part (1) is an easy consequence of [3, Lemma 5.3, Propositions 5.4, and 5.6]. The singularities $\operatorname{Sing}(C_5) = \{p_1, p_2\}$ are of type 0, in the sense of the Kummer cover π_k (see Definition 3.3), and C_5 has no singularities outside the intersection points of the axes. Moreover C_5 intersects the line L_z transversally at a point of type 1; then by Proposition 3.5 (2) and by Remark 3.6, the singularities of C_{5k} are exactly the points p_1 and p_2.

Since p_1 and p_2 are of type 0, we deduce the structure of C_{5k} at these points using Proposition 3.8 (2) (b). At p_1 one has $(C_5, L_z)_{p_1} = 5$, $(C_5, L_x)_{p_1} = 2$, and $r_{p_1} = \gcd(k, 2, 5) = 1$ for all k , and so that the number of branches of C_{5k} at p_1 is equal to k. On the other hand, to study the number of branches at p_2, we compute the intersection numbers $(C_5, L_x)_{p_2} = 2$ and $(C_5, L_y)_{p_2} = 4$ and therefore

Fig. 1 Curve C_5

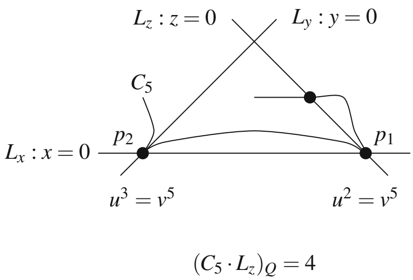


Table 1 Bases of syzygies

Ideal	First generator	Second generator
J	$R_1 \cdot D_{1,y,1}$	$R_2 \cdot D_{1,1,z}$
J_x	$R_1 \cdot D_{1,y,1}$	$R_2 \cdot D_{1,x,xz}$
J_y	R_1	$R_2 \cdot D_{y,1,yz}$
J_z	$R_1 \cdot D_{1,yz,1}$	R_2
J_{xy}	R_1	$R_2 \cdot D_{y,x,xyz}$
J_{xz}	$R_1 \cdot D_{1,yz,1}$	$R_2 \cdot D_{1,x,x}$
J_{yz}	$R_1 \cdot D_{1,z,1}$	$R_2 \cdot D_{y,1,y}$
J_{xyz}	$R_1 \cdot D_{1,z,1}$	$R_2 \cdot D_{y,x,xy}$

$r_{p_2} = \gcd(k, 2, 4) = \gcd(k, 2)$. Ik k is odd, $r_{p_2} = 1$ and the number of branches of C_{5k} at p_2 is equal to k. Otherwise $r_{p_2} = 2$ and the number of branches of C_{5k} at p_2 is equal to $2k$.

In order to prove (2), we follow the ideas of [12, Theorem 4.6]. Let us study first the syzygies of the free curve C_5. Let us denote by $D_{u,v,w}$, the diagonal matrix with entries u, v, w, and define the vectors

$$R_1 = \bigl(0,\ 2y,\ x^2 - 3yz\bigr)\,, \quad R_2 = \bigl(2(x^2 - yz), 2(5x^2 - 4xy + 15yz), 8x - 45z\bigr)\,.$$

Let us denote by J the Jacobian ideal J of f_5. Let us denote by J_x the ideal generated by $(xf_{5x}, f_{5y}, f_{5z})$. In the same way, we consider the ideals J_y, J_z, J_{xy}, J_{xz}, J_{yz}, J_{xyz}. Table 1 shows bases for the syzygies of these ideals, computed with `Singular` [9]. Note that

$$f_{5k_x} = kx^{k-1} f_{5_x}(x^k, y^k, z^k),\ f_{5k_y} = ky^{k-1} f_{5_y}(x^k, y^k, z^k),\ f_{5k_z} = kz^{k-1} f_{5_z}(x^k, y^k, z^k).$$

Let $S_k := \mathbb{C}[x^k, y^k, z^k]$. We have a decomposition

$$S = \bigoplus_{(i,j,l)\in\{0,\ldots,k-1\}} x^i y^j z^l S_k. \tag{6}$$

By construction, $f_{5k_x} \in x^{k-1} S_k$, $f_{5k_y} \in y^{k-1} S_k$ and $f_{5k_z} \in z^{k-1} S_k$. Hence, in order to compute the syzygies (a, b, c) among the partial derivatives of f_{5k}, we need to characterize the triples (a, b, c) such that each entry belongs to a factor of the decomposition (6).

Let us assume that $a \in x^{i_x} y^{j_x} z^{j_x} S_k$, $b \in x^{i_y} y^{j_y} z^{j_y} S_k$, and $c \in x^{i_z} y^{j_z} z^{j_z} S_k$. We deduce that

$$i_x + k - 1 \equiv i_y \equiv i_z \bmod k \Longrightarrow i = i_y = i_z \text{ and } i_x = \begin{cases} i+1 & \text{if } i < k-1 \\ 0 & \text{if } i = k-1. \end{cases}$$

Analogous relations hold for the other indices. We distinguish four cases:

Case 1 $i = j = l = k - 1$.

In this case $a(x, y, z) = y^{k-1}z^{k-1}\alpha(x^k, y^k, z^k)$, $b(x, y, z) = x^{k-1}z^{k-1}\beta(x^k, y^k, z^k)$, and $c(x, y, z) = x^{k-1}y^{k-1}\gamma(x^k, y^k, z^k)$. Hence, (α, β, γ) is a syzygy for the partial derivatives of f_5. We conclude that (a, b, c) is a combination of

$$R_1(x^k, y^k, z^k) \cdot D_{1,x^{k-1}y^kz^{k-1},x^{k-1}y^{k-1}} = x^{k-1}y^{k-1}R_1(x^k, y^k, z^k) \cdot D_{1,yz^{k-1},1}$$

and

$$R_2(x^k, y^k, z^k) \cdot D_{y^{k-1}z^{k-1},x^{k-1}z^{k-1},x^{k-1}y^{k-1}z^k} = z^{k-1}R_2(x^k, y^k, z^k) \cdot D_{y^{k-1},x^{k-1},x^{k-1}y^{k-1}z}.$$

Divided by common factors, we obtain syzygies of degree $2k$ and $3k - 1$.

Case 2 $i < k - 1, j = l = k - 1$.

In this case $a(x, y, z) = x^{i+1}y^{k-1}z^{k-1}\alpha(x^k, y^k, z^k)$, $b(x, y, z) = x^iz^{k-1}\beta(x^k, y^k, z^k)$, and $c(x, y, z) = x^iy^{k-1}\gamma(x^k, y^k, z^k)$. Hence, (α, β, γ) is a syzygy for the generators of the ideal J_x. It is easily seen that we obtain combination of generators of the above syzygies. The other cases are treated in the same way.

We conclude that C_{5k} is free with $d_1 = \mathrm{mdr}(f_{5k}) = 2k$ and $d_2 = d - 1 - d_1 = 5k - 1 - 2k = 3k - 1$. By Eq. (2) $\tau(C_{5k}) = 19k^2 - 8k + 1$ for all k.

In order to prove (3), we study the branched cover $\tilde{\pi}_k : \tilde{C}_{5k} \to \tilde{C}_5$ between the normalizations of the curves. The monodromy of this map as an unramified cover of $\mathbb{P}^2 \setminus \{xyz = 0\}$ is determined by an epimorphism

$$H_1(\mathbb{P}^2 \setminus \{xyz = 0\}; \mathbb{Z}) \to \mathbb{Z}_k \times \mathbb{Z}_k =: G_k$$

such that the meridians of the lines are sent to a_x, a_y, a_z, a system of generators of G_k such that $a_x + a_y + a_z = 0$. Since the singularities of C_5 are locally irreducible, then C_5 and $\tilde{C}_5$ are homeomorphic, and the covering $\tilde{\pi}_k$ is determined by the monodromy map

$$H_1(\tilde{C}_5 \setminus \{xyz = 0\}; \mathbb{Z}) \to \mathbb{Z}_k \times \mathbb{Z}_k =: G_k$$

obtained by composing using the map defined by the inclusion. Hence, $\tilde{C}_5 \setminus \{xyz = 0\}$ is isomorphic to $\mathbb{P}^1 \setminus \{\text{three points}\}$). The image of a meridian corresponding to a point P in the axes is given by

$$m_P^{L_x} a_x + m_P^{L_y} a_y + m_P^{L_z} a_z.$$

Hence, we obtain a_z (the smooth point), $3a_x + 5a_y$ (the $\mathbb{E}_8$-point), and $2a_x + 4a_z$ (the $\mathbb{A}_4$-point). In terms of the basis a_y, a_z, they read as $a_z, 2a_y - 3a_z, -2a_y + 2a_z$, i.e., the monodromy group is generated by $2a_y, a_z$. If k is even, the monodromy group is of index 2 in G_k, and hence, $\tilde{C}_{5k}$ has two connected components. Otherwise, it is equal to G_k when k is odd and $\tilde{C}_{5k}$ is connected. These properties give us the statement about the number of irreducible components.

The genus can be computed using the singularities of C_{5k} or via Riemann-Hurwitz's formula. Note that the covering $\tilde{\pi}_k$ is of degree k^2 with three ramification points: at p_2 and the smooth point in the axis where we find k preimages, while at p_1 we find k preimages if k is odd and $2k$ preimages if it is even, because of (1). Hence, for k odd, the Euler characteristic of the normalization is

$$\chi(\tilde{C}_{5k}) = -k^2 + 3k \Longrightarrow g(\tilde{C}_{5k}) = \frac{(k-1)(k-2)}{2}.$$

And for k even, where $\tilde{C}_{5k} = \tilde{C}^1_{5k} \cup \tilde{C}^2_{5k}$, the Euler characteristic is

$$\chi(\tilde{C}_{5k}) = -k^2 + 4k \Longrightarrow g(\tilde{C}^i_{5k}) = \frac{2 - \frac{\chi(\tilde{C}_{5k})}{2}}{2} = \frac{(k-2)^2}{4}. \qquad \square$$

So, for odd $k \geq 3$, the curve C_{5k} is an irreducible free curve of positive genus whose singularities have k branches each. This is a counterexample to both the free divisor part of Conjectures 1.1(ii) and 1.2(i).

Remark 3.10 Up to projective transformation, there are two quintic curves with two singular points of type $\mathbb{A}_4$ and $\mathbb{E}_8$. One is $C_5 : (yz-x^2)^2y-x^5 = 0$, which is free; the other one is defined by $D_5 : g = y^3z^2 - x^5 = 0$ (the contact of the tangent line to the $\mathbb{A}_4$-point distinguishes both curves). Moreover, the curve D_5 is nearly free; it can be computed that $\operatorname{mdr}(g) = 1$. Since both singular points are quasihomogeneous, $12 = \tau(C_5) = \mu(C_5) = \mu(D_5) = \tau(D_5)$, and we may apply Theorem 2.7(2); the pair (C_5,D_5) is a kind of counterexample to Terao's conjecture [19, Conjecture 4.138] for irreducible divisors (with constant Tjurina number); compare with [22].

3.3 *Irreducible Nearly Free Curves with Many Branches and High Genus*

The quartic curve C_4 defined by $f_4 := (yz - x^2)^2 - x^3y = 0$ has two singular points, $p_1 = [0 : 1 : 0]$ of type $\mathbb{A}_2$ and $p_2 = [0 : 0 : 1]$ of type $\mathbb{A}_4$. Therefore, it is rational and cuspidal. We will consider the Kummer transform C_{4k}, defined by $f_{4k} := (y^kz^k - x^{2k})^2 - x^{3k}y^k = 0$, of the curve C_4.

Theorem 3.11 *For any $k \geq 1$, the curve C_{4k} of degree $d = 4k$ defined by*

$$C_{4k} : \ f_{4k} := (y^kz^k - x^{2k})^2 - x^{3k}y^k = 0,$$

verifies the following properties

(1) $\mathrm{Sing}(C_{4k}) = \{p_1, p_2\}$. *The number of branches of* C_{4k} *at each* p_2 *is* k, *and at* p_1, *it equals* k *(if* k *is odd) or* $2k$ *(if* k *is even).*

(2) C_{4k} *is a nearly free divisor with exponents* $d_1 = d_2 = d_3 = 2k$ *and* $\tau(C_{4k}) = 6k(2k-1)$.

(3) C_{4k} *has two irreducible components of genus* $\frac{(k-2)^2}{4}$ *if* k *is even and it is irreducible of genus* $\frac{(k-1)(k-2)}{2}$ *otherwise.*

Proof Since $\mathrm{Sing}(C_4)(= \{p_1, p_2\})$ are points of type 0, C_4 meets $\{xyz = 0\}$ at three points p_1, p_2, and transversally at p_3 which is of type 1, therefore $\mathrm{Sing}(C_{4k}) = \{p_1, p_2\}$. To prove Part (1), it is enough to find the number of branches of C_{4k} at these points using Proposition 3.8 (2) (b). At p_1 one has $(C_4, L_z)_{p_1} = 3,\ (C_4, L_x)_{p_1} = 2$, and $r_{p_1} = \gcd(k, 2, 3) = 1$ for all k , and so that the number of branches of C_{4k} at p_1 is equal to k. In the same way, at p_2, the intersection $(C_4, L_x)_{p_2} = 2,\ (C_4, L_y)_{p_2} = 4$, and $r_{p_2} = \gcd(k, 2, 4) = \gcd(k, 2)$. If k is odd, $r_{p_2} = 1$ and the number of branches of C_{4k} at p_2 is equal to k. Otherwise $r_{p_2} = 2$ and the number of branches of C_{4k} at p_2 is equal to $2k$.

The proof of Part (2) follows the same guidelines as Theorem 3.9. With the notations of that proof, a generator system for the syzygies of J (Jacobian ideal of f_4) is given by

$$
\begin{aligned}
R_1 &:= (y(3x-4z),\ 3y(4x-3y),\ z(9y-20x)), \\
R_2 &:= \big(-x(x+2z),\ -4x^2+3xy+10yz,\ -z(3x+10z)\big), \\
R_3 &:= \big(xy,\ -3y^2,\ 2x^2+3yz\big).
\end{aligned}
\tag{7}
$$

These syzygies satisfy the relation $xR_1 + 3yR_2 + 10zR_3 = 0$. Therefore, using Dimca Sticlaru remark (see our Remark 2.5), C_4 is a nearly free divisor with exponents $d_1 = d_2 = d_3 = 2$.

For the ideal J_z, we have a similar situation. For the other ideals, their syzygy space is free of rank 2. Using these results it is not hard to prove that the syzygies of f_{4k} are generated by

$$
\begin{aligned}
R_{k,1} &:= \big(y^k(3x^k-4z^k),\ 3x^{k-1}y(4x^k-3y^k),\ x^{k-1}z(9y^k-20x^k)\big), \\
R_{k,2} &:= \big(-xy^{k-1}(x^k+2z^k),\ -4x^{2k}+3x^ky^k+10y^kz^k,\ -y^{k-1}z(3x^k+10z^k)\big), \\
R_{k,3} &:= \big(xy^kz^{k-1},\ -3y^{k+1}z^{k-1},\ 2x^{2k}+3y^kz^k\big).
\end{aligned}
$$

The results follow as in the proof of Theorem 3.9.

These syzygies satisfy the relation $xR_{k,1} + 3yR_{k,2} + 10zR_{k,3} = 0$, and therefore, using Dimca-Sticlaru Remark (see also our Remark 2.5), C_{4k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = 2k$ and by Eq. (4) $\tau(C_{4k}) = 6k(2k-1)$.

The proof of Part (3) follows the same ideas as in Theorem 3.9 (3). □

So, for odd $k \geq 3$, the curve C_{4k} is an irreducible nearly free curve of positive genus whose singularities have k branches each. This is a counterexample to both the nearly free divisor part of Conjectures 1.1(ii) and 1.2(ii).

3.4 Positive Genus Nearly Free Curves with Many Singularities

Let us consider the conic C_2 given by $f_2 = x^2 + y^2 + z^2 - 2(xy + xz + yz) = 0$. This conic is tangent to three axes, and it is very useful to produce interesting curves using Kummer covers (Fig. 2).

Theorem 3.12 *For any $k \geq 1$, the curve C_{2k} of degree $d = 2k$ defined by*

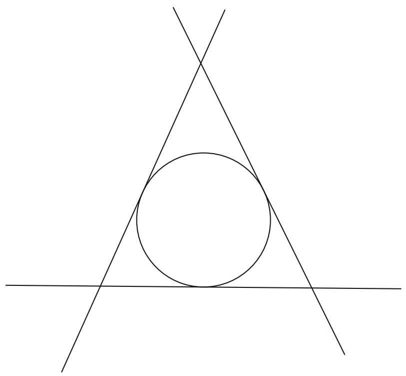
$$C_{2k} : f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0,$$

verifies the following properties:

(1) $\operatorname{Sing}(C_{2k})$ *are $3k$ singular points of type $\mathbb{A}_{k-1}$.*
(2) *C_{2k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.*
(3) *C_{2k} is irreducible of genus $\frac{(k-1)(k-2)}{2}$ if k is odd and it has four irreducible smooth components of degree $\frac{k}{2}$ if k is even.*

Proof To prove (1) it is enough to take into account that C_2 is nonsingular, and by Remark 3.6 the singularities of C_{2k} satisfy $\operatorname{Sing}(C_{2k}) \subset \{xyz = 0\}$. Moreover C_2 is tangent to the three axes at three points $\{p_1, p_2, p_3\}$ of type 1 with $(C_2, L_x)_{p_1} = (C_2, L_y)_{p_2} = (C_2, L_z)_{p_3} = 2$ at these points. For $i = 1, \ldots, 3$, the points p_i are of type 1, and by Remark 3.4, all the k preimages under π_k are analytically equivalent. By Example 3.7, over each p_i, one has k singular points of type $\mathbb{A}_{k-1}$.

Fig. 2 Conic C_2



Let us study (2). A generator system for the syzygies of J (Jacobian ideal of f_2) is given by

$$\begin{aligned} R_1 &:= (y - z,\ y,\ -z)\,, \\ R_2 &:= (-x,\ z - x,\ z)\,, \\ R_3 &:= (x,\ -y,\ x - y)\,. \end{aligned} \tag{8}$$

These syzygies satisfy the relation $xR_1 + yR_2 + zR_3 = 0$. The other ideals have free 2-rank syzygy modules. A simple computation gives the following syzygies for f_{2k}:

$$\begin{aligned} R_{k,1} &:= \left(y^k - z^k,\ x^{k-1}y,\ -x^{k-1}z\right), \\ R_{k,2} &:= \left(-xy^{k-1},\ z^k - x^k,\ y^{k-1}z\right), \\ R_{k,3} &:= \left(xz^{k-1},\ -yz^{k-1},\ x^k - y^k\right). \end{aligned}$$

These syzygies satisfy the relation $xR_{k,1} + yR_{k,2} + zR_{k,3} = 0$ and therefore, C_{2k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.

To prove (3) we follow as in the proof of Theorem 3.9 (3); the main difference is that π_2 has no ramification over C_2 and in fact C_4 is the union of four lines in general position; their preimages. If $k = 2\ell$, since $\pi_k = \pi_\ell \circ \pi_2$, each irreducible component is a smooth Fermat curve. □

For odd $k \geq 3$, these curves have positive genus and give a counterexample to the nearly free divisor part of Conjecture 1.1(ii) (with unbounded genus and number of singularities). Furthermore, if $k \geq 5$, since $d_1 = 5 \geq 4$ then by Corollary 2.6 C_{2k} is projectively rigid. Note that it is not the case for C_6, where we find it is the dual of a smooth cubic which is a nearly free divisor. A simple computation shows that the dual of a generic smooth cubic is also a nearly free divisor.

4 Pencil Associated to Unicuspidal Rational Plane Curves

In this section we are going to show that it is possible to construct a rational nearly free curve whose singular points has more than three branches, that is the condition to have high genus is not needed.

Given a curve $C \subset \mathbb{P}^2$, let $\pi : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the minimal, (not the "embedded" minimal) resolution of singularities of C. Let $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ be the strict transform of C, and let $\tilde{\nu}(C) = \widetilde{C} \cdot \widetilde{C}$ denote the self-intersection number of $\widetilde{C}$ on $\widetilde{\mathbb{P}}^2$.

A *unicuspidal rational curve* is a pair (C, P) where C is a curve and $P \in C$ satisfies $C \setminus \{P\} \cong \mathbb{A}^1$. We call P the distinguished point of C. Given a unicuspidal rational curve (C, P), D. Daigle and the last named author proved the existence of a unique pencil Λ_C on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $Bs(\Lambda_C) = \{P\}$ where $Bs(\Lambda_C)$ denotes the base locus of Λ_C on $\mathbb{P}^2$, see [7, 8].

Let $\pi_m : \widetilde{\mathbb{P}}^2_m \to \mathbb{P}^2$ be the minimal resolution of the base points of the pencil. By Bertini theorem, the singularities of the general member C_{gen} of Λ_C are contained in $Bs(\Lambda_C) = \{P\}$.

For a unicuspidal rational curve $C \subset \mathbb{P}^2$, we show (cf. [8, Theorem 4.1]) that the general member of Λ_C is a rational curve if and only if $\tilde{\nu}(C) \geq 0$. In this case

1. the general element C_{gen} of Λ_C satisfies that the weighted cluster of infinitely near points of C_{gen} and C are equal (see [7, Proposition 2.7]).
2. Λ_C has either 1 or 2 dicriticals, and at least one of them has degree 1.

In view of these results, it is worth noting that *all currently known unicuspidal rational curves* $C \subset \mathbb{P}^2$ *satisfy* $\tilde{\nu}(C) \geq 0$, see [8, Remark 4.3] for details.

Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree d and with distinguished point P. In [8, Proposition 1] it is proved that Λ_C is in fact the set of effective divisors D of $\mathbb{P}^2$ such that $\deg(D) = d$ and $i_P(C, D) \geq d^2$. Since $i_P(C, C) = \infty > d^2$, then the curve $C \in \Lambda_C$.

The main idea here is to take the general member C_{gen} of the pencil Λ_C for a nonnegative curve, i.e., $\tilde{\nu}(C) \geq 0$. Doing this one gets a rational curve C_{gen} whose singularities is $\text{Sing}(C_{\text{gen}}) = \{P\}$ and the branches of C_{gen} at P equals to the sum of the degrees of the dicriticals divisors.

The classification of unicuspidal rational plane curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$ was started by Tsunoda [27] and finished by Tono [26] (see also p. 125 in [14]).

Our next example starts with C_{49} with $\bar{\kappa}(\mathbb{P}^2 \setminus C_{49}) = 1$. Secondly we take the pencil $\Lambda_{C_{49}}$, and finally its general member $C_{49,gen}$ has degree 49 and is rational nearly free with just one singular point which has four branches.

The curve C_{49} is given by

$$f_{49} = ((f_1^s y + \sum_{i=2}^{s+1} a_i f_1^{s+1-i} x^{ia-a+1})^a - f_1^{as+1})/x^{a-1} = 0,$$

where $f_1 = x^{4-1}z + y^4$, $a = 4$, $s = 3$, $a_2 = \ldots = a_s \in \mathbb{C}$ and $a_{s+1} \in \mathbb{C} \setminus \{0\}$. We can take for instance $a_2 = \ldots = a_s = 0 \in \mathbb{C}$ and $a_{s+1} = 1$. In this case, $d = a^2 s + 1 = 49$, and the multiplicity sequence of (C_{49}, P) of the singular point $P := [0, 0, 1]$ is $[36, 12_7, 4_6]$. It is no-negative with $\tilde{\nu}(C_{49}) = 1$.

If we consider the rational curves C_4 defined by $f_1 = 0$ (resp. C_{13} defined by $f_{13} : (f_1)^3 y + x^{13} = 0$), then $i_P(C_{49}, C_4) = 4 \cdot 49$ (resp. $i_P(C_{49}, C_{13}) = 13 \cdot 49$). Thus, the curve $C_{13} C_4^{s(a-1)}$ belongs to the pencil $\Lambda_{C_{49}}$ if $s(a-1) = 9$.

If we take the curve $C_{49,gen}$ defined by $f_{49,gen} := f_{49} + 13 f_{13} f_4^9 = 0$. This curve is irreducible, rational, and $\text{Sing}(C_{49,gen}) = \{P\}$, and the number of branches of $C_{49,gen}$ at P is 4.

It is a nearly free divisor, using the computations with `Singular` [9]. A minimal resolution (3) for $f_{49,gen}$ is determined by three syzygies of degrees $d_1 = 24$ and $d_2 = d_3 = 25$. Therefore, $\text{mdr}(f_{49,gen}) = 24$. The computations yield a relation between these syzygies of multidegree $(2, 1, 1)$. Then $C_{49,gen}$ is a rational nearly free curve. Let us note that a direct computation using `Singular` [9] of the Tjurina number of the singular point of the curve fails, but the *nearly free* condition makes the computation possible via Theorem 2.7(2): $\tau(C_{49,gen}) = (49-1)(49-24-1)+24^2-1 = 1727$ which is the result in `Singular` using characteristic $p = 1666666649$.

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A Classification Algorithm for Complex Singularities of Corank and Modality up to Two

Janko Böhm, Magdaleen S. Marais, and Gerhard Pfister

Abstract In Arnold et al. (Singularities of Differential Maps, vol. I. Birkhäuser, Boston, 1985), Arnold has obtained normal forms and has developed a classifier for, in particular, all isolated hypersurface singularities over the complex numbers up to modality 2. Building on a series of 105 theorems, this classifier determines the type of the given singularity. However, for positive modality, this does not fix the right equivalence class of the singularity, since the values of the moduli parameters are not specified. In this paper, we present a simple classification algorithm for isolated hypersurface singularities of corank ≤ 2 and modality ≤ 2. For a singularity given by a polynomial over the rationals, the algorithm determines its right equivalence class by specifying a polynomial representative in Arnold's list of normal forms.

Keywords Algorithmic classification • Arnold normal forms • Hypersurface singularities • Moduli parameters

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J. Böhm (✉)
Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
e-mail: boehm@mathematik.uni-kl.de

M.S. Marais
Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa
e-mail: magdaleen.marais@up.ac.za

G. Pfister
Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
e-mail: pfister@mathematik.uni-kl.de

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1 Introduction

In his classical paper on singularities [1], Arnold has classified all isolated hypersurface singularities over the complex numbers with modality ≤ 2. He has given normal forms in the sense of polynomial families with moduli parameters such that every stable equivalence class of function germs contains at least one, but only finitely many, elements of these families. We refer to such elements as normal form equations. Two germs are stably equivalent if they are right equivalent after the direct addition of a nondegenerate quadratic form. Two function germs $f, g \in \mathfrak{m}^2 \subset \mathbb{C}[[x_1, \ldots, x_n]]$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, are right equivalent, written $f \sim g$, if there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathbb{C}[[x_1, \ldots, x_n]]$ such that $\phi(f) = g$. Using the Splitting Lemma (see, e.g., [8, Theorem 2.47]), any germ with an isolated singularity at the origin can be written, after choosing a suitable coordinate system, as the sum of two functions on disjoint sets of variables. One function that is called the nondegenerate part is a nondegenerate quadratic form, and the other part, called the residual part, is in $\mathfrak{m}^3$. The Splitting Lemma is implemented in SINGULAR [6, 7] as part of the library `classify.lib` [10].

In [2], Arnold has made this classification explicit by describing an algorithmic classifier, which is based on a series of 105 theorems. This approach determines the type of the singularity in the sense of its normal form. However, the values of the moduli parameters are not determined, that is, no normal form equation is given. Arnold's classifier is implemented in `classify.lib`.

Classification of complex singularities has a multitude of practical and theoretical applications. The classification of real singularities in [3, 12–14] is based on determining the complex type of the singularity.

In this paper, we develop a determinator for complex singularities of modality ≤ 2 and corank ≤ 2, which computes, for a given rational input polynomial, a normal form equation in its equivalence class. For singularities with nondegenerate Newton boundary, our determinator is based on a simple and uniform approach, which does not require a case-by-case analysis (except for some trivial final steps to read off the values of the moduli parameters according to Arnold's choice of the normal form). Two series of cases with degenerate Newton boundary are handled with more specific methods. Here, we use results of Luengo and Pfister [11] to compute a normal form. In this way, we obtain an approach which does not only determine the moduli parameters but also allows for an elegant implementation. We have implemented our algorithm in the SINGULAR-library `classify2.lib` [4].

It is important to note that two different normal form equations do not necessarily represent two different right equivalence classes. In [14] the complete structure of the equivalence classes for, in particular, complex singularities of modality 1 and corank 2 is determined, in the sense that all equivalences between normal form equations are described. All normal form equations in the right equivalence class of a given unimodal corank 2 singularity can, hence, be determined by combining our classifier with the results in [14]. There is not yet a similar complete description of the structure of the equivalence classes of bimodal singularities.

This paper is structured as follows: In Sect. 2, we give the fundamental definitions and provide the prerequisites on singularities and their classification. In Sect. 3, we develop a general algorithm for the classification of complex singularities of modality ≤ 2 and corank ≤ 2. Essentially, the algorithm is structured into a subalgorithm for elimination below the Newton polygon, and a subalgorithm for elimination on and above the Newton polygon, which also determines the values of the moduli parameters. The algorithm for the two series of germs of modality 2 with degenerate Newton boundary is discussed in Sect. 4.

2 Definitions and Preliminary Results

In this section, we give some basic definitions and results, as well as some notation that will be used throughout the paper.

Definition 1 Let $K \subset \mathbb{C}[[x_1, \ldots, x_n]]$ be a union of equivalence classes with respect to the relation $\sim$. A **normal form** for K is given by a smooth map

$$\Phi : B \longrightarrow \mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}[[x_1, \ldots, x_n]]$$

of a finite-dimensional $\mathbb{C}$-linear space B into the space of polynomials for which the following three conditions hold:

(1) $\Phi(B)$ intersects all equivalence classes of K,
(2) the inverse image in B of each equivalence class is finite,
(3) $\Phi^{-1}(\Phi(B) \setminus K)$ is contained in a proper hypersurface in B.

The elements of the image of Φ are called **normal form equations**.

Remark 2 Arnold has chosen a normal form for each of the corank 2 singularities of modality ≤ 2. He has also associated a type to each normal form; see Table 1. We denote the normal form corresponding to the type T by $\mathrm{NF}(T)$. For $b \in \mathrm{par}(\mathrm{NF}(T)) := \Phi^{-1}(K)$ with K as in Definition 1, we write $\mathrm{NF}(T)(b) := \Phi(b)$ for the corresponding normal form equation.

In the following, we give a short account on weighted jets, filtrations, and Newton polygons. See [1] and [5] for more details.

Definition 3 Let $w = (c_1, \ldots, c_n) \in \mathbb{N}^n$ be a weight on the variables $(x_1, \ldots, x_n)$. The w-weighted degree on $\mathrm{Mon}(x_1, \ldots, x_n)$ is given by $w\text{-}\deg(\prod_{i=1}^n x_i^{s_i}) := \sum_{i=1}^n c_i s_i$. If the weight of all variables is equal to 1, we refer to the weighted degree of a monomial m as the standard degree of m and write $\deg(m)$ for $w\text{-}\deg(m)$. We use the same notation for terms of polynomials.

We call a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ **quasihomogeneous** or weighted homogeneous of degree d with respect to the weight w if $w\text{-}\deg(t) = d$ for any term t of f.

Table 1 Normal forms of singularities of modality ≤ 2 and corank ≤ 2 as given in Arnold et al. [2]

	Complex normal form	Restrictions		Complex normal form	Restrictions
Simple			*Bimodal*		
A_k	x^{k+1}	$k \geq 1$	$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + cxy^7$	$4b^3 + 27 \neq 0$
D_k	$x^2y + y^{k-1}$	$k \geq 4$	$J_{3,p}$	$x^3 + x^2y^3 + \mathbf{a}y^{9+p}$	$p > 0, a_0 \neq 0$
E_6	$x^3 + y^4$	–	$Z_{1,0}$	$x^3y + dx^2y^3 + cxy^6 + y^7$	$4d^3 + 27 \neq 0$
E_7	$x^3 + xy^3$	–	$Z_{1,p}$	$x^3y + x^2y^3 + \mathbf{a}y^{7+p}$	$p > 0, a_0 \neq 0$
E_8	$x^3 + y^5$	–	$W_{1,0}$	$x^4 + \mathbf{a}x^2y^3 + y^6$	$a_0^2 \neq 4$
Unimodal			$W^\sharp_{1,2q}$	$(x^2 + y^3)^2 + \mathbf{a}x^2y^{3+q}$	$q > 0, a_0 \neq 0$
X_9	$x^4 + ax^2y^2 + y^4$	$a^2 \neq 4$	$W_{1,p}$	$x^4 + x^2y^3 + \mathbf{a}y^{6+p}$	$p > 0, a_0 \neq 0$
J_{10}	$x^3 + ax^2y^2 + y^6$	$4a^3 + 27 \neq 0$	$W^\sharp_{1,2q-1}$	$(x^2 + y^3)^2 + \mathbf{a}xy^{4+q}$	$q > 0, a_0 \neq 0$
J_{10+k}	$x^3 + x^2y^2 + ay^{6+k}$	$a \neq 0,\ k > 0$	E_{18}	$x^3 + y^{10} + \mathbf{a}xy^7$	–
X_{9+k}	$x^4 + x^2y^2 + ay^{4+k}$	$a \neq 0,\ k > 0$	E_{19}	$x^3 + xy^7 + \mathbf{a}y^{11}$	–
$Y_{r,s}$	$x^r + ax^2y^2 + y^s$	$a \neq 0,\ r, s > 4$	E_{20}	$x^3 + y^{11} + \mathbf{a}xy^8$	–
E_{12}	$x^3 + y^7 + axy^5$	–	Z_{17}	$x^3y + y^8 + \mathbf{a}xy^6$	–
E_{13}	$x^3 + xy^5 + ay^8$	–	Z_{18}	$x^3y + xy^6 + \mathbf{a}y^9$	–
E_{14}	$x^3 + y^8 + axy^6$	–	Z_{19}	$x^3y + y^9 + \mathbf{a}xy^7$	–
Z_{11}	$x^3y + y^5 + axy^4$	–	W_{17}	$x^4 + xy^5 + \mathbf{a}y^7$	–
Z_{12}	$x^3y + xy^4 + ax^2y^3$	–	W_{18}	$x^4 + y^7 + \mathbf{a}x^2y^4$	–
Z_{13}	$x^3y + y^6 + axy^5$	–			
W_{12}	$x^4 + y^5 + ax^2y^3$	–		where $\mathbf{a} = a_0 + a_1y$	
W_{13}	$x^4 + xy^4 + ay^6$	–			

Definition 4 Let $w = (w_1, \ldots, w_s) \in (\mathbb{N}^n)^s$ be a finite family of weights on the variables $(x_1, \ldots, x_n)$. For any monomial (or nonzero term) $m \in \mathbb{C}[x_1, \ldots, x_n]$, we define the **piecewise weight** with respect to w as

$$w\text{-}\deg(m) := \min_{i=1,\ldots,s} w_i\text{-}\deg(m).$$

We set $\deg(0) = \infty$. A polynomial f is called **piecewise homogeneous** of degree d with respect to w if $w\text{-}\deg(t) = d$ for any term t of f.

Definition 5 Let w be a (piecewise) weight on $\mathrm{Mon}(x_1, \ldots, x_n)$.

1. Let $f = \sum_{i=0}^{\infty} f_i$ be the decomposition of $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ into weighted homogeneous summands f_i of w-degree i. The **weighted j-jet** of f with respect to w is

$$w\text{-}\mathrm{jet}(f, j) := \sum_{i=0}^{j} f_i \,.$$

   The sum of terms of f of lowest w-degree is the **principal part** of f with respect to w.
2. A power series in $\mathbb{C}[[x_1, \ldots, x_n]]$ has **filtration** $d \in \mathbb{N}$ with respect to w if all its monomials are of w-weighted degree d or higher. The power series of filtration d form a sub-vector space

$$E_d^w \subset \mathbb{C}[[x_1, \ldots, x_n]] \,.$$

3. A power series $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ is **weighted k-determined** with respect to the weight w if

$$f \sim w\text{-}\mathrm{jet}(f, k) + g \qquad \text{for all } g \in E_{k+1}^w.$$

   We define the **weighted determinacy** of f as the minimum number k such that f is k-determined.

Definition 6 Let $w \in \mathbb{N}^n$ be a single weight. A power series $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ is called **semi-quasihomogeneous** with respect to w if its principal part with respect to w is nondegenerate, that is, has finite Milnor number.[1] The principal part is then called the **quasihomogeneous part** of f.

Notation 7 1. If the weight of each variable is 1, we write E_d and $\mathrm{jet}(f, j)$ instead of E_d^w and $w\text{-}\mathrm{jet}(f, j)$, respectively.

[1]We say that f is (semi-)quasihomogeneous if there exists a weight w such that f is (semi-)quasihomogeneous with respect to w.

2. If for a given type T, $w\text{-jet}(\mathrm{NF}(T)(b), j)$ is independent of $b \in \mathrm{par}(\mathrm{NF}(T))$, we denote it by $w\text{-jet}(T, j)$.

There are similar concepts of jets and filtrations for coordinate transformations:

Definition 8 Let ϕ be a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[[x_1, \ldots, x_n]]$ and let w be a weight on $\mathrm{Mon}(x_1, \ldots, x_n)$.

1. For $j > 0$ we define $w\text{-jet}(\phi, j) := \phi_j^w$ as the automorphism given by

$$\phi_j^w(x_i) := w\text{-jet}(\phi(x_i), w\text{-deg}(x_i) + j) \quad \text{for all } i = 1, \ldots, n\,.$$

If the weight of each variable is equal to 1, that is, $w = (1, \ldots, 1)$, we write ϕ_j for ϕ_j^w.

2. ϕ has filtration d if, for all $\lambda \in \mathbb{N}$,

$$(\phi - \mathrm{id})E_\lambda^w \subset E_{\lambda+d}^w\,.$$

Remark 9 Note that $\phi_0(x_i) = \mathrm{jet}(\phi(x_i), 1)$ for all $i = 1, \ldots, n$. Furthermore note that ϕ_0^w has filtration ≤ 0, and that, for $j > 0$, ϕ_j^w has filtration j if $\phi_{j-1}^w = \mathrm{id}$.

The following definition gives an infinitesimal analogue of the above definition:

Definition 10 A formal vector field $\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i}$ has filtration d with respect to a weight w, if the directional derivative of $\mathbf{v}$ raises the filtration by not less than d, that is,

$$\text{for all } g \in E_\delta^w, \quad L_{\mathbf{v}}(g) := \sum_i v_i \frac{\partial g}{\partial x_i} \in E_{\delta+d}^w.$$

In a similar way as [13, Proposition 8], one can prove:

Proposition 11 *Let $f, g \in \mathbb{C}[[x_1, \ldots, x_n]]$ be two power series with $f \sim g$. Let $w \in \mathbb{N}^n$ and suppose that the maximal weighted filtration of f with respect to w is k. Furthermore, let ϕ be a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[[x_1, \ldots, x_n]]$ such that $\phi(f) = g$. If* $\mathrm{jet}(f, k)$ *factorizes as*

$$w\text{-jet}(f, k) = f_1^{s_1} \cdots f_t^{s_t}$$

in $\mathbb{C}[x_1, \ldots, x_n]$, then $w\text{-jet}(g, k)$ *factorizes as*

$$w\text{-jet}(g, k) = \phi_0^w(f_1)^{s_1} \cdots \phi_0^w(f_t)^{s_t}\,.$$

Definition 12 Let $f = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{C}[[x, y]]$, and let T be a corank 2 singularity type. We call

$$\mathrm{supp}(f) := \{x^i y^j \mid a_{i,j} \neq 0\}$$
$$\mathrm{supp}(T) := \mathrm{supp}(\mathrm{NF}(T)(b))$$

where $b \in \mathrm{par}(\mathrm{NF}(T))$ is generic, the **support** of f and of T, respectively. Let

$$\Gamma_+(f) := \bigcup_{x^i y^j \in \mathrm{supp}(f)} ((i,j) + \mathbb{R}^2_+)$$

$$\Gamma_+(T) := \bigcup_{x^i y^j \in \mathrm{supp}(\mathrm{NF}(T)(b))} ((i,j) + \mathbb{R}^2_+)$$

where $b = 0$ if $0 \in \mathrm{par}(\mathrm{NF}(T))$ and $b \in \mathrm{par}(\mathrm{NF}(T))$ is generic, otherwise. Let $\Gamma(f)$ and $\Gamma(T)$ be the boundaries in $\mathbb{R}^2$ of the convex hulls of $\Gamma_+(f)$ and $\Gamma_+(T)$, respectively. Then:

1. $\Gamma(f)$ and $\Gamma(T)$ are called the **Newton polygons** of f and T, respectively.
2. The compact segments of $\Gamma(f)$ or $\Gamma(T)$ are called **faces**. If Δ is a face, then the set of monomials of f lying on Δ is denoted by $\mathrm{supp}(f, \Delta)$ and the sum of the terms lying on Δ by $\mathrm{jet}(f, \Delta)$. Moreover, we write $\mathrm{supp}(\Delta)$ for the set of monomials corresponding to the lattice points of Δ, and set $\mathrm{supp}(T, \Delta) := \mathrm{supp}(T) \cap \mathrm{supp}(\Delta)$. We use the same notation for a set of faces, considering the monomials lying on the union of the faces.
3. Any face Δ induces a weight $w(\Delta)$ on $\mathrm{Mon}(x, y)$ in the following way: If Δ has slope $-\frac{w_x}{w_y}$, in lowest terms, and $w_x, w_y > 0$, we set $w(\Delta)\text{-}\deg(x) = w_x$ and $w(\Delta)\text{-}\deg(y) = w_y$.
4. If $w_1, \ldots, w_s$ are the weights associated with the faces of $\Gamma(f)$, respectively $\Gamma(T)$, ordered by increasing slope, there are unique minimal integers $\lambda_1, \ldots, \lambda_s \geq 1$ such that the piecewise weight associated with $(\lambda_1 w_1, \ldots, \lambda_s w_s)$ by Definition 4 is constant on $\Gamma(f)$, respectively $\Gamma(T)$. We denote this piecewise weight by $w(f)$, respectively $w(T)$, and the corresponding constant by $d(f)$, respectively $d(T)$.
5. Let Δ_i and Δ_j be faces with weights w_1 and w_2, respectively, and let w be the piecewise weight defined by w_1 and w_2. Let d be the w-degree of the monomials on Δ_1 and Δ_2. Then $\mathrm{span}(\Delta_1, \Delta_2)$ is the Newton polygon associated with the sum of all monomials of w-degree d.
6. A monomial m lies strictly underneath, on, or above $\Gamma(f)$, if the $w(f)$-degree of m is less than, equal to, or greater than $d(f)$, respectively. We use this notation also with respect to $\Gamma(T)$, $w(T)$, and $d(T)$.

Notation 13 Given $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ and $m \in \mathrm{Mon}(x_1, \ldots, x_n)$, we write $\mathrm{coeff}(f, m)$ for the coefficient of m in f.

Definition 14 The **Jacobian ideal** $\mathrm{Jac}(f) \subset \mathbb{C}[[x_1, \ldots x_n]]$ of f is generated by the partial derivatives of $f \in \mathbb{C}[[x_1, \ldots x_n]]$. The **local algebra** of f is the residue class ring of the Jacobian ideal of f.

Definition 15 Suppose f is a nondegenerate germ, $e_1, \ldots, e_\mu$ are monomials representing a basis of the local algebra of f, and $e_1, \ldots, e_s$ are the monomials in this basis above or on $\Gamma(f)$. We then call $e_1, \ldots, e_s$ a **system** of the local algebra of f.

Lemma 16 (Arnold [1], Corollary 3.3) *Let f be a semi-quasihomogeneous function with quasihomogeneous part f_0, and let $e_1, \ldots, e_\mu$ be monomials representing a basis of the local algebra of f_0. Then $e_1, \ldots, e_\mu$ also represent a basis of the local algebra of f.*

Theorem 17 (Arnold [1], Theorem 7.2) *Let f be a semi-quasihomogeneous function with quasihomogeneous part f_0 and let $e_1, \ldots, e_s$ be a system of the local algebra of f. Then f is equivalent to a function of the form $f_0 + \sum_{k=1}^{s} c_k e_k$ with constants c_k.*

In [1], the following approach is used to extend the above results to a larger class of singularities of corank 2:

Definition 18 A piecewise homogeneous function f_0 of degree d satisfies **Condition A**, if for every function g of filtration $d + \delta > d$ in the ideal spanned by the derivatives of f_0, there is a decomposition

$$g = \sum_i \frac{\partial f_0}{\partial x_i} v_i + g',$$

where the vector field v has filtration δ and g' has filtration bigger than $d + \delta$.

Note that quasihomogeneous functions satisfy Condition A.

Theorem 19 *Suppose that the principal part f_0 of the piecewise homogeneous function f has finite Milnor number and satisfies Condition A. Let $e_1, \ldots, e_s$ be a system of the local algebra of f_0. Then f is equivalent to a function of the form $f_0 + \sum_k c_k e_k$ with constants c_k.*

Following Arnold's proof of Theorem 17, Theorem 19 can be proven by iteratively applying the following lemma:

Lemma 20 *Let $f_0 \in \mathbb{C}[[x_1, \ldots, x_n]]$ be a piecewise homogeneous function of weighted w-degree d_w that satisfies Condition A, and let $e_1, \ldots, e_r$ be the monomials of a given w-degree $d' > d_w$ in a system of the local algebra of f_0. Then, for every series of the form $f_0 + f_1$, where the filtration of f_1 is greater than d_w, we have*

$$f_0 + f_1 \sim f_0 + f_1',$$

where the terms in f_1' of degree less than d' are the same as in f_1, and the part of degree d' can be written as $c_1 e_1 + \cdots + c_r e_r$ with $c_i \in \mathbb{C}$.

Proof Let $g(\mathbf{x})$ denote the sum of the terms of degree d' in f_1. There exists a decomposition of g of the form

$$g(\mathbf{x}) = \sum_i \frac{\partial f_0}{\partial x_i} v_i(\mathbf{x}) + c_1 e_1 + \cdots + c_r e_r, \qquad v_i \in \mathbb{C}[[x_1, \ldots, x_n]],$$

since $e_1, \ldots, e_r$ represent a monomial vector space basis of the local algebra of f_0 in degree d'. Let $d(x_i)$ be the w-degree of x_i, and let $v_i' := w\text{-}\mathrm{jet}(v_i, d(x_i))$. Then

$$g(\mathbf{x}) = \sum_i \frac{\partial f_0}{\partial x_i} v_i'(\mathbf{x}) + c_1 e_1 + \cdots + c_r e_r - g'(\mathbf{x}),$$

where $g'(\mathbf{x})$ has filtration greater than d'. Applying the transformation defined by

$$x_i \mapsto x_i - v_i'(\mathbf{x})$$

to $f = f_0 + f_1$, we transform f to

$$f_0(\mathbf{x}) + (f_1(\mathbf{x}) + (c_1 e_1(\mathbf{x}) + \cdots + c_r e_r(\mathbf{x}) - g(\mathbf{x})) + R(\mathbf{x}),$$

where the filtration of R is greater than d'. □

Remark 21 A system of the local algebra is in general not unique. For his lists of normal forms of hypersurface singularities, Arnold has chosen in each case (in particular) a specific system of the local algebra. In the rest of the paper, we call these systems the **Arnold systems**.

Definition 22 (Kouchnirenko [9]) We say that $f \in \mathbb{C}[[x, y]]$ has **nondegenerate Newton boundary** if for every face Δ of $\Gamma(f)$, the saturation of $\mathrm{jet}(f, \Delta)$ has finite Milnor number.[2]

Remark 23

1. Note that if f has nondegenerate Newton boundary and finite Milnor number, then the principal part of f with respect to $w(f)$ has finite Milnor number.
2. Also note that for Arnold's normal forms $\mathrm{NF}(T)$ of corank and modality ≤ 2, the principal part with respect to $w(T)$ satisfies Condition A.
3. Suppose that f is a function of corank 2 with nondegenerate Newton boundary such that, for one of Arnold's normal forms $\mathrm{NF}(T)$ of modality ≤ 2, the support of the principal part f_0 of f with respect to $w(T)$ coincides with that of the principal part of $\mathrm{NF}(T)$. Then a system of the local algebra of f_0 is also a system of the local algebra of f.

Remark 24 It follows from Lemma 20 that all hypersurface singularities of corank ≤ 2 and modality ≤ 2 with nondegenerate Newton boundary are finitely weighted determined. Moreover, we explicitly obtain the weighted determinacy for each such singularity.

[2] We say that the singularity defined by f has non-degenerate Newton boundary if there exists a germ $\tilde{f} \in \mathbb{C}[[x, y]]$ with $f \sim \tilde{f}$ which has non-degenerate Newton boundary. We use the analogous terminology also for semi-quasihomogeneous.

3 A Classification Algorithm for Corank 2 Complex Simple, Unimodal, and Bimodal Singularities

We now describe an algorithm to determine an Arnold normal form equation for a given input polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{Q}[x, y]$ of modality ≤ 2. In this section, we limit our discussion to functions with a normal form with nondegenerate Newton boundary. In the case of normal forms with degenerate Newton boundary, our algorithm will resort to special algorithms described in Sect. 4. Figures 1, 2, 3, and 4 illustrate the modality 2 types with nondegenerate Newton boundary. The figures show in the gray shaded area all monomials which can possibly occur in a polynomial f of the given type T. The faces of the Newton polygon $\Gamma(T)$ are shown in blue (and are extended with a thin line toward the coordinate axes). The dots with a thick black circle indicate the moduli monomials in the Arnold system. Red dots indicate monomials which are not in $\mathrm{Jac}(f)$. Monomials occurring in any normal form equation with nonzero coefficients are shown as blue dots.

The structure of our algorithm consists out of two basic steps; see Algorithm 1. We first determine the complex type of f by removing all the monomials underneath $\Gamma(T)$, in the semi-quasihomogeneous cases, and all the monomials underneath and on $\Gamma(T)$ which are not in $\mathrm{NF}(T)$, in the other cases (see Algorithm 2). After that, we determine a normal form equation of f (using Algorithm 5 in the non-simple cases). More generally, we will formulate the algorithm in a way that it is applicable to any $f \in \mathfrak{m}^2$ and will recognize if f is of modality > 2, returning an error in this case.

Algorithm 1 Algorithm to classify singularities of modality ≤ 2 and corank ≤ 2

Input: A polynomial germ $f \in \mathfrak{m}^2$ over the rationals.

Output: NF(f) as well as the values of all moduli parameters occurring in a normal form equations that is equivalent to f, if f is of modality ≤ 2, corank ≤ 2; `false` otherwise.

```
Apply Algorithm 2 to f.
if T as returned by Algorithm 2 is a simple type then
    return (NF(T), ())
Apply Algorithm 5 to the output of Algorithm 2 and return the result.
```

We first discuss Algorithm 2. If f is of corank ≤ 1, then f is of type A_k, where $k = \mu(f)$. Suppose now that f is of corank 2. Determining T in the process, we remove all monomials below $\Gamma(T)$ if $\Gamma(T)$ has only one face, and all monomials on or below $\Gamma(T)$ which are not in $\mathrm{NF}(T)$, if $\Gamma(T)$ has two faces. Let d be the maximal filtration of f. If f is of type X_9, nothing has to be done. Note that f is of type X_9 if and only if the d-jet of f has four different roots over the complex numbers. If f is not of type X_9, then Algorithm 3 will transform f such that $\mathrm{supp}(T, d) = \mathrm{supp}(\mathrm{jet}(f, d))$. Using [13, Proposition 8], we find the corresponding linear transformation by factorizing $\mathrm{jet}(f, d)$. At this stage we know that $\mathrm{supp}(\mathrm{jet}(f, d)) \subset \mathrm{supp}(T)$. We

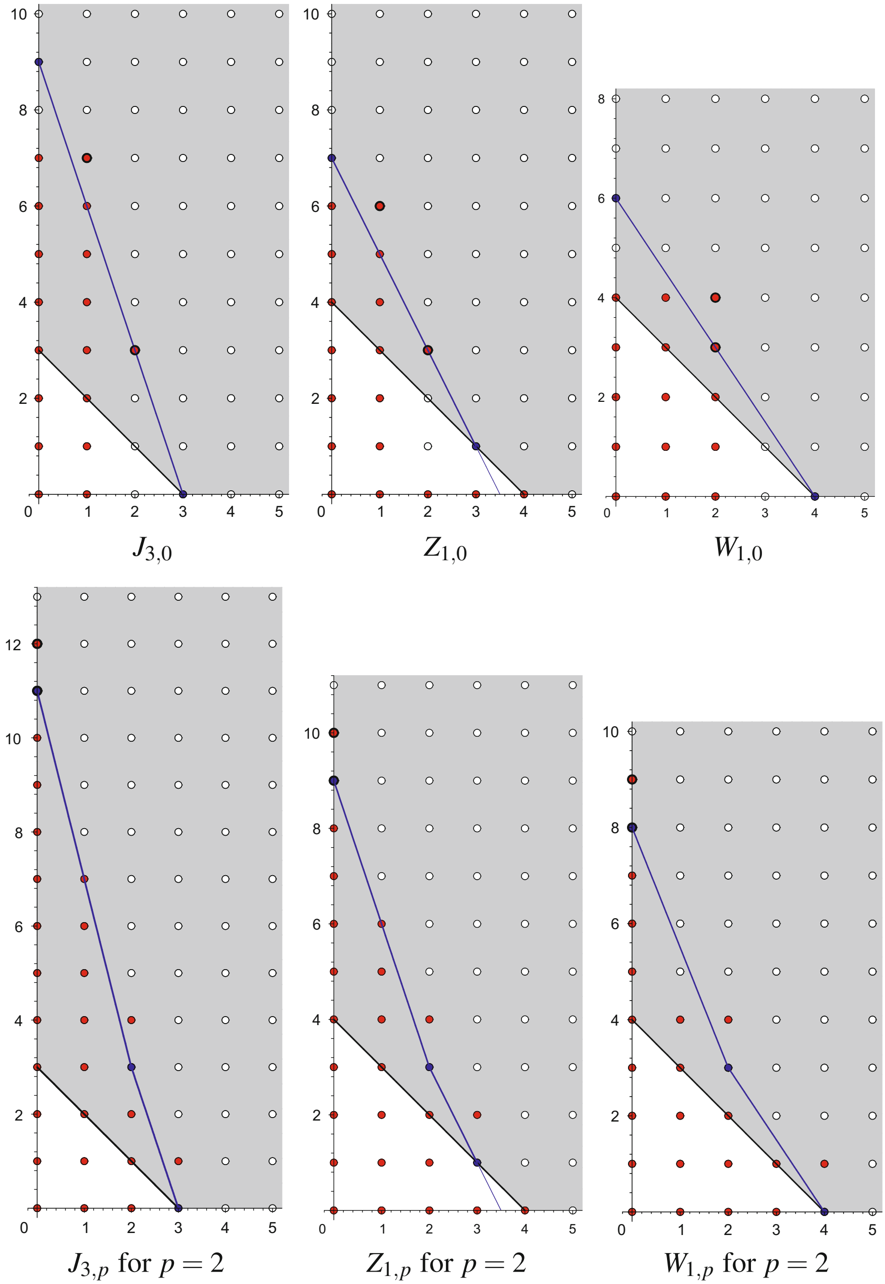

Fig. 1 Infinite series of bimodal corank 2 singularities with nondegenerate Newton boundary

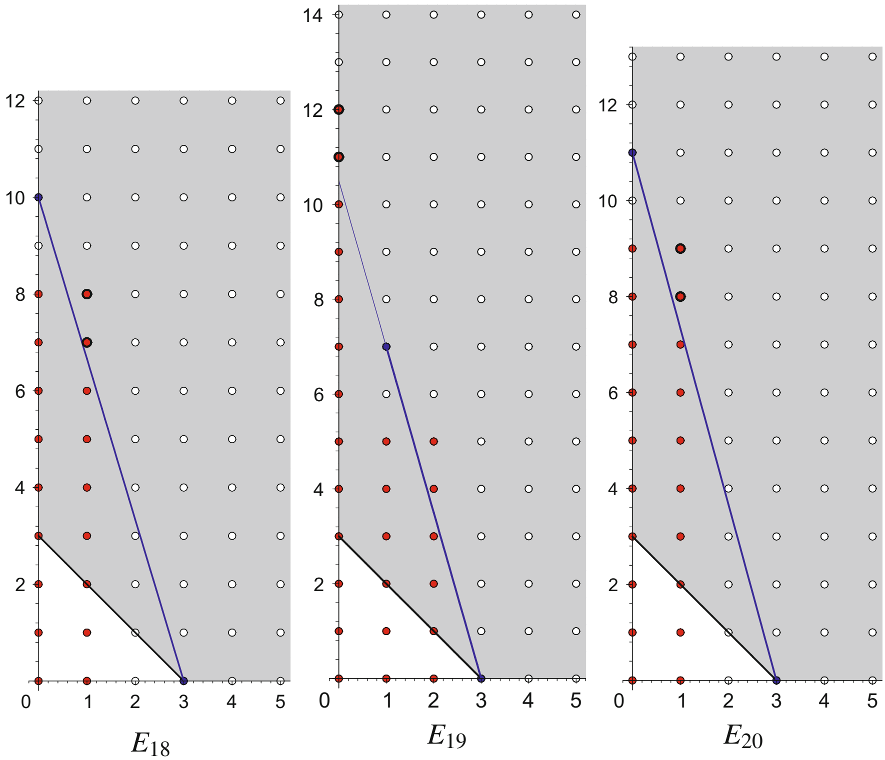

Fig. 2 Exceptional bimodal corank 2 singularities of type E

store the monomials of the d-jet of f in $S_0 = \text{supp}(\text{jet}(f, d))$. The remainder of Algorithm 2 will proceed in an iterative way, changing f and S_0 in the process: In each step of the iteration, we can have one of the following two possibilities for $\Gamma(f)$:

1. First note that monomials of the form $x^{n_1}y$ or xy^{n_2} cannot be intersection points of (finite) faces of $\Gamma(T)$. In the case that any of the monomials $m_0 \in S_0$ which is not of the form $x^{n_1}y$ or xy^{n_2} lies on two faces of $\Gamma(f)$, it is clear that $\Gamma(T)$ has at least two faces with corner point m_0. The algorithm will then stay in this case. Let Δ_i and Δ_j be the two different faces of $\Gamma(f)$ on which m_0 lies. The corner point in all modality 1 and 2 cases with a Newton polygon with two faces is either x^2y^2 or x^2y^3. It follows that if $m_0 \neq x^2y^t$, $t = 2$ or $t = 3$, then f is not of modality ≤ 2. Otherwise, using the shape of $\Gamma_0 := \text{span}(\Delta_i, \Delta_j)$ and the fact that $m_0 = x^2y^t$ is a corner point of Γ_0, all monomials in f on Γ_0 of the form xy^n or x^ny^{t-1} can be removed, replacing the corresponding terms of the given degree by higher $w(f)$-degree terms using Algorithm 4. We proceed iteratively in this way. After each iteration, f, Δ_i, Δ_j, and Γ_0 are recalculated. In each iteration, there will either be no terms of the considered form on Γ_0, in which case the process

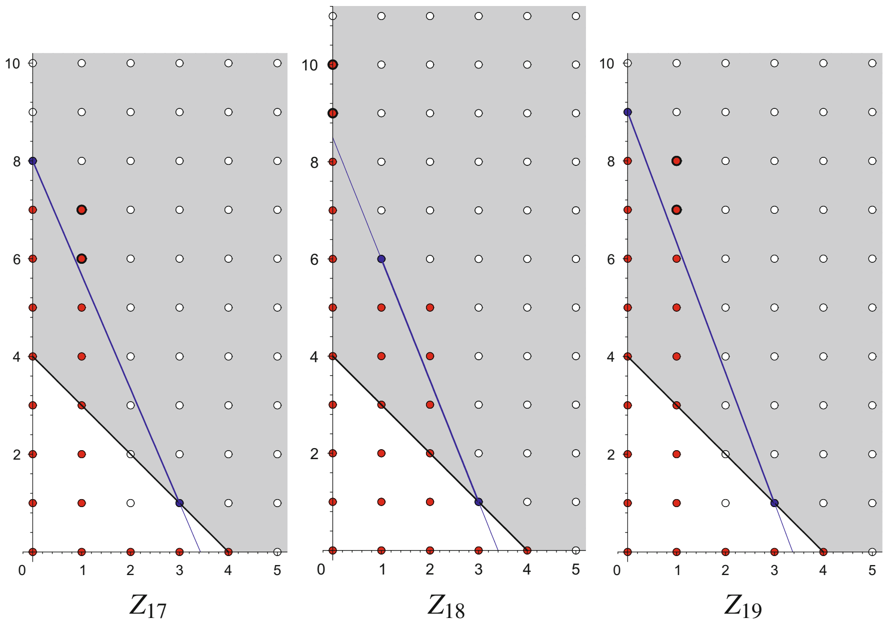

Fig. 3 Exceptional bimodal corank 2 singularities of type Z

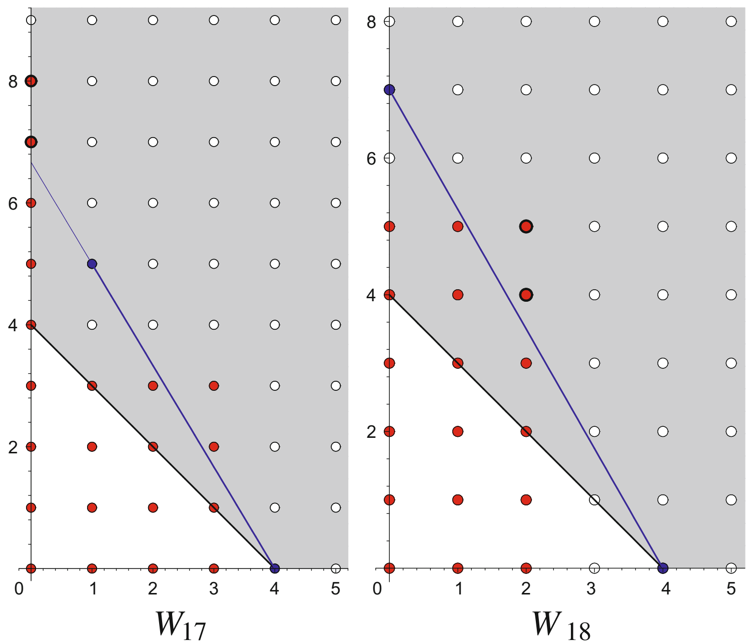

Fig. 4 Exceptional bimodal corank 2 singularities of type W

Algorithm 2 Determine the complex type of a corank ≤ 2 singularity of modality ≤ 2 with non-degenerate Newton boundary

Input: A polynomial germ $f \in \mathfrak{m}^2$ over the rationals.
Output: If f is of modality ≤ 2 and corank ≤ 2, then the complex singularity type T of f, and a polynomial g right equivalent to f such that the set of faces of $\Gamma(T)$ equals the set of faces of $\Gamma(w(T)\text{-}\mathrm{jet}(g, d(T)))$; `false` otherwise.

$f :=$ residual part given by the Splitting Lemma applied to f, as implemented in `classify.lib`.
if $\mathrm{corank}(f) \leq 1$ **then**
    return $(f, A_{\mu(f)})$
if $\mathrm{corank}(f) > 2$ **then**
    return `false`
if $f \in E_5$ **then**
    return `false` (modality > 2)
$f :=$ output of Algorithm 3 applied to $f \in \mathbb{Q}[x, y]$
$S_0 := \mathrm{supp}(\mathrm{jet}(f, d))$, where $d :=$ maximal filtration of f with respect to the standard grading.
while true do
    Let $\Delta_1, \ldots, \Delta_n$ be the faces of $\Gamma(f)$ ordered by increasing slope.
    if exist $i \neq j$ and $n_1, n_2 > 1$: $m_0 := x^{n_1} y^{n_2} \in \mathrm{supp}(\Delta_i) \cap \mathrm{supp}(\Delta_j) \subset S_0$ **then**
        if $m_0 \neq x^2y^2$ and $m_0 \neq x^2y^3$ **then**
            return `false` (modality > 2)
        $\Gamma_0 := \mathrm{span}(\Delta_i, \Delta_j)$
        $f_1 := \mathrm{jet}(f, \Gamma_0)$
        while exists a term of the form $t = c \cdot x^{n_1-1}y^r$ or $t = c \cdot x^r y^{n_2-1}$ in f_1 **do**
            $f :=$ output of Alg. 4 with input f, f_1, t, and weights $w(\Delta_i), w(\Delta_j)$
            Let $\Delta_1, \ldots, \Delta_n$ be the faces of $\Gamma(f)$.
            $\Gamma_0 := \mathrm{span}(\Delta_i, \Delta_j)$, with i, j such that $m_0 \in \mathrm{supp}(\Delta_i) \cap \mathrm{supp}(\Delta_j)$
            $f_1 := \mathrm{jet}(f, \Gamma_0)$
        if exists modal 1 or 2 type T with $\mathrm{supp}(T, \Gamma_0) = \mathrm{supp}(f_1)$ **then**
            return (f, T)
        else
            return `false` (modality > 2)
    else
        Let Δ be the face of $\Gamma(f)$ of smallest slope such that $S_0 \subset \mathrm{supp}(\Delta)$.
        $f_1 := \mathrm{jet}(f, \Delta)$
        if $\mu(f_1) = \infty$ **then**
            Let g_1 be the factor of f_1 with highest multiplicity.
            if $\deg_x(g_1) = 1$ **then**
                Replace f by $g_1 \mapsto x, y \mapsto y$ applied to f.
                $S_0 := \mathrm{supp}(\mathrm{jet}(f, \Delta))$
            else

if $\operatorname{supp}(f, \Delta) = \operatorname{supp}((y^2 - x^3)^2)$ **then**
  return $(f, W^{\sharp}_{1,\mu(f)-15})$
else
  return `false` (modality > 2)
else
  if exists modal 1 or 2 type T with $\Gamma(T) = \Gamma(f_1)$ **then**
    return (f, T)
  else
    return `false` (modality > 2)

Algorithm 3 Reverse linear jet

Input: A polynomial $f \in \mathfrak{m}^3 \subset \mathbb{Q}[x, y]$ with $\operatorname{jet}(f, 4) \neq 0$.

Output: $g \in \mathfrak{m}^3 \subset \mathbb{K}[x, y]$, where $\mathbb{K}$ is an algebraic extension field of $\mathbb{Q}$, such that $g \sim f$, and, in case f is of type $T \neq X_9$ of modality ≤ 2, then $\operatorname{supp}(\operatorname{jet}(g, d)) = \operatorname{supp}(T, d)$ where d is the maximal filtration of f w.r.t. the standard grading.

Factorize $\operatorname{jet}(f, d) = c g_1^{\alpha} g_2^{\beta} g_3^{\gamma} g_4^{\delta}$ over $\mathbb{C}$, where $0 \neq c \in \mathbb{Q}$, g_1, g_2, g_3 and g_4 are monic in x and pairwise coprime, and $4 \geq \alpha \geq \beta \geq \gamma \geq \delta \geq 0$.
if $\beta, \gamma, \delta = 0$ **then**
  if $g_1 \neq c'y$, $c' \in \mathbb{Q}$ **then**
    Replace f with $g_1 \mapsto x$, $y \mapsto y$ applied to f.
  else
    Replace f with $x \mapsto y$, $y \mapsto x$ applied to f.
if $\gamma, \delta = 0$ **then**
  Replace f with $g_1 \mapsto x$ and $g_2 \mapsto y$ applied to f.
if $\alpha = 2$ and $\beta, \gamma = 1$ and $\delta = 0$ **then**
  if $g_1 \neq c'y$, $c' \in \mathbb{Q}$ **then**
    Replace f with $g_1 \mapsto x$, $y \mapsto y$ applied to f.
  else
    Replace f with $x \mapsto y$, $y \mapsto x$ applied to f.
  Write $f = a_0x^4 + a_1x^3y + a_2x^2y^2 + R$, $a_0, a_1 \in \mathbb{Q}$, $a_2 \in \mathbb{Q}^{\times}$ and $R \in E_5$.
  Replace f with $y \mapsto y - \frac{a_1}{2a_2}x$, $x \mapsto x$ applied to f.
return f

stops, or the number of equivalence classes in the local algebra of f represented by powers of x or y underneath Γ_0 strictly increases, except possibly in the last two steps of the process (where monomials on the final Newton polygon may be removed). Note that if x^{m_1} and y^{m_2} are largest powers of x and y underneath Γ_0, then $1, x, \ldots, x^{m_1-1}, y, \ldots, y^{m_2-1}$ represent different equivalence classes. Since $\mu(f)$ is finite, the process must stop after finitely many iterations. No further monomials on Γ_0 can be removed without creating terms underneath Γ_0. Hence, in all cases in consideration, this algorithm will produce the Newton polygon of

Algorithm 4 Remove term via partials

Input: $f, f_0 \in \mathbb{K}[x,y]$ over a field $\mathbb{K}$, with t a term of f, and weights $u_1, u_2 \in \mathbb{Z}^2$.
Output: $g \in \mathbb{K}[x,y]$ such that $f \sim g$. If called with input as in Algorithms 2 or 5, then $f = g + t +$ terms of higher (u_1, u_2) -degree than t.

$m_x :=$ the sum of the terms of $\frac{\partial f_0}{\partial x}$ of lowest u_2-degree
$m_{x,y} :=$ the term of m_x of lowest u_1-degree
$m_y :=$ the sum of the terms of $\frac{\partial f_0}{\partial y}$ of lowest u_1-degree
$m_{y,x} :=$ the term of m_y of lowest u_2-degree
if $m_{x,y} | t$ **then**

$$\alpha : \mathbb{K}[x,y] \to \mathbb{K}[x,y]$$
$$x \mapsto x - t/m_{x,y}$$
$$y \mapsto y$$

  return $\alpha(f)$
if $m_{y,x} | t$ **then**

$$\alpha : \mathbb{K}[x,y] \to \mathbb{K}[x,y]$$
$$x \mapsto x$$
$$y \mapsto y - t/m_{y,x}$$

  return $\alpha(f)$
return f

the normal form. In fact, if $\operatorname{supp}(f, \Gamma_0)$ does not coincide with $\operatorname{supp}(T, \Gamma_0)$ for some type T of modality ≤ 2, then the modality of f is bigger than 2. Otherwise, f is a germ of the corresponding type T, and all monomials in f underneath or on $\Gamma(T)$ not in $\operatorname{NF}(T)$ are removed.

2. Suppose no monomials in S_0, except monomials of the form $x^{n_1}y$ or xy^{n_2}, lie on two faces of $\Gamma(f)$. Then f is not of type X_{9+k} or $Y_{r,s}$, since these cases will be recognized to have two faces in the first iteration of the above step. All the monomials in S_0 lie on only one face of $\Gamma(f)$. Let Δ be this face. If $f_1 := \operatorname{jet}(f, \Delta)$ is nondegenerate, then f is a semi-quasihomogeneous germ. Since $w\text{-}\operatorname{jet}(\phi_0^w(f), d(f)) = \phi_0^w(f_1)$ for any automorphism ϕ of filtration ≥ 0 with respect to the weight w associated with Δ, $\operatorname{span}(\Delta)$ is an invariant of the type of f. The corresponding type T can, hence, be identified. The case X_9 will already be recognized as a semi-quasihomogeneous function in the first iteration, and f will be returned by the algorithm without any change. In all other cases, the weight w associated with Δ will be such that $w\text{-}\deg(x) > w\text{-}\deg(y)$. If f_1 is degenerate, then either f has monomials underneath $\Gamma(T)$ or $\Gamma(T)$ is degenerate.

For all semi-quasihomogeneous cases of modality ≤ 2, except X_9, $\mathrm{jet}(T, d)$ is divisible by a power of x, and x has the highest multiplicity among all prime factors. Any weighted jet of $\mathrm{NF}(T)$ with respect to a face lying below $\Gamma(T)$ and intersecting $\Gamma(T)$ in $\mathrm{jet}(T, d)$ has the same property. Suppose Δ is such a face. Then $\mathrm{supp}(T, \Delta) = \{x^n y^m\}$ with $n > m$. Taking into account that the weighted degree of x is greater than the weighted degree of y, it follows that $f_1 = g_1^n y^m$ with $\deg_x(g_1) = 1$. The right equivalence $g_1 \mapsto x, y \mapsto y$ transforms f such that $\mathrm{supp}(f, \Delta) = \mathrm{supp}(T, \Delta)$. If the normal form of f has a nondegenerate Newton boundary, but is not semi-quasihomogeneous, we can proceed in the same way: Suppose Δ lies underneath or on the face of biggest slope of $\Gamma(T)$. If g_1 is the factor of highest multiplicity of f_1 with $\deg_x(g_1) = 1$, then the right equivalence $g_1 \mapsto x, y \mapsto y$ transforms f such that $\mathrm{supp}(f, \Delta) = \mathrm{supp}(T, \Delta)$. We then update $S_0 := \mathrm{supp}(f, \Delta)$ and pass to the next iteration. If f_1 does not have any x-linear factor, then the normal form of f has a degenerate Newton boundary. In this case, we resort to the algorithms described in Sect. 4. Since $\mu(f)$ is finite, the same argument as in (1) shows that the iteration terminates after finitely many steps.

We now discuss Algorithm 5, which determines the values of the moduli parameters. Let $w = w(T)$ be the weight associated with $\Gamma(T)$. If $\Gamma(T)$ has only one face Δ, then $\mathrm{supp}(f, \Delta)$ is not necessarily equal to $\mathrm{supp}(T, \Delta)$. We achieve equality by a weighted linear transformation. In the cases where $\Gamma(T)$ has two faces, equality has already been achieved in Algorithm 2. Above $\Gamma(T)$, we then use the method described in the proof of Lemma 20 to reduce f modulo $\mathrm{Jac}(f_0)$ where $f_0 = \mathrm{jet}(f, \Gamma(T))$: We iteratively apply Algorithm 4 to each term, in the two-face case only considering terms in $\mathrm{Jac}(f_0)$, proceeding weighted degree by weighted degree in increasing order (and in each weighted degree according to a total (ordinary) degree ordering). After handling a given weighted degree, if Arnold's system for type T contains a monomial m of this degree, we write the sum of the remaining terms in the form

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2 + cm,$$

where $v_1, v_2 \in \mathbb{C}[x, y]$ are weighted homogeneous, $c \in \mathbb{C}$, and as

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2,$$

otherwise. By Remark 23 this is always possible. Applying $x \mapsto x - v_1$, $y \mapsto y - v_2$, results in replacing the sum of the remaining terms by a sum of terms which are either in Arnold's system in the w-degree under consideration or of higher w-degree. Since f is weighted d'-determined, we stop the iteration when we reach degree $d'+1$, where d' is the w-degree of the highest w-degree monomial in Arnold's system.

Algorithm 5 Determine the moduli parameters of a normal form equation of a corank 2 uni- or bimodal singularities

Input: $f \in \mathfrak{m}^3 \subset \mathbb{K}[x, y]$, a germ of modality 1 or 2 and corank 2 of type T over an algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, as returned by Algorithm 2. In particular, the set of faces of $\Gamma(T)$ equals the set of faces of $\Gamma(w(T)\text{-}\mathrm{jet}(f, d(T)))$.

Output: The normal form of f, as well as the values of all moduli parameters occurring in a normal form equations that is equivalent to f, specified as elements of an algebraic extension field of $\mathbb{K}$.

1: **if** $T = W^{\sharp}_{1,\mu-15}$ for some μ **then**
2:     **return** result of Algorithm 6 applied to f
3: $w := w(T)$ and $d := d(T)$
4: **if** $\Gamma(T)$ has exactly one face Δ **then**
5:     Apply a weighted homogeneous transformation to f such that $\mathrm{supp}(f, \Delta) = \mathrm{supp}(T, \Delta)$.
6: $d' :=$ highest w-degree of a monomial in Arnold's system of T
7: $f_0 := w\text{-}\mathrm{jet}(f, d)$
8: **for** $j = d + 1, \ldots, d'$ **do**
9:     **for all** terms t of f of w-degree j, increasing w.r.t. a total degree ordering **do**
10:         **if** $\Gamma(T)$ has exactly one face **then**
11:             $f :=$ result of Algorithm 4 with input f, f_0, t and $(1, 1), (1, 1)$
12:         **else**
13:             **if** $t \in \mathrm{Jac}(f_0)$ **then**
14:                 $f :=$ result of Algorithm 4 with input f, f_0, t and w_2, w_1
15:     **if** exists monomial m of w-degree j in Arnold's system **then**
16:         Write $w\text{-}\mathrm{jet}(f, j) - w\text{-}\mathrm{jet}(f, j-1) = \frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2 + cm$ with $c \in \mathbb{K}$, $v_1, v_2 \in \mathbb{K}[x, y]$ weighted homogeneous.
17:     **else**
18:         Write $w\text{-}\mathrm{jet}(f, j) - w\text{-}\mathrm{jet}(f, j-1) = \frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2$ with $v_1, v_2 \in \mathbb{K}[x, y]$ weighted homogeneous.
19:     Apply $x \mapsto x - v_1, y \mapsto y - v_2$ to f.
20: Delete all terms in f of w-degree $> d'$.
21: Apply transformation $x \mapsto ax$, $y \mapsto by$ over an algebraic extension of $\mathbb{K}$ to transform the non-parameter terms to the terms of $\mathrm{NF}(T)$.
22: Read off the parameters a_i.
23: **return** $(\mathrm{NF}(T), (a_i))$

Remark 25 In the semi-quasihomogeneous cases, line 11 in Algorithm 5 can be omitted, since the reduction modulo $\mathrm{Jac}(f_0)$ is also handled by lines 15–18.

Remark 26 In Algorithm 5, Arnold's system can be replaced by any other choice of a system of the local algebra.

Remark 27 The algebraic extension of $\mathbb{Q}$ introduced for representing the moduli parameters can arise in two steps of the overall algorithm: Reversal of the linear jet in Algorithm 3 and rescaling of the variables at the end of Algorithm 5. Note that the transformation reversing the linear jet is obtained from the factorization $\mathrm{jet}(f,d) = cg_1^{\alpha}g_2^{\beta}g_3^{\gamma}g_4^{\delta}$. Here, a field extension can only occur if $\alpha = \beta = 2$ and $\gamma = \delta = 0$.

4 A Classification Algorithm for Corank 2, Bimodal Singularities with Degenerate Newton Boundary

In this section we give a classification algorithm for the singularities $W^{\sharp}_{1,\mu-15}$, where μ is the Milnor number, in Arnold's list. They have the property that in all coordinate systems, the Newton boundary is degenerate, which is the reason that they have to be treated separately. They are of multiplicity 4 and the 4-jet is a 4-th power of a linear homogeneous polynomial. After a suitable automorphism of $\mathbb{C}[[x,y]]$, we may assume that the corresponding polynomial is of the form

$$f = (x^2 + y^3)^2 + \sum_{3i+2j \geq 12+d} w_{ij}x^iy^j \ , \ d \geq 1.$$

This automorphism was already constructed in the previous section. Singularities of this type have been studied in [11]. It is proved that the Milnor number satisfies $\mu(f) \geq 15 + d$, and equality holds if and only if

$$\sum_{3i+2j=12+d} (-1)^{[i/2]}w_{ij} \neq 0.$$

If the Milnor number $\mu(f) = 15 + d$ is even, then the germ of the curve defined by f is irreducible with semi-group $\langle 4, 6, 12 + d\rangle$. In the odd case, the curve has two branches. Let

$$f = (x^2 + y^3)^2 + \sum_{3i+2j>12} w_{ij}x^iy^j$$

and assume $\mu := \mu(f) < \infty$. Let $>$ be the weighted degree reverse lexicographical ordering with respect to the weights $(3, 2)$ on $\mathbb{C}[[x,y]]$ with $x > y$.

In [11] it is proved that in case of μ being even the leading ideal of the Jacobian ideal, $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ is generated by $x^3, x^2y^2, xy^{\frac{\mu-2}{2}}$. If μ is odd, then the leading ideal is generated by $x^3, x^2y^2, xy^{\frac{\mu-5}{2}}, y^{\frac{\mu+1}{2}}$. We obtain a monomial basis of $\mathbb{C}[[x,y]]/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ as $\{x^iy^i\}_{(i,j)\in B}$ with

$$B = \left\{(i,j) \,\middle|\, i \leq 2, j \leq 1\right\} \cup \left\{(i,j) \,\middle|\, i \leq 1, 2 \leq j \leq \frac{\mu-4}{2}\right\}$$

in case that μ is even and

$$B = \{(i,j) \mid i \leq 2, j \leq 1\} \cup \left\{(1,j) \,\middle|\, 2 \leq j \leq \frac{\mu-7}{2}\right\} \cup \left\{(0,j) \,\middle|\, 2 \leq j \leq \frac{\mu-1}{2}\right\},$$

in case that μ is odd. Let

$$B_1 := \{(1, \frac{\mu-6}{2}), (1, \frac{\mu-4}{2})\}$$

if μ is even and

$$B_1 := \{(0, \frac{\mu-3}{2}), (0, \frac{\mu-1}{2})\}$$

if μ is odd.

In [11], the following theorem is proved.

Theorem 28 *There exists an automorphism φ of $\mathbb{C}[[x,y]]$ such that*

$$\varphi(f) = (x^2 + y^3)^2 + \sum_{(i,j)\in B_1} w_{ij} x^i y^j.$$

Note 29 In particular, it follows that these singularities are bimodal.

Remark 30 The normal form given in this way for the case that the Milnor number is odd differs from Arnold's normal form. Instead of $y^{\frac{\mu-3}{2}}$ and $y^{\frac{\mu-1}{2}}$, he used the monomials $x^2y^{\frac{\mu-9}{2}}$ and $x^2y^{\frac{\mu-7}{2}}$. From a computational point of view, our choice is better. It is easy to convert our normal form to Arnold's normal form. See Fig. 5, for an illustration of the normal forms (using our choice of parameter monomials).

The construction of the automorphism in the theorem is done separately for each weighted degree: Assume we have already

$$f = (x^2 + y^3)^2 + \sum_{3i+2j\geq 12+a} w_{ij} x^i y^j$$

for some a (with Milnor number $\mu = 15 + d$). If $a < d$, then we have

$$\sum_{3i+2j=12+a} (-1)^{[i/2]} w_{ij} = 0.$$

This implies that

$$\sum_{3i+2j=12+a} w_{ij} x^i y^j = l \cdot (x^2 + y^3).$$

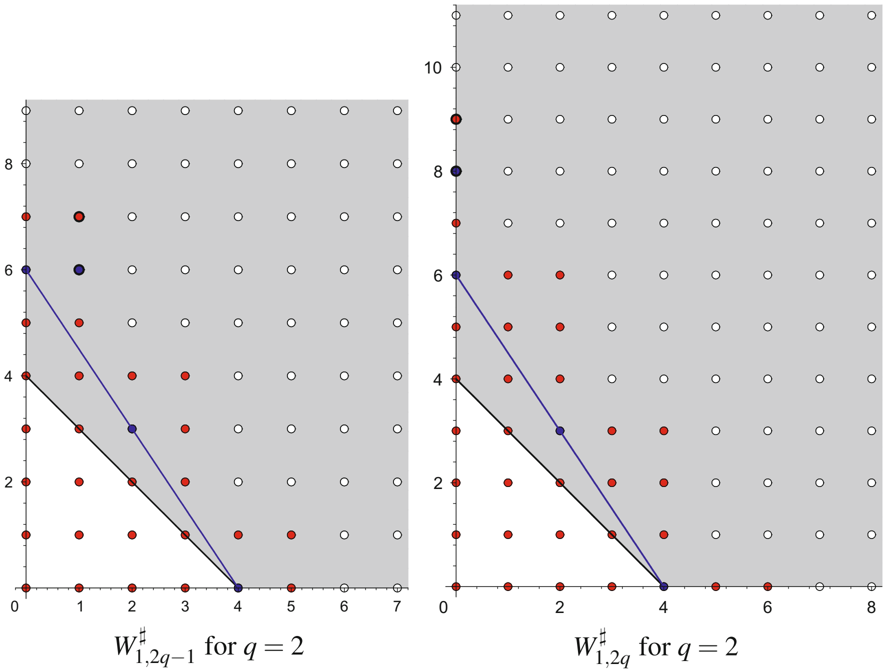

Fig. 5 Infinite series of bimodal corank 2 singularities with degenerate Newton boundary

We obtain

$$f = \left(x^2 + y^3 + \frac{1}{2}l\right)^2 + \sum_{3i+2j>12+a} \tilde{w}_{ij} x^i y^j$$

for suitable $\tilde{w}_{ij} \in \mathbb{C}$. Now we can choose an automorphism φ of $\mathbb{C}[[x,y]]$ such that

$$\varphi\left(x^2 + y^3 + \frac{1}{2}l\right) = x^2 + y^3 + \text{ terms of weighted degree } \geq \mu$$

(note that we could even find an automorphism mapping $x^2 + y^3 + \frac{1}{2}l$ to $x^2 + y^3$). We obtain

$$\varphi(f) = (x^2 + y^3)^2 + \sum_{3i+2j>12+a} \overline{w}_{ij} x^i y^j$$

for suitable $\overline{w}_{ij} \in \mathbb{C}$.

If $a = d$, then we have

$$\sum_{3i+2j=12+d} (-1)^{[i/2]} w_{ij} \neq 0.$$

Similarly as before, we can write

$$\sum_{3i+2j=12+d} w_{ij}x^i y^j = w_{i_0 j_0} x^{i_0} y^{j_0} + l \cdot (x^2 + y^3)$$

with

$$(i_0, j_0) = \begin{cases} (0, \frac{\mu-3}{2}) \text{ if } \mu \text{ is odd} \\ (1, \frac{\mu-6}{2}) \text{ if } \mu \text{ is even.} \end{cases}$$

Since the Milnor number is $15 + d$, we obtain $w_{i_0,j_0} \neq 0$. Using a similar automorphism as in the previous case, we may assume with $a_0 := w_{i_0,j_0}$ (the first modulus), that

$$f = (x^2 + y^3)^2 + a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i+2j>12+d} w_{ij} x^i y^j .$$

Note that $12 + d = \mu - 3$, and we have to compute the normal form of f up to degree $\mu - 1$. Now we can write

$$\sum_{3i+2j=13+d} w_{ij} x^i y^j = e \cdot x^{i_1} y^{j_1} + l \cdot (x^2 + y^3)$$

with

$$(i_1, j_1) = \begin{cases} (1, \frac{\mu-5}{2}) \text{ if } \mu \text{ is odd} \\ (0, \frac{\mu-2}{2}) \text{ if } \mu \text{ is even.} \end{cases}$$

Using an automorphism as before, we may assume that $l = 0$.

If $e = 0$, we are done with weighted degree $\mu - 2$.

If $e \neq 0$, we define an automorphism φ of $\mathbb{C}[[x, y]]$ by the exponential of the vector field

$$\delta = c \cdot (3y^2 \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y})$$

with

$$c = (-1)^{\mu-1} \frac{e}{(\mu-3)a_0}.$$

Since by construction, $\varphi(x^2 + y^3) = x^2 + y^3$, we obtain

$$\varphi(f) = (x^2 + y^3)^2 + a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i+2j \geq 14+d} \tilde{w}_{ij} x^i y^j$$

for suitable $\tilde{w}_{ij} \in \mathbb{C}$.

Remark 31 Note that for practical purposes, we have to compute φ only up to weighted degree 5 and apply it to $a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i+2j=13+d} w_{ij} x^i y^j$ since we know that $\varphi((x^2 + y^3)^2) = (x^2 + y^3)^2$.

Now let

$$(i_1, j_1) = \begin{cases} (0, \frac{\mu-1}{2}) \text{ if } \mu \text{ is odd} \\ (1, \frac{\mu-4}{2}) \text{ if } \mu \text{ is even} \end{cases}$$

and write

$$\sum_{3i+2j=14+d} \tilde{w}_{ij} x^i y^j = a_1 \cdot x^{i_1} y^{j_1} + l \cdot (x^2 + y^3).$$

Using an automorphism as in the first case, we may assume $l = 0$ and obtain as normal form

$$(x^2 + y^3)^2 + a_0 x^{i_0} y^{j_0} + a_1 \cdot x^{i_1} y^{j_1}.$$

We summarize the approach in Algorithm 6.

Remark 32 The approach described in Algorithm 5 in case of a nondegenerate Newton boundary can be adapted to also handle the cases $W^{\sharp}_{1,\mu-15}$. However, this strategy requires more iterations than Algorithm 6. To adapt Algorithm 5, we remove lines 1 and 2, and in line 11, we call Algorithm 7 instead of Algorithm 4 if f is of type $W^{\sharp}_{1,\mu-15}$.

Note that in these cases, Algorithm 2 does not require a field extension; hence, Algorithm 7 is called with input defined over $\mathbb{Q}$. Note also that Algorithm 7 is applicable with any choice of a system B of the local algebra.

Algorithm 6 Algorithm to determine parameters for singularities of type $W^{\sharp}_{1,\mu-15}$

Input: $f = \gamma \cdot (\alpha x^2 + \beta y^3)^2 +$ terms of weighted $(3,2)$-degree $> 12 \in \mathbb{K}[x,y]$ with $\alpha, \beta, \gamma \in \mathbb{K}$ and $\mu := \mu(f) < \infty$.

Output: A normal form of f of the form

$$\begin{array}{ll} (x^2+y^3)^2 + a_0 \cdot xy^{\frac{\mu-6}{2}} + a_1 \cdot xy^{\frac{\mu-4}{2}} & \text{if } \mu \text{ is even} \\ (x^2+y^3)^2 + a_0 \cdot y^{\frac{\mu-3}{2}} + a_1 \cdot y^{\frac{\mu-1}{2}} & \text{if } \mu \text{ is odd} \end{array}$$

with $a_0 \neq 0$, as well as the corresponding moduli parameters of a normal form equation defined over an algebraic extension field of $\mathbb{K}$.

Apply transformation $x \mapsto ax$, $y \mapsto by$ over an algebraic extension field of $\mathbb{K}$ to f to transform the weighted homogeneous part of f to $(x^2+y^3)^2$.
Let $>$ be the local weighted degree reverse lexicographical ordering with weights $(3,2)$ and $x > y$.
Compute a standard basis G of $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ with respect to $>$.
Compute μ the Milnor number of f, and set $d := \mu - 15$.
$a := 13$
while $a < 12 + d$ **do**
    $g :=$ weighted homogeneous part of f of degree a
    Write $g = l \cdot (x^2 + y^3)$.
    Construct automorphism φ with $\varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3$ up to degree $\mu - 1$.
    $f := \varphi(f)$, $a := a + 1$
$g :=$ weighted homogeneous part of f of degree $12 + d$
if μ is odd **then**
    $m_0 := y^{\frac{\mu-3}{2}}$, $m_1 := xy^{\frac{\mu-5}{2}}$, $m_2 := y^{\frac{\mu-1}{2}}$
else
    $m_0 := xy^{\frac{\mu-6}{2}}$, $m_1 := y^{\frac{\mu-2}{2}}$, $m_2 := xy^{\frac{\mu-4}{2}}$
Write $g = a_0 \cdot m_0 + l \cdot (x^2 + y^3)$.
Construct automorphism φ with $\varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3$ up to degree $\mu - 1$.
$f := \varphi(f)$
$g :=$ weighted homogeneous part of φ of degree $13 + d$
Write $g = e \cdot m_1 + l \cdot (x^2 + y^3)$.
Construct automorphism φ with $\varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3$ up to degree $\mu - 1$.
$f := \varphi(f)$
if $e \neq 0$ **then**
    $c := (-1)^{\mu-1} \frac{e}{(\mu-3)a}$
    Construct automorphism φ defined by the vector field $c \cdot (3y^2 \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y})$ up to degree 5.
    $f := (x^2 + y^3)^2 + \varphi(f - (x^2 + y^3)^2)$
    $g :=$ the weighted homogeneous part of f of degree $14 + d$
    Write $g = a_1 \cdot m_2 + l \cdot (x^2 + y^3)$.
return $(\mathrm{NF}(W^{\sharp}_{1,\mu-15}), (a_0, a_1))$

Algorithm 7 Remove terms above the diagonal in cases with degenerate Newton boundary

Input: $f, f_0 \in \mathbb{Q}[x,y]$, $t \in \mathbb{Q}[x,y]$ a term, and weights $u_1, u_2 \in \mathbb{Z}^2$.
Output: $h \in \mathbb{Q}[x,y]$ such that $f \sim h$.

$w := w(f)$ and $j := w\text{-}\deg(t)$
$g :=$ output of Algorithm 4 with input f, f_0, t and u_1, u_2
$B :=$ Arnold's system of $\mathbb{Q}[x,y]/\operatorname{Jac}(f)$
if $t \in \operatorname{Jac}(f_0)$ or $g \neq f$ or ($g = f$ and B contains an element of degree j) **then**
    return g
$m :=$ monomial in B of minimal w-degree
Factorize $f_0 = \gamma \cdot g_0^2$ over $\mathbb{Q}$ with $\gamma \in \mathbb{Q}$ and $g_0 \in \mathbb{Q}[x,y]$ linear.
$\phi :=$ automorphism defined by $(\frac{\partial g_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial g_0}{\partial x}\frac{\partial}{\partial y})$ up to w-degree 5
$s := \operatorname{coeff}(f, m) \cdot m$
$t' := w\text{-}\operatorname{jet}(\phi(s) - s, j)$
for all terms $\tilde{t}$ of t' in increasing order by standard degree **do**
    $t' := -f_0 +$ result of Algorithm 4 with input $t' + f_0, f_0, \tilde{t}$ and u_1, u_2
    $t' := w\text{-}\operatorname{jet}(t', j)$
$c := -t/t'$
$\phi_c :=$ automorphism defined by $c \cdot (\frac{\partial g_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial g_0}{\partial x}\frac{\partial}{\partial y})$ up to w-degree 5.
$h := f_0 + \phi_c(f - f_0)$
for all terms $\tilde{t}$ of h of w-degree j in increasing order by standard degree **do**
    $h :=$ result of Algorithm 4 with input $h, f_0, \tilde{t}$ and u_1, u_2
return h

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Linear Resolutions of Powers and Products

Winfried Bruns and Aldo Conca

To Gert-Martin Greuel on the occasion of his seventieth birthday

Abstract The goal of this paper is to present examples of families of homogeneous ideals in the polynomial ring over a field that satisfy the following condition: every product of ideals of the family has a linear free resolution. As we will see, this condition is strongly correlated to good primary decompositions of the products and good homological and arithmetical properties of the associated multi-Rees algebras. The following families will be discussed in detail: polymatroidal ideals, ideals generated by linear forms, and Borel-fixed ideals of maximal minors. The main tools are Gröbner bases and Sagbi deformation.

Keywords Determinantal ideal • Gröbner basis • Ideal of linear forms • Koszul algebra • Linear resolution • Polymatroidal ideal • Primary decomposition • Rees algebra • Regularity • Toric deformation

AMS Subject classes (2010) 13A30, 13D02, 13C40, 13F20, 14M12, 13P10

1 Introduction

The goal of this paper is to present examples of families of homogeneous ideals in the polynomial ring $R = K[X_1, \dots, X_n]$ over a field K that satisfy the following condition: every product of ideals of the family has a linear free resolution. As we will see, this condition is strongly correlated to "good" primary decompositions of

W. Bruns (✉)
Universität Osnabrück, Institut für Mathematik, Albrechtstr. 28a, 49069 Osnabrück, Germany
e-mail: wbruns@uos.de

A. Conca
Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso, 35, Genova, Italy
e-mail: conca@dima.unige.it

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the products and "good" homological and arithmetical properties of the associated multi-Rees algebras.

For the notions and basic theorems of commutative algebra that we will use in the following, we refer the reader to the books of Bruns and Herzog [8] and Greuel and Pfister [23]. However, at one point our terminology differs from [23]: where [23] uses the attribute "lead" (monomial, ideal, etc.), we are accustomed to "initial" and keep our convention in this note. Moreover, we draw some standard results from our survey [6].

The extensive experimental work for this paper (and its predecessors) involved several computer algebra systems: CoCoA [1], Macaulay 2 [22], Normaliz [11], and Singular [19].

Let us first give a name to the condition on free resolutions in which we are interested and as well recall the definition of ideal with linear powers from [10].

Definition 1.1

1. A homogeneous ideal I of R has *linear powers* if I^k has a linear resolution for every $k \in \mathbb{N}$.
2. A (not necessarily finite) family of homogeneous ideals $\mathscr{F}$ has *linear products* if for all $I_1, \dots, I_w \in \mathscr{F}$ the product $I_1 \cdots I_w$ has a linear resolution.

By *resolution* we mean a graded free resolution, and we call such a resolution *linear* if the matrices in the resolution have linear entries, of course except the matrix that contains the generators of the ideal. However, we assume that the ideal is generated by elements of constant degree. This terminology will be applied similarly to graded modules.

Note that in (2) we have not demanded that the ideals are distinct, so, in particular, powers and products of powers of elements in $\mathscr{F}$ are required to have linear resolutions.

Given an ideal I of R, we denote the associated Rees algebra by $R(I)$, and similarly, for ideals $I_1, \dots, I_w$ of R, we denote the associated multi-Rees algebra by $R(I_1, \dots, I_w)$. If each ideal I_i is homogeneous and generated by elements of the same degree, say d_i, then $R(I_1, \dots, I_w)$ can be given the structure of a standard $\mathbb{Z}^{w+1}$-graded algebra. In Sect. 3 we will explain this in more detail.

The two notions introduced above can be characterized homologically. In [32] and [10], it is proved that an ideal I of R has linear powers if and only if $\operatorname{reg}_0 R(I) = 0$. In Theorem 3.1 we extend this result by showing that a family of ideals $\mathscr{F}$ of R has linear products if and only if for all $I_1, \dots, I_w \in \mathscr{F}$ one has $\operatorname{reg}_0 R(I_1, \dots, I_w) = 0$. Here reg_0 refers to the Castelnuovo–Mumford regularity computed according to the $\mathbb{Z}$-graded structure of the Rees algebra induced by inclusion of R in it; see Sects. 2 and 3 for the precise definitions and for the proof of this statement.

The prototype of a family of ideals with linear products is the following:

Example 1.2 A monomial ideal I of R is *strongly stable* if $I : (X_i) \supseteq I : (X_j)$ for all i, j, $1 \le i < j \le n$. Strongly stable ideals are *Borel fixed* (i.e., fixed under the K-algebra automorphisms of R induced by upper triangular linear transformations). In characteristic 0, every Borel-fixed ideal is strongly stable. The regularity of a strongly stable ideal I is the largest degree of a minimal generator of I. Hence a strongly stable ideal generated in a single degree has a linear resolution. Furthermore it is easy to see that the product of strongly stable ideals is strongly stable. Summing up, the family

$$\mathscr{F} = \{I : I \text{ is a strongly stable ideal generated in a single degree}\}$$

has linear products.

The following example, discovered in the late 1990s by the second author in collaboration with Emanuela De Negri, shows that the Rees algebra associated with a strongly stable ideal need not be Koszul, normal, or Cohen–Macaulay.

Example 1.3 In the polynomial ring $K[X, Y, Z]$, consider the smallest strongly stable ideal I that contains the three monomials Y^6, $X^2Y^2Z^2$, X^3Z^3. The ideal I is generated by

$$X^6, X^5Y, X^4Y^2, X^3Y^3, X^2Y^4, XY^5, Y^6, X^5Z, X^4YZ, X^3Y^2Z, X^2Y^3Z, X^4Z^2, X^3YZ^2, X^2Y^2Z^2, X^3Z^3.$$

It has a non-quadratic, non-normal, and non-Cohen–Macaulay Rees algebra. Indeed, $R(I)$ is defined by 22 relations of degree $(1, 1)$, 72 relations of degree $(0, 2)$, and exactly one relation of degree $(0, 3)$ corresponding to $(X^2Y^2Z^2)^3 = (Y^6)(X^3Z^3)^2$, and its h-polynomial (the numerator of the Hilbert series) has negative coefficients. Therefore, it is not Cohen–Macaulay, and by Hochster's theorem [8, Theorem 6.3.5], it cannot be normal.

On the other hand, for a *principal strongly stable* ideal, i.e., the smallest strongly stable ideal containing a given monomial, the situation is much better. Say $u = X_1^{a_1} \cdots X_n^{a_n}$ is a monomial of R and $I(u)$ is the smallest strongly stable ideal containing u. Then

$$I(u) = \prod_{i=1}^{n}(X_1, \ldots, X_i)^{a_i} = \bigcap_{i=1}^{n}(X_1, \ldots, X_i)^{b_i}, \qquad b_i = \sum_{j=1}^{i} a_j. \tag{1}$$

Since the powers of an ideal generated by variables are primary and hence integrally closed, the primary decomposition formula (1) implies right away that the ideal $I(u)$ is integrally closed. It is an easy consequence of (1) that for every pair of monomials u_1, u_2, one has

$$I(u_1)I(u_2) = I(u_1u_2).$$

It follows that products of principal strongly stable ideals are integrally closed. Hence the multi-Rees algebra $R(I(u_1), \ldots, I(u_w))$ associated with principal strongly stable ideals $I(u_1), \ldots, I(u_w)$ is normal, which implies that it is also Cohen–Macaulay.

Furthermore, De Negri [18] proved that the fiber ring of $R(I(u_1))$ is defined by a Gröbner basis of quadrics, and very likely a similar statement is true for the multi-fiber ring of the multi-Rees algebra $R(I(u_1), \dots, I(u_w))$.

Let us formalize the properties of the primary decomposition that we have observed for principal strongly stable ideals:

Definition 1.4

1. An ideal $I \subseteq R$ is of *P-adically closed* if

$$I = \bigcap_{P \in \operatorname{Spec} R} P^{(v_P(I))}$$

where v_P denotes the P-adic valuation: $v_P(I) = \max\{k \geq 0 : P^{(k)} \supseteq I\}$.

2. The family $\mathscr{F}$ of ideals has the *multiplicative intersection property* if for every product $J = I_1 \cdots I_w$ of ideals $I_i \in \mathscr{F}$, one has

$$J = \bigcap_{P \in \mathscr{F} \cap \operatorname{Spec} R} P^{(v_P(J))}.$$

Note that a P-adically closed ideal is integrally closed. Therefore all products considered in (2) are integrally closed. The multiplicative intersection property does not necessarily imply that the prime ideals in $\mathscr{F}$ have primary powers, but this is obviously the case if $v_P(Q) = 1$ whenever $Q \subset P$ are prime ideals in the family.

Our goal is to present three families $\mathscr{F}$ of ideals that generalize the family of principal strongly stable ideals in different directions and have some important features in common:

1. $\mathscr{F}$ has linear products;
2. $\mathscr{F}$ has the multiplicative intersection property;
3. the multi-Rees algebras associated to ideals in the family have good homological and arithmetical properties and defining equations (conjecturally) of low degrees.

These families are:

(a) polymatroidal ideals (Sect. 5);
(b) ideals of linear forms (Sect. 6);
(c) Borel-fixed ideals of maximal minors (Sect. 7).

Each family has specific properties that will be discussed in detail.

2 Partial Regularity

Let R be a $\mathbb{Z}^r$-graded ring. The degree of an element x of R is denoted by $\deg x$, and if we speak of the degree of an element, it is assumed that the element is homogeneous. In the following it will be important to consider the partial $\mathbb{Z}$-degrees defined by the

$\mathbb{Z}^r$-grading: by $\deg_i x$ we denote the ith coordinate of $\deg x$ and speak of $\deg_i$ as the *i-degree*. The same terminology applies to $\mathbb{Z}^r$-graded R-modules. One calls *R a standard* $\mathbb{Z}^r$*-graded* K-algebra if R_0, its component of degree $0 \in \mathbb{Z}^r$, is the field K and R is generated as a K-algebra by homogeneous elements whose degree is one of the unit vectors in $\mathbb{Z}^r$. The ideal generated by the elements of nonzero degree is denoted $\mathfrak{m}_R$ or simply by $\mathfrak{m}$. It is a maximal ideal, and K is identified with $R/\mathfrak{m}_R$ as an R-module.

If $n = 1$, then no index is used for degree or any magnitude derived from it. Moreover, if it is convenient, we will label the coordinates of $\mathbb{Z}^r$ by $0, \dots, r-1$.

For any finitely generated graded module M over a standard $\mathbb{Z}^r$-graded polynomial ring R, we can define *(partial) Castelnuovo–Mumford regularities* $\operatorname{reg}_i(M)$, $i = 1, \dots, r$. In the bigraded case, they have been introduced by Aramova et al. [2]. First we set

$$\sup_i M = \sup\{\deg_i x : x \in M\}.$$

Next, let

$$t_{ik}(M) = \sup_i \operatorname{Tor}_k^R(K, M).$$

Since the Tor-modules are finite dimensional vector spaces, $t_{ik}(M) < \infty$ for all i and k. Moreover, $t_{ik}(M) = -\infty$ for all $k > \dim R$ by Hilbert's syzygy theorem. Now we can define

$$\operatorname{reg}_i^R(M) = \operatorname{reg}_i(M) = \sup_k \{t_{ik}(M) - k\}.$$

The Tor-modules have a graded structure since M has a minimal graded free resolution

$$\mathscr{L}:\ 0 \to \bigoplus_{g \in \mathbb{Z}^r} R(-g)^{\beta_{pg}} \to \cdots \to \bigoplus_{g \in \mathbb{Z}^r} R(-g)^{\beta_{0g}} \to M \to 0, \qquad p = \operatorname{projdim} M.$$

Then $t_{ik}(M)$ is the maximum ith coordinate of the shifts g for which $\beta_{kg} \neq 0$. The β_{kg} are called the *kth graded Betti numbers* of M.

As in the case of $\mathbb{Z}$-gradings, one can compute partial regularities from local cohomology.

Theorem 2.1 *Let R be a standard* $\mathbb{Z}^r$*-graded polynomial ring and M a finitely graded R-module. Then*

$$\operatorname{reg}_i(M) = \sup_k \{\sup_i H_{\mathfrak{m}}^k(M) + k\}$$

for all i.

For the theorem to make sense, one needs at least that the local cohomology with support in $\mathfrak{m}$ has a natural $\mathbb{Z}^r$-graded structure, and this indeed the case as one can see from its description by the Čech complex. In the $\mathbb{Z}$-graded case, Theorem 2.1 is due to Eisenbud and Goto [21]. For the proof of Theorem 2.1, one can follow [8, p. 169]. The crucial point is multigraded local duality. It can be derived from $\mathbb{Z}$-graded local cohomology since the $\mathbb{Z}$-graded components of the modules involved are direct sums of *finitely* many multigraded components. Therefore it makes no difference whether one takes graded $\operatorname{Hom}(-, K)$ in the category of $\mathbb{Z}$-graded modules or the category of $\mathbb{Z}^r$-graded modules.

We are interested in ideals with linear resolutions and for them Theorem 2.1 has an obvious consequence:

Corollary 2.2 *Let R be standard $\mathbb{Z}$-graded polynomial ring over a field K and I an ideal generated in degree d with a linear resolution. Then $\sup H^k_{\mathfrak{m}}(R/I) \le d - 1 - k$ for all k.*

In fact, if I has a linear resolution, then $\operatorname{reg} R/I = d - 1$.

An extremely useful consequence of Theorem 2.1 is that it allows change of rings in an easy way.

Lemma 2.3 *Let R be a standard $\mathbb{Z}^r$-graded polynomial ring over the field K and let x be an element of degree e_i for some i. Suppose that x is either a nonzerodivisor on the $\mathbb{Z}^r$-graded finitely generated module M or annihilates M, and let $S = R/(x)$. Then $\operatorname{reg}_i^R(M) = \operatorname{reg}_i^S(M/xM)$.*

Proof If x is a nonzerodivisor, then one can simply argue by free resolutions since $\mathscr{L} \otimes S$ is a minimal graded free S-resolution of M/xM if $\mathscr{L}$ is such an R-resolition of M. In the second case, in which of course $M/xM = M$, one uses Theorem 2.1 and the invariance of local cohomology under finite ring homomorphisms.

We need some auxiliary results. The behavior of reg_i along homogeneous short exact sequences $0 \to U \to M \to N \to 0$ is captured by the inequalities

$$\operatorname{reg}_i(M) \le \max\{\operatorname{reg}_i(U), \operatorname{reg}_i(N)\},$$
$$\operatorname{reg}_i(U) \le \max\{\operatorname{reg}_i(M), \operatorname{reg}_i(N) + 1\},$$
$$\operatorname{reg}_i(N) \le \max\{\operatorname{reg}_i(U) - 1, \operatorname{reg}_i(M)\}.$$

that follow immediately from the long exact sequence of Tor-modules.

Another very helpful result is the following lemma by Conca and Herzog [14, Proposition 1.2]:

Lemma 2.4 *Let R be standard $\mathbb{Z}$-graded and M be a finitely generated graded R-module. Suppose $x \in R$ a homogeneous element of degree 1 such that $0 :_M x$ has finite length. Set $a_0 = \sup H^0_{\mathfrak{m}}(M)$. Then $\operatorname{reg}(M) = \max\{\operatorname{reg}(M/xM), a_0\}$.*

3 The Multi-Rees Algebra

The natural object that allows us to study all products of the ideals $I_1, \dots, I_w$ in a ring R simultaneously is the *multi-Rees algebra*

$$\mathscr{R} = R(I_1, \dots, I_w) = R[I_1T_1, \dots, I_wT_w] = \bigoplus_{a \in \mathbb{Z}_+^w} I_1^{a_1} \cdots I_w^{a_w} T_1^{a_1} \cdots T_w^{a_w} \subseteq R[T_1, \dots, T_w]$$

where $T_1, \dots, T_w$ are indeterminates over R. As a shortcut we set

$$I^a = I_1^{a_1} \cdots I_w^{a_w} \qquad \text{and} \qquad T^a = T_1^{a_1} \cdots T_w^{a_w}.$$

If R is a standard graded algebra over a field K, and each I_i is a graded ideal generated in a single degree d_i, then $R(I_1, \dots, I_w)$ carries a natural standard $\mathbb{Z}^{w+1}$-grading. In fact let $e_0, \dots .e_w$ denote the elements of the standard basis of $\mathbb{Z}^{w+1}$. Then we identify $\deg x$ with $(\deg x)e_0$ for $x \in R$ and set $\deg T_i = -d_ie_0 + e_i$ for $i = 1, \dots, w$. Evidently $\mathscr{R}$ is then generated over K by its elements whose degree is one of $e_0, \dots, e_w$.

We want to consider $\mathscr{R}$ as a residue class ring of a standard $\mathbb{Z}^{w+1}$-graded polynomial ring $\mathscr{S}$ over K. To this end we choose a system $f_{i1}, \dots, f_{im_i}$ of degree d_i generators of I_i for $i = 1, \dots, w$ and indeterminates $Z_{i1}, \dots, Z_{im_i}$. Then we set

$$\mathscr{S} = R[Z_{ij} : i = 1, \dots, w,\ j = 1, \dots, m_i]$$

and define $\Phi : \mathscr{S} \to \mathscr{R}$ by the substitution

$$\Phi|R = \mathrm{id}_R, \qquad \Phi(Z_{ij}) = f_{ij}T_i.$$

We generalize a theorem of Römer [32], following the simple proof of Herzog et al. [27], and complement it by its converse:

Theorem 3.1 *Let R be a standard graded polynomial ring over the field K. The family $I_1, \dots, I_w$ of ideals in R has linear products if and only if* $\mathrm{reg}_0(R(I_1, \dots, I_w)) = 0$.

Proof For a graded module M over $\mathscr{S}$ and $h \in \mathbb{Z}^w$, we set

$$M_{(\bullet,h)} = \bigoplus_{j \in \mathbb{Z}} M_{(j,h)}$$

where $(j, h) \in \mathbb{Z} \times \mathbb{Z}^w = \mathbb{Z}^{w+1}$. Clearly $M_{(\bullet,h)}$ is an R-submodule of M. It is the degree h homogeneous component of M under the $\mathbb{Z}^w$-grading in which we ignore the 0th partial degree of our $\mathbb{Z}^{w+1}$-grading.

We apply the operator $(\bullet, h)$ to the whole free resolution $\mathscr{L}$ of $M = \mathscr{R}$ and obtain the exact sequence

$$\mathscr{L}_{(\bullet,h)}:\ 0 \to \bigoplus_{g\in\mathbb{Z}} \mathscr{S}(-g)^{\beta_{pg}}_{(\bullet,h)} \to \cdots \to \bigoplus_{g\in\mathbb{Z}} \mathscr{S}(-g)^{\beta_{0g}}_{(\bullet,h)} \to \mathscr{R}_{(\bullet,h)} \to 0.$$

Since the R-modules $\mathscr{S}(-g)_{(\bullet,h)}$ are free over R, we obtain a graded (not necessarily minimal) free resolution of $\mathscr{S}_{(\bullet,h)}$ over R. The rest of the proof of this implication is careful bookkeeping of shifts.

First, as an R-module,

$$\mathscr{R}_{(\bullet,h)} = I^h T^h \cong I^h(d\cdot h), \qquad d\cdot h = \sum d_i h_i.$$

Moreover, $\mathscr{S}(-g)_{(\bullet,h)}$ is a free module over R with basis elements in degree g_0 since the indeterminates Z_{ij} have degree 0 with respect to $\deg_0$.

Now suppose that $\operatorname{reg}_0(\mathscr{R}) = 0$. Then $g_{0k} \le k$ for all k, and we see that

$$\operatorname{reg} I^h(d\cdot h) = \operatorname{reg}_0(I^h T^h) = 0,$$

and so $\operatorname{reg}(I^h) = d\cdot h$ as desired.

The converse is proved by induction on $\dim R$. For $\dim R = 0$, there is nothing to show. In preparing the induction step, we first note that there is no restriction in assuming that the ground field K is infinite. Next we use that there are only finitely many prime ideals in R that are associated to any of the products I^h (West [34, Lemma 3.2]). Therefore we can find a linear form $z \in R_1$ that is not contained in any associated prime ideal of any I^h different from $\mathfrak{m}_R$. In other words, $I^h : z$ has finite length for all h.

Set $S = R/(z)$. It is again a standard graded polynomial ring (after a change of coordinates). Furthermore let $J_i = I_i S = (I_i + z)/(z)$. Then $(I^h + (z))/(z) = J^h$. Next we want to compute $\operatorname{reg}^S(S/J^h) = \operatorname{reg}^R(S/J^h)$. The hypotheses of Lemma 2.4 are satisfied for $M = R/I^h$. Since $M/zM = S/J^h$, we obtain

$$d\cdot h - 1 = \operatorname{reg}^R(R/I^h) = \max(\operatorname{reg}^R(S/J^h, \sup H^0_{\mathfrak{m}}(R/I^h))).$$

Unless we are in the trivial case $J^h = 0$, we have $\operatorname{reg}^S(S/J^h) = \operatorname{reg}^R(S/J^h) \ge d\cdot h - 1$ anyway. Thus $\operatorname{reg}^S(S/J^h) = d\cdot h - 1$, and therefore S/J^h has a linear resolution over S. This allows us to apply the induction hypothesis to the family $J_1, \dots, J_w$, which appears in the exact sequence

$$0 \to W \to \mathscr{R}/z\mathscr{R} \to S(J_1, \dots, J_w) \to 0.$$

In fact, by induction we conclude that $\operatorname{reg}_0(S(J_1, \dots, J_w)) = 0$ over its representing polynomial ring, namely, $\mathscr{S}/(z)$. But because of Lemma 2.3, we can measure reg_0 over $\mathscr{S}$ as well. Moreover, $\operatorname{reg}_0 \mathscr{R}/z\mathscr{R} = \operatorname{reg}_0 \mathscr{R}$.

It view of the behavior of reg_0 along short exact sequences, it remains to show that $\operatorname{reg}_0 W = 0$. As an R-module, W splits into its $\mathbb{Z}^w$-graded components $W_h = W'_h T^h$ where

$$W'_h = (I^h \cap (z))/zI^h = z(I^h : z)/zI^h.$$

We claim that W'_h is concentrated in degree $d \cdot h$ and therefore annihilated by $\mathfrak{m}_R$. Since it is a subquotient of I^h, it cannot have nonzero elements in degree $< d \cdot h$. On the other hand, the description $W'_h = z(I^h : z)/zI^h$ shows that the degree of the elements in W is bounded above by $1 + \sup(I^h : z)/I^h$. By the choice of z, this module is contained in $H^0_{\mathfrak{m}_R}(R/I^h)$ whose degrees are bounded by $d \cdot h - 1$ because of Corollary 2.2. Multiplication by T^h moves W'_h into degree 0 with respect to $\deg_0$ in $\mathscr{R}$.

Recombining the W_h to W, we see that W is generated as an $\mathscr{S}$-module by elements of 0-degree 0. Moreover, it is annihilated by $\mathfrak{m}_R\mathscr{S}$, and therefore a module over the polynomial ring $\mathscr{S}' = \mathscr{S}/\mathfrak{m}_R\mathscr{S}$ on which the 0-degree of $\mathscr{S}$ vanishes. Hence the 0-regularity of an $\mathscr{S}'$-module M is the maximum of $\deg_0 x$ where x varies over a minimal system of generators of M. But, as observed above, W is generated in 0-degree 0.

Remark 3.2

(a) The first part of the proof actually shows that $\operatorname{reg}(I^h) \le d \cdot h + \operatorname{reg}_0(\mathscr{R})$ for every h, as stated by Römer [32, Theorem 5.3] for a single ideal. It seems to be unknown whether there always exists at least one exponent h for which $\operatorname{reg}(I^h) = d \cdot h + \operatorname{reg}_0(\mathscr{R})$. By virtue of Theorem 3.1, this is the case if $\operatorname{reg}_0(\mathscr{R}) \le 1$.
(b) One can show that the Betti number of the ideals J^h over S (notation as in the proof) are determined by those of the ideal I^h over R. In the case of the powers of a single ideal, this has been proved in [10, Lemma 2.4].
(c) An object naturally associated with the multi-Rees algebra is the *multi-fiber ring*

$$F(I_1, \dots, I_w) = R(I_1, \dots, I_w)/\mathfrak{m}_R R(I_1, \dots, I_w).$$

If each of the ideals I_i is generated in a single degree, then $F(I_1, \dots, I_w)$ can be identified with the K-subalgebra of $R(I_1, \dots, I_w)$ that is generated by the elements of degrees $e_1, \dots, e_w$. As such, it is a retract of $R(I_1, \dots, I_w)$, and since the ideals of interest in this paper are generated in a single degree, we will always consider the multi-fiber ring as a retract of the multi-Rees algebra in the following.

An ideal $I \subseteq R$ that is generated by the elements $f_1, \dots, f_m$ of constant degree is said to be of *fiber type* if the defining ideal of the Rees algebra $R(I)$ is generated by polynomials that are either (1) linear in the indeterminates representing the generators $f_i T$ of the Rees algebra (and therefore are relations of the symmetric algebra) (2) belong to the defining ideal of $F(I)$. One can immediately generalize this notion and speak of a family $I_1, \dots, I_w$ of *multi-fiber type*. In the situation of the theorem, it follows that $J_1, \dots, J_w$ is of multi-fiber type if $I_1, \dots, I_w$ has this property.

More generally than in Theorem 3.1, one can consider arbitrary standard $\mathbb{Z}^r$-graded K-algebras. For them Blum [3] has proved an interesting result. Note that a standard $\mathbb{Z}^r$-graded K-algebra is a standard $\mathbb{Z}$-graded K-algebra in a natural way, namely, with respect to the total degree, the sum of the partial degrees $\deg_i$, $i = 1, \dots, r$. Therefore it makes sense to discuss properties of a standard $\mathbb{Z}$-graded algebra in the standard $\mathbb{Z}^r$-graded case.

Theorem 3.3 *Suppose the standard $\mathbb{Z}^r$-graded algebra R is a Koszul ring. Furthermore let $a, b \in \mathbb{Z}^n_+$ and consider the module*

$$M_{(a,b)} = \bigoplus_{k=0}^{\infty} R_{ka+b}$$

over the "diagonal" subalgebra

$$R_{(a)} = \bigoplus_{k=0}^{\infty} R_{ka}.$$

Then $R_{(a)}$ is a standard graded K-algebra with $(R_{(a)})_k = R_{ka}$, and $M_{(a,b)}$ has a (not necessarily finite) linear resolution over it.

Blum states this theorem only in the bigraded case, but remarks himself that it can easily be generalized to the multigraded case. Let us sketch a slightly simplified version of his proof for the general case. Let $\mathfrak{m}$ be the ideal generated by all elements of total degree 1. The "degree selection functor" that takes the direct sum of the $\mathbb{Z}^r$-graded components whose degrees belong to a subset of $\mathbb{Z}^r$ is exact. There is nothing to prove for $M_{(a,b)}$ if $b = 0$. If $b \neq 0$, then the degree selection functor for the subset $\{ka + b : k \geq 0\}$ cuts out an exact complex of $R_{(a)}$-modules from the resolution of $\mathfrak{m}$, which, by hypothesis on R, is linear over R. The modules occurring in it are not necessarily free over $R_{(a)}$, but they are under control, and an involved induction allows one to bound the shifts in their resolutions. These bounds in their turn imply that $M_{(a,b)}$ has a linear resolution over $R_{(a)}$.

We highlight the following special case of Theorem 3.3:

Theorem 3.4 *Let $R = K[X_1, \dots, X_n]$ and $I_1, \dots, I_w$ ideals of R such that $R(I_1, \dots, I_w)$ is Koszul. Then the family $I_1, \dots, I_w$ has linear products.*

Proof Note that R is the diagonal subalgebra for $a = e_0$ of $R(I_1, \dots, I_w)$ and one obtains $I^h T^h$ as a module over it for $b = (0, h)$.

Remark 3.5

(a) In [10, Example 2.6] we give an example of a monomial ideal I with linear powers that is not of fiber type and whose Rees algebra is not Koszul. But even linear powers and fiber type together do not imply that the Rees algebra is Koszul: see [10, Example 2.7].

(b) For the “if” part of Theorem 3.1 and for Theorem 3.3, one can replace the polynomial base ring R by a Koszul ring. This generalization allows relative versions in the following sense: if $R(I, J)$ is Koszul, then $R(I)$ is Koszul because retracts of Koszul rings are Koszul; since $R(I, J) = (R(I))(J)$, one obtains that $JR(I)$ has a linear resolution over $R(I)$.

4 Initial Ideals and Initial Algebras

We start again from a standard graded polynomial ring $R = K[X_1, \dots, X_n]$. It is a familiar technique to compare an ideal J to its initial ideal $\operatorname{in}_<(J)$ for a monomial order $<$ on R. The initial ideal is generated by monomials and therefore amenable to combinatorial methods. Homological properties like being Cohen–Macaulay or Gorenstein or enumerative data like the Hilbert series (if I is homogeneous) can be transferred from $R/\operatorname{in}_<(J)$ to R/J, and for others, like regularity, the value of $R/\operatorname{in}_<(J)$ bounds that of R/J.

Let us formulate a criterion that will allow us to apply Theorem 3.1 or even Theorem 3.3 in order to conclude that a family of ideals has linear products. We use the presentation of the multi-Rees algebra $\mathscr{R} = R(I_1, \dots, I_w)$ as a residue class ring of a polynomial ring $\mathscr{S}$ that has been introduced above Theorem 3.1.

Theorem 4.1 *Let $\mathscr{I} = \operatorname{Ker}\Phi$ be defining ideal of $\mathscr{R}$ as a residue class ring of the polynomial ring $\mathscr{S}$, and let $\prec$ be a monomial order on $\mathscr{S}$. Let G be the minimal set of generators of the monomial ideal $\operatorname{in}_\prec(\mathscr{I})$.*

1. *If every element of G is at most linear in the indeterminates $X_1, \dots, X_n$ of R, then $\operatorname{reg}_0(\mathscr{R}) = 0$, and $I_1, \dots, I_w$ has linear products.*
2. *If G consists of quadratic monomials, then $\operatorname{reg}_0(\mathscr{R}) = 0$, $\mathscr{R}$ is a Koszul algebra, and $I_1, \dots, I_w$ has linear products.*

Proof For the first statement(already observed in [27]), it is enough to show that $\operatorname{reg}_0(\mathscr{S}/\operatorname{in}_\prec(\mathscr{I})) = 0$. Then we obtain $\operatorname{reg}_0(\mathscr{R}) = 0$ as well and can apply Theorem 3.1. In order to estimate the maximal shifts in a minimal free resolution of $\mathscr{S}/\operatorname{in}_\prec(\mathscr{I})$, it is sufficient to estimate them in an arbitrary free resolution, and such is given by the Taylor resolution. All its matrices have entries that are at most linear in the variables $X_1, \dots, X_n$ of R.

If G consists of quadratic monomials, then $\mathscr{S}/\operatorname{in}_\prec(\mathscr{I})$ is Koszul by a theorem of Fröberg, and Koszulness of $\mathscr{S}/\operatorname{in}_\prec(\mathscr{I})$ implies the Koszulness of $\mathscr{R} = \mathscr{S}/\mathscr{I}$. In its turn this implies that $I_1, \dots, I_w$ has linear powers by Theorem 3.4, and then we obtain $\operatorname{reg}_0(\mathscr{R}) = 0$ by Theorem 3.1.

However, we can circumvent Theorem 3.4. Namely, if G consists of quadratic monomials, then it is impossible that any of them is of degree ≥ 2 in $X_1, \dots, X_n$. In this multigraded situation, this would imply that $\mathscr{I}$ contains a nonzero element of R, and this is evidently impossible.

In the previous theorem, we have replaced the defining ideal of the multi-Rees algebra by a monomial object, namely, its initial ideal. It is often useful to replace the multi-Rees algebra by a monomial algebra. In the following $<$ denotes a monomial order on R. We are interested in the products $I^h = I_1^{h_1} \cdots I_w^{h_w}$. There is an obvious inclusion, namely,

$$\mathrm{in}(I_1)^{h_1} \cdots \mathrm{in}(I_w)^{h_w} \subseteq \mathrm{in}(I^h), \tag{2}$$

and it is an immediate question whether one has equality for all h. Let us introduce the notation $\mathrm{in}(I)^h$ for the left-hand side of the inclusion.

To bring the multi-Rees algebra $\mathscr{R} = R(I_1, \ldots, I_w)$ into the play, we extend the monomial order from R to $R[T_1, \ldots, T_w]$ in an arbitrary way. There are two natural initial objects, namely, the initial subalgebra $\mathrm{in}(\mathscr{R})$ on the one side and the multi-Rees algebra

$$\mathscr{R}_{\mathrm{in}} = R(\mathrm{in}(I_1), \ldots, \mathrm{in}(I_w))$$

on the other. Since $\mathscr{R}$ is multigraded in the variables $T_1, \ldots, T_w$, $\mathrm{in}(\mathscr{R})$ does not depend on the extension of $<$ to $R[T_1, \ldots, T_w]$, and we have

$$\mathscr{R}_{\mathrm{in}} \subseteq \mathrm{in}(\mathscr{R}). \tag{3}$$

Equality in (2) for all h is equivalent to equality in (3). It is a special case of the question whether polynomials $f_1, \ldots, f_m$ form a Sagbi basis of the algebra $A = K[f_1, \ldots, f_m]$ they generate. By definition, this means that the initial monomials $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_m)$ generate the initial algebra $\mathrm{in}(A)$. Similarly to the Buchberger criterion for ordinary Gröbner bases, there is a lifting criterion for syzygies. In this case, the syzygies are polynomial equations.

We choose a polynomial ring $S = K[Z_1, \ldots, Z_n]$. Then the substitution

$$\Phi(Z_i) = f_i, \qquad i = 1, \ldots, m.$$

makes A a residue class ring of S. Simultaneously we can consider the substitution

$$\Psi(Z_i) = \mathrm{in}(f_i), \qquad i = 1, \ldots, m.$$

Roughly speaking, one has $\mathrm{in}(A) = K[\mathrm{in}(f_1), \ldots, \mathrm{in}(f_m)]$ if and only if every element of $\operatorname{Ker} \Psi$ can be lifted to an element of $\operatorname{Ker} \Phi$. More precisely:

Theorem 4.2 *With the notation introduced, let B be a set of binomials in S that generate* $\operatorname{Ker}\Psi$. *Then the following are equivalent:*

1. $\operatorname{in}(K[f_1,\dots,f_m]) = K[\operatorname{in}(F_1),\dots,\operatorname{in}(f_m)]$;
2. *For every $b \in B$ there exist monomials $\mu_1,\dots,\mu_q \in S$ and coefficients $\lambda_1,\dots\lambda_q \in K$, $q \geq 0$, such that*

$$b - \sum_{i=1}^{q} \lambda_i \mu_i \in \operatorname{Ker}\Phi \tag{4}$$

and $\operatorname{in}(\Phi(\mu_i)) \leq \operatorname{in}(\Phi(b))$ *for* $i = 1,\dots,q$.

Moreover, in this case, the elements in (4) *generate* $\operatorname{Ker}\Phi$.

The criterion was established by Robbiano and Sweedler [31] in a somewhat individual terminology; in the form above, one finds it in Conca et al. [15, Proposition 1.1]. For the reader who would expect the relation $<$ in (2): if $b = \nu_1 - \nu_2$ with monomials ν_1, ν_2, then $\operatorname{in}(\Phi(b)) < \operatorname{in}(\Phi(\nu_1)) = \operatorname{in}(\Phi(\nu_2))$, unless $\Phi(b) = 0$. The last statement of Theorem 4.2 is [15, Corollary 2.1].

A priori, there is no "natural" monomial order on S. But suppose we have a monomial order $\prec$ on S. Then we can use $\prec$ as the tiebreaker in lifting the monomial order on R to S:

$$\mu \prec_\Psi \nu \quad\iff\quad \Psi(\mu) < \Psi(\nu) \text{ or } \Psi(\mu) = \Psi(\nu),\ \mu \prec \nu.$$

Theorem 4.3 *Suppose that the generating set B of Theorem 4.2(2) is a Gröbner basis of* $\operatorname{Ker}\Psi$ *with respect to $\prec$. Then the elements in* (4) *are a Gröbner basis of* $\operatorname{Ker}\Phi$ *with respect to $\prec_\Psi$.*

See Sturmfels [33, Corollary 11.6] or [15, Corollary 2.2].

5 Polymatroidal Ideals

A monomial ideal I of $R = K[X_1,\dots,X_n]$ is *polymatroidal* if it is generated in a single degree, say d, and for all pairs $u = \prod_h X_h^{a_h}$, $v = \prod_h X_h^{b_h}$ of monomials of degree d in I and for every i such that $a_i > b_i$ there exists j with $a_j < b_j$ and $X_j(u/X_i) \in I$. A square-free polymatroidal ideal is said to be *matroidal* because the underlying combinatorial object is a matroid. Conca and Herzog [14] and Herzog and Hibi [25] proved:

Theorem 5.1 *Every polymatroidal ideal has a linear resolution, and the product of polymatroidal ideals is polymatroidal.*

Hence we may say that the family

$$\mathscr{F} = \{I : I \text{ is polymatroidal}\}$$

has linear products.

Those polymatroidal ideals that are obtained as products of ideals of variables are called *transversal*. For example, $(X_1, X_2)(X_1, X_3)$ is a transversal polymatroidal ideal and (X_1X_2, X_1X_3, X_2X_3) is polymatroidal, but not transversal.

In [26] Herzog and Vladoiu proved the following theorem on primary decomposition:

Theorem 5.2 *The family of polymatroidal ideals has the multiplicative intersection property: given a polymatroidal ideal I, one has*

$$I = \bigcap_P P^{v_P(I)}$$

where the intersection is extended over all the monomial prime ideals P (i.e., ideals generated by variables).

They also proved that $v_P(I)$ can be characterized as the "local" regularity of I and P, that is, the regularity of the ideal obtained from I by substituting 1 for the variables not in P. Of course one gets an irredundant primary decomposition by restricting the intersection to the ideals $P \in \operatorname{Ass}(R/I)$. The problem of describing the associated primes of a polymatroidal ideal in combinatorial terms is discussed by Herzog et al. in [29] where they proved:

Theorem 5.3 *Every polymatroidal ideal I satisfies*

$$\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1}) \qquad \textit{for all } k > 0.$$

Furthermore the equality

$$\operatorname{Ass}(R/I) = \operatorname{Ass}(R/I^k) \qquad \textit{for all } k > 0$$

holds, provided I is transversal.

The latter equality is not true for general polymatroidal ideals. Examples of polymatroidal ideals I such that $\operatorname{Ass}(R/I) \neq \operatorname{Ass}(R/I^2)$ are given by some polymatroidal ideals of Veronese type. For example, for $I = (X_1X_2, X_1X_3, X_2X_3)$, one has $(X_1, X_2, X_3) \in \operatorname{Ass}(R/I^2) \setminus \operatorname{Ass}(R/I)$.

From the primary decomposition formula of Theorem 5.2, it follows that a polymatroidal ideal is integrally closed. Since products of polymatroidal ideals are polymatroidal, it follows then that the multi-Rees algebra $R(I_1, \dots, I_w)$ of polymatroidal ideals $I_1, \dots, I_w$ is normal and hence Cohen–Macaulay by virtue of Hochster's theorem [8, Theorem 6.3.5]. The same is true for the fiber ring $F(I_1, \dots, I_w)$ because it is an algebra retract of $R(I_1, \dots, I_w)$. Therefore:

Theorem 5.4 *Let $I_1, \dots, I_w$ be polymatroidal ideals. Then both $R(I_1, \dots, I_w)$ and $F(I_1, \dots, I_w)$ are Cohen–Macaulay and normal.*

White's conjecture, in its original form, predicts that the fiber ring, called the *base ring* of the matroid in this context, associated to a single matroidal ideal, is defined by quadrics, more precisely by quadrics arising from exchange relations. White's conjecture has been extended to (the fiber rings of) polymatroidal ideals by Herzog and Hibi in [24] who"did not escape from the temptation" to ask also if such a ring is Koszul and defined by a Gröbner basis of quadrics. These conjectures are still open. The major progress toward a solution has been obtained by Lasoń and Michałek [30] who proved White's conjecture "up to saturation" for matroids. Further questions and potential generalizations of White's conjecture refer to the Rees algebra $R(I)$ of a matroidal (or polymatroidal) ideal I: is it defined by (a Gröbner basis) quadrics? Is it Koszul?

Note however that the fiber ring of the multi-Rees algebra associated to polymatroidal ideals need not be defined by quadrics. For example:

Example 5.5 For $I_1 = (X_1, X_2)$, $I_2 = (X_1, X_3)$, and $I_3 = (X_2, X_3)$, the fiber ring of $R(I_1, I_2, I_3)$ is

$$K[T_1X_1, T_1X_2, T_2X_1, T_2X_3, T_3X_2, T_3X_3]$$

and it is defined by a single cubic equation, namely,

$$(T_1X_1)(T_2X_3)(T_3X_2) = (T_1X_2)(T_2X_1)(T_3X_3).$$

Nevertheless for a polymatroidal ideal I Herzog, Hibi, and Vladoiu proved in [28] that the Rees algebra $R(I)$ is of fiber type, and it might be true that multi-Rees algebras $R(I_1, \dots, I_w)$ associated to polymatroidal ideals $I_1, \dots, I_w$ are of multi-fiber type.

6 Products of Ideals of Linear Forms

Let $P_1, \dots, P_w$ be ideals of $R = K[X_1, \dots, X_n]$ generated by linear forms. Each P_i is clearly a prime ideal with primary powers. One of the main results of Conca and Herzog [14] is the following:

Theorem 6.1 *The product $P_1 \cdots P_w$ has a linear resolution.*

Hence may say that the family

$$\mathscr{F} = \{P : P \text{ is generated by linear forms and } P \neq 0\}$$

has linear products. The theorem is proved by induction on the number of variables. The inductive step is based on an estimate of the 0th local cohomology of the corresponding quotient ring or, equivalently, on the saturation degree of the corresponding ideal. The latter is controlled by means of the following primary decomposition computed in [14]:

Theorem 6.2 *The family of ideals generated by linear forms has the multiplicative intersection property. In other words, for every $P_1, \dots, P_w \in \mathscr{F}$ and $I = \prod_{i=1}^{w} P_i$, one has*

$$I = \bigcap_{P \in \mathscr{F}} P^{v_P(I)}.$$

Clearly one can restrict the intersection to the primes of the form

$$P_A = \sum_{i \in A} P_i$$

for a nonempty subset A of $\{1, \dots, w\}$. Setting

$$\mathscr{P} = \{P : P = P_A \text{ for some non-empty subset } A \text{ of } \{1, \dots, w\}\},$$

one gets

$$I = \bigcap_{P \in \mathscr{P}} P^{v_P(I)}.$$

This primary decomposition need not be irredundant. So an important question is whether a given $P \in \mathscr{P}$ is associated to R/I. Inspired by results in [29], we have a partial answer:

Lemma 6.3 *With the notation above, let $P \in \mathscr{P}$ and let $V = \{i : P_i \subseteq P\}$.*

1. *Let G_P be the graph with vertices V and edges $\{i,j\}$ such that $P_i \cap P_j$ contains a nonzero linear form. If G_P is connected, then $P \in \operatorname{Ass}(R/I)$.*
2. *Assume that P can be written as $P' + P''$ with $P', P'' \in \mathscr{F}$ such that $P' \cap P''$ contains no linear form and for every $i \in V$ one has either $P_i \subseteq P'$ or $P_i \subseteq P''$. Then $P \notin \operatorname{Ass}(R/I)$.*

Proof (1) By localizing at P, we may right away assume that $P = \sum_{i=1}^{w} P_i = (X_1, \dots, X_n)$ and $V = \{1, \dots, w\}$. Since the graph G_P is connected, we can take a spanning tree T, and for each edge $\{i,j\}$ in T, we may take a linear form $L_{ij} \in P_i \cap P_j$. The product $F = \prod_{(i,j) \in T} L_{ij}$ has degree $w-1$ and, by construction, $P_i F \subseteq I$ for all i. Hence $P \subseteq I : F$. Since $F \notin I$ by degree reasons, it follows that $P = I : F$.

(2) Again we may assume $P = \sum_{i=1}^{w} P_i = (X_1, \dots, X_n)$ and we may further assume that $P' = (X_1, \dots, X_m)$ and $P'' = (X_{m+1}, \dots, X_n)$. We may also assume that $P_i \subseteq P'$ for $i = 1, \dots, c$ and $P_i \subseteq P''$ for $i = c+1, \dots, w$ for some m and

c such that $1 \le m < n$ and $1 \le c < w$. Set $J = P_1 \cdots P_c$ and $H = P_{c+1} \cdots P_w$. Then $I = JH = J \cap H$ because J and H are ideals in distinct variables. We may conclude that any associated prime of I is either contained in P' or in P'', and hence P cannot be associated to I.

When each of the P_i is generated by indeterminates, then I is a transversal polymatroid, and for a given $P \in \mathscr{P}$, either (1) or (2) is satisfied. Hence we have, as a corollary, the following results of Herzog et al. [29]. We state it in a slightly different form.

Corollary 6.4 *Let $I = P_1 \cdots P_w$ with $P_i \in \mathscr{F}$ generated by variables (i.e., I is a transversal polymatroidal ideal). Then $P \in \mathscr{P}$ is associated to R/I if and only if the graph G_P is connected.*

But in general, for $I = P_1 \cdots P_w$, a prime $P \in \mathscr{P}$ can be associated to I even when G_P is not connected:

Example 6.5 Let $R = K[X_1, \ldots, X_4]$ and let P_1, P_2, P_3 be ideals generated by two general linear forms each and $I = P_1P_2P_3$. Then $P = P_1 + P_2 + P_3 = (X_1, \ldots, X_4)$ is associated to R/I and G_P is not connected (it has no edges). That P is associated to R/I can be proved by taking a nonzero quadric q in the intersection $P_1 \cap P_2 \cap P_3$ and checking that, by construction, $qP \subseteq I$.

The general question of whether a prime ideal $P \in \mathscr{P}$ is associated to R/I can be reduced by localization to the following:

Question 6.6 Let $P_1, \ldots, P_w \in \mathscr{F}$ and $I = P_1 \cdots P_w$. Under which (possibly combinatorial) conditions on $P_1, \ldots, P_w$ is $\sum_{i=1}^{w} P_i$ associated to R/I?

Another interesting (and very much related) question is the description of the relationship between the associated primes of I and those of its powers. We have:

Lemma 6.7 *Let I and J be ideals that are products of elements in $\mathscr{F}$. Then*

$$\operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/J) \subseteq \operatorname{Ass}(R/IJ).$$

In particular, $\operatorname{Ass}(R/I^h) \subseteq \operatorname{Ass}(R/I^{h+1})$ *for all $h > 0$.*

Proof Let $P_1, \ldots, P_w, Q_1, \ldots, Q_v \in \mathscr{F}$ such that $I = P_1 \cdots P_w$ and $J = Q_1 \cdots Q_v$. Let $P \in \operatorname{Ass}(R/I)$. We know that $P = \sum_{i \in A} P_i$ for a subset A of $\{1, \ldots, w\}$. Localizing at P we may restrict our attention to the factors $P_i \subseteq P$ and $Q_j \subseteq P$. Hence we assume that $P = \sum_{i=1}^{w} P_i = (X_1, \ldots, X_n)$. Let f be a homogeneous element such that $P = I : (f)$. Since I has a linear resolution, it coincides with its saturation from degree w on. Hence f has degree $w - 1$. Then $P = (IJ) : (fJ)$ because $fJ \not\subseteq IJ$ by degree reasons.

The main question here is the following:

Question 6.8 Let $P_1, \dots, P_w \in \mathscr{F}$ and $I = P_1 \cdots P_w$. Is it true that $\mathrm{Ass}(R/I) = \mathrm{Ass}(R/I^k)$ for every $k > 0$?

We conclude the section with the following:

Theorem 6.9 *Let $P_1, \dots, P_w \in \mathscr{F}$. The multi-Rees algebra $R(P_1, \dots, P_w)$ and its multi-fiber ring $F(P_1, \dots, P_w)$ are normal and Cohen–Macaulay. Furthermore they are defined by Gröbner bases of elements of degrees bounded above by $\sum_{i=0}^{w} e_i \in \mathbb{Z}^{w+1}$.*

Proof The multiplicative intersection property implies that a product of elements in $\mathscr{F}$ is integrally closed, and this entails the normality of $R(P_1, \dots, P_w)$. The multi-fiber ring is normal as well because it is an algebra retract of the Rees algebra. By construction, $R(P_1, \dots, P_w)$ can be identified with $F(P_0, P_1, \dots, P_w)$ where $P_0 = (X_1, \dots, X_n)$. Hence it is enough to prove the Cohen–Macaulay property of the multi-fiber ring $F = F(P_1, \dots, P_w)$. Note that F is a subring of the Segre product $R * S$ of R with $S = K[T_1, \dots, T_w]$. The defining ideal of $R * S$, i.e., the ideal of 2-minors of a generic $m \times w$ matrix, has a square-free generic initial ideal with respect to the $\mathbb{Z}^w$-graded structure, as proved in [13]. So it is a Cartwright–Sturmfels ideal, a notion defined in Conca et al. [17] that was inspired by result of Cartwright and Sturmfels [12] and Conca et al. [16]. In [17] it is proved that eliminating variables from a Cartwright–Sturmfels ideal one gets a Cartwright–Sturmfels ideal. So F itself is defined by a Cartwright–Sturmfels ideal. Such an ideal has a multiplicity-free multidegree. Hence we may use Brion's theorem [4] asserting that a multigraded prime ideal with multiplicity-free multidegree defines a Cohen–Macaulay ring. Finally the statement about the degrees of Gröbner basis elements is a general fact about Cartwright–Sturmfels ideals.

As we have seen already in Example 5.5, we cannot expect $R(P_1, \dots, P_w)$ and $F(P_1, \dots, P_w)$ to be defined by quadrics. But the strategy developed in [13] together with Theorem 6.9 implies:

Theorem 6.10 *Let $P_1, \dots, P_w \in \mathscr{F}$ and $I = P_1 \cdots P_w$. Then the fiber ring $F(I)$ is Koszul.*

Remark 6.11 The result on linear products of ideals generated by linear forms has been generalized by Derksen and Sidman [20]. Roughly speaking, they show that the regularity of an ideal that is constructed from ideals of linear forms by D successive basic operations like products, intersections, and sums is bounded by D.

7 Product of Borel-Fixed Ideals of Maximal Minors

Let K be a field. Let $X = (X_{ij})$ be the matrix of size $n \times n$ whose entries are the indeterminates of the polynomial ring $R = K[x_{ij} : 1 \le i, j \le n]$. Let t and a be positive integers with that $t + a \le n + 1$ and set

$$X_t(a) = (X_{ij} : 1 \le i \le t,\ a \le j \le n),$$

and define the *northeast ideal* $I_t(a)$ associated to the pair (t, a) to be the ideal generated by the t-minors, i.e., $t \times t$-subdeterminants, of the matrix $X_t(a)$. Note that, by construction, $X_t(a)$ has size $t \times (n + 1 - t)$ and $t \le (n + 1 - t)$. Hence $I_t(a)$ is the ideal of maximal minors of $X_t(a)$. There is a natural action of $\mathrm{GL}_n(K) \times \mathrm{GL}_n(K)$ on R. Let $B_n(K)$ denote the subgroup of lower triangular matrices in $\mathrm{GL}_n(K)$ and by $B'_n(K)$ the subgroup of upper triangular matrices. Note that the ideals $I_t(a)$ are fixed by the action of the Borel group $B_m(K) \times B'_n(K)$. Hence they are Borel-fixed ideals of maximal minors.

Let $<$ be the lexicographic term order on R associated to the total order

$$X_{11} > X_{12} > \cdots > X_{1n} > X_{21} > \cdots > X_{nn}.$$

Then the initial monomial of a $t \times t$ subdeterminant of X is the product of its diagonal elements. Therefore $<$ is a *diagonal* monomial order. The statements below remain true if one replaces $<$ with another diagonal monomial order.

Set

$$J_t(a) = (X_{1b_1} \cdots X_{tb_t} : a \le b_1 < \cdots < b_t \le n).$$

It is the ideal generated by the initial monomials of the $t \times t$ minors in $I_t(a)$.

The ideal of maximal minors of a matrix of variables, such as $X_t(a)$, is a prime ideal with primary powers. Furthermore

$$\mathrm{in}_<(I_t(a)) = J_t(a).$$

The main result of Bruns and Conca [7] is the following:

Theorem 7.1 *The families*

$$\mathscr{F} = \{I_t(a) :\ \ t > 0,\ \ a > 0,\ \ t + a \le n + 1\}$$

and

$$\mathscr{F}' = \{J_t(a) : \ \ t > 0, \ \ a > 0, \ \ t + a \le n + 1\}$$

have linear products.

Furthermore, given $(t_1, a_1), \ldots, (t_w, a_w)$*, set* $I_j = I_{t_j}(a_j)$ *and* $J_j = J_{t_j}(a_j)$*. Then*

$$\operatorname{in}_<(R(I_1, \ldots, I_w)) = R(J_1, \ldots, J_w),$$

and both multi-Rees algebras $R(I_1, \ldots, I_w)$ *and* $R(J_1, \ldots, J_w)$ *as well as their multi-fiber rings* $F(I_1, \ldots, I_w)$ *and* $F(J_1, \ldots, J_w)$ *are Cohen–Macaulay and normal and defined by Gröbner bases of quadrics.*

The proof of the theorem is based on the general strategy described in Sect. 4 and on the following decomposition formulas proved in [7]:

Theorem 7.2 *For every* $S = \{(t_1, a_1), \ldots, (t_w, a_w)\}$ *set* $I_S = \prod_{i=1}^{w} I_{t_i}(a_i)$ *and* $J_S = \prod_{i=1}^{w} J_{t_i}(a_i)$*. Then*

$$I_S = \bigcap_{u,b} I_u(b)^{e_{ub}(S)}$$

and

$$J_S = \bigcap_{u,b} J_u(b)^{e_{ub}(S)}$$

where

$$e_{ub}(S) = |\{i : b \le a_i \text{ and } u \le t_i\}|.$$

The theorem allows one to define a certain normal form for elements in I_S. The passage to the normal form uses quadratic rewriting rules that represent Gröbner basis elements of the multi-Rees algebra.

The exponent $e_{ub}(S)$ can be characterized as well by the equalities

$$e_{ub}(S) = \max\{j : I \subseteq I_u(b)^j\} = \max\{j : J \subseteq J_u(b)^j\}.$$

The theorem shows that the family $\mathscr{F}$ of NE-ideals of maximal minors has the multiplicative intersection property since the representation of I as an intersection is a primary decomposition: each ideal $I_u(b)$ is prime with primary powers. The representation of J is not a primary decomposition. To get a primary decomposition for J, one uses the fact that $J_u(b)$ is radical with decomposition

$$J_u(b) = \bigcap_{F \in F_{u,b}} P_F$$

where $F_{u,b}$ denotes the set of facets of the simplicial complex associated to $J_u(b)$, and P_F the prime ideal associated to F. Moreover,

$$J_u(b)^k = \bigcap_{F \in F_{u,b}} P_F^k$$

as proved in [5, Proposition 7.2]. Therefore, while the family $\mathscr{F}'$ does not have the multiplicative intersection property, its members are nevertheless P-adically closed.

The primary decomposition of I_S given in Theorem 7.2 can be refined as follows:

Proposition 7.3 *Given $S = \{(t_1, a_1), \ldots, (t_w, a_w)\}$, let Y be the set of the elements $(t, a) \in \mathbb{N}_+^2 \setminus S$ such that there exists $(u, b) \in \mathbb{N}_+^2$ for which $(t, b), (u, a) \in S$ and $t < u$, $a < b$. Then we have the following primary decomposition:*

$$I_S = \bigcap_{(v,c) \in S \cup Y} I_v(c)^{e_{vc}(S)}.$$

Furthermore this decomposition is irredundant, provided all the points (u, b) above can be taken so that $u + b \leq n + 1$.

Note that for a given S, the primary decomposition in Proposition 7.3 is irredundant if n is sufficiently large. Therefore we obtain:

Corollary 7.4 *Given $S = \{(t_1, a_1), \ldots, (t_w, a_w)\}$ assume that n is sufficiently large. Then*

$$\mathrm{Ass}(R/I_S) = \mathrm{Ass}(R/I_S^k)$$

for all $k > 0$.

In some cases the equality stated in Corollary 7.4 holds true also for small values of n.

Remark 7.5

(a) There exist well-known families of ideals that have the multiplicative intersection property, but lack linear products. In characteristic 0 this holds for the ideals $I_t(X)$, $t = 0, \ldots, m$, that are generated by all t-minors of the full matrix X if $m > 2$. See Bruns and Vetter [9, Theorem 10.9].

(b) In [10] the authors and Varbaro proved that the ideal of maximal minors of a matrix of linear forms has linear powers under certain conditions that are significantly weaker than "full" genericity. The techniques applied in [10] are quite different from those on which the results of this note rely.

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Minors and Categorical Resolutions

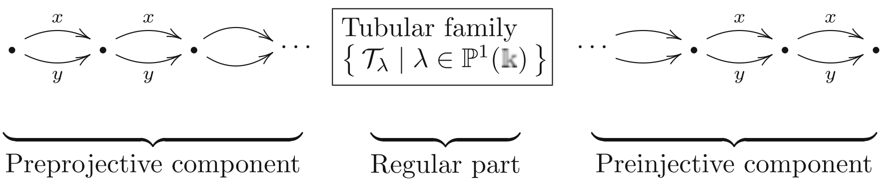
Igor Burban, Yuriy Drozd, and Volodymyr Gavran

Abstract We define minors of non-commutative schemes and study their properties. It is then applied to the study of a special class of non-commutative schemes, called quasi-hereditary, and to a construction of categorical resolutions for singular curves (maybe, non-commutative). In the rational case, this categorical resolution is realized by a finite dimensional quasi-hereditary algebra.

Keywords Bilocalization • Categorical resolution • Derived categories • Minors • Non-commutative schemes • Quasi-hereditary schemes

1 Introduction

When one compares the category of representations of the *Kronecker quiver* $\bullet \rightrightarrows \bullet$ ("matrix pencils") and the category of coherent sheaves over the projective line $\mathbb{P}^1$, one sees an astonishing resemblance. Indeed, the Auslander–Reiten quiver (describing the subcategory of indecomposable objects) of the first category looks like



I. Burban
Mathematical Institute of the University of Cologne, Weyertal 86-90, 50931 Köln, Germany
e-mail: burban@math.uni-koeln.de

Y. Drozd (✉) • V. Gavran
Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereschenkivska 3, 01601 Kiev, Ukraine
e-mail: y.a.drozd@gmail.com; vlgvrn@gmail.com

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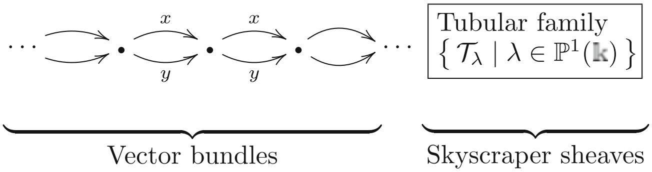
while that of the second is

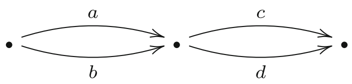


both with relations $xy = yx$. The *tubular family* here means a set of disjoint subcategories $\mathcal{T}_\lambda$ (*tubes*) parametrized by the points of the projective line and such that every $\mathcal{T}_\lambda$ is equivalent to the category of indecomposable finite dimensional modules over the algebra of formal power series $\Bbbk[[t]]$. Except the products of arrows, there are only morphisms "from the left to the right," also similar in both cases. Note that if we move the preinjective component of the first quiver to the very left and join it with the preprojective component by the arrows $\rightrightarrows$, we obtain the second quiver.

This resemblance has now a rather simple explanation. Namely, the vector bundle $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ is a so-called *tilting object* of the category $\operatorname{Coh} \mathbb{P}^1$. It means that $\operatorname{Ext}^i_{\mathbb{P}^1}(\mathcal{G}, \mathcal{G}) = 0$ for all $i > 0$ and, for every nonzero morphism $f : \mathcal{F} \to \mathcal{F}'$ of coherent sheaves, $\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{G}, f) \neq 0$ (equivalently, $\mathcal{G}$ generates the derived category $\mathscr{D}(\operatorname{Coh} \mathbb{P}^1)$). Then it is known that the derived functor $\operatorname{RHom}_{\mathbb{P}^1}(\mathcal{G}, \_)$ establishes an equivalence of the derived categories $\mathscr{D}(\operatorname{Coh} \mathbb{P}^1)$ and $\mathscr{D}(\boldsymbol{A}\text{-mod})$, where $\boldsymbol{A} = \operatorname{End}(\mathcal{G})^{\mathrm{op}}$. In our case, $\boldsymbol{A}$ is just the path algebra of the Kronecker quiver. Moreover, since both categories $\boldsymbol{A}$-mod and $\operatorname{Coh} \mathbb{P}^1$ are *hereditary* (i.e., of global dimension 1), every indecomposable object of the derived category is just a shift of a module.

Actually, Beilinson [3] proved that the category $\operatorname{Coh} \mathbb{P}^n$ has a tilting sheaf $\mathcal{G} = \bigoplus_{i=-n}^{0} \mathcal{O}_{\mathbb{P}^n}(i)$, hence is equivalent to the category of representations of the finite dimensional algebra $\boldsymbol{A} = \operatorname{End}(\mathcal{G})^{\mathrm{op}}$, which can be explicitly described. Afterwards analogues of these results were proved for a wide class of projective varieties. In particular, Hille and Perling [21] constructed a tilting vector bundle for any smooth rational surface.

Later on Greuel and the second author [18] noticed that there is a resemblance between the categories of vector bundles over a class of singular curves and the categories of representations of some finite dimensional algebras. In particular, it is so for a nodal cubic C and the algebra $\boldsymbol{A}$ with the quiver



and relations $da = cb = 0$. Certainly, this correspondence could not be a corollary of an equivalence of derived categories, since the algebra $\boldsymbol{A}$ is of global dimension 2, while $\mathcal{O}_C$ is of infinite global dimension. An explanation of this phenomenon was given by the first two authors [7]. For this purpose they considered a sheaf of non-commutative algebras $\mathcal{A}$ (called *Auslander sheaf*), which was already of global dimension 2, and constructed a tilting sheaf over $\mathcal{A}$ such that its endomorphism algebra was just the algebra mentioned in [18]. The category of coherent sheaves over the initial curve turned to be a *Serre quotient* of the category of coherent sheaves over $\mathcal{A}$ by a semi-simple subcategory; hence, their indecomposable objects were almost the same.

This paper is devoted to a generalization of the results of Burban and Drozd [7] to all singular curves. Namely, we construct for every curve X a sheaf of $\mathcal{O}_X$-algebras $\mathcal{R}$, such that $\mathcal{R}$ is of finite global dimension and there is a functor $\mathsf{F} : \operatorname{Coh}\mathcal{R} \to \operatorname{Coh}X$, which defines $\operatorname{Coh}X$ as a *bilocalization* (i.e., both localization and colocalization) of $\operatorname{Coh}\mathcal{R}$. The same is certainly true for their derived categories. Moreover, $\mathcal{R}$ has rather special properties analogous to those of *quasi-hereditary* algebras from [12, 14]. We call $\mathcal{R}$ the *König's resolution* of the curve X, since the idea of its construction goes back to the König's paper [23]. If X is rational, $\mathcal{R}$ has a tilting complex $\mathcal{T}$ which establishes an equivalence between the derived category of $\operatorname{Coh}\mathcal{R}$ and that of a finite dimensional quasi-hereditary algebra. Altogether, this construction can be considered as a *categorical resolution* of the category $\mathscr{D}(\operatorname{Coh}X)$ in the sense of [24]. If the curve X is Gorenstein, this categorical resolution is *weakly crepant* in the sense of [24]. We also show that this construction can also be applied to *non-commutative curves*.

The main tool in our considerations is the notion of *minors* of non-commutative schemes studied in Sect. 3. For the affine case (i.e., for rings), it was introduced in [15]. A minor $\mathcal{B}$ of a sheaf of algebras $\mathcal{A}$ is the endomorphism sheaf of a locally projective sheaf of $\mathcal{A}$-modules. Then the category $\operatorname{Qcoh}\mathcal{B}$ is a bilocalization of $\operatorname{Qcoh}\mathcal{A}$ and the same is true for their derived categories. We establish the main features of these bilocalizations and specialize them to the most important case arising as *endomorphism construction* (Example 3.14). The general properties of localizations and colocalizations used here are gathered in Sect. 2. In Sect. 4 we apply this technique to a special class of non-commutative schemes called *quasi-hereditary*. This notion generalizes that of quasi-hereditary algebras and has a lot of similar features. In particular, a quasi-hereditary non-commutative scheme is always of finite global dimension, and its derived category has good semi-orthogonal decompositions (see Corollary 4.23). In Sect. 5 we study some general properties of non-commutative curves and their minors, especially related with Cohen–Macaulay (or, the same, torsion-free) modules. In Sect. 6 we construct the König's resolution and prove that it is quasi-hereditary. We also show that in the commutative case, the functors of direct and inverse image arising from the normalization of the curve are actually compositions of the functors arising from the König's resolution. Finally, in Sect. 7 we construct a tilting complex for the König's resolution of a rational singular curve (maybe, non-commutative). It gives a categorical resolution of $\mathscr{D}(\operatorname{Qcoh}X)$ by a quasi-hereditary finite dimensional algebra. We also consider, as an example, the case when all singularities of a curve are of ADE types in the sense of Arnold [2].

Most results of Sects. 1–5 are contained in [8]. Sections 6 and 7 generalize the results of [9] to the non-commutative situation.

2 Bilocalizations

We recall here some general facts concerning localizations and bilocalizations of abelian and triangular categories. Their proofs are gathered in [8, Sect. 2].

Theorem 2.1 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be an exact functor between abelian categories,* $\mathsf{F}^* : \mathcal{B} \to \mathcal{A}$ *be its right (left) adjoint such that the natural morphism* $\mathsf{F}\mathsf{F}^* \to \mathbb{1}_{\mathscr{B}}$ *(respectively,* $\mathbb{1}_{\mathscr{B}} \to \mathsf{F}\mathsf{F}^*$*) is an isomorphism. Let* $\mathscr{C} = \ker \mathsf{F}$.

1. $\mathscr{C}$ *is a thick subcategory in* $\mathscr{A}$ *and* $\mathsf{F} = \bar{\mathsf{F}}\Pi_{\mathscr{C}}$*, where* $\Pi_{\mathscr{C}} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ *is the natural functor to the Serre quotient and* $\bar{\mathsf{F}} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ *is an equivalence. The quasi-inverse functor to* $\bar{\mathsf{F}}$ *is* $\Pi_{\mathscr{C}}\mathsf{F}^*$.
2. F^* *is a full embedding and its essential image* $\operatorname{Im}\mathsf{F}^*$ *coincides with the right (respectively, left) orthogonal subcategory to* $\mathscr{C}$*, i.e., the full subcategory*

$$\mathscr{C}^{\perp} = \left\{A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}_{\mathscr{A}}(C,A) = \operatorname{Ext}^1_{\mathscr{A}}(C,A) = 0 \text{ for all } C \in \mathscr{C}\right\}$$

(respectively,

$$^{\perp}\mathscr{C} = \left\{A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}_{\mathscr{A}}(A,C) = \operatorname{Ext}^1_{\mathscr{A}}(A,C) = 0 \text{ for all } C \in \mathscr{C}\right\}).$$

3. $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$ *(respectively,* $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$*).*
4. *The embedding functor* $\mathsf{I} : \mathscr{C} \to \mathscr{A}$ *has a right (respectively, left) adjoint.*

In this case they say that F is a *localizing functor*, $\mathscr{C}$ is a *localizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *localization* of the category $\mathscr{A}$ (respectively, F is a *colocalizing functor*, $\mathscr{C}$ is a *colocalizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *colocalization* of the category $\mathscr{A}$).

Theorem 2.2 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be an exact functor between triangulated categories,* $\mathsf{F}^* : \mathcal{B} \to \mathcal{A}$ *be its right (left) adjoint such that the natural morphism* $\mathsf{F}\mathsf{F}^* \to \mathbb{1}_{\mathscr{B}}$ *(respectively,* $\mathbb{1}_{\mathscr{B}} \to \mathsf{F}\mathsf{F}^*$*) is an isomorphism. Let* $\mathscr{C} = \ker \mathsf{F}$.

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3. $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$ *(respectively,* $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$*).*
4. *The embedding functor* $\mathsf{I} : \mathscr{C} \to \mathscr{A}$ *has a right (respectively, left) adjoint, which induces an equivalence* $\mathscr{A} / \operatorname{Im} \mathsf{F}^* \xrightarrow{\sim} \mathscr{C}$*.*[1]

In this case they say that F is a *localizing functor*, $\mathscr{C}$ is a *localizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *localization* of the category $\mathscr{A}$ (respectively, F is a *colocalizing functor*, $\mathscr{C}$ is a *colocalizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *colocalization* of the category $\mathscr{A}$).

Theorem 2.3 *Suppose that an exact functor* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *between abelian (or triangulated) categories has both left adjoint* F^* *and right adjoint* $\mathsf{F}^!$*. Then* F *is a localizing functor if and only if it is a colocalizing functor.*

In this case we say that F is a *bilocalizing functor*, its kernel $\mathscr{C} = \ker \mathsf{F}$ is a *bilocalizing subcategory*, and $\mathscr{B}$ is a *bilocalization* of the category $\mathscr{A}$.

If $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is an exact functor between abelian categories, we denote by DF the functor between the derived categories $\mathscr{D}\mathscr{A} \to \mathscr{D}\mathscr{B}$ which acts on complexes componentwise. It is both right and left derived functor of F.

Theorem 2.4 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be a localizing (colocalizing) functor between abelian categories,* $\mathscr{C} = \ker \mathsf{F}$*. Suppose that right (left) adjoint* F^* *of* F *has right (respectively, left) derived functor. Then* DF *is also a localizing (respectively, colocalizing) functor,* RF^* *is its right adjoint (respectively,* LF^* *is its left adjoint),*

$$\ker \mathsf{DF} = \mathscr{D}_{\mathscr{C}}\mathscr{A} = \{\, \mathcal{F}^\bullet \in \mathscr{D}\mathscr{A} \mid H^n(\mathcal{F}^\bullet) \in \mathscr{C} \text{ for all } n \,\}$$

and $\mathscr{D}\mathscr{B} \simeq \mathscr{D}\mathscr{A} / \mathscr{D}_{\mathscr{C}}\mathscr{A}$*.*

Remark 2.5 If $\mathscr{A}$ is a Grothendieck category, a right derived functor always exists, so Theorem 2.4 can always be applied. We do not know any natural "categorical" conditions for the existence of a left adjoint, though it is the case in the situation that we consider nearby.

We recall that a *semi-orthogonal* decomposition $\langle\, \mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_m \,\rangle$ of a triangulated category $\mathscr{A}$ is a sequence of subcategories $(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_m)$ such that

1. $\operatorname{Hom}_{\mathscr{A}}(A, B[m]) = 0$ if $A \in \mathscr{T}_i$, $B \in \mathscr{T}_j$ and $i > j$.
2. For every $A \in \mathscr{A}$, there is a sequence of morphisms

$$0 = T_m \xrightarrow{f_m} T_{m-1} \xrightarrow{f_{m-1}} \dots T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 = A$$

such that $\operatorname{cone}(f_i) \in \mathscr{T}_i$ $(1 \le i \le m)$ [25].

[1] In the case of abelian categories the functor $\mathscr{A} / \operatorname{Im} \mathsf{F}^* \to \mathscr{C}$ induced by the right (respectively, left) adjoint of I need not be an equivalence.

In particular, if $m = 2$, it means that there is an exact triangle $T_2 \to A \to T_1$, where $T_1 \in \mathscr{T}_1$, $T_2 \in \mathscr{T}_2$.

Corollary 2.6 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be a localizing (colocalizing) functor between triangulated categories,* F^* *be its right (respectively, left) adjoint. There is a semi-orthogonal decomposition* $(\operatorname{Im}\mathsf{F}^*, \ker\mathsf{F})$ *(respectively,* $(\ker\mathsf{F}, \operatorname{Im}\mathsf{F}^*)$*) of the category* $\mathscr{A}$*.*

3 Minors

In this paper a *non-commutative scheme* is a pair $(X, \mathcal{A})$, where X is a scheme (called the *commutative background* of the non-commutative scheme) and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, which is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. Sometimes we say "non-commutative scheme $\mathcal{A}$" not mentioning its commutative background X. We denote by X_{cl} the set of closed points of X. If $\mathcal{A} = \mathcal{O}_X$, we sometimes say that it is a *usual scheme*. We denote by $\mathcal{A}$-Mod (respectively, by $\mathcal{A}$-mod) the category of quasi-coherent (respectively, coherent) sheaves of $\mathcal{A}$-modules. We call objects of this category just $\mathcal{A}$-modules (respectively, coherent $\mathcal{A}$-modules).

A non-commutative scheme $(X, \mathcal{A})$ is said to be *affine* (*separated, quasi-compact*) if so is its commutative background X. It is said to be *reduced* if $\mathcal{A}$ has no nilpotent ideals. If X is noetherian and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module, we say that this non-commutative scheme is *noetherian*. We say that $(X, \mathcal{A})$ is *quasi-projective* if there is an ample $\mathcal{O}_X$-module $\mathcal{L}$. Note that then X is indeed a quasi-projective scheme over the ring $\boldsymbol{R} = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes n})$. In this paper we always suppose that the considered non-commutative schemes are **separated** and **quasi-compact**. In this case $\mathcal{A}$-Mod is a Grothendieck category. In particular, every quasi-coherent $\mathcal{A}$-module has an injective envelope. We denote by $\mathcal{A}$-Inj the full subcategory of $\mathcal{A}$-Mod formed by injective modules.

A *morphism* of non-commutative schemes $f : (Y, \mathcal{B}) \to (X, \mathcal{A})$ is a pair $(f_X, f^\#)$, where $f_X : Y \to X$ is a morphism of schemes and $f^\#$ is a morphism of $f_X^{-1}\mathcal{O}_X$-algebras $f_X^{-1}\mathcal{A} \to \mathcal{B}$. In what follows we usually write f instead of f_X. Such morphism defines the functor of inverse image $f^* : \mathcal{A}\text{-Mod} \to \mathcal{B}\text{-Mod}$ which maps an $\mathcal{A}$-module $\mathcal{M}$ to the $\mathcal{B}$-module $\mathcal{B} \otimes_{f^{-1}\mathcal{A}} f^{-1}\mathcal{M}$. As the map f_X is separated and quasi-compact, the functor of direct image $f_* : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ is also well-defined (cf. [20, § 0.1 and § 1.9.2]). Moreover, f^* maps coherent modules to coherent ones. Note that f^* and f_* do not coincide with $(f_X)^*$ and $(f_X)_*$. It is guaranteed only if $\mathcal{B} = f_X^*\mathcal{A}$, for instance, if Y is an open subset of X and $\mathcal{B} = \mathcal{A}|_Y$.

We call a non-commutative scheme $(X, \mathcal{A})$ *central* if $\operatorname{center}(\mathcal{A}) = \mathcal{O}_X$. Actually, we can only consider central non-commutative schemes as the following evident results show.

Proposition 3.7 *Let* $\mathcal{C} = \operatorname{center}(\mathcal{A})$*,* $\tilde{X} = \operatorname{spec}\mathcal{C}$*,* $\nu_X : \tilde{X} \to X$ *be the corresponding affine morphism, and* $\tilde{\mathcal{A}} = \nu_X^{-1}(\mathcal{A})$*. Then* ν_X *extends to a morphism* $\nu : (\tilde{X}, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$ *and* ν_* *induces equivalences* $\tilde{\mathcal{A}}\text{-Mod} \to \mathcal{A}\text{-Mod}$ *and* $\tilde{\mathcal{A}}\text{-mod} \to \mathcal{A}\text{-mod}$*.*

We denote by $\mathsf{lp}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of *locally projective* modules, i.e., such coherent modules $\mathcal{P}$ that all localizations $\mathcal{P}_x$ are projective $\mathcal{A}_x$-modules. We say that $\mathcal{A}$ *has enough locally projective modules* if for every coherent $\mathcal{A}$-module $\mathcal{M}$, there is an epimorphism $\mathcal{P} \to \mathcal{M}$, where $\mathcal{P}$ is locally projective. It is the case, for instance, if the non-commutative scheme is quasi-projective.

We denote by $\mathscr{D}\mathcal{A}$ the derived category $\mathscr{D}(\mathcal{A}\text{-Mod})$, with subscripts $^{+},^{-},^{b}$ denoting its full subcategories consisting, respectively, of left-, right-, and two-sided bounded complexes. We also denote by $\mathsf{Perf}\mathcal{A}$ the full subcategory of *small* objects from $\mathscr{D}\mathcal{A}$, i.e., such complexes $\mathcal{F}^\bullet$ that $\mathrm{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{F}^\bullet, \bigsqcup_i \mathcal{G}_i^\bullet) \simeq \bigsqcup_i \mathrm{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}_i^\bullet)$ for any coproduct $\bigsqcup_i \mathcal{G}_i^\bullet$. As X is separated and quasi-compact, small objects in $\mathscr{D}\mathcal{A}$ are just *perfect complexes*, i.e., complexes $\mathcal{F}^\bullet$ such that for every $x \in X$ the complex $\mathcal{F}_x$ is isomorphic to a finite complex of locally projective coherent modules. Moreover, $\mathsf{Perf}\mathcal{A}$ generates $\mathscr{D}\mathcal{A}$, i.e., for every complex $\mathcal{G}^\bullet$, there is a nonzero morphism from a perfect complex to $\mathcal{G}^\bullet$. It is well-known in affine and commutative cases and the proof in general case is quite analogous [8, Theorem 3.14].

Definition 3.8 Let $\mathcal{P}$ be a locally projective $\mathcal{A}$-module, $\mathcal{B} = (\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P})^{\mathrm{op}}$. We call the non-commutative scheme $(X, \mathcal{B})$ a *minor* of the non-commutative scheme $(X, \mathcal{A})$.

This notion is just a globalization of the corresponding notion from [15].

We consider $\mathcal{P}$ as *right* $\mathcal{B}$-module and denote $\mathcal{P}^\vee = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{A})$; it is a right $\mathcal{A}$-module. It is known that for every $\mathcal{P} \in \mathsf{lp}\,\mathcal{A}$, the natural map $\mathcal{P} \to \mathcal{P}^{\vee\vee}$ is an isomorphism and $\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P}^\vee \simeq \mathcal{E}nd_{\mathcal{A}}\,\mathcal{P} \simeq \mathcal{P}^\vee \otimes_{\mathcal{A}} \mathcal{P}$. The following functors play the crucial role in this paper:

$$
\begin{aligned}
\mathsf{F} &= \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \_) \simeq \mathcal{P}^\vee \otimes_{\mathcal{A}} \_ : \mathcal{A}\text{-Mod} \to \mathcal{B}\text{-Mod},\\
\mathsf{F}^* &= \mathcal{P} \otimes_{\mathcal{B}} \_ : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}, \qquad (1)\\
\mathsf{F}^! &= \mathcal{H}om_{\mathcal{B}}(\mathcal{P}^\vee, \_) : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}.
\end{aligned}
$$

The functor F is exact and both $(\mathsf{F}^*, \mathsf{F})$ and $(\mathsf{F}, \mathsf{F}^!)$ are adjoint pairs of functors. If the non-commutative scheme $(X, \mathcal{A})$ is noetherian, so is also $(X, \mathcal{B})$ and the functors $\mathsf{F}, \mathsf{F}^*, \mathsf{F}^!$ map coherent sheaves to coherent. Note that if $\mathcal{I}$ is an injective $\mathcal{B}$-module, then $\mathsf{F}^!(\mathcal{I})$ is an injective $\mathcal{A}$-module. We denote by $\mathcal{P}$-Inj the image $\mathsf{F}^!(\mathcal{B}\text{-Inj})$. We also denote by $\mu_{\mathcal{P}}$ the natural map $\mathcal{P} \otimes_{\mathcal{B}} \mathcal{P}^\vee \to \mathcal{A}$ such that $\mu(p \otimes f) = f(p)$ and $\mathcal{I}_P = \mathrm{Im}\,\mu_{\mathcal{P}}$. If $\mathcal{P} = \mathcal{A}e$, where e is an idempotent, then $\mathcal{P}^\vee \simeq e\mathcal{A}$, $(\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P})^{\mathrm{op}} \simeq e\mathcal{A}e$ and $\mathcal{I}_P = \mathcal{A}e\mathcal{A}$.

The following result plays a crucial role in this paper:

Theorem 3.9

1. F *is a bilocalizing functor and its kernel* $\mathscr{C} = \ker \mathsf{F}$ *consists of the modules* $\mathcal{M}$ *such* $\mathcal{I}_P \mathcal{M} = 0$, *so can be identified with* $\mathcal{A}/\mathcal{I}_{\mathcal{P}}$-Mod.

2. $\operatorname{Im} \mathsf{F}^* = {}^{\perp}\mathscr{C}$ *consists of all* $\mathcal{A}$*-modules* $\mathcal{M}$ *such that for every* $x \in X$ *there is an exact sequence* $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0$*, where* $\mathcal{P}_0$ *and* $\mathcal{P}_1$ *are multiples (maybe infinite) of* $\mathcal{P}_x$*. We denote this subcategory by* $\mathcal{P}$-Mod.
3. $\operatorname{Im} \mathsf{F}^! = \mathscr{C}^{\perp}$ *consists of all* $\mathcal{A}$*-modules* $\mathcal{M}$ *such that there is an exact sequence* $0 \to \mathcal{M} \to \mathcal{I}_0 \to \mathcal{I}_1$*, where* $\mathcal{I}_0$ *and* $\mathcal{I}_1$ *belong to* $\mathcal{P}$-Inj. *We denote this subcategory by* $\mathcal{P}^{\mathrm{Inj}}$-Mod.

Proof The results of the preceding section show that it is enough to prove the following statements:

Proposition 3.10

1. *The natural morphism* $\phi : \mathbb{1}_{\mathcal{B}\text{-Mod}} \to \mathsf{F}\mathsf{F}^*$ *is an isomorphism.*
2. $\operatorname{Im} \mathsf{F}^* = \mathcal{P}$-Mod.
3. $\operatorname{Im} \mathsf{F}^! = \mathcal{P}^{\mathrm{Inj}}$-Mod.
4. $\ker \mathsf{F} = \{\, \mathcal{M} \mid \mathcal{I}_P \mathcal{M} = 0 \,\}$.

Proof Evidently, all claims are local, so we can suppose that $X = \operatorname{spec} \boldsymbol{R}$ for a commutative ring $\boldsymbol{R}$; $\mathcal{A} = \boldsymbol{A}^{\sim}$ is the sheafification of an $\boldsymbol{R}$-algebra $\boldsymbol{A}$, $\mathcal{P} = P^{\sim}$, where P is a finitely generated projective $\boldsymbol{A}$-module; and $\mathcal{B} = \boldsymbol{B}^{\sim}$, where $\boldsymbol{B} = \operatorname{End}_{\boldsymbol{A}} P$. Then we can replace $\mathcal{A}$-Mod, $\mathcal{B}$-Mod, and $\mathcal{P}$-Mod, respectively, by $\boldsymbol{A}$-Mod, $\boldsymbol{B}$-Mod, and P-Mod, where P-Mod is the full subcategory of $\boldsymbol{A}$-Mod consisting of all modules M such that there is an exact sequence $P_1 \to P_0 \to M \to 0$, where P_0 and P_1 are multiples of P.

Certainly, $\phi(\boldsymbol{B})$ is an isomorphism. Hence $\phi(F)$ is an isomorphism for every free $\boldsymbol{B}$-module F. For every $\boldsymbol{B}$-module M, there is an exact sequence $F_1 \to F_1 \to M \to 0$ with free modules F_0, F_1. It induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle \phi(F_1)}\downarrow & & {\scriptstyle \phi(F_0)}\downarrow & & {\scriptstyle \phi(M)}\downarrow & & \\ \mathsf{F}\mathsf{F}^*(F_1) & \longrightarrow & \mathsf{F}\mathsf{F}^*(F_0) & \longrightarrow & \mathsf{F}\mathsf{F}^*(M) & \longrightarrow & 0 \end{array}$$

As $\phi(F_1)$ and $\phi(F_2)$ are isomorphisms, so is $\phi(M)$. It proves (*1*).

Moreover, we have an exact sequence $\mathsf{F}^*(F_1) \to \mathsf{F}^*(F_0) \to \mathsf{F}^*(M) \to 0$, where $\mathsf{F}^*(F_i)$ are multiples of $\mathsf{F}^*(\boldsymbol{B}) = P$, so $\mathsf{F}^*(M) \in P$-Mod. On the contrary, let we have an exact sequence $P_1 \to P_0 \to N \to 0$, where P_i are multiples of P. Consider the natural morphism $\psi : \mathsf{F}^*\mathsf{F} \to \mathbb{1}_{\mathcal{A}\text{-Mod}}$. Obviously, $\psi(P)$ is an isomorphism, so $\psi(P_i)$ are also isomorphisms. Again we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathsf{F}^*\mathsf{F}(P_1) & \longrightarrow & \mathsf{F}^*\mathsf{F}(P_0) & \longrightarrow & \mathsf{F}^*\mathsf{F}(N) & \longrightarrow & 0 \\ {\scriptstyle \psi(P_1)}\downarrow & & {\scriptstyle \psi(P_0)}\downarrow & & {\scriptstyle \psi(N)}\downarrow & & \\ P_1 & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

It implies that $\psi(N)$ is an isomorphism, hence $N \in \operatorname{Im} \mathsf{F}^*$. It proves (*2*). The proof of (*3*) is quite analogous.

To prove (*4*), note that $I_P P = P$ [10, Proposition VII.3.1], where $I_P = \operatorname{Im} \mu_P$. Let $M \notin \ker \mathsf{F}$ and $f : P \to M$ be a nonzero homomorphism. Then $I_P M \supseteq \operatorname{Im} f \neq 0$. On the contrary, if $I_P M \neq 0$, there is an element $z \in M$, elements $p_i \in P$, and homomorphisms $f_i : P \to \boldsymbol{A}$ such that $\sum_i f_i(p_i)z \neq 0$. Denote by g the homomorphism $\boldsymbol{A} \to M$ mapping 1 to z and set $g_i = gf_i$. Then at least one of g_i is nonzero, so $M \notin \ker \mathsf{F}$. □

As the functor F is exact, it induces a functor $\mathsf{DF} : \mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$ acting on complexes componentwise. It is both left and right derived of F. There are also left derived functor LF^* and right derived functor $\mathsf{RF}^!$, both $\mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ [29, Sect. 6]. Moreover, it follows from [29] that both $(\mathsf{LF}^*, \mathsf{DF})$ and $(\mathsf{DF}, \mathsf{RF}^!)$ are adjoint pairs (see [8] for details). Obviously, DF maps $\mathscr{D}^\sigma\mathcal{A}$ to $\mathscr{D}^\sigma\mathcal{B}$, where $\sigma \in \{+, -, b\}$; LF^* maps $\mathscr{D}^-\mathcal{B}$ to $\mathscr{D}^-\mathcal{A}$ and $\mathsf{RF}^!$ maps $\mathscr{D}^+\mathcal{B}$ to $\mathscr{D}^+\mathcal{A}$.

Theorem 3.11

1. DF *is a bilocalizing functor and* $\ker \mathsf{DF} \simeq \mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}}\mathcal{A}$*, where* $\mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}}\mathcal{A}$ *is the full subcategory of* $\mathscr{D}\mathcal{A}$ *consisting of complexes with cohomologies annihilated by* $\mathcal{I}_\mathcal{P}$ *(i.e., belonging to* $(\mathcal{A}/\mathcal{I}_\mathcal{P})$-Mod*).*
2. LF^* *maps* $\mathsf{Perf}\mathcal{B}$ *to* $\mathsf{Perf}\mathcal{A}$*.*
3. $\operatorname{Im} \mathsf{LF}^* = {}^\perp\mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}}\mathcal{A}$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}_\to$ *of* $\mathscr{D}\mathcal{A}$ *consisting of complexes quasi-isomorphic to K-flat complexes* $\mathcal{F}^\bullet$ *such that for every component* $\mathcal{F}^i$ *and every point* $x \in X$*, the localization* $\mathcal{F}^i_x$ *is a direct limit of modules from* $\operatorname{add}\mathcal{P}_x$*. The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^-$*.*
4. $\operatorname{Im} \mathsf{RF}^! = \mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}}\mathcal{A}^\perp$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}^{\mathrm{Inj}}$ *of* $\mathscr{D}\mathcal{A}$ *consisting of complexes quasi-isomorphic to K-injective complexes* $\mathcal{I}^\bullet$ *such that every component* $\mathcal{I}^i$ *belongs to* $\mathsf{F}^!(\mathcal{B}\text{-Inj})$*. The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^+$*.*

Note that the condition (*4*) can be verified locally at every point $x \in X$.

We recall that a complex $\mathcal{F}^\bullet$ is said to be *K-flat* (*K*-injective) if for every acyclic complex of right (respectively, left) $\mathcal{A}$-modules $\mathcal{C}^\bullet$, the complex $\mathcal{F}^\bullet \otimes_\mathcal{A} \mathcal{C}^\bullet$ (respectively, $\mathcal{H}om_\mathcal{A}(\mathcal{C}^\bullet, \mathcal{F}^\bullet)$) is also acyclic [29].

Proof (*1*) follows from the results of the previous section.

As $\mathcal{P}$ is coherent and locally projective, the functor DF preserves arbitrary coproducts. Therefore its left adjoint LF^* maps small objects to small ones, which gives (*2*).

(*3*) It follows from [1] that for every complex $\mathcal{M}^\bullet$ of $\mathcal{B}$-modules, there is a quasi-isomorphic *K*-flat complex $\mathcal{F}^\bullet$ with flat components. Then $\mathsf{LF}^*(\mathcal{M}^\bullet) = \mathsf{F}^*(\mathcal{F}^\bullet)$. By Bourbaki [5, Chap. X, § 1, Theorem 1], every localization $\mathcal{F}^i_x$ is a direct limit $\varinjlim_n \mathcal{L}^i_n$, where all $\mathcal{L}^i_n$ are finitely generated and projective, hence belong to $\operatorname{add}\mathcal{B}_x$. As F^* preserves direct limits and $\mathsf{F}^*(\mathcal{B}) \simeq \mathcal{P}$, $\mathsf{F}^*(\mathcal{F}_i) \simeq \varinjlim_n \mathsf{F}^*(\mathcal{L}^i_n)$ and $\mathsf{F}^*(\mathcal{L}^i_n)$ belongs to $\operatorname{add}\mathcal{P}_x$. Therefore, $\mathsf{F}^*(\mathcal{M}^\bullet) \in \mathscr{D}\mathcal{P}_\to$.

On the contrary, let $\mathcal{N}^\bullet \in \mathscr{D}\mathcal{P}_{\rightarrow}$. We can suppose that this complex is K-flat and every localization $\mathcal{N}_x^i$ is a direct limit $\varinjlim_n \mathcal{P}_n^i$, where $\mathcal{P}_n^i \in \operatorname{add} \mathcal{P}_x$. Then the complex $\mathsf{F}(\mathcal{N}^\bullet)$ is also K-flat, so $(\mathsf{LF}^*)(\mathsf{F}(\mathcal{N}^\bullet)) \simeq \mathsf{F}^*\mathsf{F}(\mathcal{N}^\bullet)$. As the natural map $\mathsf{F}^*\mathsf{F}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism, the same is true for $\mathsf{F}^*\mathsf{F}(\mathcal{P}_n^i) \to \mathcal{P}_n^i$, hence also for $\mathsf{F}^*\mathsf{F}(\mathcal{N}_x^i) \to \mathcal{N}_x^i$. Therefore, the map $(\mathsf{LF}^*)(\mathsf{DF})(\mathcal{N}^\bullet) \to \mathcal{N}^\bullet$ is an isomorphism and $\mathcal{N} \in \operatorname{Im} \mathsf{LF}^*$.

The proof of (*4*) is quite analogous. □

Corollary 3.12 *There are semi-orthogonal decompositions* $(\ker \mathsf{DF}, \operatorname{Im} \mathsf{LF}^*)$ *and* $(\operatorname{Im} \mathsf{RF}^!, \ker \mathsf{DF})$ *of the category* $\mathscr{D}\mathcal{A}$.

Note that the subcategories $\operatorname{Im} \mathsf{LF}^*$ and $\operatorname{Im} \mathsf{RF}^!$ are equivalent (both are equivalent to $\mathscr{D}\mathcal{B}$) but usually do not coincide.

The following special case is rather important:

Theorem 3.13 *Suppose that the ideal* $\mathcal{I} = \mathcal{I}_{\mathcal{P}}$ *is flat as right* $\mathcal{A}$*-module. Set* $\mathcal{Q} = \mathcal{A}/\mathcal{I}$*. Then* $\ker \mathsf{DF} = \mathscr{D}_{\mathcal{Q}}\mathcal{A} \simeq \mathscr{D}\mathcal{Q}$.

Proof Let $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a quasi-isomorphism. As $\mathcal{I}$ is flat, then $\mathcal{I}\otimes_{\mathcal{A}}\mathcal{F}^\bullet \to \mathcal{I}\otimes_{\mathcal{A}}\mathcal{G}^\bullet$ is also a quasi-isomorphism. Therefore, $\mathcal{F}^\bullet\otimes_{\mathcal{A}}\mathcal{Q} \to \mathcal{G}^\bullet\otimes_{\mathcal{A}}\mathcal{Q}$ is also a quasi-isomorphism. In particular, if $\mathcal{G}^\bullet$ consists of $\mathcal{Q}$-modules, we get a quasi-isomorphism $\mathcal{F}^\bullet\otimes_{\mathcal{A}}\mathcal{Q} \to \mathcal{G}^\bullet$. It implies that we can identify $\mathscr{D}\mathcal{Q}$ with the full triangulated subcategory of $\mathscr{D}\mathcal{A}$. Obviously $\mathscr{D}\mathcal{Q} \subseteq \mathscr{D}_{\mathcal{Q}}\mathcal{A}$. Moreover, $\mathcal{I}^2 = \mathcal{I}$. Let $\mathcal{F}^\bullet \in \mathscr{D}_{\mathcal{Q}}\mathcal{A}$. We can suppose that $\mathcal{F}^\bullet$ is K-flat. Its tensor product with the exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{Q} \to 0$ gives an exact sequence of complexes $0 \to \mathcal{I}\otimes_{\mathcal{A}}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{Q}\otimes_{\mathcal{A}}\mathcal{F}^\bullet \to 0$. As $\mathcal{I}$ is flat, $H^*(\mathcal{I}\otimes_{\mathcal{A}}\mathcal{F}^\bullet) \simeq \mathcal{I}\otimes_{\mathcal{A}}H^*(\mathcal{F}^\bullet)$. Since $\mathcal{I}\otimes_{\mathcal{A}}\mathcal{Q} \simeq \mathcal{I}/\mathcal{I}^2 = 0$, also $\mathcal{I}\otimes_{\mathcal{A}}\mathcal{M} = 0$ for every $\mathcal{Q}$-module $\mathcal{M}$. Therefore $H^*(\mathcal{I}\otimes_{\mathcal{A}}\mathcal{F}^\bullet) = 0$, i.e., $\mathcal{I}\otimes_{\mathcal{A}}\mathcal{F}^\bullet$ is acyclic, whence $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{Q}\otimes_{\mathcal{A}}\mathcal{F}$, which belongs to $\mathscr{D}\mathcal{Q}$. □

Example 3.14 (Endomorphism Construction) Let $\mathcal{F}$ be a coherent $\mathcal{A}$-module and $\mathcal{A}_{\mathcal{F}} = (\mathcal{E}nd_{\mathcal{A}}(\mathcal{A}\oplus\mathcal{F}))^{\mathrm{op}}$. We identify $\mathcal{A}_{\mathcal{F}}$ with the algebra of matrices

$$\mathcal{A}_{\mathcal{F}} = \begin{pmatrix} \mathcal{A} & \mathcal{F} \\ \mathcal{F}' & \mathcal{E} \end{pmatrix},$$

where $\mathcal{F}' = \operatorname{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$ and $\mathcal{E} = (\operatorname{End}_{\mathcal{A}} \mathcal{F})^{\mathrm{op}}$. Then $\mathcal{P} = \mathcal{P}_{\mathcal{F}} = \left(\begin{smallmatrix} \mathcal{A} \\ \mathcal{F}' \end{smallmatrix}\right)$ is a locally projective $\mathcal{A}_{\mathcal{F}}$-module and $\mathcal{A} \simeq (\operatorname{End}_{\mathcal{A}_{\mathcal{F}}} \mathcal{P})^{\mathrm{op}}$. Hence $\mathcal{A}$ is a minor of $\mathcal{A}_{\mathcal{F}}$, thus $\mathcal{A}$-Mod and $\mathscr{D}\mathcal{A}$ are bilocalizations, respectively, of $\mathcal{A}_{\mathcal{F}}$-Mod and $\mathscr{D}\mathcal{A}_{\mathcal{F}}$. The corresponding functors are

$$\begin{aligned} \mathsf{F}_{\mathcal{F}} &= \operatorname{Hom}_{\mathcal{A}_{\mathcal{F}}}(\mathcal{P}_{\mathcal{F}}, \_), \\ \mathsf{F}^*_{\mathcal{F}} &= \mathcal{P}_{\mathcal{F}}\otimes_{\mathcal{A}}\_, \\ \mathsf{F}^!_{\mathcal{F}} &= \operatorname{Hom}_{\mathcal{A}}(\mathcal{P}^\vee_{\mathcal{F}}, \_) \end{aligned}$$

and their derived functors. Note that $\mathcal{P}^\vee \simeq \begin{pmatrix}\mathcal{A} & \mathcal{F}\end{pmatrix} \simeq \mathcal{A}\oplus\mathcal{F}$ as $\mathcal{A}$-$\mathcal{A}_\mathcal{F}$-bimodule and $\mathcal{I}_\mathcal{P}$ is the ideal of matrices

$$\mathcal{I}_\mathcal{P} = \begin{pmatrix}\mathcal{A} & \mathcal{F} \\ \mathcal{F}' & \mathcal{I}'_\mathcal{F}\end{pmatrix}$$

where $\mathcal{I}'_\mathcal{F}$ is the image of the map $\mu' : \mathcal{F}'\otimes_\mathcal{A}\mathcal{F} \to \mathcal{E}$ such that $\mu'(f'\otimes f)(v) = f'(v)f$ for all $f, v \in \mathcal{F}, f' \in \mathcal{F}'$. Therefore, $\ker \mathsf{F}_\mathcal{F} \simeq (\mathcal{E}/\mathcal{I}'_\mathcal{F})$-Mod and $\ker \mathsf{DF}_\mathcal{F} = \mathscr{D}_{\mathcal{E}/\mathcal{I}'_\mathcal{F}}\mathcal{A}_\mathcal{F}$.

This construction is especially convenient when $\mathcal{A}$ is strongly Gorenstein in the sense of the following definition:

Definition 3.15 A noetherian non-commutative scheme $(X, \mathcal{A})$ is said to be *strongly Gorenstein* if X is equidimensional, $\mathcal{A}$ is a Cohen–Macaulay $\mathcal{O}_X$-module, and $\operatorname{inj.dim}_\mathcal{A} \mathcal{A} = \dim X$.

Such non-commutative schemes possess almost all usual properties of Cohen–Macaulay rings and ("usual") schemes, and their proofs are quite analogous to those from [6] (see [8, Sect. 5] for details). We need here the *Cohen–Macaulay duality*. For a noetherian non-commutative scheme $(X, \mathcal{A})$ denote by $\mathrm{CM}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of *maximal Cohen–Macaulay $\mathcal{A}$-modules*, i.e., such coherent $\mathcal{A}$-modules $\mathcal{M}$ that each localization $\mathcal{M}_x$ is a maximal Cohen–Macaulay $\mathcal{O}_{X,x}$-module. Let $^* : \mathcal{A}\text{-mod} \to \mathcal{A}^{\mathrm{op}}\text{-mod}$ be the functor mapping $\mathcal{M}$ to $\mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{A})$. If $\mathcal{A}$ is strongly Gorenstein, so is also $\mathcal{A}^{\mathrm{op}}$, and * defines an exact duality between $\mathrm{CM}\,\mathcal{A}$ and $\mathrm{CM}\,\mathcal{A}^{\mathrm{op}}$. It means that, for every $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$, $\mathcal{E}xt^i_\mathcal{A}(\mathcal{M}, \mathcal{A}) = 0$ for $i > 0$ and the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism. It also implies that the natural map $\mathcal{M}\otimes_\mathcal{A}\mathcal{L} \to \mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{L})$ is an isomorphism for every locally projective $\mathcal{A}$-module $\mathcal{L}$.

Theorem 3.16 *In the situation of Example 3.14, let $\mathcal{A}$ be strongly Gorenstein and has enough locally projective modules and $\mathcal{F} \in \mathrm{CM}\,\mathcal{A}$. Then the restrictions of the functors $\mathsf{LF}^*_\mathcal{F}$ and $\mathsf{RF}^!_\mathcal{F}$ onto $\mathsf{Perf}\mathcal{A}$ coincide. Thus these restrictions are both left and right adjoint to $\mathsf{DF}_\mathcal{F}$.*

Proof Note first that under the given conditions $\mathsf{F}^*_\mathcal{F}(\mathcal{L}) \simeq \mathsf{F}^!_\mathcal{F}(\mathcal{L})$ for every locally projective $\mathcal{A}$-module $\mathcal{L}$. As $\mathcal{A}$ has enough locally projective modules, any complex from $\mathsf{Perf}\mathcal{A}$ is quasi-isomorphic to a finite complex $\mathcal{L}^\bullet$ such that all $\mathcal{L}^i$ are from $\mathsf{lp}\,\mathcal{A}$. Then $\mathsf{LF}^*_\mathcal{F}(\mathcal{L}^\bullet) = \mathsf{F}^*_\mathcal{F}(\mathcal{L}^\bullet)$. On the other hand, $\mathsf{R}^k\mathsf{F}^!_\mathcal{F}(\mathcal{L}^i) = \mathcal{E}xt^k_\mathcal{A}(\mathcal{P}_\mathcal{F}, \mathcal{L}^i) = 0$ for $k \neq 0$. Therefore, $\mathsf{RF}^!_\mathcal{F}(\mathcal{L}^\bullet) = \mathsf{F}^!_\mathcal{F}(\mathcal{L}^\bullet) \simeq \mathsf{F}^*_\mathcal{F}(\mathcal{L}^\bullet)$. □

4 Quasi-Hereditary Schemes

In this section we generalize the notions of quasi-hereditary algebras and orders [12, 22] to non-commutative schemes. It is closely related with minors and bilocalizations. We start from the following facts. Let $(X, \mathcal{A})$ be a non-commutative scheme, $\mathcal{M}$ be an $\mathcal{A}$-module. We call $\sup\left\{ i \mid \mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M}, \_) \neq 0 \right\}$ the *local projective dimension* of the $\mathcal{A}$-module $\mathcal{M}$ and denote it by $\operatorname{lp.dim}_{\mathcal{A}} \mathcal{M}$. If the non-commutative scheme $(X, \mathcal{A})$ is noetherian and the module $\mathcal{M}$ is coherent, then $\operatorname{lp.dim}_{\mathcal{A}} \mathcal{M} = \sup\left\{ \operatorname{pr.dim}_{\mathcal{A}_x} \mathcal{M}_x \mid x \in X \right\}$.

Lemma 4.17 (Cf. [8, Lemma 4.9]) *Let* $(X, \mathcal{A})$ *be a non-commutative scheme,* $\mathcal{P}$ *be a coherent locally projective* $\mathcal{A}$*-module,* $\mathcal{B} = (\mathcal{E}nd_{\mathcal{A}} \mathcal{P})^{\mathrm{op}}$*, and* $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{I}_{\mathcal{P}}$*. Suppose that* $\mathcal{P}$ *is flat as right* $\mathcal{B}$*-module,*

$$\operatorname{lp.dim}_{\mathcal{A}} \mathcal{I}_{\mathcal{P}} = d,$$

$$\operatorname{gl.dim} \mathcal{B} = n,$$

$$\operatorname{gl.dim} \bar{\mathcal{A}} = m.$$

Then $\operatorname{gl.dim} \mathcal{A} \leq \max\{ m + d + 2,\ n \}$.

Proof Let $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{I}_{\mathcal{P}}$. Then $\operatorname{lp.dim}_{\mathcal{A}} \bar{\mathcal{A}} = d + 1$. The spectral sequence $\operatorname{Ext}^p_{\bar{\mathcal{A}}}(\mathcal{M}, \mathcal{E}xt^q_{\mathcal{A}}(\bar{\mathcal{A}}, \_)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{A}}(\mathcal{M}, \_)$ implies that $\operatorname{pr.dim}_{\mathcal{A}} \mathcal{M} \leq m + d + 1$ for every $\bar{\mathcal{A}}$-module $\mathcal{M}$. Consider the functors $\mathsf{F} = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \_)$ and $\mathsf{F}^* = \mathcal{P} \otimes_{\mathcal{B}} \_$. As the morphism $\mathsf{F}\mathsf{F}^*\mathsf{F} \to \mathsf{F}$ arising from the adjunction is an isomorphism, the kernel and the cokernel of the natural map $\alpha : \mathsf{F}^*\mathsf{F}\mathcal{M} \to \mathcal{M}$ are annihilated by F, so are actually $\bar{\mathcal{A}}$-modules. It implies that $\operatorname{Ext}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \simeq \operatorname{Ext}^i_{\mathcal{A}}(\mathsf{F}^*\mathsf{F}\mathcal{M}, \mathcal{N})$ if $i > m + d + 2$, so $\operatorname{pr.dim}_{\mathcal{A}} \mathcal{M} \leq \max\{ m + d + 2,\ \operatorname{pr.dim}_{\mathcal{A}} \mathsf{F}^*\mathsf{F}\mathcal{M} \}$. As both functors F and F^* are exact, $\operatorname{Ext}^i_{\mathcal{A}}(\mathsf{F}^*\_, \_) \simeq \operatorname{Ext}^i_{\mathcal{B}}(\_, \mathsf{F}\_)$, so $\operatorname{pr.dim}_{\mathcal{A}} \mathsf{F}^*\mathsf{F}\mathcal{M} \leq n$. □

This result motivates the following definitions:

Definition 4.18 (Cf. [8, Definition 4.9])

1. Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be two non-commutative schemes. A *relating chain* between $\mathcal{A}$ and $\mathcal{B}$ is a sequence $(\mathcal{A}_1, \mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_2, \ldots, \mathcal{P}_r, \mathcal{A}_{r+1})$, where $\mathcal{A}_1 = \mathcal{A}$; $\mathcal{A}_{r+1} = \mathcal{B}$; every $\mathcal{P}_i$ $(1 \leq i \leq r)$ is a coherent locally projective $\mathcal{A}_i$-module which is also flat as right $\mathcal{B}_i$-module, where $\mathcal{B}_i = (\mathcal{E}nd_{\mathcal{A}_i} \mathcal{P}_i)^{\mathrm{op}}$; and $\mathcal{A}_{i+1} = \mathcal{A}_i/\mathcal{I}_{\mathcal{P}_i}$ for $1 \leq i \leq r$.
2. The relating chain is said to be *flat* if, for every $1 \leq i \leq r$, $\mathcal{I}_{\mathcal{P}_i}$ is flat as right $\mathcal{A}_i$-module. Note that it is the case if the natural map $\mu_i : \mathcal{P}_i \otimes_{\mathcal{B}_i} \mathcal{P}_i^\vee \to \mathcal{A}_i$ is a monomorphism.
3. The relating chain is said to be *heredity* if, for every $1 \leq i \leq r$, $\mathcal{I}_{\mathcal{P}_i}$ is locally projective as left $\mathcal{A}_i$-module. In this case μ_i is a monomorphism (it can be proved as in [14, Statement 7]), so this chain is flat.
4. If the relating chain is heredity and all non-commutative schemes $\mathcal{B}_i$ are hereditary, i.e., $\operatorname{gl.dim} \mathcal{B}_i \leq 1$, we say that the non-commutative scheme $\mathcal{A}$ is *quasi-hereditary* of level r. (Thus quasi-hereditary of level 0 means hereditary.)

The following result is obvious:

Proposition 4.19 *If* $(\mathcal{A}_1, \mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_2, \ldots, \mathcal{P}_r, \mathcal{A}_{r+1})$ *is a relating chain between* $\mathcal{A}$ *and* $\mathcal{B}$*, then* $(\mathcal{A}_1^{\mathrm{op}}, \mathcal{P}_1^\vee, \mathcal{A}_2^{\mathrm{op}}, \mathcal{P}_2^\vee, \ldots, \mathcal{P}_r^\vee, \mathcal{A}_{r+1}^{\mathrm{op}})$ *is a relating chain between* $\mathcal{A}^{\mathrm{op}}$ *and* $\mathcal{B}^{\mathrm{op}}$ *with the same endomorphism algebras* $\mathcal{B}_i$*.*

Note that if $\mathcal{A}$ is noetherian, so are all $\mathcal{A}_i$ and $\mathcal{B}_i$. As for noetherian non-commutative schemes all flat coherent modules are locally projective, we obtain the following corollary:

Corollary 4.20 *If a noetherian non-commutative scheme* $(X,\mathcal{A})$ *is quasi-hereditary, so is also* $(X,\mathcal{A}^{\mathrm{op}})$.

We fix a relating chain $(\mathcal{A}_1,\mathcal{P}_1,\mathcal{A}_2,\mathcal{P}_2,\dots,\mathcal{P}_r,\mathcal{A}_{r+1})$ between $\mathcal{A}$ and $\mathcal{B}$ and keep the notations of Definition 4.18(1). Lemma 4.17 immediately implies an estimate for global dimensions.

Corollary 4.21 *Let* $\operatorname{gl.dim}\mathcal{B}_i \le n$ *and* $\operatorname{lp.dim}_{\mathcal{A}_i}\mathcal{I}_{\mathcal{P}_i} \le d$ *for all* $1 \le i \le r$. *Then* $\operatorname{gl.dim}\mathcal{A} \le r(d+2)+\max\{\operatorname{gl.dim}\mathcal{B},\, n-d-2\}$. *If this relating chain is heredity, then* $\operatorname{gl.dim}\mathcal{A} \le \operatorname{gl.dim}\mathcal{B} + 2r$.

Using Corollary 3.12, Theorem 3.13, and induction, we obtain the following result:

Corollary 4.22 *If this relating chain is flat, there are semi-orthogonal decompositions* $(\mathcal{T},\mathcal{T}_r,\dots,\mathcal{T}_1)$ *and* $(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_r',\mathcal{T})$ *of* $\mathcal{D}\mathcal{A}$ *such that* $\mathcal{T}_i \simeq \mathcal{T}_i' \simeq \mathcal{D}\mathcal{B}_i$ $(1 \le i \le r)$ *and* $\mathcal{T} \simeq \mathcal{D}\mathcal{B}$.

Note that, as a rule, $\mathcal{T}_i \ne \mathcal{T}_i'$.

Corollary 4.23 *If a non-commutative scheme* $\mathcal{A}$ *is quasi-hereditary of level r, then* $\operatorname{gl.dim}\mathcal{A} \le 2r+1$, *and there are semi-orthogonal decompositions* $(\mathcal{T},\mathcal{T}_r,\dots,\mathcal{T}_1)$ *and* $(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_r',\mathcal{T})$ *of* $\mathcal{D}\mathcal{A}$ *such that* $\mathcal{T}_i \simeq \mathcal{T}_i'$ $(1 \le i \le r)$ *and all categories* $\mathcal{T}_i$*, as well as* $\mathcal{T}$*, are equivalent to derived categories of some hereditary non-commutative schemes.*

The following result is evident:

Proposition 4.24 *If there is a heredity relating chain between* $\mathcal{A}$ *and* $\mathcal{B}$ *such that all* $\mathcal{B}_i$ *are hereditary and* $\mathcal{B}$ *is quasi-hereditary, then* $\mathcal{A}$ *is quasi-hereditary too.*

Corollary 4.25 *Consider the endomorphism construction of Example 3.14 (with the same notations). Suppose that* $\mathcal{F}$ *is flat as right* $\mathcal{E}$*-module,* $\mathcal{F}'$ *is locally projective as left* $\mathcal{E}$*-module and the natural map* $\mu_{\mathcal{F}}: \mathcal{F}\otimes_{\mathcal{E}}\mathcal{F}' \to \mathcal{A}$ *is a monomorphism. If both* $\mathcal{E}$ *and* $\bar{\mathcal{A}} = \mathcal{A}/\operatorname{Im}\mu_{\mathcal{F}}$ *are quasi-hereditary, so is* $\mathcal{A}_{\mathcal{F}}$.

Proof Let $\tilde{\mathcal{P}} = \begin{pmatrix}\mathcal{F}\\ \mathcal{E}\end{pmatrix}$. Then $\mathcal{I}_{\tilde{\mathcal{P}}}$ is the ideal of matrices

$$\begin{pmatrix}\mathcal{F}\otimes_{\mathcal{E}}\mathcal{F}' & \mathcal{F}\\ \mathcal{F}' & \mathcal{E}\end{pmatrix}.$$

Its first row is $\mathcal{F}\otimes_{\mathcal{E}}\begin{pmatrix}\mathcal{F}' & \mathcal{E}\end{pmatrix}$ and its first column is $\begin{pmatrix}\mathcal{F}\\ \mathcal{E}\end{pmatrix}\otimes_{\mathcal{E}}\mathcal{F}'$. Under the prescribed conditions, the first one is flat as right $\mathcal{A}_{\mathcal{F}}$-module and the second one is locally projective as left $\mathcal{A}_{\mathcal{F}}$-module. Therefore $(\mathcal{A}_{\mathcal{F}},\tilde{\mathcal{P}},\bar{\mathcal{A}})$ is a heredity relating chain relating between $\mathcal{A}$ and $\bar{\mathcal{A}}$, so we can apply Proposition 4.24. □

One can show that the definition of quasi-hereditary non-commutative schemes is indeed a generalization of the well-known definition for semiprimary rings [12, 14].

Theorem 4.26 *Let* $(X, \mathcal{A})$ *be affine:* $X = \operatorname{spec} \boldsymbol{R}$, $\mathcal{A} = \boldsymbol{A}^{\sim}$, *and the ring* $\boldsymbol{A}$ *be semiprimary. This non-commutative scheme is quasi-hereditary in the sense of Definition 4.18(4) if and only if the ring* $\boldsymbol{A}$ *is quasi-hereditary in the sense of [12].*

Proof Recall that a semiprimary ring $\boldsymbol{A}$ is called quasi-hereditary if there is a chain of ideals $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r \subset I_{r+1} = \boldsymbol{A}$ such that the following conditions hold for $\bar{I}_i = I_i/I_{i-1}$ as for an ideal in $\boldsymbol{A}_i = \boldsymbol{A}/\boldsymbol{I}_{i-1}$:

1. $\bar{I}_i^2 = \bar{I}_i$. (As $\boldsymbol{A}_i$ is semiprimary, it means that $\bar{I}_i = \boldsymbol{A}_i e_i \boldsymbol{A}_i$ for some idempotent e_i.)
2. $\bar{I}_i(\operatorname{rad} \boldsymbol{A}_i)\bar{I}_i = 0$. (It means that $\operatorname{rad}(e_i \boldsymbol{A}_i e_i) = 0$.)
3. $\bar{I}_i$ is projective as $\boldsymbol{A}_i$-module. (Under condition (2) it is equivalent to the claim that the map $\boldsymbol{A}_i e_i \otimes_{e_i A_i e_i} e_i \boldsymbol{A}_i \to \bar{I}_i$ is bijective, see [14, Statement 7].)

In other words, it means that $(\boldsymbol{A} = \boldsymbol{A}_1, P_1, \boldsymbol{A}_2, P_2, \ldots, \boldsymbol{A}_r, P_r, \boldsymbol{A}_{r+1})$, where $P_i = \boldsymbol{A}_i e_i$ and $\boldsymbol{A}_{r+1} = \boldsymbol{A}/I_r$, is a heredity relating chain between $\boldsymbol{A}$ with semisimple endomorphism rings $\boldsymbol{B}_i$ and semisimple ring $\boldsymbol{A}_{r+1}$. Thus $\boldsymbol{A}$ is a quasi-hereditary affine non-commutative scheme. On the contrary, let $\boldsymbol{A}$ be quasi-hereditary as an affine non-commutative scheme. To show that $\boldsymbol{B}$ is a quasi-hereditary ring, we can use induction and the following result.

Recall that a ring $\boldsymbol{A}$ is said to be *triangular* if it has a set of idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $\sum_{i=1}^{m} e_i = 1$, $e_i \boldsymbol{A} e_j = 0$ if $i > j$ and $\boldsymbol{A}_i = e_i \boldsymbol{A} e_i$ are *prime rings*, i.e., $IJ \neq 0$ for any two nonzero ideals of $\boldsymbol{A}_i$. If $\boldsymbol{A}$ is semiprimary, then $\boldsymbol{A}_i$ are simple artinian rings. For instance, every semiprimary hereditary ring is triangular [16].

Lemma 4.27 *Let* $\boldsymbol{A}$ *be a semiprimary ring,* $I = \boldsymbol{A}e\boldsymbol{A}$ *be an idempotent ideal such that* I *is projective as* $\boldsymbol{A}$*-module,* $\boldsymbol{A}/I$ *is quasi-hereditary, and* $\boldsymbol{E} = e\boldsymbol{A}e$ *is triangular. Then* $\boldsymbol{A}$ *is quasi-hereditary. In particular, any triangular semiprimary ring is quasi-hereditary.*

Proof According to [13], it is enough to find a heredity chain of ideals $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m = \boldsymbol{E}$ in $\boldsymbol{E}$ such that each factor $I_i e/I_{i-1}e$ is projective as $\boldsymbol{E}_i = \boldsymbol{E}/I_{i-1}$-module. Since $\boldsymbol{E}$ is triangular, it can be considered as an algebra of triangular matrices:

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E}_{11} & \boldsymbol{E}_{12} & \boldsymbol{E}_{13} & \ldots & \boldsymbol{E}_{1m} \\ 0 & \boldsymbol{E}_{22} & \boldsymbol{E}_{23} & \ldots & \boldsymbol{E}_{2m} \\ 0 & 0 & \boldsymbol{E}_{33} & \ldots & \boldsymbol{E}_{3m} \\ & & \ldots\ldots\ldots & & \\ 0 & 0 & 0 & \ldots & \boldsymbol{E}_{mm} \end{pmatrix},$$

where all rings $\boldsymbol{E}_{ii}$ are simple artinian. Let e_j $(1 \leq j \leq m)$ be the standard diagonal idempotents in this matrix ring, $\varepsilon_i = \sum_{j=1}^{i} e_j$ and $I_i = \boldsymbol{E}\varepsilon_i\boldsymbol{E}$. Then I_i is the ideal of matrices such that their first $m - i$ rows are zero. Therefore, $\boldsymbol{E}/I_{i-1}$ is the matrix ring obtained from $\boldsymbol{E}$ by crossing out the first $i-1$ rows and columns. Evidently, $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m = \boldsymbol{E}$ is a heredity chain of ideals in $\boldsymbol{E}$. One easily sees that $e_j(I_iM) = 0$ for any $\boldsymbol{E}_i$-module M and any $j > i$. Then there is an epimorphism $k\boldsymbol{E}_ie_i \to I_iM$ for some k. As the module $\boldsymbol{E}_ie_i$ is semisimple, this epimorphism splits, so I_iM is projective. In particular, $I_ie/I_{i-1}e$ is projective, so $\boldsymbol{A}$ is quasi-hereditary.

□

Just in the same way, one can show that if $X = \operatorname{spec}\boldsymbol{R}$, where $\boldsymbol{R}$ is a discrete valuation ring and $\mathcal{A} = \boldsymbol{A}^{\sim}$, where $\boldsymbol{A}$ is a semiprime $\boldsymbol{R}$-order, then the non-commutative scheme $(X, \mathcal{A})$ is quasi-hereditary if and only if $\boldsymbol{A}$ is a quasi-hereditary $\boldsymbol{R}$-order in the sense of [22].

5 Non-commutative Curves

We call a *curve* a noetherian excellent reduced scheme such that all its irreducible components are of dimension 1. We call a *non-commutative curve* a reduced non-commutative scheme $(X, \mathcal{A})$ such that X is a curve and $\mathcal{A}$ is a torsion-free finitely generated $\mathcal{O}_X$-module. We can suppose, without loss of generality, that the $\mathcal{O}_X$-module $\mathcal{A}$ is sincere. In this section $(X, \mathcal{A})$ always denotes a non-commutative curve and we suppose that $\mathcal{A}$ is a sincere $\mathcal{O}_X$-module. We denote by X_{reg} and X_{sng}, respectively, the subsets of regular and singular points of X. As X is excellent and reduced, the set X_{sng} is finite.

If $(X, \mathcal{A})$ is a non-commutative curve, the category $\operatorname{CM}\mathcal{A}$ consists of coherent $\mathcal{A}$-modules which are *torsion-free* as $\mathcal{O}_X$-modules. These modules can be defined locally. Namely, let $\mathcal{K} = \mathcal{K}_X$ be the sheaf of rational functions on X. Set $\mathcal{K}\mathcal{M} = \mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{M}$. Then $\mathcal{K}\mathcal{A}$ is a sheaf of semisimple $\mathcal{K}$-algebras and $\mathcal{A}$ is an $\mathcal{O}_X$*-order* in $\mathcal{K}\mathcal{A}$, i.e., an $\mathcal{O}_X$-subalgebra in $\mathcal{K}\mathcal{A}$ which is coherent as $\mathcal{O}_X$-module and generates $\mathcal{K}\mathcal{A}$ as $\mathcal{K}$-module. If $\mathcal{V}$ is a coherent $\mathcal{K}\mathcal{A}$-module and $\mathcal{M} \subset \mathcal{V}$ is its coherent $\mathcal{A}$-submodule which generates $\mathcal{V}$ as $\mathcal{K}$-module, we say that $\mathcal{M}$ is an $\mathcal{A}$*-lattice* in $\mathcal{V}$. Then $\mathcal{M} \in \operatorname{CM}\mathcal{A}$ and conversely, every $\mathcal{M} \in \operatorname{CM}\mathcal{A}$ is a lattice in $\mathcal{K}\mathcal{M}$. If $\mathcal{M}$ is a lattice in $\mathcal{V}$, then $\mathcal{M}_x$ is a lattice in $\mathcal{V}_x$ and $\mathcal{M}$ is completely defined by the set of lattices $\{\, \mathcal{M}_x \mid x \in X_{\text{cl}} \,\}$. The following result is an immediate consequence of its affine variant, which can be proved like in [4, Chap. 7, § 4, Théorème 3]:

Proposition 5.28

1. *If $\mathcal{M}$ and $\mathcal{N}$ are lattices in $\mathcal{V}$, then $\mathcal{M}_x = \mathcal{N}_x$ for almost all $x \in X_{\text{cl}}$.*
2. *Let $\mathcal{M}$ be a lattice in $\mathcal{V}$, $S \subseteq X_{\text{cl}}$ be a finite set, and for every $x \in S$, let $N(x)$ be an $\mathcal{A}_x$-lattice in $\mathcal{V}_x$. Then there is a lattice $\mathcal{N}$ in $\mathcal{V}$ such that $\mathcal{N}_x = N(x)$ for every $x \in S$ and $\mathcal{N}_x = \mathcal{M}_x$ for every $x \notin S$.*

Using this proposition, one can prove the following properties of non-commutative curves (see [8] for details):

Proposition 5.29 *Let $(X, \mathcal{A})$ be a non-commutative curve.*

1. *$\mathcal{A}$ has enough locally projective modules.*
2. *There is a* canonical $\mathcal{A}$-module*, i.e., such module $\omega_{\mathcal{A}}$ from* $\operatorname{CM}\mathcal{A}$ *that* $\operatorname{inj.dim}_{\mathcal{A}} \omega_{\mathcal{A}} = 1$ *and $\mathcal{E}nd_{\mathcal{A}}\, \omega_{\mathcal{A}} \simeq \mathcal{A}^{\mathrm{op}}$ (hence $\omega_{\mathcal{A}}$ is indeed an $\mathcal{A}$-bimodule). Moreover, also* $\operatorname{inj.dim}_{\mathcal{A}^{\mathrm{op}}} \omega_{\mathcal{A}} = 1$ *and $\omega_{\mathcal{A}}$ is isomorphic (as $\mathcal{A}$-bimodule) to an ideal of $\mathcal{A}$.*
3. *The functor $^* : \mathcal{M} \mapsto \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}})$ defines an exact duality between* $\operatorname{CM}\mathcal{A}$ *and* $\operatorname{CM}\mathcal{A}^{\mathrm{op}}$. *It means that, for every $\mathcal{M} \in \operatorname{CM}\mathcal{A}$, the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism and $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}}) = 0$ if $i > 0$.*

Actually, one can choose for $\omega_{\mathcal{A}}$ the module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$, and we always do so. Then $\mathcal{M}^*$ is identified with $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$. Note also that $\mathcal{A}$ is strongly Gorenstein if and only if $\mathcal{A}$ is itself a canonical $\mathcal{A}$-module.

Let $\mathcal{B}$ be a minor of the non-commutative curve $\mathcal{A}$, i.e., $\mathcal{B} = \mathcal{E}nd_{\mathcal{A}}\, \mathcal{P}$ for some coherent locally projective $\mathcal{A}$-module $\mathcal{P}$, and let $\mathsf{F}, \mathsf{F}^*, \mathsf{F}^!$ denote the corresponding functors (see formulae (1) on page 77).

$$\mathcal{A}\text{-Mod} \;\begin{matrix} \xleftarrow{\;\mathsf{F}^*\;} \\ \xrightarrow{\;\mathsf{F}\;} \\ \xleftarrow[\;\mathsf{F}^!\;]{} \end{matrix}\; \mathcal{B}\text{-Mod}$$

Obviously F and $\mathsf{F}^!$ map torsion-free modules to torsion-free. It is not true for F^*, so we modify it, setting $\mathsf{F}^\dagger(\mathcal{M}) = (\mathsf{F}^*(\mathcal{M}))^{**}$. We also set $\mathcal{P}' = (\mathcal{P}^\vee)^*$. Then $\operatorname{inj.dim}_{\mathcal{A}} \mathcal{P}' = 1$ and $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{M}, \mathcal{P}') = 0$ for every $\mathcal{M} \in \operatorname{CM}\mathcal{A}$; such $\mathcal{A}$-lattices are called *locally injective $\mathcal{A}$-lattices*. In affine case they are indeed injective in the exact category $\operatorname{CM}\mathcal{A}$. After this modification, we have results about the categories of torsion-free modules quite analogous to Theorem 3.9.

Theorem 5.30

1. *The functors $\mathsf{F}^!$ and F define an equivalence of* $\operatorname{CM}\mathcal{B}$ *and* $\operatorname{CM}'\mathcal{P}$*, where* $\operatorname{CM}'\mathcal{P}$ *is the full subcategory of* $\operatorname{CM}\mathcal{A}$ *consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an exact sequence $0 \to \mathcal{M}_x \to Q \to N \to 0$, where Q is a multiple of the $\mathcal{A}_x$-module $\mathcal{P}'_x$ and $N \in \operatorname{CM}\mathcal{A}_x$.*
2. *The restriction of the functor $\mathsf{F}^\dagger$ onto* $\operatorname{CM}\mathcal{B}$ *is left adjoint to the restriction of F onto* $\operatorname{CM}\mathcal{A}$. *Moreover, if $\mathcal{M} \in \operatorname{CM}\mathcal{B}$, the natural map $\mathsf{F}\mathsf{F}^\dagger(\mathcal{M}) \to \mathcal{M}$ is an isomorphism, and the functors $\mathsf{F}^\dagger$ and F define an equivalence of the categories* $\operatorname{CM}\mathcal{B}$ *and* $\operatorname{CM}\mathcal{P}$*, where* $\operatorname{CM}\mathcal{P}$ *is the full subcategory of* $\operatorname{CM}\mathcal{A}$ *consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an epimorphism $n\mathcal{P}_x \to \mathcal{M}$.*

Proof

1. This statement is local, so we can suppose that $X = \operatorname{spec} \boldsymbol{R}$, where $\boldsymbol{R}$ is an excellent local reduced ring of Krull dimension 1, $\mathcal{A} = \boldsymbol{A}^\sim$ for some $\boldsymbol{R}$-order $\boldsymbol{A}$, i.e., an $\boldsymbol{R}$-algebra $\boldsymbol{A}$ without nilpotent ideals which is finitely generated and Cohen–Macaulay as an $\boldsymbol{R}$-module, $\mathcal{P} = P^\sim$ for some finitely generated

projective $\boldsymbol{A}$-module P. Moreover, we can suppose that P is sincere as $\boldsymbol{A}$-module. Then $\mathcal{B} = \boldsymbol{B}^{\sim}$, where $\boldsymbol{B} = \mathrm{End}_{\boldsymbol{A}} P$. If $M \in \mathrm{CM}\,\boldsymbol{B}$, there is an exact sequence $m\boldsymbol{B} \to n\boldsymbol{B} \to M^* \to 0$, which gives an exact sequence:

$$0 \to M \to n\boldsymbol{B}^* \to m\boldsymbol{B}^*. \tag{2}$$

We denote by ψ the natural morphism $\mathbb{1}_{\boldsymbol{A}\text{-Mod}} \to \mathsf{F}^!\mathsf{F}$. One easily sees that $\mathsf{F}(P') \simeq \boldsymbol{B}^*$ and $\mathsf{F}^!(\boldsymbol{B}^*) \simeq P'$, so $\psi(P')$ is an isomorphism. The exact sequence (2) gives an exact sequence $0 \to \mathsf{F}^!(M) \to nP' \to mP'$, which shows that $\mathsf{F}^!(M) \in \mathrm{CM}'\,P$.

Let now $M \in \mathrm{CM}'\,P$. An exact sequence $0 \to M \to nP' \to N \to 0$, where $N \in \mathrm{CM}\,\boldsymbol{A}$, gives an exact sequence $0 \to \mathsf{F}(M) \to \mathsf{F}(nP') \to \mathsf{F}(N) \to 0$. For any $\boldsymbol{A}$-module N, $\psi(N)$ is the homomorphism:

$$h : N \to \mathcal{H}om_{\boldsymbol{B}}(P^\vee, \mathcal{H}om_{\boldsymbol{A}}(P, N)) \simeq \mathcal{H}om_{\boldsymbol{A}}(P \otimes_{\boldsymbol{B}} P^\vee, N)$$

such that $h(u)(\alpha \otimes \gamma) = \gamma(\alpha)u$. Tensoring with $\boldsymbol{KA}$, we obtain the map $\boldsymbol{K}N \to \mathcal{H}om_{\boldsymbol{KA}}(\boldsymbol{K}P \otimes_{\boldsymbol{KB}} \boldsymbol{K}P^\vee, \boldsymbol{K}N)$. As $\boldsymbol{KA}$ is semi-simple and $\boldsymbol{K}P$ is sincere, the natural map $\boldsymbol{K}P \otimes_{\boldsymbol{KB}} \boldsymbol{K}P^\vee \to \boldsymbol{A}$ is surjective; therefore, $\boldsymbol{K}\psi(N)$ is injective. If N is torsion-free, hence embeds into $\boldsymbol{K}N$, it implies that $\psi(N)$ is injective. So we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & nP' & \longrightarrow & N \\
 & & \Big\downarrow{\scriptstyle \psi(M)} & & \Big\downarrow{\scriptstyle \psi(nP')} & & \Big\downarrow{\scriptstyle \psi(N)} \\
0 & \longrightarrow & \mathsf{F}^!\mathsf{F}(M) & \longrightarrow & \mathsf{F}^!\mathsf{F}(nP') & \longrightarrow & \mathsf{F}^!\mathsf{F}(N).
\end{array}$$

Since $\psi(nP')$ is an isomorphism and $\psi(N)$ is a monomorphism, $\psi(M)$ is an isomorphism. As the natural map $\mathsf{F}\mathsf{F}^! \to \mathbb{1}_{\boldsymbol{B}\text{-Mod}}$ is an isomorphism, it proves the statement (*1*).

2. If $\mathcal{M} \in \mathrm{CM}\,\mathcal{B}$ and $\mathcal{N} \in \mathrm{CM}\,\mathcal{A}$, then also $\mathsf{F}\mathcal{N} \in \mathrm{CM}\,\mathcal{B}$, so

$$\mathcal{H}om_{\mathcal{A}}(\mathsf{F}^\dagger\mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathsf{F}^*\mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathsf{F}\mathcal{N}),$$

which proves the first claim. Consider now the functors

$$\mathsf{F}^{\mathrm{op}} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}^\vee, \_) : \mathrm{Qcoh}\,\mathcal{A}^{\mathrm{op}} \to \mathrm{Qcoh}\,\mathcal{B}^{\mathrm{op}},$$
$$(\mathsf{F}^{\mathrm{op}})^! = \mathrm{Hom}_{\mathcal{B}}(\mathcal{P}, \_) : \mathrm{Qcoh}\,\mathcal{B}^{\mathrm{op}} \to \mathrm{Qcoh}\,\mathcal{A}^{\mathrm{op}}.$$

As we have just proved, they establish an equivalence between the categories $\mathrm{CM}\,\mathcal{B}^{\mathrm{op}}$ and $\mathrm{CM}'\,\mathcal{P}^\vee$, where $\mathrm{CM}'\,\mathcal{P}^\vee$ consists of all right $\mathcal{A}$-modules $\mathcal{N}$ such that for every point $x \in X$ there is an exact sequence $0 \to \mathcal{N}_x \to Q \to N' \to 0$, where Q is a multiple of $\mathcal{P}_x^*$ and $N' \in \mathrm{CM}\,\mathcal{A}_x$. Equivalently, there is an epimorphism

$Q^* \to \mathcal{N}_x^*$, i.e., $\mathcal{N}^* \in \mathrm{CM}\,\mathcal{P}$. On the other hand,

$$(\mathsf{F}^{\mathrm{op}})^! \mathcal{M}^* = \mathrm{Hom}_{\mathcal{B}}(\mathcal{P}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \omega_X)) \simeq \\ \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P} \otimes_{\mathcal{B}} \mathcal{M}, \omega_X) = (\mathcal{P} \otimes_{\mathcal{B}} \mathcal{M})^* = (\mathsf{F}^\dagger \mathcal{M})^*.$$

Therefore, the statement (*2*) follows by duality. □

6 König's Resolution

A non-commutative curve $(X, \mathcal{A}')$, where $\mathcal{A} \subseteq \mathcal{A}' \subset \mathcal{K}\mathcal{A}$, is called an *over-ring* of the non-commutative curve $(X, \mathcal{A})$. If $\mathcal{A}$ has no proper over-rings, it is called *normal*. Since X is excellent and $\mathcal{A}$ is reduced, the set of over-rings of $\mathcal{A}$ satisfies the maximality condition, i.e., there are no infinite ascending chains of over-rings. (It follows, for instance, from [27, Chap. 5] or from [17].) In particular, there is always a normal over-ring of $\mathcal{A}$. In non-commutative case, such a normal over-ring is usually not unique, though all of them are locally conjugate inside $\mathcal{K}\mathcal{A}$ [27, Theorem 18.7], and every normal non-commutative curve is hereditary [27, Theorem 18.1]. Thus every non-commutative curve has a hereditary over-ring, and usually a lot of them. Actually, there is one "special" hereditary over-ring which plays an important role in this section.

Let $(X, \mathcal{A})$ be a non-commutative curve. Consider the ideal $\mathcal{J} = \mathcal{J}_A$ defined by its localizations as follows:

$$\mathcal{J}_x = \begin{cases} \mathcal{A}_x & \text{if } \mathcal{A}_x \text{ is hereditary,} \\ \mathrm{rad}\, \mathcal{A}_x & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}^\sharp = \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}} \mathcal{J}$ (the endomorphism algebra of $\mathcal{J}$ as of right $\mathcal{A}$-module). Note that $\mathcal{A}^\sharp$ can (and will) be identified with the over-ring of $\mathcal{A}$ such that its x-localization coincides with $\{\, \lambda \in \mathcal{K}\mathcal{A}_x \mid \lambda \mathcal{J}_x \subseteq \mathcal{J}_x \,\}$ for each $x \in X_{\mathrm{cl}}$. It is known [27, Theorem 39.14] that $\mathcal{A}^\sharp = \mathcal{A}$ if and only if $\mathcal{A}$ is hereditary. So there is a chain of over-rings of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} = \tilde{\mathcal{A}},$$

where $\mathcal{A}_{i+1} = \mathcal{A}_i^\sharp$ for $1 \le i \le n$ and $\tilde{\mathcal{A}}$ is hereditary. We call n the *level* of $\mathcal{A}$. For instance, a usual (commutative) curve over an algebraically closed field is of level 1 if and only if all its singular points are simple nodes or cusps. (The derived categories of such curves were investigated in [7].)

Consider the endomorphism algebra $\mathcal{R} = \mathcal{R}_{\mathcal{A}} = (\mathcal{E}nd_{\mathcal{A}} \bigoplus_{i=1}^{n+1} \mathcal{A}_i)^{\mathrm{op}}$. We call it the *König's resolution* of the non-commutative curve $\mathcal{A}$, since it is analogous to that considered in [23] (though does not coincide with it even in case if orders over discrete valuation rings) and has analogous properties.

We identify $\mathcal{R}_{\mathcal{A}}$ with the ring of matrices:

$$\mathcal{R} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1n} & \mathcal{A}_{1,n+1} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \dots & \mathcal{A}_{2n} & \mathcal{A}_{2,n+1} \\ & \dots\dots & \dots\dots & \dots\dots & \\ \mathcal{A}_{n1} & \mathcal{A}_{n2} & \dots & \mathcal{A}_{nn} & \mathcal{A}_{n,n+1} \\ \mathcal{A}_{n+1,1} & \mathcal{A}_{n+1,2} & \dots & \mathcal{A}_{n+1,n} & \mathcal{A}_{n+1,n+1} \end{pmatrix},$$

where $\mathcal{A}_{ij} = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_i, \mathcal{A}_j)$. Note that $\mathcal{A}_{ij} = \mathcal{A}_j$ if $i \le j$ and $\mathcal{A}_{i+1,i} = \mathcal{J}_{\mathcal{A}_i}$. We denote by e_j the standard diagonal idempotents in $\mathcal{R}$ and set $\mathcal{P} = \mathcal{R}e_1, \tilde{\mathcal{P}} = \mathcal{R}e_{n+1}$. Then $(\mathcal{E}nd_{\mathcal{R}}\,\mathcal{P})^{\mathrm{op}} \simeq \mathcal{A}$, so $\mathcal{A}$ is a minor of $\mathcal{R}$ and the categories $\mathcal{A}$-Mod and $\mathscr{D}\mathcal{A}$ are bilocalization, respectively, of $\mathcal{R}$-Mod and $\mathscr{D}\mathcal{R}$. The corresponding functors are $\mathsf{F} = \mathcal{H}om_{\mathcal{R}}(\mathcal{P}, \_)$ and its left derived functor LF. In the same way, $\tilde{\mathcal{A}} \simeq (\mathcal{E}nd_{\mathcal{R}}\,\tilde{\mathcal{P}})^{\mathrm{op}}$ is a minor of $\mathcal{R}$, so the categories $\tilde{\mathcal{A}}$-Mod and $\mathscr{D}\tilde{\mathcal{A}}$ are bilocalization, respectively, of $\mathcal{R}$-Mod and $\mathscr{D}\mathcal{R}$. The corresponding functors are $\tilde{\mathsf{F}} = \mathcal{H}om_{\mathcal{R}}(\tilde{\mathcal{P}}, \_)$ and its left derived functor $\mathsf{L}\tilde{\mathsf{F}}$. Thus we have a diagram of bilocalizations

$$\tilde{\mathcal{A}}\text{-Mod} \begin{array}{c} \xrightarrow{\tilde{\mathsf{F}}^*} \\ \xleftarrow{\;\tilde{\mathsf{F}}\;} \\ \xrightarrow[\tilde{\mathsf{F}}^!]{} \end{array} \mathcal{R}\text{-Mod} \begin{array}{c} \xleftarrow{\mathsf{F}^*} \\ \xrightarrow{\;\mathsf{F}\;} \\ \xleftarrow[\mathsf{F}^!]{} \end{array} \mathcal{A}\text{-Mod} \tag{3}$$

Since $\tilde{\mathcal{A}}$ is an over-ring of $\mathcal{A}$, there is a morphism $\nu : (X, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$. According to Proposition 3.7, we can replace here $(X, \tilde{\mathcal{A}})$ by $(\tilde{X}, \tilde{\mathcal{A}})$ where $\tilde{X} = \operatorname{spec}(\operatorname{center}(\tilde{\mathcal{A}}))$. In case of "usual" schemes, when $\mathcal{A} = \mathcal{O}_X$, $\tilde{X}$ is the normalization of X and ν is the normalization map. The morphism ν induces the functor of direct image $\nu_* : \tilde{\mathcal{A}}\text{-Mod} \to \mathcal{A}\text{-Mod}$ and its left and right adjoints ν^* and $\nu^!$, which are functors $\mathcal{A}\text{-Mod} \to \tilde{\mathcal{A}}\text{-Mod}$. It so happens that these functors, maybe up to twist, are compositions of the functors from the diagram (3).

Theorem 6.31

1. $\mathsf{F}\tilde{\mathsf{F}}^* \simeq \nu_*$ *and* $\tilde{\mathsf{F}}\mathsf{F}^! \simeq \nu^!$.
2. $\tilde{\mathsf{F}}\mathsf{F}^* \simeq \mathcal{C} \otimes_{\tilde{\mathcal{A}}} \nu^* \_$ *and* $\mathsf{F}\tilde{\mathsf{F}}^! \simeq \nu_*(\mathcal{C}' \otimes_{\tilde{\mathcal{A}}} \_)$*, where* $\mathcal{C} = \mathcal{H}om_{\mathcal{A}}(\tilde{\mathcal{A}}, \mathcal{A}) = \mathcal{A}_{n+1,1}$ *is the conductor of* $\tilde{\mathcal{A}}$ *in* $\mathcal{A}$ *and* $\mathcal{C}' = \mathcal{H}om_{\tilde{\mathcal{A}}}(\mathcal{C}, \tilde{\mathcal{A}})$ *is its dual* $\tilde{\mathcal{A}}$*-module.*

Proof We verify the equalities (*1*). Indeed, since $e_1\tilde{\mathcal{P}} = \tilde{\mathcal{A}}$ as $\mathcal{A}$-$\tilde{\mathcal{A}}$-bimodule,

$$\mathsf{F}\tilde{\mathsf{F}}^*(\mathcal{M}) = \mathcal{H}om_{\mathcal{R}}(\mathcal{P}, \tilde{\mathcal{P}} \otimes_{\tilde{\mathcal{A}}} \mathcal{M}) \simeq e_1 \tilde{\mathcal{P}} \otimes_{\tilde{\mathcal{A}}} \mathcal{M} \simeq \mathcal{M}$$

considered as $\mathcal{A}$-module, which is just $\nu_*(\mathcal{M})$. Also

$$\begin{aligned} \tilde{\mathsf{F}}\mathsf{F}^!(\mathcal{N}) &= \mathcal{H}om_{\mathcal{R}}(\tilde{\mathcal{P}}, \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \mathcal{N})) \simeq e_{n+1}\, \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \mathcal{N}) \\ &\simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee e_{n+1}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\tilde{\mathcal{A}}, \mathcal{N}) = \nu^!(\mathcal{N}). \end{aligned}$$

The equalities (*2*) are proved analogously (see also [9]). □

Theorem 6.32 *Let $\varepsilon_k = \sum_{j=k}^{n+1} e_j$, $\mathcal{I}_k = \mathcal{R}\varepsilon_k\mathcal{R}$, $\mathcal{Q}_k = \mathcal{R}/\mathcal{I}_{k+1}$, and $\mathcal{P}_k = \mathcal{Q}_k e_k$. Then $(\mathcal{R}, \tilde{\mathcal{P}}, \mathcal{Q}_n, \mathcal{P}_n, \mathcal{Q}_{n-1}, \mathcal{P}_{n-1}, \ldots, \mathcal{P}_2, \mathcal{Q}_1)$ is a heredity relating chain between $\mathcal{R}$ and $\mathcal{Q}_1 \simeq \mathcal{A}/\mathcal{J}_{\mathcal{A}}$. Moreover, $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_i)^{\mathrm{op}} \simeq \mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$ is a semi-simple algebra, so $\mathcal{R}$ is a quasi-hereditary non-commutative scheme of level n and* $\mathrm{gl.dim}\,\mathcal{R} \leq 2n$.

Proof A straightforward calculation shows that $\mathcal{I}_k$ is the ideal of matrices:

$$\mathcal{I}_k = \begin{pmatrix} \mathcal{A}_{k1} & \mathcal{A}_{k2} & \ldots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \ldots & \mathcal{A}_{n+1} \\ \mathcal{A}_{k1} & \mathcal{A}_{k2} & \ldots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \ldots & \mathcal{A}_{n+1} \\ & & \ldots & \ldots & \ldots & \ldots & & \\ \mathcal{A}_{k1} & \mathcal{A}_{k2} & \ldots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \ldots & \mathcal{A}_{n+1} \\ \mathcal{A}_{k+1,1} & \mathcal{A}_{k+1,2} & \ldots & \mathcal{A}_{k+1,k-1} & \mathcal{A}_{k+1,k} & \mathcal{A}_{k+1} & \ldots & \mathcal{A}_{n+1} \\ & & \ldots & \ldots & \ldots & \ldots & & \\ \mathcal{A}_{n+1,1} & \mathcal{A}_{n+1,2} & \ldots & \mathcal{A}_{n+1,k-1} & \mathcal{A}_{n+1,k} & \mathcal{A}_{n+1,k+1} & \ldots & \mathcal{A}_{n+1} \end{pmatrix}$$

Hence, $\mathcal{Q}_k$ is the algebra of $k \times k$ matrices (a_{ij}), where $a_{ij} \in \mathcal{A}_{ij}/\mathcal{A}_{k+1,j}$. In particular, $a_{ik} \in \mathcal{A}_k/\mathcal{A}_{i+1,k} = \mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$ and this algebra is semi-simple. Therefore, $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_k)^{\mathrm{op}} \simeq e_k\mathcal{Q}_k e_k = \mathcal{A}_{kk}/\mathcal{A}_{k+1,k}$ is semi-simple. Obviously, $\mathcal{I}_{\mathcal{P}_k} = \mathcal{Q}_k e_k \mathcal{Q}_k = \mathcal{I}_{k+1}/\mathcal{I}_k$, hence $\mathcal{Q}_{k-1} \simeq \mathcal{Q}_k/\mathcal{I}_{\mathcal{P}_k}$, so we have indeed a relating chain. Moreover, $\mathcal{I}_k$ is obviously projective as right $\mathcal{R}$-module, hence $\mathcal{I}_k/\mathcal{I}_{k+1}$ is projective as right $\mathcal{Q}_k$-module and this relating chain is heredity. As $\tilde{\mathcal{A}} = (\mathcal{E}nd_{\mathcal{R}} \tilde{\mathcal{P}})^{\mathrm{op}}$ is hereditary and all $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_k)^{\mathrm{op}}$ are semi-simple, $\mathcal{R}$ is quasi-hereditary and $\mathrm{gl.dim}\,\mathcal{R} \leq 2n$. □

Thus the functor $\mathsf{DF} : \mathscr{D}\mathcal{R} \to \mathscr{D}\mathcal{A}$ defines a *categorical resolution* of the derived category $\mathscr{D}\mathcal{A}$ in the sense of [24]. If $\mathcal{A}$ is strongly Gorenstein, Theorem 3.13 shows that this resolution is even *weakly crepant*, i.e., the restrictions of its left and right adjoint functors coincide on perfect complexes (small objects in $\mathscr{D}\mathcal{A}$).

We denote by $\bar{\mathcal{A}}_k$ the semi-simple algebra $\mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$.

Corollary 6.33 *The derived category $\mathscr{D}\mathcal{R}$ has two semi-orthogonal decompositions: $\mathscr{D}\mathcal{R} = \langle \mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_n, \mathscr{T} \rangle$ and $\mathscr{D}\mathcal{R} = \langle \mathscr{T}', \mathscr{T}'_n, \ldots, \mathscr{T}'_2, \mathscr{T}'_1 \rangle$, where $\mathscr{T} \simeq \mathscr{T}' \simeq \mathscr{D}\tilde{\mathcal{A}}$ and $\mathscr{T}_k \simeq \mathscr{T}'_k \simeq \mathscr{D}\bar{\mathcal{A}}_k$.*

Remark 6.34 Note that usually $\mathscr{T} \neq \mathscr{T}'$ as well as $\mathscr{T}_k \neq \mathscr{T}'_k$ for $k > 1$, though $\mathscr{T}_1 = \mathscr{T}'_1 = \mathscr{D}(\mathcal{R}/\mathcal{I}_2)$ naturally embedded into $\mathscr{D}\mathcal{R}$.

7 Tilting on Rational Curves

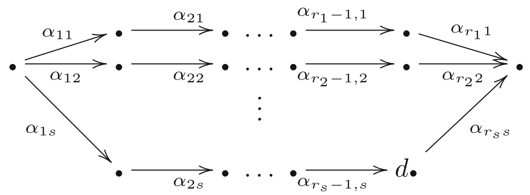
We say that a non-commutative curve $(X, \mathcal{A})$ is *rational* if X is a rational projective curve over an algebraically closed field $\Bbbk$ and $\mathcal{A}$ is central. Since the Brauer group of the field of rational functions $\Bbbk(t)$ is trivial [26], then $\mathcal{K}\mathcal{A} \simeq \mathrm{Mat}(m, \mathcal{K})$ for some

m. In this case the structure of hereditary non-commutative curves is well-known (see, for instance, [11] or [8]). Namely, if such a curve is connected, then $X = \mathbb{P}^1$, and up to Morita equivalence, this curve is given by a function $\mathbf{r} : X_{\mathrm{cl}} \to \mathbb{N}$ such that $\mathbf{r}(x) = 1$ for almost all points. A representative $\mathcal{H}(\mathbf{r})$ of the Morita class defined by this function can be defined as follows. Choose $m \in \mathbb{N}$ such that $m \geq \mathbf{r}(x)$ for all $x \in X_{\mathrm{cl}}$ and choose partitions $m = \sum_{k=1}^{\mathbf{r}(x)} m_{xk}$ for every x. Set $\hat{m}_{xk} = \sum_{l=1}^{k} m_{xl}$. Let $\mathcal{H}_x$ be the subalgebra in $\mathrm{Mat}(m, \mathcal{O}_{X,x})$ consisting of all matrices (a_{ij}) such that $a_{ij}(x) = 0$ if $i \leq \hat{m}_{xk}$ and $j > \hat{m}_{xk}$ for some k. Then $\mathcal{H}(\mathbf{r})$ is the subsheaf of $\mathrm{Mat}(m, \mathcal{O}_X)$ such that its x-stalk equals $\mathcal{H}_x$.

It is also known that $\mathcal{H}(\mathbf{r})$ has a *tilting module*, i.e., a coherent $\mathcal{H}(\mathbf{r})$-module $\mathcal{T}$ such that $\mathrm{pr.dim}\,\mathcal{T} < \infty$, $\mathrm{Ext}^q_{\mathcal{H}(\mathbf{r})}(\mathcal{T}, \mathcal{T}) = 0$ for all $q > 0$ and $\mathcal{T}$ generates the derived category $\mathscr{D}\mathcal{H}(\mathbf{r})$. Namely, let $\mathcal{H} = \mathcal{H}(\mathbf{r})$, $\mathcal{L} = \mathcal{O}_X^m$ considered as $\mathcal{H}$-module and $\mathbf{S} = \{\, x \in X_{\mathrm{cl}} \mid \mathbf{r}(x) > 1 \,\}$. If $\mathbf{S} = \{\, x_1, x_2, \dots, x_s \,\}$ with $s > 1$, we suppose that $x_1 = (1:0)$, $x_2 = (0:1)$ and $x_i = (1:\lambda_i)$ for $1 < i \leq s$, where $\lambda \in \mathbb{k} \setminus \{\, 0, 1 \,\}$, and set $r_i = \mathbf{r}(x_i)$. If $\#(\mathbf{S}) = 1$, we set $s = 2$, $r_1 = \mathbf{r}(x_1)$, $r_2 = 1$. If $\mathbf{S} = \emptyset$, then $\mathcal{H} = \mathrm{Mat}(m, \mathcal{O}_X)$ is Morita equivalent to $\mathcal{O}_X$, so $\mathcal{L} \oplus \mathcal{L}(1)$ is a tilting sheaf for $\mathcal{H}$. In this case we also set $s = 2$, $r_1 = r_2 = 1$. Consider the submodule $\mathcal{L}(x, k) \subseteq \mathcal{L}$ such that $\mathcal{L}(x, k)_y = \mathcal{L}_y$ for $y \neq x$ and $\mathcal{L}(x, k)_x$ consists of all vectors $(a_i)_{1 \leq i \leq k}$ such that $a_i(x) = 0$ for $i \leq \hat{m}_k$ and set $\mathcal{T} = \mathcal{L} \oplus \mathcal{L}(1) \oplus \big(\bigoplus_{\mathbf{r}(x)>1} \bigoplus_{k=1}^{\mathbf{r}(x)-1} \mathcal{L}(x, k) \big)$.

Theorem 7.35 (See [8][2])

1. *$\mathcal{T}$ is a tilting module for $\mathcal{H}$.*
2. *$(\mathcal{E}nd_{\mathcal{H}}\, \mathcal{T})^{\mathrm{op}} \simeq \boldsymbol{R}(\mathbf{r}, \boldsymbol{\lambda})$, where $\boldsymbol{R}(\mathbf{r}, \boldsymbol{\lambda})$ is the* canonical algebra *defined by the sequences $\mathbf{r} = (r_1, r_2, \dots, r_s)$ and $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_s)$, i.e., the algebra given by the quiver*



with relations $\alpha_j = \alpha_1 + \lambda_j \alpha_2$ for $3 \leq j \leq s$, where $\alpha_j = \alpha_{r_j j} \dots \alpha_{2j} \alpha_{1j}$ [28, Sect. 3.7].

Note that if $s = 2$, it is just the quiver algebra of the quiver $\tilde{A}_{r_1, r_2}$; if, moreover, $r_1 = r_2 = 1$, it is the Kronecker algebra. Note also that any canonical algebra is triangular, hence quasi-hereditary.

[2] It also follows from [19], since $\mathcal{H}(\mathbf{r})$ is Morita equivalent to the weighted projective line $C(\mathbf{r}, \mathbf{S})$.

Obviously, if a rational hereditary non-commutative scheme $(X, \mathcal{H})$ is not connected, it splits into a direct product of connected hereditary non-commutative schemes. Therefore it has a tilting module $\mathcal{T}$ such that $(\mathcal{E}nd_{\mathcal{H}})^{\mathrm{op}}$ is a direct product of canonical algebras.

Let now $(X, \mathcal{A})$ be a rational non-commutative curve, $\mathcal{R}$ be its König's resolution. We use the notations of the preceding section. The hereditary non-commutative curve $\tilde{\mathcal{A}}$ has a tilting module $\mathcal{T}$ such that $(\mathcal{E}nd_{\tilde{\mathcal{A}}}\,\mathcal{T})^{\mathrm{op}} = \boldsymbol{R}$ is a direct product of canonical algebras. Then $\tilde{\mathcal{T}} = \tilde{\mathsf{F}}(\mathcal{T})$ generates $\operatorname{Im}\tilde{\mathsf{F}}$ and $\operatorname{Ext}^q_{\mathcal{H}(\mathbf{r})}(\mathcal{T}, \mathcal{T}) = 0$ for all $q > 0$. As $\langle \ker\tilde{\mathsf{F}}, \operatorname{Im}\tilde{\mathsf{F}}\rangle$ is a semi-orthogonal decomposition of $\mathscr{D}\mathcal{R}$, also $\operatorname{pr.dim}\tilde{\mathcal{T}} < \infty$. As $\mathcal{Q}$ generates $\mathscr{D}\mathcal{Q}$, which can be identified with $\ker\tilde{\mathsf{F}}$, $\mathcal{Q}\oplus\tilde{\mathcal{T}}$ generates $\mathscr{D}\mathcal{R}$. Note that $\dim\operatorname{supp}\mathcal{Q} = 0$; therefore, $\operatorname{Ext}^q_{\mathcal{R}}(\mathcal{Q}, \mathcal{M}) = H^0(X, \mathcal{E}xt^q_{\mathcal{R}}(\mathcal{Q}, \mathcal{M}))$ for every quasi-coherent module $\mathcal{M}$. A locally projective resolution of $\mathcal{Q}$ is $0 \to \tilde{\mathcal{I}} \xrightarrow{\iota} \mathcal{R} \to \mathcal{Q} \to 0$. Thus $\operatorname{pr.dim}_{\mathcal{R}}\mathcal{Q} = 1$. Moreover, $\operatorname{Ext}^1_{\mathcal{R}}(\mathcal{Q}, \mathcal{N}) = 0$ for any $\mathcal{Q}$-module $\mathcal{N}$, because $\tilde{\mathcal{I}}^2 = \tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}\mathcal{N} = 0$, thus $\operatorname{Hom}_{\mathcal{R}}(\tilde{\mathcal{I}}, \mathcal{N}) = 0$. Obviously, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{Q}, \mathcal{T}) = 0$. It implies the following result:

Theorem 7.36 $\mathcal{T}^+ = \mathcal{Q}[-1]\oplus\tilde{\mathcal{T}}$ *is a* tilting complex *for* $\mathcal{R}$*, i.e., it belongs to* $\mathsf{Perf}\mathcal{R}$*, generates* $\mathscr{D}\mathcal{R}$ *and* $\operatorname{Hom}_{\mathscr{D}\mathcal{R}}(\mathcal{T}^+, \mathcal{T}^+[k]) = 0$ *if* $k \neq 0$*. Therefore* $\mathscr{D}\mathcal{R} \simeq \mathscr{D}\boldsymbol{E}$*, where* $\boldsymbol{E} = (\mathcal{E}nd_{\mathscr{D}\mathcal{R}}\,\mathcal{T}^+)^{\mathrm{op}}$.

Note that $\boldsymbol{E}$ can be considered as the algebra of triangular matrices:

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{T} \\ 0 & \boldsymbol{R} \end{pmatrix}, \tag{4}$$

where $\boldsymbol{R} = (\mathcal{E}nd_{\tilde{\mathcal{A}}}\,\mathcal{T})^{\mathrm{op}}$ is a direct product of canonical algebras and $\boldsymbol{T} = \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{Q}, \tilde{\mathcal{T}}) \simeq \operatorname{Hom}_{\mathcal{R}}(\tilde{\mathcal{I}}, \tilde{\mathcal{T}})/\iota^*\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}, \tilde{\mathcal{T}})$. Note that $\tilde{\mathcal{I}} \simeq \bigoplus_{i=1}^{n+1}\tilde{\mathsf{F}}\mathcal{A}_{n+1,i}$, whence $\boldsymbol{T} \simeq \bigoplus_{i=1}^{n+1}\operatorname{Hom}_{\tilde{\mathcal{A}}}(\mathcal{A}_{n+1,i}, \mathcal{T})$.

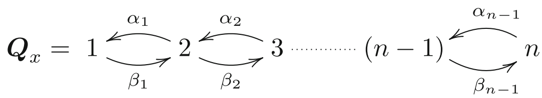
Corollary 7.37 *For every rational non-commutative curve* $(X, \mathcal{A})$*, there is a finite dimensional quasi-hereditary algebra* $\boldsymbol{E}$ *and a bilocalizing functor* $\mathscr{D}\boldsymbol{E} \to \mathscr{D}\mathcal{A}$.

Proof In the triangular presentation (4) of the algebra $\boldsymbol{E}$, let $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\boldsymbol{I} = \boldsymbol{E}e\boldsymbol{E} = \left(\begin{smallmatrix} 0 & \boldsymbol{T} \\ 0 & \boldsymbol{R} \end{smallmatrix}\right)$ is projective as $\boldsymbol{E}$-module and $e\boldsymbol{E}e \simeq \boldsymbol{R}$ is triangular. Hence $\boldsymbol{E}$ is quasi-hereditary by Lemma 4.27.

Thus every rational non-commutative curve has a categorical resolution by a finite dimensional quasi-hereditary algebra. If the curve is strongly Gorenstein, this resolution is weakly crepant. In particular, it is the case for "usual" (commutative) rational curves. Note that $\boldsymbol{Q} = \prod_x \boldsymbol{Q}_x$, where x runs through all points such that $\mathcal{A}_x$ is not hereditary (in the commutative case through singular points of X).

Example 7.38 (See [9, Sect. 8]) We consider the input $\boldsymbol{Q}_x$ for *simple singularities* of (usual) plain curves in the sense of [2]. We present it as a quiver with relations.

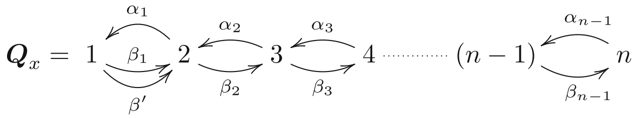
1. If x is of type A_m, $m > 2$, then



$$\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \text{if} \quad 1 \le k < n-1,$$
$$\beta_{n-1}\alpha_{n-1} = 0,$$

where $n = \left[\frac{m+1}{2}\right]$. Note that for $m \le 2$, the algebra $\boldsymbol{Q}_x$ is semisimple.

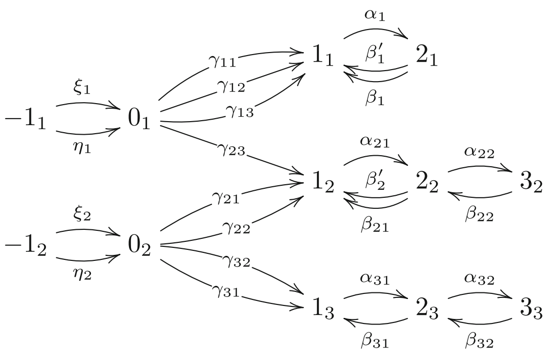
2. If x is of type D_m, $m \ge 4$, then



$$\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \text{if} \quad 1 \le k < n-1,$$
$$\beta_{n-1}\alpha_{n-1} = 0,$$
$$\beta'\alpha_1 = 0,$$
$$\beta_2\beta' = 0,$$

where $n = \left[\frac{m}{2}\right]$.

3. If x is of type E_6, $\boldsymbol{Q}_x$ is the same as for D_4, and if x is of type E_7 or E_8, $\boldsymbol{Q}_x$ is the same as for D_6.
4. Finally, we consider a "global" example, where X has two irreducible rational components X_1, X_2 and three singular points $x_1 \in X_1$ of type E_6, $x_2 \in X_1 \cap X_2$ of type D_7, and $x_3 \in X_2$ of type A_5. Then the algebra $\boldsymbol{E}$ has the quiver



It consists of "local parts" (formed by the vertices $1_i, 2_i, 3_i$) and "Kronecker parts" (formed by the vertices $0_j, -1_j$) arising from the components of $\tilde{X}$.

One can also explicitly describe the relations for the arrows γ_{ij} between these parts (they depend on the positions of the preimages of singular points on the components of $\tilde{X}$).

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Higher-Order Spectra, Equivariant Hodge–Deligne Polynomials, and Macdonald-Type Equations

Wolfgang Ebeling and Sabir M. Gusein-Zade

Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract We define notions of higher-order spectra of a complex quasi-projective manifold with an action of a finite group G and with a G-equivariant automorphism of finite order, some of their refinements and give Macdonald-type equations for them.

Keywords Group actions • Macdonald-type equations • Orbifold Euler characteristic • Spectrum

Subject Classifications:14L30, 55M35, 57R18

1 Introduction

For a "good" topological space X, say, a union of cells in a finite CW-complex or a quasi-projective complex analytic variety, the Euler characteristic $\chi(X)$, defined as the alternating sum of the dimensions of the cohomology groups with compact support, is an additive invariant. In [17], I.G. Macdonald derived a formula for the Poincaré polynomial of a symmetric product. For the Euler characteristic, this formula gives the following. Let $S^nX = X^n/S_n$ be the nth symmetric power of the

W. Ebeling (✉)
Institut für Algebraische Geometrie, Leibniz Universität Hannover, Postfach 6009, 30060 Hannover, Germany
e-mail: ebeling@math.uni-hannover.de

S.M. Gusein-Zade
Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, GSP-1, Leninskie Gory 1, Moscow 119991, Russia
e-mail: sabir@mccme.ru

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space X. Then one has [17]

$$1+\sum_{n=1}^{\infty}\chi(S^nX)t^n=(1-t)^{-\chi(X)}\,.$$

We can interpret this formula as a formula for an invariant (here the Euler characteristic) expressing the generating series of the values of the invariant for the symmetric powers of a space as a series not depending on the space (here simply $(1-t)^{-1}$) with an exponent which is equal to the value of the invariant for the space itself. We call such an equation a *Macdonald-type equation*. In [12], formulae of this type were considered for some generalizations of the Euler characteristic (with values in certain rings). If the ring of values is not a number ring ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$), to formulate these equations, one needs to use so-called power structures over the rings [10] which can be defined through (pre-)λ-ring structures on them.

Here we consider other generalizations of these formulas. We consider another additive invariant which is finer than the Euler characteristic: the (Hodge) spectrum. The (Hodge) spectrum was first defined in [20, 21] for a germ of a holomorphic function on $(\mathbb{C}^n,0)$ with an isolated critical point at the origin. It can also be defined for a pair (V,φ), where V is a complex quasi-projective variety and φ is an automorphism of V of finite order: [5]. (The spectrum of a germ of a holomorphic function is essentially the spectrum of its motivic Milnor fibre defined in [5].)

Traditionally the spectrum is defined as a finite collection of rational numbers with integer multiplicities (possibly negative ones) and therefore as an element of the group ring $\mathbb{Z}[\mathbb{Q}]$ of the group $\mathbb{Q}$ of rational numbers. Let $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck group of pairs (V,φ), where V is a quasi-projective variety and φ is an automorphism of V of finite order (with the addition defined by the disjoint union). The group $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ is a ring with the multiplication defined by the Cartesian product of varieties and with the automorphism defined by the diagonal action. The Euler characteristic can be interpreted as a ring homomorphism from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to the ring of integers $\mathbb{Z}$. The spectrum can be regarded as a sort of generalized Euler characteristic. (The spectrum of a pair (V,φ) determines the Euler characteristic of V in a natural way.) Namely, it can be viewed as a group homomorphism from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to the group ring $\mathbb{Z}[\mathbb{Q}]$, but it is not a ring homomorphism. Rational numbers (i.e., elements of the group $\mathbb{Q}$) are in one-to-one correspondence with the elements of the group $(\mathbb{Q}/\mathbb{Z})\times\mathbb{Z}$: $r \longleftrightarrow (\{r\},[r])$, where $\{r\}$ is the fractional part of the rational number r and $[r]$ is its integer part. In this way, the group rings $\mathbb{Z}[\mathbb{Q}]$ and $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Z}]$ can be identified as abelian groups. (The isomorphism is not a ring homomorphism!) This permits to consider the spectrum as an element of the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Z}]$. Moreover, the corresponding map from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Z}]$ is a ring homomorphism.

The group ring $\mathbb{Z}[\mathscr{A}]$ of an abelian group $\mathscr{A}$ has a natural λ-structure. We use this fact to show that the spectrum of a pair (V,φ) as an element of $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Z}]$ also satisfies a Macdonald-type equation.

For a quasi-projective variety with an action of a finite group, one has the orbifold Euler characteristic defined in [7, 8] (see also [1, 14]) and the higher-order Euler characteristics defined in [1, 3]. These notions can be extended to some generalizations of the Euler characteristic.

For a complex quasi-projective manifold V with an action of a finite group G and with a G-equivariant automorphism φ of finite order, one can define the notion of an orbifold spectrum as an element of the group ring $\mathbb{Z}[\mathbb{Q}]$: [9]. This spectrum takes into account not only the logarithms of the eigenvalues of the action of the transformation φ on the cohomology groups but also the so-called ages of elements of G at their fixed points (both being rational numbers). Algebraic manipulations with these two summands are different. The first ones behave as elements of $\mathbb{Q}/\mathbb{Z}$, whereas the second ones as elements of $\mathbb{Q}$. This explains why the existence of a Macdonald-type equation for this spectrum is not clear. However, if one considers the "usual" Hodge spectrum as an element of the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}]$ and applies the orbifold approach to the summand $\mathbb{Z}$ (thus, substituting it by $\mathbb{Q}$), one gets a version of the orbifold spectrum with values in the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}]$ permitting a Macdonald-type equation. This version of the orbifold spectrum determines the one from [9] in a natural way. Moreover, taking into account the weight filtration as well, one can consider a refinement of this notion with values in the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}]$. The last notion is equivalent to the notion of the equivariant orbifold Hodge–Deligne polynomial.

Applying the traditional method to define higher-order Euler characteristics through the orbifold one to the described notions, we define higher-order spectra of a triple (V, G, φ) with a quasi-projective G-manifold V, some of their refinements, and give Macdonald-type equations for them.

2 λ-Structure on the Group Ring of an Abelian Group

Let $\mathscr{A}$ be an abelian group (with the sum as the group operation) and let $\mathbb{Z}[\mathscr{A}]$ be the group ring of $\mathscr{A}$. The elements of $\mathbb{Z}[\mathscr{A}]$ are finite sums of the form $\sum\limits_{a\in\mathscr{A}} k_a\{a\}$ with $k_a \in \mathbb{Z}$. (We use the curly brackets in order to avoid ambiguity when $\mathscr{A}$ is a subgroup of the group $\mathbb{R}$ of real numbers: $\mathbb{Z}$ or $\mathbb{Q}$.) The ring operations on $\mathbb{Z}[\mathscr{A}]$ are defined by $\sum k'_a\{a\} + \sum k''_a\{a\} = \sum(k'_a + k''_a)\{a\}$, $(\sum k'_a\{a\})(\sum k''_a\{a\}) = \sum\limits_{a,b\in\mathscr{A}} (k'_a \cdot k''_b)\{a+b\}$.

Let R be a commutative ring with a unit. A λ-ring structure on R (sometimes called a pre-λ-ring structure; see, e.g., [16]) is an "additive-to-multiplicative" homomorphism $\lambda : R \to 1 + T \cdot R[[T]]$ ($a \mapsto \lambda_a(T)$) such that $\lambda_a(T) = 1 + aT + \ldots$. This means that $\lambda_{a+b}(T) = \lambda_a(T) \cdot \lambda_b(T)$ for $a, b \in R$.

The notion of a λ-ring structure is closely related to the notion of a power structure defined in [10]. Sometimes a power structure has its own good description which permits to use it, e.g., for obtaining formulae for generating series of some invariants. A power structure over a ring R is a map $(1+T\cdot R[[T]])\times R \to 1+T\cdot R[[T]]$, $(A(T), m) \mapsto (A(T))^m$ $(A(t) = 1 + a_1T + a_2T^2 + \ldots, a_i \in R, m \in R)$ possessing all the basic properties of the exponential function: see [10]. A λ-structure on a ring defines a power structure over it. On the other hand, there are, in general, many λ-structures on a ring corresponding to one power structure over it.

The group ring $\mathbb{Z}[\mathscr{A}]$ of an abelian group $\mathscr{A}$ can be considered as a λ-ring. The λ-ring structure on $\mathbb{Z}[\mathscr{A}]$ is natural and must be well known. However, we have not found its description in the literature. Therefore, we give here a definition of a λ-structure on the ring $\mathbb{Z}[\mathscr{A}]$. (A similar construction was discussed in [11] for the ring of formal "power" series over a semigroup with certain finiteness properties.)

The group ring $\mathbb{Z}[\mathscr{A}]$ can be regarded as the Grothendieck ring of the group semiring $S[\mathscr{A}]$ of maps of finite sets to the group $\mathscr{A}$. Elements of $S[\mathscr{A}]$ are the equivalence classes of the pairs (X, ψ) consisting of a finite set X and a map $\psi : X \to \mathscr{A}$. (Two pairs (X_1, ψ_1) and (X_2, ψ_2) are equivalent if there exists a bijective map $\xi : X_1 \to X_2$ such that $\psi_2 \circ \xi = \psi_1$.) The group ring $\mathbb{Z}[\mathscr{A}]$ is the Grothendieck ring of the semiring $S[\mathscr{A}]$. Elements of the ring $\mathbb{Z}[\mathscr{A}]$ are the equivalence classes of maps of finite virtual sets (i.e., formal differences of sets) to $\mathscr{A}$. For a pair (X, ψ) representing an element a of the semiring $S[\mathscr{A}]$, let its nth symmetric power $S^n(X, \psi)$ be the pair $(S^nX, \psi^{(n)})$ consisting of the nth symmetric power $S^nX = X^n/S_n$ of the set X and of the map $\psi^{(n)} : S^nX \to \mathscr{A}$ defined by $\psi^{(n)}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i)$. One can easily see that the series

$$\lambda_a(T) = 1 + [S^1(X, \psi)]T + [S^2(X, \psi)]T^2 + [S^3(X, \psi)]T^3 + \ldots$$

defines a λ-structure on the ring $\mathbb{Z}[\mathscr{A}]$ (or rather a λ-structure on the semiring $S[\mathscr{A}]$ extendable to a λ-structure on $\mathbb{Z}[\mathscr{A}]$ in a natural way).

The power structure over the ring $\mathbb{Z}[\mathscr{A}]$ corresponding to this λ-structure can be described in the following way. Let $A(T) = 1 + a_1T + a_2T^2 + \ldots$, where $a_i = [(X_i, \psi_i)]$, $m = [(M, \psi)]$ with finite *sets* X_i and M (thus, a_i and m being actually elements of the semiring $S[\mathscr{A}]$). Then

$$(A(T))^m = 1+\sum_{n=1}^{\infty}\left(\sum_{\{n_i\}:\sum in_i=n}\left[\left(((M^{\sum_i n_i} \setminus \Delta) \times \prod_i X_i^{n_i}) \,/\, \prod_i S_{n_i}\ , \psi_{\{n_i\}}\right)\right]\right)\cdot T^n ,$$

where Δ is the big diagonal in $M^{\sum_i n_i}$ (consisting of $(\sum n_i)$-tuples of points of M with at least two coinciding ones), the group $\prod_i S_{n_i}$ acts on $(M^{\sum_i n_i} \setminus \Delta) \times \prod_i X_i^{n_i}$ by permuting simultaneously the factors in $M^{\sum_i n_i} = \prod_i M^{n_i}$ and in $\prod_i X_i^{n_i}$, and the map $\psi_{\{n_i\}} : ((M^{\sum_i n_i} \setminus \Delta) \times \prod_i X_i^{n_i})/ \prod_i S_{n_i} \to \mathscr{A}$ is defined by

$$\psi_{\{n_i\}}(\{y_i^j\}, \{x_i^j\}) = \sum_i (i \cdot \psi(y_i^j) + \psi_i(x_i^j)) ,$$

where y_i^j and x_i^j, $j = 1, \ldots, n_i$, are the jth components of the point in M^{n_i} and in $X_i^{n_i}$ respectively (cf. [10, Eq. (1)]); a similar construction for the Grothendieck ring of quasi-projective varieties with maps to an abelian manifold was introduced in [18].

The ring $R[z_1, \ldots, z_n]$ of polynomials in $z_1, \ldots, z_n$ with the coefficients from a λ-ring R carries a natural λ-structure: see, e.g., [16]. The same holds for the ring $R[z_1^{1/m}, \ldots, z_n^{1/m}]$ of fractional power polynomials in $z_1, \ldots, z_n$. In terms of the corresponding power structure, one can write

$$(1-T)^{-\sum_{\underline{k}} a_{\underline{k}} \underline{z}^{\underline{k}}} = \prod_{\underline{k}} \lambda_{a_{\underline{k}}}(\underline{z}^{\underline{k}} T)\,,$$

where $\underline{z} = (z_1, \ldots, z_n)$, $\underline{k} = (k_1, \ldots, k_n)$, $\underline{z}^{\underline{k}} = z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$.

The ring $R(G)$ of representations of a group G is regarded as a λ-ring with the λ-structure defined by

$$\lambda_{[\omega]}(T) = 1 + [\omega]t + [S^2\omega]T^2 + [S^3\omega]T^3 + \ldots\,,$$

where ω is a representation of G and $S^n\omega$ is its nth symmetric power.

3 The Spectrum and the Equivariant Hodge–Deligne Polynomial

Let V be a complex quasi-projective variety with an automorphism φ of finite order. For a rational α, $0 \le \alpha < 1$, let $H_\alpha^k(V)$ be the subspace of $H^k(V) = H_c^k(V; \mathbb{C})$ (the cohomology group with compact support) consisting of the eigenvectors of φ_* with the eigenvalue $\mathbf{e}[\alpha] := \exp(2\pi\alpha i)$. The subspace $H_\alpha^k(V)$ carries a natural complex mixed Hodge structure.

Definition 1 (See, e.g., [5]) The *(Hodge) spectrum* $\mathrm{hsp}(V, \varphi)$ of the pair (V, φ) is defined by

$$\mathrm{hsp}(V,\varphi) = \sum_{k,p,q,\alpha} (-1)^k \dim(H_\alpha^k(V))^{p,q} \cdot \{p+\alpha\} \in \mathbb{Z}[\mathbb{Q}]\,.$$

The spectrum $\mathrm{hsp}(V, \varphi)$ can be identified either with the fractional power polynomial (Poincaré polynomial)

$$p_{(V,\varphi)}(t) = \sum_{k,p,q,\alpha} (-1)^k \dim(H_\alpha^k(V))^{p,q} \cdot t^{p+\alpha} \in \mathbb{Z}[t^{1/m}]$$

or with the equivariant Poincaré polynomial

$$\overline{e}_{(V,\varphi)}(t) = \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V))^{p,q} \omega_{\mathbf{e}[\alpha]} \cdot t^p \in R_f(\mathbb{Z})[t]\,,$$

where $R_f(Z)$ is the ring of finite order representations of the cyclic group $\mathbb{Z}$ and $\omega_{\mathbf{e}[\alpha]}$ is the one-dimensional representation of $\mathbb{Z}$ with the character equal to $\mathbf{e}[\alpha]$ at 1. Both rings $\mathbb{Z}[t^{1/m}]$ and $R_f(\mathbb{Z})[t]$ carry natural λ-structures and thus power structures. However, the natural power structure over $\mathbb{Z}[t^{1/m}]$ is not compatible with the multiplication of spaces: the map

$$p_\bullet : K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[t^{1/m}]$$

is not a ring homomorphism. Therefore, a natural Macdonald-type equation for the spectrum is formulated in terms of the equivariant Poincaré polynomial $\overline{e}_{(V,\varphi)}(t)$. Moreover, a stronger statement can be formulated in terms of the equivariant Hodge–Deligne polynomial of the pair (V, φ).

Definition 2 ([6], see also [19]) The *equivariant Hodge–Deligne polynomial* of the pair (V, φ) is

$$e_{(V,\varphi)}(u, v) = \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V))^{p,q} \omega_{\mathbf{e}[\alpha]} \cdot u^p v^q \in R_f(\mathbb{Z})[u, v]\,,$$

One has $\overline{e}_{(V,\varphi)}(t) = e_{(V,\varphi)}(t, 1)$.

Let S^nV be the nth symmetric power of the variety V. The transformation $\varphi : V \to V$ defines a transformation $\varphi^{(n)} : S^nV \to S^nV$ in a natural way.

Theorem 1 *One has*

$$1 + e_{(V,\varphi)}(u, v)T + e_{(S^2V,\varphi^{(2)})}(u, v)T^2 + e_{(S^3V,\varphi^{(3)})}(u, v)T^3 + \ldots = (1 - T)^{-e_{(V,\varphi)}(u,v)}\,, \qquad (1)$$

where the RHS of (1) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[u, v]$.

The *proof* is essentially contained in [4] where J. Cheah proved an analogue of (1) for the usual (non-equivariant) Hodge–Deligne polynomial. Theorem 1 can be deduced from the arguments of Cheah [4] by taking care of different eigenspaces.

Theorem 1 means that the natural map $e_\bullet$ from $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})$ to $R_f(\mathbb{Z})[u, v]$ is a λ-ring homomorphism.

Corollary 1 *One has*

$$1 + \overline{e}_{(V,\varphi)}(t)T + \overline{e}_{(S^2V,\varphi^{(2)})}(t)T^2 + \overline{e}_{(S^3V,\varphi^{(3)})}(t)T^3 + \ldots = (1 - T)^{-\overline{e}_{(V,\varphi)}(t)}\,, \qquad (2)$$

where the RHS of (2) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[t]$.

4 The Orbifold Spectrum and the Equivariant Orbifold Hodge–Deligne Polynomial

Let X be a topological G-space and G a finite group. Let G_* be the set of conjugacy classes of elements of G. For an element $g \in G$, let $X^{\langle g\rangle} = \{x \in X : gx = x\}$ be the fixed point set of g, and let $C_G(g) = \{h \in G : h^{-1}gh = g\} \subset G$ be the centralizer of g. The group $C_G(g)$ acts on the fixed point set $X^{\langle g\rangle}$. The orbifold Euler characteristic $\chi^{\text{orb}}(X, G)$ can be defined by

$$\chi^{\text{orb}}(X, G) = \sum_{[g]\in G_*} \chi(X^{\langle g\rangle}/C_G(g)) .$$

Refinements of this notion taking into account the mixed Hodge structure on the cohomology groups use the so-called ages of elements of the group as shifts of the corresponding graded components of the mixed Hodge structure (see, e.g., [2, 22]).

Let V be a complex quasi-projective manifold of dimension d with an action of a finite group G and with a G-equivariant automorphism φ of finite order. One can say that the notion of the orbifold spectrum of the triple (V, G, φ) is inspired by the notion of the orbifold Hodge–Deligne polynomial: [2].

Let G_*, $V^{\langle g\rangle}$ and $C_G(g)$ be defined as above. The group $C_G(g)$ acts on the fixed point set $V^{\langle g\rangle}$. Let $\hat{\varphi}$ be the transformation of the quotient $V^{\langle g\rangle}/C_G(g)$ induced by φ. For a point $x \in V^{\langle g\rangle}$, the *age* of g (or *fermion shift number*) is defined in the following way [15, Sect. 2.1], [23, Eq. (3.17)]. The element g acts on the tangent space T_xV as a complex linear operator of finite order. It can be represented by a diagonal matrix with the diagonal entries $\mathbf{e}[\beta_1], \dots, \mathbf{e}[\beta_d]$, where $0 \le \beta_i < 1$ for $i = 1, \dots, d$ and $\mathbf{e}[r] := \exp(2\pi i r)$ for a real number r. The *age* of the element g at the point x is defined by $\text{age}_x(g) = \sum_{i=1}^{d} \beta_i \in \mathbb{Q}_{\ge 0}$. For a rational number $\beta \ge 0$, let $V_\beta^{\langle g\rangle}$ be the subspace of the fixed point set $V^{\langle g\rangle}$ consisting of the points x with $\text{age}_x(g) = \beta$. (The subspace $V_\beta^{\langle g\rangle}$ of $V^{\langle g\rangle}$ is a union of connected components of the latter one.)

Definition 3 (cf. [9]) The *orbifold spectrum* of the triple (V, G, φ) is

$$\text{hsp}^{\text{orb}}(V, G, \varphi) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\ge 0}} \text{hsp}(V_\beta^{\langle g\rangle}/C_G(g), \hat{\varphi}) \cdot \{\beta\} \in \mathbb{Z}[\mathbb{Q}] .$$

As above the spectrum $\text{hsp}^{\text{orb}}(V, G, \varphi)$ can be identified with the orbifold Poincaré polynomial

$$p^{\text{orb}}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\ge 0}} p_{(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi})}(t) \cdot t^\beta \in \mathbb{Z}[t^{1/m}] .$$

It can be regarded as a reduction of the equivariant orbifold Poincaré polynomial

$$\overline{e}^{\mathrm{orb}}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \overline{e}_{(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi})}(t)\cdot t^\beta \in R_f(\mathbb{Z})[t^{1/m}]$$

or of the equivariant orbifold Hodge–Deligne polynomial

$$e^{\mathrm{orb}}_{(V,\varphi)}(u,v) = \sum_{k,p,q,\alpha,[g],\beta} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g)))^{p,q}\omega_{\mathbf{e}[\alpha]}\cdot u^p v^q (uv)^\beta$$

(an element of $R_f(\mathbb{Z})[u,v][(uv)^{1/m}]$).

As it was explained above, the presence of (rational) summands of different nature—elements of the quotient $\mathbb{Q}/\mathbb{Z}$ and elements of $\mathbb{Q}$ itself—leads to the situation when the existence of a Macdonald-type equation for the orbifold spectrum (and for the orbifold Poincaré polynomial) is doubtful. On the other hand, there exist Macdonald-type equations for the equivariant orbifold Poincaré polynomial and for the equivariant orbifold Hodge–Deligne polynomial (see Sect. 5). This inspires the definition of the corresponding version of the orbifold spectrum.

Definition 4 The *orbifold pair spectrum* $\mathrm{hsp}_2^{\mathrm{orb}}(V,G,\varphi)$ of (V,G,φ) is

$$\sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi}))^{p,q}\{(\alpha,p+\beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}]\,.$$

The word *pair* is used, in particular, to distinguish this notion from the one defined in [9]. Moreover, taking into account the weight filtration as well, one gets a certain refinement of this notion.

Definition 5 The *orbifold triple spectrum* $\mathrm{hsp}_3^{\mathrm{orb}}(V,G,\varphi)$ of (V,G,φ) is

$$\sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \sum_{k,p,q,\alpha} (-1)^k \dim(H^k_\alpha(V_\beta^{\langle g\rangle}/C_G(g),\hat{\varphi}))^{p,q}\{(\alpha,p+\beta,q+\beta)\}$$

(an element of $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}\times\mathbb{Q}]$).

5 Higher-Order Spectrum and Equivariant Hodge–Deligne Polynomial

The notions of the higher-order spectrum of a triple (V,G,φ) and of the higher-order equivariant Hodge–Deligne polynomial of it are inspired by the notions of the higher-order Euler characteristic [1, 3] and of the corresponding higher-order generalized Euler characteristic [12]. For a topological G-space X, the Euler characteristic χ of order k can be defined by

$$\chi^{(k)}(X,G) = \sum_{[g]\in G_*} \chi^{(k-1)}(X^{\langle g\rangle}, C_G(g)),$$

where $\chi^{(0)}(X,G) := \chi(X/G)$ (see the notations in Sect. 4). One can see that $\chi^{(1)}(X,G) := \chi^{orb}(X,G)$. As for the orbifold Euler characteristic (i.e., for the Euler characteristic of order 1), refinements of these notions taking into account the mixed Hodge structure should use the age shift.

Let (V,G,φ), $V_\beta^{\langle g\rangle}$ and $\hat{\varphi}$ be as in Sect. 4 and let $k \geq 1$.

Definition 6 The *spectrum of order k* of the triple (V,G,φ) is

$$\mathrm{hsp}^{(k)}(V,G,\varphi) = \sum_{[g]\in G_*}\sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g),\varphi)\cdot\{\beta\} \in \mathbb{Z}[\mathbb{Q}],$$

where $\mathrm{hsp}^{(0)}(V,G,\varphi) := \mathrm{hsp}(V/G,\hat{\varphi})$.

The orbifold spectrum is the spectrum of order 1.

Like above the spectrum of order k can be described by the corresponding order k Poincaré polynomial:

$$p^{(k)}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*}\sum_{\beta\in\mathbb{Q}_{\geq 0}} p^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(t)\cdot t^\beta \in \mathbb{Z}[t^{1/m}],$$

where $p^{(1)}_\bullet(t) := p^{\mathrm{orb}}_\bullet(t)$.

It can be regarded as a reduction of the *equivariant order k Poincaré polynomial*

$$\overline{e}^{(k)}_{(V,G,\varphi)}(t) = \sum_{[g]\in G_*}\sum_{\beta\in\mathbb{Q}_{\geq 0}} \overline{e}^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(t)\cdot t^\beta \in R_f(\mathbb{Z})[t^{1/m}]$$

or of the *equivariant order k Hodge–Deligne polynomial*

$$e^{(k)}_{(V,G,\varphi)}(u,v) = \sum_{[g]\in G_*}\sum_{\beta\in\mathbb{Q}_{\geq 0}} e^{(k-1)}_{(V_\beta^{\langle g\rangle}, C_G(g),\varphi)}(u,v)(uv)^\beta \in R_f(\mathbb{Z})[u,v][(uv)^{1/m}].$$

The following definition is an analogue of the definition of the orbifold pair and triple spectra in Sect. 4.

Definition 7 The *pair spectrum of order k* of (V,G,φ) is

$$\mathrm{hsp}_2^{(k)}(V,G,\varphi) = \sum_{[g]\in G_*}\sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}_2^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g),\varphi)\{(0,\beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})\times\mathbb{Q}].$$

The *triple spectrum of order k* of (V, G, φ) is

$$\mathrm{hsp}_3^{(k)}(V, G, \varphi) = \sum_{[g]\in G_*} \sum_{\beta\in\mathbb{Q}_{\geq 0}} \mathrm{hsp}_3^{(k-1)}(V_\beta^{\langle g\rangle}, C_G(g), \varphi)\{(0, \beta, \beta)\} \in \mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}] .$$

The following statement is a Macdonald-type equation for the equivariant order k Hodge–Deligne polynomial. For $n \geq 1$, the Cartesian power V^n of the manifold V is endowed with the natural action of the wreath product $G_n = G \wr S_n = G^n \rtimes S_n$ generated by the componentwise action of the Cartesian power G^n and by the natural action of the symmetric group S_n (permuting the factors). Also one has the automorphism $\varphi^{(n)}$ of V^n induced by φ. The triple $(V_n, G_n, \varphi^{(n)})$ should be regarded as an analogue of the symmetric power of the triple (V, G, φ).

Example 1 Let $f(z_1, \ldots, z_n)$ be a quasi-homogeneous function with the quasi-weights $q_1, \ldots, q_n$ (and with the quasi-degree 1), and let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group of its symmetries ($f(g\underline{z}) = f(\underline{z})$ for $g \in G$). The Milnor fiber $M_f = \{f = 1\}$ of f is an $(n-1)$-dimensional complex manifold with an action of a group G and with a natural finite-order automorphism φ (the monodromy transformation or the exponential grading operator):

$$\varphi(z_1, \ldots, z_n) = (\mathbf{e}[q_1]z_1, \ldots, \mathbf{e}[q_n]z_n) .$$

For $s \geq 1$, let $\mathbb{C}^{ns} = (\mathbb{C}^n)^s$ be the affine space with the coordinates $z_i^{(j)}$, $1 \leq i \leq n$, $1 \leq j \leq s$. The system of equations $f(z_1^{(j)}, \ldots, z_n^{(j)}) = 0$, $j = 1, \ldots, s$, defines a complete intersection in $\mathbb{C}^{ns}$. Its Milnor fiber $M = \{f(z_1^{(j)}, \ldots, z_n^{(j)}) = 1,\ \text{for}\ j = 1, \ldots, s\}$ is the sth Cartesian power of the Milnor fiber M_f of f and has a natural action of the wreath product G_s. The spectrum of a complete intersection singularity is defined by a choice of a monodromy transformation. A natural monodromy transformation on M is the sth Cartesian power $\varphi^{(s)}$ of the monodromy transformation φ. Thus, the triple $(M, G_s, \varphi^{(s)})$ can be regarded as an analogue of the sth symmetric power of the triple (M_f, G, φ).

Theorem 2 *Let V be a (smooth) quasi-projective G-manifold of dimension d with a G-equivariant automorphism φ of finite order. One has*

$$1 + \sum_{n=1}^{\infty} e^{(k)}_{(V^n, G_n, \varphi^{(n)})}(u, v) \cdot T^n = \left(\prod_{r_1,\ldots,r_k\geq 1} \left(1 - (uv)^{(r_1 r_2 \cdots r_k)d/2} \cdot T^{r_1 r_2 \cdots r_k}\right)^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-e^{(k)}_{(V,G,\varphi)}(u,v)}, \qquad (3)$$

where the RHS of (3) is understood in terms of the power structure over the ring $R_f(\mathbb{Z})[u, v][(uv)^{1/m}]$.

Proof In [13], there were defined equivariant generalized higher-order Euler characteristics of a complex quasi-projective manifold with commuting actions of two finite groups G_O and G_B as elements of the extension $K_0^{G_B}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ of the Grothendieck ring $K_0^{G_B}(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective G_B-varieties ($\mathbb{L}$ is the class of the complex affine line with the trivial action), and there were given Macdonald-type equations for them: [13, Theorem 2]. One can see that these definitions and the Macdonald-type equations can be applied when instead of an action of a finite group G_B, one has a finite order action of the cyclic group $\mathbb{Z}$. The equivariant order k Hodge–Deligne polynomial is the image of the equivariant generalized Euler characteristic of order k under the map $K_0^{\mathbb{Z}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}] \to R_f(\mathbb{Z})[u, v][(uv)^{1/m}]$. Since this map is a λ-ring homomorphism (Theorem 1), the Macdonald-type equation for the equivariant generalized Euler characteristic of order k implies (3).

Corollary 2 *In the situation described above, one has*

$$1 + \sum_{n=1}^{\infty} \mathrm{hsp}_{\nu}^{(k)}(V^n, G_n, \varphi^{(n)}) \cdot T^n$$

$$= \left(\prod_{r_1,\dots,r_k \geq 1} \left(1 - a_{r_1,\dots,r_k}^{(\nu)} T^{r_1 r_2 \cdots r_k}\right)^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-\mathrm{hsp}_{\nu}^{(k)}(V,G,\varphi)},$$

where $\nu = 2, 3$*,*

$$a_{r_1,\dots,r_k}^{(2)} = \{(0, (r_1 r_2 \cdots r_k)d/2)\},$$
$$a_{r_1,\dots,r_k}^{(3)} = \{(0, (r_1 r_2 \cdots r_k)d/2, (r_1 r_2 \cdots r_k)d/2)\},$$

and the RHS is understood in terms of the power structures over the group rings $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}]$ *for* $\nu = 2$ *and* $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q}]$ *for* $\nu = 3$.

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μ-Constant Monodromy Groups and Torelli Results for Marked Singularities, for the Unimodal and Some Bimodal Singularities

Falko Gauss and Claus Hertling

Abstract This paper is a sequel to Hertling (Ann Inst Fourier (Grenoble) 61(7):2643–2680, 2011). There a notion of marking of isolated hypersurface singularities was defined, and a moduli space M_μ^{mar} for marked singularities in one μ-homotopy class of isolated hypersurface singularities was established. One can consider it as a global μ-constant stratum or as a Teichmüller space for singularities. It comes together with a μ-constant monodromy group $G^{mar} \subset G_{\mathbb{Z}}$. Here $G_{\mathbb{Z}}$ is the group of automorphisms of a Milnor lattice which respect the Seifert form. It was conjectured that M_μ^{mar} is connected. This is equivalent to $G^{mar} = G_{\mathbb{Z}}$. Also Torelli-type conjectures were formulated. All conjectures were proved for the simple singularities and 22 of the exceptional unimodal and bimodal singularities. In this paper, the conjectures are proved for the remaining unimodal singularities and the remaining exceptional bimodal singularities.

Keywords μ-Constant monodromy group • Hyperbolic singularities • Marked singularity • Moduli space • Simple elliptic singularities • Torelli-type problem

1 Introduction

This paper is a sequel to [12]. That paper studied local objects, namely, holomorphic functions germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 (short: singularity), from a global perspective.

There a notion of marking of a singularity was defined. One has to fix one singularity f_0, which serves as reference singularity. Then a marked singularity is a pair $(f, \pm\rho)$ where f is in the μ-homotopy class of f_0 and $\rho : (Ml(f), L) \to (Ml(f_0), L)$ is an isomorphism. Here $Ml(f)$ is the Milnor lattice of f, and L is the

F. Gauss • C. Hertling (✉)
Lehrstuhl für Mathematik VI, Universität Mannheim, Seminargebäude A 5, 6, 68131 Mannheim, Germany
e-mail: gauss@math.uni-mannheim.de; hertling@math.uni-mannheim.de

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Seifert form on it (the definitions are recalled in Sect. 2). The group $G_{\mathbb{Z}}(f_0) := \mathrm{Aut}(Ml(f_0), L)$ will be important, too.

A moduli space M_μ^{mar} of right equivalence classes of marked singularities in one μ-homotopy class was established. One can consider it as a global μ-constant stratum or as a Teichmüller space for singularities. The group $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on it, and the quotient is the moduli space M_μ of right equivalence classes of unmarked singularities from [11, Chap. 13]. The *μ-constant monodromy group* $G^{mar}(f_0) \subset G_{\mathbb{Z}}(f_0)$ is the subgroup of automorphisms which map the topological component $(M_\mu^{mar}(f_0))^0$ which contains $[f_0, \mathrm{id}]$ to itself. It can also be constructed as the group of all automorphisms which can be realized modulo $\pm\,\mathrm{id}$ as transversal monodromies of μ-constant families.

Conjecture 1.1 ([12, Conjecture 3.2 (a)]) M_μ^{mar} is connected. Equivalently: $G^{mar} = G_{\mathbb{Z}}$.

Roughly this conjecture says that all abstract automorphisms come from geometry, from coordinate changes, and μ-constant families.

Also Torelli-type conjectures are formulated in [12]. Any singularity f comes equipped with its *Brieskorn lattice* $H_0''(f)$, and any marked singularity $(f, \pm\rho)$ comes equipped with a *marked Brieskorn lattice* $BL(f, \pm\rho)$. The Gauß-Manin connection for singularities and the Brieskorn lattices had been introduced in 1970 by Brieskorn and had been studied since then. The second author has a long-going project on Torelli-type conjectures around them.

In [9] a classifying space $D_{BL}(f_0)$ for marked Brieskorn lattices was constructed. It is especially a complex manifold, and $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on it. The quotient $D_{BL}/G_{\mathbb{Z}}$ is a space of isomorphism classes of Brieskorn lattices. There is a natural holomorphic period map

$$BL : M_\mu^{mar}(f_0) \to D_{BL}(f_0).$$

It is an immersion [11, Theorem 12.8] (this refines a weaker result in [21]). And it is $G_{\mathbb{Z}}$-equivariant.

Conjecture 1.2

(a) [12, Conjecture 5.3] BL is an embedding.
(b) [12, Conjecture 5.4], [11, Conjecture 12.7], [6, Kap. 2 (d)]) $LBL : M_\mu = M_\mu^{mar}/G_{\mathbb{Z}} \to D_{BL}/G_{\mathbb{Z}}$ is an embedding.

Part (b) says that the right equivalence class of f is determined by the isomorphism class of $H_0''(f)$. Part (a) would imply part (b). Both are global Torelli-type conjectures. Part (b) was proved in [6] for all simple and unimodal singularities and almost all bimodal singularities, all except three subseries of the eight bimodal series. Therefore, for the proof of part (a), in these cases it remains mainly to control $G_{\mathbb{Z}}$ well. But that is surprisingly difficult.

In [12] the Conjectures 1.1 and 1.2 were proved for all simple and those 22 of the 28 exceptional unimodal and bimodal singularities, where all eigenvalues of the monodromy have multiplicity one. In this paper, they will be proved for the remaining unimodal and exceptional bimodal singularities, that means, for the simple elliptic and the hyperbolic singularities and for those 6 of the 28 exceptional unimodal and bimodal singularities which had not been treated in [12].

A priori, logically Conjecture 1.1 comes before Conjecture 1.2. But the results in [6] are more concrete and give already some information about the action of $G_{\mathbb{Z}}$ on D_{BL} and M_μ^{mar}. Anyway, the main remaining work is a good control of the groups $G_{\mathbb{Z}}$. That presents some unexpected difficulties. For example, we need two surprising generalizations of the number theoretic fact $\mathbb{Z}[e^{2\pi i/a}] \cap S^1 = \{\pm e^{2\pi ik/a} \mid k \in \mathbb{Z}\}$ for $a \in \mathbb{N}$: one is Lemma 2.5, and the other is related to U_{16}; see Remark 4.3.

The groups $G_{\mathbb{Z}}(f_0)$ will be calculated in Sects. 3 and 4. Section 2 collects well-known background material on the topology of singularities. But it contains also an algebraic Lemma 2.5 about automorphisms of monodromy modules. Section 5 collects notions and results from [12] on marked singularities, the moduli spaces $M_\mu^{mar}(f_0)$, the groups $G^{mar}(f_0)$, and the Torelli-type conjectures. Sections 6 and 7 give the proofs of the Conjectures 1.1 and 1.2 in the cases considered. Section 8 is motivated by the paper [17] of Milanov and Shen and complements their results on (transversal) monodromy groups for certain families of simple elliptic singularities. The three principal congruence subgroups $\Gamma(3)$, $\Gamma(4)$, and $\Gamma(6)$, which turn up in [17] for certain families are shown to turn up also in the biggest possible families.

2 Review on the Topology of Isolated Hypersurface Singularities

First, we recall some classical facts and fix some notations. An *isolated hypersurface singularity* (short: *singularity*) is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. Its *Milnor number*

$$\mu := \dim \mathscr{O}_{\mathbb{C}^{n+1},0}/(\frac{\partial f}{\partial x_i})$$

is finite. For the following notions and facts, compare [2, 4] and (for the notion of an unfolding) [1]. A *good representative* of f has to be defined with some care [2, 4, 18]. It is $f : Y \to T$ with $Y \subset \mathbb{C}^{n+1}$ a suitable small neighborhood of 0 and $T \subset \mathbb{C}$ a small disk around 0. Then $f : Y' \to T'$ with $Y' = Y - f^{-1}(0)$ and $T' = T - \{0\}$ is a locally trivial C^∞-fibration, the *Milnor fibration*. Each fiber has the homotopy type of a bouquet of μn-spheres [18].

Therefore, the (reduced for $n = 0$) middle homology groups are $H_n^{(red)}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ for $\tau \in T'$. Each comes equipped with an intersection form I, which is a datum of one fiber, a monodromy M_h and a Seifert form L, which come from the Milnor fibration; see [2, I.2.3] for their definitions (for the Seifert form, there are several conventions in the literature; we follow [2]). M_h is a quasiunipotent automorphism, I and L are bilinear forms with values in $\mathbb{Z}$, I is $(-1)^n$-symmetric, and L is unimodular.

L determines M_h and I because of the formulas [2, I.2.3]

$$L(M_h a, b) = (-1)^{n+1} L(b, a), \tag{1}$$

$$I(a, b) = -L(a, b) + (-1)^{n+1} L(b, a). \tag{2}$$

The Milnor lattices $H_n(f^{-1}(\tau), \mathbb{Z})$ for all Milnor fibrations $f : Y' \to T'$ and then all $\tau \in \mathbb{R}_{>0} \cap T'$ are canonically isomorphic, and the isomorphisms respect M_h, I, and L. This follows from Lemma 2.2 in [15]. These lattices are identified and called *Milnor lattice* $Ml(f)$. The group $G_{\mathbb{Z}}$ is

$$G_{\mathbb{Z}} = G_{\mathbb{Z}}(f) := \mathrm{Aut}(Ml(f), L) = \mathrm{Aut}(Ml(f), M_h, I, L), \tag{3}$$

the second equality is true because L determines M_h and I. We will use the notation $Ml(f)_{\mathbb{C}} := Ml(f) \otimes_{\mathbb{Z}} \mathbb{C}$, and analogously for other rings R with $\mathbb{Z} \subset R \subset \mathbb{C}$, and the notations

$$\begin{aligned}
Ml(f)_\lambda &:= \ker((M_h - \lambda\,\mathrm{id})^\mu : Ml(f)_{\mathbb{C}} \to Ml(f)_{\mathbb{C}}) \subset Ml(f)_{\mathbb{C}}, \\
Ml(f)_{1,\mathbb{Z}} &:= Ml(f)_1 \cap Ml(f) \subset Ml(f), \\
Ml(f)_{\neq 1} &:= \bigoplus_{\lambda \neq 1} Ml(f)_\lambda \subset Ml(f)_{\mathbb{C}}, \\
Ml(f)_{\neq 1,\mathbb{Z}} &:= Ml(f)_{\neq 1} \cap Ml(f) \subset Ml(f).
\end{aligned}$$

The formulas (1) and (2) show $I(a, b) = L((M_h - \mathrm{id})a, b)$. Therefore, the eigenspace with eigenvalue 1 of M_h is the radical $\mathrm{Rad}(I) \subset Ml(f)$ of I. By (2) L is $(-1)^{n+1}$-symmetric on the radical of I.

In the case of a curve singularity ($n = 1$) with r branches, $f = \prod_{j=1}^r f_j$, the radical of I is a $\mathbb{Z}$-lattice of rank $r - 1$, and it is generated by the classes $l_j \in Ml(f)$ which are obtained by pushing the (correctly oriented) cycles $\partial Y \cap f_j^{-1}(0)$ from the boundary of the fiber $f^{-1}(0)$ to the boundary of the fiber $f^{-1}(\tau)$. Then

$$l_1 + \ldots + l_r = 0, \tag{4}$$

$$L(l_i, l_j) = \text{intersection number of } (f_i, f_j) \quad \text{for } i \neq j,$$

$$\text{so } L(l_i, l_j) > 0 \quad \text{for } i \neq j, \tag{5}$$

$$l_1, \ldots, \widehat{l_j}, \ldots, l_r \quad \text{is a } \mathbb{Z}\text{-basis of } \mathrm{Rad}(I) \text{ for any } j. \tag{6}$$

Kaenders proved the following result using Selling reduction. It will be useful for the calculation of $\mathrm{Aut}(\mathrm{Rad}(I), L)$, because it implies that any automorphism of $(\mathrm{Rad}(I), L)$ maps the set $\{l_1, \ldots, l_r\}$ to itself or to minus itself.

Theorem 2.1 ([13]) *In the case of a curve singularity as above, the set* $\{l_1, \ldots, l_r\}$ *is up to a common sign uniquely determined by the properties* (4)–(6). *So it is up* *to a common sign determined by the pair* $(\mathrm{Rad}(I), L)$. *Furthermore, L is negative definite on* $\mathrm{Rad}(I)$.

Example 2.2 In the following three examples $\underline{l} = (l_1, \ldots, l_r)$, and $L(\underline{l}^t, \underline{l}) = (L(l_i, l_j))$. Then $(L(l_i, l_j))_{1 \le i,j \le r-1}$ is the matrix of L on $\mathrm{Rad}(I)$ with respect to the $\mathbb{Z}$-basis $l_1, \ldots, l_{r-1}$. The examples (ii) and (iii) will be useful in Sect. 4, and the example (i) is an alternative to a calculation in the proof of Theorem 8.4 in [12].

(i) $D_{2m} : f = x^{2m-1} + xy^2 = x(x^{m-1} - iy)(x^{m-1} + iy)$,

$$L(\underline{l}^t, \underline{l}) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -m & m-1 \\ 1 & m-1 & -m \end{pmatrix}, \qquad (L(l_i, l_j))_{1 \le i,j \le 2} = \begin{pmatrix} -2 & 1 \\ 1 & -m \end{pmatrix}.$$

Obviously $|\operatorname{Aut}(\mathrm{Rad}(I), L)| = 12$ in the case $m = 2$, and $|\operatorname{Aut}(\mathrm{Rad}(I), L)| = 4$ in the cases $m \ge 3$.

(ii) $Z_{12} : f = x^3y + xy^4 = xy(x^2 + y^3)$,

$$L(\underline{l}^t, \underline{l}) = \begin{pmatrix} -4 & 1 & 3 \\ 1 & -3 & 2 \\ 3 & 2 & -5 \end{pmatrix}, \qquad (L(l_i, l_j))_{1 \le i,j \le 2} = \begin{pmatrix} -4 & 1 \\ 1 & -3 \end{pmatrix}.$$

Obviously $\operatorname{Aut}(\mathrm{Rad}(I), L) = \{\pm \mathrm{id}\}$.

(iii) $Z_{18} : f = x^3y + xy^6 = xy(x^2 + y^5)$,

$$L(\underline{l}^t, \underline{l}) = \begin{pmatrix} -6 & 1 & 5 \\ 1 & -3 & 2 \\ 5 & 2 & -7 \end{pmatrix}, \qquad (L(l_i, l_j))_{1 \le i,j \le 2} = \begin{pmatrix} -6 & 1 \\ 1 & -3 \end{pmatrix}.$$

Obviously $\operatorname{Aut}(\mathrm{Rad}(I), L) = \{\pm \mathrm{id}\}$.

Finally, in Sect. 3, distinguished bases will be used. Again, good references for them are [2] and [4]. We sketch their construction and properties.

One can choose a *universal unfolding* of f, a *good representative* F of it with base space $M \subset \mathbb{C}^\mu$, and a generic parameter $t \in M$. Then $F_t : Y_t \to T$ with $T \subset \mathbb{C}$ the same disk as above and $Y_t \subset \mathbb{C}^{n+1}$ is a *morsification* of f. It has μA_1-singularities, and their critical values $u_1, \ldots, u_\mu \in T$ are pairwise different. Their numbering is also a choice. Now choose a value $\tau \in T \cap \mathbb{R}_{>0} - \{u_1, \ldots, u_\mu\}$ and a *distinguished system of paths*. That is, a system of μ paths $\gamma_j, j = 1, \ldots, \mu$, from u_j to τ which do not intersect except at τ and which arrive at τ in clockwise order. Finally, shift from the A_1 singularity above each value u_j the (up to sign unique) vanishing cycle along γ_j to the Milnor fiber $Ml(f) = H_n(f^{-1}(\tau), \mathbb{Z})$, and call the image δ_j.

The tuple $(\delta_1, \ldots, \delta_\mu)$ forms a $\mathbb{Z}$-basis of $Ml(f)$. All such bases are called *distinguished bases*. They form one orbit of an action of a semidirect product $\mathrm{Br}_\mu \ltimes \{\pm 1\}^\mu$. Here Br_μ is the braid group with μ strings; see [2] or [4] for its action.

The *sign change group* $\{\pm 1\}^\mu$ acts simply by changing the signs of the entries of the tuples $(\delta_1, \dots, \delta_\mu)$. The members of the distinguished bases are called *vanishing cycles*.

The *Stokes matrix* S of a distinguished basis is defined as the upper triangular matrix in $M(\mu \times \mu, \mathbb{Z})$ with 1s on the diagonal and with

$$S_{ij} := (-1)^{n(n+1)/2} \cdot I(\delta_i, \delta_j) \quad \text{for all } i, j \text{ with } i < j.$$

The *Coxeter-Dynkin diagram* of a distinguished basis encodes S in a geometric way. It has μ vertices which are numbered from 1 to μ. Between two vertices i and j with $i < j$, one draws

no edge	if $S_{ij} = 0$,
$\lvert S_{ij}\rvert$ edges	if $S_{ij} < 0$,
S_{ij} dotted edges	if $S_{ij} > 0$.

Coxeter-Dynkin diagrams of many singularities were calculated by A'Campo, Ebeling, Gabrielov, and Gusein-Zade. Some of them can be found in [3, 5] and [4].

Example 2.3 The hyperbolic singularities of type T_{pqr} with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and the simple elliptic singularities of types $\widetilde{E}_6 = T_{333}, \widetilde{E}_7 = T_{442}$, and $\widetilde{E}_8 = T_{632}$ have distinguished bases with the Coxeter-Dynkin diagrams in Fig. 1 [5].

The Picard-Lefschetz transformation on $Ml(f)$ of a vanishing cycle δ is

$$s_\delta(b) := b - (-1)^{n(n+1)/2} \cdot I(\delta, b) \cdot \delta. \tag{7}$$

The monodromy M_h is

$$M_h = s_{\delta_1} \circ \dots \circ s_{\delta_\mu} \tag{8}$$

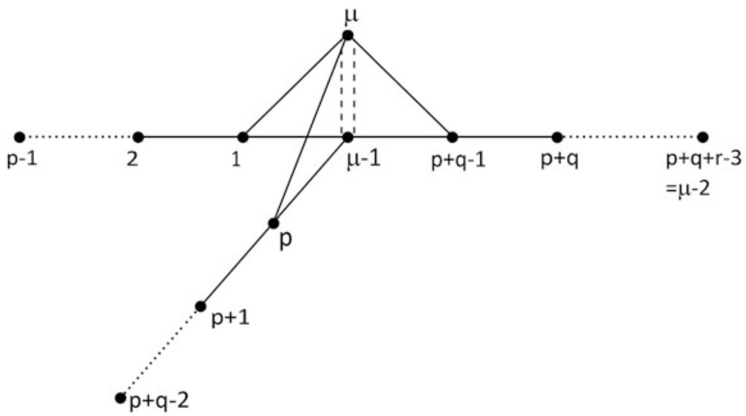

Fig. 1 A Coxeter-Dynkin diagram of the singularities of type T_{pqr}

for any distinguished basis $\underline{\delta} = (\delta_1, \dots, \delta_\mu)$. The matrices of the monodromy, Seifert form, and intersection form with respect to a distinguished basis $\underline{\delta}$ are given by the following formulas:

$$M_h(\underline{\delta}) = \underline{\delta} \cdot (-1)^{n+1} \cdot S^{-1} S^t, \tag{9}$$

$$I(\underline{\delta}^t, \underline{\delta}) = (-1)^{n(n+1)/2} \cdot (S + (-1)^n S^t), \tag{10}$$

$$L(\underline{\delta}^t, \underline{\delta}) = (-1)^{(n+1)(n+2)/2} \cdot S^t. \tag{11}$$

Remark 2.4 The Stokes matrix S of a distinguished basis is related to the matrix V in [4, Korollar 5.3 (i)] by the formula $V = L(\underline{\delta}^t, \underline{\delta}) = (-1)^{(n+1)(n+2)/2} \cdot S^t$. Thus, V is *lower* triangular, not *upper* triangular, contrary to the claim in [4, Korollar 5.3 (i)]. The matrix of Var^{-1} for a distinguished basis $\underline{\delta}$ and its dual basis $\underline{\delta}^*$ is $(-1)^{(n+1)(n+2)/2} S$, namely,

$$\mathrm{Var}^{-1}(\underline{\delta}) = \underline{\delta}^* \cdot (-1)^{(n+1)(n+2)/2} \cdot S.$$

But then the matrix for the Seifert form L with $L(a, b) = (\mathrm{Var}^{-1}(a))(b)$ is

$$L(\underline{\delta}^t, \underline{\delta}) = \mathrm{Var}^{-1}(\underline{\delta}^t)(\underline{\delta}) = (-1)^{(n+1)(n+2)/2} S^t.$$

In [2, I.2.5], this problem does not arise because there the matrix of a bilinear form with respect to a basis is the transpose of the usual matrix [2, p. 45]. This is applied to the matrices of I and L.

A result of Thom and Sebastiani compares the Milnor lattices and monodromies of the singularities $f = f(x_0, \dots, x_n)$, $g = g(y_0, \dots, y_m)$, and $f + g = f(x_0, \dots, x_n) + g(x_{n+1}, \dots, x_{n+m+1})$. There are extensions by Deligne for the Seifert form and by Gabrielov for distinguished bases. All results can be found in [2, I.2.7]. They are restated here. There is a canonical isomorphism

$$\Phi : Ml(f + g) \xrightarrow{\cong} Ml(f) \otimes Ml(g), \tag{12}$$

$$\text{with } M_h(f + g) \cong M_h(f) \otimes M_h(g) \tag{13}$$

$$\text{and } L(f + g) \cong (-1)^{(n+1)(m+1)} \cdot L(f) \otimes L(g). \tag{14}$$

If $\underline{\delta} = (\delta_1, \dots, \delta_{\mu(f)})$ and $\underline{\gamma} = (\gamma_1, \dots, \gamma_{\mu(g)})$ are distinguished bases of f and g with Stokes matrices $S(f)$ and $S(g)$, then

$$\Phi^{-1}(\delta_1 \otimes \gamma_1, \dots, \delta_1 \otimes \gamma_{\mu(g)}, \delta_2 \otimes \gamma_1, \dots, \delta_2 \otimes \gamma_{\mu(g)}, \dots, \delta_{\mu(f)} \otimes \gamma_1, \dots, \delta_{\mu(f)} \otimes \gamma_{\mu(g)})$$

is a distinguished basis of $Ml(f + g)$, that means, one takes the vanishing cycles $\Phi^{-1}(\delta_i \otimes \gamma_j)$ in the lexicographic order. Then by (11) and (14), the matrix

$$S(f + g) = S(f) \otimes S(g) \tag{15}$$

(where the tensor product is defined so that it fits to the lexicographic order) is the Stokes matrix of this distinguished basis.

In the special case $g = x_{n+1}^2$, the function germ $f + g = f(x_0, \dots, x_n) + x_{n+1}^2 \in \mathcal{O}_{\mathbb{C}^{n+2},0}$ is called *stabilization* or *suspension* of f. As there are only two isomorphisms $Ml(x_{n+1}^2) \to \mathbb{Z}$, and they differ by a sign, there are two equally canonical isomorphisms $Ml(f) \to Ml(f + x_{n+1}^2)$, and they differ just by a sign. Therefore, automorphisms and bilinear forms on $Ml(f)$ can be identified with automorphisms and bilinear forms on $Ml(f + x_{n+1}^2)$. In this sense,

$$L(f + x_{n+1}^2) = (-1)^n \cdot L(f) \quad \text{and} \quad M_h(f + x_{n+1}^2) = -M_h(f) \tag{16}$$

[2, I.2.7], and $G_{\mathbb{Z}}(f + x_{n+1}^2) = G_{\mathbb{Z}}(f)$. The Stokes matrix S does not change under stabilization.

The following algebraic lemma from [12] will be very useful in Sects. 3 and 4. It can be seen as a generalization of the number theoretic fact $\mathbb{Z}[e^{2\pi i/a}] \cap S^1 = \{\pm e^{2\pi ik/a} \mid k \in \mathbb{Z}\}$.

Lemma 2.5 ([12, Lemma 8.2]) *Let H be a free $\mathbb{Z}$-module of finite rank μ, and $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$. Let $M_h : H \to H$ be an automorphism of finite order, called monodromy, with three properties:*

(i) Each eigenvalue has multiplicity 1.
  Denote $H_\lambda := \ker(M_h - \lambda \cdot \mathrm{id} : H_{\mathbb{C}} \to H_{\mathbb{C}})$.
(ii) Denote $\mathrm{Ord} := \{\mathrm{ord}\,\lambda \mid \lambda \text{ eigenvalue of } M_h\} \subset \mathbb{Z}_{\geq 1}$. *There exist four sequences* $(m_i)_{i=1,\dots,|\mathrm{Ord}|}$, $(j(i))_{i=2,\dots,|\mathrm{Ord}|}$, $(p_i)_{i=2,\dots,|\mathrm{Ord}|}$, $(k_i)_{i=2,\dots,|\mathrm{Ord}|}$ *of numbers in* $\mathbb{Z}_{\geq 1}$ *and two numbers* $i_1, i_2 \in \mathbb{Z}_{\geq 1}$ *with* $i_1 \leq i_2 \leq |\mathrm{Ord}|$ *and with the properties:*
  $\mathrm{Ord} = \{m_1, \dots, m_{|\mathrm{Ord}|}\}$,
  p_i is a prime number, $p_i = 2$ for $i_1 + 1 \leq i \leq i_2$, $p_i \geq 3$ else,
  $j(i) = i - 1$ for $i_1 + 1 \leq i \leq i_2$, $j(i) < i$ else,

$$m_i = m_{j(i)}/p_i^{k_i}.$$

(iii) A cyclic generator $a_1 \in H$ exists, that means,

$$H = \bigoplus_{i=0}^{\mu-1} \mathbb{Z} \cdot M_h^i(a_1).$$

Finally, let I be an M_h-invariant nondegenerate bilinear form (not necessarily (± 1)*-symmetric) on $\bigoplus_{\lambda \neq \pm 1} H_\lambda$ with values in $\mathbb{C}$. Then*

$$\mathrm{Aut}(H, M_h, I) = \{\pm M_h^k \mid k \in \mathbb{Z}\}.$$

3 The Group $G_{\mathbb{Z}}$ for the Simple Elliptic and the Hyperbolic Singularities

The simple elliptic and the hyperbolic singularities are 1-parameter families of singularities, which had been classified by Arnold [1]. For each triple $(p, q, r) \in \mathbb{N}^3_{\geq 2}$ with $p \geq q \geq r$ and $\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$, one has one family, denoted T_{pqr}. The hyperbolic singularities are those with $\kappa < 1$, the simple elliptic are those with $\kappa = 1$. For the three families of simple elliptic singularities, also other symbols are used, $T_{333} = \widetilde{E}_6$, $T_{442} = \widetilde{E}_7$, $T_{632} = \widetilde{E}_8$. The singularities of types T_{pqr} with $r = 2$ exist as curve singularities, all others as surface singularities. Normal forms will be discussed in Sect. 6. Here for each family, the group $G_{\mathbb{Z}} = \mathrm{Aut}(Ml(f), L)$ will be analyzed. The result in Theorem 3.1 is completely explicit in the case $\kappa < 1$ and partly explicit in the case $\kappa = 1$. The whole section is devoted to its proof. Besides

$$\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r}, \quad \text{also} \quad \chi := \mathrm{lcm}(p, q, r) \in \mathbb{N}$$

will be used. A singularity of type T_{pqr} has Milnor number $\mu = p + q + r - 1$, and its monodromy has the characteristic polynomial

$$\frac{t^p - 1}{t - 1} \cdot \frac{t^q - 1}{t - 1} \cdot \frac{t^r - 1}{t - 1} \cdot (t - 1)^2.$$

Theorem 3.1 *Consider a surface singularity f of type T_{pqr} (with $\kappa \leq 1$) with Milnor lattice $Ml(f)$, monodromy M_h, intersection form I, and Seifert form L:*

(a) Then $\dim Ml(f)_1 = 2$, $\mathrm{rank}\,\mathrm{Rad}(I) = 1$ if $\kappa < 1$ and $= 2$ if $\kappa = 1$. Choose a $\mathbb{Z}$-basis b_1, b_2 of $Ml(f)_{1,\mathbb{Z}}$ with $b_1 \in \mathrm{Rad}(I)$ and $L(b_1, b_2) \leq 0$. Then

$$L(b_1, b_2) = -\chi \quad \text{and} \quad M_h b_2 = b_2 + \chi(\kappa - 1) \cdot b_1. \tag{17}$$

(b) The restriction map $G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$ is surjective. Denote by $T \in \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}})$ the automorphism with $T(b_1) = b_1$ and $T(b_2) = b_2 + b_1$. Denote $\underline{b} := (b_1, b_2)$:

$$\begin{aligned} \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L) &= \{\underline{b} \mapsto \underline{b} \cdot A \mid A \in SL(2, \mathbb{Z})\} \qquad (18)\\ &\cong SL(2, \mathbb{Z}) \quad \text{if} \quad \kappa = 1, \\ \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L) &= \{\pm T^k \mid k \in \mathbb{Z}\} \quad \text{if} \quad \kappa < 1. \qquad (19) \end{aligned}$$

(c) The group $G_{\mathbb{Z}}$ for $\kappa < 1$ and the subgroup $\{g \in G_{\mathbb{Z}} \mid g(b_1) = \pm b_1\} \subset G_{\mathbb{Z}}$ for $\kappa = 1$ will be described explicitly, except for the part U_2; see below. There is a monodromy invariant decomposition

$$Ml(f)_{\neq 1} = Ml^{(1)}_{\mathbb{C}} \oplus Ml^{(2)}_{\mathbb{C}} \oplus Ml^{(3)}_{\mathbb{C}} \tag{20}$$

such that the characteristic polynomial of $M_h|_{Ml^{(j)}}$ is

$$\frac{t^p-1}{t-1},\ \frac{t^q-1}{t-1},\ \frac{t^r-1}{t-1} \quad \text{for } j=1,2,3 \tag{21}$$

and such that the following holds:

$$\left.\begin{array}{rl} G_{\mathbb{Z}} & \text{for } \kappa<1 \\ \{g\in G_{\mathbb{Z}}\,|\,g(b_1)=\pm b_1\} & \text{for } \kappa=1 \end{array}\right\} = (U_1\rtimes U_2)\times\{\pm\,\mathrm{id}\}, \tag{22}$$

where U_1 is the infinite subgroup of $G_{\mathbb{Z}}$

$$\begin{aligned} U_1 = \{&T^\delta\times(M_h|_{Ml^{(1)}_{\mathbb{C}}})^\alpha\times(M_h|_{Ml^{(2)}_{\mathbb{C}}})^\beta\times(M_h|_{Ml^{(3)}_{\mathbb{C}}})^\gamma\,| \\ &(\delta,\alpha,\beta,\gamma)\in\mathbb{Z}\times\mathbb{Z}_p\times\mathbb{Z}_q\times\mathbb{Z}_r \text{ with } \frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\equiv\frac{\delta}{\chi} \mod 1\} \end{aligned} \tag{23}$$

and where U_2 is a finite subgroup of $G_{\mathbb{Z}}$ with

$$U_2\begin{cases} =\{\mathrm{id}\} \text{ if } p>q>r, \\ \cong S_2 \quad \text{if } p=q>r \text{ or } p>q=r, \\ \cong S_3 \quad \text{if } p=q=r. \end{cases} \tag{24}$$

which consists of certain automorphisms which act trivially on $Ml(f)_1$ and which permute those of the subspaces $Ml^{(j)}_{\mathbb{C}}$ which have equal dimension.

Proof Choose a distinguished basis $\underline{\delta}=(\delta_1,\ldots,\delta_\mu)$ with the Coxeter-Dynkin diagram in Example 2.3. Then the monodromy matrix M_M with $M_h(\underline{\delta})=\underline{\delta}\cdot M_M$ can be calculated either with (8) or with (9). It had been calculated in [6] with (8), and it is (here all not specified entries are 0)

$$M_M=\begin{pmatrix} M_1 & & & M_8 \\ & M_2 & & M_9 \\ & & M_3 & M_{10} \\ M_5 & M_6 & M_7 & M_4 \end{pmatrix} \tag{25}$$

with the following blocks,

$$M_1=\begin{pmatrix} 0 & & & -1 \\ 1 & \ddots & & -1 \\ & \ddots & 0 & -1 \\ & & 1 & -1 \end{pmatrix}\in M((p-1)\times(p-1),\mathbb{Z}),$$

$M_2\in M((q-1)\times(q-1),\mathbb{Z})$ and

$$M_3 \in M((r-1)\times(r-1),\mathbb{Z}) \text{ are defined analogously,}$$
$$M_4 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix},$$

and $M_5, M_6, M_7, M_8, M_9, M_{10}$ are of suitable sizes with all entries except the following being 0,

$$(M_5)_{11} = (M_6)_{11} = (M_7)_{11} = -1, \quad (M_5)_{21} = (M_6)_{21} = (M_7)_{21} = 1,$$
$$(M_8)_{11} = (M_8)_{12} = (M_9)_{11} = (M_9)_{12} = (M_{10})_{11} = (M_{10})_{12} = 1.$$

Define

$$\tilde{b}_1 := \delta_{\mu-1} - \delta_\mu, \tag{26}$$
$$\tilde{b}_2 := \chi \cdot \left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i + \sum_{i=1}^{q-1} \frac{q-i}{q}\delta_{p-1+i} + \sum_{i=1}^{r-1} \frac{r-i}{r}\delta_{p+q-2+i} + \delta_{\mu-1} \right). \tag{27}$$

Then one calculates

$$M_h(\tilde{b}_1) = \tilde{b}_1, \quad M_h(\tilde{b}_2) = \tilde{b}_2 + \chi(\kappa-1)\cdot\tilde{b}_1, \tag{28}$$

and with (11), which is here $L(\underline{\delta}^t, \underline{\delta}) = S^t$,

$$\begin{pmatrix} L(\tilde{b}_1,\tilde{b}_1) & L(\tilde{b}_1,\tilde{b}_2) \\ L(\tilde{b}_2,\tilde{b}_1) & L(\tilde{b}_2,\tilde{b}_2) \end{pmatrix} = \begin{pmatrix} 0 & -\chi \\ \chi & \frac{\chi^2}{2}(\kappa-1) \end{pmatrix}. \tag{29}$$

By (28), $\tilde{b}_1, \tilde{b}_2$ is a $\mathbb{Q}$-basis of $Ml(f)_{1,\mathbb{Q}}$ and M_h is on $Ml(f)_1$ semisimple if $\kappa = 1$, and it has a 2×2 Jordan block if $\kappa < 1$ (of course, this is well known). From the coefficients, one sees that $\tilde{b}_1, \tilde{b}_2$ is a $\mathbb{Z}$-basis of $Ml(f)_{1,\mathbb{Z}}$. Here it is important that the coefficients of $\tilde{b}_2$ have greatest common divisor 1. As Eq. (17) hold for $\tilde{b}_1, \tilde{b}_2$, they hold for any basis b_1, b_2 as in (a).

(b) If $\kappa = 1$, then (29) shows that L is on $Ml(f)_{1,\mathbb{Z}}$ up to the factor χ the standard symplectic form. Therefore, (18) holds. If $\kappa < 1$, then (29) shows (19).

The restriction map $G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}})$ contains T. This follows from (22) (whose proof below does not use this fact), because obviously $(\delta, \alpha, \beta, \gamma)$ as in (23) with $\delta = 1$ exist.

This shows (b) in the case $\kappa < 1$. In the case $\kappa = 1$, we did not calculate lifts to $G_{\mathbb{Z}}$ of other elements of $\mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$. In this case, the surjectivity of $G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$ follows in two ways: It follows from [14, III.2.6], and it follows from calculations in [6], which are discussed in the proof of Theorem 6.1.

(c) We will prove (c) for the special choice $\tilde{b}_1, \tilde{b}_2$. Then (c) holds for any b_1, b_2 as in (a) because by the surjectivity of the map $G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$, an element $g \in G_{\mathbb{Z}}$ with $g(\tilde{b}_1) = b_1, g(\tilde{b}_2) = b_2$ exists. Define

$$Ml_{\mathbb{Z}}^{[1]} := \mathbb{Z}\tilde{b}_1 \oplus \bigoplus_{i=1}^{p-1} \mathbb{Z}\delta_i, \ Ml_{\mathbb{C}}^{[1]} := Ml_{\mathbb{Z}}^{[1]} \otimes_{\mathbb{Z}} \mathbb{C}, \tag{30}$$

$$Ml_{\mathbb{C}}^{(1)} := Ml_{\mathbb{C}}^{[1]} \cap Ml(f)_{\neq 1}, \tag{31}$$

and analogously $Ml_{\mathbb{Z}}^{[2]}, Ml_{\mathbb{C}}^{[2]}, Ml_{\mathbb{C}}^{(2)}$ and $Ml_{\mathbb{Z}}^{[3]}, Ml_{\mathbb{C}}^{[3]}, Ml_{\mathbb{C}}^{(3)}$.

A look at the matrix M_M shows the following:

$$M_h : \delta_1 + \tilde{b}_1 \mapsto \delta_2 \mapsto \ldots \mapsto \delta_{p-1} \mapsto -(\delta_1 + \ldots + \delta_{p-1}) \mapsto \delta_1 + \tilde{b}_1. \tag{32}$$

Therefore, $Ml_{\mathbb{Z}}^{[1]}$ is a cyclic M_h-module with characteristic polynomial $t^p - 1$, and $Ml_{\mathbb{C}}^{[1]} = \mathbb{C}\tilde{b}_1 \oplus Ml_{\mathbb{C}}^{(1)}$, and M_h on $Ml_{\mathbb{C}}^{(1)}$ has the characteristic polynomial $(t^p - 1) / (t - 1)$.

Lemma 2.5 applies and shows

$$\mathrm{Aut}(Ml_{\mathbb{Z}}^{[1]}, L) = \{\pm (M_h|_{Ml_{\mathbb{Z}}^{[1]}})^{\alpha} \mid \alpha \in \{0, 1, \ldots, p-1\}\}. \tag{33}$$

Finally, M_h, I and L are well defined on the quotient lattice $Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z} \cdot \tilde{b}_1$, and $(Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z} \cdot \tilde{b}_1, -I)$ is a root lattice of type A_{p-1}. The last statement follows immediately from the part of the Coxeter-Dynkin diagram which corresponds to $\delta_1, \ldots, \delta_{p-1}$.

$Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$ have the same properties as $Ml_{\mathbb{Z}}^{[1]}$, with q respectively r instead of p.

Now

$$Ml(f)_{\neq 1} = Ml_{\mathbb{C}}^{(1)} \oplus Ml_{\mathbb{C}}^{(2)} \oplus Ml_{\mathbb{C}}^{(3)}$$

is clear. The $\mathbb{Z}$-lattice

$$Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]} = \mathbb{Z}\tilde{b}_1 \oplus \bigoplus_{i=1}^{\mu-2} \mathbb{Z}\delta_i = (\mathbb{C}\tilde{b}_1 \oplus Ml(f)_{\neq 1}) \cap Ml(f)$$

is a primitive sublattice of $Ml(f)$ of rank $\mu - 1$. Any $g \in G_{\mathbb{Z}}$ with $g(\tilde{b}_1) = \pm\tilde{b}_1$ maps it to itself, because it maps $\mathbb{C}\tilde{b}_1$ and $Ml(f)_{\neq 1}$ and $Ml(f)$ to themselves. g maps also the quotient lattice

$$(Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]})/\mathbb{Z}\tilde{b}_1 = Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z}\tilde{b}_1 \oplus Ml_{\mathbb{Z}}^{[2]}/\mathbb{Z}\tilde{b}_1 \oplus Ml_{\mathbb{Z}}^{[3]}/\mathbb{Z}\tilde{b}_1$$

to itself. But this is together with $-I$ an orthogonal sum of lattices of types A_{p-1}, A_{q-1}, and A_{r-1}. Therefore, g can only permute the summands, and only those summands of equal rank.

If $p = q$, a special element $\sigma_{12} \in G_{\mathbb{Z}}$ is given by

$$\sigma_{12}(\delta_i) = \delta_{p-1+i}, \quad \sigma_{12}(\delta_{p-1+i}) = \delta_i \quad \text{for } 1 \le i \le p-1,$$
$$\sigma_{12}(\delta_j) = \delta_j \quad \text{for } p+q-2 \le j \le \mu.$$

$\sigma_{12} \in G_{\mathbb{Z}}$ follows immediately from the symmetry of the Coxeter-Dynkin diagram. Similarly $\sigma_{23} \in G_{\mathbb{Z}}$ is defined if $q = r$. In any case, these elements generate a subgroup $U_2 \subset G_{\mathbb{Z}}$ with the properties in (c).

Therefore, starting with an arbitrary element $\tilde{g} \in G_{\mathbb{Z}}$ if $\kappa < 1$ respectively $\tilde{g} \in \{g \in G_{\mathbb{Z}} \,|\, g(\tilde{b}_1) = \pm\tilde{b}_1\}$ if $\kappa = 1$, one can compose it with $\pm$ id and an element of U_2, and one obtains an element $g \in G_{\mathbb{Z}}$ with $g(\tilde{b}_1) = \tilde{b}_1$ and $g(Ml_{\mathbb{Z}}^{[j]}) = Ml_{\mathbb{Z}}^{[j]}$ for $j = 1, 2, 3$. Then, $g|_{Ml_{\mathbb{Z}}^{[1]}} = (M_h|_{Ml_{\mathbb{Z}}^{[1]}})^{\alpha}$ for a unique $\alpha \in \{0, 1, \dots, p-1\}$ and similarly with $\beta \in \{0, 1, \dots, q-1\}$ and $\gamma \in \{0, 1, \dots, r-1\}$ for $Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$. Also $g(\tilde{b}_2) = \tilde{b}_2 + \delta\tilde{b}_1$ for some $\delta \in \mathbb{Z}$. One calculates, observing (32),

$$M_h\left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i\right) = \left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i\right) - (\delta_1 + \tilde{b}_1) + \frac{1}{p}\tilde{b}_1, \tag{34}$$

$$M_h^{\alpha}\left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i\right) = \left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i\right) - \left(\tilde{b}_1 + \sum_{i=1}^{\alpha}\delta_i\right) + \frac{\alpha}{p}\tilde{b}_1. \tag{35}$$

The definition (27) of $\tilde{b}_2$ shows

$$-\delta_{\mu-1} = -\frac{1}{\chi}\tilde{b}_2 + \sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i + \sum_{i=1}^{q-1}\frac{q-i}{q}\delta_{p-1+i} + \sum_{i=1}^{r-1}\frac{r-i}{r}\delta_{p+q-2+i}, \tag{36}$$

(35) gives

$$\begin{aligned} g(-\delta_{\mu-1}) = -\delta_{\mu-1} &+ \left(\frac{-\delta}{\chi} + \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r}\right)\cdot\tilde{b}_1 \\ &- \left(\tilde{b}_1 + \sum_{i=1}^{\alpha}\delta_i\right) - \left(\tilde{b}_1 + \sum_{i=p}^{p-1+\beta}\delta_i\right) - \left(\tilde{b}_1 + \sum_{i=p+q-1}^{p+q-2+\gamma}\delta_i\right). \end{aligned} \tag{37}$$

Therefore,

$$\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv \frac{\delta}{\chi} \mod 1 \tag{38}$$
$$\text{and} \quad g = T^{\delta} \times (M_h|_{Ml_{\mathbb{C}}^{(1)}})^{\alpha} \times (M_h|_{Ml_{\mathbb{C}}^{(2)}})^{\beta} \times (M_h|_{Ml_{\mathbb{C}}^{(3)}})^{\gamma}.$$

Thus, $g \in U_1$, so $G_{\mathbb{Z}} \subset (U_1 \rtimes U_2) \times \{\pm \mathrm{id}\}$.

Vice versa, we have to show $U_1 \subset G_{\mathbb{Z}}$. Fix a $g \in U_1$. It respects the decomposition

$$Ml(f)_{\mathbb{C}} = Ml(f)_1 \oplus Ml_{\mathbb{C}}^{(1)} \oplus Ml_{\mathbb{C}}^{(2)} \oplus Ml_{\mathbb{C}}^{(3)}.$$

This is a left and right orthogonal decomposition with respect to the Seifert form L. The restriction of g to each of the four blocks respects L there, so $g \in \mathrm{Aut}(Ml(f)_{\mathbb{C}}, L)$. It restricts on $Ml_{\mathbb{C}}^{[1]}$ to M_h^{α}, so it maps the lattice $Ml_{\mathbb{Z}}^{[1]}$ to itself, and analogously the lattices $Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$, thus also the sum $Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]}$. This sum is a primitive sublattice of $Ml(f)$ of rank $\mu - 1$ with

$$Ml(f) = \left(Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]}\right) \oplus \mathbb{Z}\delta_{\mu-1}.$$

The calculation above of $g(-\delta_{\mu-1})$ shows $g(\delta_{\mu-1}) \in Ml(f)$ and $g(\delta_{\mu-1}) \equiv \delta_{\mu-1}$ modulo the sublattice. Therefore, $g \in G_{\mathbb{Z}}$. □

4 The Group $G_{\mathbb{Z}}$ for 6 of the 28 Exceptional Unimodal and Bimodal Singularities

The 14 1-parameter families of exceptional unimodal singularities and the 14 2-parameter families of exceptional bimodal singularities had been classified by Arnold. Normal forms can be found in [1]. In [12, Theorem 8.3] the group $G_{\mathbb{Z}}$ was calculated for 22 of the 28 families, namely, for those families where all eigenvalues of the monodromy have multiplicity 1. In these cases, it turned out that $G_{\mathbb{Z}}$ is simply $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$. The proof used Lemma 2.5 and that the Milnor lattice is in these cases a cyclic monodromy module.

In this section, $G_{\mathbb{Z}}$ will be determined for the remaining 6 of the 28 families. These are the families $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16}$. In these cases, some eigenvalues have multiplicity 2. This is similar to the case of the singularity D_{2m}, which had also been treated in [12, Theorem 8.4]. Also the proof will be similar. It will again use Lemma 2.5 and combine that with an additional analysis of the action of $G_{\mathbb{Z}}$ on the sum of the eigenspaces with dimension = 2; see Lemma 4.2 below.

This lemma presents a surprise; it points at a funny generalization of the number theoretic fact $\mathbb{Z}[e^{2\pi i/n}] \cap S_1 = \{\pm e^{2\pi ik/n} \,|\, k \in \mathbb{Z}\}$. Also, the lemma uses at the end a calculation of $G^{mar} \subset G_{\mathbb{Z}}$, which will only come in Sect. 7. The whole Sect. 4 is devoted to the proof of the following theorem.

Theorem 4.1 *In the case of the six families of exceptional unimodal and bimodal singularities* $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}$, *and* U_{16}, *the group* $G_{\mathbb{Z}}$ *is* $G_{\mathbb{Z}} = \{\pm M_h^k \,|\, k \in \mathbb{Z}\} \times U$ *with*

	Z_{12}	Q_{12}	U_{12}	Z_{18}	Q_{16}	U_{16}
$U \cong$	$\{\mathrm{id}\}$	S_2	S_3	$\{\mathrm{id}\}$	S_2	S_3

(39)

(This is independent of the number of variables, i.e., it does not change under stabilization.)

Proof Here we consider all six families as surface singularities. Their characteristic polynomials p_{ch} have all the property $p_{ch} = p_1 \cdot p_2$ with $p_2|p_1$ and p_1 having only simple roots. They are as follows. Here Φ_m is the cyclotomic polynomial of primitive mth unit roots:

	Z_{12}	Q_{12}	U_{12}	Z_{18}	Q_{16}	U_{16}
p_{ch}	$\Phi_{22}\Phi_2^2$	$\Phi_{15}\Phi_3^2$	$\Phi_{12}\Phi_6\Phi_4^2\Phi_2^2$	$\Phi_{34}\Phi_2^2$	$\Phi_{21}\Phi_3^2$	$\Phi_{15}\Phi_5^2$
p_1	$\Phi_{22}\Phi_2$	$\Phi_{15}\Phi_3$	$\Phi_{12}\Phi_6\Phi_4\Phi_2$	$\Phi_{34}\Phi_2$	$\Phi_{21}\Phi_3$	$\Phi_{15}\Phi_5$
p_2	Φ_2	Φ_3	$\Phi_4\Phi_2$	Φ_2	Φ_3	Φ_5

(40)

Orlik [19] had conjectured that the Milnor lattice of any quasihomogeneous singularity is a sum of cyclic monodromy modules and that the characteristic polynomials of M_h on the cyclic pieces are $p_1, \ldots, p_r$ where $p_{ch} = p_1 \cdot \ldots \cdot p_r$ and $p_r | p_{r-1} | \ldots | p_1$ and p_1 has simple roots (r and $p_1, \ldots, p_r$ are uniquely determined by this). In the case of curve singularities, Michel and Weber [16] have a proof of this conjecture. In [6, 3.1] the conjecture was proved (using Coxeter-Dynkin diagrams) for all those quasihomogeneous surface singularities of modality ≤ 2 which are not stabilizations of curve singularities. So especially, the conjecture is true for the families of singularities $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16}$. There are $a_1, a_2 \in Ml(f)$ with

$$Ml(f) = \left(\bigoplus_{i=0}^{\deg p_1 - 1} \mathbb{Z} \cdot M_h^i(a_1) \right) \oplus \left(\bigoplus_{i=0}^{\deg p_2 - 1} \mathbb{Z} \cdot M_h^i(a_2) \right) =: B_1 \oplus B_2. \tag{41}$$

Denote

$$B_3 := \ker \left(p_2(M_h) : Ml(f)_{\mathbb{C}} \to Ml(f)_{\mathbb{C}} \right) \cap Ml(f). \tag{42}$$

It is a primitive sublattice of $Ml(f)$ of rank $2 \deg p_2$. Also,

$$(B_1)_{\mathbb{C}} = \ker((p_1/p_2)(M_h)) \oplus (B_1 \cap B_3)_{\mathbb{C}}, \quad B_2 \subset B_3, \tag{43}$$

and $B_1 \cap B_3$ and B_2 are both M_h-invariant primitive sublattices of B_3 of rank $\deg p_2$. Together they generate B_3.

Any $g \in G_{\mathbb{Z}}$ with $g|_{B_3} = \pm(M_h|_{B_3})^k$ for some $k \in \mathbb{Z}$ restricts because of (43) to an automorphism of B_1. Lemma 2.5 applies and shows $g|_{B_1} = \pm(M_h|_{B_1})^l$ for some $l \in \mathbb{Z}$. Now $g|_{B_3} = \pm(M_h|_{B_3})^k$ enforces $k \equiv l \mod \operatorname{lcm}(m \mid \Phi_m | p_2)$ and $g = \pm M_h^l$. Therefore,

$$\{g \in G_{\mathbb{Z}} \mid g|_{B_3} = \pm(M_h|_{B_3})^k \text{ for some } k\} = \{\pm M_h^k \mid k \in \mathbb{Z}\}. \tag{44}$$

Lemma 4.2 (c) determines $\mathrm{Aut}(B_3, L)$,

$$\mathrm{Aut}(B_3, L) = \{\pm(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \times U \quad \text{with } U \text{ as in (39)}. \tag{45}$$

Lemma 4.2 (d) shows that the map $G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)$ is surjective. Together with (44), this gives (39). □

Lemma 4.2

(a) Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-lattice of rank 2 with a $\mathbb{Z}$-basis $\underline{b} = (b_1, b_2)$ and a symmetric pairing $L_{\mathbb{Z}}$ given by

$$L_{\mathbb{Z}}(\underline{b}^t, \underline{b}) = \begin{pmatrix} 2 & -1 \\ -1 & m \end{pmatrix} \quad \text{for some } m \in \mathbb{N}_{\geq 2}.$$

Define $\xi := e^{2\pi i/l}$ for some $l \in \{3, 4, 5\}$, where we exclude the cases $(m \geq 3, l = 5)$. Define $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, $V_{\mathbb{Z}[\xi]} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\xi] \subset V_{\mathbb{C}}$, and extend $L_{\mathbb{Z}}$ sesquilinearly (=linear×semilinear) to $V_{\mathbb{C}}$. Then

$$\begin{aligned} &\{r \in V_{\mathbb{Z}[\xi]} \mid L_{\mathbb{C}}(r, r) = 2\} \\ &= \{\pm\xi^k \mid k \in \mathbb{Z}\} \times \{r \in V_{\mathbb{Z}} \mid L_{\mathbb{Z}}(r, r) = 2\}. \end{aligned} \tag{46}$$

In the case, $m \geq 3$

$$\begin{aligned} &\{r \in V_{\mathbb{Z}[\xi]} \mid L_{\mathbb{C}}(r, r) = m, r \notin \mathbb{Z}[\xi] b_1\} \\ &= \{\pm\xi^k \mid k \in \mathbb{Z}\} \times \{r \in V_{\mathbb{Z}} \mid L_{\mathbb{Z}}(r, r) = m, r \notin \mathbb{Z} b_1\}. \end{aligned} \tag{47}$$

(b) In the situation of (a)

$$\mathrm{Aut}(V_{\mathbb{Z}[\xi]}, L_{\mathbb{C}}) = \{\pm\xi^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}}) \tag{48}$$

$$(\pm\,\mathrm{id} \text{ lives on both sides, therefore not } \times)$$

$$\begin{aligned} \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}}) &\cong \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\} \\ &\cong \{\pm\,\mathrm{id}\} \times S_2 \quad \text{in the cases } m \geq 3, \end{aligned} \tag{49}$$

$$\begin{aligned} \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}}) &\cong \mathrm{Aut}(\text{root lattice of type } A_2) \\ &\cong \{\pm\,\mathrm{id}\} \times S_3 \quad \text{in the case } m = 2. \end{aligned} \tag{50}$$

(c) In the situation of the proof of Theorem 4.1,

$$\mathrm{Aut}(B_3, L) = \{\pm(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \times U \quad \text{with } U \text{ as in (39)}. \tag{51}$$

(d) In the situation of the proof of Theorem 4.1, the map $G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)$ is surjective.

Proof

(a) An element $r = r_1 b_1 + r_2 b_2$ with $r_1, r_2 \in \mathbb{Z}[\xi]$ satisfies

$$L_{\mathbb{C}}(r,r) = 2|r_1|^2 - (r_1\overline{r_2} + \overline{r_1}r_2) + m|r_2|^2 \\ = |r_1|^2 + |r_1 - r_2|^2 + (m-1)|r_2|^2. \tag{52}$$

First consider the cases $l \in \{3,4\}$. Then $\mathbb{Z}[\xi] \cap \mathbb{R} = \mathbb{Z}$. Then the three absolute values in (52) are nonnegative integers. Their sum is 2 if and only if

$$|r_1| = 1, r_2 = 0 \quad \text{in the cases } m \geq 3, \tag{53}$$

$$\left.\begin{matrix} (|r_1|,|r_2|) \in \{(1,0),(0,1),(1,1)\} \\ \text{and in the last case } r_1 = r_2 \end{matrix}\right\} \quad \text{in the case } m = 2. \tag{54}$$

In the case $m \geq 3$ and in the case of an $r \notin \mathbb{Z}[\xi]b_1$, the sum of the three absolute values in (53) is m if and only if

$$(r_1 = 0, |r_2| = 1) \text{ or } (r_1 = r_2, |r_1| = 1). \tag{55}$$

Together with $\mathbb{Z}[\xi] \cap S^1 = \{\pm\xi^k \mid k \in \mathbb{Z}\}$, this shows part (a) in the cases $l \in \{3,4\}$.

It rests to consider the case $(m,l) = (2,5)$. In that case, write

$$r_1 = r_{10} + r_{11}\xi + r_{12}\xi^2 + r_{13}\xi^3, \quad r_2 = r_{20} + r_{21}\xi + r_{22}\xi^2 + r_{23}\xi^3$$

with $r_{ij} \in \mathbb{Z}$. Then,

$$\begin{aligned} L_{\mathbb{C}}(r,r) &= 2|r_1|^2 + 2|r_2|^2 - (r_1\overline{r_2} + \overline{r_1}r_2) \\ &= 2\left[\sum_{j=0}^{3} r_{1j}^2 + (\xi+\xi^4)\sum_{j=1}^{3} r_{1j}r_{1,j-1} + (\xi^2+\xi^3)\sum_{j-k\geq 2} r_{1j}r_{1k}\right] \\ &\quad + 2\left[\sum_{j=0}^{3} r_{2j}^2 + (\xi+\xi^4)\sum_{j=1}^{3} r_{2j}r_{2,j-1} + (\xi^2+\xi^3)\sum_{j-k\geq 2} r_{2j}r_{2k}\right] \\ &\quad - \left[2\sum_{j=0}^{3} r_{1j}r_{2j} + (\xi+\xi^4)\sum_{j=1}^{3}(r_{1j}r_{2,j-1} + r_{1,j-1}r_{2j})\right. \end{aligned}$$

$$+ (\xi^2+\xi^3)\sum_{j-k\geq 2}(r_{1j}r_{2k}+r_{1k}r_{2j})\Bigg]$$
$$= A_1 + A_2\cdot\frac{\sqrt{5}}{2} \quad \text{with } A_1,A_2\in\mathbb{Z}.$$

It is not so easy to find, but easy to check that A_1 is equal to

$$\frac{1}{4}\sum_{j=0}^{3}\left[r_{1j}^2+r_{2j}^2+(r_{1j}-r_{2j})^2\right] \tag{56}$$
$$+\frac{1}{4}\sum_{j<k}\left[(r_{1j}-r_{1k}-r_{2j}+r_{2k})^2+(r_{1j}-r_{1k})^2+(r_{2j}-r_{2k})^2\right].$$

Now it is an easy exercise to find the 8-tuples $(r_{10},\dots,r_{23})\in\mathbb{Z}^8$ for which (56) takes the value 2. They are (here $e_j=(\delta_{ij})_{i=1,\dots,8}$ for $j=1,\dots,8$ is the standard basis of $\mathbb{Z}^8$)

$$\pm e_1,\dots,\pm e_8, \pm(e_1+e_5), \pm(e_2+e_6), \pm(e_3+e_7), \pm(e_4+e_8), \tag{57}$$
$$\pm(1,1,1,1,0,0,0,0), \pm(0,0,0,0,1,1,1,1), \pm(1,1,1,1,1,1,1,1).$$

Observe $1+\xi+\xi^2+\xi^3=-\xi^4$ and

$$\{r\in V_{\mathbb{Z}}\,|\,L_{\mathbb{Z}}(r,r)=2\}=\{\pm b_1,\pm b_2,\pm(b_1+b_2)\}.$$

The coefficients $(r_{10},\dots,r_{23})$ in (57) give precisely the elements $r=r_1b_1+r_2b_2$ on the right-hand side of (46). This shows part (a) for $(m,l)=(2,5)$.

(b) Any element g of $\mathrm{Aut}(V_{\mathbb{Z}[\xi]},L_{\mathbb{C}})$ will map the sets in (46) and (47) to themselves. The basis elements b_1 and b_2 are mapped to two elements in these sets with $L_{\mathbb{C}}(g(b_1)),g(b_2))=L_{\mathbb{Z}}(b_1,b_2)=-1$. Therefore, g is up to a factor in $\{\pm\xi^k\,|\,k\in\mathbb{Z}\}$, an element of $\mathrm{Aut}(V_{\mathbb{Z}},L_{\mathbb{Z}})$. This shows (48).

In the case $m\geq 3$

$$\{r\in V_{\mathbb{Z}}\,|\,L_{\mathbb{Z}}(r,r)=2\}=\{\pm b_1\}$$
$$\text{and}\quad \{r\in V_{\mathbb{Z}}\,|\,L_{\mathbb{Z}}(r,r)=m\}=\{\pm b_2,\pm(b_1+b_2)\}.$$

This shows (49). The case $m=2$ is the case of the root lattice of type A_2. Equation (50) is well known and easy to see.

(c) In the cases Z_{12} and Z_{18} as curve singularities, the Examples 2.2 (ii) and (iii) showed $\mathrm{Aut}(\mathrm{Rad}(I),L)=\{\pm\,\mathrm{id}\}$. Here $\mathrm{Rad}(I)=\ker(M_h^{curve\ case}-\mathrm{id})$. Under stabilization $\mathrm{Rad}(I)$ becomes B_3, and L changes just the sign; see (16). Thus,

$\mathrm{Aut}(B_3, L) = \{\pm\,\mathrm{id}\}$. Because of $M_h|_{B_3} = -\,\mathrm{id}$ and $U := \{\mathrm{id}\}$ in the cases Z_{12} and Z_{18}, this shows (c) in these cases.

Now consider the cases Q_{12}, Q_{16}, U_{12}, and U_{16}. Here part (b) will be used, but that has to be prepared.

The normal forms of the quasihomogeneous surface singularities, given in Sect. 7, show that they are sums of singularities in different variables of types A_l and D_{2m} with $(l, 2m)$ as follows:

	Q_{12}	Q_{16}	U_{12}	U_{16}
$(l, 2m)$	$(2, 6)$	$(2, 8)$	$(3, 4)$	$(4, 4)$
	$A_2 \otimes D_6$	$A_2 \otimes D_8$	$A_3 \otimes D_4$	$A_4 \otimes D_4$

(58)

Here the singularity A_l is in one variable and has the characteristic polynomial $p_{ch}^{A_l} = (t^{l+1} - 1)/(t - 1)$, and the singularity D_{2m} is a curve singularity and has the characteristic polynomial $p_{ch}^{D_{2m}} = (t^{2m-1} - 1)\Phi_1$. The Thom-Sebastiani results which were cited in Sect. 2 apply

$$\begin{aligned} (Ml(f), L) &\cong (Ml(A_l), L_{A_l}) \otimes (Ml(D_{2m}), L_{D_{2m}}), \\ M_h &\cong M_h^{A_l} \otimes M_h^{D_{2m}}, \end{aligned} \tag{59}$$

and show

$$\begin{aligned} p_2 &= p_{ch}^{A_l}, \\ (B_3, L) &\cong (Ml(A_l), L_{A_l}) \otimes (Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}}), \\ M_h|_{B_3} &\cong M_h^{A_l} \otimes \mathrm{id}\,. \end{aligned} \tag{60}$$

The pair $(Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}})$ was considered in Example 2.2 (i). There is a $\mathbb{Z}$-basis $\underline{b} = (b_1, b_2)$ of $Ml(D_{2m})_{1,\mathbb{Z}}$ with

$$L_{D_{2m}}(\underline{b}^t, \underline{b}) = \begin{pmatrix} -2 & 1 \\ 1 & -m \end{pmatrix}. \tag{61}$$

The pairings L and $L_{A_l} \otimes L_{D_{2m}}$ will be extended sesquilinearly from the $\mathbb{Z}$-lattices to the $\mathbb{C}$-vector spaces.

The $\mathbb{Z}$-lattice $Ml(A_l)$ is a cyclic monodromy module. Choose a generator e of it. Therefore, $Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}$ is a sum of two cyclic monodromy modules, and generators are $e \otimes b_1$ and $e \otimes b_2$. For any automorphism g of $(Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}, M_h^{A_l} \otimes \mathrm{id})$, there are unique polynomials $g_1, g_2, g_3, g_4 \in \mathbb{Z}[t]$ of degree $\leq \deg p_2 - 1$ such that

$$\begin{pmatrix} g(v \otimes b_1) \\ g(v \otimes b_2) \end{pmatrix} = \begin{pmatrix} g_1(M_h^{A_l})(v) \otimes b_1 + g_3(M_h^{A_l})(v) \otimes b_2 \\ g_2(M_h^{A_l})(v) \otimes b_1 + g_4(M_h^{A_l})(v) \otimes b_2 \end{pmatrix} \tag{62}$$

for any $v \in Ml(A_l)$.

Now choose any eigenvalue ξ of $M_h^{A_l}$. Then $\mathbb{Z}[\xi]$ is a principal ideal domain. The space $\ker(M_h^{A_l} - \xi\,\mathrm{id}) \cap Ml(A_l)_{\mathbb{Z}[\xi]}$ is a free $\mathbb{Z}[\xi]$-module of rank 1. Choose a generating vector v. This choice gives an isomorphism from this space to $\mathbb{Z}[\xi]$. The spaces

$$\ker(M_h - \xi\,\mathrm{id}) \cap Ml(f)_{\mathbb{Z}[\xi]} \cong (\ker(M_h^{A_l} - \xi\,\mathrm{id}) \cap Ml(A_l)_{\mathbb{Z}[\xi]}) \otimes Ml(D_{2m})_{1,\mathbb{Z}[\xi]}$$

are free $\mathbb{Z}[\xi]$-modules of rank 2. The space on the right-hand side has the $\mathbb{Z}[\xi]$-basis $(v \otimes b_1, v \otimes b_2) =: v \otimes \underline{b}$. Now (62) becomes

$$g(v \otimes \underline{b}) = v \otimes \underline{b} \cdot \begin{pmatrix} g_1(\xi) & g_2(\xi) \\ g_3(\xi) & g_4(\xi) \end{pmatrix}. \tag{63}$$

The pairing satisfies

$$(L_{A_l} \otimes L_{D_{2m}})(v \otimes \underline{b}) = L_{A_l}(v, v) \cdot \begin{pmatrix} -2 & 1 \\ 1 & -m \end{pmatrix}, \tag{64}$$

where $L_{A_l}(v, v) \in \mathbb{Z}[\xi] \cap \mathbb{R}_{>0}$. This space with this pairing is up to a scalar isomorphic to a pair $(V_{\mathbb{Z}[\xi]}, L_{\mathbb{C}})$ considered in the parts (a) and (b). Therefore, by part (b), its group of automorphisms is isomorphic to $\{\pm\xi^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}})$.

Thus, any element of $\mathrm{Aut}(B_3, L)$ restricts on $\ker(M_h - \xi\,\mathrm{id}) \cap Ml(f)_{\mathbb{Z}[\xi]}$ to such an automorphism. In the cases Q_{12}, Q_{16} and U_{16}, the polynomial $p_2 = \Phi_3, \Phi_3$, respectively, Φ_5 is irreducible, so all its zeros ξ are conjugate. Therefore, then

$$\begin{aligned} \mathrm{Aut}(B_3, L) &\cong \{\pm(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}}\} \\ &\cong \{\pm(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \times U, \end{aligned}$$

which proves (c) in these cases.

In the case, U_{12} the characteristic polynomial $p_{ch}^{A_3} = p_2 = \Phi_4\Phi_2$ is reducible. Consider an automorphism g of

$$(Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}, L_{A_l} \otimes L_{D_{2m}}).$$

It is determined by the polynomials $g_1, g_2, g_3, g_4 \in \mathbb{Z}[t]$ in (62). For $\xi = i$ and for $\xi = -1$, it gives an automorphism of $\mathbb{Z}[\xi]v \otimes b_1 \oplus \mathbb{Z}[\xi]v \otimes b_2$ which is given by a matrix $\begin{pmatrix} g_1(\xi) & g_2(\xi) \\ g_3(\xi) & g_4(\xi) \end{pmatrix}$ which is by part (b) in

$$\begin{aligned} \{\pm\xi^k \mid k \in \mathbb{Z}\} \cdot \Big\{ &\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \\ &\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \Big\}. \end{aligned} \tag{65}$$

By multiplying g with a suitable automorphism, we can suppose that the matrix for $\xi = i$ is the identity matrix. Then

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} 1+(t^2+1)\tilde{g}_1 & (t^2+1)\tilde{g}_2 \\ (t^2+1)\tilde{g}_3 & 1+(t^2+1)\tilde{g}_4 \end{pmatrix},$$

for some $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4 \in \mathbb{Z}[t]$, so

$$\begin{pmatrix} g_1(-1) & g_2(-1) \\ g_3(-1) & g_4(-1) \end{pmatrix} = \begin{pmatrix} 1+2\tilde{g}_1(-1) & 2\tilde{g}_2(-1) \\ 2\tilde{g}_3(-1) & 1+2\tilde{g}_4(-1) \end{pmatrix}.$$

The only two possibilities are $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the case of a minus sign, $g \circ ((M_h^{A_l})^2 \otimes \mathrm{id}_{D_{2m}}) = -\mathrm{id}$, and in the case of a plus sign, $g = \mathrm{id}$. This finishes the proof of (c) in the case U_{12}.

(d) In Sect. 7 a subgroup G^{mar} of $G_{\mathbb{Z}}$ will be calculated, and it will be shown that $G^{mar} = \{\pm M_h^k \mid k \in \mathbb{Z}\} \times U$. This shows $G_{\mathbb{Z}} \supset \{\pm M_h^k \mid k \in \mathbb{Z}\} \times U$ and that the map $G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)$ is surjective. □

Remark 4.3 The number theoretic fact $\mathbb{Z}[e^{2\pi i/a}] \cap S^1 = \{\pm e^{2\pi ik/a} \mid k \in \mathbb{Z}\}$ can be interpreted as saying that in the case of the A_1-lattice $V_{\mathbb{Z}} = \mathbb{Z}$ with $\mathbb{Z}$-basis $b_1 = 1$ and standard bilinear form $L_{\mathbb{Z}}$ with $L_{\mathbb{Z}}(b_1, b_1) = 1$ and Hermitian extension $L_{\mathbb{C}}$ to $\mathbb{C}$, the analogue of (46) holds. Now (46) for $m = 2$ can be seen as a generalization from the case A_1 to the case A_2. Above it is proved only in the cases $l = 3, 4, 5$.

5 Review on μ-Constant Monodromy Groups G^{mar}, Marked Singularities, Their Moduli Spaces M_μ^{mar}, and Torelli Type Conjectures

This paper is a sequel to [12]. That paper studied holomorphic functions germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 from a global perspective. Here we review most of the notions and results from [12].

It defined the notions of *marked singularity* and *strongly marked singularity*. The marking uses the Milnor lattice $Ml(f) \cong \mathbb{Z}^\mu$ and the Seifert form L on it, which are explained in Sect. 2.

Definition 5.1 Fix one reference singularity f_0.

(a) Then a strong marking for any singularity f in the μ-homotopy class of f_0 (i.e., there is a 1-parameter family of singularities with constant Milnor number connecting f and f_0) is an isomorphism $\rho : (Ml(f), L) \to (Ml(f_0), L)$.

(b) The pair (f, ρ) is a *strongly marked singularity*. Two strongly marked singularities (f_1, ρ_1) and (f_2, ρ_2) are right equivalent (notation: $\sim_{\mathscr{R}}$) if a coordinate change $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ with

$$f_1 = f_2 \circ \varphi \quad \text{and} \quad \rho_1 = \rho_2 \circ \varphi_{hom}$$

exists, where $\varphi_{hom} : (Ml(f_1), L) \to (Ml(f_2), L)$ is the induced isomorphism.

(c) The notion of a marked singularity is slightly weaker. If f and ρ are as above, then the pair $(f, \pm\rho)$ is a *marked singularity* (writing $\pm\rho$, the set $\{\rho, -\rho\}$ is meant, neither ρ nor $-\rho$ is preferred).

(d) Two marked singularities (f_1, ρ_1) and (f_2, ρ_2) are right equivalent (notation: $\sim_{\mathscr{R}}$) if a coordinate change φ with

$$f_1 = f_2 \circ \varphi \quad \text{and} \quad \rho_1 = \pm\rho_2 \circ \varphi_{hom}$$

exists.

Remark 5.2

(i) The notion of a marked singularity behaves better than the notion of a strongly marked singularity, because it is not known whether all μ-homotopy families of singularities satisfy one of the following two properties:

$$\text{Assumption (5.1): Any singularity in the } \mu\text{-homotopy class of } f_0 \text{ has multiplicity } \geq 3. \tag{66}$$

$$\text{Assumption (5.2): Any singularity in the } \mu\text{-homotopy class of } f_0 \text{ has multiplicity } 2. \tag{67}$$

We expect that always one of two assumptions holds. For curve singularities and singularities right equivalent to semiquasihomogeneous singularities, this is true, but in general it is not known. In a μ-homotopy family where neither of the two assumptions holds, strong marking behaves badly; see (ii).

(ii) If $\text{mult}(f) = 2$, then $(f, \rho) \sim_{\mathscr{R}} (f, -\rho)$, which is easy to see. If $\text{mult}(f) \geq 3$, then $(f, \rho) \not\sim_{\mathscr{R}} (f, -\rho)$, whose proof in [12] is quite intricate. These properties imply that the moduli space for strongly marked singularities discussed below is not Hausdorff in the case of a μ-homotopy class which satisfies neither one of the assumptions (66) or (67).

In [11] for the μ-homotopy class of any singularity f_0, a moduli space $M_\mu(f_0)$ was constructed. As a set, it is simply the set of right equivalence classes of singularities in the μ-homotopy class of f_0. But in [11], it is constructed as an analytic geometric quotient, and it is shown that it is locally isomorphic to the μ-constant stratum of a singularity modulo the action of a finite group. The μ-constant stratum of a singularity is the germ $(S_\mu, 0) \subset (M, 0)$ within the germ of the base space of a universal unfolding F of f, such that for a suitable representative

$$S_\mu = \{t \in M \,|\, F_t \text{ has only one singularity } x_0 \text{ and } F_t(x_0) = 0\}. \tag{68}$$

It comes equipped with a canonical complex structure, and M_μ inherits a canonical structure; see Chaps. 12 and 13 in [11].

In [12] analogous results for marked singularities were proved. A better property is that M_μ^{mar} is locally isomorphic to a μ-constant stratum without dividing out a finite group action. Therefore, one can consider it as a *global μ-constant stratum* or as a *Teichmüller space for singularities*. The following theorem collects results from [12, Theorem 4.3].

Theorem 5.3 *Fix one reference singularity f_0. Define the sets*

$$M_\mu^{smar}(f_0) := \{\text{strongly marked } (f,\rho) \mid \tag{69}$$
$$f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathscr{R}},$$
$$M_\mu^{mar}(f_0) := \{\text{marked } (f,\pm\rho) \mid \tag{70}$$
$$f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathscr{R}} .$$

(a) $M_\mu^{mar}(f_0)$ carries a natural canonical complex structure. It can be constructed with the underlying reduced complex structure as an analytic geometric quotient (see [12, Theorem 4.3] for details).

(b) The germ $(M_\mu^{mar}(f_0), [(f,\pm\rho)])$ with its canonical complex structure is isomorphic to the μ-constant stratum of f with its canonical complex structure (see [11, Chap. 12] for the definition of that).

(c) For any $\psi \in G_{\mathbb{Z}}(f_0) =: G_{\mathbb{Z}}$, the map

$$\psi_{mar} : M_\mu^{mar} \to M_\mu^{mar}, \quad [(f,\pm\rho)] \to [(f,\pm\psi\circ\rho)]$$

is an automorphism of M_μ^{mar}. The action

$$G_{\mathbb{Z}} \times M_\mu^{mar} \to M_\mu^{mar}, \quad (\psi, [(f,\pm\rho)] \mapsto \psi_{mar}([(f,\pm\rho)])$$

is a group action from the left.

(d) The action of $G_{\mathbb{Z}}$ on M_μ^{mar} is properly discontinuous. The quotient $M_\mu^{mar}/G_{\mathbb{Z}}$ is the moduli space M_μ for right equivalence classes in the μ-homotopy class of f_0, with its canonical complex structure. Especially, $[(f_1,\pm\rho_1)]$ and $[(f_2,\pm\rho_2)]$ are in one $G_{\mathbb{Z}}$-orbit if and only if f_1 and f_2 are right equivalent.

(e) If assumption (66) or (67) holds, then (a) to (d) are also true for M_μ^{smar} and ψ_{smar} with $\psi_{smar}([(f,\rho)]) := [(f,\psi\circ\rho)]$. If neither (66) nor (67) holds, then the natural topology on M_μ^{smar} is not Hausdorff.

We stick to the situation in Theorem 5.3 and define two subgroups of $G_{\mathbb{Z}}(f_0)$. The definitions in [12, Definition 3.1] are different; they use μ-constant families. The following definitions are a part of Theorem 4.4 in [12].

Definition 5.4 Let $(M_\mu^{mar})^0$ be the topological component of M_μ^{mar} (with its reduced complex structure) which contains $[(f_0, \pm \mathrm{id})]$. Then

$$G^{mar}(f_0) := \{\psi \in G_\mathbb{Z} \mid \psi \text{ maps } (M_\mu^{mar})^0 \text{ to itself}\} \subset G_\mathbb{Z}(f_0). \tag{71}$$

If assumption (66) or (67) holds, $(M_\mu^{smar})^0$ and $G^{smar}(f_0) \subset G_\mathbb{Z}(f_0)$ are defined analogously.

The following theorem is also proved in [12].

Theorem 5.5

(a) In the situation above, the map

$$\begin{aligned} G_\mathbb{Z}/G^{mar}(f_0) &\to \{\text{topological components of } M_\mu^{mar}\} \\ \psi \cdot G^{mar}(f_0) &\mapsto \text{the component } \psi_{mar}((M_\mu^{mar})^0) \end{aligned}$$

is a bijection.

(b) If assumption (66) or (67) holds, then (a) is also true for M_μ^{smar} and $G^{smar}(f_0)$.

(c) $-\mathrm{id} \in G_\mathbb{Z}$ acts trivially on $M_\mu^{mar}(f_0)$. Suppose additionally that assumption (66) holds for f_0. Then $\{\pm \mathrm{id}\}$ acts freely on $M_\mu^{smar}(f_0)$, and the quotient map

$$M_\mu^{smar}(f_0) \stackrel{/\{\pm \mathrm{id}\}}{\longrightarrow} M_\mu^{mar}(f_0), \quad [(f,\rho)] \mapsto [(f, \pm\rho)]$$

is a double covering.

The first main conjecture in [12] is part (a) of the following conjecture (the second main conjecture in [12] is Conjecture 5.11 (a) below).

Conjecture 5.6

(a) Fix a singularity f_0. Then M_μ^{mar} is connected. Equivalently (in view of Theorem 5.5 (a)): $G^{mar}(f_0) = G_\mathbb{Z}$.

(b) If the μ-homotopy class of f_0 satisfies assumption (66), then $-\mathrm{id} \notin G^{smar}(f_0)$.

If (a) holds, then (b) is equivalent to $M_\mu^{smar}(f_0)$ having two components. If (a) and (b) hold, there should be a natural invariant which distinguishes the index 2 subgroup $G^{smar} \subset G^{mar} = G_\mathbb{Z}$. Anyway, part (a) is the more important conjecture. Using the other definition of G^{mar} in [12], it says that up to $\pm$ id, any element of $G_\mathbb{Z}$ can be realized as transversal monodromy of a μ-constant family with parameter space S^1.

The whole Conjecture 5.6 had been proved in [12] for the simple singularities and those 22 of the 28 exceptional unimodal and bimodal singularities, where all eigenvalues of the monodromy have only multiplicity one [12, Theorems 8.3 and 8.4]. In this paper, it will be proved for the remaining unimodal and exceptional bimodal singularities.

In order to understand the stabilizers $\text{Stab}_{G_{\mathbb{Z}}}([(f,\rho)])$ and $\text{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)])$ of points $[(f,\rho)] \in M_\mu^{smar}(f_0)$ and $[(f,\pm\rho)] \in M_\mu^{mar}(f_0)$, we have to look at the *symmetries* of a single singularity. These had been discussed in [11, Chap. 13.2]. The discussion had been taken up again in [12].

Definition 5.7 Let $f_0 = f_0(x_0,\dots,x_n)$ be a reference singularity and let f be any singularity in the μ-homotopy class of f_0. If ρ is a marking, then $G_{\mathbb{Z}}(f) = \rho^{-1} \circ G_{\mathbb{Z}} \circ \rho$.

We define

$$\mathscr{R} := \{\varphi : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0) \quad \text{biholomorphic}\}, \tag{72}$$

$$\mathscr{R}^f := \{\varphi \in \mathscr{R} \,|\, f\circ\varphi = f\}, \tag{73}$$

$$R_f := j_1\mathscr{R}^f/(j_1\mathscr{R}^f)^0, \tag{74}$$

$$G_{\mathscr{R}}^{smar}(f) := \{\varphi_{hom} \,|\, \varphi \in \mathscr{R}^f\} \subset G_{\mathbb{Z}}(f), \tag{75}$$

$$G_{\mathscr{R}}^{mar}(f) := \{\pm\psi \,|\, \psi \in G_{\mathscr{R}}^{smar}(f)\}. \tag{76}$$

Again, the definition of $G_{\mathscr{R}}^{smar}$ is different from the definition in [12, Definition 3.1]. The characterization in (75) is [12, Theorem 3.3. (e)]. R_f is the finite group of components of the group $j_1\mathscr{R}^f$ of 1-jets of coordinate changes which leave f invariant. The following theorem collects results from several theorems in [12].

Theorem 5.8 *Consider the data in Definition 5.7.*

(a) *If* $\text{mult}(f) \geq 3$, *then* $j_1\mathscr{R}^f = R_f$.
(b) *The homomorphism* $()_{hom} : \mathscr{R}^f \to G_{\mathbb{Z}}(f)$ *factors through* R_f. *Its image is* $(R_f)_{hom} = G_{\mathscr{R}}^{smar}(f) \subset G_{\mathbb{Z}}(f)$.
(c) *The homomorphism* $()_{hom} : R_f \to G_{\mathscr{R}}^{smar}(f)$ *is an isomorphism.*
(d)

$$-\text{id} \notin G_{\mathscr{R}}^{smar}(f) \iff \text{mult} f \geq 3. \tag{77}$$

Equivalently: $G_{\mathscr{R}}^{mar}(f) = G_{\mathscr{R}}^{smar}(f)$ *if* $\text{mult} f = 2$, *and* $G_{\mathscr{R}}^{mar}(f) = G_{\mathscr{R}}^{smar}(f) \times \{\pm\,\text{id}\}$ *if* $\text{mult} f \geq 3$.
(e) $G_{\mathscr{R}}^{mar}(f) = G_{\mathscr{R}}^{mar}(f + x_{n+1}^2)$.
(f) $M_h \in G^{smar}(f)$. *If f is quasihomogeneous, then* $M_h \in G_{\mathscr{R}}^{smar}(f)$.
(g) *For any* $[(f,\rho)] \in M_\mu^{smar}$

$$\text{Stab}_{G_{\mathbb{Z}}}([(f,\rho)]) = \rho \circ G_{\mathscr{R}}^{smar}(f) \circ \rho^{-1}, \tag{78}$$

$$\text{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)]) = \rho \circ G_{\mathscr{R}}^{mar}(f) \circ \rho^{-1}. \tag{79}$$

((78) does not require assumption (66) or (67)). As $G_{\mathbb{Z}}$ acts properly discontinuously on $M_\mu^{mar}(f_0)$, $G_{\mathscr{R}}^{smar}(f)$ and $G_{\mathscr{R}}^{mar}(f)$ are finite. (But this follows already from the finiteness of R_f and (b).)

In the case of a quasihomogeneous singularity, the group R_f has a canonical lift to $\mathscr{R}^f$. It will be useful for the calculation of R_f.

Theorem 5.9 ([11, Theorem 13.11]) *Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and weights $w_0, \dots, w_n \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and weighted degree 1. Suppose that $w_0 \le \dots \le w_{n-1} < \frac{1}{2}$ (then $f \in \mathbf{m}^3$ if and only if $w_n < \frac{1}{2}$). Let G_w be the algebraic group of quasihomogeneous coordinate changes, that means, those which respect $\mathbb{C}[x_0, \dots, x_n]$ and the grading by the weights $w_0, \dots, w_n$ on it. Then*

$$R_f \cong \mathrm{Stab}_{G_w}(f). \tag{80}$$

Finally we need and we want to study period maps and Torelli-type problems for singularities.

This story should start with the definition of the Gauß-Manin connection and the Brieskorn lattice for an isolated hypersurface singularity. This had been developed in many papers of the second author and also much earlier by Brieskorn, K. Saito, G.-M. Greuel, F. Pham, A. Varchenko, M. Saito, and others.

As we will build here on calculations done in [6] and therefore never have to touch Brieskorn lattices explicitly, we take here a formal point of view and refer to [6, 7, 9, 11] and [12] for the definitions of the following objects.

Any singularity f comes equipped with a *Brieskorn lattice* $H_0''(f)$. It is much richer than, but still comparable to, a Hodge structure of a closed Kähler manifold.

After fixing a reference singularity f_0, a marked singularity $(f, \pm\rho)$ comes equipped with a *marked Brieskorn lattice* $BL(f, \pm\rho)$. A classifying space $D_{BL}(f_0)$ for marked Brieskorn lattices was constructed in [9]. It is especially a complex manifold, and $G_{\mathbb{Z}}$ acts properly discontinuously on it.

Theorem 5.10 *Fix one reference singularity f_0.*

(a) There is a natural holomorphic period map

$$BL : M_\mu^{mar}(f_0) \to D_{BL}(f_0). \tag{81}$$

It is $G_{\mathbb{Z}}$-equivariant.

(b) [11, Theorem 12.8] It is an immersion; here the reduced complex structure on $M_\mu^{mar}(f_0)$ is considered. (The second author has also a proof that it is an immersion where the canonical complex structure on $M_\mu^{mar}(f_0)$ is considered, but the proof is not written.)

The second main conjecture in [12] is part (a) of the following conjecture. Part (a) and part (b) are global Torelli-type conjectures.

Conjecture 5.11 Fix one reference singularity f_0.

(a) The period map $BL : M_\mu^{mar} \to D_{BL}$ is injective.
(b) The period map $LBL : M_\mu = M_\mu^{mar}/G_{\mathbb{Z}} \to D_{BL}/G_{\mathbb{Z}}$ is injective.
(c) For any singularity f in the μ-homotopy class of f_0 and any marking ρ,

$$\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)]) = \mathrm{Stab}_{G_{\mathbb{Z}}}(BL([(f,\pm\rho)])) \tag{82}$$

(only $\subset$ and the finiteness of both groups are clear).

The second author has a long-going project on Torelli-type conjectures. Already in [6], part (b) was conjectured and proved for all simple and unimodal singularities and almost all bimodal singularities (all except 3 subseries of the 8 bimodal series). This was possible without the general construction of M_μ and D_{BL}, which came later in [11] and [9]. In the concrete cases considered in [6], it is easy to identify a posteriori the spaces M_μ and D_{BL}. We will make use of that in the Sects. 6 and 7. Part (c) was conjectured in [11]. See [8] and [10] for other Torelli-type results.

The following lemma from [12] clarifies the logic between the parts (a), (b), and (c) of Conjecture 5.11.

Lemma 5.12 *In Conjecture 5.11, (a) $\iff$ (b) and (c).*

Part (a) of Conjecture 5.11 was proved in [12] for the simple and those 22 of the 28 exceptional unimodal and bimodal singularities, where all eigenvalues of the monodromy have multiplicity one. Here it will be proved for the remaining unimodal and the remaining exceptional bimodal singularities.

As part (b) of Conjecture 5.11 was already proved in all these cases in [6], the main work in [12, Sect. 8] and here is the control of the group $G_{\mathbb{Z}}$. This is carried out here in Sects. 3 and 4, and it is surprisingly difficult.

6 G^{mar}, M_μ^{mar} and a Strong Torelli Result for the Simple Elliptic and the Hyperbolic Singularities

The 1-parameter families of the hyperbolic singularities of type T_{pqr} ($p,q,r\in\mathbb{N}_{\geq 2}$, $p\geq q\geq r$, $\kappa := \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$) have as surface singularities the normal forms [1]

$$x^p + y^q + z^r + t\cdot xyz, \quad t\in X := \mathbb{C}^*. \tag{83}$$

The 1-parameter families of the simple elliptic singularities $T_{333} = \widetilde{E}_6, T_{442} = \widetilde{E}_7, T_{632} = \widetilde{E}_8$ have as surface singularities different normal forms [20]. The normal form $x^p + y^q + z^r + t\cdot xyz$ does in the case of T_{442} not contain representatives of all right equivalence classes; the class with j-invariant $j = 1$ is missing [20, 1.11, Bem. (ii)]. Therefore, we work in the following also with the Legendre normal forms:

$$\begin{aligned} T_{333}: &\ y(y-x)(y-\lambda x) - xz^2, \quad t\in X := \mathbb{C}-\{0,1\},\\ T_{442}: &\ yx(y-x)(y-\lambda x) + z^2, \quad t\in X := \mathbb{C}-\{0,1\},\\ T_{632}: &\ y(y-x^2)(y-\lambda x^2) + z^2, \quad t\in X := \mathbb{C}-\{0,1\}. \end{aligned} \tag{84}$$

They contain representatives of all right equivalence classes. Let X^{univ} denote the universal covering of X, so $X^{univ} = \mathbb{C}$ if $\kappa < 1$ and $X^{univ} = \mathbb{H}$ if $\kappa = 1$.

Theorem 6.1

(a) For the simple elliptic singularities and the hyperbolic singularities in any number of variables, the space M_μ^{mar} of right equivalence classes of marked singularities is $M_\mu^{mar} \cong X^{univ}$, so it is connected, and thus $G^{mar} = G_{\mathbb{Z}}$. The period map $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism, so the strong global Torelli Conjecture 5.11 (a) is true.

(b) Now consider the singularities of type T_{pqr} as curve singularities if $r = 2$ and as surface singularities if $r \geq 3$. Then

$$G_{\mathbb{Z}} = G^{mar} = G^{smar} \times \{\pm \mathrm{id}\}, \quad \text{equivalently: } -\mathrm{id} \notin G^{smar}. \tag{85}$$

The subgroup of G^{smar}, which acts trivially on M_μ^{mar}, is the kernel of the surjective map

$$G^{smar} \to \mathrm{Aut}(Ml(f_0)_{1,\mathbb{Z}}, L)/\{\pm \mathrm{id}\}. \tag{86}$$

It is equal to $\rho \circ (\mathscr{R}^f)_{hom} \circ \rho^{-1}$ for a generic $[(f, \rho)] \in M_\mu^{mar}$. Its size is 54, 16, and 6 for T_{333}, T_{442}, and T_{632}.

Proof

(a) The proof uses two Torelli-type results from [6].

We choose a marked reference singularity $[(f_0, \pm \mathrm{id})]$ in X^{univ}; then all elements of X^{univ} become marked singularities, because X^{univ} is simply connected. Then the period map $X^{univ} \to D_{BL}$ is well defined. The first Torelli-type result from [6] is that this map is an isomorphism.

Therefore, the marked Brieskorn lattices of the marked singularities in X^{univ} are all different. Therefore, the marked singularities in X^{univ} are all not right equivalent. This gives an embedding $X^{univ} \hookrightarrow M_\mu^{mar}(f_0)^0$.

On the other hand, we have the period map $BL : M_\mu^{mar}(f_0)^0 \to D_{BL}$, which is an immersion. As it restricts to the isomorphism $X^{univ} \to D_{BL}$, finally $X^{univ} = M_\mu^{mar}(f_0)^0$.

For part (a) it rests to show that $G^{mar} = G_{\mathbb{Z}}$. Then M_μ^{mar} is connected and $M_\mu^{mar} = X^{univ}$, and $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism.

We have to look closer at D_{BL} and the action of $G_{\mathbb{Z}}$ on it. In the case $\kappa < 1$,

$$D_{BL} \cong \{V \subset Ml(f_0)_1 \mid \dim V = 1, V \neq \ker(M_h - \mathrm{id})\}. \tag{87}$$

In the case $\kappa = 1$,

$$D_{BL} \cong \text{one component of } \{V \subset Ml(f_0)_1 \mid \dim V = 1, V \neq \overline{V}\}. \tag{88}$$

In both cases, the group $\mathrm{Aut}(Ml(f_0)_{1,\mathbb{Z}}, L)/\{\pm \mathrm{id}\}$ acts faithfully on D_{BL}.

In both cases, the second Torelli-type result from [6] which we need is that the period map

$$X^{univ}/\sim_R \to D_{BL}/\operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}},L) \tag{89}$$

is an isomorphism. Here $\sim_R$ denotes right equivalence for unmarked singularities.

Because of $X^{univ} = M_\mu^{mar}(f_0)^0$,

$$X^{univ}/\sim_R = M_\mu^{mar}(f_0)^0/G^{mar}. \tag{90}$$

The isomorphism (89) and the isomorphism $M_\mu^{mar}(f_0)^0 = X^{univ} \to D_{BL}$ show that the map

$$G^{mar} \to \operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}},L) \tag{91}$$

is surjective. This completes the proof of part (b) of Theorem 3.1.

It also shows that for proving $G^{mar} = G_\mathbb{Z}$, it is sufficient to show that the kernels of the maps to $\operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}},L)/\{\pm \operatorname{id}\}$ coincide. The kernel of the map $G_\mathbb{Z} \to \operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}},L)/\{\pm \operatorname{id}\}$ was determined in Theorem 3.1. In fact, this is the only part of Theorem 3.1 which we need here. It consists of those elements of $(U_1 \rtimes U_2) \times \{\pm \operatorname{id}\}$ in (22) for which $\delta = 0$, so it is isomorphic to the group

$$\left(\{(\alpha,\beta,\gamma) \in \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \,|\, \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod 1\} \rtimes U_2\right) \times \{\pm \operatorname{id}\}$$
$$=: (U_1^0 \rtimes U_2) \times \{\pm \operatorname{id}\}. \tag{92}$$

The kernel of the map $G^{mar} \to \operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}},L)/\{\pm \operatorname{id}\}$ is the subgroup of G^{mar} which acts trivially on $M_\mu^{mar}(f_0)^0$. It is the isotropy group in G^{mar} of a generic point $[(f,\pm\rho)] \in M_\mu^{mar}(f_0)^0$. So by Theorem 5.8 (g), it is the group

$$\rho \circ G_{\mathcal{R}}^{mar}(f) \circ \rho^{-1} = \rho \circ \{\pm \varphi_{hom} \,|\, \varphi \in \mathcal{R}^f\} \circ \rho^{-1}. \tag{93}$$

As f is generic, we can and will use now the normal form $f = x^p + y^q + z^r + t \cdot xyz$, also in the case $\kappa = 1$. The following coordinate changes generate a finite subgroup $S \subset \mathcal{R}^f$

$$\varphi^{\alpha,\beta,\gamma} : (x,y,z) \mapsto (e^{2\pi i\alpha/p}x, e^{2\pi i\beta/q}y, e^{2\pi i\gamma/r}z) \tag{94}$$
$$\text{with } \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod 1,$$
$$\varphi^{1,2} : (x,y,z) \mapsto (y,x,z) \quad \text{if } p = q,$$
$$\varphi^{2,3} : (x,y,z) \mapsto (x,z,y) \quad \text{if } q = r,$$
$$\varphi^{minus} : (x,y,z) \mapsto (x,y,-z-txy) \quad \text{if } r = 2.$$

φ^{minus} has order 2 and commutes with the other coordinate changes ($q = r = 2$ is impossible because of $\kappa \leq 1$). The group S is isomorphic (as an abstract group) to $U_1^0 \rtimes U_2$ if $r \geq 3$ and to $(U_1^0 \rtimes U_2) \times \{\pm \mathrm{id}\}$ if $r = 2$. The map to 1-jets of coordinate changes is injective,

$$S \xrightarrow{\cong} j_1 S \subset j_1 \mathcal{R}^f \subset j_1 \mathcal{R}. \tag{95}$$

Now we have to treat the cases $r \geq 3$ and $r = 2$ separately.

The case $r \geq 3$: Then $j_1 \mathcal{R}^f$ is finite and isomorphic to R_f; the map

$$()_{hom} : R_f \to G_{\mathbb{Z}}(f) = \rho^{-1} \circ G_{\mathbb{Z}} \circ \rho$$

is injective, and the image $G_{\mathcal{R}}^{smar}(f)$ does not contain $-\mathrm{id}$ (Theorem 5.8). Therefore, then $S \cong (S)_{hom} \subset G_{\mathbb{Z}}(f)$ and $-\mathrm{id} \notin (S)_{hom}$. Thus, the group $(S)_{hom} \times \{\pm \mathrm{id}\}$ is isomorphic to $(U_1^0 \rtimes U_2) \times \{\pm \mathrm{id}\}$. Now it is clear that the group in (93) is at least as big as the group in (92). But it cannot be bigger. So they are of equal size. This implies $G^{mar} = G_{\mathbb{Z}}$.

The case $r = 2$: We claim that the map $S \to (S)_{hom}$ is injective. If this is true, then $(S)_{hom} \cong (U_1^0 \rtimes U_2) \times \{\pm \mathrm{id}\}$, and this is of equal size as the group in (92). Then again the group in (93) is at least as big as the group in (92), but it cannot be bigger. So they are of equal size. This implies $G^{mar} = G_{\mathbb{Z}}$.

It rests to prove the claim. For this we consider the curve singularity

$$g := x^p + y^q - \frac{1}{4} t x^2 y^2. \tag{96}$$

Then,

$$\begin{aligned} g + z^2 &= f \circ \psi \quad \text{with } \psi(x, y, z) = (x, y, z - \frac{1}{2} txy), \\ \mathcal{R}^{g+z^2} &= \psi^{-1} \circ \mathcal{R}^f \circ \psi, \\ \psi^{-1} \circ \varphi^{\alpha,\beta,\gamma} \circ \psi &= \varphi^{\alpha,\beta,\gamma}, \\ \psi^{-1} \circ \varphi^{1,2} \circ \psi &= \varphi^{1,2}, \quad \text{if } p = q, \\ \psi^{-1} \circ \varphi^{minus} \circ \psi &= ((x, y, z) \mapsto (x, y, -z)). \end{aligned} \tag{97}$$

($q = r = 2$ is impossible because of $\kappa \leq 1$). The subgroup

$$S^{curve} := \{\varphi^{\alpha,\beta,\gamma} \circ (\varphi^{minus})^{-\gamma} \mid (\alpha, \beta, \gamma) \in U_1^0\} \rtimes U_2 \tag{98}$$

has index 2 in S, its conjugate $\psi^{-1} \circ S^{curve} \circ \psi$ restricts to $\mathcal{R}^g$, and it maps injectively to $j_1 \mathcal{R}^g \cong R_g$. By Theorem 5.8 (c), the map $S^{curve} \to (S^{curve})_{hom}$ is injective, and $-\mathrm{id}$ is not in the image. But $(\varphi^{minus})_{hom} = -\mathrm{id}$. This proves the claim.

(b) By Theorem 5.5 (c), the projection $M_\mu^{smar} \to M_\mu^{mar}$ is a twofold covering, and $-\mathrm{id}$ exchanges the two sheets of the covering. Because of $M_\mu^{mar} = \mathbb{C}$ if $\kappa < 1$ and $M_\mu^{mar} = \mathbb{H}$ if $\kappa = 1$, M_μ^{smar} has two components. Therefore, $-\mathrm{id} \notin G^{smar}$ and $G_\mathbb{Z} = G^{mar} = G^{smar} \times \{\pm \mathrm{id}\}$.

The statements right before and after (86) were already proved and used in the proof of part (a).

The group $\rho \circ (\mathscr{R}^f)_{hom} \circ \rho^{-1}$ for a generic $[(f, \rho)] \in M_\mu^{mar}$ has size 54, 16, and 6 for T_{333}, T_{442}, and T_{632}, because it is isomorphic to an index 2 subgroup of the group in (92), and that group has 108, 32, and 12 elements in the cases T_{333}, T_{442}, and T_{632}
□

7 G^{mar}, M_μ^{mar}, and a Strong Torelli Result for 6 of the 28 Exceptional Unimodal and Bimodal Singularities

Normal forms for the 1-parameter families of the exceptional unimodal and bimodal singularities of types $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}$, and U_{16} in the minimal number of variables are as follows [1]. Here Z_{12} and Z_{18} are curve singularities, and Q_{12}, U_{12}, Q_{16}, and U_{16} are surface singularities. The singularity for the parameter $t = 0$ is quasihomogeneous; the others are semiquasihomogeneous. The space of the parameter $t = t_1$ or $t = (t_1, t_2)$ is $X = \mathbb{C}^{\mathrm{mod}(f_0)}$. The weights $w = (w_x, w_y)$, respectively, $w = (w_x, w_y, w_z)$ are normalized such that $\deg_w f_0 = 1$.

Normal form	mod(f_0)	Weights	
$Z_{12} : f_t = x^3y + xy^4 + tx^2y^3$	1	$(\frac{3}{11}, \frac{2}{11})$	
$Q_{12} : f_t = x^3 + y^5 + yz^2 + txy^4$	1	$(\frac{1}{3}, \frac{1}{5}, \frac{2}{5})$	
$U_{12} : f_t = x^3 + y^3 + z^4 + txyz^2$	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{4})$	(99)
$Z_{18} : f_t = x^3y + xy^6 + (t_1 + t_2y)y^9$	2	$(\frac{5}{17}, \frac{2}{17})$	
$Q_{16} : f_t = x^3 + y^7 + yz^2 + (t_1 + t_2y)xy^5$	2	$(\frac{1}{3}, \frac{1}{7}, \frac{3}{7})$	
$U_{16} : f_t = x^3 + xz^2 + y^5 + (t_1 + t_2y)x^2y^2$	2	$(\frac{1}{3}, \frac{1}{5}, \frac{1}{3})$	

The normal form of the quasihomogeneous singularity of type Q_{12}, Q_{16}, U_{12}, and U_{16} is a sum of an A_l-singularity in one variable and a D_{2m} singularity in two variables with $(l, 2m)$ as in table (100)=(58).

	Q_{12}	Q_{16}	U_{12}	U_{16}	
$(l, 2m)$	$(2, 6)$	$(2, 8)$	$(3, 4)$	$(4, 4)$	(100)
	$A_2 \otimes D_6$	$A_2 \otimes D_8$	$A_3 \otimes D_4$	$A_4 \otimes D_4$	

The rest of this section is devoted to the proof of the following theorem.

Theorem 7.1

(a) For the six families of exceptional unimodal and bimodal singularities of types $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}$, and U_{16} in any number of variables, the space M_μ^{mar} of right equivalence classes of marked singularities is $M_\mu^{mar} \cong X = \mathbb{C}^{\mathrm{mod}(f_0)}$, so it is connected, and thus $G^{mar} = G_\mathbb{Z}$. The period map $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism, so the strong global Torelli Conjecture 5.11 (a) is true.

(b) Now consider the singularities of type Z_{12} and Z_{18} as curve singularities and the singularities of types Q_{12}, U_{12}, Q_{16}, and U_{16} as surface singularities. Then their multiplicities are ≥ 3. Then

$$G_\mathbb{Z} = G^{mar} = G^{smar} \times \{\pm \mathrm{id}\}, \quad \text{equivalently: } -\mathrm{id} \notin G^{smar}. \tag{101}$$

Proof

(a) The proof is similar to the proof of Theorem 6.1, but simpler. We need only the first of the two Torelli-type results from [6], which were used in the proof of Theorem 6.1.

We choose as marked reference singularity the quasihomogeneous singularity with trivial marking $[(f_0, \pm \mathrm{id})]$ in X. Then all elements of X become marked singularities, because X is simply connected. Then the period map $X \to D_{BL}$ is well defined. A Torelli-type result from [6] says that this map is an isomorphism. It is in fact easy, because the singularities here are semiquasihomogeneous and only f_0 is quasihomogeneous. That makes the calculations easy.

Therefore, the marked Brieskorn lattices of the marked singularities X are all different. Therefore, the marked singularities in X are all not right equivalent. This gives an embedding $X \hookrightarrow M_\mu^{mar}(f_0)^0$.

On the other hand, we have the period map $BL : M_\mu^{mar}(f_0)^0 \to D_{BL}$, which is an immersion. As it restricts to the isomorphism $X \to D_{BL}$, finally $X = M_\mu^{mar}(f_0)^0$.

For part (a), it rests to show that $G^{mar} = G_\mathbb{Z}$. Then M_μ^{mar} is connected and $M_\mu^{mar} = X$, and $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism.

The weights of the deformation parameter(s) t_1 (and t_2) equip the parameter space $X = M_\mu^{mar}$ with a good $\mathbb{C}^*$-action. It commutes with the action of $G_\mathbb{Z}$. This gives the first equality in

$$G^{mar} = \mathrm{Stab}_{G_\mathbb{Z}}([(f_0, \pm \mathrm{id})]) = G_\mathscr{R}^{mar}(f_0). \tag{102}$$

The second equality is part of Theorem 5.8 (g).

On the other hand, by Theorem 5.8 (d), $G_\mathscr{R}^{smar}(f_0) \times \{\pm \mathrm{id}\} = G_\mathscr{R}^{mar}(f_0)$. But because f_0 is a quasihomogeneous singularity of degree ≥ 3, the group $G_\mathscr{R}^{smar}(f_0)$ can be calculated easily via $\mathrm{Stab}_{G_w}(f_0)$; see Theorem 5.8 (c) and Theorem 5.9: $\mathrm{Stab}_{G_w}(f_0) \xrightarrow{\cong} G_\mathscr{R}^{smar}(f_0)$. Therefore, it is sufficient to show that $\mathrm{Stab}_{G_w}(f_0)$ has half as many elements as the group $G_\mathbb{Z}$. We postpone its proof. If it holds, then

$$G_\mathbb{Z} = G^{mar} = G_\mathscr{R}^{mar}(f_0) = G_\mathscr{R}^{smar}(f_0) \times \{\pm \mathrm{id}\} \tag{103}$$

follows, and part (a) of the theorem is proved. For part (b), the same argument as in the proof of Theorem 6.1 (b) works: M_μ^{smar} is a twofold covering of M_μ^{mar}, and the two sheets are exchanged by the action of $-\mathrm{id}$. As $X = \mathbb{C}^{\mathrm{mod}(f_0)}$, M_μ^{smar} has two components, and $-\mathrm{id} \notin M_\mu^{smar}$.

In Theorem 4.1 it was shown that $G_\mathbb{Z}$ is $G_\mathbb{Z} = \{\pm M_h^k \mid k \in \mathbb{Z}\} \times U$ with U as in table (39)=(104):

	Z_{12}	Q_{12}	U_{12}	Z_{18}	Q_{16}	U_{16}
$U \cong$	$\{\mathrm{id}\}$	S_2	S_3	$\{\mathrm{id}\}$	S_2	S_3

(104)

Now we compare $\mathrm{Stab}_{G_w}(f_0)$. It is sufficient to find enough elements so that the resulting group has half as many elements as $G_\mathbb{Z}$.

The cases Z_{12} and Z_{18}: Then

$$\varphi_1 : (x, y) \mapsto (e^{2\pi i w_x} x, e^{2\pi i w_y} y) \quad \text{satisfies} \quad (\varphi_1)_{hom} = M_h. \tag{105}$$

This is already sufficient. Here $G^{smar} = G_{\mathcal{R}}^{smar} = \{M_h^k \mid k \in \mathbb{Z}\}$.

The cases $Q_{12}, Q_{16}, U_{12}, U_{16}$: Here it is convenient to make use of the decomposition of the singularity f_0 into a sum of an A_l singularity g_0 in one variable and a D_{2m} singularity h_0 in two variables. In all four cases, the weight system w' of the A_l singularity and the weight system w'' of the D_{2m} singularity have denominators $l+1$ and $2m-1$ with $\gcd(l+1, 2m-1) = 1$. Therefore,

$$\begin{aligned} \mathrm{Stab}_{G_w}(f_0) &= \mathrm{Stab}_{G_{w'}}(g_0) \times \mathrm{Stab}_{G_{w''}}(h_0) \\ &\cong \mathbb{Z}_{l+1} \times \begin{cases} \mathbb{Z}_{2m-1} \times S_2 & \text{if } m \geq 3 \\ \mathbb{Z}_3 \times S_3 & \text{if } m = 2. \end{cases} \end{aligned} \tag{106}$$

In all four cases, this group has half as many elements as $G_\mathbb{Z}$. □

8 More on $G_\mathbb{Z}$ for the Simple Elliptic Singularities

This section is motivated by the paper [17] of Milanov and Shen. They consider the 1-parameter families

$$x^p + y^q + z^r + t \cdot xyz, \quad t \in \Sigma \subset \mathbb{C} \tag{107}$$

of the simple elliptic singularities T_{pqr} with $(p, q, r) \in \{(3, 3, 3), (4, 4, 2), (6, 3, 2)\}$ which had also been used above in Sect. 6. Here $\Sigma \subset \mathbb{C}$ is the complement of the finite set of parameters where the function in (107) has a non-isolated singularity. Remark that now $\chi := \mathrm{lcm}(p, q, r) = p$.

In [17] the groups of the transversal monodromies of these three families are studied, more precisely, the natural representations

$$\begin{aligned} \rho &: \pi_1(\Sigma) \to G_{\mathbb{Z}}, \\ \rho_1 &: \pi_1(\Sigma) \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L), \\ \rho_{\neq 1} &: \pi_1(\Sigma) \to \mathrm{Aut}(Ml(f)_{\neq 1,\mathbb{Z}}, L), \\ \overline{\rho}_{\neq 1} &: \pi_1(\Sigma) \to \mathrm{Aut}(Ml(f)_{\neq 1,\mathbb{Z}}, L)/\langle M_h\rangle. \end{aligned} \tag{108}$$

By explicit computations, they show

$$\ker(\rho_1) \subset \ker(\overline{\rho}_{\neq 1}). \tag{109}$$

They ask about a conceptual explanation of (109) and whether this might be true for other normal forms, i.e., other natural 1-parameter families of the simple elliptic singularities. Because of (109), there is an induced representation

$$\rho_W : \mathrm{Im}(\rho_1) \to \mathrm{Aut}(Ml(f)_{\neq 1}, L)/\langle M_h\rangle. \tag{110}$$

Then $\ker(\rho_W) \subset \mathrm{Im}(\rho_1) \subset \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$. One main result of [17] is the following.

Theorem 8.1 *In all three cases, under an isomorphism* $\mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L) \cong SL(2,\mathbb{Z})$ *as in* (18)*, the subgroup* $\ker(\rho_W)$ *is isomorphic to the principal congruence subgroup* $\Gamma(p)$*.*

Here $\Gamma(N) := \{A \in SL(2,\mathbb{Z}) \,|\, A \equiv \mathbf{1}_2 \mod N\}$ (not $A \equiv \pm\mathbf{1}_2 \mod N$) is the principal congruence subgroup of level N of $SL(2,\mathbb{Z})$.

In the following, we give results which complement (109) and Theorem 8.1. We consider not a special 1-parameter family, but the biggest possible family of quasihomogeneous simple elliptic singularities. Define $(w_x, w_y, w_z) := (\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$, define for any monomial its weighted degree $\deg_w(x^\alpha y^\beta z^\gamma) := \alpha w_x + \beta w_y + \gamma w_z$, and define

$$\begin{aligned} \mathbb{C}[x,y,z]_1 &:= \langle x^\alpha y^\beta z^\gamma \mid \deg_w(x^\alpha y^\beta z^\gamma) = 1\rangle_{\mathbb{C}}, \\ R &:= \{f \in \mathbb{C}[x,y,z]_1 \,|\, f \text{ has an isolated singularity at } 0\}. \end{aligned} \tag{111}$$

R is the complement of a hypersurface in the vector space $\mathbb{C}[x,y,z]_1$. The transversal monodromies of the family of singularities parametrized by R give the natural representation σ; the other representations are induced,

$$\begin{aligned} \sigma &: \pi_1(R) \to G_{\mathbb{Z}}, \\ \sigma_1 &: \pi_1(R) \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L), \end{aligned} \tag{112}$$

$$\sigma_{\neq 1}: \pi_1(R) \to \mathrm{Aut}(Ml(f)_{\neq 1,\mathbb{Z}}, L),$$
$$\overline{\sigma}_{\neq 1}: \pi_1(R) \to \mathrm{Aut}(Ml(f)_{\neq 1,\mathbb{Z}}, L)/\langle M_h \rangle.$$

By the definition of G^{smar} in [12, Definition 3.1], $G^{smar} = \mathrm{Im}(\sigma)$. Theorem 6.1 tells that the monodromy group $\mathrm{Im}(\sigma)$ is as large as possible (up to $\pm$ id in the case T_{333}).

Theorem 8.2 $\mathrm{Im}(\sigma) = G_{\mathbb{Z}}$ *in the cases* T_{442} *and* T_{632}*, and* $G_{\mathbb{Z}} = \mathrm{Im}(\sigma) \times \{\pm \mathrm{id}\}$ *in the case* T_{333} *(here it is important that the surface singularities are considered).*

The explicit information on $G_{\mathbb{Z}}$ in Theorem 3.1 allows the following conclusion.

Corollary 8.3 *The analogue* $\ker(\sigma_1) \subset \ker(\overline{\sigma}_{\neq 1})$ *of* (109) *does not hold in the cases* T_{333} *and* T_{442}*. It holds in the case* T_{632}*, and there the analogue of* (109) *holds for any* μ*-constant family.*

Proof By Theorem 3.1 (c),

$$\begin{aligned} &\{g \in G_{\mathbb{Z}} \mid g|_{Ml(f)_1} = \mathrm{id}\} \\ = &\{\mathrm{id}\,|_{Ml(f)_1} \times (M_h|_{Ml^{(1)}_{\mathbb{C}}})^{\alpha} \times (M_h|_{Ml^{(2)}_{\mathbb{C}}})^{\beta} \times (M_h|_{Ml^{(3)}_{\mathbb{C}}})^{\gamma} \mid \\ &(\alpha, \beta, \gamma) \in \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \text{ with } \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod 1\} \times U_2. \end{aligned}$$

In the cases T_{442} and T_{632}, this is isomorphic to $\ker(\sigma_1)/\ker(\sigma)$, and in the case T_{333} $\ker(\sigma_1)/\ker(\sigma)$ is isomorphic to this group or a subgroup of this group of index 2. In the cases T_{442} and T_{333}, already the factor U_2 in this group is an obstruction to the analogue of (109).

In the case T_{632}, U_2 is trivial and

$$\{g \in G_{\mathbb{Z}} \mid g|_{Ml(f)_1} = \mathrm{id}\} = \{M_h^{\alpha} \mid \alpha \in \mathbb{Z}\} \tag{113}$$

(because above α determines β and γ uniquely in the case $(p, q, r) = (6, 3, 2)$). Therefore, the analogue of (109) holds in the case T_{632} for the family parametrized by R and for any subfamily. □

Equation (109) is used in [17] in order to define ρ_W and the group $\ker(\rho_W)$. But because $M_h|_{Ml(f)_1} = \mathrm{id}$, it is obvious that the group $\ker(\rho_W)$ coincides with

$$\{g|_{Ml(f)_{1,\mathbb{Z}}} \mid g \in \mathrm{Im}(\sigma), g|_{Ml(f)_{\neq 1,\mathbb{Z}}} = \mathrm{id}\} \subset \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L). \tag{114}$$

And the analogue of this group can be defined for any μ-constant family, whether or not it satisfies the analogue of (109). Our main result in this section is Theorem 8.4. Our proof uses Theorem 3.1. A different proof of Theorem 8.4 was given by Kluitmann in [14, III 2.4 Satz, p. 66]. Theorem 8.4 shows that the group $\Gamma(p)$ turns up naturally within the maximal possible μ-constant family, which is parametrized by R, and it shows the part $\ker(\rho_W) \subset \Gamma(p)$ of the equality $\ker(\rho_W) = \Gamma(p)$ in Theorem 8.1.

Theorem 8.4 *In all three cases, under an isomorphism* $\mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L) \cong SL(2,\mathbb{Z})$ *as in* (18)*, the subgroup*

$$\{g|_{Ml(f)_{1,\mathbb{Z}}} \mid g \in G_{\mathbb{Z}}, g|_{Ml(f)_{\neq 1,\mathbb{Z}}} = \mathrm{id}\} \subset \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$$

is isomorphic to the principal congruence subgroup $\Gamma(p)$.

Proof We use the notations and objects in the proof of Theorem 3.1. $Ml_{\mathbb{C}}^{(1)}$ was defined in (31). Define

$$Ml_{\mathbb{Z}}^{(1)} := Ml_{\mathbb{C}}^{(1)} \cap Ml(f), \tag{115}$$

and analogously $Ml_{\mathbb{Z}}^{(2)}$ and $Ml_{\mathbb{Z}}^{(3)}$. Then

$$Ml_{\mathbb{Z}}^{(1)} = (M_h - \mathrm{id})(Ml_{\mathbb{Z}}^{[1]}).$$

Because of (32), this image is generated as a $\mathbb{Z}$-lattice by

$$\delta_2 - (\delta_1 + \tilde{b}_1), \delta_3 - \delta_2, \ldots, \delta_{p-1} - \delta_{p-2}, -(\delta_1 + \ldots + \delta_{p-1}) - \delta_{p-1}, \tag{116}$$

respectively, by

$$\delta_1 + (p-1)\delta_2, \delta_3 - \delta_2, \ldots, \delta_{p-1} - \delta_{p-2}, p\delta_2 - \delta_{\mu-1} + \delta_\mu. \tag{117}$$

$Ml_{\mathbb{Z}}^{(2)}$ and $Ml_{\mathbb{Z}}^{(3)}$ (if $r \geq 3$) are generated by the analogous elements. If $r = 2$, then

$$Ml_{\mathbb{Z}}^{(3)} = \mathbb{Z} \cdot (2\delta_{\mu-2} - \delta_{\mu-1} + \delta_\mu). \tag{118}$$

In any case, the sum $Ml_{\mathbb{Z}}^{(1)} \oplus Ml_{\mathbb{Z}}^{(2)} \oplus Ml_{\mathbb{Z}}^{(3)}$ is a sublattice of finite index of the primitive sublattice $Ml(f)_{\neq 1,\mathbb{Z}}$ in $Ml(f)$. Observe that $q|p$ and $r|p$ in all three cases. The lattice $Ml(f)_{\neq 1,\mathbb{Z}}$ is generated by

$$\delta_1 + (p-1)\delta_2, \delta_3 - \delta_2, \ldots, \delta_{p-1} - \delta_{p-2}, p\delta_2 - \delta_{\mu-1} + \delta_\mu, \tag{119}$$
$$\delta_p + (q-1)\delta_{p+1}, \delta_{p+2} - \delta_{p+1}, \ldots, \delta_{p+q-2} - \delta_{p+q-3}, \frac{p}{q}\delta_2 - \delta_{p+1},$$
$$\delta_{p+q-1} + (r-1)\delta_{p+q}, \delta_{p+q+1} - \delta_{p+q}, \ldots, \delta_{\mu-2} - \delta_{\mu-3}, \frac{p}{r}\delta_2 - \delta_{p+q},$$

if $r \geq 3$. If $r = 2$, then the third line has to be replaced by

$$\frac{p}{r}\delta_2 - \delta_{\mu-2}.$$

In any case, one sees

$$Ml(f) = Ml(f)_{\neq 1,\mathbb{Z}} \oplus \mathbb{Z}\cdot\delta_2 \oplus \mathbb{Z}\cdot\delta_\mu. \tag{120}$$

One also calculates

$$\tilde{b}_1 = \delta_{\mu-1} - \delta_\mu = \gamma_1 + p\delta_2 \tag{121}$$

$$\text{with} \quad \gamma_1 := -(p\delta_2 - \delta_{\mu-1} + \delta_\mu) \in Ml(f)_{\neq 1,\mathbb{Z}},$$

$$\tilde{b}_2 = \gamma_2 + p\delta_\mu, \tag{122}$$

$$\text{with} \quad \gamma_2 := \sum_{i=1}^{p-1}(p-i)\delta_i + \sum_{i=1}^{q-1}\frac{p}{q}(q-i)\delta_{p-1+i} + \sum_{i=1}^{r-1}\frac{p}{r}(r-i)\delta_{p+q-2+i} + p\delta_{\mu-1} - p\delta_\mu \in Ml(f)_{\neq 1,\mathbb{Z}}.$$

One sees also

$$Ml(f) \cap (\mathbb{Q}\cdot\gamma_1 + \mathbb{Q}\cdot\gamma_2) = \mathbb{Z}\cdot\gamma_1 \oplus \mathbb{Z}\cdot\gamma_2. \tag{123}$$

For any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, define the automorphism $f : Ml(f)_\mathbb{Q} \to Ml(f)_\mathbb{Q}$ by

$$f(\tilde{b}_1) := a\tilde{b}_1 + c\tilde{b}_2, \quad f(\tilde{b}_2) := b\tilde{b}_1 + d\tilde{b}_2, \quad f|_{Ml(f)_{\neq 1,\mathbb{Z}}} := \mathrm{id}\,. \tag{124}$$

It respects L because the decomposition $Ml(f)_\mathbb{Q} = Ml(f)_{1,\mathbb{Q}} \oplus Ml(f)_{\neq 1,\mathbb{Q}}$ is left and right orthogonal with respect to L. It is an automorphism of $Ml(f)$ if and only if $f(\delta_2)$ and $f(\delta_\mu)$ are in $Ml(f)$. One calculates

$$f(\delta_2) = a\delta_2 + c\delta_\mu + \frac{1}{p}(a-1)\gamma_1 + \frac{1}{p}c\gamma_2,$$

$$f(\delta_\mu) = b\delta_2 + d\delta_\mu + \frac{1}{p}b\gamma_1 + \frac{1}{p}(d-1)\gamma_2.$$

In view of (123), this shows

$$f \in G_\mathbb{Z} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathbf{1}_2 \mod p \overset{\text{def.}}{\iff} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p). \qquad \square$$

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Divisor Class Groups of Affine Complete Intersections

Helmut A. Hamm

Abstract Geometrically interesting examples of factorial rings are provided by the coordinate rings of certain affine complete intersections. Here one has to show that the Weil divisor class group vanishes. Using Hodge theory or Deligne-Beilinson cohomology one can prove that it is sufficient to show the vanishing of certain singular homology groups. Examples from singularity theory are given.

Keywords Deligne-Beilinson cohomology • Divisor class group • Factorial domain • Hodge theory

MSC classification numbers: 13F15, 14C22, 14M05, 14C30, 14F43

1 Introduction

Let X be a normal complex (sc. irreducible) algebraic variety. Then we can look at the group of Weil divisor classes $Cl(X)$ on X. In the smooth case it coincides with the group of Cartier divisor classes on X, i.e. the Picard group $Pic(X)$. In general, $Cl(X) \simeq Cl(X \setminus \Sigma) \simeq Pic(X \setminus \Sigma)$, where Σ is the singular locus of X (use [13, II Prop. 6.5(b), p. 133]).

The case where X is affine is particularly interesting because the coordinate ring $\mathcal{O}(X)$ is factorial if and only if $Cl(X)$ is trivial, see [13, II Prop. 6.2, p. 131].

There is a modification of this notion: an integral domain R is called almost factorial if for each $x \in R \setminus R^* \cup \{0\}$ there is an $n > 0$ such that x^n can be written as a product of primary elements.

Note that an element is called primary if its principal ideal is primary.

Then $\mathcal{O}(X)$ is almost factorial if and only if $Cl(X)$ is a torsion group. See [27, Satz 1, p. 3].

H.A. Hamm (✉)
Fachbereich Mathematik und Informatik, Mathematisches Institut, WWU Münster, Einsteinstrasse 62, 48149 Münster, Germany
e-mail: hamm@uni-muenster.de

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For concrete results it is reasonable to restrict to the case of affine complete intersections: this allows to use general vanishing results and explicit calculations.

We will show how to use topological results from singularity theory, and we will extend the tools using results about the Picard group resp. the Picard group for line bundles with connection as well as Deligne-Beilinson cohomology.

Remember that the divisor class group has been considered very early in singularity theory by Brieskorn [4, 5] but he looked at the divisor class group of the local analytic ring.

Now let us discuss the tools:

Suppose that X is smooth. First we look at the case where X is projective or, more generally, complete. Because of GAGA we may switch between the algebraic and analytic category and use the exponential sequence

$$\to H^1(X^{an};\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to Pic(X) \to H^2(X^{an};\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to$$

where X^{an} is the corresponding complex manifold. Here Hodge theory is helpful: e.g. if we know that the first Betti number $b_1(X) := b_1(X^{an})$ is 0, the first two terms vanish, so $Pic^0(X) = 0$.

Recall the definition of Deligne cohomology (also in the case where X is not complete), cf. [9, 1.1, p. 45]:

Let $\mathbb{Z}(p) := (2\pi i)^p\mathbb{Z}$ and let $\mathbb{Z}(p)_{\mathcal{D}}$ be the complex

$$\mathbb{Z}(p)_{X^{an}} \to \Omega^0_{X^{an}} \to \ldots \to \Omega^{p-1}_{X^{an}} \to 0 \to \ldots$$

Then $H^k_{\mathcal{D}}(X^{an};\mathbb{Z}(p)) := \mathbb{H}^k(X^{an};\mathbb{Z}(p)_{\mathcal{D}})$.

Note that $Pic(X) \simeq H^2_{\mathcal{D}}(X^{an};\mathbb{Z}(1))$, because of the exponential sequence.

Let $Pic_c(X)$ resp. $Pic_{ci}(X)$ be the group of isomorphism classes on X with connection resp. integrable connection.

It is known that $H^2_{\mathcal{D}}(X^{an};\mathbb{Z}(2)) = Pic_c(X)$, $H^2_{\mathcal{D}}(X^{an};\mathbb{Z}(p)) = Pic_{ci}(X)$ for $p > 2$. See [11, p. 156].

In our case, even $H^2_{\mathcal{D}}(X^{an};\mathbb{Z}(p)) = Pic_{ci}(X)$ for $p \geq 2$, because every connection on the complete variety X is integrable, see [20, Lemma 3.5].

This yields an exact sequence

$$H^0(X,\Omega^1_X) \to Pic_{ci}(X) \to Pic(X) \to H^1(X,\Omega^1_X)$$

see [20, Theorem 2.1], and $Pic_{ci}X \simeq H^1(X^{an};\mathbb{C}^*)$, see, e.g., [20, Prop. 2.12].

Now let us pass to the case where X is smooth but not complete:

Use a good smooth compactification $\bar{X}$ (i.e. $\bar{X} \supset X$ is smooth and $D := \bar{X} \setminus X$ is a divisor with normal crossings). Similarly as above, we can use (generalized) Hodge theory (cf. [7]), Deligne-Beilinson cohomology or comparison with $Pic_{cir}(X)$, where Pic_{cir} refers to regular integrable connections.

More precisely: we have an exact sequence

$$H^0(\bar{X},\Omega^1_{\bar{X}}(log\, D)) \to Pic_{cir}(X) \to Pic(X) \to H^1(\bar{X},\Omega^1_{\bar{X}}(log\, D)) \quad (1)$$

cf. [20, Theorem 3.13, Lemma 3.14], and the Deligne-Beilinson cohomology is defined as follows (cf. [9, p. 57]): $H^k_{DB}(X, \mathbb{Z}(p)) := \mathbb{H}^k(\bar{X}^{an}, \mathbb{Z}(p)_{DB})$, where $\mathbb{Z}(p)_{DB} := cone(Rj^{an}_* \mathbb{Z}(p)_{X^{an}} \to \Omega^{\leq p-1}_{\bar{X}^{an}}(log\, D))[-1]$.

Here $j : X \to \bar{X}$ is the inclusion.

Then

$$Pic(X) = ker(H^2_{DB}(X, \mathbb{Z}(1)) \to H^3(\bar{X}^{an}, X^{an}; \mathbb{Z})), \tag{2}$$

see [17, Theorem 2.4].

2 Main Results

Theorem 2.1 *Let X be a smooth complex algebraic variety.*

(a) $b_1(X) = 0 \Rightarrow Pic^0(X) = 0$*, i.e.* $Pic(X) \simeq NS(X)$*, hence* $Pic(X)$ *is finitely generated.*
(b) $b_1(X) = b_2(X) = 0 \Rightarrow Pic(X) \xrightarrow{\simeq} H^2(X^{an}; \mathbb{Z})$ *which is a finite group.*
(c) If $H_1(X^{an}; \mathbb{Z}) = 0$ *we have that* $Pic(X)$ *is free abelian of finite rank.*
(d) $H_1(X^{an}; \mathbb{Z}) = 0, b_2(X) = 0 \Rightarrow Pic(X) = 0$.

Here $NS(X) := Pic(X)/Pic^0(X)$ is the Néron-Severi group.

Proof

(a) We have different possibilities to prove this:

(i) Let $\bar{X}$ be a good compactification of X, $D := \bar{X} \setminus X$. According to [20, Theorem 3.10] we have an exact sequence

$$Pic_{cir}(X) \to Pic(X) \xrightarrow{c_1} H^2(X^{an}; \mathbb{C}) \tag{3}$$

Suppose that $[\mathcal{L}] \in Pic^0(X)$. By the sequence above it has an inverse image of the form $[\mathcal{L}, \nabla]$ in $Pic_{cir}(X)$. But $Pic_{cir}(X) \simeq Pic_{ci}(X^{an}) \simeq H^1(X^{an}; \mathbb{C}^*)$, cf. [6, II Théor. 5.9, p. 97], [20, Prop. 2.12].

Look at the exact sequence

$$H^1(X^{an}; \mathbb{C}) \to H^1(X^{an}; \mathbb{C}^*) \to H^2(X^{an}; \mathbb{Z})$$

Since $[\mathcal{L}] \in Pic^0(X)$, $[\mathcal{L}, \nabla]$ has an inverse image in $H^1(X^{an}; \mathbb{C})$ but this group vanishes since $b_1(X) = 0$.

So $[\mathcal{L}, \nabla] = 0$, hence $[\mathcal{L}] = 0$.

(ii) Or: apply [19, Prop. 2.2, p. 74].

(iii) We use Deligne-Beilinson cohomology.

We have an exact sequence

$$H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \to H^2_{DB}(X, \mathbb{Z}(1)) \to H^2(X^{an}; \mathbb{Z}) \tag{4}$$

By Hodge theory we have that the left group vanishes because $b_1(X) = 0$: The spectral sequence for the Hodge filtration on $H^*(X^{an}; \mathbb{C})$ degenerates at E_1, see [7, Cor. 3.2.13], and $Gr^0_F H^1(X^{an}; \mathbb{C}) = H^1(\bar{X}, \Omega^0_{\bar{X}}(log\, D)) = H^1(\bar{X}, \mathcal{O}_{\bar{X}})$. The rest follows from the fact that

$$Pic^0(X) = ker(H^2_{DB}(X, \mathbb{Z}(1)) \to H^2(X^{an}; \mathbb{Z}))$$

recall that $Pic(X) = ker(H^2_{DB}(X; \mathbb{Z}(1)) \to H^3(\bar{X}^{an}, X^{an}; \mathbb{Z}))$, see (2).

(b) Again we have different possibilities to prove this:

(i) Look at the exact sequence

$$H^1(X^{an}; \mathbb{C}) \to H^1(X^{an}; \mathbb{C}^*) \to H^2(X^{an}; \mathbb{Z}) \to H^2(X^{an}; \mathbb{C})$$

The first and last group are 0, so $Pic_{cir}(X) \simeq H^1(X^{an}; \mathbb{C}^*) \simeq H^2(X^{an}; \mathbb{Z})$.

By the exact sequence (3) above we get that $Pic_{cir}(X) \to Pic(X)$ is surjective. But furthermore we have bijectivity: the composition $Pic_{cir}(X) \to Pic(X) \to H^2(X^{an}; \mathbb{Z})$ is an isomorphism, as we have seen.

The isomorphism $Pic_{cir}(X) \simeq Pic(X)$ can also be obtained from the exact sequence

$$H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D)) \to Pic_{cir}(X) \to Pic(X) \to H^1(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$$

quoted in (1) because the spectral sequence for the Hodge filtration for X degenerates at E_1.

(ii) Or: by (a) $Pic(X) \simeq NS(X)$. Since $b_2(X) = 0$ we have $NS(X) \simeq Tor(NS(X))$, and $Tor(NS(X)) \simeq Tor(H^2(X^{an}; \mathbb{Z}))$ by Hamm and Lê [19, Cor. 3.2, p. 80]. Finally, $Tor(H^2(X^{an}; \mathbb{Z})) \simeq H^2(X^{an}; \mathbb{Z})$.

(iii) Finally we have the exact sequence

$$H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \to H^2_{DB}(X, \mathbb{Z}(1)) \to H^2(X^{an}; \mathbb{Z}) \to H^2(\bar{X}, \mathcal{O}_{\bar{X}})$$

whose left part was quoted in (4) above.

Since $b_1(X) = b_2(X) = 0$ the left and right group vanish, because the spectral sequence for the Hodge filtration for X degenerates at E_1.

So $H^2_{DB}(X, \mathbb{Z}(1)) \simeq H^2(X^{an}; \mathbb{Z})$. Finally $Pic(X) = H^2_{DB}(X, \mathbb{Z}(1))$ because the mapping $H^2_{DB}(X, \mathbb{Z}(1)) \to H^3(\bar{X}^{an}, X^{an}; \mathbb{Z})$ is the zero map: the last group is free abelian by Hamm [17, Lemma 1.5] and the mapping factors through the finite group $H^2(X^{an}; \mathbb{Z})$.

(c) If $H_1(X^{an}; \mathbb{Z}) = 0$, we have $b_1(X) = 0$ and $0 = Tor(H_1(X^{an}; \mathbb{Z})) \simeq Tor(H^2(X^{an}; \mathbb{Z}))$, so $NS(X) \subset H^2(X^{an}; \mathbb{Z})$ is torsion free. Now conclude by (a).

(d) follows from (b) and (c).

Let us give an application:

Let $f_1, \ldots, f_k \in \mathbb{C}[z_1, \ldots, z_{n+k}]$ be polynomials which are weighted homogeneous of degree $d_1, \ldots, d_k$ with respect to weights $w_1, \ldots, w_{n+k} > 0, n \geq 2$. We assume that the algebraic variety $X := \{f_1 = \ldots = f_{k-1} = 0\}$ is a complete intersection with isolated singularity and that $f_k|X$ has an isolated singularity at 0, too. More precisely, assume that if $X := \{f_1 = \ldots = f_{k-1} = 0\}$ we have that $X \setminus \{0\}$ is smooth of dimension $n+1$ and that $X \cap \{f_k = 0\} \setminus \{0\}$ is smooth of dimension n. Put $X_t := X \cap f_k^{-1}(\{t\})$.

Theorem 2.2 *$\mathcal{O}(X_t)$ is factorial if $n \geq 3, t \in \mathbb{C}^*$ or $n \geq 4, t = 0$.*

Proof We apply Theorem 2.1(d).

First assume that $t \neq 0, n \geq 3$. Then X_t^{an} has the homotopy type of the Milnor fibre of $f_k|X$, so of a bouquet of n-spheres. See [14, Satz 1.7]. This implies that X_t^{an} is simply connected and $b_2(X_t) = 0$.

Now assume that $t = 0$ and $n \geq 4$. Then $X_0^{an} \setminus \{0\}$ has the homotopy type of the link of X_0 at 0 which is $(n-2)$-connected, see [14, Kor. 1.3]. Hence $X_0^{an} \setminus \{0\}$ is simply connected, $b_2(X_0 \setminus \{0\}) = 0$. So $Cl(X_0) = Cl(X_0 \setminus \{0\}) = 0$.

In the following two paragraphs we will derive results about $X_t, t \neq 0$, resp. X_0 under milder hypotheses.

Now let us drop the smoothness assumption.

Lemma 2.3 *Let X be a normal complex algebraic variety, Σ the singular locus of X (which is of codimension ≥ 2).*

Suppose that $b_1(X \setminus \Sigma) = 0$ and $H^2(X^{an}; \mathbb{Z}) = 0$. Then $Pic(X) = 0$.

Proof By Theorem 2.1(a) we have that $Pic^0(X \setminus \Sigma) = 0$.

Suppose that $\mathcal{L}$ is invertible on X. Then $c_1(\mathcal{L}) = 0$, hence $[\mathcal{L}|X \setminus \Sigma] \in Pic^0(X \setminus \Sigma) = 0$.

So $\mathcal{L}|X \setminus \Sigma$ is trivial. By normality, $j_*\mathcal{O}_{X \setminus \Sigma} \simeq \mathcal{O}_X$, see [21, Theorem 2.15, p. 124]. Hence $\mathcal{H}^1_\Sigma(\mathcal{O}_X) = 0$, so $\mathcal{H}^1_\Sigma(\mathcal{L}) = 0$, too, which implies $\mathcal{L} \simeq j_*j^*\mathcal{L}$, and $j_*j^*\mathcal{L} \simeq j_*\mathcal{O}_{X \setminus \Sigma} \simeq \mathcal{O}_X$. Hence $\mathcal{L}$ is trivial, too.

3 Smooth Affine Complete Intersections

Here we look at different classes of such varieties.

(a) Let $f_1, \ldots, f_k \in \mathbb{C}[z_1, \ldots, z_{n+k}]$ be polynomials which are weighted homogeneous of degree $d_1, \ldots, d_k$ with respect to weights $w_1, \ldots, w_{n+k}, n \geq 2$. We assume that the algebraic variety $X := \{f_1 = \ldots = f_{k-1} = 0\}$ is a complete intersection with isolated singularity and that $f_k|X$ has an isolated singularity at 0, too. Put $X_t := X \cap f_k^{-1}(\{t\}), t \neq 0$: it is a smooth affine complete intersection.

Theorem 3.4 *Suppose $t \in \mathbb{C}^*$.*

(a) (see Theorem 2.2) $\mathcal{O}(X_t)$ is factorial if $n \geq 3$.
(b) $Pic(X_t)$ is free abelian of finite rank if $n = 2$.

Proof X_t^{an} has the homotopy type of the Milnor fibre of $f_k|X$, so of a bouquet of n-spheres. So X_t^{an} is simply connected for $n \geq 2$.

(a) For $n \geq 3$ we have $b_2(X_t) = 0$, too. Now apply Theorem 2.1(d). See Theorem 2.2.
(b) This follows from Theorem 2.1(c).

(b) Now we study a different case: Let $\bar{X}$ be any smooth projective complete intersection of dimension n such that the part at infinity is a smooth divisor D. Then $X := \bar{X} \setminus D$ is a smooth affine complete intersection. We have an analogue of Theorem 3.4:

Lemma 3.5

(a) If $n \geq 3$, then $\mathcal{O}(X)$ is factorial.
(b) $n = 2 \Rightarrow Pic(X)$ is free abelian of finite rank.

Proof Similar as in the case of Theorem 3.4. Instead of arguing with the Milnor fibre we use a theorem of Zariski-Lefschetz type in order to show that X^{an} is $(n-1)$-connected, see [18, Theorem 1.1.1].

Now let us study the case $n = 2$ more closely. Let $\rho(\bar{X})$ be the Picard number of $\bar{X}$, i.e. the rank of the Néron-Severi group of $\bar{X}$, and let $p_a(\bar{X}) := \sum_{i<n}(-1)^i \dim H^{n-i}(\bar{X}, \mathcal{O}_{\bar{X}})$ be the arithmetic genus of $\bar{X}$, see [13, III Exc. 5.3, p. 230]. Since $\bar{X}$ is a complete intersection of dimension n, $H^j(\bar{X}, \mathcal{O}_{\bar{X}}) = 0, 0 < j < n$, so $p_a(\bar{X}) = \dim H^n(\bar{X}, \mathcal{O}_{\bar{X}})$, see [13, III Exc. 5.5, p. 231].

Theorem 3.6 *Assume $n = 2$.*

(a) $Pic(X) \simeq \mathbb{Z}^{\rho(\bar{X})-1}$.
(b) If $p_a(\bar{X}) = 0$, $Pic(X) \simeq \mathbb{Z}^{\chi(\bar{X})-3}$.

Proof

(a) We have an exact sequence

$$0 \to \mathbb{Z} \to Pic(\bar{X}) \to Pic(X) \to 0$$

cf. [13, II Prop. 6.5, p. 133]: note that the first mapping is indeed injective, because using $\mathcal{O}_{\bar{X}}(1)$ one sees that $Pic(\bar{X}) \to Pic(X)$ is not injective. Furthermore, $\bar{X}^{an}$ is simply connected, hence $Pic(\bar{X}) = NS(\bar{X})$, similarly for $Pic(X)$, cf. Theorem 2.1(a). By Lemma 3.5(b) we have that $NS(\bar{X}) \simeq NS(X) \oplus \mathbb{Z}$. The rest is clear.
(b) The exponential sequence for $\bar{X}$ yields that $Pic(\bar{X}) \simeq H^2(\bar{X}^{an}; \mathbb{Z})$, so $\rho(\bar{X}) = rk\, H^2(\bar{X}^{an}; \mathbb{Z})$.

(c) Finally we look at the case where X is an $\{1,\ldots,n+k\}$-full affine non-degenerate complete intersection of dimension n in the sense of [23, p. 137].

This means that $X = \{f_1 = \ldots = f_k = 0\} \subset \mathbb{C}^{n+k}$, where $f_1,\ldots,f_k$ are polynomials, and that the following holds for every subset J of $\{1,\ldots,n+k\}$: Let $j := \#J$ and consider $f_l^J := f_l|\{z \in \mathbb{C}^{n+k} \mid z_\nu = 0 \text{ for } \nu \notin J\}$ as a function on $\mathbb{C}^j$. Then $f_1^J = \ldots = f_k^J = 0\} \cap (\mathbb{C}^*)^j$ is a non-degenerate complete intersection which is full, i.e. $\dim \Delta(f_l^J) = j$ for all j. Here $\Delta(f_l^J)$ is the Newton polyhedron of f_l^J, i.e. the convex hull of its support.

Part (a) of the following theorem has been proved in the hypersurface case by Dolgachev [8, Cor. 1.2] by completely different methods:

Theorem 3.7 *Suppose that X satisfies the condition introduced just before.*

(a) If $n \geq 3$, then $\mathcal{O}(X)$ is factorial.
(b) $n = 2 \Rightarrow Pic(X)$ is free abelian of finite rank.

Proof As Lemma 3.5, using the Lefschetz theorem proved by Oka [23, Main Theorem (1.1)].

Now we give examples with $n = 2$:

Examples 3.8 In Examples 3.8.1 and 3.8.2 we use the formula of [13, I Exc. 7.2(c), p. 54] for the calculation of the arithmetic genus.

3.8.1 $X := \{z \in \mathbb{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 1\}$: then $p_a(\bar{X}) = 0$, X^{an} has the homotopy type of the corresponding Milnor fibre, so $\chi(\bar{X}, D) = \chi(X) = 2$. Furthermore, $\chi(D) = 2$, so $\chi(\bar{X}) = 4$. Hence $Pic(X) \simeq \mathbb{Z}$, by Theorem 3.6(b).

3.8.2 $X := \{z \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = 1\}$: then $p_a(\bar{X}) = 0$, X^{an} has the homotopy type of the corresponding Milnor fibre, so $\chi(\bar{X}, D) = 9$, $\chi(D) = 0$, $\chi(\bar{X}) = 9$, $Pic(X) \simeq \mathbb{Z}^6$.

3.8.3 p prime ≥ 5, $X := \{z \in \mathbb{C}^3 \mid z_1 z_2^{p-1} + z_2 z_3^{p-1} + z_3 z_1^{p-1} = 1\}$. By Shioda [25, Theorem 4.1] we have $\rho(\bar{X}) = 1$, so $\mathcal{O}(X)$ is factorial by Theorem 3.6(a).

3.8.4 $X := \{z \in \mathbb{C}^3 \mid z_1^4 + z_2^4 + z_3^4 = 1\}$: then $\rho(\bar{X}) = 20$, see [24, Sect. 1, Example 17], hence $Pic(X) \simeq \mathbb{Z}^{19}$.

4 Weil Divisor Class Groups of Singular Affine Complete Intersections

(a) Let X be an affine weighted homogeneous complete intersection of dimension $n \geq 2$ with an isolated singularity. Then X is normal. Let K be the link of X at 0, it is a deformation retract of $X^{an} \setminus \{0\}$.

Theorem 4.9

(a) $n \geq 4$ *implies that* $Cl(X) = 0$*, i.e.* $\mathcal{O}(X)$ *is factorial.*
(b) $n = 3$ *implies that* $Cl(X) \simeq NS(X \setminus \{0\})$*, so* $Cl(X)$ *is free abelian of rank* $\leq b_2(K)$.
In particular, $\mathcal{O}(X)$ *is factorial if* $n = 3$ *and* $b_2(K) = 0$*, i.e.* K *is a 5-dimensional rational homology sphere.*
(c) $n = 2, b_1(K) = 0$ *implies that* $Cl(X) \simeq H_1(K; \mathbb{Z})$.
In particular, $\mathcal{O}(X)$ *is factorial (resp. almost factorial) if* K *is an integral (resp. rational) homology sphere.*

Proof

(a) This follows from Theorem 2.1(d) because K is $(n-2)$-connected, so $Cl(X) = Cl(X \setminus \{0\}) = 0$.
(b) This follows similarly from Theorem 2.1(a) and (c).
(c) Apply Theorem 2.1(b). Note that $b_2(K) = 0$ by Poincaré duality.

The Hodge numbers of $X \setminus \{0\}$ can be calculated by Hamm [16], so the Betti numbers of K, too. This enables to check whether K is a rational homology sphere.

In particular, let us look at a special case studied by the author in his thesis, cf. [15], generalizing a result of Brieskorn about hypersurfaces, see [4, Satz 1, p. 6]:

Let $a_1, \ldots, a_{n+k}$ be positive integers. Choose $\alpha_{ik} \in \mathbb{C}, i = 1, \ldots, k, j = 1, \ldots, n+k$ such that all $k \times k$ subdeterminants of (α_{ij}) do not vanish. Let $X := \{z \in \mathbb{C}^{n+k} \mid \alpha_{i1} z_1^{a_1} + \ldots + \alpha_{i,n+k} z_{n+k}^{a_{n+k}} = 0, i = 1, \ldots, k\}$. Then X is a weighted homogeneous complete intersection of dimension n with an isolated singularity.

Let us associate a graph with vertices $1, \ldots, n+k$ where i, j are joined by an edge if and only if a_i, a_j are not coprime. For any component C of the graph look at the condition

(A) : Either C consists of a single vertex, or C consists of an odd number of vertices such that for all vertices $i \neq j$ in C, the greatest common divisor of a_i, a_j is 2.

Theorem 4.10 *Let X be chosen in this way.*

(a) For $n \geq 4$ *the ring* $\mathcal{O}(X)$ *is factorial.*
(b) If $n = 3$ *and there are* k *components with condition* (B) *the ring* $\mathcal{O}(X)$ *is factorial.*
(c) If $n = 2$ *and there are* k *(resp.* $k+1$*) components with condition* (B) *the ring* $\mathcal{O}(X)$ *is almost factorial (resp. factorial).*

Proof According to [15] we have that K is a rational (resp. integral) homology sphere if there are at least k (resp. $k+1$) components with condition (B). The rest follows from Theorem 4.9.

In particular: if $a_1, \ldots, a_{n+1}$ are mutually coprime, $n \geq 2$, the ring $\mathbb{C}[z_1, \ldots, z_{n+1}]/(z_1^{a_1} + \ldots + z_{n+1}^{a_{n+1}})$ is factorial. Cf. [3, p. 99] if $n = 2$.

(b) Now let us look at a different case:

Let $\overline{X}$ be a complete intersection of dimension n in $\mathbb{P}_m$, H the hyperplane $\{z_0 = 0\}$ in $\mathbb{P}_m$. Suppose that the singular locus of $\bar{X}$ is contained in $X := \bar{X} \setminus H$ and that H intersects $\bar{X}$ transversally.

Let Σ be the singular locus of X and K_x the link of X at x, $x \in \Sigma$. Note that Σ must be 0-dimensional.

Theorem 4.11

(a) *If $n \geq 4$, we have that $Cl(X) = 0$.*
(b) *Suppose $n = 3$:*

  (i) *$Cl(X)$ is free abelian of finite rank.*
  (ii) *If all $K_x, x \in \Sigma$, are rational homology spheres: $Cl(X) = 0$.*

(c) *Suppose $n = 2$:*

  (i) *If all K_x are integral homology spheres $Cl(X)$ is free abelian of finite rank.*
  (ii) *If $b_2(X) = 0$ and all K_x are rational homology spheres: $Cl(X)$ is finite.*
  (iii) *If $b_2(X) = 0$ and all K_x are integral homology spheres: $Cl(X) = 0$.*

Proof By a Lefschetz theorem, see [18, Theor. 1.1.1], we have that $\tilde{H}^j(X^{an}; \mathbb{Z}) = 0$, $j \leq n-1$, $H^n(X^{an}; \mathbb{Z})$ free abelian.

Now look at the exact sequence

$$\ldots \to H^j(X^{an}; \mathbb{Z}) \to H^j(X^{an} \setminus \Sigma^{an}; \mathbb{Z}) \to H^{j+1}(X^{an}, X^{an} \setminus \Sigma^{an}; \mathbb{Z}) \to \ldots$$

where $H^{j+1}(X^{an}, X^{an} \setminus \Sigma^{an}; \mathbb{Z}) \simeq \oplus H^j(K_x; \mathbb{Z})$.

Note that K_x is $(n-2)$-connected, in particular, $H^{n-1}(K_x; \mathbb{Z})$ is free abelian of finite rank.

Now the statements about $Cl(X) \simeq Pic(X \setminus \Sigma)$ follow from Theorem 2.1.

(c) Now let X be as in (a) but restrict to the case that $X \subset \mathbb{C}^m$ is homogeneous. Then the closure $\bar{X}$ in the projective space $\mathbb{P}_m$ is smooth, as well as the divisor $V \subset \mathbb{P}_{m-1}$ at infinity. In fact, X is the cone over V.

On the other hand, we may start with a smooth complex projective subvariety V of $\mathbb{P}_{m-1}$ of dimension $n-1$. We suppose that V is projectively normal, i.e. that the cone X over V is normal (cf. [13, II Exc. 5.14, p. 126]).

According to [13, II Exc. 6.3, p. 146f.], we have an exact sequence

$$0 \to \mathbb{Z} \to Cl(V) \to Cl(X) \to 0$$

Lemma 4.12 *Let V be a complete intersection of dimension 2.*

(a) *$Cl(X) \simeq \mathbb{Z}^{\rho-1}$, where ρ is the Picard number of V.*
(b) *Assume that the arithmetic genus of V vanishes. Then $Cl(X) \simeq \mathbb{Z}^{\chi-3}$ where χ is the Euler characteristic of V.*

Proof

(a) By Theorem 4.9(b) we have that $Cl(X)$ is free abelian of finite rank. The rest is clear.
(b) $H^2(V, \mathcal{O}_V) = 0$, because the arithmetic genus vanishes. So $Pic(V) \simeq H^2(V^{an}; \mathbb{Z}) \simeq \mathbb{Z}^{\chi-2}$.

Now we conclude as before.

The reader will notice a similarity between Theorem 3.6 and Lemma 4.12, it is not difficult to make it more precise!

Remark 4.13 By Theorem 2.1(b), $\mathcal{O}(X)$ is almost factorial as soon as the link K of X at 0 is a rational homology sphere and $n \geq 2$.

However this can turn out to be very restrictive:

Lemma 4.14 *K is a rational homology sphere if and only if V has the same Betti numbers as the corresponding projective space.*

Proof Look at the Gysin sequence associated with the sphere bundle $K \to V$.

In particular we get:

Lemma 4.15 *Suppose that n is odd. Then K cannot be a rational homology sphere except for $V \simeq \mathbb{P}_{n-1}$.*

Proof Use [22] Remark.

(d) From now on let V be a smooth curve.

Lemma 4.16 *$Cl(X)$ is finitely generated if and only if $g(V) = 0$.*

Proof

(i) Suppose $g(V) = 0$. Then $Cl(V) = \mathbb{Z}$, hence $Cl(X)$ is cyclic, so finitely generated.
(ii) Suppose $g(V) > 0$. Then $Pic^0(V)$ is a non-trivial torus, hence not finitely generated. So $Pic(V) = Cl(V)$ and $Cl(X)$ cannot be finitely generated, too.

Therefore let us look at the case $g(V) = 0$:

Lemma 4.17 *Let $g(V) = 0$ and let d be the degree of V. Then we have that $Cl(X) \simeq \mathbb{Z}/d\mathbb{Z}$.*

So $\mathcal{O}(X)$ is almost factorial but only for $d = 1$ factorial.

Proof Note that the homomorphism $\mathbb{Z} \to Cl(V)$ above can be identified with $\mathbb{Z} \to \mathbb{Z}$: $1 \mapsto d$.

Use [13, IV Expl. 3.3.2, p. 309].

Examples 4.18 Let us start with a surface which is not weighted homogeneous:

4.18.1 $X := \{z \in \mathbb{C}^4 \,|\, f(z) = 0\}$, where $f(z) := z_1^4 + z_2^4 + z_3^4 + z_4^4 - z_1^3 - z_2^2 - z_3^2 - z_4^2$.

Then $X \setminus \{0\}$ is smooth: If $z \neq 0$ is a critical point of f we have that $f(z)$ is real and negative.

So Theorem 4.11(b)(ii) can be applied because K_0 is an integral homology sphere, hence $Cl(X) = 0$.

Note that f is semiquasihomogeneous, so K_0 is homeomorphic to the corresponding link for $-z_1^3 - z_2^2 - z_3^2 - z_4^2$ instead of f, cf. [1, Theorem 9.5], so we can apply Theorem 4.10(b).

In the following examples, X is the cone over a smooth projective complete intersection V defined by the same equation(s).

4.18.2 $X := \{z \in \mathbb{C}^3 \,|\, z_1^2 + z_2^2 + z_3^2 = 0\}$. Then $Cl(X) \simeq \mathbb{Z}/2\mathbb{Z}$, see Lemma 4.17. So $\mathcal{O}(X)$ is almost factorial but not factorial.

We can also apply [13, II Expl. 6.5.2, p. 133].

4.18.3 $X := \{z \in \mathbb{C}^3 \,|\, z_1^3 + z_2^3 + z_3^3 = 0\}$. By Hartshorne [13, I Exc. 7.2(b), p. 54], we have that V has genus 1. By Lemma 4.16, $Cl(X)$ is not finitely generated.

4.18.4

(a) $X := \{z \in \mathbb{C}^4 \,|\, z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$. The Euler characteristic of $V \simeq \mathbb{P}_1 \times \mathbb{P}_1$ is 4, the arithmetic genus is 0, so $Pic(V) \simeq \mathbb{Z}^2$.
Or by Hartshorne [13, II Expl. 6.6.1, p. 135].
So $Cl(X) \simeq \mathbb{Z}$: use Lemma 4.12(b).

(b) $X := \{z \in \mathbb{C}^4 \,|\, z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0\}$. By Hartshorne [13, V Rem. 4.7.1, p. 401, and V Exc. 3.1, p. 394], the arithmetic genus of V vanishes; in fact, V is a del Pezzo surface of degree 3. The Euler characteristic of V is 9. By Lemma 4.12(b), $Pic(X) \simeq \mathbb{Z}^6$. Note that, by Hartshorne [13, II Expl. 6.6.4, p. 136], $Pic(V) \simeq \mathbb{Z}^7$: V is $\mathbb{P}_2$ blown up in 6 points; use [13, II Exc. 8.5, p. 188].

(c) $X := \{z \in \mathbb{C}^4 \,|\, z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\}$ (Fermat quartic). Here the arithmetic genus of V is 1. V is a singular K3 surface, hence $\rho(V) = 20$, see [24, Sect. 11, Example 17], so $Cl(V) \simeq \mathbb{Z}^{19}$, by Lemma 4.12(a).

(d) p = general homogeneous polynomial of degree $d \geq 4$, $X := \{z \in \mathbb{C}^4 \,|\, p(z) = 0\}$. By the Noether-Lefschetz theorem (see [12, p. 31]), $\rho(V) = 1$, so $\mathcal{O}(X)$ is factorial.

4.18.5 $X := \{z \in \mathbb{C}^5 \,|\, z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = z_1^2 + 2z_2^2 - z_3^2 - 2z_4^2 = 0\}$. Then V is a del Pezzo surface of degree 4, see [13, V Exc. 4.13, p. 408f.], so $p_a(V) = 0$. On the other hand, V has Euler characteristic 8. So $Cl(X) \simeq \mathbb{Z}^5$ by Lemma 4.12(b).

Or use [13, III Exc. 8.5, p. 188]: X is obtained blowing up 5 points of $\mathbb{P}^2$.

5 Picard Group in the Singular Case

(a) Let X be a normal complex algebraic variety. Let Σ be the singular locus of X and $j : X \setminus \Sigma \to X$ the inclusion.

Note that there may be now a difference between Weil and Cartier divisors.
We will compare the divisor class groups of X and $X \setminus \Sigma$.
Note that $Cl(X) \simeq Cl(X \setminus \Sigma)$. So we concentrate upon the Picard group.

First we have $Pic(X) \subset Pic(X \setminus \Sigma)$ because $Pic(X) \subset Cl(X)$ and $Cl(X) = Cl(X \setminus \Sigma)$.

Lemma 5.19 $Pic(X) = \{[\mathcal{L}] \in Pic(X \setminus \Sigma) \mid j_*\mathcal{L}$ *is locally free* $\}$.

Proof "$\supset$" is trivial.

"$\subset$": Suppose that $\mathcal{L}'$ is invertible on X, $\mathcal{L} = \mathcal{L}'|X \setminus \Sigma$. Then $\mathcal{L}' \simeq j_*\mathcal{L}$ because X is normal, cf. proof of Lemma 2.3.

Let us try to replace the condition in Lemma 5.19 by a more transparent one if possible.

Let $\mathcal{L}$ be an invertible sheaf on $X \setminus \Sigma$, $x \in \Sigma^{an}$. Let $c_{1,x}(\mathcal{L}) := \lim\limits_{\leftarrow U} c_1(\mathcal{L}^{an}|U \setminus \Sigma^{an})$, where U runs through the set of all open neighbourhoods of x in X^{an}.

Note that $c_{1,x}(\mathcal{L}) = 0$ if $\mathcal{L}$ admits an invertible extension $\mathcal{L}'$ to X: choose U small enough, then $c_1((\mathcal{L}')^{an}|U) = 0$, hence $c_1(\mathcal{L}^{an}|U \setminus \Sigma^{an}) = 0$.

Recall that $prof_{\Sigma^{an}}\mathcal{O}_{X^{an}} \geq 2$ because X is normal. For the definition of depth (profondeur) with respect to a subspace, see [2, II §3, p. 63].

Lemma 5.20 *Suppose that* $prof_{\Sigma^{an}}\mathcal{O}_{X^{an}} \geq 3$. *Then* $Pic(X) = \{[\mathcal{L}] \in Pic(X \setminus \Sigma) \mid c_{1,x}(\mathcal{L}) = 0$ *for all* $x \in \Sigma^{an}\}$.

Proof "$\subset$": see above.

"$\supset$": By assumption, $\mathcal{H}^k_{\Sigma^{an}}(\mathcal{O}_{X^{an}}) = 0, k \leq 2$, so for a suitable neighbourhood U of x in X^{an}: $H^1(U \setminus \Sigma^{an}, \mathcal{O}_{X^{an}}) = 0$.

Therefore the right arrow in the following exact sequence is injective:

$$H^1(U \setminus \Sigma^{an}, \mathcal{O}_{X^{an}}) \to H^1(U \setminus \Sigma^{an}, \mathcal{O}^*_{X^{an}}) \to H^2(U \setminus \Sigma^{an}; \mathbb{Z})$$

Let $\mathcal{L}$ be invertible on $X \setminus \Sigma$, $c_{1,x}(\mathcal{L}) = 0$ for all $x \in \Sigma^{an}$.

The class of $\mathcal{L}^{an}|U \setminus \Sigma^{an}$ in $H^1(U \setminus \Sigma^{an}, \mathcal{O}^*_{X^{an}})$ is 0, because it is mapped to 0 by assumption and the right arrow is injective.

Therefore $\mathcal{L}^{an}|U \setminus \Sigma^{an} \simeq \mathcal{O}_{U \setminus \Sigma^{an}}$, so $j^{an}_*\mathcal{L}^{an}$ is locally free.

The same holds for $j_*\mathcal{L}$:

According to [13, II Exc. 5.15, p. 126], there is a coherent extension $\mathcal{L}'$ of $\mathcal{L}$ to X. By Siu [26, Theorem A and B, p. 348], we have that $\mathcal{H}^k_\Sigma\mathcal{L}'$ is coherent, $(\mathcal{H}^k_\Sigma\mathcal{L}')^{an} \simeq \mathcal{H}^k_{\Sigma^{an}}\mathcal{L}'^{an}, k = 0, 1$. Now look at the exact sequence

$$0 \to \mathcal{H}^0_\Sigma\mathcal{L}' \to \mathcal{L}' \to j_*\mathcal{L} \to \mathcal{H}^1_\Sigma\mathcal{L}' \to 0$$

First, $j_*\mathcal{L}$ is coherent.

Applying $\ldots^{an}$ resp. taking the analytic analogue we see by comparison that $(j_*\mathcal{L})^{an} \simeq j^{an}_*\mathcal{L}^{an}$. This implies that $j_*\mathcal{L}$ is locally free, so we have a locally free extension of $\mathcal{L}$. Now apply Lemma 5.19.

(b) Now let us look at the situation of Sect. 4(b):

Let $\overline{X}$ be a complete intersection of dimension n in $\mathbb{P}_m$, H the hyperplane $\{z_0 = 0\}$ in $\mathbb{P}_m$. Suppose that the singular locus of $\overline{X}$ is contained in $X \setminus H$ and that H intersects $\overline{X}$ transversally.

Let Σ be the (0-dimensional) singular locus of X and K_x the link of X at x, $x \in \Sigma$.

Theorem 5.21

(a) Suppose $n = 3$*: Then* $Pic(X) = 0$.
(b) Suppose $n = 2$, $b_2(X) = 0$ *and all* K_x *are rational homology spheres: Then* $Pic(X) = 0$.

Proof Apply Lemma 2.3.

(c) In the homogeneous case we have the following result:

Theorem 5.22 *Let* X *be the affine cone at* 0 *over a smooth projectively normal variety* V. *Then* $Pic(X) = 0$.

Proof Let $p : \tilde{X} \to X$ be the blow-up of X at the origin. Then $p_1 := p|(\tilde{X} \setminus V_0) \to X \setminus \{0\}$ is an isomorphism, where $V_0 := p^{-1}(\{0\})$. We have a projection $\pi : \tilde{X} \to V$. Then $\pi_0 := \pi|V_0 : V_0 \to V$ is an isomorphism. Furthermore, π induces an isomorphism $Pic(V) \simeq Pic(\tilde{X})$, because $\tilde{X}$ is a line bundle over V, see [10, Theorem 3.3, p. 64].

Assume now that we have an element of $Pic(X)$ which is represented by an invertible sheaf $\mathcal{L}$. Then there is a neighbourhood U of 0 in X such that $\mathcal{L}|U$ is trivial, hence $\mathcal{L}|U \setminus \{0\}$ is trivial. A trivialization of $p_1^*(\mathcal{L}|U \setminus \{0\})$, considered as a rational function on $\tilde{X} \setminus V_0$, leads to a corresponding divisor on $\tilde{X} \setminus V_0$. Its closure does not meet V_0, so it leads to an invertible extension $\mathcal{L}'$ of $p_1^*(\mathcal{L})$ to $\tilde{X}$ such that $\mathcal{L}'|V_0$ is trivial. Now we have a commutative diagram

$$\begin{array}{ccc} Pic(V) & & \\ \simeq\downarrow & \searrow\simeq & \\ Pic(\tilde{X}) & \to & Pic(V_0) \end{array}$$

hence $Pic(\tilde{X}) \simeq Pic(V_0)$. So $\mathcal{L}'$ is trivial, and $\mathcal{L}'|\tilde{X} \setminus V_0 = p_1^*(\mathcal{L}|X \setminus \{0\})$, too. So $\mathcal{L}|X \setminus \{0\}$ is trivial. Hence $\mathcal{L}$ is trivial, too, see proof of Lemma 2.3, i.e. $Pic(X) = 0$.

Corollary 5.23 *Suppose that* V *is a smooth projectively normal curve of genus* > 0. *Then* $Pic(X) = 0$, $Cl(X)$ *is not finitely generated, so* $Pic(X) \neq \{[\mathcal{L}] \in Cl(X) \,|\, c_{1,0}(\mathcal{L}) = 0\}$.

Proof Apply Lemma 4.16, too.

Example 5.24 $X := \{z \in \mathbb{C}^3 \,|\, z_1^3 + z_2^3 + z_3^3 = 0\}$ (Example 4.18(3)) satisfies the hypothesis of Corollary 5.23.

Therefore the hypothesis $prof_{\Sigma^{an}} \mathcal{O}_{X^{an}} \geq 3$ in Lemma 5.20 cannot be dropped!

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Möbius Strips, Knots, Pentagons, Polyhedra, and the SURFER Software

Stephan Klaus

Abstract The SURFER software for the visualization of real algebraic surfaces was developed from professional research software. It was adapted to the IMAGINARY exhibition of Oberwolfach under the direction of Gert-Martin Greuel in the Year of Mathematics in Germany 2008. As it is freely available and very easy to use also for nonexperts, it became one of the most successful public tools to visualize mathematical objects. Based on many discussions with Gert-Martin Greuel, the author used this software to give algebraic constructions and visualizations of some low-dimensional objects in geometry and topology. This has led to new connections and specific constructions for objects such as knots, moduli spaces of pentagons, and polyhedra.

Keywords Algebraic variable elimination • Cinquefoil knot • Dodecahedron • Icosahedron • Möbius strip • Octahedron • Pentagon moduli space • Pyrite • Real algebraic surface • Rhombic dodecahedron • Spherical harmonics • Torus knot • Trefoil knot

MSC: 14J25, 14Q10, 51N10, 52B10, 55R80, 57M25, 57N05, 57N35

1 The Year of Mathematics in Germany 2008 and the SURFER

In the Year of Mathematics in Germany 2008, Gert-Martin Greuel as former director of Oberwolfach created the international mathematical touring exhibition IMAGINARY (see www.imaginary.org). It was very successful from the beginning and has developed into one of the most successful international exhibitions in mathematics. One of the highlights of this exhibition is the SURFER software [3] which

S. Klaus (✉)
Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstr. 9-11, 77709 Oberwolfach, Germany
e-mail: klaus@mfo.de

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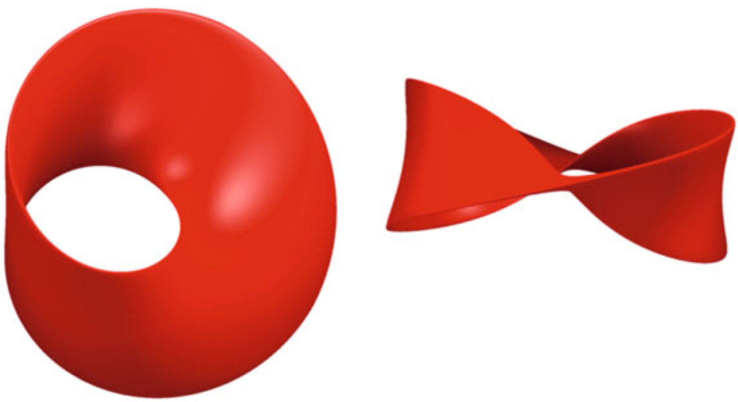
allows real-time visualizations of real algebraic surfaces by choosing a polynomial in three variables. This is a program of the Mathematisches Forschungsinstitut Oberwolfach in collaboration with the Martin Luther University Halle-Wittenberg arising from professional research software in algebraic geometry. The program is very easy to use also for nonmathematicians. In fact, many competitions for a general audience have taken place with thousands of interesting contributions from pupils, for example.

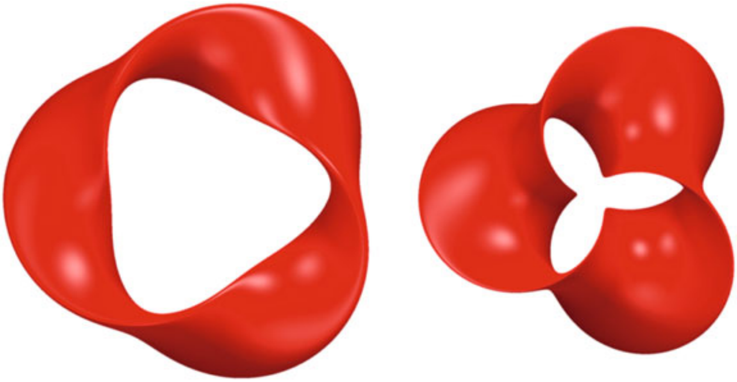
2 Impossible Möbius Strips

A classical object of topology is the Möbius strip, and in a conversation with Gert-Martin in 2008, we discussed the problem to find a suitable polynomial for the SURFER. My first reaction as a topologist was that it cannot be represented as the level set of a regular value of a smooth real function $p : \mathbb{R}^3 \to \mathbb{R}$. The reason is that by a basic lemma of differential topology, such a level set has to be a smooth surface in $\mathbb{R}^3$ without boundary and with trivial normal bundle. Nevertheless, in 2008, I constructed polynomials in [12] of degrees 6, 8, and 10 giving a Möbius strip with 1, 2, and 3 twists, respectively, as can be seen in the following pictures:



simple and double Möbius strip

$$p_1 = ((a-b)(x(x^2+y^2-z^2+1)-2yz)-(2a+2b+ab)(x^2+y^2))^2$$
$$-(x^2+y^2)((a+b)(x^2+y^2+z^2+1)+2(a-b)(yz-x))^2$$
$$p_2 = ((b^2x^2+a^2y^2)(x^2+y^2)+b^2(-x+yz)^2+a^2(y+xz)^2-a^2b^2(x^2+y^2))^2$$
$$-4(x^2+y^2)(b^2x(-x+yz)-a^2y(y+xz))^2$$



triple Möbius strips

$$p_3 = (x^2+y^2)((a+b)(x^2+y^2)(x^2+y^2+z^2+1)$$
$$+2(a-b)(z(3x^2y-y^3)-(x^3-3xy^2))-2ab(x^2+y^2))^2$$
$$-(2(a+b)(x^2+y^2)^2+(a-b)(2z(3x^2y-y^3)+(x^3-3xy^2)(z^2-x^2-y^2-1)))^2$$

The reader can easily check with the SURFER software that the given polynomials produce the pictures. The solution of this seeming contradiction to the mentioned basic result in differential topology is that the level sets of these polynomials give only *fake* Möbius strips. In fact, the idea of the construction is to take a very thin ellipse with center $(1,0)$ in a plane with coordinates t and z. Then let the ellipse rotate in the plane around its center, and, at the same time, let the plane rotate around the z-axis. Hence, the coordinates x and y appear as $x = t\cos(\phi)$ and $y = t\sin(\phi)$, whereas the ellipse is rotating around its center with an angle $\psi = \frac{k}{2}\phi$ where $k = 1, 2, 3$. Variable elimination of t, ϕ, and ψ yields the above polynomials, where a and b denote the two radii of the ellipse. Hence, $a = 0.5$ and $b = 0.01$ give pictures as above, but it is also possible to deform a torus from $a = b = 0.5$ into a fake Möbius strip by putting these formulas into the SURFER. The construction works also for higher Möbius strips and yields a polynomial of degree $4 + 2k$ for k twists. Details on the computation can be found in [6, 12].

3 Relation to Knot Theory

In 2009, José-Francisco Rodrigues asked me in Oberwolfach if a construction of the “thickened” trefoil knot $T_{2,3}$ as a real algebraic surface in $\mathbb{R}^3$ would also be possible. In fact, the idea of a doubly rotating algebraic object gives also a solution to this problem: one has just to replace the ellipse by two small circles.

Here, the computation was more complex as the object (two circles) is given by an equation of degree 4 in contrast to the case of an ellipse. Then algebraic variable elimination (which is tedious but can be done by hand) gives a polynomial for $T_{2,3}$ of degree 14:

solid trefoil knot

$$
\begin{aligned}
p = (&-8(x^2+y^2)^2(x^2+y^2+1+z^2+a^2-b^2)\\
&+4a^2[2(x^2+y^2)^2-(x^3-3xy^2)(x^2+y^2+1)]\\
&+8a^2(3x^2y-y^3)z+4a^2(x^3-3xy^2)z^2)^2\\
&+(x^2+y^2)(2(x^2+y^2)(x^2+y^2+1+z^2+a^2-b^2)^2\\
&+8(x^2+y^2)2+4a^2[2(x^3-3xy^2)-(x^2+y^2)(x^2+y^2+1)]\\
&-8a^2(3x^2y-y^3)z-4(x^2+y^2)a^2z^2)^2
\end{aligned}
$$

I gave talks on this construction in Braga and Lisbon, and the result was published in [8]; see also [7]. I would like to thank José-Francisco for the invitation to Portugal and the possibility to introduce the SURFER program to a group of gifted pupils.

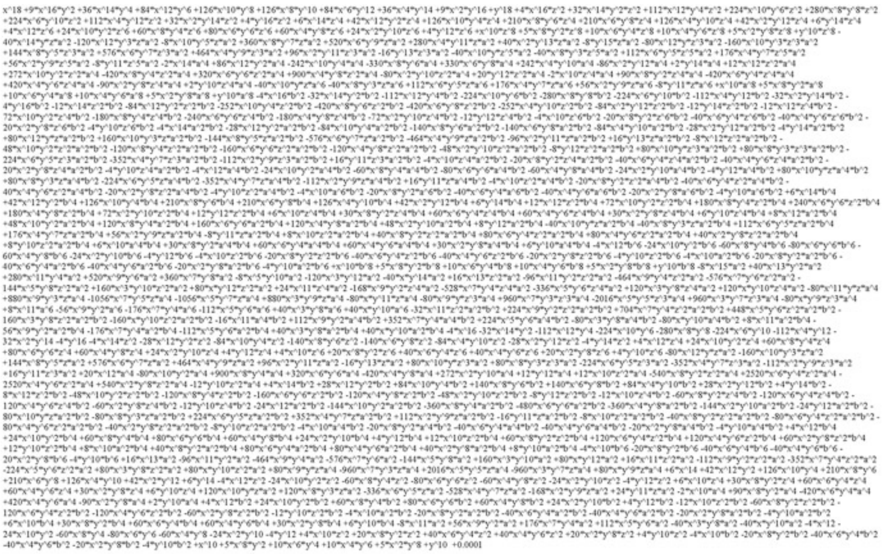
It is a natural question how one can construct such polynomials for other knots. In the paper [9], I gave two explicit methods—at least in principle—to construct such polynomials. The first method is to convert a piecewise linear knot into a singular variety formed by a product of quadratic polynomials and then to smooth out the singularities. The second method is to consider finite Fourier sums approximating a knot (a knot can be considered as a periodic function $\mathbb{R} \to \mathbb{R}^3$) and then to use algebraic variable elimination for the equations defining the tubular neighborhood of the knot. Moreover, these methods give upper bounds for the minimal degree of a polynomial representing a given knot type and other invariants of the knot such as its minimal stick number and its Fourier degree defined by Kauffman [5] and Trautwein [18]. All this is explained in the mentioned paper. I gave a talk on these topics at the 5th World Conference on 21st Century Mathematics (which was coorganized by Gert-Martin) at Abdus Salam School of Mathematical Sciences in Lahore, Pakistan, in February 2011.

I should also emphasize that the two methods generally work only "in principle": Already for such a basic example as the figure eight knot, the computation of a concrete polynomial $p(x, y, z)$ by the Fourier method stops at the elimination of the fourth variable from two polynomials in four variables of degrees 14 and 20. This task still seems to be out of reach using standard algebra software such as Singular.

4 Torus Knots

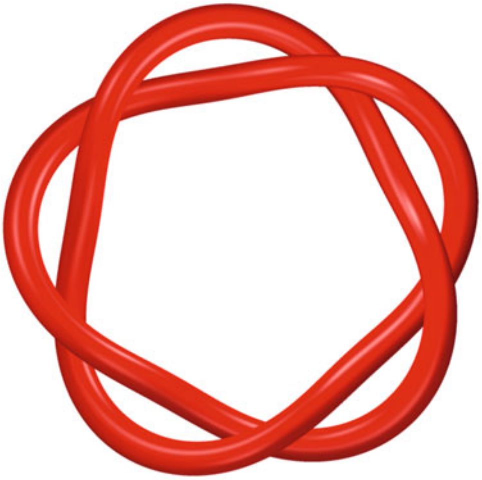
The idea of the "double rotation construction" which was used for the Möbius strips and for the trefoil knot can also be used for other torus knots because of their symmetry.

Already the next nontrivial torus knot, the cinquefoil knot $T_{2,5}$ presents a hard example of explicit computation. During the Lahore conference in 2011, I showed Gert-Martin the construction which I could carry out by hand up to a variable elimination problem for two polynomials of four variables similar to the case of the figure eight knot. In this case, however, the Singular software could do the remaining part of the job, and the output from Gert-Martin's laptop was a polynomial of degree 18 with about 500 terms:



screenshot of the polynomial of the cinquefoil knot

It appeared to us like a miracle that one can copy this huge formula from Singular into the SURFER window and immediately gets the visualization of the cinquefoil knot $T_{2,5}$:



solid cinquefoil knot

Note that the explicit formula of the polynomial is available in my personal SURFER gallery at the IMAGINARY web site, such that the reader can check the result.

During a research visit with Sofia Lambropoulou in Athens 2012, I could finally solve by hand the elimination problem for all torus knots $T_{p,q}$ [10]. Thus, there is an explicit (but long) general formula for the representing polynomial of $T_{p,q}$ which has a degree of $4p + 2q$. This result was possible because of the large symmetry of torus knots and by an explicit combinatorial variable transformation using higher dimensional pyramidal numbers. Here, I would like to thank Sofia for her hospitality and for the possibility to carry out this research on algebraic torus knots.

Moreover, the method of Fourier approximations also applies to braids and has led to new invariants and research problems; see the extended abstract [11] of my talk in the Oberwolfach Workshop "Algebraic Structures in Low-Dimensional Topology" in May 2014 and the paper [13].

5 Pentagons

In 2013, I learned in Oberwolfach from Sadayoshi Kojima that the configuration space of equilateral pentagons is a closed surface of genus four. See the article of Kapovich and Millson [4] as a reference. By fixing two consecutive points of a pentagon to the points 0 and 1 in the complex plane, this configuration space can be described by four complex variables z_1, z_2, z_3, and z_4 such that all $|z_i| = 1$ (equilateral condition) and $1 + z_1 + z_2 + z_3 + z_4 = 0$ (closure to a pentagon). Thus, there are *eight* real variables and *six* real polynomial conditions.

As a result of our discussion, we found a new visualization of the configuration space using the well-known rational parametrization

$$r : \mathbb{R} \to S^1, \qquad r(t) := \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}$$

applied to z_1, z_2, and z_3. Note that r misses the point -1 of S^1 which corresponds to the boundary $\pm\infty$ of the real line $\mathbb{R}$. Then, variable elimination of z_4 and a short computation with $|1 + z_1 + z_2 + z_3| = 1$ yields the following polynomial of order 12 which can be easily visualized using the SURFER:

$$\begin{gathered} p(x,y,z) = 4(2x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 - 1)^2 \\ +4(x(y^2+1)(z^2+1) + (x^2+1)y(z^2+1) + (x^2+1)(y^2+1)z)^2 \\ -((x^2+1)(y^2+1)(z^2+1))^2 \end{gathered}$$

configuration space of equilateral pentagons

Because of the threefold rational parametrization of the three torus $(S^1)^3$, the ambient space $\mathbb{R}^3$ in the picture above should be considered as the open cube I^3 together with a pair-wise identification of the three antipodal pairs of faces. Thus, the six open handles in the picture compactify to three closed handles which together with the hole in the center form a closed surface of genus four.

At this point, I would also like to thank Dirk Siersma who, after the Pfalz-Akademie conference, gave me further advice concerning the more general case of configuration spaces of pentagons which are not equilateral. This is work in progress where the first (equilateral) part can be found in [14].

6 Polyhedra: From Atomic Nuclei to Crystals

In 2013, David Rouvel, at this time a graduate student of theoretical physics in Strasbourg, asked me which sums of spherical harmonics $Y_l^m(\theta, \phi)$ define shapes of icosahedral symmetry. This problem was relevant for his mathematical description of certain atomic nuclei. As spherical harmonics can be expressed by harmonic polynomials, the problem can be transformed to the classical question of invariants of the icosahedral group A_5 acting by rotations on the polynomial ring $\mathbb{R}[x, y, z]$. Of course, the answer

$$\mathbb{R}[x, y, z]^{A_5} = \mathbb{R}[p_2, p_6, p_{10}]$$

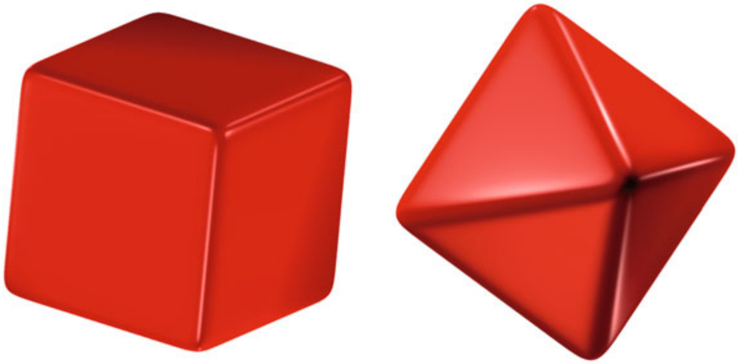
with basic invariants

$$p_2 = x^2 + y^2 + z^2$$
$$p_6 = (x^2 - \Phi^2 y^2)(y^2 - \Phi^2 z^2)(z^2 - \Phi^2 x^2)$$
$$p_{10} = (x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2)(x^2 - \Phi^4 y^2)(y^2 - \Phi^4 z^2)(z^2 - \Phi^4 x^2)$$

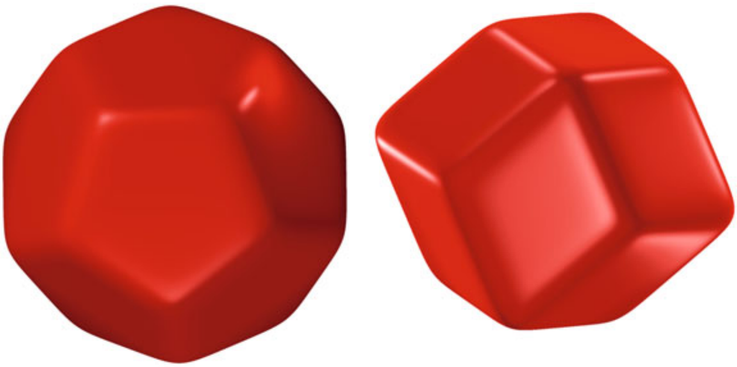
and $\Phi := \frac{1}{2}(\sqrt{5} + 1) = 1.618\ldots$ denoting the golden ratio, is a well-known result of invariant theory [1] and gives (in principle) also the answer to David's question. But David was additionally interested in the visualization of the corresponding sums of spherical harmonics. Thus, I recommended to him the SURFER software, and the reader can find a lot of SURFER pictures of deformed atomic nuclei in his thesis [17] in 2014.

I was still interested in the problem of a simple and explicit geometric construction of the dodecahedron and of the icosahedron as a real algebraic surface. During the ICM 2014 at Seoul, I asked Gert-Martin about this problem, and in the night after our conversation, I could find a short and explicit polynomial of degree $2n$ with two parameters a and b. The larger is n, the better is the quality of approximation, and the parameters a and b allow an interpolation between a cube, a dodecahedron, an octahedron, and a rhombic dodecahedron:

$$p = (ax + by + z)^{2n} + (x + ay + bz)^{2n} + (bx + y + az)^{2n} +$$
$$(-ax + by + z)^{2n} + (x - ay + bz)^{2n} + (bx + y - az)^{2n} - 1.$$



cube and octahedron



dodecahedron and rhombic dodecahedron

The idea of the construction is simple: It is well known that the icosahedron is spanned by the 12 vertices

$$(\pm 1, \pm\phi, 0), (0, \pm 1, \pm\phi), (\pm\phi, 0, \pm 1)$$

of three golden rectangles parallel to the coordinates, where $\phi := \frac{1}{2}(\sqrt{5} - 1) = 0.618\ldots$ denotes the small golden ratio. Now, a pair of antipodal vertex coordinates $\pm(a, b, c)$ gives a pair of normal planes by $(ax + by + cz)^2 - 1 = 0$ and forming the sum of squares results in a convex approximation of the dual of an icosahedron, i.e., of a dodecahedron. (Here I took the free parameter a instead of ϕ and b instead of 0 in order to have more freedom to make experiments!)

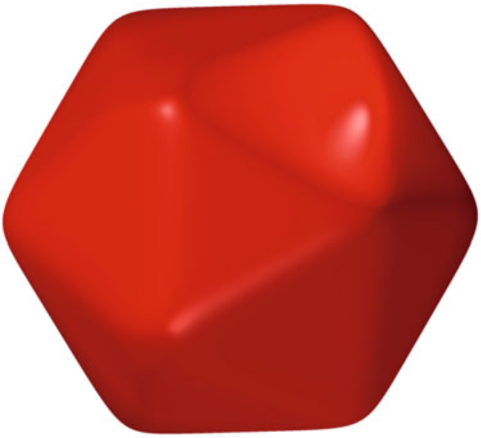
After I showed Gert-Martin the formula, Bianca Violet from the IMAGINARY team and I produced a paper [15] and a short movie [16], both with the title "Katzengold." Katzengold (cat's gold) is the German nickname for the mineral pyrite (iron sulfide, FeS_2, with English nick name "fool's gold"). Pyrite exists in many crystal forms: cubical, octahedral, as an (irregular) dodecahedron, and as a rhombic dodecahedron! Here are some pictures from Egbert Brieskorn's mineral collection which is now displayed at the Museum of Minerals and Mathematics in Oberwolfach.

some crystal forms of pyrite from Brieskorn's mineral collection

I would also like to mention the paper [2] of Gert-Martin on "Crystals and Mathematics." Both the "Katzengold" paper and movie were presented by Bianca at the Bridges Conference on Mathematics and Art in Baltimore 2015.

Finally, I should mention that also the icosahedron can be constructed in a similar way, as the vertex coordinates of the dodecahedron are given by sums of the triangle vertex coordinates for each of the 20 triangles of the icosahedron. Then, take the vertex coordinates $\pm(a, b, c)$ of the dodecahedron to form again a sum of terms $(ax + by + cz)^{2n}$. A short computation with a instead of ϕ gives

$$\begin{aligned} p = {} & ((2+a)x+y)^{2n} + ((2+a)y+z)^{2n} + ((2+a)z+x)^{2n} \\ & + ((2+a)x-y)^{2n} + ((2+a)y-z)^{2n} + ((2+a)z-x)^{2n} \\ & + (1+a)^{2n}(x+y+z)^{2n} + (1+a)^{2n}(-x+y+z)^{2n} \\ & + (1+a)^{2n}(x-y+z)^{2n} + (1+a)^{2n}(x+y-z)^{2n} - 1 \end{aligned}$$



icosahedron

With Bianca Violet, we plan to construct further polyhedra by this method, such as the Archimedean and Catalan solids.

Acknowledgements I was introduced to the SURFER software by Gert-Martin Greuel in 2008, and I got interested immediately because of possible visualizations of two-dimensional topological objects. I would like to thank Gert-Martin cordially for all his advice and help with algebraic geometry and numerous discussions on mathematics. I learned from him the importance of raising public awareness for its beauty. This written version of my talk at the Pfalz-Akademie conference 2015 in honor of Gert-Martin Greuel gives a report on the surfaces and visualizations which I studied since 2008, emphasizing his influence to my work.

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Seiberg–Witten Invariant of the Universal Abelian Cover of $S^3_{-p/q}(K)$

József Bodnár and András Némethi

Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract We prove an additivity property for the normalized Seiberg–Witten invariants with respect to the universal abelian cover of those 3-manifolds, which are obtained via negative rational Dehn surgeries along connected sum of algebraic knots. Although the statement is purely topological, we use the theory of complex singularities in several steps of the proof. This topological covering additivity property can be compared with certain analytic properties of normal surface singularities, especially with functorial behaviour of the (equivariant) geometric genus of singularities. We present several examples in order to find the validity limits of the proved property, one of them shows that the covering additivity property is not true for negative definite plumbed 3-manifolds in general.

Keywords 3-Manifolds • Abelian coverings • Geometric genus • Lattice cohomology • Links of singularities • Normal surface singularities • Plumbed 3-manifolds • $\mathbb{Q}$-Homology spheres • Seiberg–Witten invariants • Superisolated singularities • Surgery 3-manifolds

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J. Bodnár
Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
e-mail: bodnar.jozef@renyi.mta.hu; jozef.bodnar@stonybrook.edu

A. Némethi (✉)
A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15, 1053 Budapest, Hungary
e-mail: nemethi.andras@renyi.mta.hu; nemethi@renyi.hu

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1 Introduction

1.1 Main Goal

In this paper, we prove an additivity property of the three-dimensional (normalized) Seiberg–Witten invariant with respect to taking the abelian cover, valid for surgery 3-manifolds. Namely, assume that M is obtained as a negative rational surgery along connected sum of algebraic knots in the three-sphere S^3. Let Σ be its universal abelian cover. Theorem 10 states that the sum over all spin^c structures of the Seiberg–Witten invariants of M (after normalization) equals to the canonical Seiberg–Witten invariant of Σ.

1.2 Motivation

Both covers of manifolds, and manifolds of form $S^3_{-p/q}(K)$, are extensively studied. The stability of certain properties and invariants with respect to the coverings is a key classical strategy in topology; it is even more motivated by the recent proof of Thurston's virtually fibered conjecture [1, 32]. Manifolds of form $S^3_{-p/q}(K)$ can be particularly interesting due to theorem of Lickorish and Wallace [13, 31] stating that every closed oriented 3-manifold can be expressed as surgery on a *link* in S^3. Based on this result, one can ask which manifolds have surgery representations with some restrictions. For example, using Heegaard Floer homology, [11] provides necessary conditions on manifolds having surgery representation along a *knot*. In this context, Theorem 10 can be viewed also as a criterion for a manifold having surgery representation of form $S^3_{-p/q}(K)$ with K a connected sum of algebraic knots.

In fact, Seiberg–Witten (SW) invariants and Heegaard Floer homologies are closely related. The SW invariants were originally introduced by Witten in [33], but they also arise as Euler characteristics of Heegaard Floer homologies (cf. [29, 30]). In this article, we will involve another cohomology theory with similar property. Since $S^3_{-p/q}(K)$ is representable by a negative definite plumbing graph, via [22] we can view the SW invariants as Euler characteristics of lattice cohomologies introduced in [21]. The big advantage of the lattice cohomology over the classical definition of Heegaard Floer homology is that it is computable algorithmically from the plumbing graph in an elementary way. In the last section of applications and examples, the above 'covering additivity property' will be combined with results involving lattice cohomology.

Another strong motivation to study the above property is provided by the theory of complex normal surface singularities: the geometric genus of the analytic germ is conjecturally connected with the SW invariant of the link of the germ (see [23–26]). Since the geometric genus satisfies the 'covering additivity property (cf. Sect. 2.1)', it is natural to ask for the validity of similar property at purely topological level.

Furthermore, from the point of view of singularity theory, the motivation for the surgery manifolds $S^3_{-p/q}(K)$ is also strong: the link of the so-called superisolated singularities (introduced in [14]) is of this form. These singularities are key test examples for several properties and provide counterexamples for several conjectures. They embed the theory of projective plane curves to the theory of surface singularities. For their brief introduction, see Example 16; for a detailed presentation, see [14, 15].

All these connections with the analytic theory will be used deeply in several points of the proof. For consequences of the main result regarding analytic invariants, see the last sections.

1.3 Notations

We recall some facts about negative definite plumbed 3- and 4-manifolds, their spin^c structures and Seiberg–Witten invariants. For more, see [24, 26].

Let M be a 3-manifold which is a rational homology sphere ($\mathbb{Q}HS^3$). Assume that it has a negative definite plumbing representation with a decorated connected graph G with vertex set $\mathcal{V}$. In particular, M is the boundary of a plumbed 4-manifold P, which is obtained by plumbing disc bundles over oriented surfaces $E_v \simeq S^2$, $v \in \mathcal{V}$ (according to G), and which has a negative definite intersection form. A vertex $v \in \mathcal{V} = \mathcal{V}(G)$ is decorated by the *self-intersection* $e_v \in \mathbb{Z}$ of E_v in P. One can think of e_v also as the Euler number of the disc bundle over $E_v \cong S^2$ used in the plumbing construction. Since M is a $\mathbb{Q}HS^3$, the graph G is a tree. We set $\#\mathcal{V}(G)$ for the number of vertices of G.

Below all the (co)homologies are considered with $\mathbb{Z}$-coefficients.

Denote by $L = L_G = \mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}}$, the free abelian group generated by basis elements $\{E_v\}_v$, indexed by $\mathcal{V}$. It can be identified with $H_2(P)$, where $\{E_v\}_v$ represent the zero sections of the disc bundles used in the plumbing construction. It carries the negative definite intersection form $(.,.) = (.,.)_G$ (of P; readable from G too). This form naturally extends to $L \otimes \mathbb{Q}$. Denoting by $L' = L'_G = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, the dual lattice, one gets a natural embedding $L \to L'$ by $l \mapsto (\cdot, l)$. Furthermore, we can regard L' as a subgroup of $L \otimes \mathbb{Q}$, therefore $(\cdot,\cdot)$ extends to L' as well. We introduce the anti-dual basis elements E_v^* in L' defined by $(E_{v'}, E_v^*)$ being -1 if $v = v'$ and 0 otherwise. Notice that $L' \cong H^2(P) \cong H_2(P, M)$. The short exact sequence $0 \to H_2(P) \to H_2(P, M) \to H_1(M) \to 0$ identifies L'/L with $H_1(M)$, which will be denoted by H. We denote the *class* of $l' \in L'$ by $[l'] \in H$, and we call $l' \in L'$ a *representative* of $[l']$.

Assume that the intersection form in the basis $\{E_v\}_v$ has matrix I; then we define $\det(G) := \det(-I)$. It also equals the order of H. (Since I is negative definite, $\det(G) > 0$.)

For any negative definite plumbing graph G, which is a tree, and for any two vertices u, v, the following holds (see [7, Sect. 10] in the integral homology case and [24] in general):

$$\begin{cases} -\det(G)\cdot(E_u^*,E_v^*) \text{ equals the product of the determinants of the} \\ \text{connected components of that graph, which is obtained from } G \text{ by} \\ \text{deleting the shortest path connecting } u \text{ and } v \text{ and the adjacent edges.} \end{cases} \tag{1}$$

For any $h \in H$, we denote by $r_h = \sum_{v\in\mathcal{V}} c_v E_v$ 'the smallest effective representative' of its class in L', determined by the property $0 \le c_v < 1$ for all v.

Finally, we define the *canonical characteristic element* in L'. It is the unique element $k_G \in L'$ such that $(k_G, E_v) = -(E_v, E_v) - 2$ for every $v \in \mathcal{V}$. (In fact, P carries the structure of a smooth complex surface—in the case of singularities, P is a resolution (cf. Sect. 2)— and k_G is the first Chern class of its complex cotangent bundle.)

The Seiberg–Witten invariants of M associate a rational number to each spin^c structure of M. There is a 'canonical' spin^c structure $\sigma_{\text{can}} \in \text{Spin}^c(M)$, the restriction of that spin^c structure of P, which has first Chern class $k_G \in H^2(P)$. As we assumed M to be a $\mathbb{Q}HS^3$, $\text{Spin}^c(M)$ is finite. It is an H torsor: for $h \in H$, we denote this action by $\sigma \mapsto h * \sigma$.

We denote by $\mathfrak{sw}_\sigma(M) \in \mathbb{Q}$ the *Seiberg–Witten invariant* of M corresponding to the spin^c structure σ. This is the classical monopole counting Seiberg–Witten invariant of M corrected by the Kreck–Stolz invariant to make it dependent only on the manifold M. Nevertheless, we will adopt the approach from [24], and we regard $\mathfrak{sw}_\sigma(M)$ as the sum of the sign-refined Reidemeister–Turaev torsion and the Casson–Walker invariant. By Némethi and Nicolaescu [24], it can be computed from the plumbing graph G, and in this note, this combinatorial approach (and its consequences and related formulae from succeeding articles) will be used.

Next, we consider the following normalization term: for each $h \in H$, we set

$$\mathfrak{i}_h(M) := \frac{(k_G + 2r_h, k_G + 2r_h) + \#\mathcal{V}}{8}. \tag{2}$$

It does not depend on the particular plumbing representation of the manifold M (or P); hence, it is an invariant of the manifold M. Then, the normalized Seiberg–Witten invariant is defined as follows: for any $h \in H$, we set

$$\mathfrak{s}_h(M) = \mathfrak{sw}_{h*\sigma_{\text{can}}}(M) - \mathfrak{i}_h(M). \tag{3}$$

Sometimes we will also use the notations $\mathfrak{s}_h(G) = \mathfrak{s}_h(M)$ or $\mathfrak{sw}_h(G) = \mathfrak{sw}_{h*\sigma_{\text{can}}}(M)$.

In fact, $\mathfrak{s}_h(M) \in \mathbb{Z}$. This can be seen through the identity (23), where $\mathfrak{s}_h(M)$ appears as the Euler characteristic of the lattice cohomology. We also refer to Sect. 4.1 for the fact that $\mathfrak{s}_h(M)$ and $\mathfrak{i}_h(M)$ are indeed independent of the plumbing representation of the manifold.

Let Σ be the universal abelian cover (UAC) of the manifold M: it is associated with the abelianization $\pi_1(M) \to H_1(M)$. Usually the UAC of a rational homology sphere is not a rational homology sphere. However, in this note, we will consider only those situations when this happens.

Definition 1 Assume that Σ is a rational homology sphere and both M and Σ admit uniquely defined 'canonical' spin^c structures (e.g. they are plumbed 3-manifolds associated with negative definite graph; see above). We say that for the manifold M, *the 'covering additivity property' of the invariant* $\mathfrak{s}$ *holds with respect to the universal abelian cover* (shortly, 'CAP of $\mathfrak{s}$ holds') if

$$\mathfrak{s}_0(\Sigma) = \sum_{h \in H_1(M)} \mathfrak{s}_h(M).$$

(The index 0 in the left-hand side is the unit element in $H_1(\Sigma)$.)

Our main result is the following:

Theorem 2 *Let* $M = S^3_{-p/q}(K)$ *be a manifold obtained by a negative rational Dehn surgery of* S^3 *along a connected sum of algebraic knots* $K = K_1 \# \ldots \# K_\nu$ *(*$p, q > 0$, $\gcd(p,q) = 1$*). Assume that* Σ*, the UAC of* M*, is a* $\mathbb{Q}HS^3$*. Then CAP of* $\mathfrak{s}$ *holds.*

Though the statement is topological, in the proof we use several analytic steps based on the theory of singularities. These steps not only emphasize the role of the algebraic knots and of the negative definite plumbing construction, but they also provide the possibility to use certain deep results valid for singularities (which are transported into the proof).

We emphasize that the above covering additivity property is not true for general negative definite plumbed 3-manifolds (hence, for general 3-manifolds either) (cf. Example 18). In particular, we cannot expect a proof of the main theorem by a general topological machinery unless some restrictions are made.

1.4 Further Notations

It is convenient to extend the definitions (2) and (3) for any representative $l' \in L'$:

$$\mathfrak{i}_{l'}(G) := \frac{(k_G + 2l', k_G + 2l')_G + \#\mathcal{V}(G)}{8} \quad \text{and} \quad \mathfrak{s}_{l'}(M) = \mathfrak{sw}_{[l']}(G) - \mathfrak{i}_{l'}(G).$$

By a computation, for two representatives $[l'_1] = [l'_2] = h \in H$, one has

$$\mathfrak{s}_{l'_2}(G) - \mathfrak{s}_{l'_1}(G) = \chi(l'_2) - \chi(l'_1), \tag{4}$$

where

$$\chi(l') := -(l', l' + k_G)/2. \tag{5}$$

In particular,

$$\mathfrak{s}_{l'}(G) = \mathfrak{s}_{[l']}(G) + \chi(l') - \chi(r_{[l']}). \tag{6}$$

Note that for $l \in L$ one has $\chi(l) = -(l, l + k_G)/2 \in \mathbb{Z}$. In fact, By Riemann–Roch theorem, $\chi(l)$ is the topological description of the analytic Euler characteristic $\chi(\mathcal{O}_l)$ of the structure sheaf $\mathcal{O}_l$ of any non-zero effective cycle $l \in L$.

2 Preliminaries

2.1 Connection with Singularity Theory

Theorem 2 is motivated by a geometric genus formula valid for normal surface singularities. Next, we present two pieces of this connection, namely, the definition and covering properties of the equivariant geometric genera of normal surface singularities and the Seiberg–Witten invariant conjecture of Némethi and Nicolaescu [24]. For details, we refer to [20, 22–24, 26].

Let $(X, 0)$ be a complex normal surface singularity germ with link M. Let $\pi : \tilde{X} \to X$ be a good resolution with negative definite dual resolution graph G, which can be regarded also as a plumbing graph for the 4-manifold $\tilde{X}$ and its boundary M. (Hence, the E_v's in this context are the irreducible exceptional curves.) The *geometric genus* of the singularity is defined as $p_g(X) = \dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, where $\mathcal{O}_{\tilde{X}}$ is the structure sheaf of $\tilde{X}$. Although $p_g(X)$ is defined via $\tilde{X}$, it does not depend on the particular choice of the resolution. In [24], the following conjecture was formulated for certain (analytic types of) singularities, as a topological characterization of $p_g(X)$:

$$p_g(X) = \mathfrak{s}_0(M). \tag{7}$$

We say that the *Seiberg–Witten invariant conjecture* (SWIC) holds for $(X, 0)$ if (7) is valid.

It is natural to ask whether there is any similar connection involving the other Seiberg–Witten invariants? The answer is given in [20, 23]. Let $(Y, 0)$ be the universal abelian cover of the singularity $(X, 0)$ (that is, its link Σ is the regular UAC of M, $(Y, 0)$ is normal, and $(Y, 0) \to (X, 0)$ is analytic). The covering action of $H = H_1(M)$ on Y extends to an action on the resolution $\tilde{Y}$ of Y. Hence, H acts on $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ as well, and it provides an eigenspace decomposition $\oplus_{\xi \in \hat{H}} H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})_\xi$, indexed by the characters $\xi \in \hat{H} := \mathrm{Hom}(H, \mathbb{C}^*)$ of H. Set

$$p_g(X)_h = \dim_{\mathbb{C}} H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})_{\xi_h},$$

where $\xi_h \in \hat{H}$ is the character given by $h'' \mapsto e^{2\pi i(l',l'')}$, $[l'] = h$, $[l''] = h''$. The numbers $p_g(X)_h$ are called the *equivariant geometric genera* of $(X, 0)$. Note that $p_g(X)_0 = p_g(X)$.

We say that the *equivariant Seiberg–Witten invariant conjecture* (EqSWIC) holds for $(X, 0)$ if the next identity (8) is satisfied for *every* $h \in H$:

$$p_g(X)_h = \mathfrak{s}_h(M). \tag{8}$$

The EqSWIC holds for several families of singularities: rational, minimally elliptic, weighted homogeneous, or splice quotient singularities [20, 23, 25].

Observe that by the definition, $p_g(Y) = \sum_{h\in H} p_g(X)_h$. Hence, the next claim is obvious.

Claim 3 *If for a singularity* $(X, 0)$ *with* $\mathbb{Q}HS^3$ *link the EqSWIC holds, and for its (analytic) universal abelian cover* $(Y, 0)$ *with* $\mathbb{Q}HS^3$ *link the SWIC holds too, then for the link* M *of* $(X, 0)$ *the (purely topological) covering additivity property of* $\mathfrak{s}$ *also holds.*

Example 4 As we already mentioned above, by Némethi [20] and Némethi and Nicolaescu [25], the assumptions of Claim 3 are satisfied, e.g. by cyclic quotient and weighted homogeneous singularities; hence the CAP of $\mathfrak{s}$ holds, e.g. for all *lens spaces* and *Seifert rational homology sphere 3-manifolds*.

Theorem 2 of the present note proves CAP for surgery manifolds. Furthermore, Example 18 shows that CAP does *not* hold for arbitrary plumbed 3-manifolds.

The invariants $\{\mathfrak{s}_h(G)\}_h$ for many 3-manifolds (graphs) are computed. The next statement basically follows from Example 4 combined with the fact that the UAC of a lens space is S^3 (i.e. the UAC of a cyclic quotient is $(\mathbb{C}^2, 0)$ with $p_g(\mathbb{C}^2) = 0$; hence $\mathfrak{s}_h(G) = p_g(X)_h = 0$).

Proposition 5 ([20, 21, 23]) *If* G *is a (not necessarily minimal) graph of* S^3 *or of a lens space, then* $\mathfrak{s}_h(G) = 0$ *for every* $h \in H$.

2.2 The Structure of the Plumbing Graph G of $S^3_{-p/q}(K)$

In this section, we describe the plumbing graph of $S^3_{-p/q}(K)$ and we also fix some additional notations.

For $j = 1, \ldots, \nu$, let $K_j \subset S^3$ be the embedded knot of an irreducible plane curve singularity $\{f_j(x, y) = 0\} \subset (\mathbb{C}^2, 0)$, where f_j is a local holomorphic germ $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, which is locally irreducible. Let G_j be the *minimal embedded resolution graph* of $\{f_j(x, y) = 0\} \subset (\mathbb{C}^2, 0)$, which is a plumbing graph (of S^3) with several additional decorations. It has an arrowhead supported on a vertex u_j, which represents K_j (or, in a different language, it represents the strict transform $S(f_j)$ of $\{f_j = 0\}$ intersecting the exceptional (-1)-curve E_{u_j}). Furthermore, G_j has a set of multiplicity decorations, the vanishing orders $\{m_v\}_v$ of the pullback of f_j along the irreducible exceptional divisors and $S(f_j)$. We collect them in the total transform $\mathrm{div}(f_j) = S(f_j) + \sum_{v\in\mathcal{V}(G_j)} m_v E_v = S(f_j) + (f_j)$ of f_j, characterized by $(\mathrm{div}(f_j), E_v)_{G_j} = 0$ for any v, and (f_j) is its part supported on $\cup_{v\in\mathcal{V}(G_j)} E_v$. (For more on the graphs of plane curve singularities, see [6, 7].)

Define $K := K_1\#K_2\#\dots\#K_\nu$.

Next, we write the surgery coefficient in *Hirzebruch–Jung continued fraction*

$$p/q = k_0 - \cfrac{1}{k_1 - \cfrac{1}{k_2 - \cfrac{1}{\dots - \cfrac{1}{k_s}}}} =: [k_0, k_1, \dots, k_s], \tag{9}$$

where $k_i \in \mathbb{Z}$, $k_0 \geq 1, k_1, \dots, k_s \geq 2$.

Then $M = S^3_{-p/q}(K)$ can be represented by a negative definite plumbing graph G, which is constructed as follows, cf. [20, 27]. G consists of ν blocks, isomorphic to $G_1, \dots, G_\nu$ (with their Euler decorations but without the multiplicity decorations and arrowheads); a chain G_0 of length s consisting of vertices $\overline{u}_1, \dots, \overline{u}_s = u'$ with decorations $e_{\overline{u}_1} = -k_1, \dots, e_{\overline{u}_s} = -k_s$, respectively; and one 'central vertex' u. Moreover, we add $\nu + 1$ new edges: u and the vertex u_j from each block G_j is connected by an edge, and u and the first vertex $\overline{u}_1$ with decoration $-k_1$ of the chain G_0 is connected by an edge. The vertex u gets decoration $e_u = -k_0 - \sum_{j=1}^{\nu} m_{u_j}$.

Note that if $q = 1$, then G_0 is empty. In this case, we have $s = 0$ and $u = u'$.

We use the notation E_v, $v \in \mathcal{V}(G)$, for the basis of the lattice L_G associated with G. We simply write $(.,.)$ for the intersection form $(.,.)_G$, and E^*_v for the anti-dual elements in G; that is, $(E_{v'}, E^*_v) = -\delta_{v,v'}$ with the Kronecker-delta notation.

Similarly, we write $(.,.)_j = (.,.)_{G_j}$ for the intersection form of G_j ($j = 0, \dots, \nu$). For any $v \in \mathcal{V}(G_j)$, we set $E^{*,j}_v \in L'(G_j)$ for the anti-dual of E_v *in the graph* G_j; that is, $(E_{v'}, E^{*,j}_v)_j = -\delta_{v,v'}$, $v' \in \mathcal{V}(G_j)$.

We denote the canonical class of G by k_G and the canonical class of G_j by k_{G_j}.

By a general fact from the theory of surgeries, $H_1(M) = \mathbb{Z}_p$. In fact, $[E^*_{u'}]$ is a generator of this group. For the convenience of the reader, we provide a proof of these two statements (which will stay as a model for the corresponding statements valid for the UAC).

Lemma 6 *$H = \{[hE^*_{u'}]\}_h$, where $h \in \{0, 1, \dots, p-1\}$.*

Proof Consider the following element D of L_G. If on each G_j it is (f_j), we put multiplicity 1 on u, multiplicity k_0 on $\overline{u}_1$ and in general the numerator of $[k_0, \dots, k_{i-1}]$ on $\overline{u}_i$, $1 \leq i \leq s$. Furthermore, put an arrowhead on $\overline{u}_s$ with multiplicity p. If this arrowhead represents a cut S supported by $E_{\overline{u}_s}$ (that is, a disc in $P = P(G)$, which intersects $E_{\overline{u}_s}$ transversally), then $pS + D$, as an element of $H_2(P, \partial P, \mathbb{Z})$, has the property that $(pS + D, E_v) = 0$ for all $v \in \mathcal{V}(G)$. This shows that, in fact,

$E^*_{u'} = D/p$. Note also that the E_u-coefficient of D/p is $1/p$; hence the class of $E^*_{u'}$ (or of D/p) in $H = L'(G)/L(G)$ has order p.

Hence, if we show that H itself has order p, then we are done. For this, we verify that $\det(G) = p$. In determinant computations of decorated *trees* $\mathfrak{G}$, the following formula is useful; see e.g. [5, 4.0.1(d)].

$$\begin{cases} \text{Let } e \text{ be an edge of } \mathfrak{G} \text{ with end vertices } a \text{ and } b. \text{ Then} \\ \det(\mathfrak{G}) = \det(\mathfrak{G} \setminus e) - \det(\mathfrak{G} \setminus \{a, b, \text{and their adjacent edges}\}). \end{cases} \tag{10}$$

We proceed by induction over ν. In order to run the induction properly, we introduce slightly more general graphs. Let $G^\ell(x)$ be the graph constructed similarly as G above, but now we glue only G_0 and $G_1, \ldots, G_\ell$, where $1 \le \ell \le \nu$, and we put for the Euler decoration e_u of the central vertex u the general integer $e_u := -k_0 - x$ for some $x \in \mathbb{Z}$. For example, $G = G^\nu(\sum_{j=1}^{\nu} m_{u_j})$.

Note that $\det(G_j) = 1$ (since the graph represents S^3), and $\det(G_j \setminus u_j) = m_{u_j}$. Indeed, using (1), $m_{u_j} = -((f_j), E^*_{u_j})_{G_j} = -(E^*_{u_j}, E^*_{u_j})_{G_j} = \det(G_j \setminus u_j)$.

Additionally, let us define also $G_0(x)$, the string with $s + 1$ vertices with decorations $-k_0 - x, -k_1, \ldots, -k_s$. Note that from definitions $\det(G_0(0)) = p = k_0 q - r$, and by (10) (used for the first edge) $\det(G_0(x)) = (x + k_0)q - r = p + xq$.

Now, let us compute $\det(G^\ell(x))$. If $\ell = 1$, and if we take for e in (10) the edge (u_1, u), then $\det(G^1(x)) = \det(G_1) \cdot \det(G_0(x)) - \det(G_1 \setminus u_1) \cdot \det(G_0(x) \setminus u) = p + xq - m_{u_1} q$. Hence, for $x = m_{u_1}$, we get p.

If $\ell = 2$, and if we take $e = (u, u_2)$, then $\det(G^2(x)) = \det(G^1(x)) - m_{u_2} q = p + xq - m_{u_1} q - m_{u_2} q$, which equals p for $x = m_{u_1} + m_{u_2}$. For arbitrary ℓ, we run induction. □

2.3 *The Structure of Plumbing Graph* Γ *of the UAC* Σ *of* $M = S^3_{-p/q}(K)$

We construct a plumbing graph Γ as follows. Γ consists of ν blocks $\Gamma_1, \ldots, \Gamma_\nu$ with distinguished vertices $w_1, \ldots, w_\nu$ (which will be described later), a chain Γ_0 of length $q - 1$ consisting of vertices $\overline{w}_1, \ldots, \overline{w}_{q-1} = w'$, all with decoration -2, and a 'central vertex' w and some additional edges. These edges are: each vertex w_j connected by an edge to w, and $\overline{w}_1$ is also connected by an edge to w. If $q = 1$, then Γ_0 is empty and $w = w'$.

Γ_j is a plumbing graph of the link of the suspension hypersurface singularity $\{g_j = 0\}$, where $g_j(x, y, z_j) = f_j(x, y) + z_j^p$ (for its shape, see [16]). The vertex w_j of Γ_j is that vertex which supports the arrowhead (representing the strict transform $S(z_j)$ of $\{z_j = 0\}$), if we regard Γ_j as the embedded resolution graph of $\{z_j = 0\} \subset \{g_j = 0\}$ (that is, it supports the strict transform of $\{z_j = 0\}$).

The self-intersection of w is determined as follows.

Let F_v, $v \in \mathcal{V}(\Gamma)$ denote the basis elements of the lattice $L(\Gamma)$ associated with Γ. We regard Γ_j as a subgraph of Γ; hence $\{F_v\}_{v\in\mathcal{V}(\Gamma_j)}$ are regarded as generators of $L(\Gamma_j)$.

We write $\mathrm{div}(z_j) = S(z_j) + \sum_{v\in\mathcal{V}(\Gamma_j)} n_v F_v$ for the total transform of $\{z_j = 0\}$ in the embedded resolution of $\{z_j = 0\} \subset \{g_j = 0\}$ with resolution graph Γ_j. This means that $\mathrm{div}(z_j)$ topologically is characterized by $(\mathrm{div}(z_j), F_v)_{\Gamma_j} = 0$ for any $v \in \mathcal{V}(\Gamma_j)$; the strict transform $S(z_j)$ can be represented as an arrowhead on w_j.

Then we define the decoration of the central vertex w in Γ by $e_w = -1 - \sum_{j=1}^{\nu} n_{w_j}$.

Lemma 7 Γ *is a (possible) plumbing graph of the UAC* Σ.

Proof This follows basically from the topological interpretation of the cyclic covering algorithms from [16, 17]. Although we will not review the whole algorithm, we will explain how it applies (in particular, some familiarity of the reader with this algorithm is needed), and we also determine the main blocks of Γ provided by the algorithm.

Consider the element D of L_G constructed in the proof of Lemma 6. Its multiplicities can be completed with an arrowhead with multiplicity p supported on $\overline{u}_s$. Equivalently, we can put a cut S on $E_{\overline{u}_s}$, and then $pS + D \in H_2(P, \partial P, \mathbb{Z})$ has the property that $(pS + D, E_v) = 0$ for all $v \in \mathcal{V}(G)$ (cf. the proof of Lemma 6). This means that $pS + D$ is a topological analogue of the divisor of a function. The algorithm which provides the (topological) cyclic $\mathbb{Z}_p$-covering of the plumbed 4-manifold P with branch locus $pS + D$ is identical with the algorithm from [16, 17] (which provides branched cyclic covers associated with an analytic function with, say, divisor $pS + D$ in $\tilde{X}$).

The point is that S has multiplicity p; hence the $\mathbb{Z}_p$-covering will have no branching along it. This can be proved as follows (we prefer again an analytic language, but the interested reader might rewrite it in a purely topological language). Let U be a local neighbourhood of the intersection point $S \cap E_{\overline{u}_s}$, let (ζ_1, ζ_2) be local analytic coordinates in U such that $\{\zeta_1 = 0\} = S \cap U$, $\{\zeta_2 = 0\} = U \cap E_{\overline{u}_s}$. Then $pS + D$ in U is the divisor of $\zeta_1^p \zeta_2^m$ for some integer m with $(p, m) = 1$. Then the cyclic $\mathbb{Z}_p$ covering above U is the normalization of $\{\zeta_1^p \zeta_2^m = \xi^p\}$. This is smooth with coordinates (ζ_1, σ), and the normalization map is $\zeta_1 = \zeta_1,\ \zeta_2 = \sigma^p,\ \xi = \zeta_1 \sigma^m$, which projects to U as $\zeta_1 = \zeta_1$ and $\zeta_2 = \sigma^p$. Hence, above the generic point of S, there are p points.

In particular, the branch locus is supported in $\cup_v E_v$, that is, we get an unbranched covering of M. Note also that the E_u-coefficient of D/p is $1/p$; hence the class of D/p (which in fact equals $E^*_{u'}$) has order p in H, and it generates H (see also the proof of Lemma 6). This implies that this algorithm provides exactly the UAC of M.

Since the algorithm is 'local', and the multiplicity of u is 1, it follows that over the subgraphs G_j, it is identical with that one which provides the graph of the suspension singularity $f_j + z_j^p$. Moreover, over u, we will have exactly one vertex in the covering, namely, w, and this will get multiplicity 1 (see again [16, 17]).

Next, we verify its behaviour over the graph G_0. This graph is the graph of a cyclic quotient singularity of type (q, r), where $q/r = [k_1, \dots, k_s]$. This is the normalization of $xy^{q-r} = z^q$. (For details regarding cyclic quotient singularities, see [2].)

Using this coordinate choice, the strict transform of y is exactly qS; the strict transform of x is a disc S' in E_u (a disc neighbourhood of $E_u \cap E_{\overline{u}_1}$ in E_u) with multiplicity q; and finally, the strict transform of z is $S' + (q - r)S$. In particular, the cyclic covering we consider over G_0 is exactly the cyclic $\mathbb{Z}_p$-covering of the normalization of $xy^{q-r} = z^q$ along the divisor of zy^{k_0-1} (here for the S-multiplicity, use $q - r + (k_0 - 1)q = k_0q - r = p$). This is a new cyclic quotient singularity, the normalization of $\{(x, y, z, w) : xy^{q-r} = z^q,\ zy^{k_0-1} = w^p\}$.

The q-power of the second equation combined with the first one gives $xy^p = w^{pq}$; hence $t := w^q/y$ is in the integral closure with $x = t^p$. Hence, after eliminating x, the new equations are $ty = w^q,\ t^p y^{q-r} = z^q$ and $zy^{k_0-1} = w^p$. A computation shows that the integral closure of this ring is given merely by $ty = w^q$. (For this, use $zy^{k_0-1} = w^p = w^{k_0q-r} = w^{(k_0-1)q}w^{q-r} = t^{k_0-1}y^{k_0-1}w^{q-r}$, that is, $y^{k_0-1}(z - t^{k_0-1}w^{q-r}) = 0$.)

This is an A_{q-1} singularity, whose minimal resolution graph is Γ_0.

Finally, notice that the above algorithm provides a system of multiplicities, which can be identified with a homologically trivial divisor; hence, similarly as in [16, 17], we get (via intersection with F_w) from the multiplicities the last 'missing Euler number' e_w, too. □

The intersection form of Γ will be denoted by $\langle .,. \rangle = (.,.)_\Gamma$. Similarly, $\langle .,. \rangle_j = (.,.)_{\Gamma_j}$ will denote the intersection form of Γ_j. The canonical class of Γ is k_Γ; the canonical class of Γ_j is k_{Γ_j}. For any $v \in \mathcal{V}(\Gamma)$, F_v^* will denote the anti-dual of the corresponding divisor F_v in Γ. Similarly, for a vertex $v \in \mathcal{V}(\Gamma_j)$, $F_v^{*,j}$ is the anti-dual of F_v in Γ_j. Set $J = H_1(\Sigma)$ and let J_j be the first homology group of the 3-manifolds determined by Γ_j.

Lemma 8

$$J \cong J_1 \times \cdots \times J_\nu.$$

Proof Let $\rho_j \in \hat{J}_j$ be a character of Γ_j, $j \geq 1$. In [26, Sect. 6.3], it is proved that ρ_j takes value 1 on $F_{w_j}^{*,j}$ (recall that the vertex w_j of Γ_j is connected with the central vertex w). Hence, for $j \neq i$, $j, i \geq 1$, there is no edge (v_j, v_i) of Γ, such that v_j is in the support of ρ_j and v_i is in the support of ρ_i. This means that each $\rho_j \in \hat{J}_j$ can be extended to a character of J, by setting $\rho_j(F_v^*) = 1$ whenever $v \notin \mathcal{V}(\Gamma_j)$; in this way providing a monomorphism $\hat{J}_j \hookrightarrow \hat{J}$. Moreover, the same property also guarantees that in fact one has a simultaneous embedding $\prod_{j\geq 1} \hat{J}_j \hookrightarrow \hat{J}$. Therefore, if we prove that $\prod_{j\geq 1} \det(\Gamma_j) = \det(\Gamma)$, then the above embedding becomes an isomorphism; hence the statement follows.

For the determinant identity, we use similar method as in the proof of Lemma 6. Let $\Gamma^\ell(x)$ be the graph obtained by connecting (the distinguished vertices of) Γ_0 and $\Gamma_1, \dots, \Gamma_\ell$ to w, where $1 \leq \ell \leq \nu$, and we put for the Euler decoration e_w the integer

$e_w := -1-x$. Furthermore, let $\Gamma_0(x)$ be the string with q vertices and decorations $-1-x, -2, -2, \dots$. Hence $\det(\Gamma_0(x)) = 1+xq$.

We write d_j for $\det(\Gamma_j) = |J_j|$, and note that $\det(\Gamma_j \setminus w_j) = d_j n_{w_j}$ (use (1)).

Now, we are ready to compute $\det(\Gamma^\ell(x))$. If $\ell = 1$, and in (10) we take $e = (w_1, w)$, then $\det(\Gamma^1(x)) = d_1(1+xq) - d_1 n_{w_j} q$, which equals d_1 whenever $x = n_{u_1}$.

If $\ell = 2$ and $e = (u, u_2)$, one has $\det(\Gamma^2(x)) = \det(\Gamma^1(x))d_2 - d_2 n_{w_2} d_1 q = d_1 d_2(1 + xq - n_{w_1} q - n_{w_2} q)$, which equals $d_1 d_2$ whenever $x = m_{u_1} + m_{u_2}$. For arbitrary ℓ, run induction. □

Lemma 9

$$\begin{aligned}
&(a) \quad -p\cdot(E^*_u, E^*_u) = q \quad \textit{and} \quad -p\cdot(E^*_{u'}, E^*_u) = 1;\\
&(b) \quad q\cdot(E^{*j}_{u_j}, E^{*j}_v)_j = p\cdot(E^*_u, E^*_v) \quad \textit{for any } v \in \mathcal{V}(G_j),\ j\geq 1;\\
&(c) \quad -\langle F^*_w, F^*_w\rangle = q \quad \textit{and} \quad -\langle F^*_{w'}, F^*_w\rangle = 1;\\
&(d) \quad q\cdot\langle F^{*j}_{w_j}, F^{*j}_v\rangle_j = \langle F^*_w, F^*_v\rangle \quad \textit{for any } v\in\mathcal{V}(\Gamma_j),\ j\geq 1.
\end{aligned} \tag{11}$$

Proof Identity (1) applied for G and the pair of vertices u, u (and u, u') gives (a), since $\det(G) = p$, $\det(G_j) = 1$ for $j \geq 1$ and $\det(G_0) = q$. (b) follows similarly. (c) and (d) follows from this property combined with Lemma 8. □

3 Proof of Theorem 2

3.1 Proof of the Main Result

Now we are ready to prove Theorem 2. To adjust it to its proof, we recall it in a more explicit form, in the language of plumbing graphs.

Theorem 10 *Let K be a connected sum of algebraic knots, p, q coprime positive integers. Assume that both $S^3_{-p/q}(K)$ (having plumbing graph G) and its universal abelian cover Σ (with plumbing graph Γ) are rational homology spheres. Then the following additivity holds:*

$$\underbrace{\mathfrak{sw}_0(\Gamma) - \frac{\langle k_\Gamma, k_\Gamma\rangle + \#\mathcal{V}(\Gamma)}{8}}_{\mathfrak{s}_0(\Gamma)} = \sum_{h=0}^{p-1} \underbrace{\left(\mathfrak{sw}_h(G) - \frac{(k_G + 2r_h, k_G + 2r_h) + \#\mathcal{V}(G)}{8}\right)}_{\mathfrak{s}_h(G)}.$$

On the left-hand side, the index 0 of $\mathfrak{sw}_0(\Gamma)$ is the unit element of $J = H_1(\Sigma)$, and on the right-hand side, we identified elements of $H \cong \mathbb{Z}_p$ with elements of $\{0, 1, \dots, p-1\}$, 0 being the unit element and 1 being the generator $[E^*_{u'}]$, i.e. $r_h = r_{[hE^*_{u'}]}$.

In fact, the condition whether Σ is a $\mathbb{Q}HS^3$ or not is readable already from p and the plane curve singularity invariants describing the knots K_i; cf. [26, Sect. 6.2(c)].

Proof Notice that deleting from G the 'central vertex' u (and all its adjacent edges), one gets $G_0, G_1, \ldots, G_\nu$ as connected components of the remaining graph. Also, deleting from Γ the 'central vertex' w (and all its adjacent edges), one gets $\Gamma_0, \Gamma_1, \ldots, \Gamma_\nu$ as connected components of the remaining graph. (Γ_0 and G_0 are present only if $q > 1$).

We use the notations R_j, respectively, $\tilde{R}_j$, for the 'restriction' homomorphisms $L'_G \to L'_{G_j}$, respectively, $L'_\Gamma \to L'_{\Gamma_j}$, dual to the natural inclusions $L_{G_j} \to L_G$, respectively, $L_{\Gamma_j} \to L_\Gamma$. They are characterized by $R_j(E^*_v) = E^{*,j}_v$, if $v \in \mathcal{V}(G_j)$ and 0 otherwise, respectively, $\tilde{R}_j(F^*_v) = F^{*,j}_v$, if $v \in \mathcal{V}(\Gamma_j)$ and 0 otherwise. For example, $R_j(k_G) = k_{G_j}$ and $\tilde{R}_j(k_\Gamma) = k_{\Gamma_j}$ (cf. [5, Definition 3.6.1 (2)]).

We can apply the surgery formula of [5, Theorem 1.0.1] (note the sign difference due to the different sign convention about $\mathfrak{sw}$) and get the following two formulae. The new symbols $\mathcal{H}^{\rm pol}_{u,h}(1)$ and $\mathcal{F}^{\rm pol}_{w,0}(1)$ are values at $t = 1$ of certain polynomials as in [5, Sect. 3.5]; their definitions will be recalled later in (14) and (15).

$$\mathfrak{s}_h(G) = \mathcal{H}^{\rm pol}_{u,h}(1) + \mathfrak{s}_{R_0(r_h)}(G_0) + \sum_{j=1}^{\nu} \mathfrak{s}_{R_j(r_h)}(G_j), \tag{12}$$

$$\mathfrak{s}_0(\Gamma) = \mathcal{F}^{\rm pol}_{w,0}(1) + \mathfrak{s}_0(\Gamma_0) + \sum_{j=1}^{\nu} \mathfrak{s}_0(\Gamma_j). \tag{13}$$

In (12), for $j \geq 1$, $[R_j(r_h)] = 0 \in L'_{G_j}/L_{G_j}$, as the latter one is the trivial group $H_1(S^3)$. Hence, by (6), $\mathfrak{s}_{R_j(r_h)}(G_j) = \mathfrak{s}_0(G_j) + \chi_j(R_j(r_h))$, where $\chi_j(x) := -\frac{1}{2}(x, x + k_{G_j})_j$.

Furthermore, by Proposition 5, $\mathfrak{s}_0(\Gamma_0) = 0$, and $\mathfrak{s}_0(G_j) = 0$ for $j \geq 1$ (as G_j is a plumbing graph for S^3). Therefore, the desired equality $\mathfrak{s}_0(\Gamma) = \sum_{h=0}^{p-1} \mathfrak{s}_h(G)$ reduces to the proof of the following three lemmas. □

Lemma 11

$$\sum_{h=0}^{p-1} \mathcal{H}^{pol}_{u,h}(1) = \mathcal{F}^{pol}_{w,0}(1).$$

Lemma 12

$$\sum_{h=0}^{p-1} \chi_j(R_j(r_h)) = \mathfrak{s}_0(\Gamma_j) \quad (\textit{for any } j \geq 1).$$

Lemma 13

$$\sum_{h=0}^{p-1} \mathfrak{s}_{R_0(r_h)}(G_0) = 0.$$

In the next paragraphs, we recall the definition of $\mathcal{H}^{\mathrm{pol}}_{u,h}$ and $\mathcal{F}^{\mathrm{pol}}_{w,0}$ (following [5, Sect. 3.5]) adapted to the present case and notations, and then we provide the proofs of the lemmas.

First, given a rational function $\mathcal{R}(t)$ of t, one defines its *polynomial part* $\mathcal{R}^{\mathrm{pol}}(t)$ as the unique polynomial in t such that $\mathcal{R}(t) - \mathcal{R}^{\mathrm{pol}}(t)$ is either 0 or it can be written as a quotient of two polynomials of t such that the numerator has degree strictly less than the denominator.

Now $\mathcal{F}^{\mathrm{pol}}_{w,0}$ and $\mathcal{H}^{\mathrm{pol}}_{u,h}$ are polynomial parts of rational functions defined as follows.

$$\mathcal{H}_{u,h}(t) = \frac{1}{p} \cdot \sum_{\varrho \in \hat{H}} \varrho^{-1}(h) \prod_{v \in \mathcal{V}(G)} (1 - \varrho([E^*_v]) t^{-p \cdot (E^*_u, E^*_v)})^{\delta_v - 2}, \tag{14}$$

where δ_v denotes the degree (number of adjacent edges) of a vertex $v \in \mathcal{V}(G)$.

$$\mathcal{F}_{w,0}(t) = \frac{1}{|J|} \cdot \sum_{\varrho \in \hat{J}} \prod_{v \in \mathcal{V}(\Gamma)} (1 - \varrho([F^*_v]) t^{-|J| \langle F^*_w, F^*_v \rangle})^{\tilde{\delta}_v - 2}, \tag{15}$$

where $\tilde{\delta}_v$ denotes the degree of a vertex $v \in \mathcal{V}(\Gamma)$.

Proof of Lemma 11 By Fourier summation

$$\mathcal{H}_u(t) := \sum_{h=0}^{p-1} \mathcal{H}_{u,h}(t) = \prod_{v \in \mathcal{V}(G)} (1 - t^{-p \cdot (E^*_u, E^*_v)})^{\delta_v - 2}. \tag{16}$$

As taking polynomial parts of rational functions is additive, Lemma 11 follows if we prove

$$\mathcal{H}_u(t^{|J|}) = \mathcal{F}_{w,0}(t). \tag{17}$$

Let $\Delta_j = \Delta_{S^3}(K_j)$ be the Alexander polynomial of the knot K_j (defined as in [26, Sect. 2.6, (8)], or [7]). Then, since $E^{*,j}_{u_j} = (f_j) \in L_{G_j}$,

$$\frac{\Delta_j(t)}{1-t} = \prod_{v \in \mathcal{V}(G_j)} (1 - t^{-(E^{*,j}_{u_j}, E^{*,j}_v)_j})^{\delta_v - 2}.$$

Comparing (16) with the above formula for the Alexander polynomials and using the identities (a), (b) of Lemma 9, we get that

$$\mathcal{H}_u(t) = \frac{\prod_{j=1}^{\nu} \Delta_j(t^q)}{(1-t)(1-t^q)}. \tag{18}$$

Recall that $J_j = L'_{\Gamma_j}/L_{\Gamma_j}$ is the first homology group of the manifold determined by Γ_j and that $F^{*,j}_{w_j} = (z_j) \in L_{\Gamma_j}$. Let Δ_{j,Γ_j} be the Alexander polynomial of the knot K_{j,Γ_j} in the manifold of Γ_j determined by $z_j = 0$ (see [26, Sect. 2.6, (8)]). That is,

$$\frac{\Delta_{j,\Gamma_j}(t)}{1-t} = \frac{1}{|J_j|} \cdot \sum_{\varrho_j \in \widehat{J_j}} \prod_{v \in \mathcal{V}(\Gamma_j)} (1 - \varrho_j([F^{*,j}_v]) t^{-\langle F^{*,j}_{w_j}, F^{*,j}_v \rangle_j})^{\tilde{\delta}_v - 2}.$$

Recall that $J = J_1 \times \cdots \times J_\nu$. Consequently, any character $\varrho \in \hat{J}$ can be written as a ν-tuple of characters, $\varrho = (\varrho_1, \ldots, \varrho_\nu)$ with $\varrho_j \in \hat{J}_j = \mathrm{Hom}(J_j, \mathbb{C}^*)$. Furthermore, for any $v \in \mathcal{V}(\Gamma_j)$, $\varrho([F^*_v]) = \varrho_j([F^{*,j}_v])$ and $\varrho([F^*_v]) = 1$ if $v = w$ or $v \in \Gamma_0$ as in that case F^*_v represents the trivial element in L'_Γ/L_Γ (see also the proof of Lemma 8).

Comparing (15) with the above formula for the Alexander polynomials and using the identities (c), (d) of Lemma 9, we get that setting $s = t^{|J|}$,

$$\mathcal{F}_{w,0}(t) = \frac{\prod_{j=1}^{\nu} \Delta_{j,\Gamma_j}(s^q)}{(1-s)(1-s^q)}. \tag{19}$$

By [26, Prop. 6.6] $\Delta_j = \Delta_{j,\Gamma_j}$, so via (18) and (19), we obtain (17). □

Proof of Lemma 12 For any element $l' = \sum_{v \in \mathcal{V}(G)} c_v E_v \in L'_G$, let $\lfloor l' \rfloor := \sum_{v \in \mathcal{V}(G)} \lfloor c_v \rfloor E_v$, respectively, $\{l'\} := l' - \lfloor l' \rfloor$, denote the coordinatewise integer, respectively, fractional part of l' in the basis $\{E_v\}_v$. We use this notation for other graphs as well.

Using the description of $E^*_{u'}$ in the proof of Lemma 7, we have

$$hE^*_{u'} = \sum_{j \geq 1} h \cdot (f_j)/p + hE_u/p + D_0 \quad (0 \leq h < p), \tag{20}$$

where D_0 is supported on G_0. Since $r_h = \{hE^*_{u'}\}$, we obtain

$$r_h = hE^*_{u'} - \sum_{j \geq 1} \lfloor h \cdot (f_j)/p \rfloor - \lfloor D_0 \rfloor.$$

Since $R_j(E^*_{u'}) = 0$, $R_j(E_v) = E_v$ for $v \in \mathcal{V}(\Gamma_j)$, and $R_j(E_v) = 0$ for $v \notin (\mathcal{V}(\Gamma_j) \cup u)$, we get

$$R_j(r_h) = -\lfloor h \cdot (f_j)/p \rfloor. \tag{21}$$

As Γ_j is the plumbing graph of a suspension hypersurface singularity $\{g_j(x, y, z_j) = f_j(x, y) + z_j^p = 0\}$ and, as it is proved in [26], *for such suspension singularities the SWIC holds* (see Sect. 2.1), we have $\mathfrak{s}_0(\Gamma_j) = p_g(\{g_j = 0\})$. Hence, the statement of the lemma is equivalent with the following geometric genus formula valid for suspension singularities:

$$p_g(\{g_j = 0\}) = \sum_{h=0}^{p-1} \chi_j \left(-\lfloor h \cdot (f_j)/p \rfloor\right).$$

This formula has major importance even independently of the present application. We separate the statement in the following Claim.

Claim 14 *Let $f(x, y) \in \mathbb{C}\{x, y\}$ be the equation of an irreducible plane curve singularity. Let G_f be the dual resolution graph of a good embedded resolution of f, from which we delete the arrowhead (strict transform) of f and all the multiplicities. Let (f) be the part of the divisor of f supported on the exceptional curves. Then for any positive integer p the geometric genus of the suspension singularity $\{g(x, y, z) = f(x, y) + z^p = 0\}$ is*

$$p_g(\{g = 0\}) = \sum_{h=0}^{p-1} \chi(-\lfloor h \cdot (f)/p \rfloor).$$

Remark 15 A combinatorial formula (involving Dedekind sums) for the *signature* of (the Milnor fibre of) suspension singularities was presented in [16]. Recall that Durfee and Laufer type formulae imply that the geometric genus and the signature determine each other modulo the link (see e.g. [18, Theorem 6.5] and the references therein). In particular, the mentioned signature formulae provide expressions for the geometric genus as well. Nevertheless, the above formula is of different type.

Proof Let $\phi : Z \to (\mathbb{C}^2, 0)$ be the embedded resolution of f. Consider the $\mathbb{Z}_p$ branched covering $c : (\{g = 0\}, 0) \to (\mathbb{C}^2, 0)$, the restriction of $(x, y, z) \mapsto (x, y)$. Let $c_\phi : W \to Z$ be the pullback of c via ϕ and let $\hat{c}_\phi : \hat{W} \to Z$ be the composition of the normalization $n : \hat{W} \to W$ with c_ϕ. Then $\hat{W} \to W \to \{g = 0\}$ is a partial resolution of $\{g = 0\}$: although it might have some cyclic quotient singularities, since these are rational, one has $p_g(\{g = 0\}) = h^1(\mathcal{O}_{\hat{W}})$. On the other hand, we claim that

$$(\hat{c}_\phi)_*(\mathcal{O}_{\hat{W}}) = \oplus_{h=0}^{p-1} \mathcal{O}_Z(\lfloor h \cdot (f)/p \rfloor). \tag{22}$$

This follows basically from [12, Sect. 9.8]. For the convenience of the reader, we sketch the proof.

We describe the sheaves $(c_\phi)_*(\mathcal{O}_W)$ and $(\hat{c}_\phi)_*(\mathcal{O}_{\hat{W}})$ in the neighbourhood U of a generic point of the exceptional set E of ϕ. Consider such a point with local coordinates (u, v), $\{u = 0\} = E \cap U$, (f) in U is given by $u^m = 0$. Consider the covering, a local neighbourhood of type $\{(u, v, z) : z^p = u^m\}$ in W. Then $\mathcal{O}_{W,0}$ as $\mathbb{C}\{u, v\}$-module is $\oplus_{h=0}^{p-1} z^h \cdot \mathbb{C}\{u, v\}$. For simplicity, we assume $\gcd(m, p) = 1$. The $\mathbb{Z}_p$-action is induced by the monodromy on the regular part, namely, by the permutation of the z-pages, induced over the loop $u(s) = \{e^{2\pi is}\}_{0\le s\le 1}$. This is the multiplication by $\xi := e^{2\pi i m/p}$. Hence, $z^h\mathbb{C}\{u, v\}$ is the ξ^h-eigensheaf of $(c_\phi)_*(\mathcal{O}_W)$.

If we globalize $z\mathbb{C}\{u, v\}$, we get a line bundle on Z, say $\mathcal{L}$. Then the local representative of $\mathcal{L}^p$ is $z^p\mathbb{C}\{u, v\} = u^m\mathbb{C}\{u, v\} = \mathbb{C}\{u, v\}(-(f))$. Hence $\mathcal{L}^p$ is trivialized by $f \circ \phi$. Since $\mathrm{Pic}(Z) = 0$, $\mathcal{L}$ itself is a trivial line bundle on Z.

Next, we consider the normalization $\hat{W}$. Above U, it is $(\mathbb{C}^2, 0)$ with local coordinates (t, v), and the normalization is $z = t^m$, $u = t^p$. In particular, $(\hat{c}_\phi)_*(\mathcal{O}_{\hat{W},0}) = \oplus_{h=0}^{p-1} t^h \cdot \mathbb{C}\{u, v\}$, where $\mathcal{F}^{(h)} := t^h \cdot \mathbb{C}\{u, v\}$ is the $e^{2\pi ih/p}$-eigensheaf. Set the integer m' with $0 \le m' < p$ and $mm' = 1 + kp$ for certain $k \in \mathbb{Z}$. Then one has the following eigensheaf inclusions: $t^h\mathbb{C}\{u, v\} \supset z^{\{\frac{hm'}{p}\}p} \cdot \mathbb{C}\{u, v\} = \mathcal{L}^{\{\frac{hm'}{p}\}p}|_U$. Hence, for some effective cycle D, we must have $t^h\mathbb{C}\{u, v\} = \mathcal{L}^{\{\frac{hm'}{p}\}p}(D)|_U$. This, by taking m-power reads as $z^h\mathbb{C}\{u, v\} = z^{\{\frac{hm'}{p}\}pm}\mathbb{C}\{u, v\}(mD)$. This means that if $\{hm'/p\} = m_h/p$ and $mm_h = k_hp + h$ for certain integers m_h and k_h, $0 \le m_h < p$, then the local equation of mD is $z^{\{\frac{hm'}{p}\}pm-h} = z^{k_hp}$. Hence D locally is given by $t^{k_hp} = u^{k_h}$. Since $k_h = \lfloor mm_h/p\rfloor$, the global reading of this fact is $D = \lfloor m_h \cdot (f)/p\rfloor$. Hence

$$(\hat{c}_\phi)_*(\mathcal{O}_{\hat{W}}) = \oplus_{h=0}^{p-1} \mathcal{L}^{\{\frac{hm'}{p}\}p}(\lfloor m_h \cdot (f)/p\rfloor).$$

Since $\mathcal{L}$ is a trivial bundle, and $h \mapsto m_h$ is a permutation of $\{0, \dots, p-1\}$, (22) follows.

Next, from (22) we obtain $p_g(\{g = 0\}) = \sum_h h^1(\mathcal{O}_Z(\lfloor h \cdot (f)/p\rfloor))$.

Set $D' := \lfloor h \cdot (f)/p\rfloor$. Then from the cohomological exact sequence of the exact sequence of sheaves $0 \to \mathcal{O}_Z \to \mathcal{O}_Z(D') \to \mathcal{O}_{D'}(D') \to 0$, and from $h^1(\mathcal{O}_Z) = p_g((\mathbb{C}^2, 0)) = 0$, we get $h^1(\mathcal{O}_Z(D')) = h^1(\mathcal{O}_{D'}(D'))$. Since by Grauert–Riemenschneider vanishing $h^0(\mathcal{O}_{D'}(D')) = 0$, we have

$$h^1(\mathcal{O}_Z(D')) = -\chi(\mathcal{O}_{D'}(D')) = -(D', D') + (D', D' + K)/2 = \chi(-D').$$

This ends the proof of the Claim. □

Moreover, the proof of Lemma 12 is also completed. □

Proof of Lemma 13 We observe two facts. First, from the proof of Lemma 7, we obtain that $R_0(r_h)$ only depends on the value p/q (and not on the blocks G_j, $j \ge 1$). (Equivalently, from (20), we have that D_0 is the unique rational cycle on G_0 such that, when completed with an arrowhead supported on $u' = \overline{u}_s$ with multiplicity one and with an arrowhead supported on $\overline{u}_1$ with multiplicity h/p, it has the property that intersected by any $E_{\overline{u}_j}$ the result is zero.) It has the same expression even if we replace all the graph G_j by the empty graph.

Second, from Eqs. (13) and (12) and the discussion after it, we get that under the validity of Lemmas 11 and 12 (what we already proved for any situation), the main Theorem 10 (property CAP) is equivalent with Lemma 13. Putting these facts together, we get that the validity of 13 is equivalent with the validity of CAP in the case when $G_j = \emptyset$ for all $j \geq 1$. But CAP for $G_0 \cup \{u\}$ is true by Claim 3 and Example 4.

(Of course, there is also a direct argument by a combinatorial computation of the involved invariants on the string G_0.) □

4 The Invariant $\mathfrak{s}_h$ and Lattice Cohomology

4.1 *Lattice Cohomology*

The normalized SW invariant $\mathfrak{s}_h(G)$ can also be expressed as the Euler characteristic of the *lattice cohomology*. The advantage of this approach is that it provides an alternative, completely elementary way to define $\mathfrak{s}$, as the definition of the lattice cohomology is purely combinatorial from the plumbing graph G. This description is rather different than the one used in [24–26] or the one used in the above proofs.

Another advantage is that for integral surgeries, there are several computations/formulae for the lattice cohomology in the literature, which provide additional information on the main theorem or on the different surgery pieces used in its proof.

4.2 *Short Introduction into Lattice Cohomology*

We briefly recall the definition and some facts about the lattice cohomology associated with a $\mathbb{Q}HS^3$ 3-manifold with negative definite plumbing graph G. For more, see [21, 27].

Usually one starts with a lattice $\mathbb{Z}^s$ with fixed base elements $\{E_i\}_i$. This automatically provides a cubical decomposition of $\mathbb{R}^s = \mathbb{Z}^s \otimes \mathbb{R}$: the 0-cubes are the lattice points $l \in \mathbb{Z}^s$, the 1-cubes are the 'segments' with endpoints l and $l + E_i$, and more generally, a q-cube $\square = (l, I)$ is determined by a lattice point $l \in \mathbb{Z}^s$ and a subset $I \subset \{1, \dots, s\}$ with $\#I = q$, and it has vertices at the lattice points $l + \sum_{j \in J} E_j$ for different $J \subset I$.

One also takes a weight function $w : \mathbb{Z}^s \to \mathbb{Z}$ bounded below, and for each cube $\square = (l, I)$, one defines $w(\square) := \max\{w(v),\ v \text{ vertex of } \square\}$. Then, for each integer $n \geq \min(w)$, one considers the simplicial complex S_n of $\mathbb{R}^s$, the union of all the cubes $\square$ (of any dimension) with $w(\square) \leq n$. Then the *lattice cohomology associated with* w is $\{\mathbb{H}^q(\mathbb{Z}^s, w)\}_{q \geq 0}$, defined by

$$\mathbb{H}^q(\mathbb{Z}^s, w) := \oplus_{n \geq \min(w)} H^q(S_n, \mathbb{Z}).$$

Each $\mathbb{H}^q$ is graded (by n), and it is a $\mathbb{Z}[U]$-module, where the U-action consists of the restriction maps induced by the inclusions $S_n \hookrightarrow S_{n+1}$. Similarly, one defines the *reduced cohomology associated with* w by

$$\mathbb{H}^q_{\mathrm{red}}(\mathbb{Z}^s, w) := \oplus_{n\geq \min(w)}\tilde{H}^q(S_n, \mathbb{Z}).$$

In all our cases, $\mathbb{H}^q_{\mathrm{red}}(\mathbb{Z}^s, w)$ has finite $\mathbb{Z}$-rank. The *normalized Euler characteristic* of $\mathbb{H}^*(\mathbb{Z}^s, w)$ is $\mathrm{eu}\,\mathbb{H}^* := -\min(w) + \sum_{q\geq 0}(-1)^q \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^q_{\mathrm{red}}$. Formally, we also set $\mathrm{eu}\,\mathbb{H}^0 := -\min(w) + \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^0_{\mathrm{red}}$.

Given a negative definite plumbing graph G of a $\mathbb{Q}HS^3$ 3-manifold M and a representative $l' \in L'$ of an element $[l'] = h \in H$, one works with the lattice $L = L_G = \mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}(G)}$ and weight function $L \ni l \mapsto -\frac{1}{2}(l, l + k_G + 2l')$. The cohomology theory corresponding to this weight function is denoted by $\mathbb{H}^*(G; k_G + 2l')$. If for $h \in H$ we choose the minimal representative $r_h \in L'$, then the cohomology theory $\mathbb{H}^*(G; k_G + 2r_h)$ is in fact an invariant of the pair (M, h) (i.e. it does not depend on the plumbing representation) and thus can be denoted by $\mathbb{H}^*_h(M)$. It is proved in [22] that for any $l' \in L'$ and $h = [l'] \in H$

$$\mathfrak{s}_{l'}(G) = \mathrm{eu}\,\mathbb{H}^*(G; k_G + 2l') \text{ and } \mathfrak{s}_h(M) = \mathrm{eu}\,\mathbb{H}^*_h(M). \tag{23}$$

4.3 Lattice Cohomology of Integral Surgeries

The lattice cohomology of integral surgeries ($q = 1$) was treated in [3, 19, 20, 27]. We use the notation $p/q = d \in \mathbb{Z}$. Clearly $u = u'$.

Let $\Delta_j(t)$ be the Alexander polynomial of the algebraic knot K_j normalized such that $\Delta_j(t) \in \mathbb{Z}[t]$ and $\Delta_j(1) = 1$. Let δ_j be the Seifert genus of $K_j \subset S^3$ (or the delta invariant of the corresponding plane curve singularity), and write $\delta := \sum_{j=1}^{\nu}\delta_j$. Set also $\Delta(t) = \prod_{j=1}^{\nu}\Delta_j(t)$ and write it in the form

$$\Delta(t) = 1 + \delta(t-1) + (t-1)^2 Q(t) \text{ with } Q(t) = \sum_{i=0}^{2\delta-2} q_i t^i.$$

Note that $Q(1) = \Delta''(1)/2$.

Example 16 The case of integral surgeries is especially important in singularity theory, since the links of *superisolated singularities* (see [14, 15]) are of this type. They appear as follows. Let $f \in \mathbb{C}[x, y, z]$ be an irreducible homogeneous polynomial of degree d such that its zero set in $\mathbb{CP}^2$ is a rational cuspidal curve; i.e. $C = \{f = 0\}$ is homeomorphic to S^2 and all the singularities of C are locally irreducible. Let their number be ν. Assume that there is no singular point on the projective line given by $z = 0$. Then the equation $f(x, y, z) + z^{d+1} = 0$ in $(\mathbb{C}^3, 0)$ determines an isolated complex surface singularity with link homeomorphic to $S^3_{-d}(K)$, where K is the connected sum of algebraic knots given by the local topological types of the singularities on C. In this case, by genus formula, $(d-1)(d-2)=2\delta$, a relation which connects K with d.

However, we can take the surgery (and plumbed) manifold $M=S^3_{-d}(K)$ for any K and with arbitrary $d>0$, even without the 'analytic compatibility' $(d-1)(d-2)=2\delta$ (imposed in the case of superisolated singularities).

4.4 Invariants of Integral Surgeries

Next, we recall some results on lattice cohomology of $M=S^3_{-d}(K)$, which will be combined with the above proved CAP. They will also be very useful in fast computations of examples in the next section. We emphasize that in the next general discussion, the identity $(d-1)(d-2)=2\delta$ will not be assumed.

Write $s_h:=hE^*_u$, then $r_h=\{hE^*_u\}=\{s_h\}$, and set also $c_h:=\chi(r_h)-\chi(s_h)$.

From [27, Theorem 7.1.1] (see also [3, Theorem 3.1.3]), we know that

$$\mathfrak{s}_{s_h}(G)=\mathrm{eu}\,\mathbb{H}^*\left(S^3_{-d}(K);k_G+2s_h\right)=\sum_{\substack{n\equiv h(\mathrm{mod}\,d)\\ 0\leq n\leq 2\delta-2}} q_n.$$

By the surgery formula [5, Theorem 1.0.1], one has

$$\mathfrak{s}_{s_h}(G)=\mathcal{H}^{\mathrm{pol}}_{u,h}(1)+\sum_{j=1}^{\nu}\mathfrak{s}_{R_j(s_h)}(G_j).$$

Since $R_j(s_h)=0$ and $\mathfrak{s}_0(G_j)=0$ (cf. Proposition 5), we get $\mathfrak{s}_{s_h}(G)=\mathcal{H}^{\mathrm{pol}}_{u,h}(1)$; hence

$$\mathcal{H}^{\mathrm{pol}}_{u}(1)=\sum_{h=0}^{p-1}\mathfrak{s}_{s_h}(G)=\sum_{n=0}^{2\delta-2}q_n=Q(1).$$

This is related with the invariants $\mathfrak{s}_h(G)$ as follows. From (23) and (4)

$$\sum_{h=0}^{d-1}\mathfrak{s}_h(G)=\sum_{h=0}^{d-1}\mathfrak{s}_{s_h}(G)+\sum_{h=0}^{d-1}(\chi(r_h)-\chi(s_h))=Q(1)+\sum_{h=0}^{d-1}c_h.$$

Furthermore, the identity $p_g(\{g_j=0\})=\sum_h\chi_j(-\lfloor h\cdot(f_j)/d\rfloor)$ from Claim 14 has also the following addendum:

$$\sum_{j=1}^{\nu}\chi_j(-\lfloor h\cdot(f_j)/d\rfloor)=\chi(r_h)-\chi(s_h)=c_h.$$

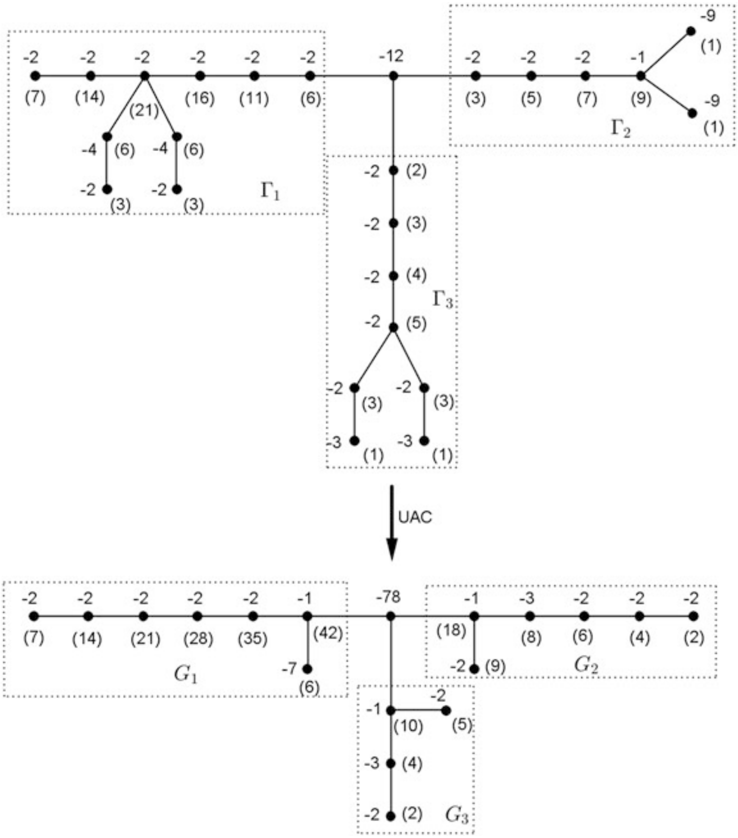
Indeed, by (20),

$$s_h = hE_u^* = hE_u/d + \sum_j h \cdot (f_j)/d;$$

hence $r_h = hE_u/d + \sum_j \{h(f_j)/d\}$, and $s_h - r_h = \sum_j \lfloor h(f_j)/d \rfloor$. This shows that $(s_h, s_h - r_h) = (hE_u^*, s_h - r_h) = 0$. Therefore $\sum_j \chi_j(-\lfloor h \cdot (f_j)/d \rfloor) = \chi(r_h - s_h) = \chi(r_h) - \chi(s_h)$.

5 Examples and Applications

Example 17 Consider the plumbing graph of a superisolated singularity corresponding to a curve of degree $d = 8$ with three singular points whose knots K_1, K_2, K_3 are the torus knots of type $(6, 7)$, $(2, 9)$, $(2, 5)$, respectively. Such a curve exists; see [10, Theorem 3.5].



One computes that

$$\mathcal{H}_u(t) = (1-t)\cdot\frac{1-t^{42}}{(1-t^6)(1-t^7)}\cdot\frac{1-t^{10}}{(1-t^2)(1-t^5)}\cdot\frac{1-t^{18}}{(1-t^2)(1-t^9)} = \frac{\Delta_1(t)\Delta_2(t)\Delta_3(t)}{(1-t)^2}$$

and $\mathcal{H}_u^{\mathrm{pol}}(1) = 293$. Correspondingly, $\sum_{h=0}^{7}\mathfrak{s}_{s_h}(G) = Q(1) = 293$. One also computes that $\sum_{h=0}^{7} c_h = 34$. Therefore, $\sum_{h=0}^{7}\mathfrak{s}_h(G) = 293 + 34 = 327$.

After computing the graph Γ of the UAC, we have $J = \mathbb{Z}_7 \times \mathbb{Z}_9 \times \mathbb{Z}_5$ (setting $s = t^{7\cdot 9\cdot 5}$ and after summation Σ_* over $\zeta_1 \in \mathbb{Z}_7$, $\zeta_2 \in \mathbb{Z}_9$, $\zeta_3 \in \mathbb{Z}_5$, where $\mathbb{Z}_l = \{e^{\frac{2\pi i m}{l}}\}_{m\in\mathbb{Z}}$ are cyclic groups), and the rational function $\mathcal{F}_{w,0}(t)$ equals

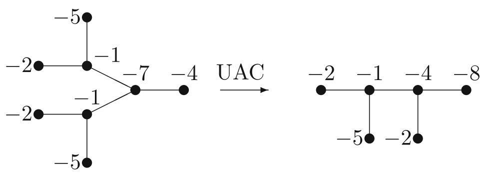
$$\frac{1-s}{7\cdot 9\cdot 5}\cdot\sum_{*}\frac{(1-s^{21})^2}{(1-s^7)(1-\zeta_1^2 s^3)(1-\zeta_1^{-2}s^3)}\,\frac{(1-s^9)}{(1-\zeta_2^5 s)(1-\zeta_2^{-5}s)}\,\frac{(1-s^5)}{(1-\zeta_3 s)(1-\zeta_3^{-1}s)} =$$

$$= \frac{\Delta_{1,\Gamma}(s)\Delta_{2,\Gamma}(s)\Delta_{3,\Gamma}(s)}{(1-s)^2}.$$

Then $\mathcal{F}_{w,0}(t) = \mathcal{H}_u(s)$ holds indeed with $s = t^{7\cdot 9\cdot 5}$. Correspondingly,

$$\mathfrak{s}_0(\Gamma) = \mathcal{F}_{w,0}^{\mathrm{pol}}(1) + \sum_{j=1}^{3} p_g(\{g_j = 0\}) = 293 + 34 = 327.$$

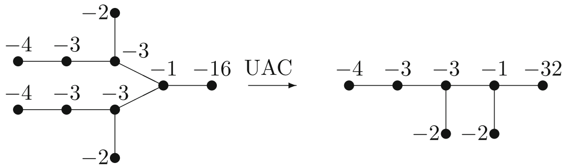
Example 18 We wish to emphasize that the covering additivity property of $\mathfrak{s}$ is not true in general, not even when restricting ourselves to integral surgeries along algebraic knots in integral homology spheres (instead of S^3). This is shown by the next example (motivated by [26, Remark 6.8.(2)]; the arrowhead of that graph is replaced by the -8 vertex below).



If we replace the (-8)-vertex of G by an arrowhead (representing a knot K), we get an integral homology sphere $\mathfrak{S}^3$; the corresponding knot has $m_{u_1} = 6$; hence $M(G) = \mathfrak{S}^3_{-2}(K)$, and G has determinant 2. One computes that $\mathfrak{s}_0(G) + \mathfrak{s}_1(G) = 15 + 14 = 29$, while $\mathfrak{s}_0(\Gamma) = 21$. In fact, when trying to copy the proof of Theorem 10, one finds that neither the polynomial identity 11 holds (this is why the example was present in [26, Remark 6.8.(2)]) nor is $\Gamma\backslash w$ of suspension type (satisfying the SWIC).

Remark 19 On the other hand, there are facts suggesting that the CAP of $\mathfrak{s}$ can hold in more general settings. Indeed, as we indicated in Claim 3, if for a given M one can find a surface singularity with link M such that the EqSWIC holds for the singularity $(X, 0)$ and the SWIC holds for its UAC, then the additivity of $\mathfrak{s}$ holds automatically. (Eq)SWIC was verified for many analytic structures, whose links are not of surgery type. On the other hand, the family of superisolated singularities is the main source of counterexamples for SWIC (and this was one of the motivations to test CAP for them).

Independently of any analytic argument, one can also find purely topological examples for which CAP still works (and in which cases not only that we cannot verify the presence of EqSWIC/SWIC, but we cannot even identify any specific analytic structure on the topological type or on certain special subgraphs). Here is one (for which the assumptions of Theorem 10 do not hold either).



One verifies that $\det(G) = 2$, and $\mathfrak{s}_0(G) + \mathfrak{s}_1(G) = 147 + 132 = 279 = \mathfrak{s}_0(\Gamma)$. This raises the interesting question that what are the precise limits of the CAP.

Remark 20 The lattice cohomology plays an intermediate role connecting the analytic invariants of a normal surface singularity X with the topology of its link $M = M(G)$. For example, one proves using [21, Proposition 6.2.2, Example 6.2.3, Theorem 7.1.3, 7.2.4] that for any h one has

$$p_g(X)_h \leq \mathrm{eu}\, \mathbb{H}^0(M(G); k_G + 2r_h).$$

Furthermore, for surgery manifolds $M(G) = S^3_{-d}(K)$, one has the vanishing $\mathbb{H}^q(M(G), k_G + 2r_h) = 0$ for $q \geq \nu$ [27]. In particular, for superisolated singularities corresponding to unicuspidal rational plane curves ($\nu = 1$), one has

$$p_g(X)_h \leq \mathrm{eu}\, \mathbb{H}^*(M(G); k_G + 2r_h) = \mathfrak{s}_h(M). \tag{24}$$

Therefore, for $M = S^3_{-d}(K)$ with $\nu = 1$, if the SWIC holds for the UAC $(Y, 0)$, that is, if $p_g(Y) = \mathfrak{s}_0(\Sigma)$, then this identity, the CAP and (24), implies $p_g(X)_h = \mathfrak{s}_h(M)$ for any h, that is, the EqSWIC for $(X, 0)$.

This is important for the following reason: for superisolated singularities, we do not know (even at conjectural level) any candidate (either topological or analytic!) for their equivariant geometric genera. It is not hard to verify that $p_g(X) = d(d-1)(d-2)/6$ (see, e.g. [15, Sect. 2.3]), but no formulae exist for $p_g(X)_h$, and no (topological or analytic) prediction exists for $p_g(Y)$ either. For further results on the geometric genus and the lattice cohomologies of superisolated singularities, see [8, 9, 28].

Example 21 Set $\nu = 1$, $d = 4$, and let K_1 be the $(3, 4)$ torus knot. This can be realized by the superisolated singularity $zx^3 + y^4 + z^5 = 0$. In this case, $M = S^3_{-4}(K_1)$.

One verifies that $\sum_{h=0}^{3} \mathfrak{s}_h(M) = 9 = \mathfrak{s}_0(\Sigma)$ correspondingly to Theorem 10.

On the other hand, the UAC $(Y, 0)$ of the *singularity* is the Brieskorn singularity $x^3 + y^4 + z^{16} = 0$, whose geometric genus is $p_g(Y) = 9$, too. Hence, by the above remark, $p_g(X)_h = \mathfrak{s}_h(M)$ for any h.

Remark 22 (Continuation of 20) It is interesting that we have two sets of invariants, an analytic package $(\{p_g(X)_h\}_h, p_g(Y))$ and a topological one $(\{\mathfrak{s}_h(M)\}_h, \mathfrak{s}_0(\Sigma))$, and both of them satisfy the additivity property. Nevertheless, in some cases, they do not agree. For example, if $p_g(Y) < \mathfrak{s}_0(\Sigma)$, then by (24) necessarily at least one of the inequalities in (24) is strict. In particular, for both topological and analytical package, the additivity property is stable, it is never damaged, but the equality of the two packages in certain cases fails.

Example 23 Set again $\nu = 1$ and $d = 4$, but this time let K_1 be the $(2, 7)$ torus knot. As usual $M = S^3_{-4}(K_1)$. By a computation $\sum_{h=0}^{3} \mathfrak{s}_h(M) = 10 = \mathfrak{s}_0(\Sigma)$.

A suitable superisolated singularity is given by $(zy - x^2)^2 - xy^3 + z^5 = 0$. By [15] (the end of Sect. 4.5.), the universal abelian cover Y satisfies the strict inequality $p_g(Y) < 10$. Therefore, $p_g(X)_h < \mathfrak{s}_h(M)$ for at least one h (in fact, not for h=0, see the main result of [4] and [28]).

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A Method to Compute the General Neron Desingularization in the Frame of One-Dimensional Local Domains

Adrian Popescu and Dorin Popescu

Abstract An algorithmic proof of General Neron Desingularization is given here for one-dimensional local domains, and it is implemented in SINGULAR. Also a theorem recalling Greenberg' strong approximation theorem is presented for one-dimensional Cohen–Macaulay local rings.

Keywords Regular morphisms • Smooth morphisms • Smoothing ring morphisms

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1 Introduction

A ring morphism $u : A \to A'$ has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e., its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$ is regular. If for all $P \in \operatorname{Spec} A$ the fraction field of A/P has characteristic 0, then the regular fibers of u are geometrically regular fibers. A flat morphism u of Noetherian rings is *regular* if its fibers are geometrically regular. If u is regular of finite type, then u is called *smooth*. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*.

A. Popescu
Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
e-mail: popescu@mathematik.uni-kl.de

D. Popescu (✉)
Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 5, University of Bucharest, P.O. Box 1-764, Bucharest 014700, Romania
e-mail: dorin.popescu@imar.ro

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In Artin approximation theory [2], an important result is the following theorem generalizing the Neron Desingularization [2, 7].

Theorem 1 (General Neron Desingularization, Popescu [8–10], André [1], Swan [13], Spivakovski [12]) *Let $u : A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is, v is a composite A-morphism $B \to C \to A'$.*

The purpose of this paper is to give an algorithmic proof of the above theorem when A, A' are one-dimensional local domains and $A \supset \mathbb{Q}$. This proof is somehow presented by the second author in a lecture given within the special semester on Artin Approximation of the Chaire Jean Morlet at CIRM, Luminy, Spring 2015 (see http://hlombardi.free.fr/Popescu-Luminy2015.pdf). The algorithm was implemented by the authors in the Computer Algebra system SINGULAR [3] and will be as soon as possible found in a development version as the library `GND.lib` at

`https://github.com/Singular/Source.`

We may take the same General Neron Desingularization for $v, v' : B \to A'$ if they are closed enough as Examples 4 and 10 show. The last section computes the General Neron Desingularization in several examples. We should point that the General Neron Desingularization is not uniquely associated to B, and it is better to speak above about **a** General Neron Desingularization.

When A' is the completion of a Cohen–Macaulay local ring A of dimension one, we show that we may have a linear Artin function as it happens in the Greenberg's case (see [5]). More precisely, the Artin function is given by $c \to 2e + c$, where e depends from the polynomial system of equations defining B (see Theorem 20).

2 The Theorem

Let $u : A \to A'$ be a flat morphism of Noetherian local domains of dimension *one*. Suppose that $A \supset \mathbb{Q}$ and the maximal ideal $\mathfrak{m}$ of A generate the maximal ideal of A'. Then u is a regular morphism. Moreover, we suppose that there exist canonical inclusions $k = A/\mathfrak{m} \to A$, $k' = A'/\mathfrak{m}A' \to A'$ such that $u(k) \subset k'$.

Let $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$. If $f = (f_1, \dots, f_r)$, $r \leq n$ is a system of polynomials from I, then we can define the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial Y_j}\right)$. After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f \big((f) : I\big)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then B_P, $P \in \operatorname{Spec} B$ is essentially smooth over A if and only if $P \not\supset H_{B/A}$ by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the nonsmooth locus of B over A.

Definition 2 B is **the standard smooth** over A if there exists f in I as above such that $1 \in \big((f):I\big)\Delta_f B$.

The aim of this paper is to give an easy algorithmic proof of the following theorem.

Theorem 3 *Any A-morphism $v : B \to A'$ factors through a standard smooth A-algebra B'.*

If A is essentially of finite type over $\mathbb{Q}$, then the ideal $H_{B/A}$ can be computed in SINGULAR by following its definition, but it is easier to describe only the ideal $\sum_f \big((f):I\big)\Delta_f B$ defined above. This is the case considered in our algorithmic part, let us say $A \cong (k[x]/F)_{(x)}$ for some variables $x = (x_1, \dots x_m)$, and the completion of A' is $k'[\![x]\!]/(f)$. When v is defined by polynomials y from $k'[x]$, then our problem is easy. Let L be the field obtained by adjoining to k all coefficients of y. Then $R = (L[x]/(f))_{(x)}$ is a subring of A' containing $\operatorname{Im} v$ which is essentially smooth over A. Then we may take B' as a standard smooth A-algebra such that R is a localization of B'. Consequently, we suppose usually that $y \notin k'[x]$.

3 Reduction to the Case When $H_{B/A} \cap A \neq 0$

We may suppose that $v(H_{B/A}) \neq 0$. Indeed, if $v(H_{B/A}) = 0$, then v induces an A-morphism $v' : B' = B/H_{B/A} \to A'$, and we may replace (B, v) by (B', v'). Applying this trick several times, we reduce to the case $v(H_{B/A}) \neq 0$. However, the fraction field of $\operatorname{Im} v$ is essentially smooth over A by separability, that is, $H_{\operatorname{Im} v/A}A' \neq 0$, and in the worst case, our trick will change B by $\operatorname{Im} v$ after several steps.

Choose $P' \in \big(\Delta_f((f):I)\big) \setminus I$ for some system of polynomials $f = (f_1, \dots, f_r)$ from I and $d' \in \big(v(P')A'\big) \cap A$, $d' \neq 0$. Moreover, we may choose P' to be from $M\big((f):I\big)$ where M is a $r \times r$-minor of $\left(\frac{\partial f}{\partial Y}\right)$. Then $d' = v(P')z \in \big(v(H_{B/A})\big) \cap A$ for some $z \in A'$. Set $B_1 = B[Z]/(f_{r+1})$, where $f_{r+1} = -d' + P'Z$, and let $v_1 : B_1 \to A'$ be the map of B-algebras given by $Z \to z$. It follows that $d' \in \big((f, f_{r+1}) : (I, f_{r+1})\big)$ and $d' \in \Delta_f$, $d' \in \Delta_{f_{r+1}}$. Then $d = d'^2 \equiv P$ modulo (I, f_{r+1}) for $P = P'^2Z^2 \in H_{B_1/A}$. For the reduction replace B by B_1 and the Jacobian matrix $J = (\partial f/\partial Y)$ will be now the new J given by $\begin{pmatrix} J & 0 \\ * & P' \end{pmatrix}$. Note that now $d \in H_{B/A} \cap A$.

Example 4 Let $a_1, a_2 \in \mathbb{C}$ be two elements algebraically independent over $\mathbb{Q}$ and ρ a root of the polynomial $T^2 + T + 1$ in $\mathbb{C}$. Then $k' = \dfrac{\mathbb{Q}(a_1, a_2)[a_3]}{(a_3^2 + a_3 + 1)} \cong \mathbb{Q}(\rho, a_1, a_2)$. Let $A = \left(\dfrac{\mathbb{Q}[x_1, x_2]}{(x_1^3 - x_2^2)}\right)_{(x_1, x_2)}$ and $B = \dfrac{A[Y_1, Y_2, Y_3]}{(Y_1^3 - Y_2^3)}$, $A' = \dfrac{k'[\![x_1, x_2]\!]}{(x_1^3 - x_2^2)}$ and the map v

defined as

$$\begin{array}{llll} v: & B & \longrightarrow & A' \\ & Y_1 & \longmapsto & a_1x_2 \\ & Y_2 & \longmapsto & a_1a_3x_2 \\ & Y_3 & \longmapsto & a_1\sum_{i=0}^{30}\frac{x_1^i}{i!} + a_2x_2\sum_{i=31}^{50}\frac{x_1^i}{i!} \end{array}$$

This is an easy example. Indeed, let $v'' : B'' = A[a_3, a_1x_2, v(Y_3)] \to A'$ be the inclusion. We have $\operatorname{Im} v \subset B'' \cong \frac{A[T, Y_1, Y_3]}{(T^2+T+1)}$, and $B''_{2a_3+1} \cong \left(\frac{A[T, Y_1, Y_3]}{(T^2+T+1)}\right)_{2T+1}$ is a smooth A-algebra, which could be taken as a General Neron Desingularization of B. Applying our algorithm we will get more complicated General Neron Desingularizations but useful for an illustration of our construction.

Then $\operatorname{Im} v$, the new B will be $\frac{B}{\operatorname{Ker} v}$, where the kernel is generated by the following polynomial:

```
ker[1]=Y1^2+Y1*Y2+Y2^2
```

Next we choose $f = Y_1^2+Y_1Y_2+Y_2^2$ and we have $M = 2Y_2+Y_1$ and $1 \in \big((f) : I\big)$ and hence $P' = Y_1 + 2Y_2$. Therefore $v(P') = (2a_1a_3 + a_1) \cdot x_2$ and $d' = x_2$, $z = \frac{1}{2a_1a_3 + a_1}$. Therefore $d = d'^2 = x_2^2$.

To be able to construct $\mathbb{Q}\left[\frac{1}{2a_1a_3+a_1}\right][x]$ in SINGULAR, we will add a new variable a, and we will factorize with the corresponding polynomial $2a_1a_3 \cdot a + a_1 \cdot a - 1$. We will see this a as a new parameter from $k' \subset A'$. Then we replace B by $B_1 = \frac{B[Y_4]}{(-d' + P'Y_4)}$ and extend v to a map $v_1 : B_1 \to A'$ given by $Y_4 \to a$. Replacing B by B_1, we may assume that $d \in H_{B/A} \cap A$.

Example 5 Note that we could use B instead $\operatorname{Im} v$. In this case we choose $f = Y_1^3 - Y_2^3$ and take $M = 3Y_2^2$ and $1 \in \big((f) : I\big)$. Therefore we obtain $P' = 3Y_2^2$, $d' = x_2^2$, and $d = x_2^4$, and the next computations are harder as we will see in Examples 17 and 25.

Remark 6 We would like to work above with $A'' = \frac{\mathbb{C}[[x_1, x_2]]}{(x_1^3 - x_2^2)}$ instead A', v being given by $v(Y_2) = a_1\rho x_2$. But this is hard since we cannot work in SINGULAR with an infinite set of parameters. We have two choices. If the definition of v involves only a finite set of parameters, then we proceed as in Example 4 using some $A' \supset \operatorname{Im} v$. Otherwise, we will see later that in the computation of the General Neron

Desingularization, we may use only a finite number of the coefficients of the formal power series defining $y = v(Y)$, and so this computation works in SINGULAR.

Remark 7 As we may see, our algorithm could go also when A' is not a domain, but there exist $P \in M((f) : I)$ as above and a regular element $d \in \mathfrak{m}$ with $d \equiv P$ modulo I. If A is Cohen–Macaulay, we may reduce to the case when there exists a regular element $d \in H_{B/A} \cap A$. However, it is hard usually to reduce to the case when $d \equiv P$ modulo I for some $P \in M((f) : I)$. Sometimes this is possible as shown in the following example.

Example 8 Let $a_1, a_2 \in \mathbb{C}$ be two elements algebraically independent over $\mathbb{Q}$. Consider $A = \left(\frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_2^3 - x_3^2, x_1^3 - x_3^2)}\right)_{(x_1,x_2,x_3)}$ and $B = \frac{A[Y_1, Y_2, Y_3]}{(Y_1^3 - Y_2^3)}$, $K' = \frac{\mathbb{Q}(a_1, a_2)[a_3]}{(a_3^2 - a_1a_2)}$, $A' = \frac{K'[\![x_1, x_2, x_3]\!]}{(x_2^3 - x_3^2, x_1^3 - x_3^2)}$ and the map v defined as

$$v: \quad B \longrightarrow A'$$
$$Y_1 \longmapsto a_3x_1$$
$$Y_2 \longmapsto a_3x_2$$
$$Y_3 \longmapsto a_1 \sum_{i=0}^{30} \frac{x_3^i}{i!} + a_2 \sum_{i=31}^{50} \frac{x_3^i}{i!}$$

Then Im v, the new B, will be $\frac{B}{\text{Ker } v}$, where the kernel is generated by six polynomials:

```
ker[1]=x2*Y1-x1*Y2
ker[2]=Y1^3-Y2^3
ker[3]=x1*Y1^2-x2*Y2^2
ker[4]=x1^2*Y1-x2^2*Y2
ker[5]=x1*x2^2*Y2-x3^2*Y1
ker[6]=x1^2*x2*Y2^2-x3^2*Y1^2
```

Next we choose $f = x_2Y_1 - x_1Y_2$ and we have $M = -x_1$. We may take $N = -x_3^2 \in \big((f) : I\big)$ and $P' = x_1x_3^2$. Note that $x_1 - x_2$ is a zero divisor in A, but $d' = P'$ is regular in A. In this example we may take $d = d' = P' = P$.

Remark 9 Replacing B by Im v can be a hard goal if let us say A' is a factor of the power series ring over $\mathbb{C}$ in some variables x and y is defined by formal power series whose coefficients form an infinite field extension L of $\mathbb{Q}$. If y are polynomials in x as in Examples 4 and 8, then it is trivial to find a General Neron Desingularization of B as we explained already in the last sentences of Sect. 1. For instance, in Example 8, B' could be a localization of $K' \otimes_{\mathbb{Q}} A$. Thus Examples 4 and 8 have no real importance, they being useful only for an illustration of our algorithm.

This is the reason that in the next examples, the field L obtained by adjoining to k of all coefficients of y will be an infinite-type field extension of k and $y \notin k'[x]$.

However, this will complicate the algorithm because we are not able to tell to the computer who is y and so how to get d'. We may choose an element $a \in \mathfrak{m}$ and find a minimal $c \in \mathbb{N}$ such that $a^c \in (v(M)) + (a^{2c})$ (this is possible because $\dim A = 1$). Set $d' = a^c$. It follows that $d' \in (v(M)) + (d'^2) \subset (v(M)) + (d'^4) \subset \ldots$ and so $d' \in (v(M))$, that is, $d' = v(M)z$ for some $z \in A'$. Certainly, we cannot find precisely z, but later it is enough to know just a kind of truncation of it modulo d'^6.

Example 10 Let $a_i \in \mathbb{C}$, $i \in \mathbb{N}$, $i \neq 3$ be elements algebraically independent over $\mathbb{Q}$ and a_3 a root of the polynomial $T^2 + T + 1$ in $\mathbb{C}$. Let $A = \left(\frac{\mathbb{Q}[x_1, x_2]}{(x_1^3 - x_2^2)}\right)_{(x_1, x_2)}$ and $B = \frac{A[Y_1, Y_2, Y_3]}{(Y_1^2 + Y_1Y_2 + Y_2^2)}$, $A' = \frac{\mathbb{C}[[x_1, x_2]]}{(x_1^3 - x_2^2)}$ and the map v defined as

$$\begin{aligned} v : \quad & B \longrightarrow A' \\ & Y_1 \longmapsto a_1\left(x_2 + \sum_{i\geq 7} a_i x_2^i\right) \\ & Y_2 \longmapsto a_1a_3\left(x_2 + \sum_{i\geq 7} a_i x_2^i\right) \\ & Y_3 \longmapsto a_1 \sum_{i=0}^{9} \frac{x_1^i}{i!} + x_2 \sum_{i=10}^{\infty} a_{i-8}\frac{x_1^i}{i!} \end{aligned}$$

As in Example 4, we may take $d' = x_2$, $d = d'^2$ and a. Our algorithm goes exactly as in Examples 4, 16, and 24 providing the same General Neron Desingularization. This time we cannot find an easy General Neron Desingularization as in the first part of Example 4.

Example 11 Let $A = \frac{\mathbb{Q}[x_1, x_2]_{(x_1,x_2)}}{(x_1^2 - x_2^3)}$, $A' = \frac{\mathbb{C}[[x_1, x_2]]}{(x_1^2 - x_2^3)}$. Then the inclusion $A \subset A'$ is regular. Let $\theta_i = \sum_{j=0}^{\infty} \alpha_{ij}x_2^j + x_1 \sum_{j=0}^{\infty} \beta_{ij}x_2^j \in \mathbb{C}[[x_1, x_2]]$ for $i = 3, 4$ with $\alpha_{i0} = 1$ and $y_1 = \frac{\theta_3^3}{\theta_4^2}$, $y_2 = \frac{\theta_4^2}{\theta_3}$, $y_3 = x_2\theta_3$, $y_4 = x_2\theta_4$. Let $f_1 = Y_3^2 - x_2^2Y_1Y_2$, $f_2 = Y_4^2 - x_2Y_2Y_3$ be polynomials in $A[Y]$, $Y = (Y_1, \ldots, Y_4)$ and set $B = A[Y]/(f)$, $f = (f_1, f_2)$.

If R is a domain and $u \in R$ is such that $Y^2 - u \in R[Y]$ has no solutions in $Q(R)$, then it is easy to see that $R[Y]/(Y^2 - u)$ is a domain too. In our case we get that $R = A[Y_1, Y_2, Y_3]/(f_1)$ and $B = R[Y_4]/(f_2)$ are domains too. Then the map $v : B \to A'$ given by $Y \to y = (y_1, \ldots, y_4)$ is injective if we suppose that

θ_3, θ_4 are algebraically independent over A. This follows since B is a domain and $\dim B = \operatorname{tr}\deg_{Q(A)} Q(B) = \operatorname{tr}\deg_{Q(A)} Q(\operatorname{Im} v) = 2 = \dim \operatorname{Im} v$. Moreover we will assume that the fields $L_i = \mathbb{Q}\big((\alpha_{ij}, \beta_{ij})_j\big), i = 3, 4$ have infinite transcendental degree over $\mathbb{Q}$. The Jacobian matrix $\left(\frac{\partial f}{\partial Y}\right)$ has a 2×2-minor $M = \det\left(\frac{\partial f_i}{\partial Y_j}\right)_{\substack{1 \le i \le 2 \\ 3 \le j \le 4}} = 4Y_3Y_4 \notin (f)$. Note that $v(M) = x_2^2 y_5$, where $y_5 = 1/(4\theta_3\theta_4)$. Then we may take $B_1 = B[Y_5]/(f_3), f_3 = -x_2^2 + MY_5$ and v_1 given by $Y_5 \to y_5$. Clearly, $P = M^2Y_5^2 \in H_{B_1/A}$ and $0 \neq d = x_2^4 = v_1(P) \in A$.

4 Proof of the Case When $H_{B/A} \cap A \neq 0$

Thus we may suppose that there exists $f = (f_1, \ldots, f_r)$, $r \le n$ a system of polynomials from I, a $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$, and $N \in ((f) : I)$ such that $0 \neq d \equiv MN$ modulo I. We may assume that $M = \det((\partial f_i/\partial Y_j)_{i,j \in [r]})$. Set $\bar{A} = A/(d^3)$, $\bar{A}' = A'/d^3A'$, $\bar{u} = \bar{A} \otimes_A u$, $\bar{B} = B/d^3B$, and $\bar{v} = \bar{A} \otimes_A v$. Clearly, $\bar{u}$ is a regular morphism of Artinian local rings.

Remark 12 The whole proof could work with $\bar{A} = A/d^2u$ for any $u \in \mathfrak{m}$. We prefer to take $u = d$ as is done in [9] and [11], but we could choose $u \neq d$, $u \in \mathfrak{m} \setminus \mathfrak{m}^2$ for easy computations.

By [6, 19,7.1.5] for every field extension L/k, there exists a flat complete Noetherian local $\bar{A}$-algebra $\tilde{A}$, unique up to an isomorphism, such that $\mathfrak{m}\tilde{A}$ is the maximal ideal of $\tilde{A}$ and $\tilde{A}/\mathfrak{m}\tilde{A} \cong L$. It follows that $\tilde{A}$ is Artinian. On the other hand, we may consider the localization A_L of $L \otimes_k \bar{A}$ in $\mathfrak{m}(L \otimes_k \bar{A})$ which is Artinian and so complete. By uniqueness we see that $A_L \cong \tilde{A}$. Set $k' = A'/\mathfrak{m}A'$. It follows that $\bar{A}' \cong A_{k'}$. Note that A_L is essentially smooth over A by base change, and $\bar{A}'$ is a filtered union of sub-$\bar{A}$-algebras A_L with L/k finite type field subextensions of k'/k.

Choose L/k a finite-type field extension such that A_L contains the residue class $\bar{y} \in \bar{A}'^n$ induced by y. In fact $\bar{y}$ is a vector of polynomials in the generators of $\mathfrak{m}$ with the coefficients c_ν in k', and we may take $L = k((c_\nu)_\nu)$. Then $\bar{v}$ factors through A_L. Assume that $k[(c_\nu)_\nu] \cong k[(U_\nu)_\nu]/\bar{J}$ for some new variables U and a prime ideal $\bar{J} \subset k[U]$. We have $H_{L/k} \neq 0$ because L/k is separable. Then we may assume that there exist $\omega = (\omega_1, \ldots, \omega_p)$ in $\bar{J}$ such that $\rho = \det(\partial\omega_i/\partial U_\nu)_{i,\nu \in [p]} \neq 0$ and a nonzero polynomial $\tau \in ((\omega) : \bar{J}) \setminus \bar{J}$. Thus L is a fraction ring of the smooth k-algebra $(k[U]/(\omega))_{\rho\tau}$. Note that $\omega, \rho, and \tau$ can be considered in A because $k \subset A$ and $c_\nu \in A'$ because $k' \subset A'$.

Then $\bar{v}$ factors through a smooth $\bar{A}$-algebra $C \cong (\bar{A}[U]/(\omega))_{\rho\tau\gamma}$ for some polynomial γ which is not in $\mathfrak{m}(\bar{A}[U]/(\omega))_{\rho\tau}$.

Lemma 13 *There exists a smooth A-algebra D such that $\bar{v}$ factors through $\bar{D} = \bar{A} \otimes_A D$.*

Proof By our assumptions $u(k) \subset k'$. Set $D = (A[U]/(\omega))_{\rho\tau\gamma}$ and $w : D \to A'$ be the map given by $U_\nu \to c_\nu$. We have $C \cong \bar{A} \otimes_A D$. Certainly, $\bar{v}$ factors through $\bar{w} = \bar{A} \otimes_A w$, but in general v does not factor through w. □

Remark 14 If $A' = \hat{A}$, then $\bar{A} \cong \bar{A}'$ and we may take $D = A$.

Remark 15 Suppose that $k \subset A$ but $L \not\subset A'$ and so $k' \not\subset A'$. Then $D = (A[U,Z]/(\omega - d^3 Z))_{\rho\tau\gamma}$, and $Z = (Z_\nu)$ is a smooth A-algebra and $\bar{D} \cong C[Z]$. Since $\bar{v}$ factors through a map $C \to \bar{A}'$ given let us say by $U \to \lambda + d^3 A'$ for some λ in A', we see that $\omega(\lambda) \equiv 0$ modulo d^3, that is, $\omega(\lambda) = d^3 z$ for some z in A'. Let $w : D \to A'$ be the A-morphism given by $(U, Z) \to (\lambda, z)$. Certainly, $\bar{v}$ factors through $\bar{w} = \bar{A} \otimes_A w$, but in general v does not factor through w. If also $k \not\subset A$, then the construction of D goes as above but using a lifting of $\omega, \tau, and \gamma$ from $k[U]$ to $A[U]$. In both cases we may use D as it follows.

Example 16 We reconsider Example 4. We already know that $d = x_2^2$. The algorithm gives us the following output:

```
This is C:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 a x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=3*a1*a+2*a3+1
_[2]=a3^2+a3+1
_[3]=x1^3-x2^2
_[4]=x2^6
This is D:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 a x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=3*a1*a+2*a3+1
_[2]=a3^2+a3+1
_[3]=x1^3-x2^2
```

Indeed,

$$C = \frac{\bar{A}[a_1, a_3, a]}{\left(3a_1 a + 2a_3 + 1, a_3^2 + a_3 + 1, x_2^6\right)}$$

and

$$D = \frac{A[a_1, a_3, a]}{\left(3a_1 a + 2a_3 + 1, a_3^2 + a_3 + 1\right)}.$$

Note that the $3a_1a + 2a_3 + 1$ comes from the standard basis computation of the ideal $(2a_1a_3a + a_1a - 1, a_3^2 + a_3 + 1)$, and in D we have $a_3, a_1, 2a_3 + 1$ invertible.

Example 17 Now we reconsider Example 5. We know that $d = x_2^4$. The algorithm gives us the following output:

```
This is C:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 a x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=a3^2+a3+1
_[2]=x1^3-x2^2
_[3]=a1^2*a-a3
_[4]=x2^12

This is D:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 a x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=a3^2+a3+1
_[2]=x1^3-x2^2
_[3]=a1^2*a-a3
```

Indeed,

$$C = \frac{\bar{A}[a_1, a_3, a]}{(a_3^2 + a_3 + 1, a_1^2a - a_3, x_2^{12})}$$

and

$$D = \frac{A[a_1, a_3, a]}{(a_3^2 + a_3 + 1, a_1^2a - a_3)}.$$

Note that a_3 and a_1 are invertible in D.

Example 18 In the case of Example 8, we obtain the following output:

```
This is C:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 x1 x2 x3
//        block   2 : ordering C
// quotient ring from ideal
_[1]=x2^3-x3^2
_[2]=x1^3-x3^2
_[3]=x3^8
```

```
This is D:
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    a1 a3 x1 x2 x3
//        block   2 : ordering C
// quotient ring from ideal
_[1]=x2^3-x3^2
_[2]=x1^3-x3^2
```

Indeed this is the case since we have $d = x_1x_3^2$ and hence

$$C = \frac{\bar{A}[a_1, a_3]}{(x_3^8)}$$

and

$$D = A[a_1, a_3].$$

Example 19 In Example 11 we consider $a_1 anda_2$ algebraically independent over $\mathbb{Q}$ and set $\theta_3' = 1 + a_1x_2$ and $\theta_4' = 1 + a_2x_2^2$. Suppose that $\theta_i' \equiv \theta_i$ modulo x_2^{12}. We have $y_3 = x_2\theta_3$, $y_4 = x_2\theta_4$, $y_1 = \dfrac{\theta_3^3}{\theta_4^2}$, $y_2 = \dfrac{\theta_4^2}{\theta_3}$, and $y_5 = \dfrac{1}{(4\theta_3\theta_4)}$. Choose y_i', $i \in [5]$ polynomials with degrees ≤ 11 in x_2 and linear in x_1 such that $y_i' \equiv y_i \operatorname{mod}(x_1^2, x_2^{12})$. We get $y_1' \equiv y_1 = \theta_3^3/\theta_4^2 \equiv \theta'^3_3/\theta'^2_4 \operatorname{mod}(x_1^2, x_2^{12})$ and similarly for y_i', $i > 1$. Here we use the fact that $\theta_4^{-2} = \sum_{j=1}^{e}(1 - \theta_4^2)^j$ for some $e >> 0$ because $1 - \theta_4^2$ is nilpotent in the ring $\bar{A}[a_1, a_2, a_3, a_4]$. Thus the coefficients of y_i', $i \in [5]$ belong to the field L obtained by adjoining to $\mathbb{Q}$ the coefficients of θ_3', θ_4'. Note that in this case, $L = Q(\mathbb{Q}[a_1, \ldots, a_4])$. Then we obtain the following output:

```
This is C:
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    a1 a2 x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=x2^3-x1^2
_[2]=x1^8
This is D:
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    a1 a2 x1 x2
//        block   2 : ordering C
// quotient ring from ideal
_[1]=x2^3-x1^2
```

Thus $C = \frac{\bar{A}[a_1, \ldots, a_4]}{(x_2^{12})} \cong \bar{A}[a_1, \ldots, a_4]$ which is smooth over $\bar{A}$. Then D is a localization of $A[a_1, \ldots, a_4]$, where θ'_3, θ'_4 must be invertible.

Back to our proof, note that the composite map $\bar{B} \to C \to \bar{D}$ is given by $Y \to y' + d^3D$ for some $y' \in D^n$. Thus $I(y') \equiv 0$ modulo d^3D. Since $\bar{v}$ factors through $\bar{w}$, we see that $\bar{w}(y' + d^3D) = \bar{y}$. Set $\tilde{y} = w(y')$. We get $y - \tilde{y} \in d^3A'^n$, let us say $y - \tilde{y} = d^2\varepsilon$ for $\varepsilon \in dA'^n$.

We have $d \equiv P = NM$ modulo I and so $P(y') \equiv d$ modulo d^3 in D because $I(y') \equiv 0$ modulo d^3D. Thus $P(y') = ds$ for a certain $s \in D$ with $s \equiv 1$ modulo d. Let H be the $n \times n$-matrix obtained adding down to $(\partial f/\partial Y)$ as a border block $(0|\mathrm{Id}_{n-r})$. Let G' be the adjoint matrix of H and $G = NG'$. We have

$$GH = HG = NM\mathrm{Id}_n = P\mathrm{Id}_n$$

and so

$$ds\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y').$$

Then $t := H(y')\varepsilon \in dA'^n$ satisfies

$$G(y')t = P(y')\varepsilon = ds\varepsilon$$

and so

$$s(y - \tilde{y}) = dw(G(y'))t.$$

Let

$$h = s(Y - y') - dG(y')T, \tag{1}$$

where $T = (T_1, \ldots, T_n)$ are new variables. The kernel of the map $\varphi : D[Y, T] \to A'$ given by $Y \to y$, $T \to t$ contains h. Since

$$s(Y - y') \equiv dG(y')T \text{ modulo } h$$

and

$$f(Y) - f(y') \equiv \sum_j \frac{\partial f}{\partial Y_j}(y')(Y_j - y'_j)$$

modulo higher-order terms in $Y_j - y'_j$, by Taylor's formula, we see that for $p = \max_i \deg f_i$, we have

$$s^p f(Y) - s^p f(y') \equiv \sum_j s^{p-1} d \frac{\partial f}{\partial Y_j}(y') G_j(y') T_j + d^2 Q = s^{p-1} dP(y')T + d^2 Q \tag{2}$$

modulo h where $Q \in T^2 D[T]^r$. This is because $(\partial f/\partial Y)G = (P\mathrm{Id}_r|0)$. We have $f(y') = d^2 b$ for some $b \in dD^r$. Then

$$g_i = s^p b_i + s^p T_i + Q_i, \qquad i \in [r] \tag{3}$$

is in the kernel of φ because $d^2\varphi(g) = d^2 g(t) \in (h(y,t), f(y)) = (0)$. Set $E = D[Y,T]/(I,g,h)$, and let $\psi : E \to A'$ be the map induced by φ. Clearly, v factors through ψ because v is the composed map $B \to B \otimes_A D \cong D[Y]/I \to E \xrightarrow{\psi} A'$.

Note that the $r \times r$-minor s' of $(\partial g/\partial T)$ given by the first r-variables, T is from $s^{rp} + (T) \subset 1 + (d,T)D[Y,T]$ because $Q \in (T)^2$. Then $U = (D[Y,T]/(h,g))_{ss'}$ is smooth over D. We claim that $I \subset (h,g)D[Y,T]_{ss's''}$ for some other $s'' \in 1 + (d,T)D[Y,T]$. Indeed, we have $PI \subset (h,g)D[Y,T]_s$ and so $P(y' + s^{-1}dG(y')T)I \subset (h,g)D[Y,T]_s$. Since $P(y' + s^{-1}dG(y')T) \in P(y') + d(T)$, we get $P(y' + s^{-1}dG(y')T) = ds''$ for some $s'' \in 1 + (d,T)D[Y,T]$. It follows that $s''I \subset (h,g)D[Y,T]_{ss'}$ because d is regular in U, the map $D \to U$ being flat, and so $I \subset (h,g)D[Y,T]_{ss's''}$. Thus $E_{ss's''} \cong U_{s''}$ is a B-algebra which is also standard smooth over D and A.

As $w(s) \equiv 1$ modulo d and $w(s'), w(s'') \equiv 1$ modulo (d,t), $d, t \in \mathfrak{m}A'$, we see that $w(s), w(s'), w(s'')$ are invertible because A' is local and ψ (thus v) factors through the standard smooth A-algebra $E_{ss's''}$.

5 A Theorem of Greenberg's Type

Let (A, m) be a Cohen–Macaulay local ring (e.g., a reduced ring) of dimension one, $A' = \hat{A}$ the completion of A, $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$ an A-algebra of finite type, and $c, e \in \mathbb{N}$. Suppose that there exist $f = (f_1, \dots, f_r)$ in I, a $r \times r$-minor M of the Jacobian matrix $(\partial f/\partial Y)$, $N \in ((f) : I)$, and an A-morphism $v : B \to A/\mathfrak{m}^{2e+c}$ such that $(v(MN)) \supset \mathfrak{m}^e/\mathfrak{m}^{2e+c}$.

Theorem 20 *Then there exists an A-morphism $v' : B \to \hat{A}$ such that $v' \equiv v$ modulo $\mathfrak{m}^c$, that is, $v'(Y + I) \equiv v(Y + I)$ modulo $\mathfrak{m}^c$.*

Proof We note that the proof of Theorem 3 can work somehow in this case. Let $y' \in A^n$ be an element inducing $v(Y + I)$. Then $\mathfrak{m}^e \subset ((MN)(y')) + \mathfrak{m}^{2e+c} \subset ((MN)(y')) + \mathfrak{m}^{3e+2c} \subset \dots$ by hypothesis. It follows that $\mathfrak{m}^e \subset ((MN)(y'))$. Since A is Cohen–Macaulay, we see that $\mathfrak{m}^e$ contains a regular element of A and so $(MN)(y')$ must be regular too.

Set $d = (MN)(y')$. Next we follow the proof of Theorem 3 with $D = A$, $s = 1$, $P = MN$, and H, G such that

$$d\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y').$$

Let

$$h = Y - y' - dG(y')T,$$

where $T = (T_1, \ldots, T_n)$ are new variables. We have

$$f(Y) - f(y') \equiv dP(y')T + d^2Q$$

modulo h where $Q \in T^2A[T]^r$. But $f(y') \in \mathfrak{m}^{2e+c}A^r \subset d^2\mathfrak{m}^cA^r$ and we get $f(y') = d^2b$ for some $b \in \mathfrak{m}^cA^r$. Set $g_i = b_i + T_i + Q_i$, $i \in [r]$ and $E = A[Y,T]/(I,h,g)$. We have an A-morphism $\beta : E \to A/\mathfrak{m}^c$ given by $(Y,T) \to (y',0)$ because $I(y') \equiv 0$ modulo $\mathfrak{m}^{2e+c}$, $h(y',0) = 0$ and $g(0) = b \in \mathfrak{m}^cA^r$.

As in the proof of Theorem 3, we have $E_{s's''} \cong U_{s''}$, where $U = (A[Y,T]/(g,h))_{s'}$. This isomorphism follows because d is regular in A and so in U. Consequently, $E_{s's''}$ is smooth over A. Note that β extends to a map $\beta' : E_{s's''} \to A/\mathfrak{m}^c$. By the implicit function theorem, β' can be lifted to a map $w : E_{s's''} \to \hat{A}$ which coincides with β' modulo $\mathfrak{m}^c$. It follows that the composite map $v', B \to E_{s's''} \xrightarrow{w} \hat{A}$ works. □

Corollary 21 *In the assumptions of the above theorem, suppose that* $(A,\mathfrak{m})$ *is excellent Henselian, then there exists an A-morphism* $v'' : B \to A$ *such that* $v'' \equiv v$ *modulo* $\mathfrak{m}^c$*, that is,* $v''(Y+I) \equiv v(Y+I)$ *modulo* $\mathfrak{m}^c$.

Proof An excellent Henselian local ring $(A,\mathfrak{m})$ has the Artin approximation property by [9], that is, the solutions in A of a system of polynomial equations f over A are dense in the set of the solutions of f in $\hat{A}$. By Theorem 20 we get an A-morphism $v' : B \to \hat{A}$ such that $v' \equiv v$ modulo $\mathfrak{m}^c$. Then there exists an A-morphism $v'' : B \to A$ such that $v'' \equiv v' \equiv v$ modulo $\mathfrak{m}^c$ by the Artin approximation property. □

Theorem 22 *Let* $(A,\mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension one,* $B = A[Y]/I$*,* $Y = (Y_1, \ldots, Y_n)$ *an A-algebra of finite type,* $e \in \mathbb{N}$*, and* $f = (f_1, \ldots, f_r)$ *a system of polynomials from I. Suppose that A is excellent Henselian and there exist a* $r \times r$*-minor M of the Jacobian matrix* $(\partial f/\partial Y)$*,* $N \in ((f) : I)$ *and* $y' \in A^n$ *such that* $I(y') \equiv 0$ *modulo* $\mathfrak{m}^e$ *and* $((NM)(y')) \supset \mathfrak{m}^e$*. Then the following statements are equivalent:*

(1) *there exists* $y'' \in A^n$ *such that* $I(y'') \equiv 0$ *modulo* $\mathfrak{m}^{3e}$ *and* $y'' \equiv y'$ *modulo* $\mathfrak{m}^e$*,*
(2) *there exists* $y \in A^n$ *such that* $I(y) = 0$ *and* $y \equiv y'$ *modulo* $\mathfrak{m}^e$.

For the proof apply the above corollary and Theorem 20.

6 Computation of the General Neron Desingularization in Examples 4, 5, 8, and 11

Example 23 We would like to compute Example 4 in SINGULAR using `GND.lib`. We quickly recall the example.

Let $a_1, a_2 \in \mathbb{C}$ be two elements algebraically independent over $\mathbb{Q}$ and ρ a root of the polynomial $T^2 + T + 1$ in $\mathbb{C}$. Then $k' = \dfrac{\mathbb{Q}(a_1, a_2)[a_3]}{(a_3^2 + a_3 + 1)} \cong \mathbb{Q}(\rho, a_1, a_2)$. Let $A = \left(\dfrac{\mathbb{Q}[x_1, x_2]}{(x_1^3 - x_2^2)}\right)_{(x_1, x_2)}$ and $B = \dfrac{A[Y_1, Y_2, Y_3]}{(Y_1^3 - Y_2^3)}$, $A' = \dfrac{k'[[x_1, x_2]]}{(x_1^3 - x_2^2)}$ and the map v defined as

$$\begin{array}{rccl} v: & B & \longrightarrow & A' \\ & Y_1 & \longmapsto & a_1 x_2 \\ & Y_2 & \longmapsto & a_1 a_3 x_2 \\ & Y_3 & \longmapsto & a_1 \displaystyle\sum_{i=0}^{30} \frac{x_1^i}{i!} + a_2 x_2 \sum_{i=31}^{50} \frac{x_1^i}{i!} \end{array}$$

For this we do the following:

```
LIB "GND.lib";                                  //load the library
ring All = 0,(a1,a2,a3,x1,x2,Y1,Y2,Y3),dp;      //define the ring
int nra = 3;                                    //number of a's
int nrx = 2;                                    //number of x's
int nry = 3;                                    //number of Y's
ideal xid = x1^3-x2^2;                          //define the ideal from A
ideal yid = Y1^3-Y2^3;                          //define the ideal from B
ideal aid = a3^2+a3+1;                          //define the ideal from k'
poly y;
int i;
for(i=0;i<=30;i++)
{
  y = y + a1*x1^i/factorial(i);
}
for(i=31;i<=50;i++)
{
  y = y + a2*x2*x1^i/factorial(i);
}
ideal f = a1*x2,a1*a3*x2,y;                    //define the map v
desingularization(All, nra,nrx,nry,xid,yid,aid,f,"debug");
```

Example 24 We continue on the idea of Examples 4 and 16. The bordered matrix H defined above is equal to

$$H = \begin{pmatrix} 2Y_1 + Y_2 & Y_1 + 2Y_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ Y_4 & 2Y_4 & 0 & Y_1 + 2Y_2 \end{pmatrix}$$

and hence $G = N \cdot G'$ is equal to

$$G = Y_4^2 \cdot \begin{pmatrix} 0 & 0 & Y_1^2 + 4Y_1Y_2 + 4Y_2^2 & 0 \\ Y_1 + 2Y_2 & 0 & -2Y_1^2 - 5Y_1Y_2 - 2Y_2^2 & 0 \\ 0 & Y_1^2 + 4Y_1Y_2 + 4Y_2^2 & 0 & 0 \\ -2Y_4 & 0 & 3Y_1Y_4 & Y_1 + 2Y_2 \end{pmatrix}$$

and $s = 1$. This is obvious in our case since $y' = y$ and always $d = P(y)$ because $I(y) = 0$ and $d \equiv P$ modulo I. Using the definition of h in Eq. (1), we get that

$$\begin{aligned} h_1 &= Y_1 - \left(x_2^4\right) \cdot T_3 - (a_1x_2), \\ h_2 &= Y_2 - \frac{x_2^3}{2a_1a_3 + a_1} \cdot T_1 + \frac{a_3x_2^4 + 2x_2^4}{2a_3 + 1} \cdot T_3 - (a_1a_3x_2), \\ h_3 &= Y_3 - \left(x_2^4\right) \cdot T_2 - \left(\frac{1}{6!}a_1x_1^6 + \frac{1}{5!}a_1x_1^5 + \frac{1}{4!}a_1x_1^4 + \frac{1}{3!}a_1x_1^3 + \frac{1}{2}a_1x_1^2 + \right. \\ &\quad \left. a_1x_1 + a_1\right), \\ h_4 &= Y_4 + \frac{2x_2^2}{(2a_1a_3 + a_1)^3} \cdot T_1 - \frac{3x_2^3}{a_1^2\,(2a_3 + 1)^3} \cdot T_3 - \frac{x_2^3}{2a_1a_3 + a_1} \cdot T_4 + \frac{1}{2a_1a_3 + a_1}. \end{aligned}$$

From Eq. (2) we get that

$$\begin{aligned} Q_1 &= \frac{x_2^2}{(2a_1a_3 + a_1)^2} \cdot T_1^2 - \frac{3x_2^3}{a_1\,(2a_3 + 1)^2} \cdot T_1T_3 + \frac{3a_3^2x_2^4 + 3a_3x_2^4 + 3x_2^4}{(2a_3 + 1)^2} \cdot T_3^2, \\ Q_2 &= -\frac{4x_2}{(2a_1a_3 + a_1)^4} \cdot T_1^2 + \frac{12x_2^2}{a_1^3\,(2a_3 + 1)^4} \cdot T_1T_3 - \frac{9x_2^3}{a_1^2\,(2a_3 + 1)^4} \cdot T_3^2 + \\ &\quad \frac{2x_2^2}{(2a_1a_3 + a_1)^2} \cdot T_1T_4 - \frac{3x_2^3}{a_1\,(2a_3 + 1)^2} \cdot T_3T_4 \end{aligned}$$

and therefore following the definition of g in Eq. (3), we have

$$\begin{aligned} g_1 &= Q_1 + T_1 + (a_1^2a_3^2 + a_1^2a_3 + a_1^2), \\ g_2 &= Q_2 + T_2. \end{aligned}$$

We print now the algorithm's debug output using the line codes from Example 23.

```
This is the bordered matrix H:
2*Y1+Y2,Y1+2*Y2,0,0,
0,      0,      1,0,
1,      0,      0,0,
Z,      2*Z,    0,Y1+2*Y2
This is G:
0,                 0,      G[1,3],    0,
Y1*Y4^2+2*Y2*Y4^2,0,      G[2,3],    0,
0,                 G[3,2],0,          0,
-2*Y4^3,           0,      3*Y1*Y4^3,Y1*Y4^2+2*Y2*Y4^2

G[1,3]=Y1^2*Y4^2+4*Y1*Y2*Y4^2+4*Y2^2*Y4^2
```

```
G[2,3]=-2*Y1^2*Y4^2-5*Y1*Y2*Y4^2-2*Y2^2*Y4^2
G[3,2]=Y1^2*Y4^2+4*Y1*Y2*Y4^2+4*Y2^2*Y4^2

s = 1
h =
_[1]=Y1+(-x2^4)*T3+(-a1*x2)
_[2]=Y2+(-x2^3)/(2*a1*a3+a1)*T1+(a3*x2^4+2*x2^4)/(2*a3+1)*T3+
(-a1*a3*x2)
_[3]=Y3+(-x2^4)*T2+(-a1*x1^6-6*a1*x1^5-30*a1*x1^4-120*a1*x1^3
-360*a1*x1^2-720*a1*x1-720*a1)/720
_[4]=Y4+(2*x2^2)/(8*a1^3*a3^3+12*a1^3*a3^2+6*a1^3*a3+a1^3)*T1+
(-3*x2^3)/(8*a1^2*a3^3+12*a1^2*a3^2+6*a1^2*a3+a1^2)*T3+
(-x2^3)/(2*a1*a3+a1)*T4-1/(2*a1*a3+a1)

m = 2
QT =
_[1]=(x2^2)/(4*a1^2*a3^2+4*a1^2*a3+a1^2)*T1^2+
(-3*x2^3)/(4*a1*a3^2+4*a1*a3+a1)*T1*T3+
(3*a3^2*x2^4+3*a3*x2^4+3*x2^4)/(4*a3^2+4*a3+1)*T3^2

_[2]=(-4*x2)/(16*a1^4*a3^4+32*a1^4*a3^3+24*a1^4*a3^2+8*a1^4*a3+a1^4)
*T1^2+(12*x2^2)/(16*a1^3*a3^4+32*a1^3*a3^3+24*a1^3*a3^2+8*a1^3*a3
+a1^3)*T1*T3+(-9*x2^3)/(16*a1^2*a3^4+32*a1^2*a3^3+24*a1^2*a3^2
+8*a1^2*a3+a1^2)*T3^2+(2*x2^2)/(4*a1^2*a3^2+4*a1^2*a3+a1^2)*T1*T4
+(-3*x2^3)/(4*a1*a3^2+4*a1*a3+a1)*T3*T4

f =
f[1]=Y1^2+Y1*Y2+Y2^2
f[2]=Y1*Y4+2*Y2*Y4+(-x2^2)

g =
_[1]=(x2^2)/(4*a1^2*a3^2+4*a1^2*a3+a1^2)*T1^2+(-3*x2^3)/(4*a1*a3^2+
 4*a1*a3+a1)*T1*T3+(3*a3^2*x2^4+3*a3*x2^4+3*x2^4)/(4*a3^2+4*a3+1)*T3^2
+T1+(a1^2*a3^2+a1^2*a3+a1^2)
_[2]=(-4*x2)/(16*a1^4*a3^4+32*a1^4*a3^3+24*a1^4*a3^2+8*a1^4*a3+a1^4)
*T1^2+(12*x2^2)/(16*a1^3*a3^4+32*a1^3*a3^3+24*a1^3*a3^2+8*a1^3*
a3+a1^3)*T1*T3+(-9*x2^3)/(16*a1^2*a3^4+32*a1^2*a3^3+24*a1^2*a3^2
+8*a1^2*a3+a1^2)*T3^2+(2*x2^2)/(4*a1^2*a3^2+4*a1^2*a3+a1^2)*T1*T4
+(-3*x2^3)/(4*a1*a3^2+4*a1*a3+a1)*T3*T4+T2
```

Thus the General Neron Desingularization is a localization of $D[Y,T]/(h,g) \cong D[T]/(g)$.

Example 25 In the case of Examples 5 and 17, we obtain that the bordered matrix

$$H = \begin{pmatrix} 3Y_1^2 & -3Y_2^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -6Y_2Y_4 & 0 & -3Y_2^2 \end{pmatrix}$$

and hence $G = N \cdot G'$ is equal to

$$G = Y_4^2 \cdot \begin{pmatrix} 0 & 0 & 9Y_2^4 & 0 \\ -3Y_2^2 & 0 & -3Y_1^2Y_2^2 & 0 \\ 0 & 9Y_2^4 & 0 & 0 \\ 6Y_2Y_4 & 0 & -18Y_1^2Y_2Y_4 & -3Y_2^2 \end{pmatrix}$$

and $s = 1$. Using the definition of h in Eq. (1), we get that

$$\begin{aligned} h_1 &= Y_1 - \left(x_2^8\right) \cdot T_3 + (-a_1x_2)\,, \\ h_2 &= Y_2 + \frac{x_2^6}{3a_1^2a_3^2} \cdot T_1 - \frac{x_2^8}{a_3^2} \cdot T_3 - (a_1a_3x_2)\,, \\ h_3 &= Y_3 - \left(x_2^8\right) \cdot T_2 - \left(\frac{1}{12!}a_1x_1^{12} + \frac{1}{11!}a_1x_1^{11} + \frac{1}{10!}a_1x_1^{10} + \frac{1}{9!}a_1x_1^9 + \frac{1}{8!}a_1x_1^8\right. \\ &\quad \left. + \frac{1}{7!}a_1x_1^7 + \frac{1}{6!}a_1x_1^6 + \frac{1}{5!}a_1x_1^5 + \frac{1}{4!}a_1x_1^4 + \frac{1}{3!}a_1x_1^3 + \frac{1}{2}a_1x_1^2 + a_1x_1 + a_1\right), \\ h_4 &= Y_4 + \frac{2x_2^5}{9a_1^5a_3^5} \cdot T_1 - \frac{2x_2^7}{3a_1^3a_3^5} \cdot T_3 + \frac{x_2^6}{3a_1^2a_3^2} \cdot T_4 + \frac{1}{3a_1^2a_3^2}. \end{aligned}$$

From Eq. (2) we get that

$$\begin{aligned} Q_1 &= \frac{x_2^{10}}{27a_1^6a_3^6} \cdot T_1^3 - \frac{x_2^{12}}{3a_1^4a_3^6} \cdot T_1^2T_3 + \frac{x_2^{14}}{a_1^2a_3^6} \cdot T_1T_3^2 + \frac{a_3^6x_2^{16} - x_2^{16}}{a_3^6} \cdot T_3^3 - \\ &\quad \frac{x_2^5}{3a_1^3a_3^3} \cdot T_1^2 + \frac{2x_2^7}{a_1a_3^3} \cdot T_1T_3 + \frac{3a_1a_3^3x_2^9 - 3a_1x_2^9}{a_3^3} \cdot T_3^2, \\ Q_2 &= \frac{2ax_2^9}{27a_1^9a_3^9} \cdot T_1^3 - \frac{2x_2^{11}}{3a_175a_3^9} \cdot T_1^2T_3 + \frac{2x_2^{13}}{a_1^5a_3^9} \cdot T_1T_3^2 - \frac{2x_2^{15}}{a_1^3a_3^9} \cdot T_3^3 + \\ &\quad \frac{x_2^{10}}{9a_1^6a_3^6} \cdot T_1^2T_4 - \frac{2x_2^{12}}{3a_1^4a_3^6} \cdot T_1T_3T_4 + \frac{x_2^{14}}{a_1^2a_3^6} \cdot T_3^2T_4 - \frac{x_2^4}{3a_1^6a_3^6} \cdot T_1^2 + \\ &\quad \frac{2x_2^6}{a_1^4a_3^6} \cdot T_1T_3 - \frac{3x_2^8}{a_1^2a_3^6} \cdot T_3^2 - \frac{2x_2^5}{3a_1^3a_3^3} \cdot T_1T_4 + \frac{2x_2^7}{a_1a_3^3} \cdot T_3T_4 \end{aligned}$$

and therefore following the definition of g in Eq. (3), we have

$$\begin{aligned} g_1 &= Q_1 + T_1, \\ g_2 &= Q_2 + T_2 \end{aligned}$$

To obtain this with SINGULAR, we use the same code lines as in Example 23, but we change the last one with

```
desingularization(All, nra,nrx,nry,xid,yid,aid,f,"injective","debug");
```

Doing this, the algorithm will not compute the kernel because of the `injective` argument.

Example 26 We do now the same computations for Examples 8 and 18. The bordered matrix H defined above is equal to

$$H = \begin{pmatrix} x_2 & -x_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1x_3^2 \end{pmatrix}$$

and hence $G = N \cdot G'$ is equal to

$$G = Y_4^2 \cdot \begin{pmatrix} 0 & 0 & x_1^2x_3^4 & 0 \\ -x_1x_3^4 & 0 & x_1x_2x_3^4 & 0 \\ 0 & x_1^2x_3^4 & 0 & 0 \\ 0 & 0 & 0 & -x_1x_3^2 \end{pmatrix}$$

and $s = 1$. Using the definition of h in Eq. (1), we get that

$$\begin{aligned} h_1 &= Y_1 + \left(x_1^3x_3^6\right) \cdot T_3 - (a_3x_1), \\ h_2 &= Y_2 - \left(x_1^2x_3^6\right) \cdot T_1 + \left(x_1^2x_2x_3^6\right) \cdot T_3 - (a_3x_2), \\ h_3 &= Y_3 + \left(x_1^3x_3^6\right) \cdot T_2 - \left(\frac{1}{7!}a_1x_3^7 + \frac{1}{6!}a_1x_3^6 + \frac{1}{5!}a_1x_3^5 \right. \\ &\quad \left. + \frac{1}{4!}a_1x_3^4 + \frac{1}{3!}a_1x_3^3 + \frac{1}{2!}a_1x_3^2 + a_1x_3 + a_1\right), \\ h_4 &= Y_4 + \left(x_1^2x_3^4\right) \cdot T_4 + 1. \end{aligned}$$

From Eq. (2) we get that

$$\begin{aligned} Q_1 &= 0, \\ Q_2 &= 0 \end{aligned}$$

and therefore following the definition of g in Eq. (3), we have

$$\begin{aligned} g_1 &= T_1 \\ g_2 &= T_2. \end{aligned}$$

To compute this with the library, we do the following:

```
ring All = 0,(a1,a2,a3,x1,x2,x3,Y1,Y2,Y3),dp;
int nra = 3;
int nrx = 3;
int nry = 3;
ideal xid = x2^3-x3^2,x1^3-x3^2;
ideal yid = Y1^3-Y2^3;
ideal aid = a3^2-a1*a2;
poly y;
int i;
```

```
for(i=0;i<=30;i++)
{
  y = y + a1*x3^i/factorial(i);
}
for(i=31;i<=50;i++)
{
  y = y + a2*x3^i/factorial(i);
}
ideal f = a3*x1,a3*x2,y;
desingularization(All, nra,nrx,nry,xid,yid,aid,f,"debug");
```

The algorithm's output is as expected:

```
This is the nice bordered matrix H:
(x2),(-x1),0,0,
0,   0,    1,0,
1,   0,    0,0,
0,   0,    0,(-x1*x3^2)
This is G:
0,              0,                (x1^2*x3^4)*Y4^2, 0,
(-x1*x3^4)*Y4^2,0,                (x1*x2*x3^4)*Y4^2,0,
0,              (x1^2*x3^4)*Y4^2,0,                 0,
0,              0,                0,                (-x1*x3^2)*Y4^2


s = 1

h =
h[1]=Y1+(x1^3*x3^6)*T3+(-a3*x1)
h[2]=Y2+(-x1^2*x3^6)*T1+(x1^2*x2*x3^6)*T3+(-a3*x2)
h[3]=Y3+(x1^3*x3^6)*T2+(-a1*x3^7-7*a1*x3^6-42*a1*x3^5-210*a1*x3^4
     -840*a1*x3^3-2520*a1*x3^2-5040*a1*x3-5040*a1)/5040
h[4]=Y4+(-x1^2*x3^4)*T4+1

m = 1

QT =
QT[1]=0
QT[2]=0
f =
f[1]=(x2)*Y1+(-x1)*Y2
f[2]=(x1*x3^2)*Y4+(-x1*x3^2)

g
_[1]=T1
_[2]=T2
```

Thus the General Neron Desingularization is a localization of $D[Y, T_3, T_4]/(h) \cong D[T_3, T_4]$.

Example 27 We do now the same computations for Example 19. In this example, the computations are much more complicated. The output is unfortunately too big but we will try to describe the result.

The bordered matrix H defined above is equal to

$$H = \begin{pmatrix} 0 & x_2 \cdot Y_3 & x_2 \cdot Y_2 & -2 \cdot Y_4 & 0 \\ x_1^2 \cdot Y_2 & x_1^2 \cdot Y_1 & -2 \cdot Y_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4Y_4Y_5 & 4Y_3Y_5 & 4Y_3Y_4 \end{pmatrix}$$

and hence $G = N \cdot G'$ is equal to

$$G = Y_5^2 \cdot \begin{pmatrix} 0 & 0 & 0 & 16Y_3^2Y_4^2 & 0 \\ 0 & 0 & 16Y_3^2Y_4^2 & 0 & 0 \\ 0 & -8Y_3Y_4^2 & 8x_1^2 \cdot Y_1Y_3Y_4^2 & 8x_1^2 \cdot Y_2Y_3Y_4^2 & 0 \\ -8Y_3^2Y_4 & -4x_2 \cdot Y_2Y_3Y_4 & G[4,3] & 4x_1^2x_2 \cdot Y_2^2Y_3Y_4 & 0 \\ 8Y_3^2Y_5 & x_2 \cdot Y_2Y_3Y_5 + 2Y_4^2Y_5 & G[5,3] & G[5,4] & 4Y_3Y_4 \end{pmatrix},$$

where

$$G[4,3] = 4x_1^2x_2 \cdot Y_1Y_2Y_3Y_4 + 2x_2 \cdot Y_3^3Y_4,$$

$$G[5,3] = -4x_1^2x_2 \cdot Y_1Y_2Y_3Y_5 - 2x_2 \cdot Y_3^3Y_5 - 2x_1^2 \cdot Y_1Y_4^2Y_5 \text{ and}$$

$$G[5,4] = -4x_1^2x_2 \cdot Y_2^2Y_3Y_5 - 2x_1^2 \cdot Y_2Y_4^2Y_5$$

and

$$\begin{aligned} s = {} & a_1^8a_2^2x_2^{12} - 2a_1^5a_2^4x_2^{13} + a_1^2a_2^6x_2^{14} - 2a_1^6a_2^3x_2^{12} + 2a_1^3a_2^5x_2^{13} - a_1^4a_2^4x_2^{12} + 2a_1a_2^6x_2^{13} + \\ & 2a_1^8a_2x_2^{10} - 4a_1^5a_2^3x_2^{11} + 4a_1^2a_2^5x_2^{12} - 4a_1^6a_2^2x_2^{10} + 4a_1^3a_2^4x_2^{11} + a_2^6x_2^{12} + 2a_1a_2^5x_2^{11} + \\ & a_1^8x_2^8 - 2a_1^5a_2^2x_2^9 + 3a_1^2a_2^4x_2^{10} - 2a_1^6a_2x_2^8 + 2a_1^3a_2^3x_2^9 + a_1^4a_2^2x_2^8 - 2a_1^4a_2x_2^6 + \\ & 2a_1a_2^3x_2^7 + 2a_1^2a_2^2x_2^6 + 2a_2^3x_2^6 - 2a_1^4x_2^4 + 2a_1a_2^2x_2^5 + 2a_1^2a_2x_2^4 + 1 \end{aligned}$$

Using the definition of h in Eq. (1), we get that

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix} = s \cdot \begin{pmatrix} Y_1 - y_1' \\ Y_2 - y_2' \\ Y_3 - y_3' \\ Y_4 - y_4' \\ Y_5 - y_5' \end{pmatrix} - x_2^4 G(y') \cdot \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix},$$

where

$$
\begin{aligned}
y_1' = {} & -18a_1^2a_2^5x_2^{12} + 5a_1^3a_2^4x_2^{11} + 7a_2^6x_2^{12} - 18a_1a_2^5x_2^{11} + 15a_1^2a_2^4x_2^{10} - 4a_1^3a_2^3x_2^9 - \\
& 6a_2^5x_2^{10} + 15a_1a_2^4x_2^9 - 12a_1^2a_2^3x_2^8 + 3a_1^3a_2^2x_2^7 + 5a_2^4x_2^8 - 12a_1a_2^3x_2^7 + 9a_1^2a_2^2x_2^6 - \\
& 2a_1^3a_2x_2^5 - 4a_2^3x_2^6 + 9a_1a_2^2x_2^5 - 6a_1^2a_2x_2^4 + a_1^3x_2^3 + 3a_2^2x_2^4 - 6a_1a_2x_2^3 + 3a_1^2x_2^2 - \\
& 2a_2x_2^2 + 3a_1x_2 + 1 \\
y_2' = {} & a_1^{12}x_2^{12} + 2a_1^{10}a_2x_2^{12} - a_1^{11}x_2^{11} + a_1^8a_2^2x_2^{12} - 2a_1^9a_2x_2^{11} + a_1^{10}x_2^{10} - a_1^7a_2^2x_2^{11} + \\
& 2a_1^8a_2x_2^{10} - a_1^9x_2^9 + a_1^6a_2^2x_2^{10} - 2a_1^7a_2x_2^9 + a_1^8x_2^8 - a_1^5a_2^2x_2^9 + 2a_1^6a_2x_2^8 - a_1^7x_2^7 + \\
& a_1^4a_2^2x_2^8 - 2a_1^5a_2x_2^7 + a_1^6x_2^6 - a_1^3a_2^2x_2^7 + 2a_1^4a_2x_2^6 - a_1^5x_2^5 + a_1^2a_2^2x_2^6 - 2a_1^3a_2x_2^5 + \\
& a_1^4x_2^4 - a_1a_2^2x_2^5 + 2a_1^2a_2x_2^4 - a_1^3x_2^3 + a_2^2x_2^4 - 2a_1a_2x_2^3 + a_1^2x_2^2 + 2a_2x_2^2 - a_1x_2 + \\
& 1 \\
y_3' = {} & a_1x_2^2 + x_2 \\
y_4' = {} & a_2x_2^3 + x_2 \\
y_5' = {} & \frac{a_2^2}{4}x_2^4 - \frac{a_1^3 + a_1a_2}{4}x_2^3 + \frac{a_1^2 - a_2}{4}x_2^2 - \frac{a_1}{4}x_2 + \frac{1}{4}.
\end{aligned}
$$

However, the output is too big to be printed. Following the idea in the above examples, we compute Q and g. This is even bigger than h, so we print the numerators and denominators of the coefficients just till degree 10 in the x_i's. However in some cases even the terms till degree 10 will be too many to write down, and hence we will print just the first terms and "...".

As a small remark, Q_3 contains also terms in degree 3 in the T_i but the numerators of the coefficients have power greater than 10, and therefore they do not appear in our shortcutting.

$$
\begin{aligned}
Q_1 = {} & \frac{3a_1^2x_1^2x_2^8 - 2a_1x_1^2x_2^7 + 4a_2x_1^2x_2^8 + x_1^2x_2^6}{4a_1a_2^2x_2^5 + 8a_1a_2x_2^3 + 4a_1x_2 + 4a_2^2x_2^4 + 8a_2x_2^2 + 4} \cdot T_1T_4 - \frac{x_2^6}{4a_2^2x_2^4 + 8a_2x_2^2 + 4} \cdot \quad T_1^2 \\
& + \frac{-a_1^4x_2^{10} + a_1^3x_2^9 - 2a_1^2a_2x_2^{10} - a_1^2x_2^8 + 2a_1a_2x_2^9 + a_1x_2^7 - a_2^2x_2^{10} - 2a_2x_2^8 - x_2^6}{4a_1a_2^2x_2^5 + 8a_1a_2x_2^3 + 4a_1x_2 + 4a_2^2x_2^4 + 8a_2x_2^2 + 4} \cdot \quad T_1T_2 \\
& + \frac{-5a_1^4x_2^{10} + 4a_1^3x_2^9 - 12a_1^2a_2x_2^{10} - 3a_1^2x_2^8 + 8a_1a_2x_2^9 + 2a_1x_2^7 - 6a_2^2x_2^{10} - 4a_2x_2^8 - x_2^6}{16a_1^2a_2^2x_2^6 + 32a_1^2a_2x_2^4 + 16a_1^2x_2^2 + 32a_1a_2^2x_2^5 + 64a_1a_2x_2^3 + 32a_1x_2 + 16a_2^2x_2^4 + 32a_2x_2^2 + 16} \cdot \quad T_2^2 \\
& + \frac{a_1^2x_1^2x_2^8 + 2a_1^2x_2^{10} + 2a_1x_1^2x_2^7 + 4a_1x_2^9 + x_1^2x_2^6 + 2x_2^8}{4a_1a_2^2x_2^5 + 8a_1a_2x_2^3 + 4a_1x_2 + 4a_2^2x_2^4 + 8a_2x_2^2 + 4} \cdot \quad T_1T_3 \\
& + \frac{a_1x_1^2x_2^7 - 2a_1x_2^9 + 2a_2x_1^2x_2^8 - 4a_2x_2^{10} + x_1^2x_2^6 - 2x_2^8}{8a_1^2a_2^2x_2^6 + 16a_1^2a_2x_2^4 + 8a_1^2x_2^2 + 16a_1a_2^2x_2^5 + 32a_1a_2x_2^3 + 16a_1x_2 + 8a_2^2x_2^4 + 16a_2x_2^2 + 8} \cdot \quad T_2T_3 \\
& + \frac{-x_1^4x_2^6 + 4x_1^2x_2^8 - 4x_2^{10}}{16a_1^2a_2^2x_2^6 + 32a_1^2a_2x_2^4 + 16a_1^2x_2^2 + 32a_1a_2^2x_2^5 + 64a_1a_2x_2^3 + 32a_1x_2 + 16a_2^2x_2^4 + 32a_2x_2^2 + 16} \cdot \quad T_3^2 \\
& + \frac{6a_1^2x_1^2x_2^8 - 3a_1x_1^2x_2^7 + 6a_2x_1^2x_2^8 + x_1^2x_2^6}{8a_1^2a_2^2x_2^6 + 16a_1^2a_2x_2^4 + 8a_1^2x_2^2 + 16a_1a_2^2x_2^5 + 32a_1a_2x_2^3 + 16a_1x_2 + 8a_2^2x_2^4 + 16a_2x_2^2 + 8} \cdot \quad T_2T_4
\end{aligned}
$$

$$+\frac{-x_1^4x_2^6+2x_1^2x_2^8}{8a_1^2a_2^2x_2^6+16a_1^2a_2x_2^4+8a_1^2x_2^2+16a_1a_2^2x_2^5+32a_1a_2x_2^3+16a_1x_2+8a_2^2x_2^4+16a_2x_2^2+8}\cdot \qquad T_3T_4$$

$$+\frac{-x_1^4x_2^6}{16a_1^2a_2^2x_2^6+32a_1^2a_2x_2^4+16a_1^2x_2^2+32a_1a_2^2x_2^5+64a_1a_2x_2^3+32a_1x_2+16a_2^2x_2^4+32a_2x_2^2+16}\cdot \qquad T_4^2$$

$$Q_2=\frac{-x_2^6}{4a_1^2x_2^2+8a_1x_2+4}\cdot T_2^2+\frac{3a_1^2x_1^2x_2^8+3a_1x_1^2x_2^7-2a_2x_1^2x_2^8+x_1^2x_2^6}{2a_1^2x_2^2+4a_1x_2+2}\cdot T_2T_3+\frac{-x_1^4x_2^6}{4a_1^2x_2^2+8a_1x_2+4}\cdot \qquad T_3^2$$

$$+\frac{a_1^2x_1^2x_2^8-a_1x_1^2x_2^7+2a_2x_1^2x_2^8+x_1^2x_2^6}{2a_1^2x_2^2+4a_1x_2+2}\cdot T_2T_4+\frac{-x_1^4x_2^6+2x_1^2x_2^8}{2a_1^2x_2^2+4a_1x_2+2}\cdot T_3T_4+\frac{-x_1^4x_2^6}{4a_1^2x_2^2+8a_1x_2+4}\cdot \qquad T_4^2$$

$$Q_3=\frac{2a_1^3x_2^9-4a_1^2a_2x_2^{10}-2a_1^2x_2^8+2a_1a_2x_2^9+2a_1x_2^7-2a_2x_2^8-2x_2^6}{4a_1a_2^3x_2^7+12a_1a_2^2x_2^5+12a_1a_2x_2^3+4a_1x_2+4a_2^3x_2^6+12a_2^2x_2^4+12a_2x_2^2+4}\cdot \qquad T_1T_2$$

$$+\frac{7a_1^3x_2^9-28a_1^2a_2x_2^{10}-7a_1^2x_2^8+21a_1a_2x_2^9+7a_1x_2^7-21a_2^2x_2^{10}-21a_2x_2^8-7x_2^6}{\ldots+48a_1^2x_2^2+48a_1a_2^3x_2^7+144a_1a_2^2x_2^5+144a_1a_2x_2^3+48a_1x_2+16a_2^3x_2^6+48a_2^2x_2^4+48a_2x_2^2+16}\cdot \qquad T_2^2$$

$$+\frac{2a_1^2x_1^2x_2^8+2a_1^2x_2^{10}+4a_1x_1^2x_2^7+4a_1x_2^9-2a_2x_1^2x_2^8-2a_2x_2^{10}+2x_1^2x_2^6+2x_2^8}{4a_1a_2^3x_2^7+12a_1a_2^2x_2^5+12a_1a_2x_2^3+4a_1x_2+4a_2^3x_2^6+12a_2^2x_2^4+12a_2x_2^2+4}\cdot \qquad T_1T_3$$

$$+\frac{7a_1^2x_1^2x_2^8+4a_1^2x_2^{10}+14a_1x_1^2x_2^7+8a_1x_2^9+7a_2x_1^2x_2^8+4a_2x_2^{10}+7x_1^2x_2^6+4x_2^8}{\ldots+24a_1^2x_2^2+24a_1a_2^3x_2^7+72a_1a_2^2x_2^5+72a_1a_2x_2^3+24a_1x_2+8a_2^3x_2^6+24a_2^2x_2^4+24a_2x_2^2+8}\cdot \qquad T_2T_3$$

$$+\frac{-7x_1^4x_2^6-8x_1^2x_2^8-4x_2^{10}}{\ldots+48a_1^2x_2^2+48a_1a_2^3x_2^7+144a_1a_2^2x_2^5+144a_1a_2x_2^3+48a_1x_2+16a_2^3x_2^6+48a_2^2x_2^4+48a_2x_2^2+16}\cdot \qquad T_3^2$$

$$+\frac{6a_1^2x_1^2x_2^8-4a_1x_1^2x_2^7+6a_2x_1^2x_2^8+2x_1^2x_2^6}{4a_1a_2^3x_2^7+12a_1a_2^2x_2^5+12a_1a_2x_2^3+4a_1x_2+4a_2^3x_2^6+12a_2^2x_2^4+12a_2x_2^2+4}\cdot \qquad T_1T_4$$

$$+\frac{21a_1^2x_1^2x_2^8-14a_1x_1^2x_2^7+35a_2x_1^2x_2^8+7x_1^2x_2^6}{\ldots+24a_1^2x_2^2+24a_1a_2^3x_2^7+72a_1a_2^2x_2^5+72a_1a_2x_2^3+24a_1x_2+8a_2^3x_2^6+24a_2^2x_2^4+24a_2x_2^2+8}\cdot \qquad T_2T_4$$

$$+\frac{-7x_1^4x_2^6-4x_1^2x_2^8}{\ldots+24a_1^2x_2^2+24a_1a_2^3x_2^7+72a_1a_2^2x_2^5+72a_1a_2x_2^3+24a_1x_2+8a_2^3x_2^6+24a_2^2x_2^4+24a_2x_2^2+8}\cdot \qquad T_3T_4$$

$$+\frac{-7x_1^4x_2^6}{\ldots+48a_1^2x_2^2+48a_1a_2^3x_2^7+144a_1a_2^2x_2^5+144a_1a_2x_2^3+48a_1x_2+16a_2^3x_2^6+48a_2^2x_2^4+48a_2x_2^2+16}\cdot \qquad T_4^2$$

$$+\frac{a_1^4x_2^{10}-a_1^2a_2x_2^{10}-a_2^2x_2^{10}+a_2x_2^8-x_2^6}{4a_2^3x_2^6+12a_2^2x_2^4+12a_2x_2^2+4}\cdot T_1^2+\frac{-x_2^6}{2a_2^2x_2^4+4a_2x_2^2+2}\cdot \qquad T_1T_5$$

$$+\frac{-3a_2^2x_2^{10}-6a_2x_2^8-3x_2^6}{4a_1^2a_2^2x_2^6+8a_1^2a_2x_2^4+4a_1^2x_2^2+8a_1a_2^2x_2^5+16a_1a_2x_2^3+8a_1x_2+4a_2^2x_2^4+8a_2x_2^2+4}\cdot \qquad T_2T_5$$

$$+\frac{9a_1^2x_1^2x_2^8+6a_1^2x_2^{10}+9a_1x_1^2x_2^7+6a_1x_2^9+3x_1^2x_2^6+2x_2^8}{4a_1^2a_2^2x_2^6+8a_1^2a_2x_2^4+4a_1^2x_2^2+8a_1a_2^2x_2^5+16a_1a_2x_2^3+8a_1x_2+4a_2^2x_2^4+8a_2x_2^2+4}\cdot \qquad T_3T_5$$

$$+\frac{3a_1^2x_1^2x_2^8-3a_1x_1^2x_2^7+12a_2x_1^2x_2^8+3x_1^2x_2^6}{4a_1^2a_2^2x_2^6+8a_1^2a_2x_2^4+4a_1^2x_2^2+8a_1a_2^2x_2^5+16a_1a_2x_2^3+8a_1x_2+4a_2^2x_2^4+8a_2x_2^2+4}\cdot \qquad T_4T_5$$

The denominators of the coefficients from Q are invertible in D because they follow from θ_3' and θ_4' which are invertible in D. Thus $Q \in D[T_1, \ldots, T_5]$. Having Q_i we obtain g_i:

$$\begin{aligned}
g_1 &= Q_1 + \left(\ldots -6a_1^4x_2^4+30a_1^3a_2^3x_2^9+45a_1^2a_2^4x_2^{10}+6a_1^2a_2^2x_2^6+6a_1^2a_2x_2^4+6a_1a_2^3x_2^7+6a_1a_2^2x_2^5+6a_2^3x_2^6+1\right) \cdot T_1 \\
g_2 &= Q_2 + \left(\ldots -6a_1^4x_2^4+30a_1^3a_2^3x_2^9+45a_1^2a_2^4x_2^{10}+6a_1^2a_2^2x_2^6+6a_1^2a_2x_2^4+6a_1a_2^3x_2^7+6a_1a_2^2x_2^5+6a_2^3x_2^6+1\right) \cdot T_2 \\
&\quad + \left(\ldots +6a_1^6x_2^4-30a_1^5a_2^3x_2^9-45a_1^4a_2^4x_2^{10}-6a_1^4a_2^2x_2^6-6a_1^4a_2x_2^4-6a_1^3a_2^3x_2^7-6a_1^3a_2^2x_2^5-6a_1^2a_2^3x_2^6-a_1^2\right) \\
g_3 &= Q_3 + \left(\ldots -6a_1^4x_2^4+30a_1^3a_2^3x_2^9+45a_1^2a_2^4x_2^{10}+6a_1^2a_2^2x_2^6+6a_1^2a_2x_2^4+6a_1a_2^3x_2^7+6a_1a_2^2x_2^5+6a_2^3x_2^6+1\right) \cdot T_3 \\
&\quad + \left(\ldots +24a_1^2a_2^5x_2^{10}+18a_1^2a_2^4x_2^8+a_1^2a_2^2x_2^4+a_1^2a_2x_2^2+12a_1a_2^5x_2^9+a_1a_2^3x_2^5+a_1a_2^2x_2^3+6a_2^6x_2^{10}+a_2^3x_2^4\right).
\end{aligned}$$

The General Neron Desingularization is a localization of $D[Y, T]/(h, g)$. For this example we will need a function

```
invp(poly p, int bound,string param,string variab)
```

which computes the inverse of `p` till order `bound` in $\mathbb{Q}$(`param`)[`variab`].

The input for this example is the following:

```
ring All = 0,(a1,a2,x1,x2,Y1,Y2,Y3,Y4),dp;
int nra = 2;
int nrx = 2;
int nry = 4;
ideal xid = x1^2-x2^3;
ideal yid = Y3^2-x1^2*Y1*Y2,Y4^2-x2*Y2*Y3;
ideal aid = 0;
poly y1,y2,y3,y4;
y3 = 1+a1*x2;
y4 = 1+a2*x2^2;
string as,xs;
if(nra != 0)
{
  as = string(var(1));
  for( int i=2;i<=nra;i++)
  {
    as = as+","+string(var(i));
  }
}
if(nrx!=0)
{
  xs = string(var(nra+1));
  for(int i=nra+2;i<=nra+nrx;i++)
  {
    xs = xs+","+string(var(i));
  }
}
y1 = y3^3*invp(y4^2,12,as,xs);
y2 = y4^2*invp(y3,12,as,xs);
y3 = x2*y3;
y4 = x2*y4;
ideal f = y1,y2,y3,y4;
desingularization(All, nra,nrx,nry,xid,yid,aid,f,"injective","debug");
```

Remark 28 Our algorithm works mainly for local domains of dimension one. If A' is not a domain but a Cohen–Macaulay ring of dimension one, then we can build an algorithm in the idea of the proof of Theorem 20. In this case it is necessary to change B by an Elkik's trick [4] (see [8, Lemma 3.4], [13, Proposition 4.6], [11, Corollary 5.10]). The algorithm and as well Theorem 20 might be also build when A' is not Cohen–Macaulay substituting in the proofs d by a certain power d^r such that $(0 :_A d^r) = (0 :_A d^{r+1})$. Such algorithm could be too complicated to work really.

On the other hand, if we restrict our present algorithm to the case when A' is the completion of A then we might get a faster algorithm using the idea of the proof of Theorem 20. This algorithm could be useful in the arc frame.

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Coherence of Direct Images of the De Rham Complex

Kyoji Saito

Dedicated to the memory of Egbert Brieskorn (7.7.1937–19.7.2013)

Abstract We show the coherence of the direct images of the De Rham complex relative to a flat holomorphic map with suitable boundary conditions. For this purpose, a notion of bi-dg-algebra called the Koszul-De Rham algebra is developed.

Keywords Coherence of direct image sheaf • Critical sets • De Rham complex • Koszul complex

1 Introduction

In the present paper, we prove the following theorem.

Main Theorem *Let $\Phi : Z \to S$ be a flat holomorphic map between complex manifolds.*[1] *Assume that there exists an open subset $Z' \subset Z$ with smooth boundary satisfying (1) Z' contains the critical set C_Φ of Φ, (2) the closure $\bar{Z}'$ in Z is proper over S, and (3) Z' is a weak deformation retract of Z along the fibers of Φ, and (4) $\partial Z'$ is transversal to all fibers $\Phi^{-1}(t)$. Then, the direct images $\mathbb{R}^k\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ of the relative De Rham complex $\Omega^\bullet_{Z/S} := \Omega^\bullet_Z/\Phi^*(\Omega^1_S) \wedge \Omega^{\bullet-1}_Z$ on Z over S are $\mathcal{O}_S$-coherent modules.*

The main Theorem is well known for a proper and/or projective morphism Φ, since (1) the ${}'E_1$-term $\mathrm{R}^q\Phi_*(\Omega^p_{Z/S})$ of the spectral sequence defining the direct image, the so-called *Hodge to De Rham spectral sequence* (1), is already $\mathcal{O}_S$-coherent due to the proper mapping theorem of Grauert and/or Grothendieck, and

[1]We assume that a manifold is connected, paracompact, Hausdorff and, hence, metrizable.

K. Saito (✉)
Kavli IPMU (WPI), UTIAS, The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan
e-mail: kyoji.saito@ipmu.jp

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(2) the differentials on the spectral sequence (induced from the relative De Rham differential $d_{Z/S}$) are $\mathcal{O}_S$-homomorphisms so that the limit of the spectral sequence is also $\mathcal{O}_S$-coherent (see [11, 12]).

Therefore, our main interest is the study of the case when Φ is a non-proper morphism between open manifolds. We give a direct and down to the earth proof of the Main Theorem by introducing the notion of a *Koszul-De Rham algebra*, which seems to detect information of the singularities of the morphism Φ and to be of interest by itself (see Step 3). In such a non-proper mapping setting, we also remark that the result has a close connection to a general theorem for coherent $\mathcal{D}_Z$-modules which are non-characteristic on the boundary by Houzel–Schapira [10], and its generalization to elliptic systems by Schapira–Schneiders (Theorem 4.2 in [18]), since *the relative de-Rham system is an elliptic system*.

If the range S of Φ is one-dimensional, i.e. Φ is a function, and Z is a suitably small neighborhood of an isolated critical point of Φ, then the main Theorem was shown by Brieskorn [1] and then by Greuel [8] (see Hamm [9] for what happens if non-isolated singularities are admitted). Namely, in the case of an isolated critical point, Φ is locally analytically equivalent to a polynomial map, and one proves the coherence by extending Φ to a projective morphism and then applying Grothendieck's coherence theorem for projective morphisms. The result was generalized by the author in [15, 16] to the complete intersection case for higher dimensional base space S, where he did not use the above-mentioned algebro-geometric method in [1] but used a complex analytic method developed by Forster and Knorr [5] who gave a new proof of the Grauert proper mapping theorem [6]. Recently, jointly with Changzheng Li and Si Li, the author studied in [14] morphisms Φ which may no longer be defined locally in a neighborhood of an isolated critical point but may have multiple critical points as in the Main Theorem. Then, Φ may no longer be equivalent to a polynomial map and the algebraic method in [1] seems to be no longer applicable. However the analytic method in [15] can be generalized for this new setting, as will be presented in the present paper, where we study the De Rham cohomology group by the Čech cohomology group with respect to an atlas (9) of relative charts due to Forster and Knorr.

In the present new setting, the morphism Φ may also no longer necessarily have only isolated critical points but may have higher dimensional critical sets in the fibers of Φ. For such semi-global settings, the vanishing cycles in the nearby fibers of Φ are no longer purely middle dimensional but mixed dimensional, and the De Rham cohomology groups are no longer pure but mixed dimensional. Then, we need to solve some topological problems. We also need to find a suitable Stein open covering of the fibration Φ in order to apply the Forster–Knorr result to the Čech complex. This is achieved in the present paper by showing an existence of some enhanced structure on the atlas of relative charts Z (Lemma 27).

The proof of the Main Theorem is divided into the following four steps.

Step 1. We describe two (including Hodge to De Rham) spectral sequences, describing the direct images $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ and see that the restriction from Z to Z' induces an isomorphism: $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S}) \simeq \mathbb{R}\Phi_*(\Omega^\bullet_{Z'/S}, d_{Z'/S})$.

Step 2. For any point $t \in S$, we find a Stein open neighborhood $S^* \subset S$ such that $Z' \cap \Phi^{-1}(S^*)$ is covered by atlases of *relative charts* in the sense of Forster–Knorr, which satisfy an additional condition, called complete intersection, and which form a family of atlases parametrized by the radius r ($^\exists r^* \le r \le 1$) of polydiscs.

Step 3. We introduce the Koszul-De Rham algebra $\mathcal{K}^{\bullet,\star}_{D(r)\times S^*/S^*,\mathbf{f}}$ on each relative chart $D(r) \times S^*$, as a sheaf of double dg-algebras over the dg-algebra $\Omega^{\bullet}_{D(r)\times S^*/S^*}$ of the relative De Rham complex, which gives an $\mathcal{O}_{D(r)\times S^*}$-free "resolution" of the relative De Rham complex $(\Omega^{\bullet}_{Z/S}, d_{Z/S})$ up to the critical set C_Φ, where the "gap," i.e. the cohomology groups of $\mathcal{K}^{\bullet,\star}_{D(r)\times S^*/S^*,\mathbf{f}}$ w.r.t. $\star$, is given by a sequence, indexed by $s \in \mathbb{Z}_{\ge 0}$, of complexes $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ of coherent $\mathcal{O}_Z$-modules supported in C_Φ.

Step 4. The Čech cohomology groups of the De Rham complex $(\Omega^{\bullet}_{Z/S}, d_{Z/S})$ and the lifted Čech cohomology groups of the Koszul-De Rham algebra appear periodically in the first and the second terms of a long exact sequence of cohomology groups, where the second terms are coherent near $t \in S$ due to Forster–Knorr's result [5]. The third terms of the sequence, described by the complexes $\mathcal{H}^{\bullet,s}_{\Phi}$ in Step 3, are also coherent on S, since C_Φ is proper over S. This shows that the first terms, i.e. the direct images of the De Rham complex, are also coherent near $t \in S$.

Since the coherence is a local property on S, this completes the proof.

Remark 1

(i) A flat map Φ is an open map and defines a family of constant

$$n := \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} S$$

dimensional fibers. So, if $n = 0$, the map Φ is proper finite and hence the Main Theorem is trivial. Therefore, in the present paper, we shall assume $n > 0$.

(ii) We introduce in the present note some tools which seem to be unknown in the literature:

a) The *atlases of some special intersection nature* (Lemmas 8 and 9) in Step 2,
b) The *sequence of chain complexes* $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ $(s \in \mathbb{Z}_{>0})$ of coherent $\mathcal{O}_Z$-modules supported in the critical set C_Φ of Φ in Step 3.

Both are essential for our purpose to give an analytic proof of the Main Theorem.

Notation We use cohomologies of three kinds: 1. De Rham complex, 2. derived functor of direct image Φ_*, and 3. Koszul complex. According to them, when it is possible, we distinguish their indices by the following choices: 1. "$\bullet$" or "p" for $p \in \mathbb{Z}_{\ge 0}$, 2. "$*$" or "q" for $q \in \mathbb{Z}_{\ge 0}$, and 3. "$\star$" or "s" for $s \in \mathbb{Z}_{\ge 0}$, respectively.

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Si Li clarified the construction of the atlas in Sect. 3, the discussions with Mikhail Kapranov and Alexander Voronov clarified the Koszul-de Rham algebra in Sect. 4, and the discussions with Alexey Bondal and Tomoyuki Abe clarified the homological algebras in Sect. 5. The author expresses gratitude to Dmytro Shklyarov, who after the present paper was submitted, informed author the paper [2][2] and pointed out some sign problem in the present paper. He expresses also his gratitude to Scott Carnahan and Simeon Hellerman for their reading of the early version of the paper. Finally, the author thanks deeply an anonymous referee for carefully reviewing the manuscript.

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2 Step 1: Hodge to De Rham Spectral Sequence

Throughout the present paper, we keep the setting and notation of the Main Theorem. Recall that the direct image is given by the hypercohomology $\mathbb{R}^{\star}\Phi_*(\Omega^{\bullet}_{Z/S}, d_{Z/S})$ and is described by the limit of the following two spectral sequences:

$$\begin{aligned} {}'E_2^{p,q} &:= \mathrm{H}^p(\mathrm{R}^q\Phi_*(\Omega^{\bullet}_{Z/S}), d_{Z/S}) \\ {}''E_2^{q,p} &:= \mathrm{R}^q\Phi_*(\mathrm{H}^p(\Omega^{\bullet}_{Z/S}, d_{Z/S})). \end{aligned} \tag{1}$$

The E_1-term ${}'E_1^{p,q} = \mathrm{R}^q\Phi_*(\Omega^p_{Z/S})$ of the first spectral sequence is sometimes called *Hodge to De Rham (or, Frölicher) spectral sequence* for the De Rham cohomology relative to Φ.

Let us consider the second spectral sequence ${}''E_2^{q,p}$ (1), which we shall denote also by ${}''E_2^{q,p}(Z/S)$ when we stress its dependence on the space Z/S. We first remark that $\mathrm{Supp}(\mathrm{H}^p(\Omega^{\bullet}_{Z/S}, d_{Z/S})) \subset C_\Phi$ for $p > 0$ (here we recall that C_Φ is the critical set of Φ so that $\Phi\,|_{C_\Phi}$ is a proper morphism), since the Poincaré complex $(\Omega^{\bullet}_{Z/S}, d_{Z/S})$ relative to Φ is exact outside the critical set of Φ. On the other hand, we have $\mathrm{H}^0(\Omega^{\bullet}_{Z/S}, d_{Z/S}) \simeq \Phi^{-1}\mathcal{O}_S$ (since $n > 0$). That is, $\mathrm{H}^0(\Omega^{\bullet}_{Z/S}, d_{Z/S})$ is constant along fibers of Φ. Therefore, we observe (cf. [19]):

Fact 2 *Let Z' be an open subset of Z satisfying 1. $C_F \subset Z'$ and 2. Z' is a deformation retract of Z along fibers of Φ. Then, the inclusion map $Z' \to Z$ induces bijection ${}''E_2^{q,p}(Z/S) \simeq {}''E_2^{q,p}(Z'/S)$ and, hence, of the hypercohomology groups $\mathbb{R}^k\Phi_*(\Omega^{\bullet}_{Z/S}, d_{Z/S}) \simeq \mathbb{R}^k\Phi_*(\Omega^{\bullet}_{Z'/S}, d_{Z'/S})$ as $\mathcal{O}_S$-module.*

[2] In [2] (Theorem 4.1), some algebra similar to the (but differently graded) Koszul-De Rham algebra in the present paper was introduced in order to calculate the Hochshild and cyclic homology of a complete intersection affine variety (similar to the complete intersection variety U in a Stein manifold W in the present paper). It should be of interest to find a relation of the present work with the Hochshild and cyclic homology. However, since the description in [2] misses the parameter space S in the present paper, the relation seems not promptly apparent.

3 Step 2: Atlas of Complete Intersection Relative Charts

We construct an atlas consisting of charts relative to the map Φ (called *relative charts* by Forster–Knorr [5]), which satisfy an additional condition called a complete intersection (Lemma 8). The atlas shall be used in Step 4 to calculate the limit of the spectral sequence ${}'E_1^{p,q}$ by a generalization of Čech cohomology. The construction of the atlas asks the existence of a certain covering of the manifold Z of quite general nature (Lemma 9). Since the proof of the existence of such a covering is rather of technical nature and is independent of the other part of the paper, hurrying readers are suggested to skip the present section and to go to Sect. 4 after looking at definitions and results, and to come back to the proofs if necessary.

Definition 3 ([5]) A *relative chart* for a flat family Φ is a closed embedding

$$j \,:\, U \longrightarrow D(r) \times S_U \tag{2}$$

where U is an open subset of Z (which may be empty), S_U is an open subset of S with $\Phi(U) \subset S_U$, and $D(r)$ is a *polycylinder of the radius* r[3] in some $\mathbb{C}^m$ ($m \in \mathbb{Z}_{\geq 0}$) such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & D(r) \times S_U \\ \Phi\,|_U \searrow & & \swarrow pr_{S_U} \\ & S_U & \end{array} \tag{3}$$

commutes. We sometimes call the embedding j a relative chart, for simplicity.

Definition 4 A relative chart is called a *complete intersection* if the j-image of U is a complete intersection subvariety in $D(r) \times S_U$. That is, there exists a sequence $f_1, \cdots, f_l$ of holomorphic functions on $D(r) \times S_U$, where l is the codimension of U in $D(r) \times S_U$:

$$l := m + \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} Z = m - n$$

such that j induces a natural isomorphism $j^* : \mathcal{O}_{D(r)\times S_U}/(f_1, \cdots, f_l) \simeq \mathcal{O}_U$.

Lemma 5 *Let $j_k : U_k \to D_k(r) \times S_k$ ($k \in K$) be a finite system of relative charts. Then the fiber product*

$$j_K : U_K \to D_K(r) \times S_K \tag{4}$$

of the morphisms j_k ($k \in K$) over $S_K := \cap_{k\in K} S_k$, where we set $D_K(r) := \prod_{k\in K} D_k(r)$, is a relative chart.

[3] A polycylinder of radius r is by definition a domain of the form $\{(z_1, \cdots, z_m) \in \mathbb{C}^m \mid |z_i - a_i| < r \ (i = 1, \cdots, m)\}$ where $(a_1, \cdots, a_m) \in \mathbb{C}^m$ is called the *center* of the polycylinder.

Proof The morphism j_K is obviously a local embedding. We need to show that its image is closed. Suppose there is a sequence $z_i \in U_K$ $(i = 1, 2, \cdots)$ such that the sequence $j_K(z_i)$ converges to a point in $D_K(r) \times S_K$. Then the projection sequence $j_k(z_i)$ also converges in $D_k \times S_k$, implying that the sequence z_i converges in U_k for all $k \in K$. Then $\lim_i \{x_i\}$ belongs to $\cap_{k\in K} U_k =: U_K$ (cf. [5, Corollary 3.2]). □

Definition 6 We shall call j_K (4) the *intersection of relative charts* j_k $(k \in K)$.

Remark 7 Let $\dim D_k = m_k$ and $l_k = m_k - n$ for $k \in K$. Then, $j_K(U_K)$ has codimension equal to $l_K := \sum_{k\in K} m_k - n = \sum_{k\in K} l_k + (\#K - 1)n$. Even if all j_k $(k \in K)$ are complete intersections, their intersection j_K may not necessarily be a complete intersection. Therefore, the following lemma is non-trivial.

Lemma 8 *Let* $\Phi : Z \to S$ *be any flat holomorphic map. Then there exists a function* $r : Z \to \mathbb{R}_{>0}$ *and a relative chart* $j_z : U_z(r) \to D_z(r) \times S_z$ *for all* $z \in Z$ *and* $0 < r < r(z)$ *such that 1)* $j_z(z)$ *is independent of r and 2)* $p_1 \circ j_z : U_z(r) \to D_z(r)$ *is a bijection, mapping z to the center of the polycylinder of radius r. Furthermore, any finite intersection of these relative charts is complete intersection.*

Proof We first provide the following lemma of a quite general nature.

Lemma 9 *Any complex manifold M of dimension N admits an atlas (= a collection of open charts covering M) such that, for any point of M, the union of charts containing the point is holomorphically embeddable into an open set in* $\mathbb{C}^N$.[4]

Proof By the assumption on manifolds (see Footnote 1), M is metrizable, and let d be a metric on M. For $p \in M$ and $r \in \mathbb{R}_{\geq 0}$, let $B(p, r) := \{q \in M \mid d(p, q) < r\}$ be the ball neighborhood of a point p of radius r. We define a function on $p \in M$ by

$R(p) := \sup\{r \in \mathbb{R}_{\geq 0} \mid B(p, r)$ is holomorphically embeddable[5] in a domain in $\mathbb{C}^N\}$. Actually, R is a positive valued continuous function on M except if it takes constant value ∞. For any fixed real number b with $0 < b < 1/3$, we show that the atlas $\{(B(p, R(p)b), \varphi_p\}_{p\in M}$, where φ_p is a holomorphic embedding of $B(p, R(p)b)$ into $\mathbb{C}^N$, has the desired property.[6] Proof: Suppose $p \in M$ belongs to the chart $B(q, R(q)b)$ centered at $q \in M$. That means $d(p, q) < R(q)b$ and then $B(p, R(q)(1-b)) \subset B(q, R(q)(1 - b + b'))$ where $b' := d(p, q)/R(q) < b$ so that $1 - b + b' < 1$. Hence, the ball $B(q, R(q)(1 - b + b'))$ is embeddable in $\mathbb{C}^N$, and so is $B(p, R(q)(1 - b))$. This implies $R(p) \geq R(q)(1 - b)$. On the other hand, for any small $\varepsilon > 0$, $B(q, R(p) - d(p, q) - \varepsilon) \subset B(p, R(p) - \varepsilon)$ is embeddable in $\mathbb{C}^N$, one gets $R(q) \geq \lim_{\varepsilon\downarrow 0}(R(p)-d(p, q)-\varepsilon) = R(p)-d(p, q)$ and, hence, $(1+b)R(q) > (1 + b')R(q) = R(q) + d(p, q) \geq R(p)$. Note that the chart $B(q, R(q)b)$ is contained

[4]A parallel statement obtained by replacing the terminologies: complex manifold, holomorphically and $\mathbb{C}^N$ by C^∞-manifold, differentiably and $\mathbb{R}^N$, respectively, holds by the same proof.

[5]Here, embeddings are not necessarily isometric.

[6]For our later application, we may assume furthermore that φ_p is extendable to a holomorphic embedding of the ball $B(p, R(p)(1 - (1 - 3b)/(1 + b))$ into $\mathbb{C}^N$ since $3b < 1$ and $1 - (1 - 3b)/(1 + b)) < 1$.

in the ball $B(p, R(q)2b)$ of radius $R(q)2b = R(q)(1-b) - R(q)(1-3b)$. Recalling $1-3b > 0$ and inequalities $R(q)(1-b) \le R(p),\ R(q) > R(p)/(1+b)$, the radius is less than $R(p)1 - R(p)(1-3b)/(1+b)) = R(p)(1-(1-3b)/(1+b))$ which is a constant ($< R(p)$) independent of the point q. That is, all charts containing p are covered by the same ball $B(p, R(p)(1-(1-3b)/(1+b))$ which is embeddable in $\mathbb{C}^N$. □

We return to the proof of Lemma 8. Let $\{(B(\underline{z}, R(\underline{z})b), \varphi_{\underline{z}}\}_{\underline{z}\in Z}$ be the atlas of Z described in Lemma 9 (with the additional assumption of Footnote 5). For any point $\underline{z} \in Z$, let $S_{\underline{z}}$ be a local coordinate neighborhood of $\Phi(\underline{z})$ in S.

Then, one finds easily a positive real number $r(\underline{z})$ such that for any real r with $0 < r < r(\underline{z})$, the polycylinder $D(r)$ of radius r centered at $\varphi_{\underline{z}}(\underline{z})$ is contained in the domain $\varphi_{\underline{z}}(B(\underline{z}, R(\underline{z})b)) \subset \mathbb{C}^N$ and $\Phi(\varphi_{\underline{z}}^{-1}(D(r))) \subset S_z$. Then,

$$\begin{array}{rcl} j_{\underline{z}} : U_{\underline{z}}(r) := \varphi_{\underline{z}}^{-1}(D(r)) & \longrightarrow & D(r) \times S_{\underline{z}} \\ \underline{z}' & \longmapsto & (\varphi_{\underline{z}}(\underline{z}'), \Phi(\underline{z}')) \end{array}$$

gives a family (parametrized by r) of relative chart centered at z. The codimension l of the image $j_{\underline{z}}(U_z(r))$ in $D(r) \times S_{\underline{z}}$ is equal to $m - n = N - n = \dim_{\mathbb{C}} S$. Actually, the image is determined by a system of equations:

$$\{t_i - \Phi_i \circ \varphi_{\underline{z}}^{-1} = 0\}_{i=1}^{\dim_{\mathbb{C}} S},$$

where $(t_1, \cdots, t_{\dim_{\mathbb{C}} S})$ is a local coordinate system of $S_{\underline{z}}$ and Φ_i is the ith coordinate component of the morphism Φ. Thus $j_{\underline{z}}$ is complete intersection.

Let us show that, for any finite set $K = \{(\underline{z}, r_{\underline{z}})$ of $\underline{z} \in Z$ and $0 < r_{\underline{z}} < r(\underline{z})$ such that $U_K := \cap_{(\underline{z}, r_{\underline{z}})\in K} U_{\underline{z}}(r_{\underline{z}})$ (and, hence, $S_K := \cap_{(\underline{z}, r_{\underline{z}})\in K} S_{\underline{z}}$) is non-empty, the intersection relative chart $j_K : U_K \to D_K(r) \times S_K$ is complete intersection.

Recall that j_K is given by the fiber product morphism:

$$j_K \ : \ \underline{z}' \in U_K \longmapsto ((\varphi_{\underline{z}}(\underline{z}'))_{(\underline{z}, r_{\underline{z}})\in K}, \Phi(\underline{z}')) \in \prod_{(\underline{z}, r_{\underline{z}})\in K} D_{\underline{z}}(r_{\underline{z}}) \times S_K,$$

where the codimension of $j_K(U_K)$ is equal to $l_K = \#K \cdot \dim_{\mathbb{C}} S + (\#K - 1)n$.

In case of $U_K \neq \emptyset$, the existence of a point $\underline{z}_0 \in U_K$ implies the inclusion:

$$\bigcup_{(\underline{z}, r_{\underline{z}})\in K} U_{\underline{z}}(r_{\underline{z}}) \ \subset \ \bigcup_{(\underline{z}, r_{\underline{z}})\in K} B(\underline{z}, R(\underline{z})b) \ \subset \ B(\underline{z}_0, R(\underline{z}_0)(1-\varepsilon))$$

for $\varepsilon := (1-3b)/(1+b)$ (Lemma 8). Let $z^1, \cdots, z^N$ be the coordinates of $\mathbb{C}^N$ where the ball $B(\underline{z}_0, R(\underline{z}_0)(1-\varepsilon))$ is embedded by extending the domain of $\varphi_{\underline{z}_0}$. We also denote by $\varphi_{\underline{z}}^{-1}$ ($\underline{z} \in K$) the composition map: $D_K \times S_K \to D_{\underline{z}} \to U_{\underline{z}} \subset Z$.

Then, the image $j_K(U_K)$ is determined by the following two types of equations:

1) System equations for identifying polycylinders $D_{\underline{z}}(r_{\underline{z}})$ ($\underline{z} \in K$) with each other. That is, for each fixed j with $1 \le j \le N$, all $z_j \circ \varphi_{\underline{z}}^{-1}$ ($\underline{z} \in K$) are equal to each other. There are $(\#K-1)N = (\#K-1)(n + \dim_{\mathbb{C}} S)$ number of equations:

$$z^j \circ \varphi_{\underline{z}_0}^{-1} = z^j \circ \varphi_{\underline{z}_1}^{-1} = \cdots = z^j \circ \varphi_{\underline{z}_{k^*}}^{-1} \qquad 1 \le j \le N\,.$$

2) System equations for the graph of Φ on each polycylinder $D_{\underline{z}}(r_{\underline{z}})$ ($\underline{z} \in K$). That is, for each fixed i with $1 \le i \le \dim_{\mathbb{C}} S$, $t_i = \Phi_i \circ \varphi_{\underline{z}}^{-1}$ for all $\underline{z} \in K$. There are $\#K \cdot \dim_{\mathbb{C}} S$ number of equations. However, after the identifications in 1), we do not need all equations but only for one point $\underline{z} \in K$: $t_i = \Phi_i \circ \varphi_{\underline{z}}^{-1}$ ($1 \le i \le \dim_{\mathbb{C}} S$), that is, the number of necessary equation is equal to $\dim_{\mathbb{C}} S$.

Thus the total number of necessary equations is $(\#K-1)(n + \dim_{\mathbb{C}} S) + \dim_{\mathbb{C}} S = \#K \cdot \dim_{\mathbb{C}} S_K + (\#K-1)n = \dim_{\mathbb{C}}(D_K \times S_K) - \dim_{\mathbb{C}} U_K$, showing that the image $j_K(U_K)$ is a complete intersection subvariety of $D_K \times S_K$. It is also clear that the Jacobian of this system of defining equations has constant maximal rank.[7]

This completes the proof of Lemma 8. □

Recall the domain $Z' \subset Z$ in the Main Theorem in Sect. 1. We assume that $\partial Z'$ in Z is smooth and transversal to all fibers $\Phi^{-1}(t)$ for all $t \in S$.

Fact 10 *For any point t of S, there exist a Stein open neighborhood S^*, a finite number of relative charts over S^**

$$j_k \;:\; U_k \longrightarrow D_k(1) \times S^*, \quad 0 \le k \le k^* \tag{5}$$

and a real number $0 < r^* < 1$ with the properties: for all r with $r^* \le r \le 1$, set

$$U_k(r) := j_k^{-1}(D_k(r) \times S^*) \quad \textit{and} \quad Z'(r) := \cup_{k=0}^{k^*} U_k(r). \tag{6}$$

Then, we have the following.

1. *One has the inclusions: $Z_{S^*} := \Phi^{-1}(S^*) \supset Z'(r) \supset Z'_{S^*} := \Phi^{-1}(S^*) \cap Z'$.*
2. *$Z'(r)$ is retractable to Z'_{S^*} along fibers of Φ.*
3. *For any $K \subset \{0, \cdots, k^*\}$, the relative chart j_K is a complete intersection.*

Corollary *For r with $r^* \le r \le 1$, we have $\mathcal{O}_{S^*}$-isomorphisms*

$$\mathbb{R}^k\Phi_*(\Omega^\bullet_{Z_{S^*}/S^*}, d_{Z_{S^*}/S^*}) \simeq \mathbb{R}^k\Phi_*(\Omega^\bullet_{Z(r)/S^*}, d_{Z(r)/S^*}). \tag{7}$$

Proof For each point $z \in \bar{Z}' \cap \Phi^{-1}(t)$, we consider a relative chart $j_z : U_z(r) \to D_z(r) \times S_z$ of Lemma 8. We consider two cases.

[7] To be precise, one need to show that any point in $D_K \times S_K$ satisfying the relations (1) and (2) is in the image of j_K. But this can be shown by a routine work so that we omit it.

Case 1. $z \in Z'$: Choose any real r such that $0 < r < r(z)$ and $U_z(r) \subset Z'$.
Case 2. $z \in \partial Z'$: Choose any real r such that $0 < r < r(z)$ and $U_z(r')$ (as a manifold with corners) is transversal to $\Phi^{-1}(t)$ for all real r' with $0 < r' \leq r$.

Since $\bar{Z}' \cap \Phi^{-1}(t)$ is compact, we can find a finite number of relative charts $\tilde{j}_k : \tilde{U}_k \to D_k(r_k) \times \tilde{S}_k$ $(0 \leq k \leq k^*)$ centered at points $\underline{z}_0, \cdots, \underline{z}_{k^*}$ on $\bar{Z}' \cap \Phi^{-1}(t)$ so that the union $\cup_{k=0}^{k^*} \tilde{U}_k$ contains the compact closure $\bar{Z}' \cap \Phi^{-1}(t)$. Then, we can find a Stein open neighborhood S^* of t such that (1) its compact closure $\bar{S}^*$ is contained in $\cap_{k=0}^{k^*} S_k$, (2) $\bar{Z}' \cap \Phi^{-1}(\bar{S^*})$ is contained in $\cup_{k=0}^{k^*} \tilde{U}_k(r)$, and (3) all fibers $\Phi^{-1}(t')$ for $t' \in \bar{S}^*$ and $U_k(r')$ $(0 < r' \leq r_k)$ for the chart j_k whose central point z_k is on the boundary $\partial Z'$. By a suitable rescaling of the coordinate system of charts, we may assume that all radii r_k $(0 \leq k \leq k^*)$ are equal to 1. Then, due to the compactness of $\bar{S}^*$, there exists a real number r^* with $0 < r* < 1$ such that $\bar{Z}' \cap \Phi^{-1}(\bar{S^*})$ is contained in $\cup_{k=0}^{k^*} \tilde{U}_k(r')$ for all r' with $r^* \leq r' \leq 1$. Then, we introduce the relative chart (5) by setting $U_k := U_k \cap j_k^{-1}(D_k(1) \times S^*)$ and define $Z'(r)$ as in (6). Then, (1) is trivial by definition, (2) is a routine work, for instance due to Thom [19], and (3) is true since the system of relative charts $\{\tilde{j}_k\}_{k=0}^{k^*}$ has already this property (Lemma 8). To see (7), we recall the argument done in Fact 2. □

Let us briefly describe how these relative charts shall be used in the sequel.

For any Stein open subset $S' \subset S^*$ and any real number r with $r^* \leq r \leq 1$, we first consider the atlas (a collection of charts)

$$\mathcal{U}(r, S') := \{(U_k(r, S') := j_k^{-1}(D_k(r) \times S'), \varphi_k)\}_{k=0}^{k^*} \tag{8}$$

of $Z'(r, S') := \cup_{k=0}^{k^*} U_k(r, S')$. Actually, this is a Stein open covering, since the intersection $U_K(r, S') := \cap_{k \in K} U_k(r, S')$ for any subset $K \subset \{0, \cdots, k^*\}$ is isomorphic to a closed submanifold of $D_K(r) \times S'$ and, hence, is Stein. Therefore, the $'E_1$-term of the Hodge to De Rham spectral sequence $\mathrm{H}^q(Z'(r, S'), \Omega^p_{Z/S'})$ is given by the Čech complex $(\check{\mathrm{C}}^*(\mathcal{U}(r, S'), \Omega^p_{Z/S'}), \check{\delta})$ with respect to the atlas $\mathcal{U}(r, S')$.

The atlas $\mathcal{U}(r, S')$ is lifted to an atlas of relative charts:

$$\mathfrak{U}(r, S') := \{j_k|_{U_k(r,S')} : U_k(r, S') \to D_k(r) \times S'\}_{k=0}^{k^*}. \tag{9}$$

In Sect. 4, we construct double dg-algebras $\mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}}$ on $D_K(1) \times S'$ (depending on a choice of bases $\mathbf{f}$ of the defining ideal of $U_K(r, S')$ in $D_K(r) \times S'$) and a natural epimorphism $\pi : \mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}} \to \Omega^{\bullet}_{U_K(r,S')}$, where the kernel of π is described by the complex $(\mathcal{H}^{\bullet,s}_{\Phi})_{s>0}$ of coherent sheaves, whose support is contained in the critical set C_Φ (we use here the complete intersection property of the relative charts). Then, in Sect. 5 we construct a "lifting" $\check{\mathrm{C}}^*(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}^{\bullet,\star}_{D(r)\times S'/S',\mathbf{f}})$ of the Čech complex (here, we need once again to "lift" the atlas $\mathfrak{U}(r, S')$ to a based lifting atlas $\widetilde{\mathfrak{U}}(r, S')$ (see Lemma 27)), whose cohomology groups induce a coherent module in a neighborhood of $t \in S^*$ due to the Forster–Knorr Lemma (see Lemma 30). Since $\mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}}$, $\Omega^{\bullet}_{U_K(r,S')}$ and $(\mathcal{H}^{\bullet,s}_{\Phi})_{s>0}$ form an exact triangle, we obtain also the coherence of the direct image of $\Omega^{\bullet}_{U_K(r,S')}$.

4 Step 3: Koszul-De Rham Algebras

We introduce the key concept of the present paper, called the *Koszul-De Rham algebra*, which is a double complex of locally free sheaves over a relative chart and gives a free resolution of the relative De Rham complex $\Omega^{\bullet}_{U/S}$ up to C_{Φ}.

More precisely, we slightly generalize the relative chart (3) $j : U \to D(r) \times S_U$ to (10) $j : U \to W$,[8] and the *Koszul-De Rham-algebra*, denoted by $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}})$,[9] is a sheaf on W of bi-graded $\Omega^{\bullet}_{W/S}$-algebras equipped with (1) the double-complex structure: De Rham operator d_{DR} and Koszul operator $\partial_{\mathcal{K}}$ and (2) a natural epimorphism: $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}}) \to (\Omega^{\bullet}_{U/S}, d_{U/S})$. If the chart (10) is a complete intersection as in Step 2, then the morphism gives a bounded $\mathcal{O}_W$-free resolution of $\Omega^{\bullet}_{U/S}$ up to some "error terms" $(\mathcal{H}^{\bullet,s}_{\Phi})_{s>0}$.

We first slightly generalize the concept of the relative chart (2), (3).

Definition 11 A *based relative chart* $(j, \mathbf{f})$ is a pair of a holomorphic closed embedding $j : U \to W$ of a complex variety U into a Stein variety W with a commutative diagram over a Stein variety S:

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ & \Phi_U \searrow \quad \swarrow \Phi_W & \\ & S & \end{array} \tag{10}$$

and a finite generator system $\mathbf{f} = \{f_1, \cdots, f_l\} \subset \Gamma(W, \mathcal{O}_W)$ of the defining ideal $\mathcal{I}_U$ of the image subvariety $j(U)$ in W (i.e. $\mathcal{I}_U := \ker(j_* j^*|_{\mathcal{O}_W}) = \sum_i \mathcal{O}_W f_i$).[10]

In this setting, for $p \in \mathbb{Z}_{\geq 0}$, there is a natural epimorphism $\pi = j_* j^*|_{\Omega^p_{W/S}}$

$$\Omega^p_{W/S} \xrightarrow{\pi} j_*(\Omega^p_{U/S}) \ (\simeq \Omega^p_{U/S}) \ \to \ 0, \tag{11}$$

between the Kähler differentials, whose kernel, depending only on $\mathcal{I}_U$, is given by

$$\sum_{i=1}^{l} f_i \cdot \Omega^p_{W/S} + \sum_{i=1}^{l} df_i \wedge \Omega^{p-1}_{W/S}.$$

[8] The generalization is done mainly for notational simplification replacing $D(r) \times S_U$ by W. In application in Sect. 5, we shall use relative charts only in the form (3).

[9] The notation might have better been $\mathcal{K}^{\bullet,\star}_{j,\mathbf{f}}$ than $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$.

[10] For a notational simplicity, we shall sometimes confuse the sheaf $\Omega^p_{U/S}$ on U with its j-direct image $j_*(\Omega^p_{U/S})$ on W. For instance, we shall write $\Omega^p_{W/S}/\Sigma_{i=1}^l (f_i \cdot \Omega^p_{W/S} + df_i \wedge \Omega^{p-1}_{W/S}) \simeq \Omega^p_{U/S}$.

We want to construct $\mathcal{O}_W$-free resolution of this ideal generated by f_i $(1 \le i \le l)$ and by df_i $(1 \le i \le l)$. We answer this problem, up to the critical set C_Φ, by introducing the *Koszul-De Rham-algebra* $(\mathcal{K}_{W/S,\mathbf{f}}, d_{DR}, \partial_\mathcal{K})$.[11]

Definition The *Koszul-De Rham-algebra* associated with the based relative chart (10) is a sheaf of *bi-dg-algebra*s $\mathcal{K}_{W/S,\mathbf{f}}$ over the dg-algebra $\Omega^\bullet_{W/S}$ on W equipped with two (co-)boundary operators $\partial_\mathcal{K}$, d_{DR} and with bi-degrees, described below.

Consider a sheaf on W of *graded commutative algebra*s over the dg algebra $\Omega^\bullet_{W/S}$

$$\mathcal{K}_{W/S,\mathbf{f}} := \Omega^\bullet_{W/S}\langle \xi_1, \cdots, \xi_l\rangle[\eta_1, \cdots, \eta_l] \;/\; \mathcal{I} \tag{12}$$

generated by indeterminates $\xi_1, \cdots, \xi_l, \eta_1, \cdots, \eta_l$, where ξ_i's (resp. η_i's) are considered as graded commutative odd (resp. even) variables in the following sense.

1. η_i's and even degree differential forms on W are commuting with all variables,
2. ξ_i's and odd degree differentials forms on W are anti-commuting with each other, and $\mathcal{I}$ is the both sided ideal generated by

$$\xi_i\xi_j + \xi_j\xi_i = 0 \quad \text{and} \quad \xi_i\omega + \omega\xi_i = 0 \quad \text{for } 1 \le i, j \le l \text{ and } \omega \in \Omega^1_W. \tag{13}$$

We equip the algebra $\mathcal{K}_{W/S,\mathbf{f}}$ with the following three structures.

1. **the Koszul structure**: We define Koszul boundary operator $\partial_\mathcal{K}$ on $\mathcal{K}_{W/S,\mathbf{f}}$ as the $\Omega^\bullet_{W/S}$-endomorphism of the algebra defined by the relations

$$\partial_\mathcal{K}\xi_i = f_i, \quad \partial_\mathcal{K}\eta_i = -df_i \quad \text{and} \quad \partial_\mathcal{K}1 = 0.$$

They automatically satisfy the relation: $\partial^2_\mathcal{K} = 0$.

Proof The endomorphism $\partial_\mathcal{K}$ is well defined on the free algebra generated by ξ_i's and η_i's. Then, one checks that the endomorphism preserves the ideal $\mathcal{I}$ generated by relations (13) (since $\partial_\mathcal{K}(\xi_i\xi_j + \xi_j\xi_i) = f_i\xi_j - \xi_i f_j + f_j\xi_i - \xi_j f_i = 0$ and $\partial_\mathcal{K}(\xi_i\omega + \omega\xi_i) = f_i\omega - \omega f_i = 0$), and, hence, induces the action $\partial_\mathcal{K}$ on the quotient $\mathcal{K}_{W/S,\mathbf{f}}$. The relation $\partial^2_\mathcal{K} = 0$ follows immediately from the facts $\partial^2_\mathcal{K}\xi_i = \partial_\mathcal{K} f_i = 0$ and $\partial^2_\mathcal{K}\eta_i = -\partial_\mathcal{K} df_i = 0$. □

2. **De Rham structure**: We regard $\mathcal{K}_{W/S,\mathbf{f}}$ as De Rham complex of the Grassmann algebra $\mathcal{O}_W\langle \xi_1, \cdots, \xi_l\rangle$ where ξ_i's satisfy the first half of the Grassmann relations (13). Then the De Rham differential operator, denoted by d_{DR}, acting on $\mathcal{K}_{W/S,\mathbf{f}}$ is given as an extension of the classical De Rham operator $d_{W/S}$ on $\Omega^\bullet_{W/S}$

[11] Usually, Koszul resolution is defined for even elements f_i's, but here we construct a resolution for odd elements df_i's together. The interpretation to regard it as the Koszul resolution for the odd elements df_i's and to introduce the variables η_i was pointed out by M. Kapranov, to whom the author is grateful.

by setting

$$d_{DR} = d_{W/S} + \sum_{j=1}^{l} \eta_j \partial_{\xi_j},$$

where ∂_{ξ_j} is the derivation of the Grassmann algebra with respect to the variable ξ_j. One, first, defines this operator as an endomorphism of the free algebra before dividing by the ideal $\mathcal{I}$. Then one checks directly that the endomorphism preserves the ideal $\mathcal{I}$ (since $d_{DR}(\xi_i\xi_j + \xi_j\xi_i) = \eta_i\xi_j - \xi_i\eta_j + \eta_j\xi_i - \xi_j\eta_i = 0$ and $d_{DR}(\xi_i\omega + \omega\xi_i) = \eta_i\omega - \omega\eta_i = 0$) so that it induces the required one acting on $\mathcal{K}_{W/S,\mathbf{f}}$. The second term of d_{DR} switches odd variables ξ_i to even variables η_i. We see easily the property $(d_{DR})^2 = 0$ follows from

$$d_{DR}(\xi_j) = \eta_j, \ d_{DR}(\eta_j) = 0 \ \text{ and } \ d_{DR}^2(\mathcal{O}_S) = 0.$$

De Rham differential and Koszul differentials are anti-commuting with each other

$$\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}} = 0$$

(since $(\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}})\xi_i = \partial_{\mathcal{K}}\eta_i + d_{DR} f_i = -df_i + df_i = 0$ and $(\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}})\eta_i = 0 + d_{W/S}(df_i) = 0$) so that the pair $(d_{DR}, \partial_{\mathcal{K}})$ forms a double complex structure on $\mathcal{K}_{W/S,\mathbf{f}}$.[12]

3. **Bi-degree decomposition**: We give an $\mathcal{O}_W$-direct sum decomposition

$$\mathcal{K}_{W/S,\mathbf{f}} := \mathcal{K}_{W/S,\mathbf{f}}^{\bullet,\star} := \oplus_{p\in\mathbb{Z}} \oplus_{s\in\mathbb{Z}} \mathcal{K}_{W/S,\mathbf{f}}^{p,s}$$

such that

(1) $\mathcal{K}_{W/S,\mathbf{f}}^{p,0} = \Omega_{W/S}^{p} \ (p \in \mathbb{Z})$ and $\mathcal{K}_{W/S,\mathbf{f}}^{p,s} = 0$ for either $p < 0$ or $s < 0$.
(2) $\partial_{\mathcal{K}} : \mathcal{K}_{W/S}^{p,s} \to \mathcal{K}_{W/S}^{p,s-1}$ and $d_{DR} : \mathcal{K}_{W/S}^{p,s} \to \mathcal{K}_{W/S}^{p+1,s} \ (p, s \in \mathbb{Z})$.

In order to achieve this, we introduce De Rham degree and Koszul degree on $\mathcal{K}_{W/S,\mathbf{f}}$. Namely, for a monomial of the form $\omega\Xi\mathcal{E}$ where $\omega \in \Omega_{W/S}^{p}$ $(p \in \mathbb{Z}_{\geq 0})$ and Ξ, $\mathcal{E}$ are monomials in $\xi_1, \cdots, \xi_l$ and $\eta_1, \cdots, \eta_l$, respectively, we define degree maps

$$\begin{aligned} \deg_{DR}(\omega\Xi\mathcal{E}) &:= \text{ the total degree as a differential form} = p + \deg(\mathcal{E}) \\ \deg_{\mathcal{K}}(\omega\Xi\mathcal{E}) &:= \text{ the total degree of the monomial } \Xi\mathcal{E} = \deg(\Xi) + \deg(\mathcal{E}) \end{aligned},$$

where one should note that ξ_j's are Grasmann variables and $\deg(\Xi)$s are bounded by l, but η_i's are even variables and $\deg(\mathcal{E})$s are un-bounded. If some monomials have the same degree with respect to $\deg_{DR}$ and/or $\deg_{\mathcal{K}}$, then we also call the sum of them homogeneous of the same degree with respect to $\deg_{DR}$ and/or $\deg_{\mathcal{K}}$.

[12]That this construction of De Rham structure is the universal construction of dg-structure on $\mathcal{K}_{W/S,\mathbf{f}}$ extending that on $\Omega_{W/S}^{\bullet}$ was pointed out by A. Voronov, to whom the author is grateful.

The degree maps are additive with respect to the product in $\Omega^\bullet_{W/S}\langle \xi_1, \cdots, \xi_l\rangle [\eta_1, \cdots, \eta_l]$. Since the ideal $\mathcal{I}$ is generated by bi-homogeneous elements (13), the bi-degrees $\deg_{DR}$ and $\deg_{\mathcal{K}}$ are induced on the quotient algebra $\mathcal{K}_{W/S,\mathbf{f}}$ (12). So, we set

$$\mathcal{K}^{p,s}_{W/S,\mathbf{f}} := \{\omega \in \mathcal{K}_{W/S,\mathbf{f}} \mid \omega \text{ is bi-homogeneous with } (\deg_{DR}, \deg_{\mathcal{K}}) = (p, s)\}.$$

for $p, s \in \mathbb{Z}$, where $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = 0$ if either p or s is negative. Each graded piece $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ is an $\mathcal{O}_W$-coherent module (since it is isomorphic to a direct sum of some $\Omega^\bullet_{W/S}$).

This completes the definition of Koszul-De Rham algebra. For short, we sometimes call the combination of these 3 structures the *bi-dg-algebra* structure. In the following sections (A)–(E), we describe some basic properties and applications of Koszul-De Rham algebras.

(A) Functoriality.

The *functoriality of* Koszul-De Rham *algebra*, formulated in the following Lemma, shall play a key role in Step 4 when we construct the lifted Čech coboundary operator $\check{\delta}$ (see (34)).

Lemma 12 *A morphism* $\mathbf{w} : (j' : U' \to W', \mathbf{f}') \to (j : U \to W, \mathbf{f})$ *between two based charts over the same base space S is a pair (w, h) of 1) a holomorphic map $w : W' \to W$ over S, whose restriction $u := w|_{U'}$ induces a holomorphic map $U' \to U$ so that we have the commutative diagram:*

$$\begin{array}{ccccc} U' & & \xrightarrow{\ j'\ } & & W' \\ \Big\downarrow & \searrow & & \swarrow & \Big\downarrow \\ u & & S & & w \\ \Big\downarrow & \nearrow & & \nwarrow & \Big\downarrow \\ U & & \xrightarrow{\ j\ } & & W \end{array} ,$$

and a matrix $\{h^k_i\}^{k=1,\cdots,l'}_{i=1,\cdots,l}$ with coefficients in $\Gamma(W', \mathcal{O}_{W'})$ such that $w^*(f_i) = \sum_{k=1}^{l'} h^k_i f'_k$. Then, the correspondence

$$\mathbf{w}^\diamond(\xi_i) := \sum_k h^k_i \xi'_k \quad \textit{and} \quad \mathbf{w}^\diamond(\eta_i) := \sum_k h^k_i \eta'_k + \sum_k dh^k_i \xi'_k. \tag{14}$$

induces a bi-dg-algebra morphism

$$\mathbf{w}^\diamond \ : \quad w^*\mathcal{K}_{W/S,\mathbf{f}} \longrightarrow \mathcal{K}_{W'/S,\mathbf{f}'} \tag{15}$$

over the dg-algebra morphism $w^* : w^*\Omega^\bullet_{W/S} \to \Omega^\bullet_{W'/S}$. The morphism is functorial in the sense that for a composition $\mathbf{w_1} \circ \mathbf{w_2}$ of morphisms, we have

$$(\mathbf{w_1} \circ \mathbf{w_2})^\diamond \quad = \quad \mathbf{w_2}^\diamond \circ \mathbf{w_1}^\diamond$$

Proof The correspondence (12) induces a morphism between free algebras before dividing by the ideal $\mathcal{I}$, which preserves the parities of the variables and matches with the degree counting by $\deg_{DR}$ and $\deg_{\mathcal{K}}$. We have the correspondences

$$\xi_i\xi_j + \xi_j\xi_i \quad \mapsto \quad \sum_{k,k} h_i^k h_j^l \left(\xi_k'\xi_l' + \xi_l'\xi_k'\right)$$

and

$$\xi_i\omega + \omega\xi_i \quad \mapsto \quad \sum_k h_i^k \left(\xi_k' w^*(\omega) + w^*(\omega)\xi_k'\right)$$

so that the defining ideal $\mathcal{I}$ is preserved and the bi-graded algebra homomorphism $\mathbf{w}^\diamond$ (15) is well defined.

The commutativity of $\mathbf{w}^\diamond$ with $\partial_{\mathcal{K}}$:

$$\mathbf{w}^\diamond\partial_{\mathcal{K}}(\xi_i) = \mathbf{w}^\diamond(f_i) = \textstyle\sum_k h_i^k f_k' = \sum_k h_i^k \partial_{\mathcal{K}}\xi_k' = \partial_{\mathcal{K}}(\sum_k h_i^k\xi_k') = \partial_K \mathbf{w}^\diamond(\xi_i),$$

$$\begin{aligned}\mathbf{w}^\diamond\partial_{\mathcal{K}}(\eta_i) &= \mathbf{w}^\diamond(df_i) = \textstyle\sum_k d(h_i^k f_k') \\ &= \textstyle\sum_k (h_i^k df_k' + dh_i^k f_k') = \sum_k \partial_{\mathcal{K}} \sum_k (h_i^k\eta_k' + dh_i^k\xi_k') = \partial_K \mathbf{w}^\diamond(\eta_i).\end{aligned}$$

The commutativity of $\mathbf{w}^\diamond$ with d_{DR}:

$$\mathbf{w}^\diamond d_{DR}(\xi_i) = \mathbf{w}^\diamond(\eta_i) = \textstyle\sum_k w_i^k\eta_k' + \sum_k dw_i^k\xi_k' = d_{DR}(\sum_k w_i^k\xi_k') = d_{DR}(\mathbf{w}^\diamond(\xi_i)),$$

$$\mathbf{w}^\diamond d_{DR}(\eta_i) = 0,\ d_{DR}\mathbf{w}^\diamond(\eta_i) = d_{DR}(\textstyle\sum_k (w_i^k\eta_k' + dw_i^k\xi_k') = \sum_k (dw_i^k\eta_k' - dw_i^k\eta_k') = 0.$$

The functoriality of $\mathbf{w}^\diamond$: consider a composition $(w, h) = (w_1, h_1) \circ (w_2, h_2)$ of two morphisms, where $w_1 : W_2 \to W_1$ and $w_2 : W_3 \to W_2$ and $w_1^*(f_{i,1}) = \sum_k h_{i,1}^k f_{k,2}$, $w_2^*(f_{k,2}) = \sum_l h_{k,2}^l f_{l,3}$. So, $w = w_1 \circ w_2$ and $h_i^l = \sum_k w_2^*(h_{i,1}^k) h_{k,2}^l$. Then, obviously,

$$\mathbf{w}^\diamond(\xi_{i,1}) = \textstyle\sum_l \left((\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l)\xi_{l,3}\right) = \sum_k w_2^*(h_{i,1}^k\xi_{k,2}) = w_2^\diamond(w_1^\diamond(\xi_{i,1})).$$

$$\begin{aligned}\mathbf{w}^\diamond(\eta_{i,1}) &= \textstyle\sum_l \Big(\big(\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l\big)\eta_{l,2} + d\big(\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l\big)\xi_{l,2}\Big) \\ &= \textstyle\sum_l \sum_k w_2^*(h_{i,1}^k)\big(h_{k,2}^l\eta_{l,2} + dh_{k,2}^l\xi_{l,2}\big) + \sum_k d(w_2^*(h_{i,1}^k))h_{k,2}^l)\xi_{l,2})\big) \\ &= w_2^\diamond(\textstyle\sum_k h_{i,1}^k\eta_{k,2} + \sum_k dh_{i,1}^k\xi_{k,2}) = w_2^\diamond(w_1^\diamond(\eta_{i,1})).\end{aligned}$$

□

(B) Comparison π with the De Rham complex $(\boldsymbol{\Omega}^\bullet_{\mathbf{U/S}}, \mathbf{d_{U/S}})$.

We compare the dg-algebras $(\Omega^\bullet_{U/S}, d_{U/S})$ and $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}})$. We summarize and reformulate well-known facts in terms of $\deg_{\mathcal{K}}$, $\partial_{\mathcal{K}}$ and d_{DR} as follows.

Lemma 13 *The morphism* π (11) *satisfies the following properties.*

1. The morphism π *induces an exact sequence:*

$$\mathcal{K}^{\bullet,1}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} \xrightarrow{\pi} \Omega^{\bullet}_{U/S} \longrightarrow 0. \tag{16}$$

2. The morphism π *commutes with De Rham differentials:*

$$\begin{array}{ccc} \mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & \Omega^{\bullet}_{U/S} \\ d_{DR}\downarrow & & \downarrow d_{U/S} \\ \mathcal{K}^{\bullet+1,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & \Omega^{\bullet+1}_{U/S} \end{array} \tag{17}$$

3. If there is a morphism $\mathbf{w} : (j', \mathbf{f}') \to (j, \mathbf{f})$ *between two based relative charts, then the morphism* π *gives natural transformation between the functors* $\mathbf{w}^{\diamond}$ *and* u^*.

$$\begin{array}{ccc} w^*\mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & u^*\Omega^{\bullet}_{U/S} \\ w^{\diamond}\downarrow & & \downarrow u^* \\ \mathcal{K}^{\bullet,0}_{W'/S,\mathbf{f}'} & \xrightarrow{\pi'} & \Omega^{\bullet}_{U'/S} \end{array} \tag{18}$$

Proof We have only to check the following facts.

1) The complex $(K^{\bullet,0}_{W/S,\mathbf{f}}, d_{DR})$ coincides with the De Rham complex $(\Omega^{\bullet}_{W/S}, d_{W/S})$.
2) The image $\partial_{\mathcal{K}}(\mathcal{K}^{\bullet,1}_{W/S,\mathbf{f}})$ in $\mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}}$ of the $\deg_{\mathcal{K}} = 1$ part of the algebra is equal to the ideal generated by $f_1, \cdots, f_l$ and $df_1, \cdots, df_l$ in $(\Omega^{\bullet}_{W/S}, d_{W/S})$ so that as $\mathcal{O}_U$-module, they are isomorphic.
3) The relative De Rham differential $d_{U/S}$ on U coincides with the one induced from the relative De Rham differential $d_{W/S}$ on W.
4) The morphism $w^{\diamond}$ on the $\deg_{\mathcal{K}} = 0$ part of Koszul-De Rham algebra coincides

with the pull-back morphism w^* of differential forms. □

The first and second properties of Lemma 13 means that the morphism π induces a quasi equivalence of the Koszul-De Rham double complex with the De Rham complex, and the third property 3. means the naturality of π, which shall be used, in the next section, when we compare the Čech-triple complex with coefficients in $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ with the Čech-double complex with coefficients in $\Omega^{\bullet}_{U/S}$.

(C) Boundedness of the Koszul-De Rham algebra.

We discuss a certain *boundedness* of the Koszul-De Rham complexes, which is crucially used in the study of triple complex in the next Sect. 5.

Since there are no relations mixing ξ and η, by definition, the bi-degree (p, s) term of the Koszul-De Rham algebra has the following direct sum decomposition.

$$\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = \bigoplus_{a=0}^{\min\{p,s\}} \bigoplus_{\substack{\Xi \text{ is a monomial in} \\ \xi_i\text{'s of } \deg(\Xi)=s-a.}} \bigoplus_{\substack{\mathcal{E} \text{ is a monomial in} \\ \eta_i\text{'s of } \deg(\mathcal{E})=a.}} \Omega^{p-a}_{W/S}\ \Xi\ \mathcal{E}. \tag{19}$$

Lemma 14 *The set* $\{(p, s) \in \mathbb{Z}^2 \mid \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \neq 0\}$ *is contained in the strip*

$$\{(p, s) \in \mathbb{Z}^2 \mid -l \leq p - s \leq \dim_{\mathbb{C}} W\} \tag{20}$$

Proof Suppose that there exists a nontrivial element $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} \ni \omega\Xi\mathcal{E} \neq 0$. Then, $p - s = \deg(\omega) - \deg(\Xi)$ (recall the definition of bi-degrees), where $0 \leq \deg(\omega) \leq \dim_{\mathbb{C}} W$ and $0 \leq \deg(\Xi) \leq l$. This gives the bound in the formula. □

Remark 15 Lemma implies that the total Koszul-De Rham complex $\mathcal{K}^{\tilde{\bullet}}_{W/S,\mathbf{f}}$ is bounded. However each term $\bigoplus_{p-s=\tilde{\bullet}} \mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ of the total complex is an infinite sum, since η_i's are even variables and the multiplication of any high power of them is non-vanishing and increases simultaneously the degrees p and s. Nevertheless, such simple repetition of same terms (in stable area) seems harmless as we shall see, in the next Step 4, that by taking a lifting of Čech cohomology groups with coefficients in Koszul-De Rham algebras, they can be truncated in (40).

(D) $\partial_{\mathcal{K}}$-cohomology group of Koszul-De Rham algebra.

For each fixed $p \in \mathbb{Z}_{\geq 0}$, we study the cohomology of the bounded complex:

$$0 \to \mathcal{K}^{p,p+l}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \cdots \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{p,3}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{p,2}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{p,1}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{p,0}_{W/S,\mathbf{f}} \to 0. \tag{21}$$

Let us first fix a notation: for $p, s \in \mathbb{Z}$, set

$$\mathcal{H}^{p,s}_{W/S,\mathbf{f}} := \mathrm{Ker}\big(\partial_{\mathcal{K}} : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s-1}_{W/S,\mathbf{f}}\big) \,/\, \partial_{\mathcal{K}}\big(\mathcal{K}^{p,s+1}_{W/S,\mathbf{f}}\big), \tag{22}$$

and call it *Koszul-cohomology*, or $\partial_{\mathcal{K}}$*-cohomology*.

We first recall some functorial properties of them.

Lemma 16

i) The De Rham operator d_{DR} on the Koszul-De Rham algebra induces

$$d_{DR} \ :\quad \mathcal{H}^{p,s}_{W/S,\mathbf{f}} \longrightarrow \mathcal{H}^{p+1,s}_{W/S,\mathbf{f}}$$

such that $d^2_{DR} = 0$, which we shall call the De Rham operator on $\partial_{\mathcal{K}}$-cohomology.

ii) Let $\mathbf{w} = (w, h) : (j', \mathbf{f}') \to (j, \mathbf{f})$ *be a morphism between based relative charts, and set* $u := w|'_U : U' \to U$. *Then, the morphism* $\mathbf{w}^\diamond$ (15) *induces a morphism*

$$u^\diamond \;:\quad \mathcal{H}^{p,s}_{W/S,\mathbf{f}} \longrightarrow \mathcal{H}^{p,s}_{W'/S',\mathbf{f}'}$$

which commutes with the De Rham operators on $\partial_{\mathcal{K}}$*-cohomologies, and has the functorial property:* $(u_1 \circ u_2)^\diamond = u_2^\diamond \circ u_1^\diamond$.

Proof All these facts are immediate consequences of the fact that $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ is a double complex with respect to ∂_K and d_{DR} shown in Sect. 4, and the fact that $\mathbf{w}^\diamond$ is a bi-dg-algebra homomorphism from $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ to $\mathcal{K}^{\bullet,\star}_{W'/S,\mathbf{f}'}$ (Lemma 12). □

Note that the ∂_K-cohomology groups are $\mathcal{O}_W$-coherent modules, since the modules $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ are $\mathcal{O}_W$-coherent and the $\partial_{\mathcal{K}}$ are $\mathcal{O}_W$-homomorphisms (19). We analyze the $\partial_{\mathcal{K}}$-cohomologies in that context. The first basic fact is that they are defined on U.

Lemma 17 *The* $\partial_{\mathcal{K}}$*-cohomology* $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ *(*$p, s \in \mathbb{Z}$*) is an* $\mathcal{O}_U$*-coherent module.*

Proof Recall that the defining ideal of U is given by $\mathcal{I}_U = \sum_i \mathcal{O}_W f_i$. Therefore, to be an $\mathcal{O}_U$-module, we have only to show that $f_i \mathcal{H}^{p,s}_{W/S,\mathbf{f}} = 0$. Let $\omega \in \mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ be a representative of an element $[\omega] \in \mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ such that $\partial_{\mathcal{K}}\omega = 0$. Then we calculate that $\partial_{\mathcal{K}}(\xi_i\omega) = \partial_{\mathcal{K}}(\xi_i)\omega + \xi_i\partial_{\mathcal{K}}\omega = f_i\omega$. That is, the class $[f_i\omega] \in \mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ is equal to zero. □

We know already by (16) that the zero-th $\partial_{\mathcal{K}}$-cohomology is naturally given by

$$\mathcal{H}^{p,0}_{W/S,\mathbf{f}} \quad\simeq\quad \Omega^p_{U/S} \tag{23}$$

which is compatible with the De Rham operator action.

In order to analyze the support of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ more carefully, recall the direct sum expression of the Koszul-De Rham algebra (19). We observe that the Koszul boundary operator $\partial_{\mathcal{K}}$ splits into a sum $\partial + \tilde{\partial}$, where each ∂ and $\tilde{\partial}$ is defined as $\Omega^\bullet_{W/S}$-linear endomorphisms such that

$$\partial\xi_i = f_i\ ,\ \partial\eta_i = 0\ ,\ \partial 1 = 0 \qquad \text{and} \qquad \tilde{\partial}\eta_i = df_i\ ,\ \tilde{\partial}\xi_i = 0\ ,\ \tilde{\partial}1 = 0\ .$$

We see immediately the relations $\partial, \tilde{\partial} : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s-1}_{W/S,\mathbf{f}}$ for all $p, s \in \mathbb{Z}$ and $\partial^2 = \tilde{\partial}^2 = \partial\tilde{\partial} + \tilde{\partial}\partial = 0$. That is, for each fixed $p \in \mathbb{Z}$, the subcomplex $(\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial_{\mathcal{K}})$ can be regarded as the total complex of a double complex $(\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial, \tilde{\partial})$.

More precisely, let us denote by $\Omega^a_{W/S}\xi^b\eta^c$ the space spanned by those elements of the form $\omega\Xi\mathcal{E}$ with $\omega \in \Omega^a_{W/S}$, and Ξ and $\mathcal{E}$ are monomials of ξ_j's and η_j's of degree $\deg(\Xi) = b$ and $\deg(\mathcal{E}) = c$, respectively. Then, we have $\partial : \Omega^a_{W/S}\xi^b\eta^c \to \Omega^a_{W/S}\xi^{b-1}\eta^c$ and $\tilde{\partial} : \Omega^a_{W/S}\xi^b\eta^c \to \Omega^{a+1}_{W/S}\xi^b\eta^{c-1}$. So, by putting $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}} := \Omega^{p-c}_{W/S}\xi^b\eta^c$ for $p, b, c \in \mathbb{Z}$, we get double complex $(\mathcal{K}^{p,\{\star,\tilde{\star}\}}_{W/S,\mathbf{f}}, \partial, \tilde{\partial})$ (where $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}} \neq 0$ only when

$0 \leq b \leq l$ and $0 \leq c \leq p$), and we have the identification of the total complex with the original Koszul complex:

$$(\oplus_{b+c=\star}\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}}, \partial + \tilde{\partial}) \quad = \quad (\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial_{\mathcal{K}}) \tag{24}$$

for each fixed $p \in \mathbb{Z}$. Explicitly, the double complex is given in the following table.

Table 1 Double complex $(\mathcal{K}^{p,\{\star,\tilde{\star}\}}_{W/S,f}, \partial, \tilde{\partial})$

$$\begin{array}{ccccccccc}
\Omega^p_{W/S} & \xleftarrow{\tilde{\partial}} & \Omega^{p-1}_{W/S}\eta^1 & \xleftarrow{\tilde{\partial}} \cdots \xleftarrow{\tilde{\partial}} & \Omega^1_{W/S}\eta^{p-1} & \xleftarrow{\tilde{\partial}} & \mathcal{O}_W\eta^p & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & \\
\Omega^p_{W/S}\xi^1 & \xleftarrow{\tilde{\partial}} & \Omega^{p-1}_{W/S}\xi^1\eta^1 & \xleftarrow{\tilde{\partial}} \cdots \xleftarrow{\tilde{\partial}} & \Omega^1_{W/S}\xi^1\eta^{p-1} & \xleftarrow{\tilde{\partial}} & \mathcal{O}_W\xi^1\eta^p & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & \\
\cdots & & \cdots & \cdots & \cdots & & \cdots & \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & \\
\Omega^p_{W/S}\xi^{l-1} & \xleftarrow{\tilde{\partial}} & \Omega^{p-1}_{W/S}\xi^{l-1}\eta^1 & \xleftarrow{\tilde{\partial}} \cdots \xleftarrow{\tilde{\partial}} & \Omega^1_{W/S}\xi^{l-1}\eta^{p-1} & \xleftarrow{\tilde{\partial}} & \mathcal{O}_W\xi^{l-1}\eta^p & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & \\
\Omega^p_{W/S}\xi^l & \xleftarrow{\tilde{\partial}} & \Omega^{p-1}_{W/S}\xi^l\eta^1 & \xleftarrow{\tilde{\partial}} \cdots \xleftarrow{\tilde{\partial}} & \Omega^1_{W/S}\xi^l\eta^{p-1} & \xleftarrow{\tilde{\partial}} & \mathcal{O}_W\xi^l\eta^p & \leftarrow 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & & 0 & & 0 & & 0 &
\end{array}$$

Definition 18 A based relative chart $(j, \mathbf{f})$ is called a *complete intersection* if its underlying relative chart (10) satisfies the following (1)–(3).

1) The varieties U, W, and S are smooth.
2) The map $\Phi_W : W \to S$ is a submersion, i.e. there are no critical points.
3) The U is a complete intersection subvariety of W and $f_1, \cdots, f_l$ is a minimal system of equations for U, i.e. $f_1, \cdots, f_l$ form a regular sequence on W.

From now on through the end of the present paper, we study only based relative charts which is complete intersection. For such relative chart, we say that the morphism $\Phi_U : U \to S$ is critical at a point in U if Φ_U is not submersive at the point. That is, the variety of critical set is given by

$$C_{\Phi_U} := \{x \in U \mid \text{the rank of the Jacobian of } \Phi_U \text{ at } x \text{ is less than } \dim_{\mathbb{C}} S\}, \tag{25}$$

whose defining ideal $\mathcal{I}_{C_{\Phi_U}}$ in $\mathcal{O}_U$ is the one generated by the minors of size $\dim_{\mathbb{C}} S$ of the Jacobian matrix of Φ_U. We now prove some basic properties of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ which we shall use in the next section seriously.

Lemma 19 *Suppose that a based relative chart (10) is a complete intersection. Then, we have*

(1) *the $\mathcal{O}_U$-module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s, p \in \mathbb{Z}$ together with the action of De Rham operator d_{DR} is independent of the choice of basis $\boldsymbol{f}$ but depends only on the morphism Φ_U,*
(2) *the support of the module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s > 0$ is contained in the critical set C_{Φ_U}.*

Proof Before we start to prove this Lemma, we visit the double complex $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}}$ given in Table 1 under the complete intersection assumption.

By Definition (1) and (2) of complete intersection, $\Omega^{\bullet}_{W/S}$ is an $\mathcal{O}_W$-locally free modules of finite rank. The ith vertical direction (w.r.t. the coboundary operator ∂) of the diagram for $i = 0, 1, \cdots, p$ is the classical Koszul complex on the locally free module $\Omega^{p-i}_{W/S}$ for the regular sequence $f_1, \cdots, f_l$ (recall that ξ_i's are odd variables), which is exact except at the zeroth stage, and the cokernel module at the zeroth stage is an $\mathcal{O}_U$ locally free module isomorphic to $\Omega^{p-i}_{W/S}\eta^i/(f_1, \cdots, f_l)\Omega^{p-i}_{W/S}\eta^i = \big(\overset{p-i}{\wedge}\Omega^1_{W/S}\eta^i\big)\otimes_{\mathcal{O}_W}\mathcal{O}_U$. Between the modules, $\tilde{\partial}$ induces a cochain complex structure, denoted again by $\tilde{\partial}$. In view of (24), this chain complex

$$\Big(\big(\overset{p-\star}{\wedge}\ \Omega^1_{W/S}\big)\eta^{\star}\otimes_{\mathcal{O}_W}\mathcal{O}_U,\ \tilde{\partial}\ \Big) \tag{**}$$

is quasi-isomorphic to the Koszul complex $(\mathcal{K}^{p,\star}_{W/s,\mathbf{f}}, \partial_{\mathcal{K}})$. Therefore, we show that the cohomology groups of (**) does not depend on the choice of the bases $\mathbf{f}$.

We provide, now, the following elementary but quite useful reduction lemma.

Lemma 20 *Let f_1 be the first element of the basis $\mathbf{f} = \{f_1, \cdots, f_l\}$. Suppose df_1 is a part of $\mathcal{O}_W$-free basis of the module $\Omega^1_{W/S}$. Consider the hypersurface $W' := \{f_1 = 0\} \subset W$ and set $\mathbf{f}' = \{f_2, \cdots, f_l\}$. Then, we have*

(1) *$(j' : U \to W', \mathbf{f}')$ is also a complete intersection based relative chart,*
(2) *the inclusion map $\iota : W' \subset W$ together with the correspondence $\eta_1 \mapsto 0$ induces a morphism between the based relative charts and a quasi-isomorphism of chain complexes of $\mathcal{O}_U$-modules:*

$$\Big(\big(\overset{p-\star}{\wedge}\ \Omega^1_{W/S}\big)\eta^{\star}\otimes_{\mathcal{O}_W}\mathcal{O}_U, \tilde{\partial}\Big) \to \Big(\big(\overset{p-\star}{\wedge}\ \Omega^1_{W'/S}\big)\eta'^{\star}\otimes_{\mathcal{O}_{W'}}\mathcal{O}_U, \tilde{\partial}\Big).$$

(3) *The $\mathcal{O}_U$-isomorphism: $\mathcal{H}^{p,s}_{W/S,\mathbf{f}} \simeq \mathcal{H}^{p,s}_{W'/S,\mathbf{f}'}$ ($p, s \in \mathbb{Z}$) obtained from this quasi-isomorphism coincides with $(\iota|_U)^{\diamond}$ (recall Lemma 16* ii*). In particular, the isomorphism commutes with the De Rham operator action.*

Proof (1) The fact that df_1 is a part of $\mathcal{O}_W$-free basis $\Omega^1_{W/S}$ implies that W' is a smooth variety and that the restriction $\Phi'_U := \Phi_U|_{W'}$ is still submersive.

(2) On the space W, the two chain complexes of sheaves $\Big(\big(\overset{p-\star}{\wedge}\ \Omega^1_{W'/S}\big)\eta'^{\star}\otimes_{\mathcal{O}_{W'}}\mathcal{O}_U, \tilde{\partial}\Big)$ and $\Big(\big(\overset{p-\star}{\wedge}\ \Omega^1_{W/S}/\mathcal{O}_W df_1\big)\eta'^{\star}\otimes_{\mathcal{O}_W}\mathcal{O}_U, \tilde{\partial}\Big)$ are naturally isomorphic, since

$\mathcal{O}_U \simeq \mathcal{O}_W/(f_1,\cdots,f_l) \simeq \mathcal{O}_{W'}/(f_2,\cdots,f_l)$. Therefore, in order to show (2), it is sufficient to show the following general linear algebraic facts (cf. [4, 17]).

Proposition *Let M be a free module of finite rank over a noetherian commutative unitary ring R. Let $\wedge^* M$ be the Grassmann algebra of M over R. For given elements $\omega_1,\cdots,\omega_k$ of M, consider the polynomial ring $\wedge^* M[\eta]$ of k variables $\eta_1,\cdots,\eta_k$ equipped with a Koszul differential $\tilde{\partial}$ defined by setting $\tilde{\partial}(\eta_i)=\omega_i$ on it.*

(a) Let $\mathfrak{a}$ be the ideal in R generated by the coefficients of $\omega_1 \wedge \cdots \wedge \omega_k$. Then, i-th cohomology group of $(\wedge^* M[\eta],\tilde{\partial})$ vanishes for $i <$ depth$(\mathfrak{a})$.
(b) If ω_1 is a part of some R-free basis system of M, then the natural chain morphism $(\wedge^* M[\eta],\tilde{\partial}) \to (\wedge^* M/R\omega_1[\eta'],\tilde{\partial}')$ (where η' is the indeterminates $\eta_2,\cdots,\eta_k$ such that $\tilde{\partial}'(\eta_i)=\omega_i$ $(i=2,\cdots,k)$ and η_1 is mapped to 0) is quasi-isomorphic.

(3) First, we note that there is a slight abuse of notation. Namely, we have needed to fix the coefficient matrix h of the transformation $\iota^*(f_1)=0$ and $\iota^*(f_i)=f_i$ for $i=2,\cdots,l$ in order that $(\iota,h)^\diamond : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s}_{W'/S,\mathbf{f}'}$ is defined (recall Lemma 12). Once $(\iota,h)^\diamond$ is defined in this manner, then we have $(\iota,h)^\diamond(\xi_1)=(\iota,h)^\diamond(\eta_1)=0$ and $(\iota,h)^\diamond(\xi_i)=\xi_i$, $(\iota,h)^\diamond(\eta_i)=\eta_i$ for $i=2,\cdots,l$. Then, we observe that $(\iota,h)^\diamond$ is compatible with the double complex $\mathcal{K}^{p,b,c}_{W/S,\mathbf{f}}$ decomposition, inducing morphism $(\iota,h)^{\diamond,double} : \mathcal{K}^{p,b,c}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,b,c}_{W'/S,\mathbf{f}'}$ for all $p,b,c \in \mathbb{Z}$. Then, the chain map in (2) obviously coincides with the one induced from $(\iota,h)^{\diamond,double}$.

□

Let us come back to the proof of Lemma 19.

Proof of Lemma 19 (1) Suppose that there are two complete intersection based charts $(j^1 : U_1 \to W_1,\mathbf{f}^1)$ and $(j^2 : U_2 \to W_2,\mathbf{f}^2)$ over the same base set S and points $z_1 \in U_1$ and $z_2 \in U_2$ such that there is a local bi-holomorphic map $(U^1,z_1) \simeq (U^2,z_2)$ which commutes with the maps Φ_{U^1} and Φ_{U^2} in neighborhoods of z_1 and z_2. Then we show that there is a natural $\mathcal{O}_{U_1,z_1}$-O_{U_2,z_2}-isomorphism of the stalks:

$$\mathcal{H}^{p,s}_{W^1/S,\mathbf{f^1},z_1} \simeq \mathcal{H}^{p,s}_{W^2/S,\mathbf{f^2},z_2}$$

which is equivariant with the De-Rham actions. By shrinking the relative charts j^i $(i=1,2)$ suitably, we may assume $U_1 \simeq U_2$, and, furthermore, that W_i is a Stein domain of $U_i \times \mathbb{C}^{l_i}$ such that i) the embedding j^i is realized by the isomorphism $U_i \simeq U_i \times 0 \subset W_i \subset U_i \times \mathbb{C}^{l_i}$ and ii) the composition of the embedding of W_i in $U_i \times \mathbb{C}^{l_i}$ with the projection to the j-th component of $\mathbb{C}^{l_i}$ is equal to the j-th component, say f^i_j, of $\mathbf{f}^i$ $(i=1,2)$ (however, the compositions of the embedding $W_i \to U_i \times \mathbb{C}^{l_i}$ with the projection to U_i and with Φ_{U_i} may not necessarily coincide with the morphism $\Phi_{W_i} : W_i \to S$).

The proof is achieved by introducing an auxiliarily third based relative chart (j,W). Namely, set $U := U^1 \simeq U^2$ and let $z \in U$ be the point corresponding to

$z_i \in U_i$. Then, $W := W_1 \times_U W_2$ may naturally be considered as a Stein domain in $U \times \mathbb{C}^{l_1+l_2}$ such that $W_i = (U \times \mathbb{C}^{l_i}) \cap W$ $(i = 1, 2)$. Since W is Stein and the maps $\Phi_{W_1} : W_1 \to S$ and $\Phi_{W_2} : W_2 \to S$ coincide with Φ_U on the intersection $W^1 \cap W^2 = U$, we can find a holomorphic map $\Phi_W : W \to S$ (up to some ambiguity) which coincides with Φ_{W_i} on each W_i (e.g. $p^*_{W_1}\Phi_{W_1} + p^*_{W_2}\Phi_{W_2} - p^*_U\Phi_U$). We shall denote again by f^1_j (resp. f^2_j) the j-th (resp. $l_1 + j$-th) component of the coordinate of $\mathbb{C}^{l_1+l_2}$.Then, $\mathbf{f} := \mathbf{f}_1 \cup \mathbf{f}_2$ forms a basis of the defining ideal $\mathcal{I}_U$ of $U \simeq U \times 0$ in W. Thus, we obtain a complete intersection based relative chart $(j : U \to W, \mathbf{f})$.

Let us show the existence of natural $\mathcal{O}_{U,z}$-isomorphisms:

$$\mathcal{H}^{p,s}_{W^i/S,\mathbf{f}^i,z_i} \simeq \mathcal{H}^{p,s}_{W/S,\mathbf{f},z} \qquad (***)$$

commuting with De-Rham action for $i = 1, 2$. We show only the $i = 1$ case (the other case follows similarly). For the end, we explicitly analyze the chain complex (**) (see Proof of Lemma 19 in p. 240) in a neighborhood of each point $z \in U$. Let $\underline{z} = (z^0, \cdots, z^n)$ be a local coordinate system of U at z so that $(\underline{z}, \mathbf{f})$ form a coordinate system of W at z. Let $\mathbf{t} = (t^1, \cdots, t^{\dim S})$ be a local coordinate system of S at the image of z, so that the morphism $\Phi_W : W \to S$ is expressed by the coordinates as $\mathbf{t} = \Phi_W(\underline{z}, \mathbf{f}^1, \mathbf{f}^2)$ so that $\Phi_{W_1} = \Phi_W(\underline{z}, \mathbf{f}^1, 0)$, $\Phi_{W_2} = \Phi_W(\underline{z}, 0, \mathbf{f}^2)$ and $\Phi_U = \Phi_W(\underline{z}, 0)$.

The fact that the restriction $\Phi_W|_{W_1} = \Phi_{W_1}$ is submersive over S implies that already a $\dim_{\mathbb{C}} S$-minor of the part of Jacobi matrix of $\Phi_W(\underline{z}, \mathbf{f}^1, \mathbf{f}^2)$ corresponding to the derivations by the coordinates z^j $(j = 0, \cdots, n)$ and $f^1_1, \cdots, f^1_{l_1}$ is invertible (in a neighborhood of z). Then, in the quotient module $\Omega^1_{W/S} = \Omega^1_W / \sum_{i=1}^{\dim_{\mathbb{C}} S} \mathcal{O}_W d\Phi_{W,i}$, the differentials $df^2_1, \cdots, df^2_{l_2}$ of the remaining coordinates $f^2_1, \cdots, f^2_{l_2}$ form part of an $\mathcal{O}_W$-free basis in a neighborhood of z. Then, again shrinking the charts W_i $(i = 1, 2)$ and W suitably, we can apply Lemma 20 repeatedly, and we obtain the $\mathcal{O}_U$-isomorphism (***).

To show the independence of De Rham operator from a choice of basis $\mathbf{f}$, we cannot use the complex (**) (there does not seem to exist a morphism d_{DR} : $(**)^p \to (**)^{p+1}$ which induces the De Rham operator: $\mathcal{H}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{H}^{p+1,s}_{W/S,\mathbf{f}}$. However, Lemma 20 (3) together with the naturality of $\iota^\diamond$ [Lemma 16 ii)] implies the compatibility of the De Rham operation with the isomorphism (***), and, hence, the independence from a choice of basis $\mathbf{f}$ of the De Rham operator on $\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}$.

Proof of Lemma 19 (2) It is sufficient to show that the stalk of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ at a point, say z, of U, where Φ_U is submersive, vanishes for $s > 0$. The assumption on the point z means that the Jacobi matrix of Φ_U with respect to the derivations by $z^0, \cdots, z^n$ has a non-vanishing minor at the point $z \in U$. So, in a neighborhood of z in W, the corresponding minor of the Jacobi matrix of Φ_W does not vanish. This means that $df_1, \cdots, df_l$ form a part of $\mathcal{O}_W$-free basis of $\Omega^1_{W/S}$. Then applying Lemma 20 inductively for a small neighborhood, we reduce to the relative chart of the form $j : U \to U$, and we conclude that $\mathcal{H}^{p,s}_{W/S,\mathbf{f},z}$ is quasi-isomorphic to a single module $\Omega^p_{U/S,z}$ at z. That is, $\mathcal{H}^{p,0}_{W/S,\mathbf{f},z} \simeq \Omega^p_{U/S,z}$ and $\mathcal{H}^{p,s}_{W/S,\mathbf{f},z} = 0$ for $s > 0$.

This completes the proof of Lemma 19. □

Notation As a consequence of Lemma 19, under the assumption that the relative chart $(j, \mathbf{f})$ is a complete intersection, the module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$, as an $\mathcal{O}_U$-module on U with De Rham differential operator, depends only on the morphism $\Phi_U : U \to S$ but not on $\mathbf{f}$. Therefore, we shall denote the module also by $\mathcal{H}^{p,s}_{\Phi_U}$ (see Lemma 22).

Remark 21

1. The support of the modules $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s > 0$ is contained in the critical set C_{Φ_U} (i.e., locally, we have $\mathcal{I}^m_{C_{\Phi_U}} \mathcal{H}^{p,s}_{W/S,\mathbf{f}} = 0$ for some positive integer m), does not imply that the module may not be an $\mathcal{O}_{C_{\Phi_U}}$-module.
2. In view of [17], $\mathcal{H}^{p,s}_{W/S,\mathbf{f}} = 0$ for $s < \mathrm{depth}(\mathcal{I}_{\Phi_U})$. But we do not use this fact in the present paper.

(E) The complex $(\mathcal{H}^{\bullet,\mathbf{s}}_{\mathbf{\Phi}}, \mathbf{d_{DR}})$ on Z.

As an important consequence of **(A)–(D)**, we introduce complexes $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ of $\mathcal{O}_Z$-coherent sheaves for $s \in \mathbb{Z}$.

Lemma 22 *Let $\Phi : Z \to S$ be a flat holomorphic map between complex manifolds and let C_Φ be its critical set loci as given in the Main Theorem. For $s \in \mathbb{Z}$, there exists a chain complex $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ of $\mathcal{O}_Z$-coherent modules such that, for any based relative chart $(j : U \to W, \mathbf{f})$, there is a natural isomorphism:*

$$(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})|_U \simeq (\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}, d_{DR}).$$

In particular, this implies

i) *For $s < 0$, $\mathcal{H}^{\bullet,s}_{\Phi} = 0$.*
ii) *For $s = 0$, there is a natural isomorphism:*

$$(\mathcal{H}^{\bullet,0}_{\Phi}, d_{DR}) \quad \simeq \quad (\Omega^{\bullet}_{Z/S}, d_{DR}).$$

iii) *For $s > 0$ and $p \in \mathbb{Z}$, we have*

$$\mathrm{Supp}(\mathcal{H}^{p,s}_{\Phi}) \quad \subset \quad C_\Phi.$$

Proof Let $(j : U \to W, \mathbf{f})$ be any based relative chart, which is a complete intersection. Applying the construction of (22), on the open subset U of Z, we obtain a sequence for $s \in \mathbb{Z}$ of complexes $(\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}, d_{DR})$ of $\mathcal{O}_Z$-coherent modules equipped with the De Rham operator action. Let $(j' : U' \to W', \mathbf{f}')$ be another complete intersection based relative chart, which introduces the complexes $(\mathcal{H}^{\bullet,s}_{W'/S,\mathbf{f}'}, d_{DR})$ on the open set U'. Then Lemma 19 together with (***) says that, on the intersection $U \cap U'$, they patch each other naturally so that we obtain the complexes of sheaves on $U \cup U'$. Obviously, Z is covered by charts which extends to complete intersection relative charts, there exists a global sheaf $\mathcal{H}^{p,s}_{\Phi}$ on Z together with the action of a De

Rham operator as stated. The statement i) follows from the definition (22) and the fact $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = 0$ for $s < 0$, ii) follows from (23), and iii) follows from Lemma 19 (2). □

Remark 23 As we see, the chain complexes $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ themselves are independent of the choices of relative charts. However, for its construction, we have used the relative charts. Can they be constructed without using the relative charts (or, without using Koszul-De Rham algebras)? (See the following Remark 24).

Remark 24 It is also possible to consider a quotient algebra $\overline{\mathcal{K}}_{W/S}$ of the Koszul-De Rham algebra $\mathcal{K}_{W/S,\mathbf{f}}$ as follows. Namely, suppose the defining ideal $\mathcal{I}_U$ of U in W has the following finite presentation.

$$\oplus\ \mathcal{O}_W^{l_1} \longrightarrow \oplus\ \mathcal{O}_W^{l_0} \longrightarrow \mathcal{I}_U \longrightarrow 0. \tag{26}$$

Explicitly, let $f_1, \cdots, f_{l_0} \in \Gamma(W, \mathcal{O}_W)$ be a system generators of $\mathcal{I}_U$ (i.e., the image of the basis of $\oplus\ \mathcal{O}_W^{l_0}$) and let $(g_j^1, \cdots, g_j^{l_0}) \in \Gamma(W, \mathcal{O}_W^{l_0})$ $(j = 1, \cdots, l_1)$ be a generating system of relations $g_j^1 f_1 + \cdots + g_j^{l_0} f_{l_0} = 0$ (i.e., the image of the basis of $\oplus\ \mathcal{O}_W^{l_1}$). Then, we define

$$\overline{\mathcal{K}}_{W/S} := \Omega^{\bullet}_{W/S}\langle \xi_1, \cdots, \xi_{l_0}\rangle[\eta_1, \cdots, \eta_{l_0}]/\mathcal{I} \tag{27}$$

where $\mathcal{I}$ is the both sided ideal generated by the relations (13) and

$$\begin{array}{ll} g_j^1 \xi_1 + \cdots + g_j^{l_0} \xi_{l_0} & (j = 1, \cdots, l_1) \\ g_j^1 \eta_1 + \cdots + g_j^{l_0} \eta_{l_0} + d g_j^1 \xi_1 + \cdots + d g_j^{l_0} \xi_{l_0} & (j = 1, \cdots, l_1). \end{array} \tag{28}$$

Then, as the notation indicates, the algebra (27) does not depend on a choice of the presentation (26) of the ideal $\mathcal{I}_U$. Furthermore, it is not hard to show that all the three structures Koszul differential ∂_K, De Rham differential d_{DR} and the bi-degree structure $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ are preserved on the quotient algebra $\overline{\mathcal{K}}^{p,s}_{W/S}$, and that a parallel statement of the functoriality Lemmas 12 and 13 hold, too. Then, for each fixed $p \in \mathbb{Z}$, we may also consider the cohomology of the $\partial_{\mathcal{K}}$. The following question has quite likely a positive answer.

Question Are the cohomology groups of $(\overline{\mathcal{K}}^{p,\star}_{W/S}, \partial_{\mathcal{K}})$ naturally isomorphic to those of $(\mathcal{K}^{p,\star}_{W/S}, \partial_{\mathcal{K}})$ (i.e., to the groups $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ $(s \in \mathbb{Z}_{ge0})$)?

Remark 25 In the present paper, we use the complexes $\mathcal{H}^{\bullet,s}_{\Phi}$ $(s \in \mathbb{Z}_{\geq 0})$ only as a supporting actor for the proof of the coherence of the relative De Rham cohomology group of Φ (see Case 3 of Sect. 5 **(D)**). But, for their definition, the condition that $\Phi|_{C_\Phi}$ is a proper map is unnecessary. Therefore, we may expect a wider use of the complexes in future.

5 Step 4: Lifting of Čech Cohomology Groups

In this section, we give a final step of a proof of the Main Theorem: the coherence of the direct image $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ in a neighborhood of any point $t \in S$ for a flat map $\Phi : Z \to S$ with a suitable boundary conditions.

We recall that at Fact 10 of Step 2, we showed that, for any point $t \in S$, there exists a Stein open neighborhood $S^* \subset S$ of t and a finite system of relative charts $\mathfrak{U} := \{j_k : U_k \to D_k(1) \times S^*\}_{k=0}^{k^*}$ and a real number $0 < r^* < 1$ such that the following holds:

1. the intersection relative chart j_K for $K \subset \{0, \cdots, k^*\}$ is complete intersection,
2. for any Stein open subset $S' \subset S^*$ and $r^* \leq \forall r \leq 1$, consider the atlas $\mathcal{U}(r, S') := \{U_k(r, S') := j_k^{-1}(D_k(r) \times S')\}_{k=0}^{k^*}$ (8) and the manifold $Z(r, S') := \cup_{k=0}^{k^*} U_k(r, S')$ covered by them. Then the direct image $\mathbb{R}\Phi(\Omega^\bullet_{Z(r,S')/S'})$ are isomorphic to each other for r in $r^* \leq r \leq 1$.

The plan of the proof is the following.

- **(A)** We express the Hodge to De Rham spectral sequence over any Stein open subset $S' \subset S^*$ in terms of Čech cohomology groups with coefficients in $\Omega^\bullet_{Z/S}$ with respect to the atlas $\mathcal{U}(r, S')$ (8).
- **(B)** We "lift" the Čech complex to the lifted atlas $\mathfrak{U}(r, S')$ (9) of relative charts. To be exact, in order to lift the coefficient to Koszul-De Rham algebra $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$, we need to enhance the atlas to a based lifted atlas $\tilde{\mathfrak{U}}(r, S')$. The existence of such enhancement shown in Lemma 27 is a quite non-trivial step in the proof.
- **(C)** We compare the Čech complex of $\Omega^\bullet_{Z/S}$ with that of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ by the morphism π (16) and obtain a short exact sequence where the third term is described again by a Čech complex with respect to the atlas $\mathcal{U}(r, S')$ and coefficient in the sheaf $\mathcal{H}^{\bullet,*}$ whose support is contained in the critical set C_Φ.
- **(D)** In the long exact sequence of cohomology groups of the above three Čech complexes, two terms (namely, the first and the third) are independent of the radius r. So the cohomology groups of the third, i.e. of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$, are also independent of r.
- **(E)** We apply the Forster–Knorr Lemma (see [5] and also Lemma 30 of the present paper) to the Čech cohomology groups of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ and see that they give coherent direct image sheaves on a neighborhood S_m of $t \in S$. On the other hand, the third term (the cohomology of $\mathcal{H}^{\bullet,*}$) is already coherent on S_m since C_Φ is proper over S. Thus, the remaining term in the long exact sequence of the cohomologies, that is, the direct images of the relative De Rham complex are also coherent on S_m.

We start the proof now.

(A) Čech complex

We consider the Čech chain complex of the relative De Rham complex $\Omega^p_{Z(r,S')/S'}$ with respect to a Stein covering $\mathcal{U}(r,S') := \{U_k(r,S')\}_{k=0}^{k^*}$ (8) of $Z(r,S')$ over any Stein open subset $S' \subset S^*$. As usual, the qth cochain module ($q \in \mathbb{Z}$) is given by

$$\check{C}^q(\mathcal{U}(r,S'), \Omega^p_{Z(r,S')/S'}) := \bigoplus_{\substack{K \subset \{0,\cdots,k^*\} \\ \#K=q+1}} \Gamma(U_K(r,S'), \Omega^p_{Z'(r,S')/S'}). \tag{29}$$

(where the summation index K runs also over the cases when $U_K(r,S') = \emptyset$). The Čech coboundary operator is the alternating sum

$$\check{\delta} := \sum_{K \subset K'} \pm(\rho_K^{K'})^* : \check{C}^q(\mathcal{U}(r,S'), \Omega^p_{Z(r,S')/S'}) \to \check{C}^{q+1}(\mathcal{U}(r,S'), \Omega^p_{Z(r,S')/S'}) \tag{30}$$

of pull-back morphisms associated with the inclusion map $\rho_K^{K'} : U_{K'} \to U_K$, where K and $K' \subset \{0,\cdots,k^*\}$ are indices satisfying $\#K = q+1$, $\#K' = q+2$ and $K \subset K'$.

(B) Based lifting atlas

Recall the lifting atlas $\mathfrak{U}(r,S') := \{j_k|_{U_k(r,S')} : U_k(r,S') \to D_k(r) \times S'\}_{k=0}^{k^*}$ (9) of the atlas $\mathcal{U}(r,S')$ (8), where each j_k is a pair (φ_k, Φ) of maps such that φ_k is a local isomorphism of a neighborhood $B(\underline{z}_k, R(\underline{z}))$ of $\underline{z}_k \in Z$ to a domain in $\mathbb{C}^N$. We attach one more structure, i.e. base (recall Definition 11), to the atlas $\mathfrak{U}(r,S')$ as in the following definition.

Definition 26 A *based lifting atlas* of $\mathcal{U}(r,S')$ is a triplet

$$\widetilde{\mathfrak{U}}(r,S') := (\mathfrak{U}(r,S'), \mathbf{f}_K, \mathbf{\Pi}_K^{K'})$$

such that

1) $\mathfrak{U}(r,S') = \{j_k \mid 0 \le k \le k^*\}$ is the relative atlas already given in (9),
2) $\mathbf{f}_K$ is a minimal generator system of the ideal $\mathcal{I}_{U_K(r,S*)}$ for $K \subset \{0,1,\cdots,k^*\}$. That is, $(j_K, \mathbf{f}_K)$ is a based relative chart in the sense of Definition 11.
3) $\mathbf{\Pi}_K^{K'}$ is a based morphism: $(j_{K'}, \mathbf{f}_{\mathbf{K}'}) \to (j_K, \mathbf{f}_{\mathbf{K}})$ (in the sense of Lemma 12) for $K, K' \subset \{0,1,\cdots,k^*\}$ with $K \subset K'$ such that

$$\mathbf{\Pi}_K^{K''} = \mathbf{\Pi}_K^{K'} \circ \mathbf{\Pi}_{K'}^{K''}$$

for any $K, K', K'' \subset \{0,1,\cdots,k^*\}$ with $K \subset K' \subset K''$.

We remark that any based relative chart in a based lifting atlas is automatically a complete intersection in the sense of Definition 18. The following existence of based lifting atlases is one crucial step towards the proof of the Main Theorem.

Lemma 27 *For the atlas* $\mathfrak{U}(r,S')$*, there exists a based lifting* $\widetilde{\mathfrak{U}}(r,S')$.

Proof We construct the based lifting explicitly in the following 1)–3).

1) Recall the notation of the proof of Lemma 8. For any subset $K \subset \{0, 1, \cdots, k^*\}$, we have $j_K : \underline{z}' \in U_K \mapsto ((\varphi_{z_i}(\underline{z}'))_{i=0}^{k^*}, \Phi(\underline{z}')) \in \prod_{i=0}^{k^*} D_{z_i}(r_{z_i}) \times S_K$.
2) As was suggested already by 1) and 2) in the proof of Lemma 8 (page 228), we choose $\mathbf{f}_K$ as follows.

   1. If $U_K = \emptyset$, then we set $\mathbf{f}_K = \{\ 1\ \}$.
   2. If $U_K \neq \emptyset$, then $\mathbf{f}_K$ is the union of two parts $\mathbf{f}_{K,I}$ and $\mathbf{f}_{K,II}$ where

$$\mathbf{f}_{K,I} = \left\{ z^j \circ \varphi_k^{-1} - z^j \circ \varphi_{k'}^{-1} \right\}_{j=1,k,k' \in K}^{N} \ \ \& \ \ \mathbf{f}_{K,II} = \left\{ t_i - \Phi_i \circ \varphi_{k_0}^{-1} \right\}_{i=1}^{\dim_{\mathbb{C}} S},$$

   where k' =the least element of K which is larger than k, and $k_0 = \min\{K\}$.

3) Let $K, K' \subset \{0, 1, \cdots, k^*\}$ such that $K \subset K'$. We construct a morphism $\Pi_K^{K'} = (\pi_K^{K'}, h_K^{K'}) : (j_{K'}, \mathbf{f}_{K'}) \to (j_K, \mathbf{f}_K)$. As a map $\pi_K^{K'}$ from the relative chart $j_{K'}$ to j_K, we consider the pair consisting of natural projection: $\pi_K^{K'} : D_{K'}(r) \times S' \to D_K(r) \times S'$ and the natural (induced) inclusion: $U_{K'} \to U_K$.

   Let us choose and fix a morphism $h_K^{K'}$ between two basis $\mathbf{f}_K$ and $\mathbf{f}_{K'}$.
   In case $U_{K'} = \emptyset$, $\mathbf{f}_{K'} = \{1\}$ and we set $h_K^1 = \mathbf{f}_K$.
   In case $U_{K'} \neq \emptyset$, then, according to the two groups of basis of $\mathbf{f}_K$ and $\mathbf{f}_{K'}$ in the above (1), we decompose the matrix $h_K^{K'}$ into 4 blocks $\begin{pmatrix} h_{K,I}^{K',I}, & h_{K,I}^{K',II} \\ h_{K,II}^{K',I}, & h_{K,II}^{K',II} \end{pmatrix}$, and fix the morphism blockwise in the following steps 1, 2, and 3

1. There is a unique way to express any element of $\mathbf{f}_{K,I}$ as a sum of elements of $\mathbf{f}_{K',I}$, then $h_{K,I}^{K',I}$ is its coefficients matrix. Thus we get: $\mathbf{f}_{K,I} = h_{K,I}^{K',I} \mathbf{f}_{K',I}$.
2. We put $h_{K,I}^{K',II} = 0$.
3. We express $\mathbf{f}_{K,II} = h_{K,II}^{K',I} \mathbf{f}_{K',I} + h_{K,II}^{K',II} \mathbf{f}_{K',II}$, where $h_{K,II}^{K',II}$ is the identity matrix of size $\dim_{\mathbb{C}} S$. In order to fix the part $h_{K,II}^{K',I}$, we prepare some functions.

For each i with $1 \leq i \leq \dim_{\mathbb{C}} S$, we express, locally in a Stein coordinate neighborhood, Φ_i (the ith component of the map Φ) as a function $\Phi_i(\underline{z})$ of N variables $\underline{z} = (z^1, \cdots, z^N)$. Consider a copy $\Phi_i(\underline{z}')$ of the function for a coordinate system $\underline{z}' = (z'^1, \cdots, z'^N)$. Then, on the product domain of the coordinate neighborhood, we can find functions $F_{ij}(\underline{z}, \underline{z}')$ $(j = 1, \cdots, N)$ such that

$$\Phi_i(\underline{z}') - \Phi_i(\underline{z}) \quad = \quad \sum_{j=1}^{N} F_{ij}(\underline{z}', \underline{z})(z'^j - z^j), \tag{31}$$

since the product domain is Stein where the ideal defining the diagonal is globally generated by $z'^j - z^j$ $(j = 1, \cdots, N)$. Then, again taking a copy $\Phi_i(\underline{z}'')$ on the triple product domain and summing up two copies of above formula, we obtain a formula

$$\sum_{j=1}^{N} F_{ij}(\underline{z}'', \underline{z})(z''^j - z^j) = \sum_{j=1}^{N} F_{ij}(\underline{z}', \underline{z})(z'^j - z^j) + \sum_{j=1}^{N} F_{ij}(\underline{z}'', \underline{z})(z''^j - z'^j). \qquad (32)$$

We return to the construction of the matrix $h^{K',I}_{K,II}$. That is, we need to express the difference: $(t_i - \Phi_i \circ \varphi_{k_0}^{-1}) - (t_i - \Phi_i \circ \varphi_{k_0'}^{-1}) = \Phi_i \circ \varphi_{k_0'}^{-1} - \Phi_i \circ \varphi_{k_0}^{-1}$ as a linear combination of $z^j \circ \varphi_{k_0'}^{-1} - z^j \circ \varphi_{k_0}^{-1}$. The formula (31) gives an answer:

$$\Phi_i \circ \varphi_{k_0'}^{-1} - \Phi_i \circ \varphi_{k_0}^{-1} \quad = \quad \sum_{j=1}^{N} F_{ij}(\varphi_{k_0'}^{-1}, \varphi_{k_0}^{-1})(z^j \circ \varphi_{k_0'}^{-1} - z^j \circ \varphi_{k_0}^{-1}),$$

and we obtain the definition: $h^{K',I}_{K,II} = \{F_{ij}(\varphi_{k_0'}^{-1}, \varphi_{k_0}^{-1})\}_{i=1,\cdots,\dim_{\mathbb{C}} S,\, j=1,\cdots,N}$.

Finally, we need to show that the above defined matrix satisfies the functoriality $h_K^{K''} = h_K^{K'} h_{K'}^{K''}$. We can prove this again by decomposing the matrix into 4 blocks, where the cases of the blocks $\begin{pmatrix} I \\ I \end{pmatrix}$, $\begin{pmatrix} II \\ I \end{pmatrix}$ and $\begin{pmatrix} II \\ II \end{pmatrix}$ are trivial. The case of block $\begin{pmatrix} I \\ II \end{pmatrix}$ follows from the addition formula (32).

This completes the proof of an existence of based lifting of the atlas $\mathcal{U}(r, S')$. □

Remark 28 The above construction does not give a canonical lifting, but depends on the choices of the decomposition (31) which is based on rather an abstract existence theorem (cf. [7]). We don't know the meaning of this freedom to the De Rham cohomology group we are studying. As we see in sequel, for the proof of coherence, any choice of the lifting does work. See also Remark 23.

From now on, we consider the base lifted atlas $\widetilde{\mathfrak{U}}(r, S')$ for all $r^* \le r \le 1$ and Stein open subset $S' \subset S^*$, depending on a choice of functions F_{ij} in (31). Since we, later on, want to compare them for different r and S', we first fix the functions F_{ij} and hence a based lifting $(\mathbf{f}_K, \mathbf{\Pi}_K^{K'})$ on the largest atlas $\mathfrak{U}(1, S^*)$, then we consider the induced based lifting to any atlas $\mathfrak{U}(r, S')$. We lift the Čech (co)chain complex (29) to the following triple chain complex. Namely, for $p, q, s \in \mathbb{Z}_{\ge 0}$, we define the cochain module

$$\check{C}^q(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}_\Phi^{p,s}) := \bigoplus_{\substack{K \subset \{0,\cdots,k^*\} \\ \#K = q+1}} \Gamma(D_K(r) \times S', \mathcal{K}^{p,s}_{D_K(r)\times S'/S', \mathbf{f}_K}).^{13} \qquad (33)$$

[13] In the notation of LHS, we replaced the subscript like $D_K(r) \times S/S$ indicating where the module is defined by Φ, since we may regard $\mathcal{K}_\Phi^{p,s}$ to be a sheaf satisfying the functoriality (Lemma 12) defined on all relative charts, depending on the choice of a based lifting in Lemma 27.

The actions of (co-)boundary operators d_{DR} and $\partial_{\mathcal{K}}$ on the coefficient $\mathcal{K}_\Phi^{\bullet,\star}$ preserve the chart, so that they induce a double complex structure $(\check{C}^q(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{\bullet,\star}), d_{DR}, d_{\mathcal{K}})$. We now lift the Čech coboundary operator on (29) to the lifted module (33).

For $p,q,s \in \mathbb{Z}$, we introduce an $\Gamma(S',\mathcal{O}_S)$-*homomorphism*

$$\check{\delta} := \sum_{K\subset K'} \pm(\Pi_K^{K'})^\diamond \ : \ \check{C}^q(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{p,s}) \longrightarrow \check{C}^{q+1}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{p,s}), \tag{34}$$

where $(\Pi_K^{K'})^\diamond$ is the pull-back morphism (15) in Lemma 12 associated with the morphism $\Pi_K^{K'} = (\pi_K^{K'}, h_K^{K'})$ given in 3) of the proof of Lemma 27, and the sign and the running index K and K' are the same as those for the Čech coboundary operator (30). We shall call this morphism the *lifted Čech coboundary operator*. The lifted Čech coboundary operator satisfies the relations:

$$\check{\delta}^2 = 0, \quad \check{\delta}d_{DR} + d_{DR}\check{\delta} = 0, \quad \text{and} \quad \check{\delta}\partial_{\mathcal{K}} + \partial_{\mathcal{K}}\check{\delta} = 0.$$

Proof To show that $\check{\delta}^2 = 0$ is the same calculation as the standard Čech coboundary case. Other relations follow from the fact that the pull-back homomorphism $(\pi_K^{K'})^\diamond$ commutes with d_{DR} and $\partial_{\mathcal{K}}$ (Lemma 12). □

(C) Comparison of the triple Čech-complex of $\mathcal{K}_\Phi^{\bullet,\star}$ with the double Čech-complex of $\Omega_\Phi^\bullet$

We compare the triple complex (33) with the double complex (29). More exactly, for our restricted purpose (to calculate the second page of the Hodge to De Rham spectral sequence), we fix the index p for the chain complex for De Rham differential operator. That is, we compare only the remaining double complex of the two coboundary operators $(\check{\delta}, \partial_{\mathcal{K}})$ with the Čech (co)chain complex of the coboundary operator $\check{\delta}$. The comparison is achieved by the morphism π (recall Sect. 4 **(B)**).

$$\cdots \xrightarrow{\partial_{\mathcal{K}}} \check{C}^q(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{p,1}) \xrightarrow{\partial_{\mathcal{K}}} \check{C}^q(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{p,0}) \xrightarrow{\pi} \check{C}^q(\mathcal{U},\Omega^p_{Z(r,S')/S'}) \to 0.$$

The commutativity of π with the lifted and un-lifted Čech coboundary operator is, termwise, equivalent to the commutativity $\rho_{U_{K'}}^{U_K} \circ \pi = \pi' \circ (\pi_K^{K'})^\diamond$ (18).

Let us consider the total complex of (33) with respect to $\check{\delta}$ and $\partial_{\mathcal{K}}$ by putting $\tilde{*} := * - \star$ and $\tilde{\partial} := \check{\delta} + \partial_{\mathcal{K}}$:

$$(Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_\Phi^{p,\cdot}),\tilde{\partial}) \tag{35}$$

where

$$Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\cdot}) := \underset{*-\star=\tilde{*}}{\oplus} \check{C}^{*}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\star}). \tag{36}$$

In view of (16), for each fixed $p \in \mathbb{Z}$, the chain morphism

$$(Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\cdot}),\tilde{\partial}) \xrightarrow{\pi} (\check{C}^{*}(\mathcal{U},\Omega^{p}_{Z(r,S')/S'}),\check{\delta}) \tag{37}$$

is an epimorphism in the category of cochain complexes. So, using the kernel of it, we obtain a short exact sequence:

$$\begin{aligned} 0 \to (Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi,\ker(\pi)}^{p,\cdot}),\tilde{\partial}) &\xrightarrow{\iota} (Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\cdot}),\tilde{\partial}) \\ &\xrightarrow{\pi} (\check{C}^{*}(\mathcal{U},\Omega^{p}_{Z(r,S')/S'}),\check{\delta}) \to 0, \end{aligned} \tag{38}$$

where the kernel (the first term) is again the total complex of a lifted Čech chain complex of the atlas $\widetilde{\mathfrak{U}}(r,S')$ with coefficients in a complex $\mathcal{K}_{\Phi,\ker(\pi)}^{p,s}$ (for fixed p), which is the sub-complex of $\mathcal{K}_{\Phi}^{p,s}$ obtainded by replacing the first term $\mathcal{K}_{\Phi}^{p,0}$ by the term $\partial_{\mathcal{K}}(\mathcal{K}_{\Phi}^{p,1}) = \ker(\pi : \mathcal{K}_{\Phi}^{p,0} \to \Omega_{\Phi}^{p})$, and ι is the map induced from the natural inclusion $\mathcal{K}_{\Phi,\ker(\pi)}^{p,s} \subset \mathcal{K}_{\Phi}^{p,s}$.

Due to the commutativity of π with the De Rham differential operator (17), the chain maps (37) commute with De Rham operator action between the modules for the indices p and $p+1$. That is, by taking the direct sum over the index $p \in \mathbb{Z}$, we may regard π as an epimorphism from the double complex of $(\tilde{\partial}, d_{DR})$ to the double complex of $(\check{\delta}, d_{Z/S})$. Then, similarly, by taking the direct sum of the sequences (38) over the index $p \in \mathbb{Z}$, we obtain a short exact sequence of double complexes.

Before calculating cohomology long exact sequence of the short exact sequence, in the following Lemma we show some *finiteness* and *boundedness* of the total complex (35) (considered as a double complex of the indices $\tilde{*}$ and p), which makes big contrast with the case of Lemma 19. Namely, in case of the total complex of $\partial_{\mathcal{K}}$ and d_{DR}, we did not get such finiteness and boundedness (see Remark 15). This finiteness, which holds for the total complex of ∂_K and $\check{\delta}$, is one of the most subtle but the key point where Koszul-De Rham algebra works mysteriously.

Lemma 29 *The complex* (35) *is finite and bounded in the following two senses.*

i) The RHS of (36) *for fixed p and* $\tilde{*}$ *is a finite direct sum of the form*

$$\bigoplus_{q=-1}^{k^*-1} \check{C}^{q}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,q-\tilde{*}}) = \bigoplus_{K\subset\{0,\cdots,k^*\}} \Gamma(D_K(r)\times S', K^{p,\#K-\tilde{*}-1}_{D_K(r)\times S'/S',\mathbf{f}_K}).$$

ii) The set $\{(p,\tilde{*}) \in \mathbb{Z}^2 \mid Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\cdot}) \neq 0\}$ *is contained in a strip*

$$-(k^*+1)(N-1)+n-1 \le \tilde{*}+p \le (k^*+1)(N+1)-1. \tag{39}$$

Proof

i) The summation index $K \subset \{0, \cdots, k^*\}$ (33) runs over a finite set so that $* = \#K - 1$ is bounded. Then the condition that $* - \star = \tilde{*}$ is fixed means that the range of $\star$ is bounded.

ii) Recall $Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}_{\Phi}^{p,\cdot}) := \bigoplus_{*-\star=\tilde{*}} \bigoplus_{\#K=*+1} \Gamma(D_K(r) \times S', \mathcal{K}^{p,\star}_{D_K(r)\times S'/S'})$. If there is a non-vanishing term in RHS for some $*$, $\star$, and K, then due to (20), one has $-l_K \le p - \star \le \dim_{\mathbb{C}} D_K(r)$. Then, adding $* = \star + \tilde{*}$ in both hand sides, we have $-l_K + * \le \tilde{*} + p \le \dim_{\mathbb{C}} D_K(r) + *$. Since $* = \#K - 1$ and $l_K = \#K \cdot \dim_{\mathbb{C}} S + (\#K - 1)n$, $\dim_{\mathbb{C}} D_K(r) = \#K \cdot N$ (recall Sect. 3) and $\dim_{\mathbb{C}} Z = N = m = n + \dim_{\mathbb{C}} S$, we get

$$-\#K(N-1) + n - 1 \ \le\ \tilde{*} + p \ \le\ \#K(N+1) - 1$$

Since the index K runs over all subsets of $\{0, 1, \ldots, k^*\}$, we obtain the formula. □

According to i) and ii) of Lemma 29, we have two important consequences: (1) the cohomology groups is described by a finite chain complex where each chain module is a finite direct sum of the spaces of holomorphic functions on some relative charts (this description is necessary to apply the Forster–Knorr Lemma), and (2) for each fixed p, the complex is bounded. This observation leads us to introduce the following truncation of the double complexes.

$$\begin{aligned} TR^{p,\tilde{*}} &:= \begin{cases} Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}_{\Phi}^{p,\cdot}) & \text{if } 0 \le p \le \dim_{\mathbb{C}} Z \\ 0 & \text{otherwise} \end{cases} \\ TR^{p,\tilde{*}}_{\ker(\pi)} &:= \begin{cases} Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}_{\Phi,\ker(\pi)}^{p,\cdot}) & \text{if } 0 \le p \le \dim_{\mathbb{C}} Z \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{40}$$

For the truncated double complexes, we have

i) The complexes $TR^{p,\tilde{*}}$ and $TR^{p,\tilde{*}}_{\ker(\pi)}$ are bounded for the both indices p and $\tilde{*}$, and also from above and below.

ii) The following is an exact sequence of bounded double complexes:

$$\begin{aligned} 0 \to (TR^{p,\tilde{*}}_{\ker(\pi)}, d_{DR}, \tilde{\partial}) &\stackrel{\iota}{\longrightarrow} (TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})^{\pi} \\ &\longrightarrow (\check{C}^{*}(\mathcal{U}, \Omega^{p}_{Z(r,S')/S'}), d_{Z/W}, \check{\delta}) \ \to\ 0, \end{aligned} \tag{41}$$

which is the goal of our construction. From now on, we start to analyze the sequence.

(D) Long exact sequence of images on S.

We consider now the long exact sequence of the cohomology group associated with the short exact sequence obtained from (41) by taking the total complex for each of the three double complexes. Recalling the construction of the atlases $\mathcal{U}(r, S')$ (8) and $\mathfrak{U}(r, S')$ (9), we note that each term of the sequence depends on the choice of a set S' and a real number r with $r^* \le r \le 1$. By fixing r and running S' over all Stein open subset of S^*, we obtain a sheaf on S^* (depending on r).

In the following, we analyze the module (sheaf or its sections over S') of the cohomology groups with the three coefficients cases separately.

Case 1. $(\check{C}^*(\mathcal{U}, \Omega^p_{Z(r,S')/S'}), d_{Z/W}, \check{\delta})$.

The module is exactly the module of relative De Rham hyper-cohomology group $\mathbb{R}\Phi_*(\Omega^\bullet_{Z_{S^*}/S^*}, d_{Z_{S^*}/S^*})$ (7) of the morphism $\Phi : Z(r) \to S^*$. It is shown that the module is independent of the choice of r with $r^* \le r \le 1$.

Case 2. $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$.

Due to the finiteness Lemma 29 i) and the boundedness of the double complex, the cohomology group is expressed as a cohomology group of a finite complex, where each chain module is a finite direct sum of a module of the form $\Gamma(D(r) \times S', \mathcal{O}_{D(r)\times S'})$ for some polydisc $D(r)$ of radius r.

Proof Recall the direct sum decomposition (19). Noting that W/S is given by $D_K(r) \times S'/S'$ and we obtain $\Omega^p_{W/S} = \oplus_{i_1<\dots<i_p} \mathcal{O}_{D_K(r)\times S'} dz_{i_1} \wedge \dots \wedge dz_{i_p}$ for a coordinate system $\underline{z}$ of the polydisc $D_K(r)$. □

Case 3. $(TR^{p,\tilde{*}}_{\ker(\pi)}, d_{DR}, \tilde{\partial})$.

We approach the cohomology group of this case by the use of the spectral sequence of the double complex w.r.t. d_{DR} and $\tilde{\partial}$. Let us first calculate the cohomology group of the double complex with respect to the coboundary operator $\tilde{\partial}$ (in order to avoid a confusion, let us call the spectral sequence E_I). Thus, each entry of the first page of the spectral sequence E_I is again the total cohomology group of the total complex $(Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r, S'), \mathcal{K}^{\bullet,\cdot}_{\Phi,\ker(\pi)}), \tilde{\partial} = \check{\delta} + \partial_K)$. Again we approach the group from the spectral sequence of the double complex w.r.t. $\check{\delta}$ and ∂_K.

Let us first consider the spectral sequence obtained by considering the cohomology group with respect to the coboundary operator ∂_K first, for the reason below (let us call this spectral sequence E_{II}).

Recall Lemma 22 that it was shown that there exists a sequence of complexes $\mathcal{H}^{\bullet,s}_\Phi$ ($s \in \mathbb{Z}$) of coherent $\mathcal{O}_Z$–modules such that (1) the restriction of the s-th complex to U_W induces a natural isomorphism to the s-th cohomology group of the Koszul-De Rham double complex $(\mathcal{K}^{\bullet,\star}_{W/S}, d_{DR}, \partial_K)$ with respect to the coboundary operator ∂_K, and (2) the support of the module for $s > 0$ is contained in the critical set C_Φ. Thus, the (q, s)-entries of E_{II} is given by direct images $\check{C}^q(\widetilde{\mathfrak{U}}(r, S'), \mathcal{H}^{\bullet,s}_\Phi)$ of coherent sheaves $\mathcal{H}^{\bullet,s}_\Phi$ (the fact that the pair d_{DR} and $\check{\delta}$ forms a double complex structure on $\oplus_{p,q}\check{C}^q(\mathcal{U}(r, S'), \mathcal{H}^{p,s}_\Phi)$ is verified by a routine). In view of the fact that $C_\Phi \subset Z'$ is proper over the base space S, this, in particular, implies that (1) the entry is independent of r, and (2) the sheaf obtained by running S' over all Stein open subset of S^* is an $\mathcal{O}_{S^*}$-coherent module. Then, these two properties should be inherited by the limit of the spectral sequence E_{II} and the associated total cohomology group.

Coming back to the spectral sequence E_I, we see that all the entries of the first page of E_I have the above properties (1) and (2). Thus the cohomology group of the total complex of the double complex $(TR^{p,\tilde{*}}_{\ker(\pi)}, d_{DR}, \tilde{\partial})$ should have the property. Then in view of the long exact sequence, we started, two terms Case 1 and 3 of them (as a triangle) are independent of r. Thus, we conclude that the third term Case 2 satisfies:

The total complex of the double complex $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$ *is quasi-isomorphic to each other for* r *and* r' *with* $r^* \leq r, r' \leq 1$.

(E) Application of the Forster–Knorr Lemma.

We are now able to apply the following key Lemma due to Forster and Knorr [5, 13] (the formulation here of the result is taken from their unpublished note which is slightly modified from the published one, however can be deduced).

Lemma 30 (Forster–Knorr) *Let* m *be a given integer,* S *a smooth complex manifold,* 0 *a point in* S. *Suppose that* $(C^*(r), d)$ *is a complex of* $\mathcal{O}_S$*-modules bounded from the left such that*

i) for any Stein open subset $S' \subset S$ *and* $q \in \mathbb{Z}$, *we have an isomorphism*

$$C^q(r)(S') \simeq \prod_{\textit{finite}} \Gamma(D(r) \times S', \mathcal{O}_{D(r)\times S'})$$

together with the Fréchet topology. Here, $D(r)$ *is a polycylinder of radius* $r \in \mathbb{R}_{>0}$ *whose dimension varies depending on each factor.*

ii) $d : C^q(r) \to C^{q+1}(r)$ *is an* $\mathcal{O}_S$*-homomorphism, which is continuous with respect to the Fréchet topology.*

iii) There exist r_1 *and* r_2 *such that, for any* r, $r_1 \geq r \geq r' \geq r_2 > 0$, *the restriction* $C^*(r) \to C^*(r')$ *is a quasi-isomorphism.*

Then, there exists a small neighborhood S_m *of 0 in* S *(depending on* $m \in \mathbb{Z}$*) such that, for* $q \geq m$, $\mathrm{H}^q(C^*(r))\big|_{S_m}$ *is an* $\mathcal{O}_{S_m}$*-coherent module.*

We apply Lemma 30 to the total complex of complex $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$.

Let us check that the complex satisfies the assumptions in the Forster–Knorr Lemma by putting $S = S^*$ (and run S' over all Stein open subset of S^* in order to make $\mathcal{O}_{S^*}$-module structure), 0 to be $t \in S^*$ and $r_1 = 1,\ r_2 = r^*$.

i) The condition i) is satisfied due to the description in **(D), Case** $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$.

ii) The condition ii) is verified as follows. The coboundary operator here is a mixture of ∂_K, $\check{\delta}$, and d_{DR}, all of them are obviously $\mathcal{O}_S$-homomorphisms. That they are continuous w.r.t. the Fréchet topology can be seen as follows.

It is well known that the holomorphic function ring (Stein algebra) $\Gamma(D_K(r) \times S', \mathcal{O}_{D_K(1)\times S^*})$ carries naturally a Fréchet topology [3, 7, p. 266] with respect to the compact open convergence. Then, the operators $\check{\delta}$ and $\partial_{\mathcal{K}}$ are $\mathcal{O}_{D_K(1)\times S^*}$-

homomorphisms and induce continuous morphisms on the modules. The operator d_{DR} is no longer an $\mathcal{O}_{D_K(1)\times S^*}$-homomorphism but is only an $\mathcal{O}_{S^*}$-homomorphism. Nevertheless, it is also well known that differentiation operators on a Stein algebra are also continuous w.r.t. the Fréchet topology.

iii) The quasi-isomorphisms between the complexes for r and r' with $r* \leq r, r' \leq 1$ were shown in the last step of **(D)**.

Finally, choosing $m = -1$, we obtain the coherence of the direct image sheaf of the total complex of $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$ in a neighborhood of $t \in S^*$. Then, we return to the long exact sequence studied in **(D)**. Two terms Case 2 and 3 of them (as a triangle) are $\mathcal{O}_S$-coherent near at $t \in S$. Therefore, the third term Case 3, the direct image of the double complex, that is, *the hyper-cohomology group* $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ *is also* $\mathcal{O}_S$*-coherent in a neighborhood of* $t \in S$.

This completes the proof of the Main Theorem given in Sect. 1. □

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Remarks on the Topology of Real and Complex Analytic Map-Germs

José Seade

Dedicated to Gert-Martin Greuel in his 70th Birthday Anniversary

Abstract We study the topology of analytic map-germs $X^n \xrightarrow{f} K^p$, $n > p$, near an isolated singularity, where K is either $\mathbb{R}$ or $\mathbb{C}$ and X is (accordingly) real or complex analytic. We do it in a way, now classical, that springs from work by Gert-Martin Greuel and Lê Dũng Tráng and somehow goes back to Lefschetz, namely by comparing the topology of the fibres of f with that of the functions one gets by dropping one of the components of the map-germ f.

Keywords Index • Lê-Greuel formula • Milnor fibre and number • Stratified vector fields

1 Introduction

A basic problem is studying the behaviour and topology of the local non-critical levels (in some appropriate sense) of a real analytic map-germ $X^n \to \mathbb{R}^p$, $n > p$, near a critical point.

When we focus on complex analytic map-germs $(X^n, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ with singular point at 0 with respect to some Whitney stratification, we have the Milnor-Lê fibration theorem: Assume the germ X is embedded in some affine space $\mathbb{C}^m$ equipped with the usual Hermitian metric. If $\varepsilon \gg \delta > 0$ is sufficiently small and we set $N(\varepsilon, \delta) = f^{-1}(\partial \mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$, then:

$$f : N(\varepsilon, \delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1,$$

J. Seade (✉)
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación Científica, Circuito exterior Ciudad Universitaria, 04510 México D.F., Mexico
e-mail: jseade@matcuer.unam.mx

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is a locally trivial fibration. The variety $N(\varepsilon, \delta)$ is usually called *a Milnor tube* for f and a typical fibre $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ is known as the *Milnor fibre* F_f. Of course it is important to know more about the topology of the fibre F_f; this is a classical subject with a vast literature.

In particular (see [21, 26]), if $(\mathbb{C}^{m+p}, \underline{0}) \xrightarrow{f} (\mathbb{C}^p, 0)$ is an isolated complete intersection singularity germ (an ICIS for short), one has a local fibration *à la* Milnor, and a well-defined Milnor number: The rank of the middle homology of the Milnor fibre. Since in the hypersurface case, where $p = 1$, this important invariant coincides with the intersection number

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{m+1,\underline{0}}}{\left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{m+1}}\right)},$$

it was natural to search for an algebraic expression for it in the case of ICIS: that is the celebrated Lê-Greuel formula for the Milnor number, proved independently by Lê [24] and Greuel [20]. This expresses the Milnor number of an ICIS determined by certain functions in terms of the Milnor number of the ICIS we get by dropping one of the defining functions, together with a certain contribution from that coordinate function (see Remark 6.1 below).

The Lê-Greuel formula was extended in [5] to holomorphic map-germs $(X^{n+k}, \underline{0}) \xrightarrow{f} (\mathbb{C}^k, 0)$ with X singular (cf. [12]), and in [9] to the real analytic setting with X smooth (cf. [11]). The articles [29, 30] are also along the same spirit we consider here. Both papers [5] and [9] are somehow reminiscents of [2, 3] and make a heavy use of the theory of indices of vector fields on singular varieties. A key point in [5] was extending the notion of the GSV-index (see [4, 18]) to the case of stratified vector fields on hypersurfaces in varieties with non-isolated singularities. In this article we describe the main ideas in [9] and [5] in a parallel way, thus highlighting the differences between the real and the complex case.

The paper is arranged as follows. Section 2 discusses the existence of Milnor type fibrations for real and complex singularities. In Sect. 3 we look at the radial and GSV indices of vector fields on singular varieties. These are the two types of indices that are most relevant for the sequel. Section 4 explains the main ideas in the generalization in [5] of the Lê-Greuel formula to holomorphic map-germs $(X^{n+k}, \underline{0}) \xrightarrow{f} (\mathbb{C}^k, 0)$ with X singular. The analogous considerations are done in Sect. 5 for the real case, following [9]. In Sect. 6 we give some concluding remarks and possible lines of further research.

2 Milnor-Lê Fibrations

We look first at the complex case. Throughout this work, singular spaces are embedded in an affine space and one considers the balls in that affine space.

2.1 Milnor-Lê Fibrations for Complex Singularities

The basic references are [5, 6, 21, 25, 28]. Let X be a complex analytic singular variety of dimension $n + k$ in an open set containing the origin $\underline{0}$ in some complex space $\mathbb{C}^m$. Let $f : (X, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ be holomorphic and assume it is generically a submersion with respect to some complex analytic Whitney stratification $\{S_\alpha\}$ of X. We assume further that the zero set $V := V(f)$ has dimension more than 0 and f has the Thom a_f-property with respect to the above stratification.

We recall (see, for instance, [19]) that a point $x \in X$ is a critical point of f, in the stratified sense, if the restriction of f to the corresponding stratum has a critical point at x. These are the points such that $\mathrm{Jac}_k(f)(x) = 0$, where $\mathrm{Jac}_k(f)$ denotes the ideal generated by the determinants of all the k minors of the Jacobian matrix of the restriction of f to the corresponding stratum.

We denote by $\mathrm{Crit}(f)$ the critical locus of f, which is the union of all its critical points. The points in its complement are the regular points of f, and we denote by $\Delta_f := f(\mathrm{Crit}(f))$ the discriminant of f. This is an analytic subset of $\mathbb{C}^k$.

Given $\varepsilon > \delta > 0$ sufficiently small, we denote by $N(\varepsilon, \delta)$ the tube

$$N(\varepsilon, \delta) := [\mathbb{B}(\varepsilon) \cap X] \cap f^{-1}(\mathbb{D}_\delta),$$

where $\mathbb{B}(\varepsilon)$ is the ball of radius ε around $\underline{0}$ in $\mathbb{C}^m$ and $\mathbb{D}_\delta$ is the ball in $\mathbb{C}^k$ of radius δ around 0.

The theorem below, taken from [5], extends to this setting the Milnor-Lê fibration theorem which corresponds to the case $k = 1$. We recall that by Hironaka's theorem in [22], holomorphic map-germs into $\mathbb{C}$ always have an isolated critical value and therefore the discriminant Δ_f consists of a single point.

Theorem 2.1 *Let $f : (X^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0)$ be as above, and $g : (X^{n+k}, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical point at $\underline{0}$ in the stratified sense, both in X and also in $V = f^{-1}(0)$. Then for every $\varepsilon \geq \varepsilon' > 0$ sufficiently small, there exists $\delta > 0$ small enough with respect to ε' such that:*

1. *One has a locally trivial fibration:*

$$f : N(\varepsilon, \delta) \setminus f^{-1}(\Delta_f) \longrightarrow \mathbb{D}_\delta \setminus \Delta_f \,.$$

2. *Each fibre $F_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$, $t \in \mathbb{D}_\delta \setminus \Delta_f$ inherits a Whitney stratification from that in X, by intersecting F_t with the strata of X.*
3. *The critical points of g in each fibre F_t are all contained in the interior of the ball $\mathbb{B}_{\varepsilon'}$*

The proof is straightforward and we refer to [5] for details. We remark that the condition that f has the Thom property can be relaxed, demanding only that for each $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that for all $t \in \mathbb{D}_\delta \setminus \Delta_f$ the fibre $f^{-1}(t)$ meets transversally the sphere $\partial\mathbb{B}_\varepsilon$ (cf. Proposition 2.2 below).

The proof of this theorem relies on the fact that in the complex case, every Whitney stratification is Whitney strong. This allows using Verdier's theory of rugose vector fields, which play a key role in this discussion. In the real case things are more complicated and we must impose restrictions in order to be able to use Verdier's work. Yet, everything works fine in the real case if we restrict ourselves to the case where the source and the target are both smooth.

2.2 Milnor-Lê Fibrations for Real Singularities

The basic references for this section are [7, 9, 28, 32] (see also [8, 10, 11]). Here we follow [9]. We now let U be an open neighbourhood of $\underline{0} \in \mathbb{R}^n$ and $f\colon (U, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$, be a real analytic map-germ with a critical point at $\underline{0}$. Let $\overline{\mathbb{B}}_\varepsilon$ be a closed ball in U centred at $\underline{0}$ of sufficiently small radius $\varepsilon > 0$. We see $\overline{\mathbb{B}}_\varepsilon$ as a stratified set where the strata are the interior $\mathbb{B}_\varepsilon$ and the boundary $\mathbb{S}_\varepsilon = \partial\overline{\mathbb{B}}_\varepsilon$ of $\overline{\mathbb{B}}_\varepsilon$. Consider the restriction $f|_{\overline{\mathbb{B}}_\varepsilon}$ which to simplify notation we still denote just by f. Denote by $\mathcal{C}_f(\mathbb{B}_\varepsilon)$ the set of critical points of f in $\mathbb{B}_\varepsilon$ and denote by $\mathcal{C}_f(\mathbb{S}_\varepsilon)$ the set of critical points in $\mathbb{S}_\varepsilon$ of the restriction $f|_{\mathbb{S}_\varepsilon}$. Let $\mathcal{C}_f = \mathcal{C}_f(\mathbb{B}_\varepsilon) \cup \mathcal{C}_f(\mathbb{S}_\varepsilon)$ be the *set of critical points of f in* $\overline{\mathbb{B}}_\varepsilon$ and denote by $\Delta_f^\varepsilon = f(\mathcal{C}_f)$ the *discriminant* of f. We have the following proposition (see [31, IV.4]):

Proposition 2.2 *The restriction*

$$f\colon E_f(\varepsilon) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{R}^p \setminus \Delta_f^\varepsilon) \to f(\overline{\mathbb{B}}_\varepsilon) \setminus \Delta_f^\varepsilon$$

is a locally trivial fibre bundle.

The proof is, again, straightforward and we refer to [9] for details. Now let $\mathbb{D}_\delta^p$ be an open ball in $\mathbb{R}^p$ centred at 0 of radius $0 < \delta \ll \varepsilon$. Let $\hat{N}_f(\varepsilon, \delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon)$ and $N_f(\varepsilon, \delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\partial\mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon)$ be the restrictions of the fibre bundle of Proposition 2.2 to $\mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon$ and $\partial\mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon$, respectively. We call $\hat{N}_f(\varepsilon, \delta)$ a *solid Milnor tube* and $N_f(\varepsilon, \delta)$ a *Milnor tube* for f. Proposition 2.2 obviously implies:

Corollary 2.3 *Let* $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$ *be as before. Then the restrictions*

$$f\colon \hat{N}_f(\varepsilon, \delta) \to \mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon,$$

and

$$f\colon N_f(\varepsilon, \delta) \to \partial\mathbb{D}_\delta^p \setminus \Delta_f^\varepsilon,$$

are locally trivial fibrations.

We call the fibrations in Corollary 2.3 the *Milnor-Lê type fibrations*. Of course one also has these two fibrations in the complex case.

Remark 2.4 It can happen that the discriminant of f splits a neighbourhood of the origin in $\mathbb{R}^p$ into several connected components. In that case one has a topological (actually differentiable) type for the Milnor fibre on each such component, which can change from one sector to another.

Now we impose a further condition on f which ensures that the Milnor-Lê type fibration does not depend on ε.

As before, consider a real analytic map-germ $f: (U, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$ with a critical point at $\underline{0}$ and $V(f) = f^{-1}(0)$ has dimension greater than 2. Let $\{S_\alpha\}_{\alpha \in A}$ be a Whitney stratification of U with $V(f)$ union of strata, and let $\{R_\gamma\}_{\gamma \in G}$ be a Whitney stratification of $f(U)$ such that both stratifications give a *stratification of f*, i.e., for every $\alpha \in A$ there exists $\gamma \in G$ such that f induces a submersion from S_α to R_γ. We further assume that f satisfies the Thom a_f-property with respect to such stratification of f: Let S_α and S_β be strata such that $S_\alpha \subset \bar{S}_\beta$, let $x \in S_\alpha$ and let $\{x_i\}$ be a sequence of points in S_β converging to x. Set $f_x^\alpha = f|_{S_\alpha}^{-1}(f(x))$, the fibre of $f|_{S_\alpha}$ which contains x and $f_{x_i}^\beta = f|_{S_\beta}^{-1}(f(x_i))$ the fibre of $f|_{S_\beta}$ which contains x_i. Let T be the limit of the sequence of tangent spaces $T_{x_i} f_{x_i}^\beta$. Then $T_x f_x^\alpha \subset T$.

Proposition 2.5 *Consider $f: (U, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$ with a critical point at $\underline{0}$ and $V(f) = f^{-1}(0)$ has dimension greater than 2. Suppose there is a stratification of f which satisfies the Thom a_f-property. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every $0 < \delta \ll \varepsilon$ there is no contribution to the discriminant Δ_f^ε coming from the boundary sphere, that is, $\mathcal{C}_f(\mathbb{S}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta^p)) = \emptyset$. Therefore, the two Milnor-Lê type fibrations do not depend on ε.*

All the previous discussion extends easily to map-germs where the source is itself singular provided we restrict to cases where the maps in question satisfy *Strict Thom* (w_f) *condition* (see [6, 35]).

3 Indices of Vector Fields on Singular Varieties

In this section we discuss indices of vector fields on singular varieties. We focus on the GSV and the radial (or Schwartz) indices, which are the most relevant indices for this work. We refer to the literature, and particularly to [4] for a complete account. The radial index springs from [23, 33], while the GSV index is defined in [18].

3.1 The Radial Index

The radial index of a vector field was introduced by H. King and D. Trotmann in the early 1990s in [23] under a different name. This was also studied by W. Ebeling and S. Gusein-Zade in several articles (e.g. [13]), as well as by Aguilar, Seade and Verjovsky in [1]. The presentation we make here basically comes from [4] where this is called the Schwartz index. For this index we do not use the complex structure, so if we have a complex analytic germ, we think of it as being real analytic.

Let V be a real analytic space with a (possibly non-isolated) singularity at a point x_o in an open set U is some $\mathbb{R}^m$. We equip U with a Whitney stratification adapted to V, i.e., X is a union of strata. A *stratified vector field* on X means a section v of the tangent bundle $TU|_V$ restricted to V, such that for each $x \in V$ the vector $v(x)$ is tangent to the corresponding stratum. We further assume that all vector fields in this article are continuous.

Let v be a stratified vector field on V with an isolated singularity at x_o. We want to define its radial index at x_o. Since the question is local we may assume that there are no more singularities of v in U.

If the vector field v is transversal to every small sphere in $\mathbb{R}^m$ centred at x_o, then we say that v is radial. We denote such a vector field by $v_{\rm rad}$ and we define its radial index to be 1. Otherwise, if v is not radial, consider two balls $\mathbb{B}_\varepsilon$, $\mathbb{B}_{\varepsilon'}$ centred at x_o, with $\varepsilon > \varepsilon' > 0$ small enough so that their boundaries are transverse to all strata. Inside the smaller $\mathbb{B}_{\varepsilon'}$ we consider a stratified radial vector field $v_{\rm rad}$ with centre x_o and pointing outwards the ball. On the boundary $\partial\mathbb{B}_\varepsilon$ of the larger one, we consider the vector field v.

Let us consider the cylinder $V_{\varepsilon,\varepsilon'} = (\mathbb{B}_\varepsilon \setminus {\rm Int}\,\mathbb{B}_{\varepsilon'}) \cap V$. On the boundaries $V \cap \mathbb{S}_{\varepsilon'} = V \cap \partial\mathbb{B}_{\varepsilon'}$ and $V \cap \mathbb{S}_\varepsilon = V \cap \partial\mathbb{B}_\varepsilon$ of $V_{\varepsilon,\varepsilon'}$ one has a vector field w defined by $v_{\rm rad}$ and v, respectively. One can always extend w to the interior of $V_{\varepsilon,\varepsilon'}$ by the classical radial extension process of M. H. Schwartz (see, for instance, [4, Chapter 2, Section 3]), so that we get a stratified vector field on $V_{\varepsilon,\varepsilon'}$ with isolated singularities in the interior. At each singular point of the extension, the index in the stratum coincides with the index in the ambient space:

$$\mathrm{Ind}_{\rm PH}(w, p_j; V_\beta) = \mathrm{Ind}_{\rm PH}(w, p_j; \mathbb{R}^m),$$

where V_β is the stratum containing p_j.

Definition 3.1 The *difference* of v and $v_{\rm rad}$ is defined as:

$$d(v, v_{\rm rad}) = \sum_\beta \sum_j \mathrm{Ind}_{\rm PH}(w, p_j; V_\beta),$$

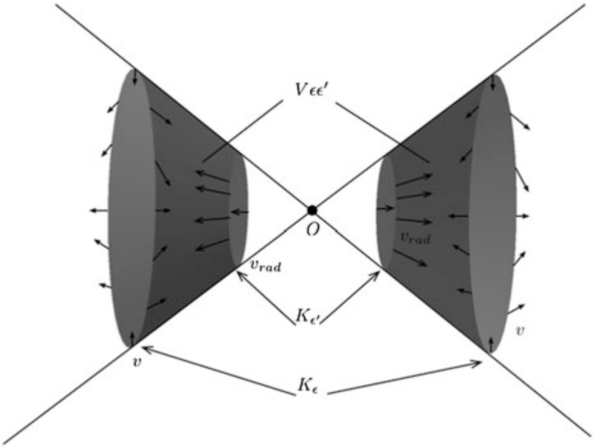
where the sum on the right runs over the Poincaré Hopf indices at the singular points of the restriction of w to each stratum V_β in $V_{\varepsilon,\varepsilon'}$.

One has that this integer does not depend on the choice of w provided this extension is done by radial extension (see [4, p. 38] or [14, p. 441]). Notice also that $d(v_{\text{rad}}, v_{\text{rad}}) = 0$.

Definition 3.2 *The radial or Schwartz index* of v at $x_o \in V$ is defined as:

$$\text{ind}_{\text{Sch}}(v, x_i; X) = 1 + d(v, v_{\text{rad}})\,;$$

Now let X be an arbitrary real analytic space in $\mathbb{R}^m$ with arbitrary singularities and equipped with a Whitney stratification; let v be a continuous stratified vector field on X with isolated singularities $x_1, \cdots, x_s$. Then define



Definition 3.3 The total radial (or Schwartz) index of v in X is the sum of the local indices $\text{ind}_{\text{Sch}}(v, x_i; X)$.

Notice that if some singular point x_i of v is at a smooth point of X, then its radial index is the usual Poincaré-Hopf index.

Definition 3.4 Let $\mathbb{B}_r$ be a closed ball in $\mathbb{R}^m$ such that each stratum in X meets transversally the boundary sphere $\mathbb{S}_r = \partial\mathbb{B}_r$. Then we say that the intersection $X_r := X \cap \mathbb{B}_r$ is a *singular variety with boundary* $\partial X_r := X \cap \mathbb{S}_r$.

The following theorem, which is used in the sequel, extends one of the fundamental properties of the Schwartz Index for compact complex varieties. The proof is exactly as that of Theorem 2.1.1 in [4] and we leave the details to the reader.

Theorem 3.5 *Let X_r be a compact, real analytic variety with boundary in $\mathbb{R}^m$, which is equipped with a Whitney stratification adapted to X. Let v be a continuous, stratified vector field defined on a neighbourhood of the boundary ∂X_r in $\mathbb{C}^m$, with* *no singularities. Then:*

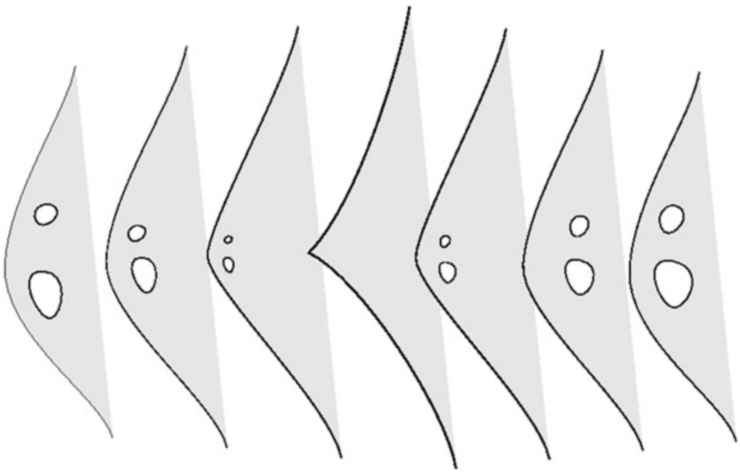
1. *Radial extension of* v *yields a stratified vector field on* X_r *with isolated singularities* $x_1, \cdots, x_s$*, and its total Schwartz index in* X_r *is independent of the choice of the extension.*
2. *If* v *is transversal to the boundary* ∂X_r *everywhere, then:*

$$\chi(V) = \sum_{i=1}^{s} \mathrm{Ind}_{\mathrm{Sch}}(v, x_i; V) \;,$$

where $\chi(V)$ *is the Euler-Poincaré characteristic.*

3.2 The GSV Index

We now consider the GSV-index of vector fields, introduced in 1987 in [34], which coincides with the local Poincaré-Hopf index when the space is smooth. Unlike the radial index, now the complex structure does play an important role because this index is closely related to the Milnor number. Since this is a well-defined integer for holomorphic map-germs, the GSV index is a well-defined integer. Yet, for real analytic map-germs, the Milnor number is defined only modulo 2, a problem that we address in the last section of this article.



We now focus on the complex case. For the sequel we actually need an extension of this index as explained below, but we discuss first the classical setting (see [18, 34]).

We consider a hypersurface (or a complete intersection) germ $(V, \underline{0})$ defined by a holomorphic function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical point at $\underline{0}$. Then, given a vector field v on V with isolated singularity at $\underline{0}$, one can always extend it to a vector field w on the Milnor fibre $F = F_f$ of f, with no singularities near the boundary. The total number of these singularities in F, counted with their local Poincaré-Hopf indices, is independent of the extension, and this number is, by definition, the GSV-index of v at $\underline{0}$. That is:

$$\mathrm{Ind}_{\mathrm{GSV}}(v, \underline{0}; V) := \mathrm{Ind}_{\mathrm{PH}}(w; F),$$

where the term on the right is the total index.

In other words we may think of this index as follows. We envisage the singular variety V as being a limit of the complex manifolds $V_t = f^{-1}(t)$, $t \neq 0$, as $t \to 0$. We may as well envision the vector field v on V as being a limit of a family of vector fields v_t on the V_t, each v_t having isolated singularities, which degenerate into $\underline{0}$ as the v_t converge to v. Then the GSV index of v equals the sum of the Poincaré-Hopf indices of each v_t at the singularities that converge to $\underline{0}$.

The original definition of the GSV index is as the degree of a certain map from the link into a Stiefel manifold, generalizing the classical definition of the local Poincaré-Hopf index. The map in question is determined by the vector field and the gradients of the functions that define the ICIS germ.

The GSV index has the following basic properties, the first of these being the one just discussed; the second property is an immediate consequence of this (see [2] for details).

Theorem 3.6

(1) *The GSV index of v at 0 equals the Poincaré-Hopf index of v in the Milnor fibre:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, 0) = \mathrm{Ind}_{\mathrm{PH}}(v, \mathcal{F}) .$$

(2) *If v is everywhere transverse to the link, then*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, 0) = 1 + (-1)^n \mu ,$$

where n is the complex dimension of V and μ is the Milnor number of 0.

(3) *One has:*

$$\mu \;=\; (-1)^n \big(\mathrm{Ind}_{\mathrm{GSV}}(v, 0) \;-\; \mathrm{Ind}_{\mathrm{Sch}}(v, 0)\big) ,$$

independently of the choice of v.

The definition of the GSV index extends immediately to vector fields on complex ICIS, and it was extended in [4, Chapter 3] to the more general setting where V can have non-isolated singularities, but the ambient space still is non-singular. In the sequel we need a further extension of this index to the case when the ambient space is itself singular. More precisely, we want to consider isolated complete intersection germs $(X,\underline{0}) \xrightarrow{f} (\mathbb{C}^k,\underline{0})$, where X has non-isolated singularities. In this setting the Milnor fibre of f is itself singular, so we cannot define the GSV index of a vector field on the special fibre as being the Poincaré-Hopf index of an extension of it to the Milnor fibre. We follow [5].

Let X, f and g be as above, i.e., $(X,\underline{0})$ is a complex analytic variety of dimension $n+k$ in $\mathbb{C}^m$ with a singular point at the origin $\underline{0}$; $f : (X,\underline{0}) \to (\mathbb{C}^k,\underline{0})$ is a holomorphic function which is generically a submersion with respect to some Whitney stratification $\{S_\alpha\}$ of X, and such that its zero set $V := V(f)$ has dimension more than 0. We assume that f has the Thom a_f-property with respect to this stratification. Finally, $g : (X,\underline{0}) \to (\mathbb{C},0)$ is a holomorphic map with an isolated critical point at $\underline{0}$ in the stratified sense, both in X and also in V.

Let us describe now the construction in [3] of a stratified vector field on V, denoted by $\overline{\nabla}_V(g)$. Let us denote by $\overline{\nabla}\tilde{g}(x)$ the gradient vector field of $\tilde{g} : U \to \mathbb{C}$ at a point x in the neighbourhood U of $\underline{0}$ in $\mathbb{C}^m$ with $\tilde{g}|_X = g$, defined by

$$\overline{\nabla}\tilde{g}(x) := \left(\overline{\frac{\partial \tilde{g}}{\partial x_1}}, \ldots, \overline{\frac{\partial \tilde{g}}{\partial x_m}}\right),$$

where the bar denotes complex conjugation. Consider V with Whitney stratification obtained by intersecting this variety with the strata of X, denoted by $\{V_\alpha\}$, and denote by $V_\alpha(x)$ the stratum containing x. Since $g : (X,\underline{0}) \to (\mathbb{C},0)$ has an isolated singularity at $\underline{0}$, the projection of $\overline{\nabla}\tilde{g}(x)$ on $T_x V_\alpha(x)$, denoted by $\hat{\zeta}_\alpha(x)$, is not zero. Gluing together the vector fields $\hat{\zeta}_\alpha$, obtain a stratified vector field on V, denoted by $\overline{\nabla}_V(g)$. This vector field is homotopic to $\overline{\nabla}\tilde{g}|_V$ (the justification for this can be seen in [3, Section 2]). Now consider g restricted to Milnor fibre F_f with the Whitney stratification obtained by intersecting this variety with the strata of X. Just in the same way, obtain a stratified vector field on F_f using, for each $x \in F_f$, the projection of $\overline{\nabla}\tilde{g}(x)$ on the tangent space to each stratum of F_f containing x, denoted by $\overline{\nabla}_{F_f}(g)$. This vector field is homotopic to $\overline{\nabla}\tilde{g}|_{F_f}$.

Now we define (compare with Definition 3.4.1 of [4]):

Definition 3.7 The *GSV-index* of g on V relative to the function f is, by definition, the total Schwartz index of the conjugate gradient vector field $\overline{\nabla}_{F_f}(g)$ on the Milnor fibre F_f:

$$\mathrm{Ind}_{\mathrm{GSV}}(g,\underline{0};f) := \mathrm{Ind}_{\mathrm{Sch}}(\overline{\nabla}_{F_f}(g); F_f)\,.$$

It is clear that the same definition adapts easily to define the corresponding index for 1-forms, in the vein of [14, 15]. In fact in that case the definition is actually simpler because one does not need to extend vector fields from the singular variety to the Milnor fibre, but we only need to consider the restriction of the 1-form dg to the boundary of F_f.

4 The Lê-Greuel Formula in the Complex Case

This section is based on [5]. We consider again a complex analytic space X of pure dimension $n + k$. Let $f : (X, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ be a holomorphic function which is generically a submersion with respect to some complex analytic Whitney stratification $\{S_\alpha\}$ of X for which the zero set $V := V(f)$ is a union of strata. We assume further that V has dimension more than 0 and f has the Thom a_f-property with respect to the above stratification.

Let $g : X \to \mathbb{C}$ be holomorphic with an isolated critical point in X (and hence also in V), with respect to this stratification. As before, we assume $V(f)$ has dimension ≥ 2.

We know from Sect. 2 that f has a Milnor-Lê fibration; we denote by F_f the corresponding fibre. Notice that the fact that g has an isolated critical point in V implies that (f, g) is also generically a submersion and it has the Thom property with respect to the stratification $\{S_\alpha\}$. Also, the variety $V(f, g)$ has dimension at least 1. So, by the same arguments as in Sect. 2, the map (f, g) has a Milnor-Lê fibration. We denote by $F_{f,g}$ the corresponding Milnor fibre. We have:

Theorem 4.1 (The Lê-Greuel Formula) *One has:*

$$\chi(F_f) = \chi(F_{f,g}) + \mathrm{Ind}_{\mathrm{GSV}}(g, \underline{0}; f) .$$

The first step for proving this theorem is choosing appropriate representatives of the Milnor fibres F_f and $F_{f,g}$. For this we consider first a Milnor sphere $\mathbb{S}_\varepsilon$ for X and $V(f)$ at $\underline{0}$, and the Milnor-Lê fibration of f given by Theorem 2.1.

Since g has an isolated critical point in $V := V(f)$, by Lê [25] there exists $\delta' > 0$ such that if we let $\mathbb{D}^2_{\delta'}$ be the disc in $\mathbb{C}$ of radius δ' about 0 and set

$$N_V(\varepsilon, \delta') = g^{-1}(\mathbb{D}^2_{\delta'}) \cap V(f) \cap \mathbb{B}_\varepsilon,$$

then

$$g : N_V(\varepsilon, \delta') \setminus V(g) \longrightarrow \mathbb{D}^2_{\delta'} \setminus \{0\},$$

is a locally trivial fibre bundle, where $V(g) := g^{-1}(0)$. Choose a typical fibre $g^{-1}(s_o) \cap V(f)$ of this fibration, so $s_o \neq 0$ with $|s_o| \leq \delta'$.

Now consider $\varepsilon' > 0$ small enough, so that the ball $\mathbb{B}_{\varepsilon'}$ in $\mathbb{C}^m$ does not meet the fibre $g^{-1}(s_o)$. We use Theorem 2.1 again to choose a Milnor fibre $F_f := f^{-1}(t_o) \cap \mathbb{B}_\varepsilon$ for f, such that the restriction of g to F_f has no critical points away from $\mathbb{B}_{\varepsilon'}$. Then the hypersurface $g^{-1}(s_o)$ meets F_f transversally and therefore the intersection $F_{f,g} := F_f \cap g^{-1}(s_o)$ serves as a model for the Milnor fibre of the Milnor-Lê fibration of (f, g).

From now on we set $F_{f,g} := F_f \cap g^{-1}(s_o)$ with $F_f = f^{-1}(t_o) \cap \mathbb{B}_\varepsilon$, and we equip F_f and $F_{f,g}$ with the Whitney stratifications obtained by intersecting these varieties with the strata of X. We have:

Lemma 4.2 (Main Lemma) *There exists a continuous stratified vector field v on F_f with the following properties:*

1. *Restricted to a neighbourhood of its boundary ∂F_f, it is transversal to the boundary, pointing outwards.*
2. *It is tangent to the hypersurface $F_{f,g}$.*
3. *Within the ball $\mathbb{B}_{\varepsilon'}$, v is the vector field $\overline{\nabla}_V(g)$.*
4. *Away from $\mathbb{B}_{\varepsilon'}$, v has only isolated singularities, all contained in $F_{f,g}$, and at each of these singular points, the vector field is transversally radial to the stratum containing the singular point of v* (that is, v is transversal to the boundary of every tubular neighbourhood of the stratum).

Notice that property 4 implies that the Schwartz index of v at each singularity in $F_{f,g}$ equals the local Poincaré-Hopf index of the restriction of v to the stratum.

Theorem 4.1 follows easily from this Lemma. In fact, the first property implies that the total Schwartz index of v in F_f is $\chi(F_f)$, by Theorem 3.5. Similarly, properties 1 and 2, together with Theorem 3.5, imply that the total Schwartz index of v in $F_{f,g}$ is $\chi(F_{f,g})$. Thus, again by Theorem 3.5, together with property 4, we get that the difference $\chi(F_f) - \chi(F_{f,g})$ equals the sum of the Schwartz indices of v away from $F_{f,g}$, and this is the GSV-index of $\overline{\nabla}_V(g)$ by property 3, since v has no other singularities away from $F_{f,g}$ but those in $\mathbb{B}_{\varepsilon'}$, where v coincides with $\overline{\nabla}_V(g)$.

Lemma 4.2 and its proof are very much inspired by Brasselet et al. [2, 3], where the authors prove similar statements to get Lefschetz type theorems for the local Euler obstruction. This lemma is in fact an immediate consequence of the three lemmas below. We refer to [5] for the proofs of these lemmas, which are inspired by Schwartz [33].

Lemma 4.3 *Let $\eta' > 0$ be small enough with respect to δ', so that the disc $\mathbb{D}_{\eta'}(s_o) \subset \mathbb{C}$ centred at s_o is contained in the interior of $\mathbb{D}_{\delta'}$ and $g^{-1}(\mathbb{D}_{\eta'}(s_o))$ does not intersect $\mathbb{B}_{\varepsilon'}$. Then, there exists a stratified vector field w_r on F_f satisfying (Fig. 1):*

1. *Its restriction to $g^{-1}(\mathbb{D}_{\delta'}) \cap \mathbb{S}_\varepsilon$ is tangent (stratified) to all the fibres $g^{-1}(s)$, and it is transversal to $\mathbb{S}_\varepsilon$, pointing outwards.*
2. *It is tangent to the fibre $g^{-1}(s_o)$, where it has only isolated singularities, and at each singularity, w_r is transversally radial in $g^{-1}(s_o)$, to the stratum that contains that zero of w_r*
3. *It is tangent to each fibre $g^{-1}(s)$, for $s \in \mathbb{D}_{\eta'}(s_o)$.*

Lemma 4.4 *There exists a stratified vector field u defined on $F_f \cap g^{-1}(\mathbb{D}_{\delta'})$ minus the interior of the ball $\mathbb{B}_{\varepsilon'}$, satisfying the following (Fig. 2):*

1. *u is tangent to $\mathbb{S}_\varepsilon$;*
2. *its zero set is $g^{-1}(s_o)$, and u is transversally radial to $g^{-1}(s_o)$ in F_f;*

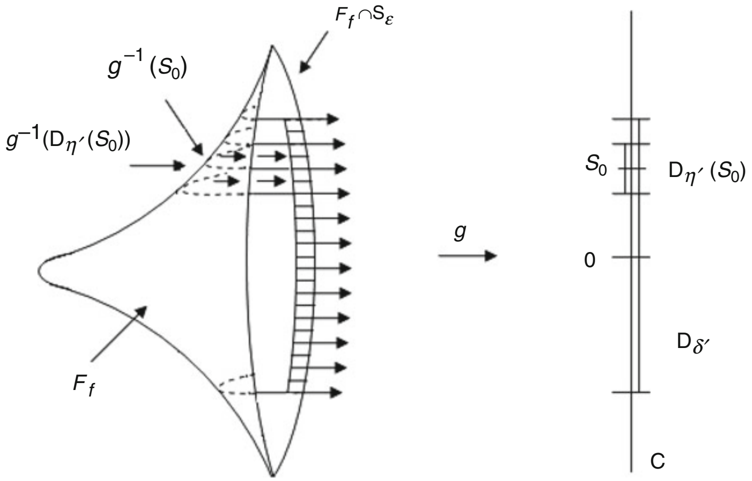


Fig. 1 The vector field w_r

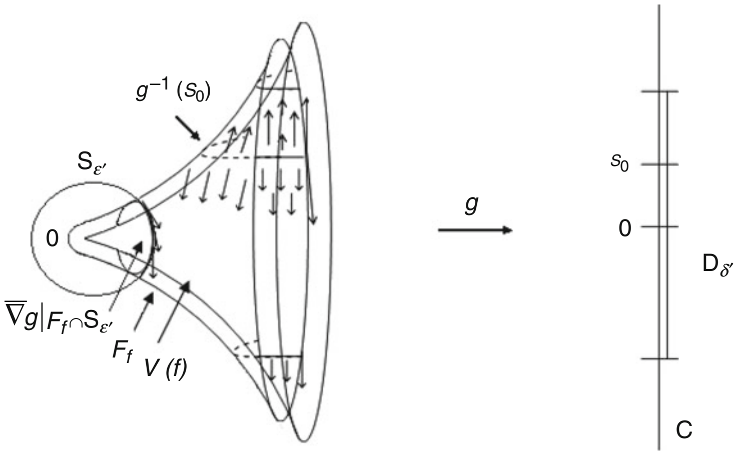


Fig. 2 The vector field u

3. *u is transversal to $F_f \cap g^{-1}(\partial \mathbb{D}_{\delta'})$, pointing outwards;*
4. *restricted to $F_f \cap \partial \mathbb{B}_{\varepsilon''}$ it coincides with the gradient vector field $\overline{\nabla} g|_{F_f}$.*

Lemma 4.5 *Let w be a stratified vector field on $F_f \cap g^{-1}(\partial \mathbb{D}_{\delta'})$, which is transverse to both $F_f \cap \mathbb{S}_\varepsilon$ and $F_f \cap g^{-1}(\partial \mathbb{D}_{\delta'})$, pointing outwards. Then there exists an extension of w to a stratified vector field on $F_f \setminus g^{-1}(\mathbb{D}_{\delta'})$, which is transverse to $\partial F_f = F_f \cap \mathbb{S}_\varepsilon$ and pointing outwards.*

Lemma 4.5 then follows from this and the fact that f has the Thom property.

We remark that if in this statement we replace F_f by $V(f)$, then the lemma is a special case of Theorem 2.3 in [2].

5 The Lê-Greuel Formula in the Real Analytic Case

This section is based on [9]. Consider now a real analytic map $(\mathbb{R}^n, \underline{0}) \xrightarrow{f} (\mathbb{R}^p, 0)$, $n > p \geq 2$, with arbitrary critical locus. Let $\overline{\mathbb{B}}_\varepsilon$ be the closed ball in $\mathbb{R}^n$ centred at $\underline{0}$ of radius ε, $\mathbb{D}^p_\delta$ be the open ball in $\mathbb{R}^p$ centred at 0 of radius δ and Δ^ε_f be the discriminant of f. We already know that f has an associated locally trivial fibration of the Milnor-Lê type

$$f\colon \hat{N}_f(\varepsilon, \delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{D}^p_\delta \setminus \Delta^\varepsilon_f) \to \mathbb{D}^p_\delta \setminus \Delta^\varepsilon_f.$$

As noted before, in this general setting the topology of the Milnor fibre is not always unique: the discriminant of f may split a neighbourhood of the origin in $\mathbb{R}^p$ into several connected components, and one has a topological (actually differentiable) model for the Milnor fibre on each such component (cf. [27]).

We further require that the map f satisfies the Thom a_f-property with respect to some Whitney stratification $\{S_\alpha\}$ such that its zero-set $V(f)$ has dimension ≥ 2 and it is union of strata. We also consider another real analytic map germ $(\mathbb{R}^n, \underline{0}) \xrightarrow{g} (\mathbb{R}^k, 0)$ with an isolated critical point in $\mathbb{R}^n$ with respect to the stratification $\{S_\alpha\}$. By the previous discussion, the map-germ (f, g) also has an associated locally trivial fibration of the Milnor-Lê type.

We have the following Lê-Greuel formula from [9]:

Theorem 5.1 *Let F_f be a Milnor fibre of f* (any Milnor fibre, regardless of the discriminant of f). *Then one has:*

$$\chi(F_f) = \chi(F_{f,g}) + \mathrm{Ind}_{\mathrm{PH}}\, \nabla\tilde{g}|_{F_f \cap \mathbb{B}_{\varepsilon'}},$$

where $\tilde{g}\colon \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{g}(x) = \|g(x) - t_0\|^2$ with $t_0 \in \mathbb{R}^k$ such that $F_{f,g} = g|^{-1}_{F_f}(t_0)$ and $\mathbb{B}_{\varepsilon'}$ is a small ball in $\mathbb{R}^n$ centred at the origin.

The term $\mathrm{Ind}_{\mathrm{PH}}\, \nabla\tilde{g}|_{F_f \cap \mathbb{B}_{\varepsilon'}}$ on the right, which by definition is the total Poincaré-Hopf index in F_f of the vector field $\nabla\tilde{g}|_{F_f}$, can be expressed also in the following equivalent ways:

1. *As the Euler class of the tangent bundle of F_f relative to the vector field $\nabla\tilde{g}|_{F_f \cap \mathbb{B}_{\varepsilon'}}$ on its boundary;*
2. *As a sum of polar multiplicities relative to $\tilde{g}$ on $F_f \cap \mathbb{B}_{\varepsilon'}$.*

The proof is in the same vein as that in the complex case, and we refer to [9] for details. The idea is constructing appropriate vector fields, and for that we use the auxiliary function $\tilde{g}$.

As noted in [9, Remark 3.8], when $k = 1$ we may replace $\tilde{g}$ by the original map g and get a similar result. We notice too that a Lê-Greuel formula for real analytic ICIS germs was recently obtained in [11].

Consider again real analytic complete intersection germs with an isolated critical point in the ambient space $f = (f_1, \cdots, f_k)$, and (f, g) and a Milnor-Lê fibration

$$(f, g)\colon N(\varepsilon, \delta) \setminus V(f, g) \longrightarrow \mathbb{D}_\delta \setminus \{0\},$$

which determines a locally trivial fibre bundle

$$\phi\colon \mathbb{S}^{n-1}_\varepsilon \setminus V(f, g) \longrightarrow \mathbb{S}^p\,.$$

As noticed in [7, 32], one has that the projection map ϕ can always be taken as $\frac{(f,g)}{\|(f,g)\|}$ in a neighbourhood of the link $L_{(f,g)} := V(f, g) \cap \mathbb{S}^{n-1}_\varepsilon$. Following [7] we say that the map germ (f, g) is d-regular if the projection map ϕ can be taken as $\frac{(f,g)}{\|(f,g)\|}$ everywhere. In this case we notice that there is a relation between the topology of f and that of (f, g), which is deeper than the one given by Theorem 5.1. To state this result, it is convenient to write g as f_{k+1}, then we have the following immediate application of Corollaries 5.4 and 5.5 in [7]:

Theorem 5.2 *Let $f = (f_1, \cdots, f_{p+1})\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^{p+1}, 0)$ be a complete intersection germ with an isolated critical point at $\underline{0}$ and which is d-regular. Let $V = f^{-1}(0)$ and for each $i = 1, \cdots, p+1$ let V_i be the singular variety $V_i = (f_1, \cdots, \widehat{f_i}, \cdots, f_{p+1})^{-1}(0)$, where $\widehat{f_i}$ means that we are removing this component and looking at the corresponding map germ into $\mathbb{R}^p$. Then the topology of $V_i \setminus V$ is independent of the choice of i and its link, which is a smooth manifold, is diffeomorphic to the disjoint union of two copies of the interior of the Milnor fibre of f.*

This motivates the following question: Is the topology of V_i independent of i as in the complex case? (see [6, Thm. 1 (i)]). In this case the link of V_i would be diffeomorphic to the double of the Milnor fibre of f. This has recently been answered positively by A. Menegon Neto and should be published soon.

6 Concluding Remarks

Remark 6.1 If $f_1, \cdots, f_k$ and g are holomorphic map-germs $(\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}, 0)$ such that $f = (f_1, \cdots, f_k)$ and (f, g) define isolated complete intersection germs, then the classical Lê-Greuel formula can be expressed as:

$$\chi(F_f) = \chi(F_{f,g}) + \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,\underline{0}}}{(f, \mathrm{Jac}_{k+1}(f, g))}, \tag{1}$$

where $\mathrm{Jac}_{k+1}(f,g)$ denotes the ideal generated by the determinants of all $(k+1)$ minors of the corresponding Jacobian matrix. To deduce that formula from Theorem 4.1 we notice that by classical intersection theory, the invariant $\mathrm{Ind}_{\mathrm{GSV}}(g,\underline{0},f)$ in (4.1) can be expressed as

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{m,\underline{0}}}{(f-t, \mathrm{Jac}_{k+1}(f,g))},$$

where $f = t$ defines the Milnor fibre F_f. In this setting the fibres of f are a flat family, so one can specialize this invariant to the fibre over 0, thus arriving to the formula above. Of course it would be interesting to find a similar algebraic interpretation of the invariant $\mathrm{Ind}_{\mathrm{GSV}}(g,\underline{0},f)$ in a more general setting (cf. [5, Section 6]).

Remark 6.2 In [12, Theorem 5.2], N. Dutertre and N. G. Grulha prove a Lê-Greuel formula which holds in the same general setting we envisage in this article. They express the difference $\chi(F_f) - \chi(F_{f,g})$ appearing in Theorem 4.1 as a sum:

$$\chi(F_f) - \chi(F_{f,g}) = \sum (-1)^{d_\alpha} \sum \mu_{ij}\big(1 - \chi(\mathrm{lk}^{\mathbb{C}}(S_\alpha, X))\big),$$

where the first sum on the right runs over the strata S_α in the Milnor fibre of f, the number d_α being the dimension of the stratum; the μ_{ij} are certain multiplicities that can be described as the number of critical points, in each stratum of the Milnor fibre F_f, of a Morsification of g, and $\chi(\mathrm{lk}^{\mathbb{C}}(S_\alpha, X))$ is the Euler characteristic of the complex link of the corresponding stratum. It would be interesting to find a direct path to pass from that formula to Theorem 4.1 above.

Remark 6.3 If $f_1, \cdots, f_k$ and $g_1, \ldots, g_r$ are holomorphic map germs $(\mathbb{C}^{n+k+r}, \underline{0}) \to (\mathbb{C}, 0)$ such that $f = (f_1, \cdots, f_k)$, $g = (g_1, \cdots, g_r)$and (f,g) define isolated complete intersection germs, then the classical Lê-Greuel formula tells us an inductive way to determine the Milnor number of (f,g) from that of f plus contributions of the g_i's obtained by dropping one by one the g_i. On the other hand, I believe that the techniques of [9] explained in Sect. 5 can be easily adapted to express the Milnor number of (f,g) in terms of that of f plus a contribution of an auxiliary function $\tilde{g}\colon \mathbb{C}^{n+k+r} \to \mathbb{R}$ given by $\tilde{g}(x) = \|g(x) - t_0\|^2$ for some appropriate $t_0 \in \mathbb{C}^r$.

Remark 6.4 As pointed out in Remark 6.1, the classical formulation of the Lê-Greuel theorem can be stated as:

$$\chi(F_f) = \chi(F_{f,g}) + \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,\underline{0}}}{(f, \mathrm{Jac}_{k+1}(f,g))} \tag{2}$$

Since the fibres of f give rise to a flat family, the invariant $\mathrm{Ind}_{\mathrm{GSV}}(g,\underline{0},f)$, which essentially is the Poincaré-Hopf index of a holomorphic vector field on the Milnor fibre, specializes to the fibre over the critical point and one gets the algebraic expression above. It is natural to ask what happens in the real analytic setting for

ICIS in $\mathbb{R}^m$ since one also has in this case a flat family. This is somehow equivalent to asking for an algebraic expression for the Poincaré-Hopf index of a real analytic vector field on the Milnor fibre, and that is in itself a very interesting field of research, with a large literature that goes back to work by Arnold, Eisenbud-Levine, Khimshiashvili, Gómez-Mont and others. In fact, a positive answer to the question above was given in [9, Section 5] for real analytic ICIS defined by two equations. This was based on [16, 17], where the authors give algebraic expressions for the index of analytic vector fields on real analytic hypersurfaces. An extension of the results in [16, 17] for vector fields on real analytic ICIS would provide an algebraic formulation for the corresponding invariant in the Lê-Greuel forumla. And perhaps this viewpoint can be used for extending to ICIS the interesting results in [16, 17], at least for gradient vector fields.

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On Welschinger Invariants of Descendant Type

Eugenii Shustin

Dedicated to Gert-Martin Greuel in occasion of his 70th birthday

Abstract We introduce enumerative invariants of real del Pezzo surfaces that count real rational curves belonging to a given divisor class, passing through a generic conjugation-invariant configuration of points and satisfying preassigned tangency conditions to given smooth arcs centered at the fixed points. The counted curves are equipped with Welschinger-type signs. We prove that such a count does not depend neither on the choice of the point-arc configuration nor on the variation of the ambient real surface. These invariants can be regarded as a real counterpart of (complex) descendant invariants.

Keywords del Pezzo surfaces • Descendant invariants • Real enumerative geometry • Real rational curves • Welschinger invariants

Subject Classification: Primary 14N35; Secondary 14H10, 14J26, 14P05

1 Introduction

Welschinger invariants of real rational symplectic manifolds [17–19, 21] serve as genus zero open Gromov–Witten invariants. In dimension four and in the algebraic-geometric setting, they are well defined for real del Pezzo surfaces (cf. [12]), and they count real rational curves in a given divisor class passing through a generic conjugation-invariant configuration of points and are equipped with weights ± 1. An important outcome of Welschinger's theory is that, whenever Welschinger invariant does not vanish, there exists a real rational curve of a given divisor class matching an appropriate number of arbitrary generic conjugation-invariant constraints.

E. Shustin (✉)
School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel
e-mail: shustin@post.tau.ac.il

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There are several extensions of the original Welschinger invariants: modifications for multicomponent real del Pezzo surfaces [9, 12], mixed and relative invariants [10, 20] (R. Rasdeaconu and J Solomon, Relative open Gromov–Witten invariants, unpublished), invariants of positive genus for multicomponent real del Pezzo surfaces [15], and for $\mathbb{P}^{2k+1}$, $k \geq 1$ [4, 5]. The goal of this paper is to introduce Welschinger-type invariants for real del Pezzo surfaces, which count real rational curves passing through some fixed points and tangent to fixed smooth arcs centered at the fixed points. They can be viewed as a real counterpart of certain descendant invariants (cf. [6]).

The main result of this note is Theorem 1 in Sect. 2, which states the existence of invariants independent of the choice of constraints and of the variation of the surface. Our approach in general is similar to that in [12], and it consists in the study of codimension one bifurcations of the set of curves subject to imposed constraints when one varies either the constraints or the real and complex structure of the surface. In Sect. 5, we show a few simple examples. The computational aspect and quantitative properties of the invariants will be treated in a forthcoming paper.

2 Invariants

Let X be a real del Pezzo surface with a nonempty real point set $\mathbb{R}X$ and $F \subset \mathbb{R}X$ a connected component. Pick a conjugation-invariant class $\varphi \in H_2(X \setminus F; \mathbb{Z}/2)$. Denote by $\text{Pic}^{\mathbb{R}}_{+}(X) \subset \text{Pic}(X)$ the subgroup of real effective divisor classes. Pick a nonzero class $D \in \text{Pic}^{\mathbb{R}}(X)$, which is F-compatible in the sense of [11, Sect. 5.2]. Observe that any real rational (irreducible) curve $C \in |D|$ has a one-dimensional real branch (see, e.g., [12, Sect. 1.2]), and hence we can define C_+, C_-, the images of the components of $\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$ by the normalization map.

Given a smooth (complex) algebraic variety Σ, a point $z \in \Sigma$, and a positive integer s, the space of s-arcs in Σ at z is

$$\text{Arc}_s(\Sigma, z) = \text{Hom}(\text{Spec}\,\mathbb{C}[t]/(t^{s+1}), (\Sigma, z))/\text{Aut}(\mathbb{C}[t]/(t^{s+1}))\ .$$

Denote by $\text{Arc}^{\text{sm}}_s(\Sigma, z) \subset \text{Arc}_s(\Sigma, z)$ the (open) subset consisting of smooth s-arcs, i.e., of those which are represented by an embedding $(\mathbb{C}, 0) \to (\Sigma, z)$.

Choose two collections of positive integers $\boldsymbol{k} = \{k_i,\ 1 \leq i \leq r\}$ and $\boldsymbol{l} = \{l_j,\ 1 \leq j \leq m\}$, where $r, m \geq 0$ and

$$\sum_{i=1}^{r} k_i + 2\sum_{j=1}^{m} l_j = -DK_X - 1\ , \qquad (1)$$

and all $k_1, \ldots, k_r$ are odd. Pick distinct points $z_1, \ldots, z_r \in F$ and real arcs $\alpha_i \in \mathrm{Arc}^{\mathrm{sm}}_{k_i}(X, z_i)$, $1 \le i \le r$, and also distinct points $w_1, \ldots, w_m \in X \setminus \mathbb{R}X$ and arcs $\beta_j \in \mathrm{Arc}^{\mathrm{sm}}_{l_j}(X, w_j)$. Denote $\boldsymbol{z} = (z_1, \ldots, z_r)$, $\boldsymbol{w} = (w_1, \overline{w}_1, \ldots, w_m, \overline{w}_m)$ and

$$\mathscr{A} = (\alpha_1, \ldots, \alpha_r) \in \prod_{i=1}^{r} \mathrm{Arc}^{\mathrm{sm}}_{k_i}(X, z_i) \ , \tag{2}$$

$$\mathscr{B} = (\beta_1, \overline{\beta}_1, \ldots, \beta_m, \overline{\beta}_m) \in \prod_{j=1}^{m} \left(\mathrm{Arc}^{\mathrm{sm}}_{l_j}(X, w_j) \times \mathrm{Arc}^{\mathrm{sm}}_{l_j}(X, \overline{w}_j) \right) \ . \tag{3}$$

In the moduli space $\mathscr{M}_{0,r+2m}(X, D)$ of stable maps of rational curves with $r + 2m$ marked points, we consider the subset $\mathscr{M}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))$ consisting of the elements $[\boldsymbol{n} : \mathbb{P}^1 \to X, \boldsymbol{p}]$, $\boldsymbol{p} = (p_1, \ldots, p_r, q_1, \ldots, q_m, q'_1, \ldots, q'_m) \subset \mathbb{P}^1$, such that

$$\boldsymbol{n}^* \left(\bigcup \mathscr{A} \cup \bigcup \mathscr{B} \right) \ge \sum_{i=1}^{r} k_i p_i + \sum_{j=1}^{m} l_j (q_j + q'_j) \ .$$

Let $\mathscr{M}^{im,\mathbb{R}}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B})) \subset \mathscr{M}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))$ be the set of elements $[\boldsymbol{n} : \mathbb{P}^1 \to X, \boldsymbol{p}]$ such that $\boldsymbol{n}$ is a conjugation-invariant immersion, the points $p_1, \ldots, p_r \in \mathbb{P}^1$ are real, and $q_j, q'_j \in \mathbb{P}^1$ are complex conjugate, $j = 1, \ldots, m$. For a generic choice of point sequences $\boldsymbol{z}$ and $\boldsymbol{w}$, and arc sequences $\mathscr{A}$ and $\mathscr{B}$ in the arc spaces indicated in (2) and (3), the set $\mathscr{M}^{im,\mathbb{R}}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))$ is finite (cf. Proposition 1(1) below).

Given an element $\xi = [\boldsymbol{n} : \mathbb{P}^1 \to X, \boldsymbol{p}] \in \mathscr{M}^{im,\mathbb{R}}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))$, denote $C = \boldsymbol{n}(\mathbb{P}^1)$ and define the *Welschinger sign* of ξ by (cf. [12, Formula (1)])

$$W_\varphi(\xi) = (-1)^{C_+ \circ C_- + C_+ \circ \varphi} \ .$$

Notice that, if C is nodal, then $C_+ \circ C_-$ has the same parity as the number of real solitary nodes of C (i.e., nodes locally equivalent to $x^2 + y^2 = 0$).

Finally, put

$$W(X, D, F, \varphi, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B})) = \sum_{\xi \in \mathscr{M}^{im,\mathbb{R}}_{0,r+2m}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))} W_\varphi(\xi) \ . \tag{4}$$

Theorem 1

(1) Let X be a real del Pezzo surface with $\mathbb{R}X \neq \emptyset$, $F \subset \mathbb{R}X$ a connected component, $\varphi \in H_2(X \setminus F, \mathbb{Z}/2)$ a conjugation-invariant class, $D \in \mathrm{Pic}^{\mathbb{R}}_+(X)$ a nef and big, F-compatible divisor class, $\boldsymbol{k} = (k_1, \ldots, k_r)$ a (possibly empty)

sequence of positive odd integers such that

$$\max\{k_1, \dots, k_r\} \le 3 \ , \tag{5}$$

and $\boldsymbol{l} = (l_1, \dots, l_m)$ *a (possibly empty) sequence of positive integers satisfying (1),* $\boldsymbol{z} = (z_1, \dots, z_r)$ *a sequence of distinct points of* F*,* $\boldsymbol{w} = (w_1, \dots, w_m, \overline{w}_1, \dots, \overline{w}_m)$ *a sequence of distinct points of* $X \setminus \mathbb{R}X$*, and, at last,* $\mathscr{A}$*,* $\mathscr{B}$ *are arc sequences as in (2), (3). Then the number* $W(X, D, F, \varphi, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}, \boldsymbol{w}), (\mathscr{A}, \mathscr{B}))$ *does not depend neither on the choice of generic point configuration* $\boldsymbol{z}$*,* $\boldsymbol{w}$ *nor on the choice of arc sequences* $\mathscr{A}$*,* $\mathscr{B}$ *subject to conditions indicated above.*

(2) If tuples (X, D, F, φ) *and* (X', D', F', φ') *are deformation equivalent so that* X *and* X' *are joined by a flat family of real smooth rational surfaces, then we have (omitting* $(\boldsymbol{z}, \boldsymbol{w})$ *and* $(\mathscr{A}, \mathscr{B})$ *in the notation)*

$$W(X, D, F, \varphi, (\boldsymbol{k}, \boldsymbol{l})) = W(X', D', F', \varphi', (\boldsymbol{k}, \boldsymbol{l})) \ .$$

Remark 1

(1) If $k_i = l_j = 1$ for all $1 \le i \le r$, $1 \le j \le m$, then we obtain original Welschinger invariants in their modified form [9], and hence the required statement follows from [12, Proposition 4 and Theorem 6]. This, in particular, yields that we have to consider the only case $-DK_X - 1 \ge 3$.

(2) In general, one cannot impose even tangency conditions at real points $z_1, \dots, z_r$. Indeed, suppose that $r \ge 1$ and $k_1 = 2s$ is even. Suppose that $-DK_X - 1 \ge 2s$ and $p_a(D) = (D^2 + DK_X)/2 + 1 \ge s$. In the linear system $|D|$, the curves, which intersect the arc A_1 at z_1 with multiplicity $\ge s$ and have at least s nodes, form a subfamily of codimension $3s$. On the other hand, the family of curves, having singularity A_{2s} at z_1 and $(s-1)$ additional infinitely near to z_1 points lying on the arc α_1, has codimension $3s + 1$, and it lies in the boundary of the former family. Over the reals, this wall-crossing results in the change of the Welschinger sign of the curve that undergoes the corresponding bifurcation. Indeed, take local coordinates x, y such that $z_1 = (0, 0)$ and $\alpha_1 = \{y = 0\}$, and consider the family of curves

$$y = t^{2s}, \quad x = \varepsilon t + t^2 + t^3, \quad \varepsilon \in (\mathbb{R}, 0) \ .$$

For $\varepsilon = 0$, the curve has singularity A_{2s} at z_1 and its next $(s-1)$ infinitely near to z_1 points belong to α_1. In turn, for $\varepsilon \ne 0$, the node, corresponding to the values $t = \pm\sqrt{-\varepsilon}$, is solitary as $\varepsilon > 0$ and non-solitary as $\varepsilon < 0$, whereas the remaining $(s-1)$ nodes stay imaginary or solitary.

Conjecture 1 Theorem 1 is valid without restriction (5).

The proof of Theorem 1 in general follows the lines of [12], where we verify the constancy of the introduced enumerative numbers in one-dimensional families of constraints and families of surfaces. The former verification requires a classification of codimension one degenerations of the curves in count, while the latter verification is based on a suitable analogue of the Abramovich–Bertram–Vakil formula [1, 16]. Restriction (5) results from the lack of our understanding of nonreduced degenerations of the counted curves.

3 Degeneration and Deformation of Curves on Complex Rational Surfaces

3.1 Auxiliary Miscellanies

(1) Tropical limit. For the reader's convenience, we shortly remind what is the tropical limit in the sense of [14, Sect. 2.3], which will be used below. In the field of complex Puiseux series $\mathbb{C}\{\{t\}\}$, we consider the non-Archimedean valuation $\text{val}(\sum_a c_a t^a) = -\min\{a\ :\ c_a \ne 0\}$. Given a polynomial (or a power series) $F(x, y) = \sum_{(i,j)\in\Delta} c_{ij}x^iy^j$ over $\mathbb{C}\{\{t\}\}$ with Newton polygon Δ, its tropical limit consists of the following data:

- A convex piecewise linear function $N_F : \Delta \to \mathbb{R}$, whose graph is the lower part of the polytope $\text{Conv}\{(i, j, -\text{val}(c_{ij}))\ :\ (i, j) \in \Delta\}$, the subdivision S_F of Δ into linearity domains of N_F, and the tropical curve T_F, the closure of $\text{val}(F = 0)$;
- Limit polynomials (power series) $F^{\delta}_{\text{ini}}(x, y) = \sum_{(i,j)\in\delta} c^0_{ij}x^iy^j$, defined for any face δ of the subdivision S_F, where $c_{ij} = t^{N_F(i,j)}(c^0_{ij}+O(t^{>0}))$ for all $(i, j) \in \Delta$.

(2) Rational curves with Newton triangles.

Lemma 1

(1) For any integer $k \ge 1$, there are exactly k polynomials $F(x, y) = \sum_{i,j} c_{ij}x^iy^j$ with Newton triangle $T = \text{Conv}\{(0, 0), (0, 2), (k, 1)\}$, whose coefficients $c_{00}, c_{01}, c_{02}, c_{11}$ are given generic nonzero constants and which define plane rational curves. Furthermore, in the space of polynomials with Newton triangle T, the family of polynomials defining rational curves intersects transversally with the linear subspace given by assigning generic nonzero constant values to the coefficients $c_{00}, c_{01}, c_{02}, c_{11}$. If the coefficients $c_{00}, c_{01}, c_{02}, c_{11}$ are real, then,

- *For an odd k, there is an odd number of real polynomials F defining rational curves, and each of these curves has an even number of real solitary nodes,*
- *For an even k, there exists an even number (possibly zero) of polynomials F defining rational curves, and half of these curves have an odd number of real solitary nodes while the other half an even number of real solitary nodes.*

(2) *For any integer* $k \ge 1$*, there are exactly* k *polynomials* $F(x,y) = \sum_{i,j} c_{ij}x^iy^j$ *with Newton triangle* $T = \text{Conv}\{(0,0),(0,2),(k,1)\}$*, whose coefficients* c_{00}, c_{02}, c_{11} *are given generic nonzero constants and the coefficient* $c_{k-1,1}$ *vanishes and which define plane rational curves. Furthermore, in the space of polynomials with Newton triangle* T *and vanishing coefficient* $c_{k-1,1}$*, the family of polynomials defining rational curves intersects transversally with the linear subspace given by assigning generic nonzero constant values to the coefficients* c_{00}, c_{02}, c_{11}*. If the coefficients* c_{00}, c_{02}, c_{11} *are real, then,*

- *For an odd* k*, there is a unique real polynomial* F *defining a rational curve, and this curve either has* $k-1$ *real solitary nodes or has no real nodes at all,*
- *For an even* k*, either there are no real polynomials defining rational curves or there are two real polynomials, one defining a rational curve with* $k-1$ *real solitary nodes and the other defining a rational curve without real solitary nodes.*

Proof Both statements can easily be derived from [14, Lemma 3.9]. ■

(3) Deformations of singular curve germs. Our key tool in the estimation of dimension of families of curves will be [8, Theorem 2] (see also [7, Lemma II.2.18]). For the reader's convenience, we remind it here. Let C be a reduced curve on a smooth surface Σ, and $z \in C$. By $\text{mt}(C,z)$, we denote the intersection multiplicity at z of C with a generic smooth curve on Σ passing through z, by $\delta(C,z)$ the δ-invariant, and by $\text{br}(C,z)$ the number of irreducible components of (C,z).

Lemma 2 *Let* C_t*,* $t \in (\mathbb{C},0)$ *be a flat family of reduced curves on a smooth surface* Σ*, and* $z_t \in C_t$*,* $t \in (\mathbb{C},0)$ *a section such that the family of germs* (C_t,z_t)*,* $t \in (\mathbb{C},0)$*, is equisingular. Denote by* U *a neighborhood of* z_0 *in* Σ *and by* $(C \cdot C')_U$ *the total intersection number of curves* C, C' *in* U*. The following lower bounds hold:*

(i) $(C_0 \cdot C_t)_U \ge \text{mt}(C_0,z_0) - \text{br}(C_0,z_0) + 2\delta(C_0,z_0)$ *for* $t \in (\mathbb{C},0)$*;*
(ii) If L *is a smooth curve passing through* $z_0 = z_t$*,* $t \in (\mathbb{C},0)$*, and* $(C_t \cdot L)_{z_0} = \text{const}$*, then*

$$(C_0 \cdot C_t)_U \ge (C_0 \cdot L)_{z_0} + \text{mt}(C_0,z_0) - \text{br}(C_0,z_0) + 2\delta(C_0,z_0)$$

for $t \in (C,0)$.
(iii) If L *is a smooth curve containing the family* z_t*,* $t \in (\mathbb{C},0)$*, and* $(C_t \cdot L)_{z_t} = \text{const}$*, then*

$$(C_0 \cdot C_t)_U \ge (C_0 \cdot L)_{z_0} - \text{br}(C_0,z_0) + 2\delta(C_0,z_0)$$

for $t \in (C,0)$.

Let $x, y \in (\mathbb{C},0)$ be local coordinates in a neighborhood of a point z in a smooth projective surface Σ. Let $L = \{y = 0\}$, and $(C,z) \subset (\Sigma,z)$ a reduced, irreducible curve germ such that $(C\cdot L)_z = s \geq 1$. Denote by $\mathfrak{m}_z \subset \mathscr{O}_{\Sigma,z}$ the maximal ideal and introduce the ideal $I^{L,s}_{\Sigma,z} = \{g \in \mathfrak{m}_z\ :\ \mathrm{ord} g\big|_{L,z} \geq s\}$. The semiuniversal deformation base of the germ (C,z) in the space of germs (C',z) subject to condition $(C'\cdot L)_z \geq s$ can be identified with the germ at zero of the space

$$B_{C,z}(L,s) := I^{L,s}_{\Sigma,z}\Big/\Big\langle f, \frac{\partial f}{\partial x}\cdot \mathfrak{m}_z, \frac{\partial f}{\partial y}\cdot I^{L,s}_{\Sigma,z}\Big\rangle\ ,$$

where $f \in \mathscr{O}_{\Sigma,z}$ locally defined the germ (C,z) (cf. [7, Corollary II.1.17]).

Lemma 3

(1) The stratum $B^{eg}_{C,z}(L,s) \subset B_{C,z}(L,s)$ parameterizing equigeneric deformations of (C,z) is smooth of codimension $\delta(C,z)$, and its tangent space is

$$T_0B^{eg}_{C,z}(L,s) = I^{L,s}_{C,z}\Big/\Big\langle f, \frac{\partial f}{\partial x}\cdot \mathfrak{m}_z, \frac{\partial f}{\partial y}\cdot I^{L,s}_{\Sigma,z}\Big\rangle\ , \tag{6}$$

where

$$I^{L,s}_{C,z} = \{g \in \mathscr{O}_{\Sigma,z}\ :\ \mathrm{ord} g\big|_{C,z} \geq s + 2\delta(C,z)\}\ .$$

(2) If Σ, (C,z), and L are real, and s is odd, then a generic member of $B^{eg}_{C,z}(L,s)$ is smooth at z and has only imaginary and real solitary nodes; the number of solitary nodes is $\delta(C,z) \mod 2$.

Proof

(1) In [10, Lemma 2.4], we proved a similar statement for the case $s = 2$ and (C,z) of type A_{2k}, $k \geq 1$, and we worked with equations. Here, we settle the general case, and we work with parameterizations. First, observe that a general member of $B^{eg}_{C,z}(L,s)$ has $\delta(C,z)$ nodes as its singularities and is smooth at z. Thus, $\mathrm{codim}_{I^{L,s}_{\Sigma,z}} B^{eg}_{C,z}(L,s) = \delta(C,z)$, the tangent space to $B^{eg}_{C,z}(L,s)$ at its generic point C', is formed by the elements $g \in \mathscr{O}_{\Sigma,z}$, which vanish at the nodes of C' and whose restriction to (L,z) has order s. Clearly, the limits of these tangent spaces as $C' \to (C,z)$ contain the space $I^{L,s}_{C,z}\big/\langle f, \frac{\partial f}{\partial x}\mathfrak{m}_z, \frac{\partial f}{\partial y}I^{L,s}_{\Sigma,z}\rangle$. On the other hand, $\dim I^{L,s}_{\Sigma,z}/I^{L,s}_{C,z} = \delta(C,z)$ (see, e.g., [13, Lemma 6]). Let us show the smoothness of $B^{eg}_{C,z}(L,s)$. Notice that the germ (C,z) admits a uniquely defined parameterization $x = t^s$, $y = \varphi(t)$, $t \in (\mathbb{C},0)$, where $\varphi(0) = 0$, and each element $C' \in B^{eg}_{C,z}(L,s)$ admits a unique parameterization $x = t^s$, $y = \varphi(t) + \sum_{i=1}^m a_it^i$, where $m = \dim B^{eg}_{C,z}(L,s)$, $a_1,\dots,a_m \in (\mathbb{C},0)$. Choose m distinct generic values $t_1,\dots,t_m \in (\mathbb{C},0)\setminus\{0\}$ and take the germs of the lines $L_i = \{(t_i^s,y)\ :\ y \in (\mathbb{C},\varphi(t_i)\}$, $i = 1,\dots,m$. It follows that the stratum $B^{eg}_{C,z}(L,s)$ is diffeomorphic to $\prod_{i=1}^m L_i$, hence the smoothness and (6).

(2) The second statement follows from the observation that the equation $t_1^s = t_2^s$ has no real solutions $t_1 \neq t_2$. ■

Let $C^{(1)}, C^{(2)} \subset \Sigma$ be two distinct immersed rational curves, $z \in C^{(1)} \cap C^{(2)}$ a smooth point of both $C^{(1)}$ and $C^{(2)}$, and $W_z \subset \Sigma$ a sufficiently small neighborhood of z. Denote by $V \subset |C^{(1)} + C^{(2)}|$ the germ at $C^{(1)} \cup C^{(2)}$ of the family of curves, whose total δ-invariant in $\Sigma \setminus U$ coincides with that of $C^{(1)} \cup C^{(2)}$.

Lemma 4

(1) The germ V is smooth of dimension

$$c = (C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma - C^{(2)}K_\Sigma - 2 ,$$

and its tangent space isomorphically projects onto the space $\mathscr{O}_{\Sigma,z}/I_z$*, where*

$$I_z = \{f \in \mathscr{O}_{\Sigma,z} \ : \ \mathrm{ord} f\big|_{(C^{(i)},z)} \ge (C^{(1)} \cdot C^{(2)})_z - C^{(i)}K_\Sigma - 1,\ i = 1,2\} .$$

(2) Let $f_1, \dots, f_c, f_{c+1}, \dots$ *be a basis of the tangent space to* $|C^{(1)} + C^{(2)}|$ *at* $C^{(1)} \cup C^{(2)}$ *such that* $f_1, \dots, f_c$ *project to a basis of* $\mathscr{O}_{\Sigma,z}/I_z$*, and* $f_j \in I_z$*,* $j > c$*, satisfy*

$$\mathrm{ord} f_{c+1}\big|_{(C^{(1)},z)} = (C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma - 1 ,$$
$$\mathrm{ord} f_j\big|_{(C^{(1)},z)} \ge (C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma, \quad j > c+1 ,$$

and let

$$\sum_{i=1}^{c} t_i f_i + \sum_{j>c} a_j(\bar{t}) f_j, \quad \bar{t} = (t_1, \dots, t_c) \in (\mathbb{C}^c, 0) ,$$

be a parameterization of V, where $C^{(1)} \cup C^{(2)}$ *corresponds to the origin, and* a_j*,* $j > c$ *are analytic functions vanishing at zero. Then*

$$\frac{\partial a_{c+1}}{\partial t_i}(0) \neq 0 \quad \text{for all } 1 \le i \le c \text{ with } \mathrm{ord} f_i\big|_{(C^{(1)},z)} \le (C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma - 2 . \tag{7}$$

Proof Let $\nu^{(i)} : \mathbb{P}^1 \to C^{(i)} \hookrightarrow \Sigma$ be the normalization, $p_i = (\nu^{(i)})^*(z)$, $i = 1,2$. Note that by Riemann–Roch

$$h^k(\mathbb{P}^1, \mathscr{N}_{\mathbb{P}^1}^{\nu^{(i)}}(-(-C^{(i)}K_\Sigma - 1)p_i)) = 0, \quad k = 0,1,\ i = 1,2 ,$$

where $\mathscr{N}$ denotes the normal bundle of the corresponding map, and observe that the codimension of I_z in $\mathscr{O}_{\Sigma,z}$ equals c. The first statement of lemma follows.

For the second statement, we note that a generic irreducible element $C \in V$ satisfies

$$(C \cdot C^{(1)})_{W_z} \le C^{(1)}C^{(2)} + (C^{(1)})^2 - (C^{(1)}C^{(2)} - (C^{(1)} \cdot C^{(2)})_z) \\ -((C^{(1)})^2 + C^{(1)}K_\Sigma + 2) = (C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma - 2\,. \quad (8)$$

Next, we choose $i \in \{1, \dots, c\}$ as in (7) and take $C \in V$ given by the parameter values $t_i = t$, $t_j = t^s$ with some $s > 1$ for all $j \in \{1, \dots, c\} \setminus \{i\}$. Then, if $a_{c+1} = O(t^m)$ with $m > 1$, one encounters at least $(C^{(1)} \cdot C^{(2)})_z - C^{(1)}K_\Sigma - 1$ intersection points of C and $C^{(1)}$ in W_z. Thus, (7) follows. ■

(4) Geometry of arc spaces. Let Σ be a smooth projective surface. Given an integer $s \ge 0$, denote by $\mathrm{Arc}_s(\Sigma)$ the vector bundle of s-arcs over Σ and by $\mathrm{Arc}_s^{\mathrm{sm}}(\Sigma)$ the bundle of smooth s-arcs over Σ (particularly, $\mathrm{Arc}_0(\Sigma) = \mathrm{Arc}_0^{\mathrm{sm}}(\Sigma) = \Sigma$). For any smooth curve $C \subset \Sigma$, we have a natural map $\mathrm{arc}_s : C \to \mathrm{Arc}_s^{\mathrm{sm}}(\Sigma)$, sending a point $z \in C$ to the s-arc at z defined by the germ (C, z). The following statement immediately follows from basic properties of ordinary analytic differential equations:

Lemma 5 *Let $s \ge 1$, U a neighborhood of a point $z \in \Sigma$, and σ a smooth section of the natural projection* $\mathrm{pr}_s : \mathrm{Arc}_s^{\mathrm{sm}}(U) \to \mathrm{Arc}_{s-1}^{\mathrm{sm}}(U)$. *Then there exists a smooth analytic curve Λ passing through z, defined in a neighborhood $V \subset U$ of z, and such that* $\mathrm{arc}_s(\Lambda) \subset \sigma(\mathrm{Arc}_{s-1}^{\mathrm{sm}}(V))$.

Now, let Σ be a smooth rational surface, $\boldsymbol{n} : \mathbb{P}^1 \to \Sigma$ an immersion, $C = \boldsymbol{n}(\mathbb{P}^1) \in |D|$, where $-DK_\Sigma = k > 0$. Pick a point $p \in \mathbb{P}^1$ such that $z = \boldsymbol{n}(p)$ is a smooth point of C. Denote by $U \subset \mathrm{Arc}_{k-1}(\Sigma)$ the natural image of the germ of $\mathscr{M}_{0,1}(\Sigma, D)$ at $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, p]$. Choose coordinates x, y in a neighborhood of z so that $z = (0,0)$, $C = \{y + x^k = 0\}$, and introduce two one-parameter subfamilies $\Lambda' = (z'_t, \alpha'_t)_{t\in(\mathbb{C},0)}$ and $\Lambda'' = (z''_t, \alpha''_t)_{t\in(\mathbb{C},0)}$ of $\mathrm{Arc}_{k-1}(\Sigma)$:

$$z'_t = (t, 0),\ \alpha'_t = \{y = (x-t)^l\}, \quad z''_t = (0,0),\ \alpha''_t = \{y = tx^{k-1}\}, \quad t \in (\mathbb{C}, 0)\,,$$

where $l > k$.

Lemma 6 *The germ U is smooth of codimension one in* $\mathrm{Arc}_{k-1}(\Sigma)$*, and it transversally intersects both Λ' and Λ''.*

Proof It follows from Riemann–Roch and from Lemma 2(iii) that V admits the following parameterization:

$$((x_0, y_0), \{y{=}y_0 + a_1(x-x_0) + \ldots + a_{k-2}(x-x_0)^{k-2} {+} \varphi(x_0, y_0, \overline{a})(x-x_0)^{k-1}\})\,,$$

$$x_0, y_0, a_1, \dots, a_{k-2} \in (\mathbb{C}, 0), \quad \overline{a} = (a_1, \dots, a_{k-2}), \quad \varphi(0) = 0, \quad \frac{\partial\varphi}{\partial x_0}(0) \ne 0\,.$$

Thus, V is a smooth hypersurface. The required intersection transversality results from a routine computation. ∎

3.2 Families of Curves and Arcs on Arbitrary del Pezzo Surfaces

Let Σ be a smooth del Pezzo surface of degree 1, and $D \in \text{Pic}(\Sigma)$ be an effective divisor such that $-DK_\Sigma - 1 > 0$. Fix positive integers $n \le -DK_\Sigma - 1$ and $s \gg -DK_\Sigma - 1$. Denote by $\mathring{\Sigma}^n \subset \Sigma^n$ the complement of the diagonals and by $\text{Arc}_s(\mathring{\Sigma}^n)$ the total space of the restriction to $\mathring{\Sigma}^n$ of the bundle $(\text{Arc}_s(\Sigma))^n \to \Sigma^n$. In this section, we stratify the space $\text{Arc}_s(\mathring{\Sigma}^n)$ with respect to the intersection of arcs with rational curves in $|D|$, and we describe all strata of codimension zero and one.

Introduce the following spaces of curves: given $(\boldsymbol{z}, \mathscr{A}) \in \text{Arc}_s(\mathring{\Sigma}^n)$, $\boldsymbol{z} = (z_1, \ldots, z_n)$, $\mathscr{A} = (\alpha_1, \ldots, \alpha_n)$, and a sequence $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n_{>0}$ summing up to $|\boldsymbol{s}| \le s$, put

$$\mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) = \{[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}(\Sigma, D) \ : \ \boldsymbol{n}(p_i) = z_i, \quad \boldsymbol{n}^*(\alpha_i) \ge s_i p_i, \quad i = 1, \ldots, n\},$$

$$\mathscr{M}^{br}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) = \{[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \ : \ \boldsymbol{n} \text{ is birational onto its image}\},$$

$$\mathscr{M}^{im}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) = \{[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}^{br}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \ : \ \boldsymbol{n} \text{ is an immersion}\},$$

$$\mathscr{M}^{sing,1}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) = \{[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}^{br}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \ : \ \boldsymbol{n} \text{ is singular in } \mathbb{P}^1 \setminus \boldsymbol{p} \text{ and smooth at } \boldsymbol{p}\},$$

$$\mathscr{M}^{sing,2}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) = \{[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}^{br}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \ : \ \boldsymbol{n} \text{ is singular at some point } p_i \in \boldsymbol{p}\}.$$

We shall consider the following strata in $\text{Arc}^{\text{sm}}_s(\mathring{\Sigma}^n)$:

(i) The subset $U^{im}(D) \subset \text{Arc}^{\text{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following conditions: For any sequence $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n_{>0}$ summing up to $|\boldsymbol{s}| \le s$ and for any element $(\boldsymbol{z}, \mathscr{A}) \in U^{im}(D)$, where $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\Sigma}^n$, $\mathscr{A} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \text{Arc}_s(\Sigma, z_i)$, the family $\mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ is empty if $|\boldsymbol{s}| \ge -DK_\Sigma$ and is finite if $|\boldsymbol{s}| = -DK_\Sigma - 1$. Furthermore, in the latter case, all elements $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ are represented by immersions $\boldsymbol{n} : \mathbb{P}^1 \to \Sigma$ such that $\boldsymbol{n}^*(\alpha_i) = s_i p_i$, $1 \le i \le n$.

(ii) The subset $U^{im}_+(D)\subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following condition:
For any element $(z,\mathscr{A})\in U^{im}_+(D)$, there exists $s\in\mathbb{Z}^n_{>0}$ with $|s|\ge -DK_\Sigma$ such that $\mathscr{M}^{im}_{0,n}(\Sigma,D,s,z,\mathscr{A})\ne\emptyset$.

(iii) The subset $U^{sing}_1(D)\subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following condition:
For any element $(z,\mathscr{A})\in U^{im}_+(D)$, there exists $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$ such that $\mathscr{M}^{sing,1}_{0,n}(\Sigma,D,s,z,\mathscr{A})\ne\emptyset$.

(iv) The subset $U^{sing}_2(D)\subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following condition:
For any element $(z,\mathscr{A})\in U^{sing}_2(D)$, there exists $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$ such that $\mathscr{M}^{sing,2}_{0,n}(\Sigma,D,s,z,\mathscr{A})\ne\emptyset$.

(v) The subset $U^{mt}(D)\subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following condition:
For any element $(z,\mathscr{A})\in U^{mt}(D)$, there exists $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$ and $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\mathscr{M}_{0,n}(\Sigma,D,s,z,\mathscr{A})$ such that $\boldsymbol{n}$ is a multiple cover of its image.

Proposition 1

(1) The set $U^{im}(D)$ is Zariski open and dense in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$.

(2) If $U\subset U^{im}_+(D)$ is a component of codimension one in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$, then, for a generic element $(z,\mathscr{A})\in U$ and any sequence $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma$, the set $\mathscr{M}^{im}_{0,n}(\Sigma,D,s,z,\mathscr{A})$ is either empty or finite, and all of its elements $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]$ are presented by immersions and satisfy $\boldsymbol{n}^*(z_i)=s_ip_i$, $i=1,\dots,n$.

(3) If $U\subset U^{sing}_1(D)$ is a component of codimension one in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$, then, for a generic element $(z,\mathscr{A})\in U$ and any sequence $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$, the set $\mathscr{M}^{sing,1}_{0,n}(\Sigma,D,s,z,\mathscr{A})$ is either empty or finite, whose all elements $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]$ satisfy $\boldsymbol{n}^*(z_i)=s_ip_i$, $i=1,\dots,n$.

(4) If $U\subset U^{sing}_2(D)$ is a component of codimension one in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$, then, for a generic element $(z,\mathscr{A})\in U$ and any sequence $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$, the set $\mathscr{M}^{sing,2}_{0,n}(\Sigma,D,s,z,\mathscr{A})$ is either empty or finite, whose all elements $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]$ satisfy $\boldsymbol{n}^*(z_i)=s_ip_i$, $i=1,\dots,n$.

(5) If $U\subset U^{mt}(D)$ is a component of codimension one in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$, then, for a generic element $(z,\mathscr{A})\in U$ and any sequence $s\in\mathbb{Z}^n_{>0}$ with $|s|=-DK_\Sigma-1$, the following holds: Each element $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\mathscr{M}_{0,n}(\Sigma,D,s,z,\mathscr{A})$ satisfying $C'=\boldsymbol{n}(\mathbb{P}^1)\in|D'|$, where $D=kD'$, $k\ge 2$, admits a factorization

$$\boldsymbol{n}:\mathbb{P}^1\xrightarrow{\rho}\mathbb{P}^1\xrightarrow{\nu}C'\hookrightarrow\Sigma$$

with ρ a k-multiple ramified covering, ν the normalization, $\boldsymbol{p}'=\rho(\boldsymbol{p})$, for which one has

$$[\nu:\mathbb{P}^1\to\Sigma,\boldsymbol{p}']\in\mathscr{M}_{0,n}(\Sigma,D',s',z,\mathscr{A})\ ,$$

where $|s'|=-D'K_\Sigma$, and all branches $\nu\big|_{\mathbb{P}^1,p'_i}$ are smooth.

Proof

(1) A general element of $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\mathscr{M}_{0,n}(\Sigma,D)$ is represented by an immersion sending $\boldsymbol{p}$ to n distinct points of Σ (cf. [12, Lemma 9(1ii)]). Let $(\boldsymbol{z},\mathscr{A})\in\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$, and a sequence $\boldsymbol{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n_{>0}$ satisfy $|\boldsymbol{s}|=-DK_\Sigma-1$. The fiber of the map $\mathrm{arc}_{\boldsymbol{s}}:\mathscr{M}_{0,n}(\Sigma,D)\to\prod_{i=1}^n\mathrm{Arc}^{\mathrm{sm}}_{s_i-1}(\Sigma)$, sending an element $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]$ to the collection of arcs defined by the branches $\boldsymbol{n}\big|_{\mathbb{P}^1,p_i}$, is either empty or finite. Indeed, otherwise, by Lemma 2(ii), we would get a contradiction:

$$D^2\ge(D^2+DK_\Sigma+2)+|\boldsymbol{s}|=D^2+1>D^2\ .$$

On the other hand,

$$\dim\mathscr{M}_{0,n}(\Sigma,D)=\dim\prod_{i=1}^n\mathrm{Arc}^{\mathrm{sm}}_{s_i-1}(\Sigma)=-DK_\Sigma-1+n\ ,$$

and hence the map $\mathrm{arc}_{\boldsymbol{s}}$ is dominant. It follows, that, for a generic element $(\boldsymbol{z},\mathscr{A})\in\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$ and any sequence $\boldsymbol{s}\in\mathbb{Z}^n_{\ge0}$ such that $|\boldsymbol{s}|\le s$, one has: $\mathscr{M}^{im}_{0,n}(\Sigma,D,\boldsymbol{s},\boldsymbol{z},\mathscr{A})$ is empty if $|\boldsymbol{s}\ge-DK_\Sigma$ and $\mathscr{M}^{im}_{0,n}(\Sigma,D,\boldsymbol{s},\boldsymbol{z},\mathscr{A})$ is finite nonempty if $|\boldsymbol{s}|=-DK_\Sigma-1$. The same argument proves Claims (2) and (3) together with the fact that $U^{im}_+(D)$ and $U^{sing}_1(D)$ have positive codimension in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$.

Next, we will show that the sets $U^{sing}_2(D)$ and $U^{mt}(D)$ have positive codimension in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$, thereby completing the proof of Claim (1), and will prove Claims (4) and (5).

(2) To proceed further, we introduce additional notations. Let $f:(\mathbb{C},0)\to(C,z)\hookrightarrow(\Sigma,z)$ be the normalization of a reduced, irreducible curve germ (C,z), and let $m_0,m_1,\dots$ be the multiplicities of (C,z) and of its subsequent strict transforms under blowups. We call this (infinite) sequence the *multiplicity sequence* of $f:(\mathbb{C},0)\to\Sigma$ and denote it $\overline{m}(f)$. Note that, if $z_0=z$ and the infinitely near points $z_1,\dots,z_j$, $0\le j\le s$, of (C,z) belong to an arc from $\mathrm{Arc}_s^{\mathrm{sm}}(\Sigma,z)$, then $m_0=\ldots=m_{j-1}$ (see, for instance, [2, Chap. III]). Such sequences $m_0,\dots,m_j$ will be called *smooth sequences*. Given smooth sequences $\overline{m}_i=(m_{0i},\dots,m_{j(i),i})$ such that $|\overline{m}_i|:=\sum_l m_{li}\le s$, $i=1,\dots,n$, denote by $\mathscr{M}_{0,n}(\Sigma,D,\{\overline{m}_i\}_{i=1}^n)$ the family of elements $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\mathscr{M}_{0,n}(\Sigma,D)$ such that $\boldsymbol{n}$ is birational onto its image and $\overline{m}(\boldsymbol{n}\big|_{\mathbb{P}^1,p_i})$ contains $\overline{m}_i$ for every $i=1,\dots,n$. Put

$$\begin{aligned}\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n,D,\{\overline{m}_i\}_{i=1}^n)=\{(\boldsymbol{z},\mathscr{A})\in\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)\ :\ &\text{there exists}\\ &[\boldsymbol{n}:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\mathscr{M}_{0,n}(\Sigma,D,\{\overline{m}_i\}_{i=1}^n)\\ &\text{such that }\boldsymbol{n}(\boldsymbol{p})=\boldsymbol{z}\text{ and }\boldsymbol{n}^*(\alpha_i)\ge|\overline{m}_i|p_i,\ i{=}1,\dots,n\}\end{aligned}$$

(3) We now prove Claim (4) together with the fact that $U_2^{sing}(D)$ has positive codimension in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$.

Let $(z, \mathscr{A})$ be a generic element of a top-dimensional component $U \subset U_2^{sing}(D)$, a sequence $\boldsymbol{s} \in \mathbb{Z}_{>0}^n$ satisfy $|\boldsymbol{s}| = -DK_\Sigma - 1$, and $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}^{br}(\Sigma, D, \boldsymbol{s}, z, \mathscr{A})$ have singular branches among $\boldsymbol{n}|_{\mathbb{P}^1, p_i}$, $i = 1, \ldots, n$. Let $\overline{m}_i = (m_{0i}, \ldots, m_{j(i),0})$ be a smooth multiplicity sequence of the branch $\boldsymbol{n}|_{\mathbb{P}^1, p_i}$ such that $|\overline{m}_i| \ge s_i$. Denote by $\mathscr{V}$ the germ at $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}]$ of a top-dimensional component of $\mathscr{M}_{0,n}(\Sigma, D, \{\overline{m}_i\}_{i=1}^n)$. Without loss of generality, we can suppose that $\mathscr{M}_{0,n}^{br}(\Sigma, D, \boldsymbol{s}, z, \mathscr{A}) \subset \mathscr{M}_{0,n}(\Sigma, D, \{\overline{m}_i\}_{i=1}^n)$ and $U \subset \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n, D, \{\overline{m}_i\}_{i=1}^n)$.

Note that $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}]$ is isolated in $\mathscr{M}_{0,n}^{br}(\Sigma, D, \boldsymbol{s}, z, \mathscr{A})$. Indeed, otherwise Lemma 2(ii) would yield a contradiction:

$$D^2 \ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^n (m_{0i} - 1 + |\overline{m}_i|) \ge (D^2 + DK_\Sigma + 2) + |\boldsymbol{s}| = D^2 + 1 > D^2 \,.$$

Next, we can suppose that $m_{0i} \ge 2$ as $1 \le i \le r$ for some $1 \le r \le n$ and that $m_{0i} = 1$ for $r < i \le n$.

Consider the case when $|\overline{m}_i| = s_i$ for all $i = 1, \ldots, n$. We claim that then

$$\dim \mathscr{V} \le \sum_{i=1}^n j(i) + n + r - 1 \,. \tag{9}$$

If so, we would get

$$\dim U \le \sum_{i=1}^n (s - j(i)) + n - r + \dim \mathscr{V} \le n(s+2) - 1 = \dim \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n) - 1 \,,$$

and the equality would yield $(\boldsymbol{n}')^*(z, \mathscr{A}) = \sum_{i=1}^n s_i = -DK_\Sigma - 1$ for each element $[\boldsymbol{n}' : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}'] \in \mathscr{M}_{0,n}^{sing,2}(\Sigma, D, \boldsymbol{s}, z, \mathscr{A})$ with generic $(z, \mathscr{A}) \in U$, as required in Claim (3). To prove (9), we show that the assumption

$$\dim \mathscr{V} \ge \sum_{i=1}^n j(i) + n + r \tag{10}$$

leads to contradiction. Namely, we impose $\sum_{i=1}^n j(i) + n + r - 1$ conditions, defining a positive-dimensional subfamily of $\mathscr{V}$ containing $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}]$, and apply Lemma 2. It is enough to consider the following situations:

(a) $1 \le r < n$;
(b) $1 < r = n, j(1) > 0$;
(c) $1 = r = n, j(1) > 0, m_{01} > m_{j(1),1}$;

(d) $r = n, j(1) = \ldots = j(n) = 0$;
(e) $1 = r = n, j(1) > 0, m_{01} = \ldots = m_{j(1),1}$.

In case (a), we fix the position of z_i and of the next $j(i)$ infinitely near points for $i = 1, \ldots, r$, and the position of additional $\sum_{i=r+1}^{n} j(i) + n - r - 1$ smooth points on $C = \boldsymbol{n}(\mathbb{P}^1)$, obtaining a positive-dimensional subfamily of U and a contradiction by Lemma 2:

$$D^2 \geq (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{r}(m_{0i} - 1 + |\overline{m}_i|) + \sum_{i=r+1}^{n} j(i) + n - r - 1$$
$$= D^2 + \sum_{i=1}^{r}(m_{0i} - 1) > D^2 \ .$$

In case (b), we fix the position of $\boldsymbol{z}$ and of additional infinitely near points: $j(1) - 1$ points for z_1, and $j(i)$ points for all $2 \leq i \leq n$. These conditions define a positive-dimensional subfamily of U, which implies a contradiction by Lemma 2:

$$D^2 \geq (D^2 + DK_\Sigma + 2) + \sum_{i=2}^{r}(m_{0i} - 1 + |\overline{m}_i|) + (m_{01} - 1) + |\overline{m}_1| - m_{j(1),1}$$
$$\geq D^2 + \sum_{i=2}^{n}(m_{0i} - 1) > D^2 \ .$$

In case (c), the same construction similarly leads to a contradiction:

$$D^2 \geq (D^2 + DK_\Sigma + 2) + (m_{01} - 1) + \sum_{0 \leq k < j(1)} m_{k1} \geq (D^2 + DK_\Sigma + 2) + |\overline{m}_1| = D^2 + 1 > D^2 \ .$$

In case (d), we fix the position of z_i, $1 < i \leq n$ and of one more smooth point of $C = \boldsymbol{n}(\mathbb{P}^1)$. Thus, Lemma 2, applied to the obtained positive-dimensional family, yields a contradiction:

$$D^2 \geq (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n}(m_{0i} - 1) + \sum_{1 < i \leq n} m_{0i} + 1 = D^2 + \sum_{1 < i \leq n}(m_{0i} - 1) + 1 > D^2 \ .$$

In case (e), relation (10) reads $\dim \mathcal{V} \geq j(1) + 2 = \dim \mathrm{Arc}_{j(1)}(\Sigma)$. As noticed above, the map $\mathrm{arc}_{j(1)} : \mathcal{V} \to \mathrm{Arc}_{j(1)}(\Sigma)$ is finite. Hence, $\dim \mathcal{V} = j(1) + 2$, and (due to the general choice of $\xi = [\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, p] \in \mathcal{V}$) the germ $(\mathcal{V}, \xi)$ diffeomorphically maps onto the germ of $\mathrm{Arc}_{j(1)}(\Sigma)$ at $\pi(\xi)$. Observe that the fragment $(m_{01}, \ldots, m_{j(1),1}, m_{j(1)+1,1})$ of the multiplicity sequence of $\boldsymbol{n}|_{\mathbb{P}^1,p}$ is a smooth sequence. That means, the map of $(\mathcal{V}, \xi)$ to $\mathrm{Arc}_{j(1)+1}(\Sigma)$ defines a section $\sigma : (\mathrm{Arc}_{j(1)}(\Sigma), \pi(\xi)) \to \mathrm{Arc}_{j(1)+1}(\Sigma)$, satisfying the hypotheses of Lemma 5. So, we take the curve Λ, defined in Lemma 5, and apply Lemma 2(iii):

$$\begin{aligned}D^2 &\ge (D^2 + DK_\Sigma + 2) + (m_{01} + \ldots + m_{j(1),1} + m_{j(1)+1,1}) - 1\\ &\ge (D^2 + DK_\Sigma + 2) + |\overline{m}_1| = D^2 + 1 > D^2 ,\end{aligned}$$

which completes the proof of (9).

Consider the case when $\sum_{i=1}^n |\overline{m}_i| > -DK_\Sigma - 1$ and show that then $\dim U \le \dim \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n) - 2$. The preceding consideration reduces the problem to the case

$$r = n \quad \text{and} \quad \sum_{i=1}^n |\overline{m}_i| - m_{j(n),n} < -DK_\Sigma - 1 < \sum_{i=1}^n |\overline{m}_i| ,$$

in which we need to prove that

$$\dim \mathscr{V} \le \sum_{i=1}^n j(i) + 2n - 2 . \qquad (11)$$

We assume that

$$\dim \mathscr{V} \ge \sum_{i=1}^n j(i) + 2n - 1 \qquad (12)$$

and derive a contradiction in the same manner as for (10). We shall separately treat several possibilities:

(a) $j(n) = 0$;
(b) $j(n) > 0$.

In case (a), we fix the position of z_i and of the additional $j(i)$ infinitely near points for all $i = 1, \ldots, n-1$, thereby cutting off $\mathscr{V}$ a positive-dimensional subfamily, and hence by Lemma 2 we get a contradiction:

$$\begin{aligned}D^2 &\ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n-1}(m_{0i} - 1 + |\overline{m}_i|) + m_{0n} - 1\\ &\ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n} |\overline{m}_i| - 1 \ge D^2 + 1 > D^2 .\end{aligned}$$

In case (b), we again fix the position of z_i and of the additional $j(i)$ infinitely near points for all $i = 1, \ldots, n-1$, thereby cutting off $\mathscr{V}$ a subfamily $\mathscr{V}'$ of dimension $\ge j(n) + 1$. Consider the map $\mathrm{arc}_{j(n)-1} : \mathscr{V}' \to \mathrm{Arc}_{j(n)-1}(\Sigma)$ and note that $\dim \mathrm{Arc}_{j(n)-1}(\Sigma) = j(n) + 1 \le \dim \mathscr{V}'$. If $\dim \pi(\mathscr{V}') \le j(n)$, fixing the position of z_n and of $j(n)-1$ additional infinitely near points, we obtain a positive-dimensional subfamily of $\mathscr{V}'$ and hence a contradiction by Lemma 2:

$$D^2 \ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n-1}(m_{0i} - 1 + |\overline{m}_i|) + (m_{0n} - 1) + |\overline{m}_n| - m_{j(n),n}$$

$$\ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n} |\overline{m}_i| - 1 \ge D^2 + 1 > D^2 \ .$$

If $\dim \pi(\mathscr{V}') = j(n)+1$, the preceding argument yields that $\dim \mathscr{V}' = j(n) + 1$, and we can suppose that the germ of $\mathscr{V}'$ at the initially chosen element $\xi = [\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{V}$ is diffeomorphically mapped onto the germ of $\mathrm{Arc}_{j(n)-1}(\Sigma)$ at $\mathrm{arc}_{j(n)-1}(\xi)$. Thus, we obtain a section σ : $(\mathrm{Arc}_{j(n)-1}(\Sigma), \pi(\xi)) \to \mathrm{Arc}_{j(n)}(\Sigma)$ defined by the map $(\mathscr{V}', \xi) \to \mathrm{Arc}_{j(n)}(\Sigma)$. It satisfies the hypotheses of Lemma 5, which allows one to construct a smooth curve Λ as in Lemma 5 and apply Lemma 2(iii):

$$D^2 \ge (D^2 + DK_\Sigma + 2) + \sum_{i=1}^{n-1}(m_{0i} - 1 + |\overline{m}_i|) + |\overline{m}_n| - 1 \ge D^2 + 1 > D^2 \ ,$$

a contradiction.

The proof of Claim (4) is completed.

(4) It remains to consider the set $U^{mt}(D)$. Let $(\boldsymbol{z}, \mathscr{A}) \in U^{mt}(D)$, $\boldsymbol{s} \in \mathbb{Z}^n_{>0}$ satisfy $|\boldsymbol{s}| = -DK_\Sigma - 1$, and $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ be such that $\boldsymbol{n}$ is a k-multiple (ramified) covering of its image $C = \boldsymbol{n}(\mathbb{P}^1)$, $k \ge 2$. We have $C \in |D'|$, where $kD' = D$, and $\nu^*(\alpha_i) \ge s'_i p'_i$, $\rho^*(p'_i) \ge l_i p_i$, where $l_i s'_i \ge s_i$ for all $i = 1, \dots, n$. Since $l_i \le k$ for all $i = 1, \dots, n$, it follows that

$$\sum_{i=1}^{n} s'_i \ge \frac{|\boldsymbol{s}|}{k} = \frac{-DK_\Sigma - 1}{k} = -D'K_\Sigma - \frac{1}{k} > -D'K_\Sigma - 1 \ .$$

This yields that $U^{mt}(D)$ has positive codimension in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$, and, furthermore, if not all branches $\nu\big|_{\mathbb{P}^1, p'_i}$, $i = 1, \dots, n$, are smooth, the codimension of $U^{mt}(D)$ in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is at least 2. The proof of Claim (4) and thereby of Claim (1) is completed. ■

3.3 Families of Curves and Arcs on Generic del Pezzo Surfaces

Let Σ be a smooth del Pezzo surface of degree 1 satisfying the following condition:

(GDP) There are only finitely many effective divisor classes $D \in \mathrm{Pic}(\Sigma)$ satisfying $-DK_\Sigma = 1$, and for any such divisor D, the linear system $|D|$ contains only finitely many rational curves, all these rational curves are immersed, and any two curves $C_1 \neq C_2$ among them intersect in C_1C_2 distinct points.

By Itenberg et al. [12, Lemmas 9 and 10], these del Pezzo surfaces form an open dense subset in the space of del Pezzo surfaces of degree 1.

Let us fix an effective divisor $D \in \mathrm{Pic}(\Sigma)$ such that $-DK_\Sigma - 1 \geq 3$.

Proposition 2 *In the notation of Sect. 3.2, let* $(z_0, \mathscr{A}_0)$ *be a generic element of a component* U *of* $U^{mt}(D)$ *having codimension one in* $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$*, a sequence* $\boldsymbol{s} \in \mathbb{Z}^n_{>0}$ *satisfy* $|\boldsymbol{s}| = -DK_\Sigma - 1$*, and* $[\boldsymbol{n}_0 : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}_0] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, z_0, \mathscr{A}_0)$ *be such that* $\boldsymbol{n}_0$ *covers its image with multiplicity* $k \geq 2$ *so that* $\boldsymbol{n}_0(\mathbb{P}^1) \in |D'|$*, where* $D = kD'$*, and* $\boldsymbol{n}_0 = \nu \circ \rho$ *with* $\nu : \mathbb{P}^1 \to C'$ *the normalization,* $\rho : \mathbb{P}^1 \to \mathbb{P}^1$ *a* k*-fold ramified covering. Assume that* $(z_t, \mathscr{A}_t) \in \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$*,* $t \in (\mathbb{C}, 0)$*, is the germ at* $(z_0, \mathscr{A}_0)$ *of a generic one-dimensional family such that* $(z_t, \mathscr{A}_t) \notin U^{mt}(D)$ *as* $t \neq 0$*, and assume that there exists a family* $[\boldsymbol{n}_t : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}_t] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, z_t, \mathscr{A}_t)$ *extending the element* $[\boldsymbol{n}_0 : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}_0]$*. Then* $n = 3$*,* $k = 2$*,* $-D'K_\Sigma = 3$*,* $\boldsymbol{s} = (2, 2, 1)$*, and* $[\nu : \mathbb{P}^1 \to C' \hookrightarrow \Sigma, \boldsymbol{p}'_0] \in \mathscr{M}_{0,3}(\Sigma, D', \boldsymbol{s}', z_0, \mathscr{A}_0)$*, where* $\boldsymbol{p}' = \rho(\boldsymbol{p}_0)$ *and* $\boldsymbol{s}' = (1, 1, 1)$*. Furthermore, the family* $[\boldsymbol{n}_t : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}_t]$*,* $t \in (\mathbb{C}, 0)$*, is smooth and isomorphically projects onto the family* $(z_t, \mathscr{A}_t)$*,* $t \in (\mathbb{C}, 0)$*.*

Proof Note, first, that by the assumption (GDP) and Proposition 1(2, 5), the map $\boldsymbol{n}_0 : \mathbb{P}^1 \to \Sigma$ is an immersion, and (in the notation of Proposition 1(5))

$$\nu^*(\alpha_i) = s'_i p'_i, \ i = 1, \ldots, n, \quad \sum_{i=1}^n s'_i = -D'K_\Sigma \ . \tag{13}$$

Furthermore, if $C' = \boldsymbol{n}_0(\mathbb{P}^1) \in |D'|$, where $D = kD'$, then $(D')^2 > 0$, since the assumption $-DK_\Sigma \geq 4$ yields $D^2 \geq 2$ by the adjunction formula. Hence, in the deformation $\boldsymbol{n}_t : \mathbb{P}^1 \to \Sigma$, $t \in (\mathbb{C}.0)$, in a neighborhood of each singular point z of C', there appear singular points of $C_t = \boldsymbol{n}_t(\mathbb{P}^1)$, $t \neq 0$, with total δ-invariant at least $k^2\delta(C', z)$, which implies

$$k^2\left(\frac{(D')^2 + D'K_\Sigma}{2} + 1\right) \leq \frac{k^2(D')^2 + kD'K_\Sigma}{2} + 1 \ , \tag{14}$$

and hence

$$-D'K_\Sigma \geq \frac{2k+2}{k} \quad \text{or, equivalently,} \quad -D'K_\Sigma \geq 3 \ . \tag{15}$$

Let $\rho^*(p_i') \geq l_i p_i$, $i = 1, \ldots, n$. We can suppose that $k \geq l_1 \geq \ldots \geq l_n$. Then

$$\sum_{i=1} l_i s_i' \geq -kD'K_\Sigma - 1 \quad \Longrightarrow \quad \sum_{i=1}^{n} (l_i - 1)s_i' \geq -(k-1)D'K_\Sigma - 1 \ . \tag{16}$$

If $l_1 \leq k-1$, then (13) and (16) yield

$$-(k-2)D' \geq -(k-1)D'K_\Sigma - 1 \quad \Longrightarrow \quad -D'K_\Sigma \leq 1 \ ,$$

forbidden by (15), and hence

$$l_1 = k \ . \tag{17}$$

By Riemann–Hurwitz, $\sum_{i>1}(l_i - 1) \leq k - 1$, and then it follows from (16) that

$$(k-1)(-D'K_\Sigma - (n-1)) + (k-1) \geq -(k-1)D'K_\Sigma - 1 \ , \tag{18}$$

or, equivalently

$$(n-2)(k-1) \leq 1 \ , \tag{19}$$

which in view of Riemann–Hurwitz and (17)–(19) leaves the following options:

- either $n = 1$,
- or $n = 2$, $\boldsymbol{s} = (k(-D'K_\Sigma - 1), (k-1))$,
- or $n = 2$, $\boldsymbol{s} = (ks_1', ks_2')$, $s_1' + s_2' = -D'K_\Sigma$,
- or $n = 3$, $\boldsymbol{s} = (2(-D'K_\Sigma - 2), 2, 1)$.

Let us show that $s_1' > 1$ is not possible. Indeed, otherwise, in suitable local coordinates x, y in a neighborhood of z_1 in Σ, we would have $z_1 = (0,0)$, $C' = \{y = 0\}$, $\boldsymbol{n}_0 : (\mathbb{P}^1, p_1) \to (\Sigma, z_1)$ acts by $\tau \in (\mathbb{C}, 0) \simeq (\mathbb{P}^1, p_1) \mapsto (\tau^k, \tau)$, and we also may assume that the family of arcs $\alpha_{1,t}$ is centered at z_1 and given by $y = \sum_{i \geq s_1'} a_i(t)x^i$ with $a_i(0) \neq 0$, $i \geq s_1'$. Then $\boldsymbol{n}_t : (\mathbb{P}^1, p_{1,t}) \to (\Sigma, z_1)$ can be expressed via $\tau \in (\mathbb{C}, 0) \simeq (\mathbb{P}^1, p_{1,t}) \mapsto (\tau^k + tf(t,\tau), tg(t,\tau))$, which contradicts the requirement $\boldsymbol{n}_t^*(\alpha_{1,t}) \geq (ks_1' - 1)p_{1,t}$ equivalently written as

$$t \cdot g(t,\tau) \equiv \sum_{i \geq s_1'} a_i(t)(\tau^k + tf(t,\tau))^i \mod (\tau^k + tf(t,\tau))^{ks_1' - 1} \ ,$$

since the term $a_{s_1'}(0)\tau^{ks_1'}$ does not cancel out here in view of $k \geq 2$.

Thus, in view of (15), we are left with $n = 3$, $k = 2$, $\boldsymbol{s}' = (1,1,1)$, and $\boldsymbol{s} = (2,2,1)$. Without loss of generality, for $(z_t, \mathscr{A}_t)$, $t \in (\mathbb{C}, 0)$, we can choose the family consisting of two fixed points $z_{1,0}, z_{2,0}$ and fixed arcs $\alpha_{1,0}, \alpha_{2,0}$ (transversal to C') and of a point $z_{3,\tau}$ moving along the germ Λ of a smooth curve transversally intersecting

C' at $z_{3,0}$ (τ being a regular parameter on Λ). We then claim that the evaluation

$$[\boldsymbol{n}_t : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}_t] \mapsto \boldsymbol{n}_t(p_{3.t}) = z_{3,\tau(t)}$$

is one-to-one, completing the proof of Proposition 2. So, we establish the formulated claim arguing on the contrary: If some point $z_{3,\tau}$, $\tau \neq 0$, has two preimages, then the curves $C_1 = \boldsymbol{n}_{t_1}(\mathbb{P}^1)$, $C_2 = \boldsymbol{n}_{t_2}(\mathbb{P}^1)$ intersect with total multiplicity ≥ 5 at $z_{1,0}, z_{2.0}, z_{3,\tau}$ and intersect with multiplicity $\geq \delta(C', z)$ in a neighborhood of each point $z \in \mathrm{Sing}(C')$, which altogether leads to a contradiction:

$$C_1C_2 \geq 5 + 4((D')^2 + D'K_\Sigma + 2) = 5 + D^2 - 4 = D^2 + 1\ . \qquad \blacksquare$$

The compactification $\overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ of the space $\mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ is obtained by adding the elements $[\boldsymbol{n} : \widehat{C} \to \Sigma, \boldsymbol{p}]$, where

- $\widehat{C}$ is a tree formed by $k \geq 2$ components $\widehat{C}^{(1)}, \ldots, \widehat{C}^{(k)}$ isomorphic to $\mathbb{P}^1$;
- the points of $\boldsymbol{p}$ are distinct but allowed to be at the nodes of $\widetilde{C}$;
- $[\boldsymbol{n} : \widehat{C}^{(j)} \to \Sigma, \widehat{C}^{(j)} \cap \boldsymbol{p}] \in \mathscr{M}_{0,|C^{(j)} \cap \boldsymbol{p}|}(\Sigma, D^{(j)}, \boldsymbol{s}^{(j)}, \boldsymbol{z}, \mathscr{A})$, where we suppose that the integer vector $\boldsymbol{s}^{(j)} \in \mathbb{Z}^n_{\geq 0}$ has coordinates $s_i^{(j)} > 0$ or $s_i^{(j)} = 0$ according as p_i belongs to $\widehat{C}^{(j)}$ or not, $j = 1, \ldots, k$;
- $\sum_{j=1}^k D^{(j)} = D$, where $D^{(j)} \neq 0, j = 1, \ldots, k$, and $\sum_{j=1}^k \boldsymbol{s}^{(j)} = \boldsymbol{s}$.

One can view this compactification as the image of the closure of $\mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ in the moduli space of stable maps $\overline{\mathscr{M}}_{0,n}(\Sigma, D)$ under the morphism, which contracts the components of the source curve that are mapped to points. Notice that in our compactification, the source curves $\widehat{C}$ may be not nodal, and the marked points may appear at intersection points of components of a (reducible) source curve.

Introduce the set $U^{red}(D) \subset \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$ defined by the following condition: For any element $(\boldsymbol{z}, \mathscr{A}) \in U^{red}(D)$, there exists $\boldsymbol{s} \in \mathbb{Z}^n_{>0}$ with $|\boldsymbol{s}| \geq -DK_\Sigma - 1$ such that $\overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \setminus \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \neq \emptyset$.

Proposition 3 *The set $U^{red}(D)$ has positive codimension in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$. Let $(\boldsymbol{z}, \mathscr{A})$ be a generic element of a component of $U^{red}(D)$ having codimension one in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$, and let $(\boldsymbol{z}_t, \mathscr{A}_t) \in \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$, $t \in (\mathbb{C}, 0)$, be a generic family which transversally intersects $U^{red}(D)$ at $(\boldsymbol{z}_0, \mathscr{A}_0) = (\boldsymbol{z}, \mathscr{A})$.*

(1) Given any vector $\boldsymbol{s} \in \mathbb{Z}^n_{>0}$ such that $|\boldsymbol{s}| = -DK_\Sigma - 1$, the set $\overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \setminus \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ is either empty or finite. Moreover, let

$$[\boldsymbol{n} : \widehat{C} \to \Sigma, \boldsymbol{p}] \in \overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \setminus \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$$

extend to a family

$$[\boldsymbol{n}_\tau : \widehat{C}_\tau \to \Sigma, \boldsymbol{p}_\tau] \in \overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}_{\varphi(\tau)}, \mathscr{A}_{\varphi(\tau)}), \quad \tau \in (\mathbb{C}, 0)\ , \tag{20}$$

for some morphism $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. *Then* $[\boldsymbol{n} : \widehat{C} \to \Sigma, \boldsymbol{p}]$ *is as follows:*

(1i) either $\widehat{C} = C^{(1)} \cup C^{(2)}$, *where* $C^{(1)} \simeq C^{(2)} \simeq \mathbb{P}^1$, $\boldsymbol{n}(C^{(1)}) \neq \boldsymbol{n}(C^{(2)})$, *and*

- *the map* $\boldsymbol{n} : \widehat{C}^{(j)} \to \Sigma$ *is an immersion and* $\boldsymbol{z} \cap \mathrm{Sing}(C^{(j)}) = \emptyset$ *for* $j = 1, 2$,
- $|\boldsymbol{p} \cap \widehat{C}^{(1)} \cap \widehat{C}^{(2)}| \leq 1$,
- $[\boldsymbol{n} : \widehat{C}^{(j)} \to \Sigma, \boldsymbol{p} \cap \widehat{C}^{(j)}] \in \mathscr{M}_{0,|\boldsymbol{p} \cap \widehat{C}^{(j)}|}(\Sigma, D^{(1)}, \boldsymbol{s}^{(j)}, \boldsymbol{z}, \mathscr{A})$, $j = 1, 2$, *where* $D^{(1)} + D^{(2)} = D$, $\boldsymbol{s}^{(1)} + \boldsymbol{s}^{(2)} = \boldsymbol{s}$, $|\boldsymbol{s}^{(1)}| = -D^{(1)}K_\Sigma$, $|\boldsymbol{s}^{(2)}| = -D^{(2)}K_\Sigma - 1$, *and, moreover,* $(\boldsymbol{n}|_{C^{(j)}})^*(\mathscr{A}) = \sum_{i=1}^n s_i^{(j)} p_i$ *for* $j = 1, 2$;

(1ii) or $n = 1$, $\boldsymbol{z} = z_1 \in \Sigma$, $\mathscr{A} = \alpha_1 \in \mathrm{Arc}_s^{\mathrm{sm}}(\Sigma, z)$, $\boldsymbol{p} = p_1 \in \widehat{C}$, $D = kD'$, *where* $k \geq 2$ *and* $-D'K_\Sigma \geq 3$, *and the following holds*

- $\widehat{C}$ *consists of few components having* p_1 *as a common point, and each of them is mapped onto the same immersed rational curve* $C \in |D'|$;
- z_1 *is a smooth point of* C, *and* $(C \cdot \alpha_1) = -D'K_\Sigma$.

(1iii) or $D = kD' + D''$, *where* $k \geq 2$, $-D'K_\Sigma \geq 2$, $D'' \neq 0$, $\widetilde{C} = \widetilde{C}' \cup \ldots \cup \widetilde{C}''$, *where*

- $\widehat{C}' \simeq \mathbb{P}^1$, $\boldsymbol{n} : \widehat{C}'' \to CX'' \hookrightarrow \Sigma$ *is an immersion, where* $C'' \in |D''|$,
- *the components of* $\widehat{C}'$ *have a common point* p_1 *and are disjoint from* $p_2, \ldots, p_n$, *and each of them is mapped onto the same immersed rational curve* $C' \in |D'|$,
- z_1 *is a smooth point of* C', *and* $(C' \cdot \alpha_1) = -D'K_\Sigma$.

(2) In case (1i),

- *if* $\boldsymbol{p} \cap \widehat{C}^{(1)} \cap \widehat{C}^{(2)} = \emptyset$, *there is a unique family of type (20), and it is smooth, parameterized by* $\tau = t$;
- *if* $\widehat{C}^{(1)} \cap \widehat{C}^{(2)} = \{p_1\}$, *then there are precisely* $\kappa = \min\{s_1^{(1)}, s_1^{(2)}\}$ *families of type (20), and for each of them* $t = \tau^{\kappa/d}$, *where* $d = gcd(s_1^{(1)}, s_1^{(2)})$.

Proof If $[\boldsymbol{n} : \widehat{C} \to \Sigma, \boldsymbol{p}] \in \overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ with a generic $(\boldsymbol{z}, \mathscr{A}) \in \mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$ and $\widehat{C}$ consisting of $m \geq 1$ components, then by Propositions 1 and 2 one obtains $m = 1$ and $\boldsymbol{n}$ immersion. Hence, $U^{red}(D)$ has positive codimension in $\mathrm{Arc}_s^{\mathrm{sm}}(\mathring{\Sigma}^n)$. Suppose that $(\boldsymbol{z}, \mathscr{A})$ satisfies the hypotheses of proposition. Then the finiteness of $\overline{\mathscr{M}}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A}) \setminus \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ and the asserted structure of its elements follows from Propositions 1 and 2, provided we show that

(a) There are no two components $\widehat{C}', \widehat{C}''$ of $\widehat{C}$ such that $\boldsymbol{n}(\widehat{C}') \neq \boldsymbol{n}(\widehat{C}'')$, $\boldsymbol{n}_*(\widehat{C}') \in |D'|$, $\boldsymbol{n}_*(\widehat{C}'') \in |D''|$, and $\deg(\boldsymbol{n}|_{\widehat{C}'})^*\mathscr{A} \geq -D'K_\Sigma$, $\deg(\boldsymbol{n}|_{\widehat{C}''})^*\mathscr{A} \geq -D''K_\Sigma$,

(b) In cases (1ii) and (1iii), we have inequalities $-D'K_\Sigma \ge 3$ and $-D'K_\Sigma \ge 2$, respectively.

The proof of Claim (a) can easily be reduced to the case when $\boldsymbol{n}\big|_{\widehat{C}'}$ and $\boldsymbol{n}\big|_{\widehat{C}''}$ are immersions, and $\deg(\boldsymbol{n}\big|_{\widehat{C}'})^*\alpha_1 = -D'K_\Sigma = \deg(\boldsymbol{n}\big|_{\widehat{C}''})^*\alpha_1 = -D''K_\Sigma$. However, in such a case, the dimension and generality assumptions yield that there exists the germ at C'' of the family of rational curves $C''_t \in |D''|$, $t \in (\mathbb{C}, 0)$, such that $(C''_t \cdot C')_{y_t} \ge -D''K_\Sigma$ for some family of points $y_t \in (C', z_1)$, $t \in (\mathbb{C}, 0)$, which together with Lemma 2(iii) implies a contradiction:

$$(D'')^2 \ge ((D'')^2 + D''K_\Sigma + 2) + (-D''K_\Sigma - 1) = (D'')^2 + 1\ .$$

Claim (b) in the case (1ii) follows from inequalities (14) and (15). In case (1iii), we perform similar estimations. If the curves C' and C'' intersect at z_1, then $(C' \cdot C'')_{z_1} = \min\{-D'K_\Sigma, -D''K_\Sigma - 1\}$, and we obtain

$$\frac{(kD' + D'')^2 + (kD' + D'')K_\Sigma}{2} + 1 \ge k^2\left(\frac{(D')^2 + D'K_\Sigma}{2} + 1\right)$$
$$+k(D'D'' - (C' \cdot C'')_{z_1}) + \frac{(D'')^2 + D''K_\Sigma}{2} + 1$$
$$\Longleftrightarrow \quad \begin{cases} (k-1)(-D'K_\Sigma) + 2(-D''K_\Sigma - 1) \ge 2k, & \text{if } -D'K_\Sigma \ge -D''K_\Sigma - 1, \\ (k+1)(-D'K_\Sigma) \ge 2k, & \text{if } -D'K_\Sigma \le -D''K_\Sigma - 1 \end{cases}$$
$$\Longrightarrow \quad -D'K_\Sigma \ge 2\ .$$

If the curves C' and C'' do not meet at z_1, then we obtain

$$\frac{(kD' + D'')^2 + (kD' + D'')K_\Sigma}{2} + 1 \ge k^2\left(\frac{(D')^2 + D'K_\Sigma}{2} + 1\right)$$
$$+k(D'D'' - 1) + \frac{(D'')^2 + D''K_\Sigma}{2} + 1 \quad \Longleftrightarrow \quad -D'K_\Sigma \ge 2\ .$$

Let us prove statement (2) of Proposition 3. If $\boldsymbol{p} \cap \widehat{C}^{(1)} \cap \widehat{C}^{(2)} = \emptyset$, then the (immersed) curves $C^{(1)} = \boldsymbol{n}(\widehat{C}^{(1)})$ and $C^{(2)} = \boldsymbol{n}(\widehat{C}^{(2)})$ intersect transversally and outside $\boldsymbol{z}$, and the point $\widehat{z} = \widehat{C}^{(1)} \cap \widehat{C}^{(2)}$ is mapped to a node of $C^{(1)} \cup C^{(2)} \setminus \boldsymbol{z}$. Then the uniqueness of the family $[\boldsymbol{n}_t : \widehat{C}_t \to \Sigma, \boldsymbol{p}_t]$, $t \in (\mathbb{C}, 0)$, and its smoothness follows from the standard properties of the deformation smoothing out a node (see, e.g., [12, Lemma 11(ii)]). Suppose now that the point $\widehat{C}^{(1)} \cap \widehat{C}^{(2)}$ belongs to $\boldsymbol{p}$. We prove statement (2) under condition $n = 1$, leaving the case $n > 1$ to the reader as a routine generalization with a bit more complicated notations. Denote $\xi := s_1^{(1)} = -D^{(1)}K_\Sigma$, $\eta := s_1^{(2)} = -D^{(2)}K_\Sigma - 1$. We have three possibilities:

- Suppose that $\xi < \eta$. In suitable coordinates x, y in a neighborhood of $z_1 = (0, 0)$, we have

$$\alpha_1 \equiv y - \lambda x^\eta \mod \mathfrak{m}_{z_1}^s, \quad C^{(1)} = \{y + x^\xi + \text{h.o.t.} = 0\}, \quad C^{(2)} = \{y = 0\}\ ,$$

where $\lambda \neq 0$ is generic. Without loss of generality, we can define the family of arcs $(z_t, \mathscr{A}_t)_{t \in (\mathbb{C},0)}$ by $z_t = (t, 0)$, $\mathscr{A}_t = \{y \equiv \lambda(x-t)^\eta \mod \mathfrak{m}_{z_t}^s\}$ (cf. Lemma 6). The ideal I_{z_1} from Lemma 6 can be expressed as $\langle y^2, yx^{\xi-1}, x^{\xi+\eta}\rangle$. Furthermore, by Lemma 6, for any family (20), the curves $C_\tau = \boldsymbol{n}(\widehat{C}_\tau) \in |D|$ are given, in a neighborhood of z_1, by

$$y^2(1 + O(x, y, \overline{c})) + yx^\xi(1 + O(x, \overline{c})) + \sigma(\overline{c})yx^{\xi-1} + \sum_{i=0}^{\xi-2} c_{i1}(\tau)yx^i + \sum_{i=0}^{\xi+\eta-1} c_{0i}(\tau)x^i + O(x^{\xi+\eta}, \overline{c}) = 0\ , \tag{21}$$

where $\overline{c}$ denotes the collection of variables $\{c_{i1},\ 0 \le i \le \xi - 2,\ c_{i0},\ 0 \le i \le \xi + \eta - 1\}$, the functions $c_{ij}(\tau)$ vanish at zero for all i, j in the summation range, and $\sigma(0) = 0$. Changing coordinates $x = x' + t$, where $t = \varphi(\tau)$, we obtain the family of curves

$$y^2(1 + O(x', y, t, \overline{c})) + y(x')^\xi(1 + O(x', t, \overline{c})) + \sigma' y(x')^{\xi-1} + \sum_{i=0}^{\xi-2} c'_{i1}y(x')^i + \sum_{i=0}^{\xi+\eta-1} c'_{0i}(x')^i + t \cdot O((x')^{\xi+\eta}, t, \overline{c}) = 0\ , \tag{22}$$

where

$$\begin{cases} c'_{i1} &= \sum_{0 \le u \le \xi-2-i} \binom{i+u}{i} t^u c_{i+u,1} + \binom{\xi-1}{i} t^{\xi-1-i}\sigma \\ & \quad + t^{\xi-i}\left(\binom{\xi}{i} + O(t)\right) + O(t^{\xi-i}, \overline{c}), \quad i = 0, \dots, \xi - 2, \\ c'_{i1} &= \sum_{u \ge 0} \binom{i+u}{i} t^u c_{i+u,0}, \quad i = 0, \dots, \xi + \eta - 1, \\ \sigma' &= \sigma + t(\xi + O(t, \overline{c})). \end{cases} \tag{23}$$

Next, we change coordinates $y = y' + \lambda(x')^\eta$ and impose the condition $(C_\tau \cdot (z_{\varphi(\tau)}, \mathscr{A}_{\varphi(\tau)})) \ge \xi + \eta$, which amounts in the following relations on the variables $\overline{c}' = \{c'_{i1},\ 0 \le i \le \xi - 2,\ c'_{i0},\ 0 \le i \le \xi + \eta - 1\}$:

$$\begin{cases} c'_{i0} = 0,\ i = 0, \dots, \eta - 1, \quad c'_{i0} + \lambda c'_{i-\eta,1} = 0,\ i = l, \dots, \eta + \xi - 2, \\ c'_{\xi+\eta-1} + \lambda\sigma' = 0. \end{cases} \tag{24}$$

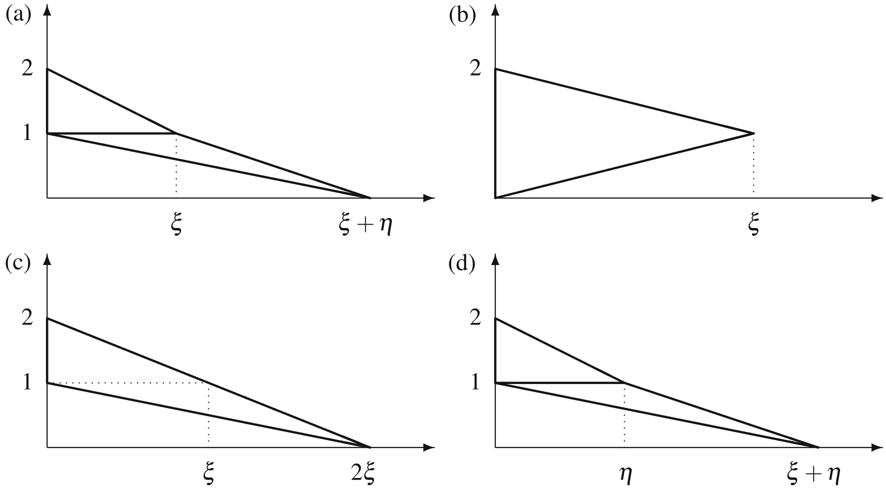

Fig. 1 Tropical limits: (a) the case $\xi < \eta$, (b) refinement, (c) the case $\xi = \eta$, (d) the case $\xi > \eta$

The new equation for the considered family of curves is then

$$F(x,y) = (y')^2(1 + O(x',y',t,\overline{c})) + y'(x')^{\xi}(1 + O(x',t,\overline{c}))$$
$$+(x')^{\xi+\eta}(a + O(x',t,\overline{c})) + y'\left(\sum_{i=0}^{\xi-2} c'_{i1}(x')^i + \sigma'(x')^{\xi-1}\right) = 0. \quad (25)$$

with some constant $a \neq 0$. Consider the tropical limit of the family (25) (see [14, Sect. 2.3] or Sect. 3.1). The corresponding subdivision of Δ must be as shown in Fig. 1a. Indeed, first, $c'_{01} \neq 0$, since otherwise the curves C_τ would be singular at z_t contrary to the general choice of $(z_t, \mathscr{A}_t)$. Second, no interior point of the segment $[(0,1),(\xi,1)]$ is a vertex of the subdivision, since otherwise the curves C_τ would have a positive genus: The tropicalization of C_τ would then be a tropical curve with a cycle which lifts to a handle of C_τ (cf. [14, Sects. 2.2 and 2.3, Lemma 2.1]). By a similar reason, the limit polynomial $F^{\delta}_{\rm ini}/y' = \sum_{i=0}^{\xi} c^0_{i1}(x')^i$, where δ is the segment $[(0,1),(\xi,1)]$, must be the ξ-th power of a binomial. The latter conclusion and relations (22) and (23) yield that $N_F(i,1) = \xi - i$ for $i = 0,\dots,\xi$ and

$$c'_{i1} = t^{\xi-i}(c^0_{i1} + c''_{i1}(t)),\ i = 0,\dots,\xi-2, \quad c'_{\xi+\eta-1,0} = t(c^0_{\xi+\eta-1,0} + c''_{\xi+\eta-1,0}),$$

where c^0_{i1}, $i = 0,\dots,\xi-2$, and $c^0_{\xi+\eta-1,0}$ are uniquely determined by the given data, the functions c''_{i1}, $0 \le i \le \xi-2$, vanish at zero, and $c''_{\xi+\eta-1,0}$ is a function of t and c''_{i1}, $0 \le i \le \xi-2$, that is determined by the given data and vanishes at zero too. To meet the condition of rationality of C_τ and to find the functions $c''_{i1}(t)$, $0 \le$

$i \leq \xi - 2$, we perform the refinement procedure as described in [14, Sect. 3.5]. It consists in further coordinate change and tropicalization, in which one encounters a subdivision containing the triangle $\mathrm{Conv}\{(0,0),(0,2),(\xi,1)\}$ (see Fig. 1b). The corresponding convex piecewise linear function N' is linear along that triangle and takes values $N'(0,2) = N'(\xi,1) = 0$, $N'(0,0) = \eta - \xi$. By Shustin [14, Lemma 3.9 and Theorem 5], there are ξ distinct solutions $\{c''_{i1}(t),\ 0 \leq i \leq \xi - 2\}$ of the rationality relation. More precisely, the initial coefficient $(c''_{i1})^0$ is nonzero only for $0 \leq i \leq \xi - 2$, $i \equiv \xi \mod 2$. The common denominator of the values of N' at these point is ξ/d, where $d = \gcd(\xi,\eta)$, and hence c''_{i1} are analytic functions of $t^{d/\xi}$. It follows thereby that $t = \tau^{\xi/d}$.

- Suppose that $\xi = \eta$ (see Fig. 1c). In this situation, the argument of the preceding case $\xi < \eta$ applies in a similar way and, after the coordinate change $x = x' + t$, $y = y' + \lambda(x')^\xi$, leads to Eq. (25), whose Newton polygon is subdivided with a fragment $\mathrm{Conv}\{(0,1),(0,2),(2\xi,0)\}$ on which the function N_F is linear with values $N_F(0,2) = N_F(2\xi,0) = 0$, $N_F(0,1) = \xi$. By Lemma 1, we get ξ solutions $\{c'_{i1}(t),\ i = 0,\dots,\xi-2\}$, which are analytic functions of t. Then, in particular, $t = \tau$.
- Suppose that $\xi > \eta$. In suitable coordinates x, y in a neighborhood of $z_1 = (0,0)$, we have

$$\alpha_1 \equiv y \mod \mathfrak{m}_{z_1}^s, \quad C^{(1)} = \{y + \lambda x^\xi + O(x^{\xi+1}) = 0\}, \quad C^{(2)} = \{y + x^\eta = 0\}\ ,$$

where $\lambda \neq 0$. Without loss of generality, we can define the family of arcs $(z_t, \mathscr{A}_t)_{t\in(\mathbb{C},0)}$ by $z_t = (0,0)$, $\mathscr{A}_t = \{y \equiv tx^{\xi-1} \mod \mathfrak{m}_{z_1}^s\}$ (cf. Lemma 6). The ideal I_{z_1} from Lemma 6 can be expressed as $\langle y^2, yx^\xi, x^{\xi+\eta-1}\rangle$. Thus, by Lemma 6, for any family (20), the curves $C_\tau = \boldsymbol{n}(\widehat{C}_\tau) \in |D|$ are given in a neighborhood of z_1 by

$$\begin{aligned} y^2(1 + O(x,y,\overline{c})) + yx^\eta(1 + O(x,\overline{c})) + \lambda x^{\xi+\eta}(1 + O(x,\overline{c})) \\ +\sigma(\overline{c})x^{\xi+\eta-1} + \sum_{i=0}^{\eta-1} c_{i1}(\tau)yx^i + \sum_{i=0}^{\xi+\eta-2} c_{0i}(\tau)x^i = 0\ , \end{aligned} \tag{26}$$

where $\overline{c}$ now denotes the collection of variables $\{c_{i1},\ 0 \leq i \leq \eta - 1,\ c_{i0},\ 0 \leq i \leq \xi + \eta - 2\}$, the functions $c_{ij}(\tau)$ vanish at zero for all i, j in the summation range, and $\sigma(0) = 0$. Inverting $t = \varphi(\tau)$, changing coordinates $y = y' + tx^{\xi-1}$, and applying the condition $(C_\tau \cdot \mathscr{A}_{\varphi(\tau)}) \geq k + l$, we obtain an equation of the curves C_τ in the form

$$\begin{aligned} F(x,y') = (y')^2(1 + O(t,x,y',\overline{c}')) + y'x^\eta(1 + O(t,x,\overline{c}')) \\ +\lambda x^{\xi+\eta}(1 + O(t,x,\overline{c}')) + \sum_{i=0}^{\eta-1} c_{i1}(t)y'x^i = 0\ , \end{aligned} \tag{27}$$

where $\overline{c}' = \{c_{i1},\ 0 \le i \le \eta - 1\}$, and the following relations must hold:

$$\begin{cases} c_{i0} = 0,\ i = 0, \dots, \eta - 2, \\ c_{i0} + tc_{i-\xi+1,1} = 0,\ i = \xi - 1, \dots, \eta + \xi - 2, \\ \sigma + t(1 + O(t, \overline{c}')) = 0\ . \end{cases} \tag{28}$$

By Lemma 4(2), $\frac{\partial \sigma}{\partial c_{\eta-1,1}}(0) \neq 0$. The rationality of the curves C_τ yields that the subdivision S_F of the Newton polygon of $F(x, y')$ given by (27) must contain two triangles $\mathrm{Conv}\{(0,1), (\eta, 1), (0, 2)\}$ and $\mathrm{Conv}\{(0, 1), (\eta, 1), (\xi + \eta, 0)\}$ (see Fig. 1d), and, furthermore, $F^{\delta}_{\mathrm{ini}}/y'$ must be the η-th power of a binomial, where $\delta = [(0, 1), (\eta, 1)]$ (cf. the argument in the treatment of the case $\xi < \eta$ above). These two conclusions and Eq. (28) uniquely determine the initial coefficients c^0_{i1} as well as the values $N_F(i, 1) = \eta - i$ for all $i = 0, \dots, \eta - 1$, and leave the final task to find the functions $c''_{i1}(t)$, $i = 0, \dots, \eta - 2$, which appear in the expansion $c_{i1}(t) = t^{\eta - i}(c^0_{i1} + c''_{i1}(t))$, $i = 0, \dots, \eta - 2$ (notice here that the last equation in (28) allows one to express $c''_{\eta-1,1}$ via c''_{i1}, $i = 0, \dots, \eta - 2$). To this extent, we again use the argument of the case $\xi < \eta$, performing the refinement procedure along the edge $\delta = [(0, 1), (\eta, 1)]$ (see [14, Sect. 3.5]) and apply the rationality requirement to draw the conclusion: There are exactly η families (20), and, for each of them, $t = \tau^{\eta/d}$, where $d = \gcd\{\xi, \eta\}$.

Statement (2) of proposition is proven. ∎

3.4 Families of Curves and Arcs on Uninodal del Pezzo Surfaces

A smooth rational surface Σ is called a uninodal del Pezzo surface if there exists a smooth rational curve $E \subset \Sigma$ such that $E^2 = -2$ and $-CK_\Sigma > 0$ for each irreducible curve $C \subset \Sigma$ different from E. Observe that $EK_\Sigma = 0$. Denote by $\mathrm{Pic}_+(\Sigma, E) \subset \mathrm{Pic}(\Sigma)$ the semigroup generated by irreducible curves different from E. Assume that Σ is of degree 1 and fix $D \in \mathrm{Pic}_+(\Sigma, E)$ such that $-DK_\Sigma - 1 \ge 3$. Fix positive integers $n \le -DK_\Sigma - 1$ and $s \gg -DK_\Sigma - 1$.

Accepting notations of Sect. 3.2, we introduce the set $U^{im}(D, E) \subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ is defined by the following conditions. For any sequence $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n_{>0}$ summing up to $|\boldsymbol{s}| \le s$ and for any element $(\boldsymbol{z}, \mathscr{A}) \in U^{im}(D, E)$, where $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\Sigma}^n$, $\boldsymbol{z} \cap E = \emptyset$, $\mathscr{A} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathrm{Arc}_s(\Sigma, z_i)$, the family $\mathscr{M}^{im}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ is empty if $|\boldsymbol{s}| \ge -DK_\Sigma$ and is finite if $|\boldsymbol{s}| = -DK_\Sigma - 1$. Furthermore, in the latter case, all elements $[\boldsymbol{n} : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathscr{M}_{0,n}(\Sigma, D, \boldsymbol{s}, \boldsymbol{z}, \mathscr{A})$ are represented by immersions $\boldsymbol{n} : \mathbb{P}^1 \to \Sigma$ such that $\boldsymbol{n}^*(\alpha_i) = s_i p_i$, $1 \le i \le n$, and $\boldsymbol{n}^*(E)$ consists of DE distinct points.

Proposition 4 *The set $U^{im}(D,E)$ is Zariski open and dense in* $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$.

Proof The statement that $U^{im}(D)$ is Zariski open and dense in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{\Sigma}^n)$ can be proved in the same way as Proposition 1(1). We will show that $U^{im}(D,E)$ is dense in $U^{im}(D)$, since the openness of $U^{im}(D,E)$ is evident. For, it is enough to show that any immersion $\boldsymbol{n}:\mathbb{P}^1\to\Sigma$ such that $\boldsymbol{n}_*(\mathbb{P}^1)=D$ can be deformed into an immersion with an image transversally crossing E at DE distinct points.

Suppose, first, that a generic element $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma]\in\mathscr{M}_{0,0}(\Sigma,D)$ is such that the divisor $\boldsymbol{n}^*(E)\subset\mathbb{P}^1$ contains an m-multiple point, $m\ge 2$. Since $\dim\mathscr{M}_{0,0}(\Sigma,D)=-DK_\Sigma-1\ge 3$, we fix the images of $-DK_\Sigma-2$ points p_i, $i=1,\dots,-DK_\Sigma-2$, obtaining a one-dimensional subfamily of $\mathscr{M}_{0,0}(\Sigma,D)$, for which one derives a contradiction by Lemma 2(iii):

$$D^2\ge(D^2+DK_\Sigma+2)+(-DK_\Sigma-2)+(m-1)=D^2+m-1>D^2\ .$$

Hence, for a generic $[\boldsymbol{n}:\mathbb{P}^1\to\Sigma]\in\mathscr{M}_{0,0}(\Sigma,D)$, the divisor $\boldsymbol{n}^*(E)$ consists of DE distinct points. Suppose that $m\ge 2$ of them are mapped to the same point in E. Fixing the position of that point on E, we define a subfamily $V\subset\mathscr{M}_{0,0}(\Sigma,D)$ of dimension

$$\dim V\ge\dim\mathscr{M}_{0,0}(\Sigma,D)-1=-DK_\Sigma-2\ge 2\ .$$

As above, we fix the images of $-DK_\Sigma-3$ additional point of $\mathbb{P}^1$ and end up with a contradiction due to Lemma 2(ii):

$$D^2\ge(D^2+DK_\Sigma+2)+(-DK_\Sigma-3)+m=D^2+m-1>D^2$$

■

Let $\mathfrak{X}\to(\mathbb{C},0)$ be a smooth flat family of smooth rational surfaces such that $\mathfrak{X}_0=\Sigma$ is a nodal del Pezzo surface with the (-2)-curve E and $\mathfrak{X}_t$, $t\ne 0$, are del Pezzo surfaces. We can naturally identify $\mathrm{Pic}(\mathfrak{X}_t)\simeq\mathrm{Pic}(\Sigma)$, $t\in(\mathbb{C},0)$. Fix a divisor $D\in\mathrm{Pic}_+(\Sigma,E)$ such that $-DK_\Sigma-1\ge 3$. Given $n\ge 1$ and $s\gg -DK_\Sigma-1$, fix a vector $\boldsymbol{s}\in\mathbb{Z}^n_{>0}$ such that $|\boldsymbol{s}|=-DK_\Sigma-1$. Denote by $\mathrm{Arc}^{\mathrm{sm}}_s(\mathfrak{X})\to\mathfrak{X}\to(\mathbb{C},0)$ the bundle with fibres $\mathrm{Arc}^{\mathrm{sm}}_s(\mathfrak{X}_t)$, $t\in(\mathbb{C},0)$. Pick n disjoint smooth sections $z_1,\dots,z_n:(\mathbb{C},0)\to\mathfrak{X}$ covered by n sections $\alpha_1,\dots,\alpha_n:(\mathbb{C},0)\to\mathrm{Arc}^{\mathrm{sm}}_s(\mathfrak{X})$ such that $(\boldsymbol{z}(0),\mathscr{A}(0))\in U^{im}(\Sigma,E)$, and $(\boldsymbol{z}(t),\mathscr{A}(t))\in U^{im}(\mathfrak{X}_t)$, $t\ne 0$.

Proposition 5 *Each element* $[\nu:\widehat{C}\to\Sigma,\boldsymbol{p}]\in\overline{\mathscr{M}_{0,n}(\Sigma,D,\boldsymbol{s},\boldsymbol{z}(0),\mathscr{A}(0))}$ *such that*

- *either $\widehat{C}\simeq\mathbb{P}^1$, or $\widehat{C}=\widehat{C}'\cup\widehat{E}_1\cup\ldots\cup\widehat{E}_k$ for some $k\ge 1$, where $\widehat{C}'\simeq\widehat{E}_1\simeq\ldots\simeq\widehat{E}_k\simeq\mathbb{P}^1$, $\widehat{E}_i\cap\widehat{E}_j=\emptyset$ for all $i\ne j$, and $\#(\widehat{C}'\cap\widehat{E}_i)=1$ for all $i=1,\dots,k$;*
- *$\boldsymbol{p}\subset\widehat{C}'$ and $[\nu:\widehat{C}'\to\Sigma,\boldsymbol{p}]\in\mathscr{M}^{im}_{0,n}(\Sigma,D-kE,\boldsymbol{s},\boldsymbol{z}(0),\mathscr{A}(0))$, and each of the $\widehat{E}_i$ is isomorphically taken onto E;*

extends to a smooth family $[\nu_t : \widehat{C}_t \to \mathfrak{X}_t, \boldsymbol{z}(t)] \in \overline{\mathscr{M}_{0,n}}(\mathfrak{X}_t, D, \boldsymbol{s}, \boldsymbol{z}(t), \mathscr{A}(t))$, $t \in (\mathbb{C}, 0)$, *where* $\widehat{C}_t \simeq \mathbb{P}^1$ *and* ν_t *is an immersion for all* $t \neq 0$, *and, furthermore, each element of* $\mathscr{M}_{0,n}(\mathfrak{X}_t, D, \boldsymbol{s}, \boldsymbol{z}(t), \mathscr{A}(t))$, $t \in (\mathbb{C}, 0) \setminus \{0\}$ *is included into some of the above families.*

Proof The statement follows from [16, Theorem 4.2] and from Proposition 4, which applies to all divisors $D - kE$, since $-(D - kE)K_\Sigma = -DK_\Sigma$ for any k. ■

4 Proof of Theorem 1

By blowing up additional real points if necessary, we reduce the problem to consideration of del Pezzo surfaces X of degree 1.

(1) To prove the first statement of Theorem 1, it is enough to consider only del Pezzo surfaces satisfying property (GDP) introduced in Sect. 3.3 (cf. [12, Lemma 17]) and real divisors satisfying $-DK_X - 1 \geq 3$ (cf. Remark 1(1)). So, let a real del Pezzo surface X satisfy property (GDP) and have a nonempty real part. Let $F \subset \mathbb{R}X$ be a connected component. Denote by $\mathscr{P}_{r,m}(X, F)$ the set of sequences $(\boldsymbol{z}, \boldsymbol{w})$ of $n = r + 2m$ distinct points in Σ such that $\boldsymbol{z}$ is a sequence of r points belonging to the component $F \subset \mathbb{R}X$, and $\boldsymbol{w}$ is a sequence of m pairs of complex conjugate points. Fix an integer $s \gg -DK_X$ and denote by $\mathbb{R}\mathrm{Arc}^{\mathrm{sm}}_s(X, F, r, m) \subset \mathrm{Arc}^{\mathrm{sm}}_s(\mathring{X}^n)$ the space of sequences of arcs $(\mathscr{A}, \mathscr{B})$ centered at $(\boldsymbol{z}, \boldsymbol{w}) \in \mathscr{P}_{t,m}(X, F)$ such that $\mathscr{A} = (\alpha_1, \ldots, \alpha_r)$ is a sequence of real arcs $\alpha_i \in \mathrm{Arc}_s(X, z_i)$, $z_i \in \boldsymbol{z}$, $i = 1, \ldots, r$, and $\mathscr{B} = (\beta_1, \overline{\beta}_1, \ldots, \beta_m, \overline{\beta}_m)$ is a sequence of m pairs of complex conjugate arcs, where $\beta_i \in \mathrm{Arc}_s(X, w_i)$, $\overline{\beta}_i \in \mathrm{Arc}_s(X, \overline{w}_i)$, $i = 1, \ldots, m$, and $\boldsymbol{w} = (w_1, \overline{w}_1, \ldots, w_m, \overline{w}_m)$.

We join two elements of $\mathbb{R}\mathrm{Arc}_s(X, F, r, m) \cap U^{im}(D)$ by a smooth real analytic path $\Pi = \{(\boldsymbol{z}_t, \boldsymbol{w}_t), (\mathscr{A}_t, \mathscr{B}_t)\}_{t \in [0,1]}$ in $\mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ and show that along this path, the function $W(t) := W(X, D, F, \varphi, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}_t, \boldsymbol{w}_t), (\mathscr{A}_t, \mathscr{B}_t))$, $t \in [0, 1]$, remains constant. By Propositions 1 and 3, we need only to verify the required constancy when the path Π crosses sets $U^{im}_+(D)$, $U^{sing}_1(D)$, $U^{sing}_2(D)$, $U^{mt}(D)$, and $U^{red}(D)$ at generic elements of their components of codimension 1 in $\mathrm{Arc}^{\mathrm{sm}}_s(\mathring{X}^n)$. Let $t^* \in (0, 1)$ correspond to the intersection of Π with some of these walls.

If is clear that crossing of the wall $U^{sm}_+(D) \cap \mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ does not affect $W(X, D, F, \varphi, (\boldsymbol{k}, \boldsymbol{l}), (\boldsymbol{z}_t, \boldsymbol{w}_t), (\mathscr{A}_t, \mathscr{B})_t)$.

The constancy of $W(t)$ in a crossing of the wall $U^{sing}_1(D) \cap \mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ follows from Proposition 1(3) and [12, Lemmas 13(2), 14 and 15]. The transversality hypothesis in [12, Lemma 15] can be proved precisely as [12, Lemma 13(1)].

The constancy of $W(t)$ in a crossing of the wall $U^{sing}_2(D) \cap \mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ follows from Proposition 1(4) and Lemma 3.

The constancy of $W(t)$ in a crossing of the wall $U^{mt}(D) \cap \mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ follows from Propositions 1(5) and 2. Indeed, by Proposition 2 exactly one real element of the set $\mathscr{M}_{0,n}(X, D, (\boldsymbol{k}, \boldsymbol{l}), (z_t, \boldsymbol{w}_t), (\mathscr{A}_t, \mathscr{B}_t))$ undergoes a bifurcation. Furthermore, the ramification points of the degenerate map $\boldsymbol{n} : \mathbb{P}^1 \to X$ are complex conjugate. Hence, the real part of a close curve doubly covers the real part of $C = \boldsymbol{n}(\mathbb{P}^1)$, which means that the number of solitary nodes is always even.

At last, the constancy of $W(t)$ in a crossing of the wall $U^{red}(D) \cap \mathbb{R}\mathrm{Arc}_s(X, F, r, m)$ we derive from Proposition 3. Notice that the points $p_1 \in \widehat{C}$ and $z_1 \in X$ must be real, and hence the cases (1ii) and (1iii) are not relevant, since we have the lower bound $-kD'K_X \geq 2k \geq 4$ contrary to (5). In the case (1i), we use Proposition 3(2):

- if $\boldsymbol{p} \cap \widehat{C}^{(1)} \cap \widehat{C}^{(2)} = \emptyset$, then the germ of the real part of the family (20) is isomorphically mapped onto the germ $(\mathbb{R}, t^*)$ so that the central curve deforms by smoothing out a node both for $t > t^*$ and $t < t^*$, and hence $W(t)$ remains unchanged;
- if $\boldsymbol{p} \cap \widehat{C}^{(1)} \cap C^{(2)} = \{p_1\}$, then $p_1 \in \mathbb{P}^1$ and $z_1 \in X$ must be real, and hence $\xi + \eta$ must be odd, in particular, $d = \gcd\{\xi, \eta\}$ is odd too, where $\xi = s_1^{(1)}$, $\eta = s_1^{(2)}$; if $\kappa = \min\{\xi, \eta\}$ is odd, then the real part of each real family (20) is homeomorphically mapped onto the germ $(\mathbb{R}, t^*)$, and, in the deformation of the central curve both for $t > t^*$ and $t < t^*$, one obtains in a neighborhood of z_1 an even number of real solitary nodes, which follows from Lemma 1(2); if κ is even, then either the real part of a real family (20) is empty or the real part of a real family (20) doubly covers one of the halves of the germ $(\mathbb{R}, t^*)$, so that in one component of $(\mathbb{R}, t^*) \setminus \{t^*\}$, one has no real curves in the family (20), and in the other component of $(\mathbb{R}, t^*) \setminus \{t^*\}$, one has a couple or real curves, one having an odd number $\kappa - 1$ real solitary nodes, and the other having no real solitary nodes [see Lemma 1(2)], and hence $W(t)$ remains constant in such a bifurcation.

(2) By Itenberg et al. [12, Proposition 1], in a generic one-dimensional family of smooth rational surfaces of degree 1 all but finitely many of them are del Pezzo and the exceptional one are uninodal. Hence, to prove the second statement of Theorem 1 it is enough to establish the constancy of

$$W(t) = W(\mathfrak{X}_t, D, F_t, \varphi, (\boldsymbol{k}, \boldsymbol{l}), (z(t), \boldsymbol{w}(t)), (\mathscr{A}(t), \mathscr{B}(t)))$$

in germs of real families $\mathfrak{X} \to (\mathbb{C}, 0)$ as in Proposition 5, where the parameter is restricted to $(\mathbb{R}, 0) \subset (\mathbb{C}, 0)$. It follows from Proposition 5 that the number of the real curves in count does not change, and real solitary nodes are not involved in the bifurcation. Hence, $W(t)$ remains constant.

5 Examples

We illustrate Theorem 1 by a few elementary examples. Consider the case of plane cubics, for which new invariants can easily be computed via integration with respect to the Euler characteristic in the style of [3, Proposition 4.7.3].

Let $r_1 + 3r_3 + 2(m_1 + 2m_2 + 3m_3 + 4m_4) = 8$, where $r_1, r_3, m_1, m_2, m_3, m_4 \geq 0$. Define integer vectors $\boldsymbol{k} = (r_1 \times 1, r_3 \times 3)$, $\boldsymbol{l} = (m_1 \times 1, m_2 \times 2, m_3 \times 3, m_4 \times 4)$. Denote by L the class of line in $\mathrm{Pic}(\mathbb{P}^2)$. Then

$$W(\mathbb{P}^2, 3L, (\boldsymbol{k}, \boldsymbol{l})) = r_1 - r_3 \ .$$

As compared with the case of usual Welschinger invariants, in the real pencil of plane cubics meeting the intersection conditions with a given collection of arcs, in addition to real rational cubics with a node outside the arc centers, one encounters rational cubics with a node at the center of an arc of order 3. Notice that this real node is not solitary since one of its local branches must be quadratically tangent to the given arc. We also remark that, in a similar computation for a collection of arcs containing a real arc of order 2, one also encounters rational cubics with a node at the center of such an arc, but this node can be solitary or non-solitary depending on the given collection of arcs, and hence the count or real rational cubics will also depend on the choice of a collection of arcs.

Of course, the same argument provides formulas for invariants of any real del Pezzo surface and $D = -K$, or, more generally, for each effective divisor with $p_a(D) = 1$.

We plan to address the computational aspects in detail in a forthcoming paper.

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Milnor Fibre Homology via Deformation

Dirk Siersma and Mihai Tibăr

Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract In case of one-dimensional singular locus, we use deformations in order to get refined information about the Betti numbers of the Milnor fibre.

Keywords Betti numbers • Milnor fibre • Singularities • One-dimensional singular set

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1 Introduction and Results

We study the topology of Milnor fibres F of function germs on $\mathbb{C}^{n+1}$ with a one-dimensional singular set. Well known is that F is a $(n-2)$ connected n-dimensional CW complex. What can be said about $H_{n-1}(F)$ and $H_n(F)$? In this paper we use deformations in order to get information about these groups. It turns out that the constraints on F yield only small numbers $b_{n-1}(F)$, for which we give upper bounds which are in general sharper than the known ones from [9]. We pay special attention to classes of singularities where $H_{n-1}(F) = 0$, where the homology is concentrated in the middle dimension.

The admissible deformations of the function have a singular locus Σ consisting of a finite set R of isolated points and finitely many curve branches. Each branch Σ_i of Σ has a generic transversal type (of transversal Milnor fibre $F_i^{\pitchfork}$ and Milnor

D. Siersma (✉)
Institute of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands
e-mail: D.Siersma@uu.nl

M. Tibăr
Université Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
e-mail: tibar@math.univ-lille1.fr

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number denoted by $\mu_i^{\pitchfork}$) and also contains a finite set Q_i of points with non-generic transversal type, which we call *special points*. In the neighbourhood of each such special point $q \in \Sigma_i$ with Milnor fibre denoted by $\mathcal{A}_q$, there are two monodromies which act on $F_i^{\pitchfork}$: the *Milnor monodromy* of the local Milnor fibration of $F_i^{\pitchfork}$ and the *vertical monodromy* of the local system defined on the germ of $\Sigma_i \setminus \{q\}$ at q.

In our topological study, we work with homology over $\mathbb{Z}$ (and therefore we systematically omit $\mathbb{Z}$ from the notation of the homology groups). We provide a detailed expression for $H_{n-1}(F)$ through a topological model of F from which we derive some results that we roughly outline here:

(a) If for every component Σ_i there exist one vertical monodromy A_s, which has no eigenvalues 1, then $b_{n-1}(F) = 0$. More generally, $b_{n-1}(F)$ is bounded by the sum, taken over the components, of the minimum (over that component) of $\dim \ker(A_s - I)$ (Theorem 4.4).

(b) Assume that for each irreducible component Σ_i, there is a special singularity at q such that $H_{n-1}(\mathcal{A}_q) = 0$. Then $H_{n-1}(F) = 0$.

More generally, let $Q' := \{q_1, \ldots, q_m\} \subset Q$ be a subset of special points such that each branch Σ_i contains at least one of its points. Then (Theorem 4.6b)

$$b_{n-1}(F) \le \dim H_{n-1}(\mathcal{A}_{q_1}) + \cdots + \dim H_{n-1}(\mathcal{A}_{q_m}).$$

Note that the choice of a good subset of special points may yield the sharpest bound.

In [12] we have studied the vanishing homology of projective hypersurfaces with a one-dimensional singular set. Similar type of methods work in the local case. We keep the notations close to those in [12] and refer to it for the proof of certain results. In the proof of the main theorems, we use the Mayer–Vietoris theorem to study local and (semi)global contributions separately. We construct a CW complex model of two bundles of transversal Milnor fibres (in Sects. 3.5 and 3.6) and their inclusion map (Sect. 4). Moreover we use the full strength of the results on local one-dimensional singularities [6, 8–10], cf also [4, 5, 14, 17].

We discuss known results such as De Jong's [1] and also compute several new examples in Sect. 5.

2 Local Theory of One-Dimensional Singular Locus

We work with local data of function germs with one-dimensional singular locus and recall some facts from [9, 10] and [11, 12].

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with singular locus Σ of dimension 1, and let $\Sigma = \bigcup_{i \in I} \Sigma_i$ be its decomposition into irreducible curve components. Let $E := B_\varepsilon \cap f^{-1}(D_\delta)$ be the Milnor neighbourhood and F be the local Milnor fibre of f, for small enough ε and δ. The only non-trivial reduced homology groups are $H_n(F) = \mathbb{Z}^{\mu_n}$, which is free, and $H_{n-1}(F)$ which can have torsion.

There is a well-defined local system on $\Sigma_i \setminus \{0\}$ having as fibre the homology of the transversal Milnor fibre $\tilde{H}_{n-1}(F_i^{\pitchfork})$, where $F_i^{\pitchfork}$ is the Milnor fibre of the restriction of f to a transversal hyperplane section at some $x \in \Sigma_i \setminus \{0\}$. This restriction has an isolated singularity whose equisingularity class is independent of the point x and of the transversal section, in particular $\tilde{H}_*(F_i^{\pitchfork})$ is concentrated in dimension $n-1$. It is on this group that acts the *local system monodromy* (also called *vertical monodromy*):

$$A_i : \tilde{H}_{n-1}(F_i^{\pitchfork}) \to \tilde{H}_{n-1}(F_i^{\pitchfork}).$$

After [9], one considers a tubular neighbourhood $\mathcal{N} := \bigsqcup_{i=}^{m} \mathcal{N}_i$ of the link of Σ and decomposes the boundary $\partial F := F \cap \partial B_\varepsilon$ of the Milnor fibre as $\partial F = \partial_1 F \cup \partial_2 F$, where $\partial_2 F := \partial F \cap \mathcal{N}$. Then $\partial_2 F = \bigsqcup_{i=1}^{m} \partial_2 F_i$, where $\partial_2 F_i := \partial_2 F \cap \mathcal{N}_i$.

Each boundary component $\partial_2 F_i$ is fibred over the link of Σ_i with fibre $F_i^{\pitchfork}$. Let then $E_i^{\pitchfork}$ denote the transversal Milnor neighbourhood containing the transversal fibre $F_i^{\pitchfork}$, and let $\partial_2 E_i$ denote the total space of its fibration above the link of Σ_i. Therefore, $E_i^{\pitchfork}$ is contractible and $\partial_2 E_i$ retracts to the link of Σ_i. The pair $(\partial_2 E_i, \partial_2 F_i)$ is related to $A_i - I$ via the following exact relative Wang sequence ([12], Lemma 3.1) $(n \geq 2)$:

$$0 \to H_{n+1}(\partial_2 E_i, \partial_2 F_i) \to H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \stackrel{A_i - I}{\to} H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \to H_{n-1}(\partial_2 E_i, \partial_2 F_i) \to 0. \quad (1)$$

3 Deformation and Vanishing Homology

3.1 Admissible Deformations

Consider a one-parameter family $f_s : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ where $f_0 = \hat{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a given germ with singular locus $\hat{\Sigma}$ of dimension 1, with Milnor data $(\hat{E}, \hat{F})$ and similar notations for all the other objects defined in Sect. 2. We use the notation with "hat" since we reserve the notation without "hat" for the deformation f_s.

We fix a ball $B := B_\varepsilon \subset \mathbb{C}^{n+1}$ centred at 0 and a disc $\Delta := \Delta_\delta \subset \mathbb{C}$ at 0 such that for small enough radii ε and δ, the restriction to the punctured disc $\hat{f}_| : B \cap (\hat{f})^{-1}(\Delta^*) \to \Delta^*$ is the Milnor fibration of $\hat{f}$.

We say that the deformation f_s is *admissible* if it has good behaviour at the boundary, i.e. if for small enough s, the family $f_{s|} : \partial B \cap f_s^{-1}(\Delta) \to \Delta$ is stratified topologically trivial. Such a situation occurs, e.g. in the case of an "equi-transversal deformation" considered in [2].

We choose a value of s which satisfies the above conditions and write from now on $f := f_s$. It then follows that the pair $(E, F) := (B \cap f^{-1}(\Delta), f^{-1}(b))$, where $b \in \partial\Delta$, is topologically equivalent to the Milnor data $(\hat{E}, \hat{F})$ of $\hat{f}$. Note that for f, we consider the semi-local singular fibration inside B and not just its Milnor fibration at the origin.

Let $\Sigma \subset B$ be the one-dimensional singular part of the singular set Sing $(f) \subset B$. The circle boundaries $\partial B \cap \hat{\Sigma}$ of $\hat{\Sigma}$ can be identified with the circle boundaries $\partial B \cap \Sigma$ of Σ. Also the corresponding vertical monodromies are the same. Note that $\hat{\Sigma}$ and Σ can have a different number of irreducible components.

3.2 Notations

We use notations similar to [12] (cf also Fig. 1).

A point q on Σ is called *special* if the transversal Milnor fibration is not a trivial local system in the neighbourhood of q.

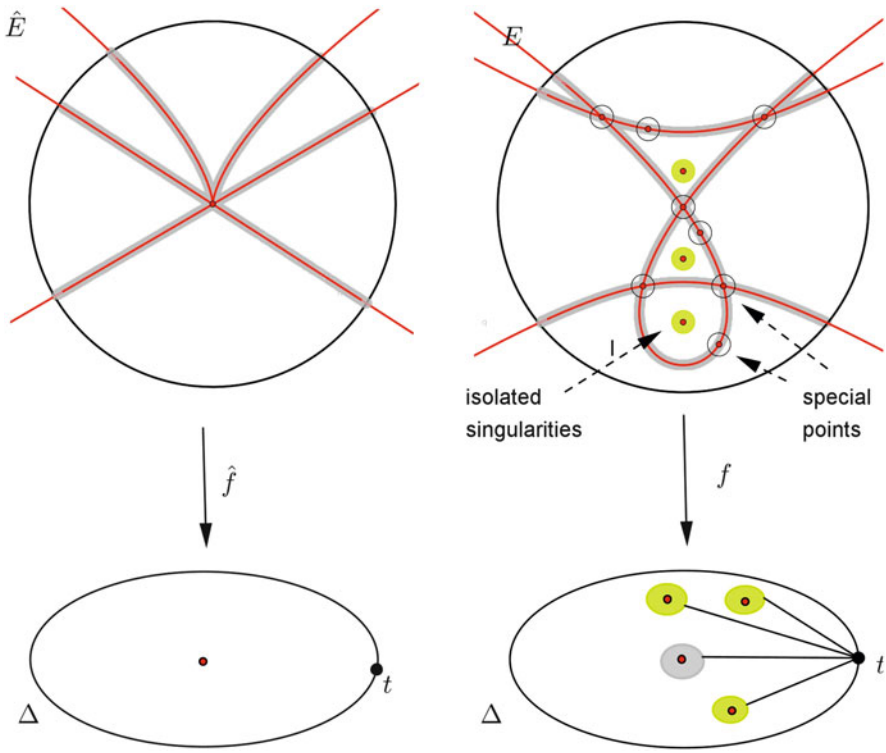

Fig. 1 Admissible deformation

Q_i := the set of special points on Σ_i; $Q := \cup_{i\in I} Q_i$,
R := the set of isolated singular points; $R = R_0 \cup R_1$, where R_0 is the set of critical points on $f^{-1}(0)$ and R_1 the set of critical points outside $f^{-1}(0)$,
B_q, B_r = small enough disjoint Milnor balls within E at the points $q \in Q, r \in R$ resp.
$B_Q := \bigsqcup_q B_q$ and $B_R := \bigsqcup_r B_r$, and similar notation for B_{R_0} and B_{R_1},
$\Sigma_i^* := \Sigma_i \setminus \mathrm{int}(B_Q)$; $\Sigma^* = \cup_{i\in I} \Sigma_i^*$ (closed sets),
$\mathcal{U}_i$:= small enough tubular neighbourhood of Σ_i^*; $\mathcal{U} = \cup_i \mathcal{U}_i$,
$\pi_\Sigma : \mathcal{U} \to \Sigma^*$ is the projection of the tubular neighbourhood.
$T = \{f(r) | r \in R\} \cup \{f(\Sigma)\}$ is the set of critical values of f and we assume without loss of generality that $f(\Sigma) = 0$.

Let $\{\Delta_t\}_{t\in T}$ be a system of nonintersecting small discs Δ_t around each $t \in T$. For any $t \in T$, choose $t' \in \partial\Delta_t$. If $t = f(r)$ then we denote by $t'(r)$ the point $t' \in \Delta_{f(r)}$. For $t = 0$ we use the notations t_0 and t'_0, respectively.

Let $E_r = B_r \cap f^{-1}(\Delta_{f(r)})$ and $F_r = B_r \cap f^{-1}(t'(r))$ be the Milnor data of the isolated singularity of f at $r \in R$. We use next the additivity of vanishing homology with respect to the different critical values and the connected components of Sing f. By homotopy retraction and by excision, we have

$$H_*(E, F) \simeq \oplus_{t\in T} H_*((f^{-1}(\Delta_t), f^{-1}(t')) = \tag{2}$$

$$= \oplus_{r\in R_0} H_*(E_r, F_r) \oplus H_*(E_0, F_0) \oplus \oplus_{r\in R_1} H_*(E_r, F_r), \tag{3}$$

where $(E_0, F_0) = (f^{-1}(\Delta_0) \cap (\mathcal{U} \cup B_Q), f^{-1}(t'_0) \cap (\mathcal{U} \cup B_Q)$. We introduce the following shorter notations:

$$(\mathcal{X}_q, \mathcal{A}_q) := (f^{-1}(\Delta_0) \cap B_q, f^{-1}(t'_0) \cap B_q)$$

$$\mathcal{X} = \bigsqcup_Q \mathcal{X}_q \ , \ \mathcal{A} = \bigsqcup_Q \mathcal{A}_q$$

$$\mathcal{Y} = \mathcal{U} \cap f^{-1}(\Delta_0) \ , \ \mathcal{B} := f^{-1}(t'_0) \cap \mathcal{Y}$$

$$\mathcal{Z} := \mathcal{X} \cap \mathcal{Y} \ , \ \mathcal{C} := \mathcal{A} \cap \mathcal{B}$$

In these new notations, we have

$$H_*(E, F) \simeq H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \oplus \oplus_{r\in R} H_*(E_r, F_r). \tag{4}$$

Note that each direct summand $H_*(E_r, F_r)$ is concentrated in dimension $n + 1$ since it identifies to the Milnor lattice $\mathbb{Z}^{\mu_r}$ of the isolated singularities germs of $f - f(r)$ at r, where μ_r denotes its Milnor number. We deal from now on with the term $H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$ from the direct sum of (4).

We consider the relative Mayer–Vietoris long exact sequence:

$$\cdots \to H_*(\mathcal{Z},\mathcal{C}) \to H_*(\mathcal{X},\mathcal{A}) \oplus H_*(\mathcal{Y},\mathcal{B}) \to H_*(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) \stackrel{\partial_s}{\to} \cdots \quad (5)$$

of the pair $(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B})$, and we compute each term of it in the following. The description follows closely [12] where we have treated deformations of projective hypersurfaces.

3.3 Homology of $(\mathcal{X},\mathcal{A})$

Since $\mathcal{X}$ is a disjoint union, one has the direct sum decomposition $H_*(\mathcal{X},\mathcal{A}) \simeq \oplus_{q\in Q} H_*(\mathcal{X}_q,\mathcal{A}_q)$. The pairs $(\mathcal{X}_q,\mathcal{A}_q)$ are local Milnor data of the hypersurface germs $(f^{-1}(t_0),q)$ with one-dimensional singular locus, and therefore the relative homology $H_*(\mathcal{X}_q,\mathcal{A}_q)$ is concentrated in dimensions n and $n+1$, cf Sect. 2.

3.4 Homology of $(\mathcal{Z},\mathcal{C})$

The pair $(\mathcal{Z},\mathcal{C})$ is a disjoint union of pairs localized at points $q \in Q$. For such points we have one contribution for each *locally irreducible branch of the germ* (Σ, q). Let S_q be the index set of all these branches at $q \in Q$. By abuse of notation, we write $s \in S_q$ for the corresponding small loops around q in Σ_i. For some $q \in \Sigma_{i_1} \cap \Sigma_{i_2}$, the set of indices S_q runs over all the local irreducible components of the curve germ (Σ, q). Nevertheless, when we are counting the local irreducible branches at some point $q \in Q_i$ on a specified component Σ_i, then the set S_q will tacitly mean only those local branches of Σ_i at q. We get the following decomposition:

$$H_*(\mathcal{Z},\mathcal{C}) \simeq \oplus_{q\in Q} \oplus_{s\in S_q} H_*(\mathcal{Z}_s,\mathcal{C}_s). \quad (6)$$

More precisely, one such local pair $(\mathcal{Z}_s,\mathcal{C}_s)$ is the bundle over the corresponding component of the link of the curve germ Σ at q having as fibre the local transversal Milnor data $(E_s^{\pitchfork}, F_s^{\pitchfork})$, with transversal Milnor numbers denoted by $\mu_s^{\pitchfork}$. These data depend only on the branch Σ_i containing s, and therefore if $s \subset \Sigma_i$ we sometimes write $(E_i^{\pitchfork}, F_i^{\pitchfork})$ and $\mu_i^{\pitchfork}$. In the notations of Sect. 2, we have $\partial_2\mathcal{A}_q = \bigsqcup_{s\in S_q} \mathcal{C}_s$.

The relative homology groups in the above direct sum decomposition (6) depend on the *local system monodromy* A_s via the Wang sequence (1) which takes here the following shape:

$$0 \to H_{n+1}(\mathcal{Z}_s,\mathcal{C}_s) \to H_n(E_s^{\pitchfork}, F_s^{\pitchfork}) \stackrel{A_s - I}{\to} H_n(E_s^{\pitchfork}, F_s^{\pitchfork}) \to H_n(\mathcal{Z}_s,\mathcal{C}_s) \to 0. \quad (7)$$

From this we obtain:

Lemma 3.1 *At $q \in Q$, for each $s \in S_q$ one has $H_k(\mathcal{Z}_s, \mathcal{C}_s) = 0$, $k \neq n, n+1$ and*

$$H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \cong \ker\,(A_s - I), \quad H_n(\mathcal{Z}_s, \mathcal{C}_s) \cong \text{coker}\,(A_s - I).$$

□

We therefore conclude that $H_*(\mathcal{Z}, \mathcal{C})$ is concentrated in dimensions n and $n+1$ only.

3.5 The CW Complex Structure of $(\mathcal{Z}, \mathcal{C})$

The pair $(\mathcal{Z}_s, \mathcal{C}_s)$ has the following structure of a relative CW complex, up to homotopy type. Each bundle over some circle link can be obtained from a trivial bundle over an interval by identifying the fibres above the end points via the geometric monodromy A_s. In order to obtain $\mathcal{Z}_s$ from $\mathcal{C}_s$, one can start by first attaching n-cells $c_1, \ldots, c_{\mu_s^{\pitchfork}}$ to the fibre $F_s^{\pitchfork}$ in order to kill the $\mu_s^{\pitchfork}$ generators of $H_{n-1}(F_s^{\pitchfork})$ at the identified ends and next by attaching $(n+1)$-cells $e_1, \ldots, e_{\mu_s^{\pitchfork}}$ to the preceding n-skeleton. The attaching of some $(n+1)$-cell goes as follows: consider some n-cell a of the n-skeleton and take the cylinder $I \times a$ as an $(n+1)$-cell. Fix an orientation of the circle link, attach the base $\{0\} \times a$ over a, then follow the circle bundle in the fixed orientation by the monodromy A_s and attach the end $\{1\} \times a$ over $A_s(a)$. At the level of the cell complex, the boundary map of this attaching identifies to $A_s - I : \mathbb{Z}^{\mu_s^{\pitchfork}} \to \mathbb{Z}^{\mu_s^{\pitchfork}}$.

3.6 The CW Complex Structure of $(\mathcal{Y}, \mathcal{B})$

The curve Σ_i has as boundary components the intersection $\partial B \cap \Sigma_i$ with the small Milnor balls B. These are all topological circles, and we denote them by $u \in U_i$, $U := \sqcup_i U_i$, and call them *outside* loops. Note that over any such loop $u \in U_i$, we have a local system monodromy $A_u : \mathbb{Z}^{\mu_i^{\pitchfork}} \to \mathbb{Z}^{\mu_i^{\pitchfork}}$. In fact this monodromy did not change in the admissible deformation from $\hat{f}$ to f.

We choose the following sets of loops in Σ_i (where we identify the loops with their index sets):

$G_i :=$ the $2g_i$ loops (called *genus loops* in the following) which are generators of π_1 of the normalization $\tilde{\Sigma}_i$ of Σ_i, where g_i denotes the genus of this normalization (which is a Riemann surface with boundary),
$S_i :=$ the loops $s \in S_q$ around the branches of Σ_i at the special points $q \in Q_i$,
$U_i =$ the outside loops,

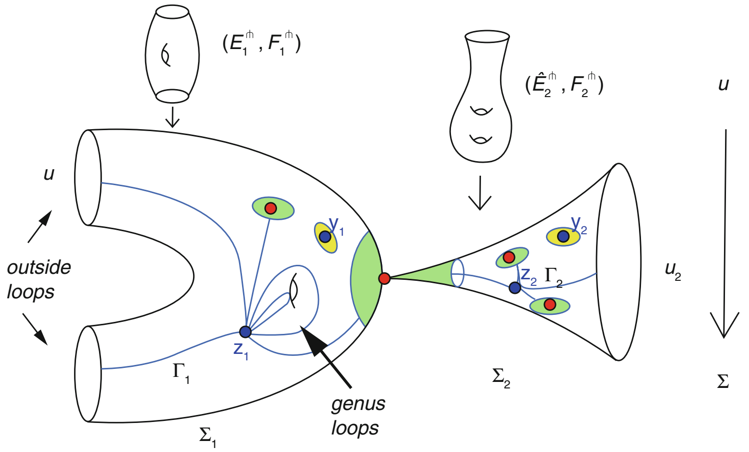

Fig. 2 Critical set and the cell models for $(\mathcal{Z}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{B})$

and define $W_i = G_i \bigsqcup S_i \bigsqcup U_i$ and $W = \bigsqcup W_i$.

We introduce one more puncture y_i on Σ_i and next redefine $\Sigma_i^* := \Sigma \setminus \text{int}(B_Q \cup B_{y_i})$. Moreover we use notations $(\mathcal{X}_y, \mathcal{A}_y)$ and $(\mathcal{Z}_y, \mathcal{C}_y)$. By enlarging "the hole" defined by the puncture y_i, we retract Σ_i^* to a configuration of loops connected by nonintersecting paths to some point z_i, denoted by Γ_i (see Fig. 2). The number of loops is $\#W_i = 2g_i + \tau_i + \gamma_i$, where $\tau_i := \#U_i$ and $\gamma_i := \sum_{q \in Q_i} \#S_q$. Note that $\tau_i > 0$ since there must be at least one outside loop.

Each pair $(\mathcal{Y}_i, \mathcal{B}_i)$ is then homotopy equivalent (by retraction) to the pair $(\pi_\Sigma^{-1}(\Gamma_i), \mathcal{B} \cap \pi_\Sigma^{-1}(\Gamma_i))$. We endow the latter with the structure of a relative CW complex as we did with $(\mathcal{Z}, \mathcal{C})$ at Sect. 3.5, namely, for each loop the similar CW complex structure as we have defined above for some pair $(\mathcal{Z}_s, \mathcal{C}_s)$. The difference is that the pairs $(\mathcal{Z}_s, \mathcal{C}_s)$ are disjoint, whereas in Σ_i^* the loops meet at a single point z_i. We take as reference the transversal fibre $F_i^{\pitchfork} = \mathcal{B} \cap \pi_\Sigma^{-1}(z_i)$ above this point, namely, we attach the n-cells (thimbles) only once to this single fibre in order to kill the $\mu_i^{\pitchfork}$ generators of $H_{n-1}(F_i^{\pitchfork})$. The $(n+1)$-cells of $(\mathcal{Y}_i, \mathcal{B}_i)$ correspond to the fibre bundles over the loops in the bouquet model of Σ_i^*. Over each loop, one attaches a number of $\mu_i^{\pitchfork}$ $(n+1)$-cells to the fixed n-skeleton described before, more precisely one $(n+1)$-cell over one n-cell generator of the n-skeleton. We extend for $w \in W$ the notation $(\mathcal{Z}_g, \mathcal{C}_g)$ to genus loops and $(\mathcal{Z}_u, \mathcal{C}_u)$ to outside loops, although they are not contained in $(\mathcal{Z}, \mathcal{C})$ but in $(\mathcal{Y}, \mathcal{B})$.

The attaching map of the $(n+1)$-cells corresponding to the bundle over a genus loop, or over an outside loop, can be identified with $A_g - I : \mathbb{Z}^{\mu_i^{\pitchfork}} \to \mathbb{Z}^{\mu_i^{\pitchfork}}$, or with $A_u - I : \mathbb{Z}^{\mu_i^{\pitchfork}} \to \mathbb{Z}^{\mu_i^{\pitchfork}}$, respectively. We have seen that the monodromy A_u over some outside loop indexed by $u \in U_i$ is necessarily one of the vertical monodromies of the original function $\hat{f}$.

From this CW complex structure, we get the following precise description in terms of the monodromies of the transversal local system, the proof of which is similar to that of Siersma and Tibăr [12, Lemma 4.4]:

Lemma 3.2

(a) $H_k(\mathcal{Y}, \mathcal{B}) = \oplus_{i \in I} H_k(\mathcal{Y}_i, \mathcal{B}_i)$ *and this is* $= 0$ *for* $k \neq n, n+1$.
(b) $H_n(\mathcal{Y}_i, \mathcal{B}_i) \simeq \mathbb{Z}^{\mu_i^{\pitchfork}} / \langle \mathrm{Im}(A_w - I) \mid w \in W_i \rangle$,
(c) $\chi(\mathcal{Y}_i, \mathcal{B}_i) = (-1)^{n-1}(2g_i + \tau_i + \gamma_i - 1)\mu_i^{\pitchfork}$.

□

If we apply χ to (4) and (5) and take into account that $\chi(\mathcal{Z}, \mathcal{C}) = 0$, we get $\chi(E, F) = \chi(\mathcal{X}, \mathcal{A}) + \chi(\mathcal{Y}, \mathcal{B}) + \sum_r \chi(E_r, F_r)$. From this one may derive the Euler characteristic of the Milnor fibre F (already computed in [2]):

Proposition 3.3

$$\chi(F) = 1 + \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1) + (-1)^n \sum_{i \in I} (2g_i + \tau_i + \gamma_i - 2)\mu_i^{\pitchfork} + (-1)^n \sum_{r \in R} \mu_r.$$

□

Proposition 3.4 *The relative Mayer–Vietoris sequence* (5) *is trivial except of the following 6-terms sequence:*

$$\begin{aligned} 0 \to H_{n+1}(\mathcal{Z}, \mathcal{C}) \to H_{n+1}(\mathcal{X}, \mathcal{A}) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B}) \to H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to \\ \to H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to H_n(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to 0. \end{aligned} \tag{8}$$

□

Proof Lemma 3.1, Sect. 3.3 and Lemma 3.2 show that the terms $H_*(\mathcal{X}, \mathcal{A})$, $H_*(\mathcal{Y}, \mathcal{B})$ and $H_*(\mathcal{Z}, \mathcal{C})$ of the Mayer–Vietoris sequence (5) are concentrated in dimensions n and $n+1$ only. Following (4) and since $\tilde{H}_*(F)$ is concentrated in levels $n-1$ and n, we obtain that $H_{n+2}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) = 0$.

The first three terms of (8) are free. By the decomposition (4), in order to find the homology of F, we thus need to compute $H_k(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$ for $k = n, n+1$, since the others are zero.

In the remainder of this paper, we collect information about $H_n(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$. The knowledge of its dimension is then enough for determining $H_n(F)$, by only using the Euler characteristic formula (Proposition 3.3).

4 The Homology Group $H_{n-1}(F)$

We concentrate on the term $H_n(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \simeq \tilde{H}_{n-1}(F)$. We need the relative version of the "variation ladder", an exact sequence found in [9, Theorem 5.2, pp. 456–457]. This sequence has an important overlap with our relative Mayer–Vietoris sequence (8).

Proposition 4.1 ([12, Proposition 5.2]) *For any point $q \in Q$, the sequence*

$$0 \to H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \to \oplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \to H_{n+1}(\mathcal{X}_q, \mathcal{A}_q) \to \\ \to H_n(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \to \oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q) \to 0$$

is exact for $n \geq 2$. □

4.1 The Image of j

We focus on the map $j = j_1 \oplus j_2$ which occurs in the 6-term exact sequence (8), more precisely on the following exact sequence:

$$H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to \tilde{H}_{n-1}(F) \to 0. \quad (9)$$

since we have the isomorphism:

$$\tilde{H}_{n-1}(F) \simeq \operatorname{coker} j. \quad (10)$$

Therefore, full information about j makes it possible to compute $H_{n-1}(F)$. But although j is of geometric nature, this information is not always easy to obtain. Below we treat its two components separately. After that we will make two statements (Theorems 4.4 and 4.6) of a more general type.

4.1.1 The First Component $j_1 : H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{X}, \mathcal{A})$

Note that, as shown above, we have the following direct sum decompositions of the source and the target:

$$H_n(\mathcal{Z}, \mathcal{C}) = \oplus_{q \in Q} \oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \oplus \oplus_{i \in I} H_n(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}), \\ H_n(\mathcal{X}, \mathcal{A}) = \oplus_{q \in Q} H_n(\mathcal{X}_q, \mathcal{A}_q) \oplus \oplus_{i \in I} H_n(\mathcal{X}_{y_i}, \mathcal{A}_{y_i}).$$

As shown in Proposition 4.1, at the special points $q \in Q$, we have surjections $\oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q)$, and moreover $H_n(\mathcal{Z}_y, \mathcal{C}_y) \to H_n(\mathcal{X}_y, \mathcal{A}_y)$ is an isomorphism. We conclude that the map j_1 is surjective and that there is no contribution of the points y_i to $\operatorname{coker} j$.

4.1.2 The Second Component $j_2 : H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{Y}, \mathcal{B})$

Both sides are described with a relative CW complex as explained in Sect. 3.6. At the level of n-cells, there are $\mu_s^{\pitchfork}$ n-cell generators of $H_n(\mathcal{Z}_s, \mathcal{C}_s)$ for each $s \in S_q$ and any $q \in Q$. Each of these generators is mapped bijectively to the single cluster of n-cell generators attached to the reference fibre $F_i^{\pitchfork}$ (which is the fibre above the common point z_i of the loops). The restriction $j_{2|} : H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{Y}_i, \mathcal{B}_i)$ is a projection for any loop s in Σ_i and $q \in Q_i$, or if instead of s we have y_i, since we add extra relations to $\mathbb{Z}^{\mu^{\pitchfork}}/\langle A_s - I\rangle$ in order to get $\mathbb{Z}^{\mu_i^{\pitchfork}}/\langle \operatorname{Im}(A_w - I) \mid w \in W_i\rangle = H_n(\mathcal{Y}_i, \mathcal{B}_i)$. We summarize the above surjections as follows:

Lemma 4.2 (Strong Surjectivity)

(a) *Both j_1 and j_2 are surjective.*
(b) *The restriction $j_{2|} : H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{Y}_i, \mathcal{A}_i)$ is surjective for any $s \in S_q$ such that $q \in Q \cap \Sigma_i$.*
(c) *The restriction $j_1| \oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q)$ is surjective, for any $q \in Q$.*

□

Corollary 4.3

(a) *If the restriction $j_2| \ker j_1$ is surjective, then j is surjective.*
(b) *If for each $i \in I$, there exist $q_i \in Q \cap \Sigma_i$ and some $s \in S_{q_i}$ such that $H_n(\mathcal{Z}_s, \mathcal{C}_s) \subset \ker j_1$, then j is surjective.*

□

Proof

(a) More generally, let $j_1 : M \to M_1$ and $j_2 : M \to M_2$ be morphisms of $\mathbb{Z}$-modules such that j_1 is surjective, and consider the direct sum of them $j := j_1 \oplus j_2$. We assume that the restriction $j_2| \ker j_1$ is surjective onto M_2 and want to prove that j is surjective.
  Let then $(a, b) \in M_1 \oplus M_2$. There exists $x \in M$ such that $j_1(x) = a$, by the surjectivity of j_1. Let $b' := j_2(x)$. By our surjectivity assumption, there exists $y \in \ker j_1$ such that $j_2(y) = b - b'$. Then $j(x + y) = a + b$, which proves the surjectivity of j.
(b) follows immediately from Lemma 4.2(b) and from the above (a). □

4.2 Effect of Local System Monodromies on $H_n(F)$

Recall that $w \in W_i$ stands for some loop s, or g, or u in Σ_i^*.

Theorem 4.4

(a) *If there is* $w \in W_i$ *such that* $\det(A_w - I) \neq 0$*, then* $\dim H_n(\mathcal{Y}_i, \mathcal{B}_i) = 0$.
If such $w \in W_i$ *exists for any* $i \in I$*, then* $b_{n-1}(F) = 0$.
(b) *If there is* $w \in W_i$ *such that* $\det(A_w - I) = \pm 1$*, then* $H_n(\mathcal{Y}_i, \mathcal{B}_i) = 0$.
If such $w \in W_i$ *exists for any* $i \in I$*, then* $H_{n-1}(F) = 0$.
(c) *The following upper bound holds:*

$$b_{n-1}(F) \leq \sum_{i \in I} \min_{w \in W_i} \dim \operatorname{coker}(A_w - I) \leq \sum_{i \in I} \mu_i^{\pitchfork}.$$

Proof By Lemma 3.2(b), we have $H_n(\mathcal{Y}_i, \mathcal{B}_i) \simeq \mathbb{Z}^{\mu_i^{\pitchfork}} / \langle \operatorname{Im}(A_w - I) \mid w \in W_i \rangle$; thus, the first parts of (a) and (b) follow. For the second part of (a), we have that $\dim H_n(\mathcal{Y}, \mathcal{B}) = 0$; hence, $\operatorname{corank} j = \operatorname{corank} j_1 = 0$. For the second part of (b), we have that $H_n(\mathcal{Y}, \mathcal{B}) = 0$, and the surjectivity of the map j of (9) is equivalent to the fact that j_1 is surjective.

To prove (c), we consider homology groups with coefficients in $\mathbb{Q}$. Since j_1 is surjective, the image of j contains all the generators of $H_n(\mathcal{X}, \mathcal{A}; \mathbb{Q})$. Hence $\dim \operatorname{coker} j \leq \dim H_n(\mathcal{Y}, \mathcal{B})$. □

Remark 4.5 Notice the *effect of the strongest bound* in the above theorem. On each Σ_i one could take an optimal loop, e.g. one with $\det(A_w - I) = \pm 1$. Since in the deformed case there may be less branches Σ_i, and more special points and hence more vertical monodromies, these bounds may become much stronger than those in [9].

4.3 Effect of the Local Fibres $\mathcal{A}_q$

Theorem 4.6 *Let* $n \geq 2$.

(a) *Assume that for each irreducible one-dimensional component* Σ_i *of* Σ*, there is a special singularity* $q \in Q_i$ *such that the* $(n-1)$*th homology group of its Milnor fibre is trivial, i.e.* $H_{n-1}(\mathcal{A}_q) = 0$*. Then* $H_{n-1}(F) = 0$.
If in the above assumption we replace $H_{n-1}(\mathcal{A}_q) = 0$ *by* $b_{n-1}(\mathcal{A}_q) = 0$*, then we get* $b_{n-1}(F) = 0$.

(b) *Let* $Q' := \{q_1, \ldots, q_m\} \subset Q$ *be some (minimal) subset of special points such that each branch* Σ_i *contains at least one of its points. Then*

$$b_{n-1}(F) \leq \dim H_n(\mathcal{X}_{q_1}, \mathcal{A}_{q_1}) + \cdots + \dim H_n(\mathcal{X}_{q_m}, \mathcal{A}_{q_m}).$$

Proof

(a) We use (9) in order to estimate the dimension of the image of $j = j_1 \oplus j_2$. If there is a $q \in Q$ such that $H_n(\mathcal{X}_q, \mathcal{A}_q) = 0$, then $\ker j_1$ contains $\oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s)$. Since Q' meets all components Σ_i, statement (a) follows from Corollary 4.3(b). The second claim of (a) follows by considering homology over $\mathbb{Q}$.

(b) We work again with homology over $\mathbb{Q}$. We consider the projection on a direct summand $\pi : H_n(\mathcal{X}, \mathcal{A}) \to \oplus_{q \notin Q'} H_n(\mathcal{X}_q, \mathcal{A}_q)$ and the composed map $J_1 := \pi \circ j_1$. Then the restriction $j_2 | \ker J_1$ is surjective, which by Corollary 4.3(a), means that $J_1 \circ j_2$ is surjective. Then the result follows from the obvious inequality $\dim(\mathrm{Im} J_1 \circ j_2) \leq \dim \mathrm{Im} j$ by counting dimensions. □

Remark 4.7 Also here we have the *effect of the strongest bound*. This works the best if one chooses an optimal or minimal Q'. In the irreducible case, $H_{n-1}(\mathcal{A}_q) = 0$ for at least one $q \in Q$ implies the triviality $H_{n-1}(F) = 0$.

Corollary 4.8 (Bouquet Theorem) *If* $n \geq 3$ *and*

(a) *If for any* $i \in I$*, there is* $w \in W_i$ *such that* $\det(A_w - I) = \pm 1$ *or*

(b) *If for any* $i \in I$*, there is a special singularity* $q \in Q_i$ *such that* $H_{n-1}(\mathcal{A}_q) = 0$ *, then*

$$F \overset{\text{ht}}{\simeq} S^n \vee \cdots \vee S^n.$$

Proof From Theorems 4.4(b) and 4.6(a), respectively, it follows that $H_{n-1}(F) = 0$. Since F is a simply connected n-dimensional CW complex, the statement follows from Milnor's argument [3, Theorem 6.5] and Whitehead's theorem. □

5 Examples

5.1 Singularities with Transversal Type A_1

The case when Σ is a smooth line was considered in [6] and later generalized to Σ a one-dimensional complete intersection (icis) [7]. It uses an admissible deformation with only D_∞-points. The main statement is:

(a) $F \overset{\text{ht}}{\simeq} S^{n-1}$ if $\# D_\infty = 0$,

(b) $F \overset{\text{ht}}{\simeq} S^n \vee \cdots \vee S^n$ else.

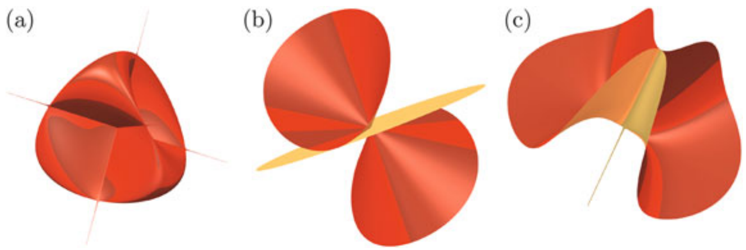

Fig. 3 Several singularites (produced with Surfer software). (**a**) Steiner Surface. (**b**) Singularity F_1A_3. (**c**) Singularity F_2A_3

Since D_∞-points have $H_{n-1}(\mathcal{A}_q) = 0$, our Theorem 4.6 provides a proof of this statement on the level of homology. If Σ is not an icis, more complicated situations may occur. The next known examples show how our results apply (see [7] for details):

1. $f = xyz$, called $T_{\infty,\infty,\infty}$. Here Σ is the union of three coordinate axis, $F \cong S^1 \times S^1$; thus, $b_1(F) = 2$, $b_2(F) = 1$ and $A_u = I$ for all u.
2. $f = x^2y^2 + y^2z^2 + x^2z^2$ has $F \cong S^2 \vee \cdots \vee S^2$. The admissible deformation $f_s = f + sxyz$ has the same Σ as $f = xyz$, but now with three D_∞-points on each component of Σ and one $T_{\infty,\infty,\infty}$-point in the origin. Our Theorem 4.6 yields $H_1(F) = 0$. A real picture of $f_s = 0$ contains the Steiner surface, for $s \neq 0$ small enough (Fig. 3a). That $H_2(F) = \mathbb{Z}^{15}$ follows from $\chi(F) = 16$ computed via Proposition 3.3.

5.2 Transversal Type $A_2, A_3, D_4, E_6, E_7, E_8$, De Jong's List

In [1] there is a detailed description of singularities with singular set a smooth line and transversal type $A_2, A_3, D_4, E_6, E_7, E_8$. De Jong's list illustrates and confirms our statements at the level of homology.

We will treat below in more detail the case $f : \mathbb{C}^3 \to \mathbb{C}$ with transversal type A_3 (to which one may add squares to become $f : \mathbb{C}^{n+1} \to \mathbb{C}$). Any singularity of this type can be deformed into

$$F_1A_3: \quad f = xz^2 + y^2z \; ; \; F \overset{\text{ht}}{\simeq} S^1 \text{ (Fig. 3b)}$$
$$F_2A_3: \quad f = xy^4 + z^2 \; ; \; F \overset{\text{ht}}{\simeq} S^2 \text{ (Fig. 3c)}$$

De Jong's observation is that for any line singularity of transversal type A_3, we have:

(a) $F \overset{\text{ht}}{\simeq} S^{n-1} \vee S^n \cdots \vee S^n$ if $\#F_2A_3 = 0$,
(b) $F \overset{\text{ht}}{\simeq} S^n \vee \cdots \vee S^n$ else.

In homology, (b) follows directly from our Theorem 4.6. The homology version of (a) takes more efforts. We demonstrate this in the following example only. First we mention that for F_1A_3 the vertical monodromy A is equal to the Milnor monodromy h. This follows from the fact that $f = xz^2 + y^2z$ is homogeneous of degree $d = 3$ and Steenbrink's remark [13] that $Ah^d = I$ and that $h^4 = I$. The matrix of h is:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

It follows: $\ker(h-I) = \mathbb{Z}$, $\mathrm{Im}\,(h-I) = \mathbb{Z}^2$, and $\mathrm{coker}(h-I) = \mathbb{Z}$.

Next consider as example the deformation $f := f_s = (x^k - s)z^2 + yz^2 + y^2z$ for some fixed small enough $s \neq 0$, which has transversal type A_3. This deformation has $\#F_1A_3 = k$ and $\#F_2A_3 = 0$ and moreover one isolated critical point of type A_k. Note that

$$H_n(\mathcal{Y}, \mathcal{B}) = \mathbb{Z}^3/\langle h-I, \cdots, h-I, A_u - I\rangle = \mathbb{Z}^3/\langle h-I\rangle = \mathbb{Z}$$

since for the outside loop u, we have $A_u = A_{s_1} \circ \cdots \circ A_{s_k} = h^k$ (all A_s are equal to h) and therefore $A_u - I = (h-1)(h^{k-1} + \cdots + h + I)$. We compare now the fundamental sequence for j in case F_1A_3 and f, respectively (we distinguish the Milnor fibres by a subscript):

$$j = j_1 \oplus j_2 : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_{n-1}(F_{F_1A_3}) = \mathbb{Z} \to 0 \tag{11}$$

$$j = j_1 \oplus j_2 : \mathbb{Z}^k \to \mathbb{Z}^k \oplus \mathbb{Z} \to H_{n-1}(F_f) \to 0 \tag{12}$$

The map j_2 for f can now be identified with: $j_2(\xi_1, \cdots, \xi_k) = \xi_1 + \cdots \xi_k$.

We conclude $H_1(F_f) = \mathbb{Z}$. Then $H_2(F_f) = \mathbb{Z}^{3k-1}$ follows from $\chi(F_f) = 3k - 1$ computed via Proposition 3.3.

We illustrate this example with Fig. 4a,b.

5.3 More General Types

We show next that the above method is not restricted to the De Jong's classes. Consider $f = z^2x^m - z^{m+2} + zy^{m+1}$. It has the properties $F \simeq S^1$; Σ is smooth; transversal type is A_{2m+1}; $A = h^m$, where h is the Milnor monodromy of A_{2m+1}.

Note that $\dim\ker(A - I) \geq 1$, and $= 1$ in many cases, e.g. $m = 2, 3, 4, 5$. This function f appears as "building block" in the following deformation:

$$g_s = z^2(x^2 - s)^m - z^{m+2} + zy^{m+1}.$$

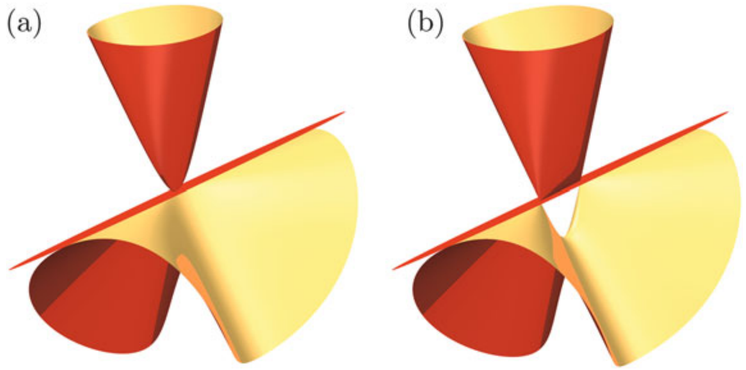

Fig. 4 Deformation $f_s = (x^k - s)z^2 + yz^2 + y^2z$ (produced with Surfer software). (**a**) Original surface. (**b**) Deformed surface

This deformation contains two special points of the type f (and no others, except isolated singularities). If one applies the same procedure as above, one gets (in the $= 1$ cases) $b_1(G) = 1$ where G is the Milnor fibre of g_0. Details are left to the reader.

Remark 5.1 The fact that the first Betti number of the Milnor fibre is nonzero can also be deduced from Van Straten's [15, Theorem 4.4.12]: *Let* $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be a germ of a function without multiple factors, and let F be the Milnor fibre of* f*. Then*

$$b_1(F) \geq \#\{\text{irreducible components of } f = 0\} - 1.$$

5.4 Deformation with Triple Points

Let $f_s = xyz(x+y+z-s)$. This defines a deformation of a central arrangement with four hyperplanes. We get $\Sigma_i = \mathbb{P}^1$ (six copies). There are four triple points $T_{\infty,\infty,\infty}$ and one A_1-point. The maps $j_{1,q} : \mathbb{Z}^3 \to \mathbb{Z}^2$ can be described by $j_{1,q}(a, b, c) = (a + c, b + c)$. The map j_2 restricts to an isomorphism $\mathbb{Z} \to \mathbb{Z}$ on each component. We have all information of the resulting map $j : \mathbb{Z}^{12} \to \mathbb{Z}^{14}$ up to the signs of the isomorphisms. From this we get $H_1(F; \mathbb{Z}_2) = \mathbb{Z}_2^3$. One may compare with the dissertation [16], where Williams showed in particular that $H_1(F; \mathbb{Z}) = \mathbb{Z}^3$.

5.5 The Class of Singularities with $b_n = 0$

Most of the singularities above have $b_{n-1} = 0$ or small b_{n-1}. One of the natural questions is what happens if $b_n = 0$. Examples are products of an isolated singularity with a smooth line (such as A_∞) and some of the functions mentioned above (e.g. F_1A_3). Very few is known about this class; let us show here the following "non-splitting property" w.r.t. isolated singularities:

Proposition 5.2 *If $\hat{f}$ has the property that $b_n(\hat{F}) = 0$, then any admissible deformation has no isolated critical points.*

Proof From (4) we get $H_{n+1}(E, F) = 0$. It follows that $H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) = 0$ and $\oplus_{r \in R} H_{n+1}(E_r, F_r) = 0$, and therefore the set R is empty. □

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Some Remarks on Hyperresolutions

J.H.M. Steenbrink

Abstract We give an example of a cubical variety which does not admit a weak resolution in the sense of Guillén et al. (Hyperrésolutions Cubiques et Descente Cohomologique. Springer Lecture Notes in Mathematics, vol 1335. Springer, Berlin, 1988). We introduce the notion of a very weak resolution of a cubical variety, and we show that it always exists in characteristic zero. This suffices for the proof of the existence of cubical hyperresolutions.

Keywords Cubical hyperresolution

MSC classification: 14E15

1 Introduction

The theory of cubical hyperresolutions is due to V. Navarro Aznar and co-authors; see [3]. This construction was motivated by Deligne's theory of mixed Hodge structures. In [1] and [2], Deligne constructs a functor from the category of reduced $\mathbb{C}$-schemes of finite type to the category of mixed Hodge structures, whose value for a smooth projective complex variety is the Hodge structure on its cohomology ring. To this end, Deligne constructs simplicial resolutions for reduced $\mathbb{C}$-schemes. These simplicial resolutions inherently have an infinite number of irreducible components. In [3] these simplicial resolutions are replaced by cubical hyperresolutions, in which only a finite number of irreducible components occur. The dimensions of these components are under control. This leads to vanishing theorems for singular varieties analogous to those of Kodaire and Akizuki-Nakano.

In the fall of 2014, the author gave lectures on mixed Hodge theory at the Mathematical Sciences Center of Tsinghua University in Beijing. Among other topics I treated the construction of cubical hyperresolutions. In the inductive

J.H.M. Steenbrink (✉)
IMAPP, Radboud University Nijmegen, Nijmegen, The Netherlands

MSC, Tsinghua University, Beijing, China
e-mail: j.steenbrink@math.ru.nl

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procedure for their construction, the notion of a *weak resolution* of a cubical variety is needed. When dealing with these, I discovered a counterexample to [3, Theorem I.2.6], which states that a weak resolution of a cubical variety always exists. In the definition of a weak resolution, there is a condition on the dimension of the preimage of the discriminant. We describe this condition and give an example where a weak resolution with this property does not exist.

2 Resolutions of Cubical Varieties

For any $n \in \mathbb{N}$ let $[n] = \{0, \ldots, n\}$. The category $\square_n^+$ is the category whose objects are the subsets $I \subset [n]$ and with $\mathrm{Hom}(I, J) = \emptyset$ unless $I \subset J$, in which case it consists of one element, the inclusion map $I \hookrightarrow J$. Let k be a field, and let Var_k be the category of reduced k-schemes of finite type. The fibre product in this category is the reduction of the scheme-theoretic fibre product.

Definition 1 An n-cubical variety over k is a contravariant functor

$$X_\bullet : \square_{n-1}^+ \to \mathrm{Var}_k.$$

Definition 2 A morphism $f : X_\bullet \to S_\bullet$ of n-cubical varieties is called *proper* if for all $I \subset [n-1]$ the morphism f_I is proper.

Definition 3 The *discriminant* of a proper morphism $f : X_\bullet \to S_\bullet$ of n-cubical varieties is the smallest closed n-cubical subvariety $D_\bullet$ of $S_\bullet$ such that f induces isomorphisms $X_I \setminus f^{-1}(D_I) \to S_I \setminus D_I$ for all $I \subset [n-1]$.

Definition 4 (See [3, Déf. I,2,5], [5, Definition 5.21], [4, Definition 10.73]) Let $S_\bullet$ be an n-cubical variety, $f : X_\bullet \to S_\bullet$ a proper morphism, and $D_\bullet$ the discriminant of f. We say that f is a *weak resolution* of S if all X_I are smooth and $\dim f_I^{-1}(D_I) < \dim S_I$ for all $I \subset [n-1]$. If in addition $\dim D_I < \dim S_I$ for all I, then f is called a resolution.

Example 1 We consider the 1-cubical variety $S_{\{0\}} \to S_\emptyset$ with $S_{\{0\}} = \mathrm{Spec}(k[x,y]/(xy))$ and $S_\emptyset = \mathrm{Spec}(k)$. Let $X_\bullet \to S_\bullet$ be any (weak) resolution in the sense of Definition 4, and let $D_\bullet$ be its discriminant. Because $S_{\{0\}}$ is not smooth, $D_{\{0\}} \neq \emptyset$; hence, $D_\emptyset \neq \emptyset$. So $D_\emptyset = S_\emptyset$. Also $X_{\{0\}} \neq \emptyset$ so $X_\emptyset \neq \emptyset$ and hence $f_\emptyset^{-1}(D_\emptyset)$ has dimension ≥ 0. So $\dim f_\emptyset^{-1}(D_\emptyset) = \dim S_\emptyset$, which contradicts the definition of a weak resolution. So $S_\bullet$ has no weak resolution in the sense of Definition 4.

Of course, a natural candidate for something like a resolution in this example is the 1-cubical variety $X_\bullet$ where $X_{\{0\}}$ is the normalisation of $S_{\{0\}}$ and $X_\emptyset$ consists of one or two copies of $S_\emptyset$.

The idea of a resolution is that a variety X is replaced by a smooth variety $\tilde{X}$ of the same dimension such that the difference between X and $\tilde{X}$ is a combination of varieties of lower dimension. Inspired by this idea, we introduce the concept of a *very weak resolution*. Moreover we show the following two facts:

1. For every n-cubical variety $S_\bullet$, there exists a very weak resolution $f : X_\bullet \to S_\bullet$;
2. Very weak resolutions are useful in the proof of the existence of cubical hyperresolutions.

Recall that the dimension of a cubical variety $X_\bullet$ is defined as the maximum of the dimensions of all X_I [3, Déf. I.2.9].

Definition 5 Let $S_\bullet$ be an n-cubical variety, $f : X_\bullet \to S_\bullet$ a proper morphism, and D the discriminant of f. We say that f is a *very weak resolution* of S if all X_I are smooth, $\dim D_I < \dim S$ and $\dim f_I^{-1}(D_I) < \dim S$ for all $I \subset [n-1]$.

Theorem 1 *Let $S_\bullet$ be an n-cubical variety. Then there exists a very weak resolution $f : X_\bullet \to S_\bullet$ of $S_\bullet$.*

Proof The proof of [3, Theorem I.2.6] should be slightly adapted to the new definition. Referring to the text in the proof of [5, Theorem 5.25]), the word "resolution" has to be replaced by "very weak resolution". Moreover the statement $\dim f_I^{-1}(D_I) < \dim S_I$ has to be replaced by the following argument.

Let $s = \dim S$. Then the construction procedure guarantees that $\dim X_I \leq s$ for all I. We show that $\dim D_I < s$. If not, there exists a maximal I for which D_I has dimension s. Then $D_I = \Delta(f_I)$. Let $X_{I,\alpha}$ be a connected component of X_I such that $\dim X_{I,\alpha} = s$. Let $D_{I,\alpha}$ be the union of the discriminant of $X_{I,\alpha} \to S_{I,\alpha}$ and the components $S_{I,\beta}$ for $\beta \neq \alpha$. Then $X_{I,\alpha} \setminus f_I^{-1}(D_{I,\alpha}) \to S_{I,\alpha} \setminus D_{I,\alpha}$ is an isomorphism; hence, $D_I \subset D_{I,\alpha}$ and D_I does not contain $S_{I,\alpha}$. However, if $\dim D_I = s$ then it must contain an irreducible component $S_{I,\alpha}$ of dimension s. This is a contradiction.

The same reasoning shows that a connected component of dimension s of X_I is not mapped inside D_I. Hence $f_I^{-1}(D_I)$ does not contain any irreducible component of X_I of dimension s, so it must be of dimension $< s$. We conclude that $f : X_\bullet \to S_\bullet$ is a very weak resolution in the sense of Definition 5. □

3 Cohomological Descent

For any complex variety Y, we let $C_Y^\bullet$ denote the Godement resolution of the constant sheaf $\mathbb{Z}_X$. Let $X_\bullet$ be an n-cubical complex variety. For $I \subset [n-1]$ let $\epsilon_I : X_I \to X_\emptyset$ denote the morphism corresponding to the inclusion $\emptyset \hookrightarrow I$. We let $C(X_\bullet)$ denote the associated single complex of the double complex

$$0 \to C^\bullet_{X_\emptyset} \to \bigoplus_{\sharp I=1} (\epsilon_I)_* C^\bullet_{X_I} \to \bigoplus_{\sharp I=2} (\epsilon_I)_* C^\bullet_{X_I} \to \dots$$

(see [5, Sect. 5.1.3] for more details).

Definition 6 A cubical variety $X_\bullet$ is *of cohomological descent* of the sheaf complex $C(X_\bullet)$ is acyclic.

Example 2 Let $f : X_\bullet \to Y_\bullet$ be a proper morphism of k-cubical varieties with discriminant $D_\bullet$. These data determine a $k+2$-cubical variety $Z_\bullet$ as follows. For $I \subset [k-1]$ we let

$$Z_I = Y_I,\ Z_{I\cup\{k\}} = X_I,\ Z_{I\cup\{k+1\}} = D_I\ Z_{I\cup\{k,k+1\}} = f_I^{-1}(D_I)$$

with the obvious maps. Then $Z_\bullet$ is called the *discriminant square* of f. By [5, Lemma 5.20], this cubical variety $Z_\bullet$ is of cohomological descent.

Example 3 Let $g : X_\bullet \to Y_\bullet$ be a proper morphism of k-cubical varieties. It defines a $k+1$-cubical variety W by

$$W_I = Y_I,\ W_{I\cup\{k\}} = X_I$$

for each $I \subset [k-1]$. Then $W_\bullet$ is of cohomological descent if and only if the natural morphism of complexes

$$f^\sharp : C(Y_\bullet) \to f_* C(X_\bullet)$$

is a quasi-isomorphism. This is a consequence of [5, Lemma 5.27].

4 Cubical Hyperresolutions

We now consider the proof of the existence of cubical hyperresolutions in [3, Sect. 1]. We formulate a small variation on it.

Theorem 2 *Let X be an algebraic variety of dimension n. Then there exists an $(n+1)$-cubical variety $X_\bullet$ with the following properties:*

1. *$X_\emptyset = X$ and X_I is smooth for $I \neq \emptyset$;*
2. *all morphisms $d_{IJ} : X_J \to X_I$ are proper;*
3. *$X_\bullet$ is of cohomological descent;*
4. *if $m \in I \subset [n]$ then $\dim X_I \leq n - m$.*

Proof Considering the proof of [5, Theorem 5.26], one observes that everything works when one replaces "resolution" by "very weak resolution". This proof also already contains the desired dimension estimate. □

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Deforming Nonnormal Isolated Surface Singularities and Constructing Threefolds with $\mathbb{P}^1$ as Exceptional Set

Jan Stevens

To Gert-Martin Greuel on the occasion of his 70th birthday.

Abstract Normally one assumes isolated surface singularities to be normal. The purpose of this paper is to show that it can be useful to look at nonnormal singularities. By deforming them interesting normal singularities can be constructed, such as isolated, non-Cohen-Macaulay threefold singularities. They arise by a small contraction of a smooth rational curve, whose normal bundle has a sufficiently positive subbundle. We study such singularities from their nonnormal general hyperplane section.

Keywords Nonnormal singularities • Simultaneous normalisation • Small modifications

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1 Introduction

Suppose we are interested in a germ $(X, 0) \subset (\mathbb{C}^N, 0)$ of a complex space, which has some salient features. Then we would like to describe the singularity X as explicit as possible. This can be done by giving generators of the local ring $\mathscr{O}_X$, or by giving equations for $X \subset \mathbb{C}^N$. But in general it is too difficult to do this directly. Instead we first replace the singularity by a simpler one. To recover the original one is then a deformation problem. In a number of situations, the simplification process leads to nonnormal singularities.

We formulate the most important simplification process as general principle.

J. Stevens (✉)
Matematik, Göteborgs universitet and Chalmers tekniska högskola, SE 412 96 Göteborg, Sweden
e-mail: stevens@chalmers.se

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The Hyperplane Section Principle *The (general) hyperplane section of a singularity has a local ring with the same structure as the original singularity, but one embedding dimension lower, and which is much easier to describe.*

A nonnormal surface singularity occurs as general hyperplane of a normal three-dimensional isolated singularity, if this singularity is not Cohen-Macaulay. Such singularities can occur as a result of small contractions. In higher dimensions a resolution (with normal crossings exceptional divisor) is in general not the correct tool for understanding the singularity. But it may happen that a small resolution exists, meaning (in dimension three) that the exceptional set is only a curve. The simplest case is that the curve is a smooth rational curve. Nevertheless, such a singularity can be quite complicated, as it need not be Cohen-Macaulay. This happens if the normal bundle of the curve is $\mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a > 1$. One has always that $2a + b < 0$ [3, 20], and Ando has given examples of the extremal case $(a, b) = (n, -2n - 1)$ by exhibiting transition functions. We study the contraction of such a curve using the hyperplane section principle.

The first example of a manifold containing an exceptional $\mathbb{P}^1$ with normal bundle with positive subbundle, namely, $\mathcal{O}(1) \oplus \mathcal{O}(-3)$, was given by Laufer [16]. Pinkham gave a construction as total space of a 1-parameter smoothing of a partial resolution of a rational double point [21]. We consider smoothings of partial resolutions of non-rational singularities. In this case the total space does blow down, but not to a smoothing of the original singularity. Instead, the special fibre is a nonnormal surface singularity. We retrieve Ando's examples using the canonical model of the hypersurface singularity $z^2 = f_{2n+1}(x, y)$. We also give new examples.

Using SINGULAR [11] it is possible to give explicit equations for some cases. We do not compute deformations of the canonical model but deformations of the nonnormal surface singularity X. We study in detail the simplest case, where X differs not too much from its normalisation $\widetilde{X}$, meaning that $\delta(X) = \dim \mathcal{O}_{\widetilde{X}}/\mathcal{O}_X = 1$. We take the equation of $\widetilde{X}$ of the form $z^2 = f(x, y)$. It turns out that it is in fact possible to give general formulas.

A 1-parameter deformation of a resolution of a normal surface singularity blows down to a deformation of the singularity if and only if the geometric genus is constant. Otherwise the special fibre is nonnormal. Given a two-dimensional hypersurface singularity, the general singularity with the same resolution graph is not Gorenstein, and not quasi-homogeneous in the case that hypersurface is quasi-homogenous. Again, deforming a nonnormal quasi-homogeneous surface singularity gives a method to find equations for surface singularities with a given star-shaped graph. We give an example using the same general formulas as for small contractions (Example 17).

The structure of this paper is as follows. In the first section, we discuss invariants for nonnormal surface singularities. In the next section, we compute deformations for a nonnormal model of a surface singularity of multiplicity two. In Sect. 3 we recall in detail the relation between deformations of a (partial) resolution and of the singularity itself. The final section treats $\mathbb{P}^1$ as exceptional curve, with explicit formulas based on the previous calculations.

2 Invariants of Nonnormal Singularities

2.1 Normalisation

Definition 1 A reduced ring R is *normal* if it is integrally closed in its total ring of fractions. For an arbitrary reduced ring R, its *normalisation* $\overline{R}$ is the integral closure of R in its total ring of fractions. A singularity $(X, 0)$ (i.e. the germ of a complex space) is *normal* if its local ring $\mathcal{O}_{(X,0)}$ is normal. The *normalisation* of a reduced germ $(X, 0)$ is a multigerm $(\overline{X}, \overline{0})$ with semi-local ring $\mathcal{O}_{(\overline{X},\overline{0})} = \overline{\mathcal{O}}_{(X,0)}$. The normalisation map is $\nu\colon (\overline{X}, \overline{0}) \to (X, 0)$, or in terms of rings $\nu^*\colon \mathcal{O}_{(\overline{X},\overline{0})} \to \mathcal{O}_{(X,0)}$.

We have the following function-theoretic characterisation of normality; see, e.g. [10, p. 143]. Let Σ be the singular locus of a reduced complex space X and set $U = X \setminus \Sigma$, with $j\colon U \to X$ the inclusion map. Then X is normal at $p \in X$ if and only if for arbitrary small neighbourhoods $V \ni p$ every bounded holomorphic function on $U \cap V$ has a holomorpic extension to $X \cap V$. If codim $\Sigma \geq 2$, then $\overline{\mathcal{O}}_X = j_*\mathcal{O}_U$.

2.2 Cohen-Macaulay Singularities

For a two-dimensional isolated singularity, normal is equivalent to Cohen-Macaulay, but in higher dimensions, this is no longer true. A local ring is *Cohen-Macaulay* if there is a regular sequence of length equal to the dimension of the ring. A d-dimensional germ $(X, 0)$ is Cohen-Macaulay, if its local ring is Cohen-Macaulay. An equivalent condition is that there exists a finite projection $\pi\colon (X, 0) \to (\mathbb{C}^d, 0)$ with fibres of constant multiplicity (i.e. the map π is flat); see, e.g. [10, Kap. III § 1]. From both descriptions it follows directly that a singularity is Cohen-Macaulay if and only if its general hyperplane section is so. In particular, a general hyperplane section of a normal but not Cohen-Macaulay isolated threefold singularity is not normal.

A cohomological characterisation, in terms of local cohomology, of isolated Cohen-Macaulay singularities is that $H^q_{\{0\}}(X, \mathcal{O}_X) = 0$ for $q < d$. Normality implies only the vanishing for $q < 2$. The local cohomology can be computed from a resolution $\widetilde{X} \to X$, as $H^q(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^{q+1}_{\{0\}}(X, \mathcal{O}_X)$ for $1 \leq q \leq n-2$ [12, Prop. 4.2]. If $\widetilde{X}$ is a good resolution of an isolated singularity with exceptional divisor E, then the map $H^i(\mathcal{O}_{\widetilde{X}}) \to H^i(\mathcal{O}_E)$ is surjective for all i [24, Lemma 2.14]. This implies that a threefold singularity is not Cohen-Macaulay as soon as the exceptional divisor of a good resolution has an irregular surface F as component (meaning that $q = h^1(\mathcal{O}_F) > 0$). The easiest example of such a singularity is the cone over an irregular surface.

Example 2 The equations of the known families of smooth irregular surfaces in $\mathbb{P}^4$ are discussed in [5, Sect. 4]. They admit a large symmetry group, the Heisenberg group. The lowest degree case is that of elliptic quintic scrolls. Their homogeneous coordinate ring has a minimal free resolution of type

$$0 \longleftarrow \mathcal{O}_S \longleftarrow \mathcal{O}(-3)^5 \overset{L}{\longleftarrow} \mathcal{O}(-4)^5 \overset{x}{\longleftarrow} \mathcal{O}(-5) \longleftarrow 0 \, ,$$

where L is a matrix

$$\begin{pmatrix} 0 & -s_1x_4 & -s_2x_3 & s_2x_2 & s_1x_1 \\ s_1x_2 & 0 & -s_1x_0 & -s_2x_4 & s_2x_3 \\ s_2x_4 & s_1x_3 & 0 & -s_1x_1 & -s_2x_0 \\ -s_2x_1 & s_2x_0 & s_1x_4 & 0 & -s_1x_2 \\ -s_1x_3 & -s_2x_2 & s_2x_1 & s_1x_0 & 0 \end{pmatrix}$$

and x is the vector $(x_0, x_1, x_2, x_3, x_4)^t$. The constants $(s_1 : s_2)$ are homogeneous coordinates on the modular curve $X(5) \cong \mathbb{P}^1$. The i-th column of the matrix $\bigwedge^4 L$ is divisible by x_i, and the equations of the scroll are the five resulting cubics, which can be obtained from the following one by cyclic permutation of the indices:

$$s_1^4x_0x_2x_3 - s_1^3s_2(x_1x_2^2 + x_3^2x_4) - s_1^2s_2^2x_0^3 + s_1s_2^3(x_1^2x_3 + x_2x_4^2) + s_2^4x_0x_1x_4 \, .$$

A general hyperplane section is a quintic elliptic curve in $\mathbb{P}^3$, which is not projectively normal. In fact, the linear system of hyperplane sections is not complete, and therefore the curve is not a (linear) normal curve.

2.3 The δ-Invariant

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an isolated singularity. One measures how far the singularity is from being normal with the *δ-invariant*:

$$\delta(X, 0) = \dim(\mathcal{O}_{\overline{X},\nu^{-1}(0)}/\mathcal{O}_{X,0}) \, .$$

For plane curves this is the familiar δ-invariant, which is also called the number of virtual double points. In higher dimensions it is not the correct double point number. One has to consider the double point locus of the composed map $\varphi\colon \overline{X} \to X \to \mathbb{C}^N$ (see [13] for the general theory of double point schemes). The expected dimension of the double point locus is $2\dim X - N$. As X has an isolated singularity, necessarily $N \geq 2\dim X$, and the best results are in case of equality.

Consider a map $\varphi\colon \overline{X} \to Y$ with $\overline{X}$ a complete intersection and Y smooth of dimension twice the dimension of $\overline{X}$. As measure of degeneracy of the map φ, Artin and Nagata [4] introduced (following Mumford) half the size of source double point locus:

$$\Delta(\varphi) = \tfrac{1}{2} \dim \operatorname{Ker}\left(\mathcal{O}_{\overline{X}\times_Y\overline{X}} \longrightarrow \mathcal{O}_X\right) \, .$$

This dimension is stable under deformations of φ, and by deforming to an immersion with only nodes, one sees that $\Delta(\varphi)$ is an integer, as in that case $\overline{X} \times_Y \overline{X}$ splits into the diagonal and a finite set with free $\mathbb{Z}/2$-action.

For plane curve singularities, it follows by deforming to a curve with only nodes that $\delta(X) = \Delta(\varphi)$, cf. [27, 3.4]. As the image of φ is given by one equation, the images of a family of maps form a flat family and $\mathscr{O}_{\overline{X}_S}/\varphi^*\mathscr{O}_{Y_S}$ is flat over the base S. In higher dimensions the images of a family of maps need not form a flat family and $\delta(X)$ may be larger than $\Delta(\varphi)$.

Example 3 (cf. [4, (5.8)]) Map three copies of $(\mathbb{C}^2, 0)$ generically to $(\mathbb{C}^4, 0)$, say by $(x, y, z, w) = (s_1, t_1, 0, 0) = (0, 0, s_2, t_2) = (s_3, t_3, s_3, t_3)$. The image of this map φ lies on the quadric $xw = yz$, and there are four more cubic equations. The singularity is rigid; one computes that $T^1 = 0$. The δ-invariant is equal to 4. On the other hand, the double point number $\Delta(\varphi)$ is 3, and the general deformation of the map is obtained by moving the third plane. The ideal of the image is then the intersection of the ideals (z, w), (x, y) and $(x - z - a, y - w - b)$. There are eight cubic equations, obtained by multiplying the generators of the three ideals in all possible ways. Specialising to $a = b = 0$, one obtains the product of the ideals (z, w), (x, y) and $(x - z, y - w)$. This ideal has an embedded component. The same ideal is obtained if one does not consider the image with its reduced structure, but with its Fitting ideal structure, as in [27, §1]; indeed, that construction commutes with base change.

2.4 Simultaneous Normalisation

Definition 4 Let $f: \mathscr{X} \to S$ be a flat map between complex spaces, such that all fibres are reduced. A *simultaneous normalisation* of f is a finite map $\nu: \overline{\mathscr{X}} \to \mathscr{X}$ such that all fibres of the composed map $f \circ \nu$ are normal and that for each $s \in S$ the induced map on the fibre $\nu_s: \overline{\mathscr{X}}_s = (f \circ \nu)^{-1}(s) \to \mathscr{X}_s = f^{-1}(s)$ is the normalisation.

Criteria for the existence of a simultaneous normalisation are given by Chiang-Hsieh and Lipman [6]; see also [11, II.2.6]. For a family $f: \mathscr{X} \to S$ of curves over a normal base S, the normalisation of $\mathscr{X}$ is a simultaneous normalisation if and only if $\delta(\mathscr{X}_s)$ (defined as the sum over the δ-invariants of the singular points) is constant, a result originally due to Teissier and Raynaud. In case S is smooth one-dimensional, the δ-constant criterion holds also in higher dimensions, if the nonnormal locus of $\mathscr{X}$ is finite over the base S; in the algebraic case, this follows from [6, Corollary 3.3.1]. The proof of Greuel et al. [11, Theorem II.2.24] extends to this case: the fact that the fibres are curves is only used twice, firstly is get that $\mathscr{O}_{\overline{\mathscr{X}}}$ has depth (at least) two, which follows by Serre's criterion from normality, and secondly in appealing to Proposition 2.55, which is only formulated and proved for families of curves, but here one can use [6, Corollary 5.4.3].

2.5 The Geometric Genus

The geometric genus of a normal surface singularities was introduced by Wagreich [28], using a resolution $\pi\colon(\widetilde{X},E) \to (X,0)$, as $p_g = \dim H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$. In terms of cycles on the resolution, one has $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}) = \varprojlim H^1(Z,\mathcal{O}_Z)$, where Z runs over all effective divisors with support on the exceptional set. This means that p_g is the maximal value of $h^1(\mathcal{O}_Z)$; see [22, 4.8]. Wagreich also defined the arithmetic genus of the singularity as the maximal value of $p_a(Z)$, where $p_a(Z) = 1 - h^0(\mathcal{O}_Z) + h^1(\mathcal{O}_Z)$. This is a topological invariant. One computes $p_a(Z)$ by the adjunction formula: $p_a(Z) = 1 + \frac{1}{2}Z(Z+K)$. The geometric genus has an interpretation independent of a resolution, as $\dim(H^0(U,\Omega^2_U)/L^2(U,\Omega^2_U))$, where $L^2(U,\Omega^2_U)$ is the subspace of square-integrable 2-forms on $U = \widetilde{X}\setminus E = X\setminus 0$ [15].

For a not necessarily normal isolated singularity $(X,0)$ the geometric genus is a combination of the δ-invariant and invariants from the resolution. This makes sense, as the resolution factors over the normalisation. In any dimension we define, following Karras [12]:

Definition 5 Let $(X,0)$ be an isolated singularity of pure dimension n with resolution $(\widetilde{X},E)$. The *geometric genus* is

$$p_g(X,0) = -\delta(X,0) + \sum_{q=1}^{n-1}(-1)^{q-1}\dim H^q(\widetilde{X},\mathcal{O}_{\widetilde{X}})\ .$$

The dimension of $H^q(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ does not depend on the chosen resolution and is therefore an invariant of the singularity; one way to see this is using an intrinsic characterisation: one has $H^q(\widetilde{X},\mathcal{O}_{\widetilde{X}}) \cong H^{q+1}_{\{0\}}(X,\mathcal{O}_X)$ for $1 \le q \le n-2$, and $H^{n-1}(\widetilde{X},\mathcal{O}_{\widetilde{X}}) \cong H^0(U,\Omega^n_U)/L^2(U,\Omega^n_U)$ [12, Prop. 4.2]. We remark that $\delta(X,0) = \dim H^1_{\{0\}}(X,\mathcal{O}_X)$. For isolated Cohen-Macaulay singularities, all terms except the last one vanish, so $p_g = (-1)^n \dim H^{n-1}(\widetilde{X},\mathcal{O}_{\widetilde{X}})$, which is the direct generalisation of Wagreich's formula.

By the results of Elkik [9], the geometric genus is semicontinuous under deformation. More precisely, let $\pi\colon\widetilde{X} \to X$ be a resolution of a pure dimensional space and let the complex $M^\bullet_X$ be the third vertex of the triangle constructed on the natural map $i\colon\mathcal{O}_X \to R^\bullet\pi_*\mathcal{O}_{\widetilde{X}}$:

$$\begin{array}{ccc} & M^\bullet_X & \\ {}^{+1}\swarrow & & \nwarrow \\ \mathcal{O}_X & \longrightarrow & R^\bullet\pi_*\mathcal{O}_{\widetilde{X}} \end{array}$$

Then $M^{-1}_X = \operatorname{Ker}\mathcal{O}_X \to \mathcal{O}_{X_{\mathrm{red}}}$, $M^0_X = \mathcal{O}_{\overline{X}}/\mathcal{O}_X$, $M^i_X = R^i\pi_*\mathcal{O}_{\widetilde{X}}$ for $0 < i < n = \dim X$ and all other M^i_X are zero. If X has isolated singularities, define the partial

Euler–Poincaré characteristics:

$$\psi_i(X) = \sum_{j=0}^{n-i} (-1)^j \dim M_X^{n-j-i-1}, \qquad 0 \le i \le n \, .$$

Proposition 6 ([9, Théorème 1]) *For an equidimensional flat morphism $f\colon \mathscr{X} \to S$ with $\mathscr{X}$ smooth outside a closed set, finite over the base, the functions $s \mapsto \psi_i(\mathscr{X}_s)$ are upper semicontinuous.*

This result has the following corollaries, which are relevant for us.

Corollary 7 ([19, p. 255]) *If a nonnormal reduced isolated surface singularity X is smoothable, then $\delta(X) \le p_g(\widetilde{X})$.*

Corollary 8 ([14, (14.2)]) *Let $f\colon X \to T$ be a morphism from a normal threefold to the germ of a smooth curve. If $X_0 = f^{-1}(0)$ has only isolated singularities and the normalisation $\overline{X}_0$ has only rational singularities, then $\overline{X}_0 = X_0$.*

For a 1-parameter deformation of an isolated nonnormal surface singularity with rational normalisation, semicontinuity of $\psi_0 = -\delta$ and $\psi_1 = \delta$ implies that δ is constant. Therefore there is a simultaneous normalisation. The same is not necessarily true for infinitesimal deformations. In the next section, we give an example, where there exist obstructed deformations without simultaneous normalisation.

3 Computations

In this section we describe equations and deformations for surface singularities with $\delta = 1$, whose normalisation is a double point, so given by an equation of the form $z^2 = f(x, y)$, with $f \in \mathfrak{m}^2$.

We recall the set-up for deformations of singularities (for details see [25]). One starts from a system of generators $(g_1, \ldots, g_k)$ of the ideal of the singularity X. We also need generators of the module of relations, which we write as matrix (r_{ij}), $i = 1, \ldots, k$, $j = 1, \ldots, l$. So we have l relations $\sum g_i r_{ij} = 0$. We perturb the generators to $G_i(x, t)$ with $G_i(x, 0) = g_i(x)$. These describe a (flat) deformation of X if it is possible to lift the relations: there should exist a matrix $R(x, t)$ with $R(x, 0) = r(x)$ such that $\sum G_i R_{ij} = 0$ for all j. One can take this as definition of flatness. In particular, for an infinitesimal deformation $G_i(x, \varepsilon) = g_i(x) + \varepsilon g_i'(x)$ (with $\varepsilon^2 = 0$), one needs the existence of a matrix $r'(x)$ such that $\sum (g_i + \varepsilon g_i')(r_{ij} + \varepsilon r_{ij}') = \varepsilon \sum (g_i' r_{ij} + g_i r_{ij}') = 0$ or equivalently that $\sum g_i' r_{ij}$ lies in the ideal generated by the g_i. Deformations, induced by coordinate transformations, are considered to be trivial. To find the versal deformation, one takes representatives for all possible nontrivial infinitesimal deformations and tries to lift to higher order. The obstructions to do this define the base space of the versal deformation.

We consider a subring $\mathscr{O}$ of $\overline{\mathscr{O}} = \mathbb{C}\{x, y, z\}/(z^2 - f(x, y))$ with $\delta = \dim \overline{\mathscr{O}}/\mathscr{O} = 1$. We need a system of generators for the defining ideal. This and the possible deformations depend on the subring in question. We write $\mathscr{O} = \mathbb{C} + L + \mathfrak{m}^2$, where L is a two-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$, which can be given as kernel of a linear form $l\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}$, $l = ax + by + cz$.

First suppose that $c \neq 0$; we may assume that $c = 1$. Generators of $\mathscr{O}$ are then

$$\begin{aligned} \xi_1 &= x - az & \eta_1 &= y - bz & \zeta_2 &= z^2 \\ \xi_2 &= z(x - az) & \eta_2 &= z(y - bz) & \zeta_3 &= z^3 \end{aligned} \tag{1}$$

This system of generators is not minimal, as we not yet have taken the relation $z^2 = f(x, y)$ into account. Because $f \in \mathfrak{m}^2$, it can be written in terms of the generators; for example, $x^2 = \xi_1^2 + 2az\xi_1 + a^2z^2 = \xi_1^2 + 2a\xi_2 + a^2\zeta_2$, and $x^3 = \xi_1^3 + 3a\xi_1\xi_2 + 3a^2\zeta_2\xi_1 + a^3\zeta_3$. The relations $z^2 = f(x, y)$ and $z^3 = zf(x, y)$ lead to equations

$$\begin{aligned} \zeta_2 &= \varphi_2(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3) \ , \\ \zeta_3 &= \varphi_3(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3) \ . \end{aligned}$$

Therefore the variables ζ_2 and ζ_3 can be eliminated and the embedding dimension of $\mathscr{O}$ is four. Coordinates are ξ_1, ξ_2, η_1 and η_2, and equations for the corresponding singularity X are

$$\begin{aligned} \xi_1\eta_2 &= \xi_2\eta_1 \\ \xi_2^2 &= \xi_1^2\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2) \\ \xi_2\eta_2 &= \xi_1\eta_1\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2) \\ \eta_2^2 &= \eta_1^2\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2) \end{aligned} \tag{2}$$

where $\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2)$ is obtained from $\varphi_2(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_2, \zeta_3)$ by eliminating ζ_2 and ζ_3.

Proposition 9 *The nonnormal singularity with Eq. (2) has only deformations with simultaneous normalisation.*

Proof We write down the four relations between the generators:

$$\begin{aligned} (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\xi_1 - (\xi_2^2 - \xi_1^2\varphi_2)\eta_1 - (\xi_1\eta_2 - \xi_2\eta_1)\xi_2 &= 0 \\ (\eta_2^2 - \eta_1^2\varphi_2)\xi_1 - (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\eta_1 - (\xi_1\eta_2 - \xi_2\eta_1)\eta_2 &= 0 \\ (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\xi_2 - (\xi_2^2 - \xi_1^2\varphi_2)\eta_2 - (\xi_1\eta_2 - \xi_2\eta_1)\xi_1\varphi_2 &= 0 \\ (\eta_2^2 - \eta_1^2\varphi_2)\xi_2 - (\xi_2\eta_2 - \xi_1\eta_1\varphi_2)\eta_2 - (\xi_1\eta_2 - \xi_2\eta_1)\eta_1\varphi_2 &= 0 \end{aligned}$$

All perturbations of the equations lie in the maximal ideal. By using coordinate transformations, we can assume that the first equation $\xi_1\eta_2 - \xi_2\eta_1$ is not perturbed at all. We perturb the other equations as $\xi_2^2 - \xi_1^2\varphi_2 - \varepsilon_{\xi\xi}$, $\xi_2\eta_2 - \xi_1\eta_1\varphi_2 - \varepsilon_{\xi\eta}$ and

$\eta_2^2 - \eta_1^2\varphi_2 - \varepsilon_{\eta\eta}$. We get four equations holding in $\mathcal{O}$, which can be written as

$$\operatorname{rk}\begin{pmatrix} \xi_2 & \xi_1 & \varepsilon_{\xi\xi} & \varepsilon_{\xi\eta} \\ \eta_2 & \eta_1 & \varepsilon_{\xi\eta} & \varepsilon_{\eta\eta} \end{pmatrix} \leq 1 \; .$$

The first minor is the equation $\xi_1\eta_2 - \xi_2\eta_1$, and the last minor vanishes identically, as we are considering infinitesimal deformations. Thus the perturbations can be written as $\varepsilon_{\xi\xi} = \xi_1^2\varepsilon_{11} + \xi_1\xi_2\varepsilon_{12}$ (without ξ_2^2-term, as $\xi_2^2 = \xi_1^2\varphi_2$), $\varepsilon_{\xi\eta} = \xi_1\eta_1\varepsilon_{11} + \xi_1\eta_2\varepsilon_{12}$ and $\varepsilon_{\eta\eta} = \eta_1^2\varepsilon_{11} + \eta_1\eta_2\varepsilon_{12}$ with ε_{11} and ε_{12} the same functions of the variables in all three perturbations. We can arrange that ε_{11} only depends on ξ_1 and η_1, and not on ξ_2 and η_2, by collecting terms in ε_{12}. The coordinate transformation $\xi_2 \mapsto \xi_2 + \frac{1}{2}\xi_1\varepsilon_{12}$, $\eta_2 \mapsto \eta_2 + \frac{1}{2}\eta_1\varepsilon_{12}$ gets rid of the terms with ε_{12}. The resulting equations

$$\begin{aligned} \xi_1\eta_2 &= \xi_2\eta_1 \\ \xi_2^2 &= \xi_1^2(\varphi_2 + \varepsilon_{11}) \\ \xi_2\eta_2 &= \xi_1(\varphi_2 + \varepsilon_{11}) \\ \eta_2^2 &= \eta_1^2(\varphi_2 + \varepsilon_{11}) \end{aligned}$$

define not only an infinitesimal deformation but also a genuine deformation. We conclude that the base space of the versal deformation is smooth (even though T^2 is not zero).

The simultaneous normalisation is given by $z^2 = f + \varepsilon_{11}(x - az, y - bz)$. □

To obtain interesting other deformations, we have to assume that $c = 0$. Then the subspace L of $\mathfrak{m}/\mathfrak{m}^2$ is given as kernel of a linear form $l = ax + by$. Assuming $a = 1$ we find z and $y - bx$ as generators of degree 1. As we have not yet specified the form of $f(x, y)$, we can apply a coordinate transformation to achieve that $b = 0$. So we take the linear form $l = x$. Generators of the ring $\mathcal{O}$ are now z, y, $w = zx$, $v = yx$, $x_2 = x^2$ and $x_3 = x^3$. If none of these monomials occur in $f(x, y)$, then the embedding dimension is 6.

The formulas

$$\begin{aligned} x_2 &= x^2 & y &= y & z &= z \\ x_3 &= x^3 & v &= xy & w &= xz \end{aligned} \tag{3}$$

define an injective map $\nu\colon \mathbb{C}^3 \to \mathbb{C}^6$. Let Y be the image, which is an isolated three-dimensional singularity. As it is not even normal, its nine equations cannot directly be given in determinantal format, but this is possible by allowing some redundancy. Consider the maximal minors of the 2×6 matrix

$$\begin{pmatrix} z & y & x_2 & w & v & x_3 \\ w & v & x_3 & zx_2 & yx_2 & x_2^2 \end{pmatrix} \; . \tag{4}$$

There are three equations occurring twice, like $vw - zyx_2$, while the last three are obtained by multiplying the first three by x_2. Therefore we get nine generators of the ideal. The determinantal format gives relations between the generators, and a computation with SINGULAR [8] shows that there are no other relations. A further computation gives $\dim T^1_Y = 1$. The 1-parameter deformation is given by the maximal minors of

$$\begin{pmatrix} z & y & x_2 & w & v & x_3 \\ w & v & x_3 & z(x_2+s) & y(x_2+s) & x_2(x_2+s) \end{pmatrix} . \tag{5}$$

It comes from deforming the map ν to

$$\begin{aligned} x_2 &= x^2 - s \qquad & y &= y \qquad & z &= z \\ x_3 &= x(x^2-s) \qquad & v &= xy \qquad & w &= xz \end{aligned} \tag{6}$$

The singularity of the general fibre is isomorphic to the one-point union of two 3-spaces in 6-space.

Now we restrict the map ν to a hypersurface $\{z^2 = f(x,y)\} \subset \mathbb{C}^3$, with $z^2 - f \in \nu^*\mathfrak{m}_6$. We assume that $f \in (y^2, yx^2, x^4)$. We get two additional equations by writing $z^2 - f$ and $x(z^2 - f)$ in the coordinates on $\mathbb{C}^6$. We write

$$\begin{aligned} z^2 &= y\alpha + x_2\beta \ , \\ zw &= v\alpha + x_3\beta \ . \end{aligned} \tag{7}$$

The second equation is obtained from the first by *rolling factors* using the matrix (4), i.e. replacing in each monomial one occurrence of an entry of the upper row by the entry of the lower row in the same column. One can roll once more, to give an expression for w^2, but as we have the equation $w^2 = x_2z^2$, the resulting equation is just the z^2-equation multiplied by x_2.

The singularity has a large component with simultaneous normalisation. For this just perturb $z^2 - f$ with elements of $\nu^*\mathfrak{m}_6$. That means that we can write the two additional equations, using rolling factors. This works also for the deformation of $\nu(\mathbb{C}^3)$ given by Eq. (5) and the map (6). But there is also another deformation direction.

Proposition 10 *The singularity X with normalisation of the form $z^2 = f(x,y)$, where $f \in (y^2, yx^2, x^4)$, and local ring with generators (3) has an infinitesimal deformation, not tangent to the component with simultaneous normalisation.*

Proof The existence is suggested by a SINGULAR [8] computation in examples. The result can be checked by hand.

We first give the relations between the equations. We write g_i for the equations of Y, coming from the matrix (4), and h_k for the two additional equations (7). A relation has the form $\sum g_i r_{ij} + h_k s_{kj} = 0$. We can pull it back to $\mathbb{C}^3$ with the map ν. It then reduces to $(z^2-f)(s_{1j} + xs_{2j}) = 0$. Therefore we find six relations, generating all relations with non-zero s_{kj}. They are found by reading the matrix product

$$\begin{pmatrix} w & -z \\ v & -y \\ x_3 & -x_2 \\ zx_2 & -w \\ yx_2 & -v \\ x_2^2 & -x_3 \end{pmatrix} \begin{pmatrix} z & y & x_2 \\ w & v & x_3 \end{pmatrix} \begin{pmatrix} z \\ -\alpha \\ -\beta \end{pmatrix}$$

in two different ways: the product of the last two matrices is a column vector containing the two equations h_1, h_2, while the product of the first two is a 6×3 matrix, with antisymmetric upper half containing the minors of the middle matrix [the first half of the matrix (4)] and symmetric lower half, containing the remaining six generators. The other relations, with $s_{kj} = 0$, are the determinantal relations between the equations g_i of the three-dimensional singularity Y.

To find a solution to $\sum g_i' r_{ij} + h_k' s_{kj} = 0 \in \mathscr{O}_X$, it suffices to compute on the normalisation. We start with the determinantal relations for the g_i. As the second row of the matrix is just the first one multiplied with $x \in \overline{\mathscr{O}}_X$, it suffices to consider only those relations obtained by doubling the first row. Consider the relation

$$0 = \begin{vmatrix} z & y & x_2 \\ z & y & x_2 \\ w & v & x_3 \end{vmatrix} = (yx_3 - vx_2)z - (zx_3 - wx_2)y + (zv - yw)x_2 \ .$$

The perturbation of the three equations involved, obtained from $z \cdot z - \alpha \cdot y - \beta \cdot x_2 = 0 \in \overline{\mathscr{O}}_X$, cannot be extended to the other equations. It is possible to extend after multiplication with $x \in \overline{\mathscr{O}}_X$. Let $\overline{\alpha} \in \mathscr{O}_X$ be the element with $\overline{\alpha} = x\alpha \in \overline{\mathscr{O}}_X$ and likewise $\overline{\beta} = x\beta$. Then $zw = y\overline{\alpha} + x_2\overline{\beta}$. We do not perturb the equations h_1 and h_2 nor the equation $x_3^2 - x_2^3$. We solve for the perturbations of the remaining equations and check that all equations $\sum g_i' r_{ij} = 0 \in \mathscr{O}_X$ described above are satisfied. The result is the following infinitesimal deformation (written as column vector):

$$G^t = g^t + \varepsilon g'^t = \begin{pmatrix} zv - yw \\ zx_3 - wx_2 \\ yx_3 - vx_2 \\ w^2 - x_2 z^2 \\ wv - x_2 zy \\ v^2 - x_2 y^2 \\ wx_3 - zx_2^2 \\ vx_3 - yx_2^2 \\ x_3^2 - x_2^3 \end{pmatrix} + \varepsilon \begin{pmatrix} \overline{\beta} \\ -\overline{\alpha} \\ -w \\ 2z\alpha \\ 2y\alpha + x_2\beta \\ 2zy \\ x_2\alpha \\ zx_2 \\ 0 \end{pmatrix}$$

□

For the extension to higher order, one needs further divisibility properties of α and β. Indeed, if $p_g(\overline{X}) = 0$, then by Corollaries 7 and 8, the deformation of the proposition has to be obstructed.

Example 11 Let the normalisation be a rational double point. Specifically, we take $\overline{X}$ of type A_3, given by $z^2 = y^2 + x^4$. For the nonnormal singularity X, the dimension of T^1 is equal to 8. There is a seven-dimensional component with simultaneous normalisation: six parameters are seen in the equation $z^2-y^2-x_2^2+a_1z+a_2y+a_3x_2+a_4w + a_5v + a_6x_3$, and s is a parameter for the deformation (6) of the map $\nu\colon \mathbb{C}^3 \to \mathbb{C}^6$. Finally let t be the coordinate for infinitesimal deformation of Proposition 10. A computation of the versal deformation with SINGULAR [8, 18] shows that the equations for the base space are $st = a_1t = a_2t = a_3t = a_4t^2 = a_5t^2 = a_6t^2 = t^3 = 0$.

For $\overline{X}$ given by $z^2 = f(x, y)$ with $f \in \mathfrak{m}^k$, the structure of the versal deformation stabilises for large k. Computations in examples with SINGULAR [8] suggest that T^2 always has dimension 16 (this is also true for the singularity of Example 11) and that there are in general 11 equations for the base space. There is one component of codimension 1 with simultaneous normalisation and two other components: a singularity $z^2 = f(x, y)$ with $f \in \mathfrak{m}^k$, $k \geq 6$ can be deformed into $\widetilde{E}_7$ or $\widetilde{E}_8$.

We compute the component related to $\widetilde{E}_7$. This can be done by determining the versal deformation in negative degrees of the singularity $z^2 = -ay^4 + bx^5$, where a and b are parameters, using [18]. After a coordinate transformation, the equations of the base space do not depend on the parameters. We find the component. For other singularities we have just to substitute suitable functions of space and deformation variables for the parameters in the formulas we find.

The result is rather complicated, so we do not give all equations, but use

$$G_3 = x_3y - x_2v + tw$$

to eliminate the variable w. Four of the original equations do not involve w. They are

$$\begin{aligned}
G_6 &= v(v + a_1t^2) - x_2y^2 - 2tzy - bt^2x_3 + a_2t^2y^2 + a_0t^2(x_2 + a_2t^2) - a_3bt^4y\ , \\
G_8 &= x_3v - x_2^2y - tzx_2 - 2a_4t^2y(y^2 + a_0t^2) \\
&\qquad + ba_4t^4(v + a_1t^2) - (a_3v + a_2x_2)t^2y - a_2t^3z - a_3bt^4(x_2 + a_2t^2)\ , \\
G_9 &= x_3^2 - x_2(x_2 + a_2t^2)^2 - a_3t^2(x_2 + a_2t^2)(v + a_1t^2) \\
&\qquad + a_4t^2(v + a_1t^2)^2 - 4a_4t^2x_2y^2 - a_3^2t^4y^2\ , \\
H_1 &= z^2 + a_4(y^2 + a_0t^2)^2 - bx_3x_2 \\
&\qquad + (a_3v + a_2x_2)(y^2 + a_0t^2) + a_1x_2v + a_0x_2^2 + a_3bt^3z - a_4b^2t^4x_2\ .
\end{aligned}$$

The ideal with w eliminated has three more generators, which we give the name of the original generators leading to them:

$$\begin{aligned} G_1 &= x_3(y^2 + a_0t^2) - x_2yv + zt(v + a_1t^2) \\ &\quad - bt^2x_2(x_2 + a_2t^2) + a_3t^2y(y^2 + a_0t^2) + a_1t^2x_2y \ , \\ G_2 &= x_3x_2y - x_2v(x_2 + a_2t^2) + tzx_3 - a_4t^2(v + a_1t^2)(y^2 + a_0t^2) \\ &\quad + 2a_4bt^4x_2y + a_3a_2t^4(y^2 + a_0t^2) - a_3t^3zy + a_3a_0t^4x_2 \ , \\ H_2 &= x_2z(v + a_1t^2) - x_3yz + a_4t(v + a_1t^2)y(y^2 + a_0t^2) \\ &\quad + a_3tx_2y^3 + a_2tx_2vy + ta_1x_2^2y + ta_0x_3x_2 \\ &\quad - btx_2^2(x_2 + a_2t^2) + a_3t^2zy^2 - 2a_4bt^3x_2y^2 - a_3a_2t^3y(y^2 + a_0t^2) \ . \end{aligned}$$

To obtain the full ideal, one has to add the equation used to eliminate w and saturate with respect to the variable t.

Example 12 We use our equations to write down the deformation in the case that $z^2 - f$ is a surface singularity of type $\widetilde{E}_7$. We start from $z^2 = y^4 - \nu x^2y^2 + x^4$. There is a second modulus, coming from changing the generator y to $y + \lambda x$. By a coordinate transformation, we can keep y as generator and take

$$z^2 = y^4 - \mu xy^3 - \nu x^2y^2 + x^4 \tag{8}$$

as normalisation. Fixing these moduli the component is one-dimensional. We describe its total space. Its equations are obtained by putting $b = a_1 = 0$, $a_0 = a_4 = -1$, $a_3 = \mu$ and $a_2 = \nu$ in the formulas above. The equation H_1 becomes

$$H_1 = z^2 - (y^2 - t^2)^2 - x_2^2 + (\mu v + \nu x_2)(y^2 - t^2) \ .$$

It is reducible if $y^2 = t^2$. If $y^2 \neq t^2$, the equation G_1 shows that x_3 also can be eliminated. There is one more equation not involving x_3:

$$G_6 = v^2 - x_2(y^2 + t^2) - 2tzy + \nu t^2(y^2 - t^2) \ .$$

The local ring of the total space is a section ring $\oplus H^0(V, nL)$ for some ample line bundle on a projective surface V. The dimension of $H^0(V, L)$ is two. We look at the normalisation of a general hyperplane section $t = \lambda y$. We assume that $\lambda^2 \neq 1$; Equation G_6 shows that v/y is in the normalisation. We set it equal to $(1 - \lambda^2)x$.

If $\lambda^2 + 1 \neq 0$, we can eliminate x_2 and find (after dividing by $(1 - \lambda^2)^2$) that that the normalisation is given by

$$(z + 2\lambda t^2 - \nu\lambda y^2)^2 = (\lambda^2 + 1)^2(y^4 - \mu xy^3 - \nu x^2y^2 + x^4) \ ,$$

which for all λ (with $\lambda^2 + 1 \neq 0$) is isomorphic to (8). The sections with $\lambda^2 = 1$ are reducible. One sees that V is a ruled surface over the elliptic curve with Eq. (8), with two sections of self-intersection zero, E_1 and E_{-1}, and L is given by the linear system $|E_1 + f|$, where f is a fibre. The general element of the linear system is a section of the ruled surface with self-intersection 2.

Remark 13 Each nonnormal singularity $X \subset \mathbb{C}^N$ is the image of its normalisation $\overline{X}$, giving rise to a map $\overline{X} \to \mathbb{C}^N$. Not every deformation of this map gives rise to a flat deformation of X. To give an example for X as above with normalisation $z^2 = f(x, y)$, we observe that we can deform the map by using the same map $\mathbb{C}^3 \to \mathbb{C}^6$ and perturbing the equation arbitrarily, say $z^2 = f(x, y) + u$. For flatness of the images, one needs to perturb both equations $z^2 = y\alpha + x_2\beta$, $zw = v\alpha + x_3\beta$ with elements in the local ring of the nonnormal singularity, where the second is obtained from the first by multiplying with x (on the normalisation). The perturbation $z^2 = f(x, y) + u$ is not of this type.

4 Deformations of a Resolution

A deformation of a resolution of a normal surface singularity blows down to a deformation of the singularity if and only if $h^1(\mathcal{O}_{\widetilde{X}})$ is constant [23, 29]. If not, the total space of a 1-parameter deformation still blows down to a three-dimensional singularity, but the special fibre is no longer normal.

Let more generally $\pi_0 \colon (Y, E) \to (X, 0)$ be the contraction of an exceptional set E to a point, with (Y, E) not necessarily smooth, of dimension n. In principle Y is a germ along E, but we work always with a strictly pseudo-convex representative, which we denote with the same symbol Y. Then $\mathcal{O}_X = (\pi_0)_*\mathcal{O}_Y$. In particular, X is normal if Y is normal. Consider now a deformation $\tilde{f} \colon \mathscr{Y} \to S$ of Y over a reduced base space $(S, 0)$. One can assume that $\tilde{f}$ has a 1-convex representative. All the exceptional sets in all fibres can be contracted: let $\pi \colon \mathscr{Y} \to \mathscr{X}$ be the Remmert reduction, so $\mathcal{O}_{\mathscr{X}} = \pi_*\mathcal{O}_{\mathscr{Y}}$ with $\tilde{f} = f \circ \pi$. Then $f \colon \mathscr{X} \to S$ is a deformation of $\mathscr{X}_0 := f^{-1}(0)$. The question is whether f also is a deformation of X, i.e. whether $X \cong \mathscr{X}_0$. The answer is the following [23, Satz 3], cf. [29] for the algebraic case.

Theorem 14 *Let* $\mathscr{Y} \xrightarrow{\pi} \mathscr{X} \xrightarrow{f} S$ *be the Remmert reduction of the deformation* $\tilde{f} \colon \mathscr{Y} \to S$ *of* Y*, over a reduced base space* $(S, 0)$*. Then the special fibre* $\mathscr{X}_0$ *of* $f \colon \mathscr{X} \to S$ *is the Remmert reduction of* $Y = \mathscr{Y}_0$ *if and only if the restriction map* $H^0(Y, \mathcal{O}_{\mathscr{Y}}) \to H^0(Y, \mathcal{O}_Y)$ *is surjective. This is the case if* $\dim H^1(\mathscr{Y}_s, \mathcal{O}_{\mathscr{Y}_s})$ *is constant on* $(S, 0)$*.*

The converse of the last clause holds if $\dim H^2(\mathscr{Y}_s, \mathcal{O}_{\mathscr{Y}_s})$ is constant [23, Satz 5]. This is automatically satisfied in the case of interest to us, when $Y \to X$ is a modification of a normal surface singularity. The result then says that a deformation of the modification blows down to a deformation of the singularity if and only if p_g is constant.

In the proof one reduces to the case of a 1-parameter deformation. Let us consider what happens in that case, so we have a diagram

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\pi} & \mathscr{X} \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

Let t be the coordinate on T and consider multiplication with t on $\mathscr{O}_{\mathscr{Y}}$ and $\mathscr{O}_{\mathscr{X}}$. We get the commutative diagram

$$\begin{array}{ccccccccc} 0 \to H^0(\mathscr{O}_{\mathscr{Y}}) & \xrightarrow{\cdot t} & H^0(\mathscr{O}_{\mathscr{Y}}) & \longrightarrow & H^0(\mathscr{O}_Y) & \longrightarrow & H^1(\mathscr{O}_{\mathscr{Y}}) & \xrightarrow{\cdot t} & H^1(\mathscr{O}_{\mathscr{Y}}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & & & \\ 0 \to \mathscr{O}_{\mathscr{X}} & \xrightarrow{\cdot t} & \mathscr{O}_{\mathscr{X}} & \longrightarrow & \mathscr{O}_{\mathscr{X}_0} & \longrightarrow & 0 & & \end{array}$$

It shows that $\mathscr{O}_{\mathscr{X}_0}$ is equal to $H^0(\mathscr{O}_Y) = \mathscr{O}_X$ if and only if the restriction map $H^0(\mathscr{O}_{\mathscr{Y}}) \to H^0(\mathscr{O}_{\mathscr{Y}})$ is surjective. If $\dim H^1(\mathscr{Y}_t, \mathscr{O}_{\mathscr{Y}_t})$ is constant, then $H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}})$ is a free $\mathscr{O}_{\mathscr{Y}}$-module, on which multiplication with t is injective, so the restriction map $H^0(\mathscr{O}_{\mathscr{Y}}) \to H^0(\mathscr{O}_Y)$ is surjective.

To study the converse, and what happens if $\dim H^1(\mathscr{Y}_t, \mathscr{O}_{\mathscr{Y}_t})$ is not constant, we restrict to the case that $H^2(\mathscr{O}_{\mathscr{Y}}) = 0$.

Proposition 15 *Let $\mathscr{Y} \xrightarrow{\pi} \mathscr{X} \xrightarrow{f} T$ be the Remmert reduction of the 1-parameter deformation $\tilde{f}\colon \mathscr{Y} \to T$ of the space Y, with $H^2(\mathscr{O}_{\mathscr{Y}}) = 0$. Then*

$$\dim H^1(\mathscr{Y}_0, \mathscr{O}_{\mathscr{Y}_0}) = \dim H^1(\mathscr{Y}_t, \mathscr{O}_{\mathscr{Y}_t}) - \dim\left((\pi_0)_*\mathscr{O}_Y/\mathscr{O}_{\mathscr{X}_0}\right),$$

where $t \neq 0$.

Proof The upper line in the commutative diagram extends as

$$0 \to H^0(\mathscr{O}_{\mathscr{Y}}) \xrightarrow{\cdot t} H^0(\mathscr{O}_{\mathscr{Y}}) \longrightarrow H^0(\mathscr{O}_Y) \longrightarrow$$
$$\longrightarrow H^1(\mathscr{O}_{\mathscr{Y}}) \xrightarrow{\cdot t} H^1(\mathscr{O}_{\mathscr{Y}}) \longrightarrow H^1(\mathscr{O}_Y)\,.$$

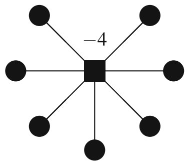
The generic rank of the $\mathscr{O}_T$-module $H^1(\mathscr{O}_{\mathscr{Y}})$ is equal to $\dim H^1(\mathscr{Y}_t, \mathscr{O}_{\mathscr{Y}_t})$ but also equal to $\dim \operatorname{Coker}(\cdot t) - \dim \operatorname{Ker}(\cdot t)$. This proves the formula. □

In particular, if Y is a resolution of the normal surface singularity X, then the formula of the proposition says that $p_g(\mathscr{X}_t) = p_g(X) - \delta = p_g(\mathscr{X}_0)$. Here we use essentially that we have a 1-parameter deformation: over a higher dimensional base space, δ will be larger than $p_g(X) - p_g(\mathscr{X}_t)$, as for a 1-parameter curve in the base, $p_g(X) - p_g(\mathscr{X}_t)$ gives how many functions fail to extend, but it will depend on the curve which ones do not extend.

Remark 16 If we have a smoothing, or more generally if $p_g(\mathscr{X}_t) = 0$, then $H^1(\mathscr{O}_{\mathscr{Y}})$ is t-torsion and isomorphic to $H^1(\mathscr{O}_Y)$.

The above proposition gives a way to construct normal surface singularities with the same resolution graph as a given hypersurface singularity but with lower p_g. If the drop in p_g is equal to δ, we start from a nonnormal model of the hypersurface singularity with δ-invariant δ and compute deformations without simultaneous resolution. If the singularity is quasi-homogeneous, deformations of positive weight will have constant topological type of the resolution.

Example 17 Consider the hypersurface singularity $z^2 = x^7 + y^7$ with resolution graph:



This is a singularity with $p_g = 3$, but its arithmetic genus is equal to two. The general, non-Gorenstein singularity with the same graph has indeed $p_g = 2$. We can use the computations of the previous section. We get a weighted homogeneous deformation by putting $b = x_2$, $a_0 = a_1 = a_2 = a_3 = 0$ and $a_4 = -y^3$. The equation $G_3 = x_3y - x_2v + tw$ shows that the w-deformation has positive weight $-(8-9) = 1$.

We give the equations for the fibre at $t = 1$. Then the equation H_2 lies in the ideal of the other ones, and we obtain the following six equations:

$$
\begin{aligned}
G_6 &= v^2 - x_2y^2 - 2zy - x_2x_3 \ , \\
G_8 &= x_3v - x_2^2y - zx_2 + 2y^6 - x_2y^3v \ , \\
G_9 &= x_3^2 - x_2^3 - v^2y^3 + 4x_2y^5 \ , \\
H_1 &= z^2 - y^7 - x_2^2x_3 + y^3x_2^3 \ . \\
G_1 &= x_3y^2 - x_2yv + zv - x_2^3 \ , \\
G_2 &= x_3x_2y - x_2^2v + zx_3 + vy^5 - 2x_2^2y^4 \ .
\end{aligned}
$$

One checks that this ideal indeed defines a singularity with the above resolution graph by resolving it; one possible method is to blow up a canonical ideal.

5 $\mathbb{P}^1$ as Exceptional Set

In understanding normal surface singularities, the resolution is a very important tool. For threefold singularities this is not the case for several reasons. First of all, there is no unique minimal resolution. The combinatorics of a good resolution (i.e.

the exceptional divisor has normal crossings) seems prohibitive in general. But now there is a new phenomenon that there may exist resolutions in which the exceptional set is not a divisor but an analytic set of lower dimension. This is called a small resolution. It means that in a certain sense the singularity is not too singular. For threefold singularities, we are talking about resolutions with as exceptional set a curve.

If the exceptional curve C is rational, then its normal bundle splits as $\mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a \geq b$. Rather surprisingly, the number a can be positive. Laufer gave in [16] an example of a curve $C \subset \widetilde{X}$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$, which even contracts to a hypersurface singularity X.

Generalising earlier results of Ando (see [2]) and Nakayama[20], that contractability limits the value of (a, b) to $2a + b < 0$, Ando proved [3]:

Theorem 18 *Let C be a smooth exceptional curve in an m-dimensional manifold $\widetilde{X}$, and let M be a subbundle of the normal bundle $N_{C/\widetilde{X}}$ of maximal degree a and put $b = \deg N_{C/\widetilde{X}} - a$. Then $2a + b < 0$ and $a + b < 0$. Moreover, if C is rational, then $a + b \leq 1 - m$.*

Ando [1, 3] has also existence results. In particular, in dimension 3 he exhibits examples with the maximal normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(-2n-1)$, by giving, in the style of Laufer, transition functions between two copies of $\mathbb{C}^3$. The resulting singularity is not Cohen-Macaulay for $n > 1$. Consider more generally a rational curve with normal bundle ot type (a, b) with $a > 1$. To see that $H^2_{\{0\}}(X, \mathcal{O}_X) \cong H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \neq 0$, let $\mathcal{J}$ be the ideal sheaf of C in $\widetilde{X}$ and look at the exact sequences:

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_C \longrightarrow 0\,,$$
$$0 \longrightarrow \mathcal{J}^2 \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow 0\,.$$

We have a surjection $H^0(\mathcal{O}_{\widetilde{X}}) \twoheadrightarrow H^0(\mathcal{O}_C) = \mathbb{C}$. As C is rational, $H^1(\mathcal{O}_C) = 0$ and therefore $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^1(\widetilde{X}, \mathcal{J})$. Because C is a curve, $H^2(\widetilde{X}, \mathcal{J}^2) = 0$ and we get a surjection $H^1(\widetilde{X}, \mathcal{J}) \twoheadrightarrow H^1(C, \mathcal{J}/\mathcal{J}^2)$. As $\mathcal{J}/\mathcal{J}^2$ is the dual of the normal bundle, we have that $h^1(C, \mathcal{J}/\mathcal{J}^2) = a - 1$ and therefore $h^1(\mathcal{O}_{\widetilde{X}}) \geq a - 1$.

Pinkham gave a construction for C with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ [21], using smoothings of partial resolutions of rational double points. The easiest example, starting from a D_4-singularity, is described in detail in [26].

Here we generalise Pinkham's construction to exceptional curves with other normal bundles. Let $\overline{H}$ be a normal surface singularity (in the end H will be a general hyperplane section of a threefold singularity), and let $\widehat{H}$ be a partial resolution of $\overline{H}$ with irreducible exceptional locus C, such that the only singularities of $\widehat{H}$ are hypersurface singularities.

The deformation space of $\widehat{H}$ is smooth. Indeed, the sheaf $\mathcal{T}^1_{\widehat{H}}$ is concentrated in the singular points, and $\mathcal{T}^2_{\widehat{H}} = 0$, as there are only hypersurface singularities. The local-to-global spectral sequence for $T^{\bullet}_{\widehat{H}}$ gives that $T^2_{\widehat{H}} = 0$, so all deformations are unobstructed, and moreover we get the exact sequence:

$$0 \longrightarrow H^1(\mathscr{T}^1_{\widehat{H}}) \longrightarrow T^1_{\widehat{H}} \longrightarrow \bigoplus_p T^1_{\widehat{H},p} \longrightarrow 0\,.$$

Therefore all singular points $p \in \widehat{H}$ can be smoothed independently.

We take $\widetilde{X}$ to be a 1-parameter smoothing of $\widehat{H}$ with smooth total space (this is possible, as all singularities are hypersurfaces). Moreover, we can arrange that the general fibre does not contain exceptional curves. Then the contraction $\pi\colon \widetilde{X} \to X$ with the curve C as exceptional locus gives an isolated threefold singularity. In general its hyperplane section H is a nonnormal surface singularity, with $\delta(H) = p_g(\overline{H})$, by Proposition 15.

Example 19 Let $\overline{H}$ be a singularity, whose resolution has a central rational curve of self-intersection $-n-1$, intersected by $2n+1$ (-2)-curves. A quasi-homogeneous singularity with this resolution is the hypersurface singularity Y_{2n+1} with equation $z^2 = f_{2n+1}(x,y)$, where f_{2n+1} is a square-free binary form. The partial resolution $\widehat{H}$ to be considered is obtained by blowing down the (-2)-curves, intersecting the central curve. For $n > 1$, this is the canonical model, while for $n = 1$, we have D_4. The next theorem shows that the construction yields a three-dimensional manifold $\widetilde{X}$ with an exceptional rational curve, whose normal bundle is $\mathscr{O}(n) \oplus \mathscr{O}(-2n-1)$.

Rational double points are absolutely isolated, i.e. they can be resolved by blowing up points. Each sequence of blowing ups gives a partial resolution. We define the *resolution depth* of an exceptional component E_i as the minimal number of blow ups required to obtain a partial resolution on which the curve E_i appears. This is the desingularisation depth of Lê and Tosun [17], shifted by one. It is easily computed from the resolution graph. The fact that C is smooth restricts the possible curves E_i in a rational double point configuration, which intersect C, to those with multiplicity one in the fundamental cycle of the configuration.

Theorem 20 *Let $\widetilde{X}$ be a 1-parameter smoothing with smooth total space of a partial resolution $\widehat{H}$ of a normal surface singularity $\overline{H}$ with exceptional set a smooth rational curve C and k rational double points as singularities. Let $-c$ be the self-intersection of the curve C on the minimal resolution $\widetilde{H}$ of $\overline{H}$. Suppose that C intersects a curve of resolution depth b_j in the jth rational double point configuration on $\widetilde{H}$. Put $b = b_1 + \cdots + b_k$. Then the normal bundle of the exceptional curve $C \subset \widehat{H}$ in $\widetilde{X}$ is $\mathscr{O}(b-c) \oplus \mathscr{O}(-b)$.*

Proof Let $\sigma\colon \widetilde{Y} \to \widetilde{X}$ be an embedded resolution of $\widehat{H}$. Denote by $\widetilde{H}$ the strict transform of $\widehat{H}$ and by $\widetilde{C}$ the strict transform of the curve C (which is isomorphic to C). As we are only interested in a neighbourhood of $\widetilde{C}$, it actually suffices to blow up the threefold $\widetilde{X}$ in points lying on C until the strict transform of $\widehat{H}$ is smooth along the strict transform of C. The number of blow ups needed is b.

Let $P_j \in C$ be the jth singular point of $\widehat{H}$; identifying $\widetilde{C}$ with C, it is also the intersection point on $\widetilde{H}$ of $\widetilde{C}$ and the jth rational double point configuration. The normal bundle $N_{\widetilde{C}/\widetilde{Y}}$ is isomorphic to $N_{C/\widetilde{X}} \otimes \mathcal{O}_C(-D)$, where we write D for the divisor $\sum b_j P_j$.

On $\widetilde{Y}$ we have the exact sequence:

$$0 \longrightarrow N_{\widetilde{C}/\widetilde{H}} \longrightarrow N_{\widetilde{C}/\widetilde{Y}} \longrightarrow N_{\widetilde{H}/\widetilde{Y}}|_{\widetilde{C}} \longrightarrow 0\ .$$

Correspondingly there is an exact sequence on $\widetilde{X}$:

$$0 \longrightarrow N' \longrightarrow N_{C/\widetilde{X}} \longrightarrow N'' \longrightarrow 0\ , \tag{9}$$

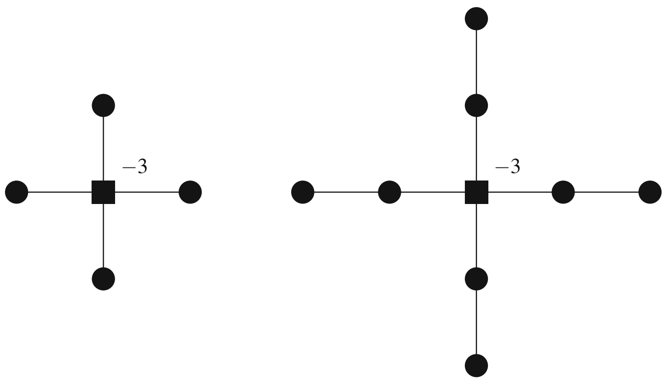
with $N' \cong N_{\widetilde{C}/\widetilde{H}} \otimes \mathcal{O}_C(D)$ a bundle, which outside the singular points coincides with the normal bundle $N_{C/\widetilde{H}}$; note that C is not a Cartier divisor in $\widetilde{H}$ at the singular points.

As $\widetilde{C}$ is a rational curve, $N_{\widetilde{C}/\widetilde{H}} \cong \mathcal{O}_{\widetilde{C}}(-c)$. To compute $N_{\widetilde{H}/\widetilde{Y}}|_{\widetilde{C}}$, we note that the total transform of $\widehat{H}$ is of the form $\widetilde{H} + \sum f_i F_i$, with F_i the exceptional divisors and that it is the divisor $t = 0$ with trivial normal bundle. The only divisors, which intersect $\widetilde{C}$, are the ones coming from the last blow up in the points P_j and occur with multiplicity $2b_j$. Therefore $N_{\widetilde{H}/\widetilde{Y}}|_{\widetilde{C}} \cong \mathcal{O}_{\widetilde{C}}(-\sum 2 f_j F_j) = \mathcal{O}_{\widetilde{C}}(-2b)$. It follows that the exact sequence (9) has the form

$$0 \longrightarrow \mathcal{O}(b-c) \longrightarrow N_{C/\widetilde{X}} \longrightarrow \mathcal{O}(-b) \longrightarrow 0.$$

As $H^0(N_{C/\widetilde{X}}) = H^0(\mathcal{O}(b-c))$, the sequence splits. □

Example 21 As noted by Ando [3], there exist exceptional rational curves with normal bundle $(1, -4)$, which contract to Cohen-Macaulay singularities and others which do not. We obtain this normal bundle starting from a RDP resolution with a central rational curve of self-intersection -3 (on the minimal resolution) with four A_i singularities on it. Consider the following two resolution graphs:



The graph on the left is a rational quadruple point graph, while a normal singularity with the second graph is minimally elliptic and equisingular to $z^3 = x^4 + y^4$. The exceptional $(1, -4)$-curve comes from a nonnormal model with $\delta = 1$.

Example 22 (Example 19 Continued) Ando's examples [1, 3] of the extremal case $(n, -2n-1)$ are of type in the example. With adapted variable names, his exceptional $\mathbb{P}^1$ is covered by two charts having coordinates (x, η, ζ_x) and (ξ, y, ζ_y) with transition functions:

$$x = \xi^{2n+1}y + {\zeta_y}^2 + \xi^{2n}{\zeta_y}^3$$
$$\eta = \xi^{-1}$$
$$\zeta_x = \zeta_y \xi^{-n}$$

So x is a global function, as is $\xi y + {\zeta_y}^3 = \eta^{2n}x - {\zeta_x}^2$. Other functions are more complicated. A general hyperplane section is obtained by setting a linear combination of these two functions to zero. In the first chart, we get ${\zeta_x}^2 = x(a+\eta^{2n})$ and in the second ${\zeta_y}^2 + (1/a + \xi^{2n})(\xi y + {\zeta_y}^3) = 0$. This is indeed the canonical model of a singularity of type $z^2 = f_{2n+1}(x, y)$.

Remark 23 For $n = 2$, we have the singularity $z^2 = f_5$, which is minimally elliptic, and therefore every singularity with the same resolution graph is a double point. For $n > 2$, this is no longer true. For $n = 3$, we gave in Example 17 equations for a singularity with the same resolution graph as $z^2 = f_7$, which is not Gorenstein. As we constructed it as deformation of a nonnormal model, with $\delta = 1$, of a hypersurface singularity, a nonnormal $\delta = 2$ model of this singularity is a deformation of a $\delta = 3$ model of the hypersurface.

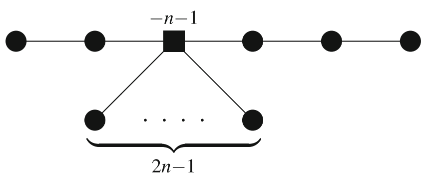
Before giving other new examples with normal bundle $(n, -2n - 1)$, we recall Kollár's length invariant [7, Lecture 16]:

Definition 24 The length l of the small contraction $\pi : (\widetilde{X}, C) \to (X, p)$ with irreducible exceptional curve C is

$$l = \lg \mathcal{O}_{\widetilde{X}}/\pi^*\mathfrak{m}_{X,p} \ .$$

The length is equal to the multiplicity of the maximal ideal cycle of $\overline{H}$ at the strict transform of the exceptional curve C.

Example 25 Consider the graph:



An example of a normal surface singularity with this graph is

$$z^2 = y(y^{4n-2} + x^{6n-3}) \ .$$

These singularities can be thought of as generalisations of E_7, just as those of the type $z^2 = f_{2n+1}(x, y)$ are generalisations of D_4. A smooth total space of a 1-parameter smoothing of the canonical model has as exceptional set a rational curve with normal bundle $(n, -2n-1)$. The invariant l has the value 4.

For $n = 2$, the singularity has $p_g = 5$.

From our computations in Sect. 3 we can draw the following conclusion.

Proposition 26 *The singularity obtained by contracting a rational curve with normal bundle* $(2, -5)$ *has embedding dimension at least 7.*

Explicit equations for the case that the normalisation $\overline{H}$ is given by $z^2 = y^5 + x^5$ can be obtained from the equations G_i, H_j of Sect. 3 by putting $b = 1$, $a_4 = a_4 - y$, $a_3 = a_3 + b_4 y$, $a_2 = a_2 + b_3 y$, $a_1 = a_1 + b_2 y + c_2 y^2$ and $a_0 = a_0$.

The equation

$$\begin{aligned} H_1 = z^2 - (y-a_4)(y^2+a_0t^2)^2 - x_3x_2 + ((a_3+b_4y)v + (a_2+b_3y)x_2)(y^2+a_0t^2) \\ + (a_1+b_2y+c_3y^2)x_2v + a_0x_2^2 + (a_3+b_4y)t^3z + (y-a_4)t^4x_2 \ . \end{aligned}$$

shows that restricted to $t = 0$ one has the simultaneous normalisation:

$$\begin{aligned} &z^2 - y^5 - x^5 \\ &\quad + a_4y^4 + a_3xy^3 + a_2x^2y^2 + a_1x^3y + a_0x^4 + b_4xy^4 + b_3x^2y^3 + b_2x^3y^2 + c_3x^3y^3. \end{aligned}$$

To compute the simultaneous canonical model, we first eliminate w and x_3, assuming $t \neq 0$. To simplify the formulas, we suppress the b_i and c_3. We write the resulting equations in determinantal form:

$$\begin{pmatrix} v(y^2+a_0t^2) - x_2yt^2 - zt^3 - a_3t^6 & v^2 - (y-a_4)t^6 \\ -(y^2+a_0t^2)^2 + x_2t^4 & -v(y^2+a_0t^2) - x_2yt^2 - zt^3 \\ 2y(y^2+a_0t^2) - vt^2 - a_1t^4 & 2vy + x_2t^2 + a_2t^4 \end{pmatrix} . \tag{10}$$

We apply a Tjurina modification followed by normalisation. On the first chart $\mathscr{U}_y$, we have the hypersurface

$$\zeta_y{}^2 - y\xi^5 - y + a_0\xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 - \tfrac{1}{4}t^2\xi^6$$

and the map to the singularity H is given by quite complicated formulas:

$$\begin{aligned} x_2 &= y^2\xi^2 - 2yt(\zeta_y - t\xi^3) + -t^3\xi(\zeta_y - \tfrac{1}{2}t\xi^3) - a_0t^2\xi^2 - a_1t^2\xi - a_2t^2 \\ v &= (y^2+a_0t^2)\xi - yt^2\xi^2 + t^3(\zeta_y - \tfrac{1}{2}t\xi^3) \end{aligned}$$

$$z = y^2(\zeta_y - \tfrac{5}{2}t\xi^3) + yt^2\xi(3\zeta_y - \tfrac{5}{2}t\xi^3) + t^4\xi^2(\zeta_y - \tfrac{1}{2}t\xi^3)$$
$$- a_0t^2(\zeta_y - \tfrac{3}{2}t\xi^3) + 2ya_0t\xi^2 + ya_1t\xi + a_1t^3\xi^2 + ya_2t + a_2t^3\xi$$

The expressions for w and x_3 are even longer; they can be computed from the equations used to eliminate these variables.

In the second chart $\mathscr{U}_x$, we have the hypersurface

$$\zeta_x{}^2 - x - x\eta^5 + a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 - \tfrac{1}{4}t^2\eta^8$$

and the transition functions $y = x\eta + t\zeta_x + \frac{1}{2}t^2\eta^4$, $\xi = \eta^{-1}$ and $\zeta_y = \zeta_x\eta^{-2} + \frac{1}{2}t\eta^{-3} + \frac{1}{2}t\eta^2$.

We see that for $a_i = 0$ the family of curves given in $\mathscr{U}_y$ by $y = \zeta_y - \frac{1}{2}t\xi^3 = 0$ and in $\mathscr{U}_x$ by $x = \zeta_x + \frac{1}{2}t\eta^4 = 0$ is exceptional. Then the determinant (10) describes for $t \neq 0$ a singularity isomorphic to the cone over the rational normal curve of degree three. The general 1-parameter smoothing of the canonical model is obtained by taking the a_i as functions of t. We can take $a_0 = t$ and $a_i = 0$ for $i > 0$. Then in the second chart $\tau = t(1 - \frac{1}{4}t\eta^8)$ can be eliminated, as $\tau = x(1 + \eta^5) - \zeta_y{}^2$, so (x, η, ζ_y) are coordinates. We can write the equation in the first chart as $\zeta_y{}^2 + \xi^4(t - \frac{1}{4}t\xi^2 - y\xi) = y$, so $(\zeta_y, \xi, \sigma = t - \frac{1}{4}t\xi^2 - y\xi)$ are coordinates. The transition functions are power series.

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On a Theorem of Greuel and Steenbrink

Duco van Straten

To Gert-Martin Greuel on the occasion of his 70th birthday.

Abstract A famous theorem of Greuel and Steenbrink states that the first Betti number of the Milnor fibre of a smoothing of a normal surface singularity vanishes. In this paper we prove a general theorem on the first Betti number of a smoothing that implies an analogous result for weakly normal singularities.

Keywords Singularities • Topology of smoothings • Weakly normal spaces

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1 Introduction

By a *singularity* we usually mean a germ $(X, p) \subset (\mathbb{C}^N, p)$ of a complex space, but in order to study its topology, it is customary to pick an appropriate contractible Stein representative X of the germ in question. By a *deformation* of the singularity (X, p) over $(S, 0)$, we understand a pullback diagram of the form

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

where the map $f : \mathscr{X} \longrightarrow S$ is flat and all spaces are appropriate representatives of the corresponding germs. We say the deformation is a *smoothing* if the general fibre $X_t = f^{-1}(t)$, $t \in S$, is smooth, in which case we say that X_t is the *Milnor fibre* of the smoothing under consideration. In the classical case of isolated hypersurface

D. van Straten (✉)
Institut für Mathematik, Johannes Gutenberg Universität, Staudingerweg 9, 55128 Mainz, Germany
e-mail: straten@mathematik.uni-mainz.de

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singularities, Milnor [4] has shown that this fibre has the homotopy type of a bouquet of spheres of dimension equal to the complex dimension of $(X,0)$. It is of considerable interest to find topological properties of Milnor fibres in more general situations.

In the paper [3] of Greuel and Steenbrink, one finds an overview of some of the basic results and in particular a proof of the following result that was conjectured by Wahl in [10].

Theorem 1 *Let $\mathscr{X} \xrightarrow{f} S$ be a smoothing of an isolated normal singularity*

$$X := f^{-1}(0) \text{ and let } X_t := f^{-1}(t),\ t \neq 0,$$

denote its Milnor fibre. Then:

$$b_1(X_t) := \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = 0\ .$$

Note that for a normal surface singularity, the fundamental group $\pi_1(X_t)$ or even $H_1(X_t)$ of the Milnor fibre need not to be trivial.

When one looks for a similar simple statement for non-isolated singularities, one soon runs into difficult problems. In [11] Zariski described two types of 6-cuspidal sextics in $\mathbb{P}^2$, for which the complements have different fundamental groups. By taking the cone over such a curve, we get a surface in $\mathbb{C}^3$, whose Milnor fibre appears as the cyclic sixfold cover of the complement of this curve. Its first Betti number depends on the position of the cusps: when they are on a conic, then $b_1(X_t) = 2$, when they are not, then $b_1(X_t) = 0$, [2]. This shows that the first Betti number b_1 is a subtle invariant.

In this paper we will show the following general theorem.

Theorem 2 *Let $\mathscr{X} \xrightarrow{f} S$ be a smoothing of a reduced and equidimensional germ (X,p). Let $X_t = f^{-1}(t)$andt $\neq 0$, its Milnor fibre. Let $X^{[0]} = \sqcup_{i=1}^{r} X_i$, where the X_i are the irreducible components of X. Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X}, p)$ be the map that associated with a divisor supported on X its class in the local class group. Then one has:*

1. $b_1(X_t) \geq \operatorname{rank}(\ker\gamma) - 1$.
2. *When X is weakly normal, then one has equality:*

$$b_1(X_t) = \operatorname{rank}(\ker\gamma) - 1.$$

In this case the action of the monodromy in $H^1(X_t)$ is trivial.

We spell out two useful corollaries of this general result:

Corollary 1 *If (X,p) is a hypersurface singularity with r irreducible components, then $b_1(X_t) \geq r-1$, with equality in the case that (X,p) is weakly normal.*

Corollary 2 *Let* $\mathscr{X} \xrightarrow{f} S$ *be a smoothing of a reduced, equidimensional and weakly normal space germ* $X = f^{-1}(0)$ *and let* $X_t := f^{-1}(t)$ *and* $t \neq 0$ *denote its Milnor fibre. Then*

$$b_1(X_t) \leq r - 1.$$

where r *denotes the number of irreducible components of* X*. For a hypersurface equality holds.*

Recall that a complex space germ X is called *weakly normal*, if every function that is continuous and holomorphic outside the singular set of X is in fact holomorphic on all of X. The union of coordinate axis in $\mathbb{C}^n$ is the unique weakly normal curve singularity with multiplicity n and a weakly normal surface has such a curve singularity as generic transversal type. If in addition X is Cohen-Macaulay, then also the converse holds. In particular, the cone over a plane curve $\Gamma \subset \mathbb{P}^2$ is weakly normal precisely when Γ has only ordinary double points. In this case the first Betti number is independent of the exact position of the double points: one has $b_1(X_t) = r - 1$, where r denotes the number of irreducible components of Γ.

Our proof of **Theorem 2** is given in the following sections and runs along the lines of the paper [3].

2 Embedded Resolution

Let X be a fixed contractible Stein representative of a *reduced and equidimensional* germ (X, p). We consider a smoothing over a smooth curve germ S:

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

The following fact is well-known:

Lemma 1 *The total space of the smoothing* $\mathscr{X}$ *of a reduced and equidimensional germ is* normal.

Proof As we are dealing with a smoothing over a smooth curve germ, the singular locus $Sing(\mathscr{X})$ of $\mathscr{X}$ is a subset of the singular locus $\Sigma := Sing(X)$ of X, so this set is of codimension ≥ 2 in $\mathscr{X}$. Furthermore, as X is reduced, we have $\text{depth}_\Sigma(X) \geq 1$, so that from the flatness of the family we obtain $\text{depth}_\Sigma(\mathscr{X}) \geq 2$; hence, $\mathscr{X}$ is normal. ◇

To study the Milnor fibre $X_t := f^{-1}(t), t \neq 0$ of a smoothing, we will make use of an *embedded resolution* of X in $\mathscr{X}$. By Hironaka's theorem there exists an appropriate sequence of blow-ups that produces a space $\mathscr{Y}$ together with a proper map:

$$\pi : \mathscr{Y} \longrightarrow \mathscr{X}$$

which has the following properties:

- $\mathscr{Y}$ is smooth.
- $Y := (f \circ \pi)^{-1}(0)$ is a normal crossing divisor.
- $\pi : \mathscr{Y} \setminus \pi^{-1}(\Sigma) \longrightarrow X \setminus \Sigma$ is an isomorphism.

By the semi-stable reduction theorem, we may and will assume, after first performing an appropriate finite base change on S, that the divisor Y in addition is *reduced*.

The divisor Y can be decomposed into two parts:

(a) $\widetilde{X}$, the strict transform of X.
(b) $F = \cup_i F_i$, a noncompact divisor, mapping properly onto Σ.

The component of F can be grouped further according to the stratum of Σ they map to, but in this paper will not need to do so.

2.1 Leray Sequence

Via the map π, the fibre $Y_t := (f \circ \pi)^{-1}(t) \subset \mathscr{Y}$ is isomorphic to the Milnor fibre X_t:

$$\pi : Y_t \xrightarrow{\simeq} X_t.$$

In a semi-stable family, the divisor Y is reduced, so that the space Y_t "passes along every component of Y just once". More precisely, one can find a contraction map

$$c : Y_t \longrightarrow Y$$

of the Milnor fibre Y_t onto the special fibre Y, which is an isomorphism on the preimage of the subset of regular points of Y; see, for example, [1]. One can try to compute the cohomology of Y_t using the Leray spectral sequence for the map c. It is easy to verify from the local model of the maps f and c that

$$c_*(\mathbb{Z}_{Y_t}) = \mathbb{Z}_Y,$$
$$R^1 c_*(\mathbb{Z}_{Y_t}) = \mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y.$$

Here $Y^{[0]} := \sqcup_i Y_i$ denotes the disjoint union of the irreducible components of Y, which naturally maps to Y. Via this map we consider the constant sheaf $\mathbb{Z}_{Y^{[0]}}$ as a sheaf on Y. Indeed, the fibre of c over a point in Y is homotopy equivalent to a real torus of dimension equal $k-1$, where k is the number of irreducible components of Y passing through the point. From this description one obtains

$$(\mathbb{Z} \simeq) H^0(Y) \xrightarrow{\simeq} H^0(Y_t)$$

and the beginning of an exact sequence of cohomology groups (always with $\mathbb{Z}$-coefficients, unless stated otherwise):

Leray sequence:

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(Y_t) \longrightarrow H^0(\mathbb{Z}_{Y[0]}/\mathbb{Z}_Y) \longrightarrow H^2(Y) \longrightarrow \cdots \tag{1}$$

Note that we also have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_Y \longrightarrow \mathbb{Z}_{Y[0]} \longrightarrow \mathbb{Z}_{Y[0]}/\mathbb{Z}_Y \longrightarrow 0$$

of sheaves on Y. From the associated long exact sequence of cohomology groups, we obtain the beginning of an exact sequence:

$$0 \longrightarrow H^0(Y) \longrightarrow H^0(Y^{[0]}) \longrightarrow H^0(\mathbb{Z}_{Y[0]}/\mathbb{Z}_Y) \longrightarrow H^1(Y) \longrightarrow \cdots \tag{2}$$

2.2 Two Further Sequences and a Diagram

There are two further exact sequences in which $H^1(Y_t)$ appears:

Milnor-Wang sequence (see [4, p. 67]):

$$0 \longrightarrow H^0(Y_t) \longrightarrow H^1(B) \longrightarrow H^1(Y_t) \xrightarrow{h^*-Id} H^1(Y_t) \longrightarrow \cdots \tag{3}$$

Here $B := \mathscr{X} \setminus X$, the total space of the Milnor fibration over $S \setminus \{0\}$ and h^* denotes the cohomological monodromy transformation.

The cohomology sequence of the pair: $\mathscr{Y} \setminus Y \hookrightarrow \mathscr{Y}$ reads

$$\cdots \longrightarrow H^1(Y) \longrightarrow H^1(\mathscr{Y} \setminus Y) \longrightarrow H^2(\mathscr{Y}, \mathscr{Y} \setminus Y) \longrightarrow H^2(Y) \longrightarrow \cdots$$

We use the isomorphism

$$\mathscr{Y} \setminus Y \xrightarrow{\simeq} \mathscr{X} \setminus X = B.$$

Furthermore, from the homoptopy equivalence $\mathscr{Y} \simeq Y$ and the Lefschetz isomorphism, we obtain

$$H^1(\mathscr{Y}, \mathscr{Y} \setminus Y) = 0, \quad H^2(\mathscr{Y}, \mathscr{Y} \setminus Y) \simeq H^0(Y^{[0]}),$$

so the sequence of the pair is seen to reduce to the exact sequence:

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(B) \xrightarrow{\alpha} H^0(Y^{[0]}) \xrightarrow{\beta} H^2(Y) \longrightarrow \cdots \tag{4}$$

We will describe the map β in detail in Sect. 4.

These four sequences fit into a single commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & H^0(Y_t) & \simeq & H^0(Y) & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(B) & \xrightarrow{\alpha} & H^0(Y^{[0]}) & \xrightarrow{\beta} & H^2(Y) & \longrightarrow & \cdots \\
 & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(Y_t) & \longrightarrow & H^0(\mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y) & \longrightarrow & H^2(Y) & \longrightarrow & \cdots \\
 & & & & \downarrow h^* - Id & & \downarrow & & & & \\
 & & & & H^1(Y_t) & & H^1(Y) & & & &
\end{array}$$

Lemma 2 *In the above situation, we have:*

1. rank $H^1(B) =$ rank $H^1(Y) +$ rank $(\ker \beta)$.
2. rank $H^1(Y_t) \geq$ rank $H^1(B) - 1$.
3. If $H^1(Y) = 0$, *then the monodromy acts trivially on* $H^1(Y_t)$.
4. If $H^1(Y) = 0$, *then* rank $H^1(Y_t) =$ rank $H^1(B) - 1$.

Proof First recall that we have rank $H^0(Y) = 1 =$ rank $H^0(Y_t)$. From the first exact row of the diagram, coming sequence (4), we read off the first statement. The first column of the diagram, coming from sequence (3), gives the second statement. If $H^1(Y)$ is assumed to be zero, the diagram simplifies, and a diagram chase learns that the map $H^1(B) \longrightarrow H^1(Y_t)$ is surjective, so that $h_* - Id$ is the zero map on $H^1(Y_t)$, which is the third statement. The last statement then follows by looking again at the first column of the diagram. $\diamond$

We now study the parts $H^1(Y)$ and $\ker \beta$ separately.

3 The Group $H^1(Y)$

If X is a plane curve singularity, then it is easy to determine rank $H^1(Y)$. The result is

$$\text{rank}\, H^1(Y) = 2g + b\ ,$$

where g is the sum of the genera of the compact components of Y and b is the number of cycles in the dual graph of Y. These numbers g and b are in fact invariants of the limit mixed Hodge structure on $H^1(X_t)$; one has $b = \dim_0^W Gr_F^1 H^1(X_t)$ and $g = \dim Gr_1^W Gr_F^1 H^1(X_t)$; see [7]. By taking $X \times \mathbb{C}$, we obtain in a trivial way examples of irreducible surfaces with arbitrary high Betti number. Only in the case that X is an ordinary double point, one has $H^1(Y) = 0$. It turns out that in general it is exactly the *weak normality* of X that forces $H^1(Y)$ to vanish.

Proposition 1 *Let* $\mathscr{X} \xrightarrow{f} S$ *be a flat deformation of a weakly normal space* $X = f^{-1}(p)$. *Let* $\mathscr{Y} \xrightarrow{\pi} \mathscr{X}$ *be a map such that*

1. $\mathscr{Y} \setminus \pi^{-1}(\Sigma) \xrightarrow{\simeq} \mathscr{X} \setminus \Sigma$, $\Sigma := Sing(X)$ *is an isomorphism.*
2. $\pi_* \mathscr{O}_{\mathscr{Y}} \simeq \mathscr{O}_{\mathscr{X}}$.

Then one has

$$R^1 \pi_* \mathscr{O}_{\mathscr{Y}} = 0 \ .$$

Proof The argument is basically the same as in [3]. First look at the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{Y}} \xrightarrow{t\cdot} \mathscr{O}_{\mathscr{Y}} \longrightarrow \mathscr{O}_Y \longrightarrow 0.$$

Here t is a local parameter on S and $t\cdot$ is the map obtained from multiplication by t. The space Y is defined by the equation $t = 0$; it is the fibre over $0 \in S$. When we take the direct image of this sequence under π, we obtain a diagram

$$\begin{array}{ccccccccccc} 0 \longrightarrow & \mathscr{O}_{\mathscr{X}} & \xrightarrow{t\cdot} & \mathscr{O}_{\mathscr{X}} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 & & & \\ & \downarrow & & \downarrow & & \downarrow & & & & & \\ 0 \longrightarrow & \pi_*\mathscr{O}_{\mathscr{Y}} & \xrightarrow{t\cdot} & \pi_*\mathscr{O}_{\mathscr{Y}} & \longrightarrow & \pi_*\mathscr{O}_Y & \longrightarrow & R^1\pi_*\mathscr{O}_{\mathscr{Y}} & \xrightarrow{t\cdot} & R^1\pi_*\mathscr{O}_{\mathscr{Y}} & \cdots \end{array}$$

By assumption, the natural map $\mathscr{O}_{\mathscr{X}} \longrightarrow \pi_* \mathscr{O}_{\mathscr{Y}}$ is an isomorphism. From this it follows that the sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \pi_* \mathscr{O}_Y \longrightarrow R^1 \pi_* \mathscr{O}_{\mathscr{Y}} \xrightarrow{t\cdot} R^1 \pi_* \mathscr{O}_{\mathscr{Y}} \longrightarrow \cdots$$

is also exact. We claim that we also have an isomorphism

$$\mathscr{O}_X \simeq \pi_* \mathscr{O}_Y \ .$$

Note that it follows from condition (2) that the fibres of π are *connected*. Consider a section $g \in \pi_* \mathscr{O}_Y$, or, what amounts to the same, a function on Y. As the π-fibres are compact and connected, this function is *constant* along the π-fibres. Hence g can be considered as a *continuous* function on X, which is holomorphic on $Y \setminus \pi^{-1}(\Sigma) = X \setminus \Sigma$. Because we assumed X to be weakly normal, it follows that g is holomorphic on X: $g \in \mathcal{O}_X$. So the map $\mathcal{O}_X \hookrightarrow \pi_*\mathcal{O}_Y$ is indeed an isomorphism. Because the map π is an isomorphism outside Σ, the coherent sheaf $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$ has as support a set contained in Σ. But the last exact sequence now tells us that $t\cdot$ acts injective on $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$. As t vanishes on $\Sigma \subset X$, we conclude that $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$. ◇

For any weakly normal surface singularity (X, p), one can construct an *improvement* $\pi : Y \longrightarrow X$, which is an isomorphism over $X \setminus \{p\}$ and which has only certain basic weakly normal singularities, called partition singularities. Such an improvement plays a role analogous to that of the resolution for normal singularities. Weakly rational singularities are defined by the vanishing of $R^1\pi_*\mathcal{O}_Y$. For more details we refer to [9]. Proposition 1 implies the following statement:

Let C be a weakly normal curve singularity and X the total space of a flat deformation $X \longrightarrow S$ of C. Then X is weakly rational.

This follows from the above proposition by applying it to $X = C$, $\mathcal{X} = X$, and $\mathcal{Y} = Y$. Note that for Proposition 1, we did not assume $\mathcal{Y}$ to be smooth.

We return to the general situation of a smoothing of a reduced equidimensional space X.

Proposition 2 *With the same notations as before, we have for the smoothing of a reduced, equidimensional space X the following implication:*

$$X \quad \text{weakly normal} \quad \Longrightarrow \quad H^1(Y) = 0\,.$$

Proof The embedded resolution map $\mathcal{Y} \longrightarrow \mathcal{X}$ clearly satisfies the condition (1) of Proposition 1. It follows from Lemma 1 that $\mathcal{X}$ is normal; hence, $\mathcal{O}_{\mathcal{X}} \xrightarrow{\simeq} i_*\mathcal{O}_{\mathcal{X}\setminus\Sigma}$. Because $\mathcal{Y} \setminus \pi^{-1}(\Sigma) \longrightarrow X \setminus \Sigma$ is an isomorphism, it follows that $\pi_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$. Hence also the second condition of Proposition 1 is fulfilled, so we can conclude that $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$; in other words we get

$$H^1(\mathcal{O}_{\mathcal{Y}}) = 0.$$

From the exponential sequence on $\mathcal{Y}$

$$0 \longrightarrow \mathbb{Z}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}^*_{\mathcal{Y}} \longrightarrow 0\,,$$

the similar sequence for $\mathcal{X}$ and the fact the $\mathcal{O}_{\mathcal{X}} \simeq \pi_*\mathcal{O}_{\mathcal{Y}}$ it follows that

$$H^1(\mathcal{Y}, \mathbb{Z}_{\mathcal{Y}}) = 0$$

As $\mathcal{Y}$ contracts onto Y, we have $H^1(Y, \mathbb{Z}) = 0$. ◇

4 The Kernel of β

In the big diagram, there was a map β:

$$H^0(Y^{[0]},\mathbb{Z}) \stackrel{\beta}{\longrightarrow} H^2(\mathscr{Y},\mathbb{Z})(= H^2(Y,\mathbb{Z}))$$

This map works as follows: Elements of the first group can be considered as *divisors* $\sum n_i Y_i$ supported on Y. Each such divisor determines a *line bundle* $\mathscr{O}(\sum_i n_i Y_i)$. Then one has

$$\beta(\sum n_i Y_i) = c_1(\mathscr{O}(\sum n_i Y_i)).$$

So the map β factorises over the map ψ which associates to a divisor its line bundle. From the exponential sequence on $\mathscr{Y}$, we obtain the following diagram:

$$\begin{array}{ccccccc} & & H^0(Y^{[0]}) & = & H^0(Y^{[0]}) & & \\ & & \downarrow \psi & & \downarrow \beta & & \\ \cdots \longrightarrow H^1(\mathscr{O}_{\mathscr{Y}}) & \longrightarrow & H^1(\mathscr{O}^*_{\mathscr{Y}}) & \longrightarrow & H^2(\mathscr{Y},\mathbb{Z}) & \longrightarrow \cdots \end{array}$$

One immediately obtains

Lemma 3 rank $(\ker\beta) \geq$ rank $(\ker\psi)$. *If* $H^1(\mathscr{O}_{\mathscr{Y}}) = 0$, *then* $\ker\psi = \ker\beta$. $\diamond$

Definition Let $(\mathscr{X},p)$ be a germ of a normal analytic space. The *local class group* is defined as

$$Cl(X,p) := We(X,p)/Ca(X,p)\ .$$

Here $We(X,p)$ is the free abelian group spanned by the (germs of) Weil divisors on X and $Ca(X,p)$ the subgroup spanned by the (germs of) Cartier divisors.

Lemma 4 *With the notations of Sect. 2, there is a diagram with exact rows and columns:*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \ker\psi & \stackrel{\simeq}{\longrightarrow} & \ker\gamma & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^0(Y^{[0]}) & \longrightarrow & H^0(X^{[0]}) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \psi & & \downarrow \gamma & & \\ 0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^1(\mathscr{O}^*_{\mathscr{Y}}) & \longrightarrow & Cl(\mathscr{X},p) & \longrightarrow & 0 \end{array}$$

Here $F^{[0]} := Y^{[0]} \setminus \widetilde{X}$ *is the disjoint union of the divisors F of Sect. 1 and* $X^{[0]} = \sqcup_{i=1}^{r} X_i$*, where the* X_i *are the irreducible components of X. The maps are the canonical ones.*

Proof The surjection $H^1(\mathcal{O}^*_{\mathcal{Y}})$ (=$R^1\pi_*\mathcal{O}_{\mathcal{Y}}$) to the local class group works as follows: by pulling back a Weil divisor on $\mathcal{X}$, we obtain a Cartier divisor on $\mathcal{Y}$ (and hence a line bundle) that maps back to the original Weil divisor on $\mathcal{X}$, as the map π is a modification in codimension ≥ 2 (cf. [5]). The main point is to show that the kernel of the map $H^1(\mathcal{O}^*_{\mathcal{Y}}) \longrightarrow Cl(\mathcal{X}, p)$ is precisely $H^0(F^{[0]})$, or what amounts to the same, that $\ker\psi = \ker\gamma$. Let $A = \sum n_i Y_i$ be in the kernel of ψ. We may assume that $n_i \geq 0$. Hence there is a function $g \in H^0(\mathcal{O}_{\mathcal{Y}})$ with $(g) = A$. By the normality of $\mathcal{X}$, we have $\mathcal{O}_{\mathcal{X}} = \pi_*\mathcal{O}_{\mathcal{Y}}$, so g can be considered as a holomorphic function on $\mathcal{X}$, having as divisor on $\mathcal{X}$ just the image of that part of A that does not involve the divisors of $F^{[0]}$. This gives the map $\ker\psi \longrightarrow \ker\gamma$. This map is injective, because if the divisor of g (on $\mathcal{X}$) would be zero, g would be a unit; hence, $A = 0$. Surjectivity of $\ker\psi \longrightarrow \ker\gamma$ can be shown as follows: $A = \sum_i n_i X_i$ is an element in $\ker\gamma$ if it is the divisor on $\mathcal{X}$ of some function $g \in \mathcal{O}_{\mathcal{X}}$. The divisor of $g \circ \pi$ is a Cartier divisor on $\mathcal{Y}$ supported on Y, so it produces an element of $\ker\psi$ mapping to A. ◇

The use of Lemma 4 is that it allows us to get rid of $\ker\psi$, that depends on the global object $\mathcal{Y}$ over which we have not much control, and replace it with the map:

$$\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathcal{X}, p)$$

that maps each irreducible component of X_i to its class of the corresponding Weil divisor on $\mathcal{X}$.

There is one further issue: in Sect. 2 we first performed a base change to arrive at a semi-stable family. The following lemma shows that kernel of the map γ is essentially independent of base change.

Lemma 5 *Consider a normal space* $\mathcal{X}$ *and a reduced principal divisor* $X \subset \mathcal{X}$*. Let* $X^{[0]} = \sqcup_{i=1}^{r} X_i$*, where the* X_i *are the irreducible components of X. Let* $\widetilde{\mathcal{X}}$ *be obtained from* $\mathcal{X}$ *by taking a d-fold cyclic covering ramified along X.*

Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathcal{X}, p)$ *and* $\widetilde{\gamma} : H^0(X^{[0]}) \longrightarrow Cl(\widetilde{\mathcal{X}}, p)$ *be the maps discussed above. Then*

$$\mathrm{rank}(\ker\gamma) = \mathrm{rank}(\ker\widetilde{\gamma}).$$

Proof Let $\phi : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be the cyclic d-fold covering map. We consider the irreducible components X_i as divisors both on $\mathcal{X}$ and on $\widetilde{\mathcal{X}}$ If $A = \sum_i n_i X_i \in \ker\gamma$, then $A = (g)$ for some $g \in \mathcal{O}_{\mathcal{X}}$. On the covering space $\widetilde{\mathcal{X}}$, the function $g \circ \phi$ now has $d \cdot A$ as divisor. Conversely, if $B = \sum_i m_i X_i \in \ker\widetilde{\gamma}$, then $B = (h)$ for some $h \in \mathcal{O}_{\widetilde{\mathcal{X}}}$. Then the norm $N(h) \in \mathcal{O}_{\mathcal{X}}$ has $d \cdot B$ as divisor. ◇

5 Proof of Theorem 2

In the introduction we formulated the following theorem.

Theorem 2 *Let* $\mathscr{X} \xrightarrow{f} S$ *be a smoothing of a reduced equidimensional germ* (X,p). *Let* $X_t = f^{-1}(t), t \neq 0$ *be its Milnor fibre. Let* $X^{[0]} = \sqcup X_i$ *be where the* X_i *are the irreducible components of* X. *Let* $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X}, p)$ *be the map that associated with a divisor supported on* X *its class in the local class group. Then one has:*

1. $b_1(X_t) \geq \mathrm{rank}(\ker \gamma) - 1$.
2. *When* X *is weakly normal, then one has equality:*

$$b_1(X_t) = \mathrm{rank}(\ker \gamma) - 1.$$

In this case the action of the monodromy is trivial.

Proof From Lemma 2, (1) and (2), we get

$$\mathrm{rank}\, H^1(Y_t) \geq \mathrm{rank}\, H^1(B) - 1 = \mathrm{rank}\, H^1(Y) + \mathrm{rank}\, (\ker \beta) - 1 \geq \mathrm{rank}\, (\ker \beta) - 1$$

Furthermore, from Lemma 3, we have

$$\mathrm{rank}\, (\ker \beta) \geq \mathrm{rank}\, (\ker \psi).$$

From Lemmas 4 and 5, the number on the right-hand side is the same as

$$\mathrm{rank}(\ker \gamma),$$

so that we get

$$\mathrm{rank}\, H^1(Y_t) \geq \mathrm{rank}(\ker \gamma) - 1$$

which is the first statement of the theorem. For equality it is necessary that $H^1(Y) = 0$, and by Lemma 3, the equality $\mathrm{rank}\, (\ker \beta) = \mathrm{rank}\, (\ker \psi)$ is implied by the vanishing of $H^1(\mathscr{O}_{\mathscr{Y}})$. If we are considering a smoothing of a weakly normal space, this follows from Propositions 2 and 1, respectively. The triviality of the monodromy is Lemma 2, (3). ◇

In particular, when X is a hypersurface or, more generally, if $Cl(\mathscr{X}, p)$ is finite, then $\mathrm{rank}(\ker \gamma)$ is equal to the number r of irreducible components of X, so $b_1(X_t) \geq r - 1$, with equality in the weakly normal case.

Remark For a hypersurface germ X in $\mathbb{C}^3$ with a complete intersection as singular locus and transversal type A_1, it is known that the first Betti number is zero or one; see [6, 8]. So the number of irreducible components of X is one or two. To put it in another way, the singular locus of a weakly normal hypersurface in $\mathbb{C}^3$ which has more than three components is *never* a complete intersection.

Question J. Stevens has shown that all degenerate cusps are smoothable. What is the first Betti number for these smoothings? Is the first Betti number an invariant of X? Maybe not, but I do not have computed any non-trivial example. This seems to be an interesting topic for further investigations.

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A Kirwan Blowup and Trees of Vector Bundles

G. Trautmann

Abstract In the paper (Markushevich et al., Cent Eur J Math 10:1331–1355, 2012) a conceptual description of compactifications of moduli spaces of stable vector bundles on surfaces has been given, whose boundaries consist of vector bundles on trees of surfaces. In this article a typical basic case for the projective plane is described explicitly including the construction of a relevant Kirwan blowup.

Keywords Blowups • GIT quotients • Moduli • Vector bundles

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1 Introduction

To some extent, the replacement of limit sheaves in a compactification of a space of vector bundles by vector bundles on trees of surfaces is very natural, being in analogy to bubbling phenomena in geometric analysis and Yang-Mills theory in the work of Taubes, Uhlenbeck, and Feehan. There the degeneration of connections and fields is described by a process where data are preserved by shifting them partially to a system of attached 4-spheres. In the analogous situation of algebraic moduli spaces of vector bundles, the attached 4-spheres can be replaced by projective planes $\mathbb{P}_2$ hanged in at exceptional lines after blowing up points in a given surface. Then a limit sheaf can be transformed eventually to a vector bundle on the new reducible surface or on repeatedly constructed trees of surfaces. In [5] the trees of surfaces and vector bundles have been defined so that these objects can be the points of a compactification of the moduli spaces of rank 2 vector bundles on a given algebraic surface and are minimal for that purpose. The original basic example of such a compactification is the moduli space $M(2; 0, 2)$ of stable rank 2 vector bundles with

G. Trautmann (✉)
Fachbereich Mathematik, Universität Kaiserslautern, Erwin-Schrödinger-Straße, 67663 Kaiserslautern, Germany
e-mail: trm@mathematik.uni-kl.de

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Chern classes $c_1 = 0, c_2 = 2$ on $\mathbb{P}_2$ which has partially been treated in [5]. In this paper an explicit construction of the Kirwan blow up of a relevant parameter space is given together with the construction of a universal family. In Sect. 2 we recall shortly the definitions and the main theorem of [5] and in Sect. 3 the typical limit trees are explicitly constructed. Notation: All varieties in this article shall be defined over an algebraically closed field k of characteristic zero. $P(V)$ denotes the projective space of lines in the k-vector space V, whereas $\mathbb{P}_n = P(k^{n+1})$. The points of $P(V)$ are written as [v].

2 Trees of Surfaces and Bundles

2.1 Trees

A **tree** T in this article is a finite graph, oriented by a partial order $\leq$ and satisfying:

- There is a unique minimal vertex $\alpha \in T$, the root of T.
- For any $a \in T,\ a \neq \alpha$, there is a unique maximal vertex $b < a$, the predecessor of a, denoted by a^-.
- By $a^+ := \{b \in T \mid b^- = a\}$, we denote the set of direct successors of $a \in T$. We let T_{top} denote the vertices of T without successor.

A tree of surfaces over a given smooth projective surface S, modeled by a tree T, is a union

$$S_T = S_\alpha \cup \bigcup_a S_a$$

where

- S_α is a blowup of S in finitely many points.
- For $a \in T_{top}$, S_a is a projective plane $P_a = \mathbb{P}_2$.
- If $\alpha \neq a \notin T_{top}$, S_a is a blown-up projective plane $P_a = \mathbb{P}_2$ in finitely many simple points not on a line $l_a \subset P_a$.
- If $a \neq \alpha$, $S_a \cap S_{a^-} = l_a$ and l_a is an exceptional line in S_{a^-}.

Such trees can be constructed by consecutive blowups of simple points, hanging in a $\mathbb{P}_2(k)$ in each exceptional line of the previous surface and then blowing up points in the new $\mathbb{P}_2$, the whole starting with the given surface S.

By the construction of S_T, all or a part of its components can be contracted. In particular, there is the morphism

$$S_T \xrightarrow{\sigma} S$$

which contracts all the components except S_α to the points of the blown-up finite set of S_α.
Note that:

1. There are no intersections of the components other than the lines l_a.
2. If $T = \{\alpha\}$ is trivial, then $S_T = S$.
3. After contracting the lines l_a topologically (when defined over $\mathbb{C}$), one obtains bubbles of attached 4-spheres.

2.2 Treelike Vector Bundles

A *weighted tree* is a pair (T, c) of a tree T with a map c which assigns to each vertex $a \in T$ an integer $n_a \geq 0$, called the *weight* or *charge* of the vertex, subject to

$$\#a^+ \geq 2 \text{ if } n_a = 0 \text{ and } a \neq \alpha.$$

The total weight or total charge of a weighted tree is the sum $\Sigma_{a \in T}\ n_a = n$ of all the weights. We denote by $\mathbf{T}_n$ the set of all trees which admit a weighting of total charge n. It is obviously finite.

In the following we consider only pairs (S_T, E_T), called $\mathbf{T}_n$**-bundles** or simply **tree bundles**, where $T \in \mathbf{T}_n$, S_T is a tree of surfaces, and E_T is a rank 2 vector bundle on S_T, such that $c_1(E_T|S_a) = 0$, $c_2(E_T|S_a) = n_a$ for all weights n_a and such that the bundles $E_a = E_T|S_a$ are **"admissible,"** replacing a lacking stability condition; see [5].

In case $S_T = S$, this includes that the bundle E on S belongs to $M^b_{S,h}(2; 0, n)$, the quasi-projective Gieseker-Maruyama moduli scheme of χ-stable rank 2 vector bundles on S with respect to a polarization h and of Chern classes $c_1 = 0, c_2 = n$. The bundles in the special case of this article will all be admissible.

In particular, an indecomposable bundle E_a on $P_a = \mathbb{P}_2$ will be admissible if $c_1 = 0, c_2 = 1$. Such a bundle is not semistable on P_a. It is represented in homogeneous coordinates by exact sequences:

$$0 \to \mathcal{O}_{P_a}(-2) \xrightarrow{(z_0^2, z_1, z_2)} \mathcal{O}_{P_a} \oplus 2\mathcal{O}_{P_a}(-1) \to E_a \to 0.$$

We call the so defined tree bundles also $\mathbf{T}_n$-bundles. There is a natural notion of isomorphism for the pairs (S_T, E_T). They consist of isomorphisms of the surfaces with the base surface fixed and of isomorphisms of the lifted bundles.

2.3 Families of Tree Bundles

A $\mathbf{T}_n$-family of tree bundles is a triple $(\mathbf{E}/\mathbf{X}/Y)$, where $\mathbf{X}$ is flat family of $\mathbf{T}_n$-surfaces $X_y,\ y \in Y$, and $\mathbf{E}$ is a rank 2 vector bundle on $\mathbf{X}$ such that each $E_y = \mathbf{E}|X_y$ is a $\mathbf{T}_n$-bundle.

One can then consider the moduli stack $\mathbb{M}_n$ defined by

$$\mathbb{M}_n(Y) := \text{ set of families } (\mathbf{E}/\mathbf{X}/Y)$$

such that any bundle $E_y = \mathbf{E}|X_y$ is a 1-parameter limit of bundles in $M^b_{S,h}(2;0,n)$. Let

$$\mathbf{M}_n(Y) = \mathbb{M}_n(Y)/\sim .$$

be the associated functor. The following theorem is stated in [5].

Theorem *There is a separated algebraic space $M_n(S)$ of finite type over k corepresenting the functor $\mathbf{M}_n$.*

However the following questions are still open:

- Is $M_n(S)$ complete?
- When is $M_n(S)$ a (projective) scheme?
- Is $M_n(\mathbb{P}_2)$ a projective compactification of $M_{\mathbb{P}_2}(2;0,n)$?
- Classification of limit tree bundles for $M_{\mathbb{P}_2}(2;0,n)$ for $n \geq 3$?
- What about higher rank bundles on $\mathbb{P}_2$?
- Limit treelike bundles for instanton bundles on $\mathbb{P}_3$?

3 Limit Trees for $M^b(0,2)$

Let $M(2;0,2)$ be the moduli space of semistable sheaves on $\mathbb{P}_2$ with Chern classes $c_1 = 0, c_2 = 2$ and rank 2 and let $M^b(0,2)$ be its open part of (stable) bundles. It is well known that $M(2;0,2)$ is isomorphic to the $\mathbb{P}_5$ of conics in the dual plane, the isomorphism being given by $[\mathcal{F}] \leftrightarrow C(\mathcal{F})$, where $[\mathcal{F}]$ is the isomorphism class of $\mathcal{F}$ and $C(\mathcal{F})$ is the conic of jumping lines of $[\mathcal{F}]$ in the dual plane.

It is also well known that any sheaf $\mathcal{F}$ from $M(2;0,2)$ has two Beilinson resolutions on $P = \mathbb{P}_2 = P(V)$:

$$0 \to 2\,\Omega^2_P(2) \xrightarrow{A} 2\,\Omega^1_P(1) \to \mathcal{F} \to 0 \tag{1}$$

$$0 \to 2\,\mathcal{O}_P(-2) \xrightarrow{B} 4\,\mathcal{O}_P(-1) \to \mathcal{F} \to 0,$$

where the matrices A (of vectors in V) and B (of vectors in V^*) are related by the exact sequence

$$0 \to k^2 \xrightarrow{A} k^2 \otimes V \xrightarrow{B} k^4 \to 0.$$

The conic $C(\mathcal{F})$ in the dual plane has the equation $\det(A)$.

$\mathcal{F}$ is locally free if and only if $C(\mathcal{F})$ is smooth or if and only if $\mathcal{F}$ is stable. If $C(\mathcal{F})$ decomposes into a pair of lines, then A is equivalent to a matrix of the form $\left(\begin{smallmatrix} x & 0 \\ z & y \end{smallmatrix}\right)$, and then $\mathcal{F}$ is an extension

$$0 \to \mathcal{I}_{[x]} \to \mathcal{F} \to \mathcal{I}_{[y]} \to 0,$$

whose extension class is represented by the entry z.

Notice here that the sheaf is still locally free at the point [y] if the extension class is non-zero, i.e. $z \notin \operatorname{Span}(x, y)$. In any case $\mathcal{F}$ is S-equivalent to the direct sum $\mathcal{I}_{[x]} \oplus \mathcal{I}_{[y]}$.

3.1 Type 1 Degeneration

In the following let $e_0,\ e_1,\ e_2$ be basis of V and denote by $x_0,\ x_1,\ x_2$ its dual basis. For the first example, consider the 1-parameter deformation $\left(\begin{smallmatrix} e_0 & tae_1 \\ tbe_2 & e_0 \end{smallmatrix}\right)$ with second Beilinson resolution

$$0 \to 2\,\mathcal{O}_C \boxtimes \mathcal{O}_P(-2) \xrightarrow{B(t)} 4\,\mathcal{O}_C \boxtimes \mathcal{O}_P(-1) \to \mathbb{F} \to 0,$$

$$B(t) = \left(\begin{smallmatrix} x_1 & x_2 & tax_0 & 0 \\ 0 & tbx_0 & x_1 & x_2 \end{smallmatrix}\right)$$

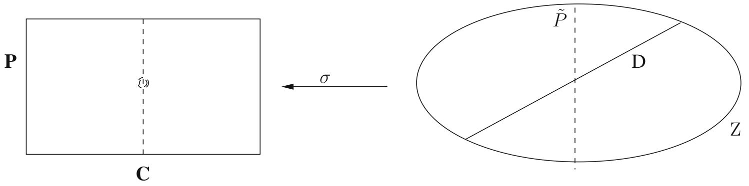
with parameters a, b, where $C = \mathbb{A}^1(k)$. For $t = 0$, the sheaf $\mathbb{F}_0$ is singular at $p = [e_0]$, $\mathbb{F}_0 = \mathcal{I}_p \oplus \mathcal{I}_p$. The blowing-up $\sigma : Z \to C \times P$ at $(0, p)$ is the subvariety of $C \times P \times \mathbb{P}_2$ given by the equations

$$tx_0u_1 - x_1u_0 = 0, \quad tx_0u_2 - x_2u_0 = 0, \quad x_1u_2 - x_2u_1 = 0,$$

where the u_ν are the coordinates of the third factor $\mathbb{P}_2$. We consider the following divisors on Z:

- $\tilde{P}$, the proper transform of $\{0\} \times P$, isomorphic to the blowup of P at p
- D, the exceptional divisor of σ
- H, the lift of $C \times h$, where h is a general line in P
- F, the divisor defined by $\mathcal{O}_Z(F) = pr_3^*\mathcal{O}_{\mathbb{P}_2}(1)$

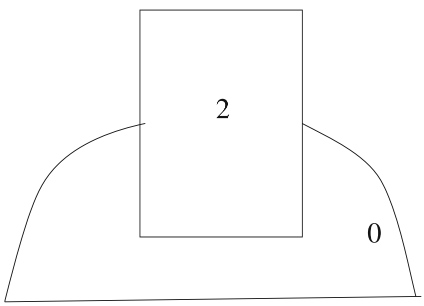
as shown in the figure.



Then $D \sim H - F$, and we let x_ν resp. u_ν denote the sections of $\mathcal{O}_X(H)$ resp. $\mathcal{O}_Z(F)$ lifting the above coordinates. Using the equations of Z, we see that the canonical section s of $\mathcal{O}_Z(D)$ is a divisor of the sections x_ν, such that $tx_0 = su_0$, $x_1 = su_1$, $x_2 = su_2$, and gives rise to the diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 2\,\mathcal{O}_D(-1) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & 2\,\mathcal{O}_Z(-2H) & \xrightarrow{\sigma^* B(t)} & 4\,\mathcal{O}_Z(-H) & \longrightarrow & \sigma^*\mathbb{F} & \longrightarrow & 0 \\
 & & \downarrow s & & \| & & \downarrow & & \\
0 & \longrightarrow & 2\,\mathcal{O}_Z(-H-F) & \xrightarrow{B_Z} & 4\,\mathcal{O}_Z(-H) & \longrightarrow & \mathbf{F} & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 2\,\mathcal{O}_D(-1) & & & & & &
\end{array}
$$

with $B_Z = \begin{pmatrix} u_1 & u_2 & au_0 & 0 \\ 0 & bu_0 & u_1 & u_2 \end{pmatrix}$. Thus B_Z represents a locally free sheaf $\mathbf{F}$ on Z, but its first Chern class has been modified by blowing up and removing the torsion. To correct this, consider the twisted bundle $\mathbf{E} := \mathbf{F}(D)$. Then $\mathbf{E}|\tilde{P} \simeq 2\,\mathcal{O}_{\tilde{P}}$, and the restriction $\mathbf{E}|D$ belongs to $M_D^b(2;0,2)$, $D \simeq \mathbb{P}_2$. Moreover, Z is flat over C, and $\mathbf{E}$ is a flat family of vector bundles over C with the limit tree bundle $\mathbf{E}|Z_0$ on the fiber $Z_0 = \tilde{P} \cup D$ over $0 \in C$. This can be symbolized by



type 1

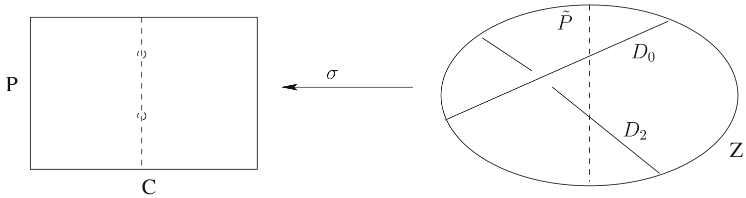
the numbers indicating the second Chern classes of the bundles on the components. The isomorphism class of this limit depends on the chosen parameters a, b which determine a normal direction to the Veronese surface in $\mathbb{P}_5$. This leads to blowing it up and to the Kirwan blowup of the parameter space; see Sect. 4.

3.2 Type 2 Degeneration

Let now a family on $C \times P$ be given by $\left(\begin{smallmatrix} e_0 & -te_1 \\ -te_1 & e_2 \end{smallmatrix}\right)$, defining a deformation of the sheaf of $\left(\begin{smallmatrix} e_0 & 0 \\ 0 & e_2 \end{smallmatrix}\right)$. Similarly to the previous case, the deforming sheaf $\mathbb{F}$ is the cokernel of the matrix

$$B(t) = \begin{pmatrix} x_2 & x_1 & tx_0 & 0 \\ 0 & tx_2 & x_1 & x_0 \end{pmatrix}$$

Blowing up $C \times P$ in the two singular points $(0, p_0)$ and $(0, p_2)$, $p_\nu = [e_\nu]$, leads to the figure:



The blown-up variety Z has the standard embedding into $(C \times P) \times \mathbb{P}_2 \times \mathbb{P}_2$ with divisors:

- H, the pullback of the divisor $C \times h$ in $C \times P$
- $\tilde{P}$, the blowup of $\{0\} \times P$ in the two points
- D_0, D_2, the two exceptional divisors
- F_0, F_2, whose invertible sheaves are the pullbacks of $\mathcal{O}_{\mathbb{P}_2}(1)$ from the third and fourth factor

Letting x_ν, u_ν, v_ν and s_0, s_2 denote the basic sections of the sheaves of H, F_0, F_2, D_0, D_2, we have the equations (as homomorphisms between invertible sheaves) $tx_0 = s_0u_0$, $x_1 = s_0u_1$, $x_2 = s_0u_2$, and $x_0 = s_2v_0$, $x_1 = s_2v_1$, $tx_2 = s_2v_2$.

By that we have the matrix decomposition

$$\begin{pmatrix} x_2 & x_1 & tx_0 & 0 \\ 0 & tx_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} s_0 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} u_2 & u_1 & u_0 & 0 \\ 0 & v_2 & v_1 & v_0 \end{pmatrix}.$$

Using this, the torsion of $\sigma^*\mathbb{F}$ can be removed as in the diagram of the previous section. Then $\mathbf{F} = \sigma^*\mathbb{F}/torsion$ has the resolution

$$0 \to \mathcal{O}_Z(-H - F_0) \oplus \mathcal{O}_Z(-H - F_2) \xrightarrow{B_Z} 4\mathcal{O}_Z(-H) \to \mathbf{F} \to 0,$$

where B_Z is the right-hand matrix. The tree components of $\mathbf{F}$ are $\mathbf{F}|D_i = \mathcal{T}_{D_i}(-1), \mathbf{F}|\tilde{P} = \mathcal{O}_{\tilde{P}}(-l_0) \oplus \mathcal{O}_{\tilde{P}}(-l_2)$, where l_0, l_2 are the exceptional lines on $\tilde{P}$. However, there is no way by twist or elementary transformation to make the first Chern classes c_1 vanish.

But starting with $\begin{pmatrix} e_0 & -t^2 e_1 \\ -t^2 e_1 & e_2 \end{pmatrix}$, we get by the same procedure a sheaf $\mathbf{F}$ on Z whose resolution matrix is

$$B_Z = \begin{pmatrix} u_2 & u_1 & tu_0 & 0 \\ 0 & tv_2 & v_1 & v_0 \end{pmatrix}.$$

This resolution implies that $\mathbf{F}$ is reflexive and singular in exactly two points $q_0 = \{u_1 = u_2 = t = 0\}$ and $q_2 = \{v_1 = v_0 = t = 0\}$ and that its restrictions to the components of $Z_0 = \tilde{P} \cup D_0 \cup D_2$ are

$$\mathbf{F}|\tilde{P} = \mathcal{O}_{\tilde{P}}(-l_0) \oplus \mathcal{O}_{\tilde{P}}(-l_2) \text{ and } \mathbf{F}|D_i = \mathcal{O}_{D_i} \oplus \mathcal{I}_{q_i,D_i}(1).$$

Hence there is an elementary transform on Z:

$$0 \to \mathbf{F}' \to \mathbf{F} \to \mathcal{O}_{D_0} \oplus \mathcal{O}_{D_2} \to 0.$$

The resolution of $\mathbf{F}'$ can be computed as follows. There is a decomposition $tu_0 = s_0\tilde{u}_0$ because tu_0 vanishes on the divisor D_0. Similarly we have $tv_2 = s_2\tilde{v}_2,\ u_1 = s_2\bar{u}_1,\ v_1 = s_0\bar{v}_1$ and from this the matrix decomposition

$$\begin{pmatrix} u_2 & \bar{u}_1 & \tilde{u}_0 & 0 \\ 0 & \tilde{v}_2 & \bar{v}_1 & v_0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & s_2 & & \\ & & s_0 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} u_2 & u_1 & tu_0 & 0 \\ 0 & tv_2 & v_1 & v_0 \end{pmatrix}$$

It follows by diagram chasing that the left-hand matrix gives the resolution

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathbf{F}' \to 0,$$

where $\mathcal{E}_1 = \mathcal{O}_Z(-H - F_0) \oplus \mathcal{O}_Z(-H - F_2)$ and
$\mathcal{E}_0 = \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z(-H - S_2) \oplus \mathcal{O}_Z(-H - S_0) \oplus \mathcal{O}_Z(-H)$.

This resolution shows that $\mathbf{F}'$ is locally free on Z. In order to determine its restrictions to the components, one should use the identities, $u_0^2 = x_0\tilde{u}_0$, $v_2^2 = x_2\tilde{v}_2$, which follow from the previous identities. Using these, one can determine the restrictions of the twisted bundle $\mathbf{E} := \mathbf{F}'(D_0 + D_2)$:

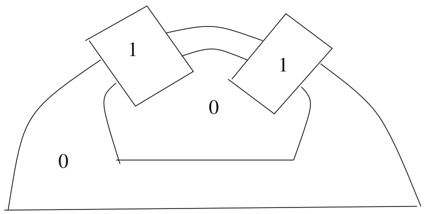
$\mathbf{E}|\tilde{P} = 2\mathcal{O}_{\tilde{P}}$ and $\mathbf{E}|D_i$ is a bundle on $D_i \cong \mathbb{P}_2$ with Chern classes $c_1 = 0, c_2 = 1$ (see the description of bundles with these Chern classes in Sect. 2.2).

Since the elementary transform and the twisting do not affect the bundle on the part of Z over $C \setminus \{0\}$, the sheaf is a limit tree bundle on the fiber $Z_0 = \tilde{P} \cup D_0 \cup D_2$.

3.3 Type 3 Degeneration

Let $\tilde{\mathbb{P}}_5$ be the blowup of $P(S^2V) = \mathbb{P}_5$ of the Veronese surface in $\mathbb{P}_5$; let $\Sigma_2 \subset \tilde{\mathbb{P}}_5$ be the exceptional divisor and $\Sigma_1 \subset \tilde{\mathbb{P}}_5$ the proper transform of the divisor of degenerate conics; see also Proposition 5.

By the above, type 1 limit tree bundles belong to $\Sigma_2 \setminus \Sigma_1$ and type 2 limit tree bundles belong to $\Sigma_1 \setminus \Sigma_2$. There is a third type of limit tree bundle belonging to $\Sigma_2 \cap \Sigma_1$ with symbolic tree.



Examples can be obtained as limits of families of type $\begin{pmatrix} e_0 & -t^3e_1 \\ -t^3e_1 & e_0+te_2 \end{pmatrix}$ and two consecutive blowups. In this case the family $\mathbb{F}$ on $C \times P$ is given as the cokernel in

$$0 \to 2\,\mathcal{O}_C \boxtimes \mathcal{O}_P(-2) \xrightarrow{B(t)} 4\,\mathcal{O}_C \boxtimes \mathcal{O}_P(-1) \to \mathbb{F} \to 0,$$

$$B(t) = \begin{pmatrix} x_2 & x_1 & t^3x_0 & 0 \\ 0 & t^2x_2 & x_1 & tx_0-x_2 \end{pmatrix}.$$

This sheaf $\mathbb{F}$ is singular in $(0,p)$, $p = [e_0]$. Let then

$$\sigma : Z \to C \times P$$

be the blowup as in Sect. 3.1, described as subvariety of $Z \subset C \times P \times \mathbb{P}_2$ with divisors $\tilde{P},\ H,\ D,\ F,\ D \sim H-F$. Let s be the standard section of $\mathcal{O}_Z(D)$ for the exceptional divisor, and let x_ν and y_ν be the basic sections of $\mathcal{O}_Z(H)$ and $\mathcal{O}_Z(F)$, respectively, with equations $tx_0 = sy_0,\ x_1 = sy_1,\ x_2 = sy_2$. It follows as in 3.1 that the sheaf $\mathbf{F} = \sigma^*\mathbb{F}/\mathit{torsion}$ has the resolution

$$0 \to 2\mathcal{O}_Z(-H-F) \xrightarrow{B_Z} 4\mathcal{O}_Z(-H) \to \mathbf{F} \to 0,$$

$$B_Z = \begin{pmatrix} y_2 & y_1 & t^2y_0 & 0 \\ 0 & t^2y_2 & y_1 & y_0-y_2 \end{pmatrix}.$$

This sheaf and its syzygy are of the same type as in Sect. 3.2. It is reflexive and singular exactly in the points $p_0,\ p_2 \in D \smallsetminus \tilde{P},\quad p_0 = \{t = y_1 = y_2 = 0\}$ and $p_2 = \{t = y_1 = y_0 - y_2 = 0\}$. Again one can verify that the sheaf $\mathbf{F}' := \mathbf{F}(D)$ has the restrictions

$$\mathbf{F}'|\tilde{P} = 2\mathcal{O}_{\tilde{P}} \quad \text{and} \quad \mathbf{F}'|D = \mathcal{I}_{p_0,D} \oplus \mathcal{I}_{p_2,D}.$$

on the components of $Z_0 = \tilde{P} \cup D$.

In order to construct a locally free limit tree bundle, we blow up Z in the two points $p_0,\ p_2$ to get

$$\tau : W \to Z$$

with exceptional divisors S_0 and S_2, the proper transform $\tilde{D}$ of D, the lifted divisors $\tilde{P}$ and F, and the two divisors F_0 and F_2 coming from the embedding.

As in Sect. 3.2, one concludes that the sheaf $\mathbf{F}'' = \tau^*\mathbf{F}'/\mathit{torsion}$ is reflexive and the cokernel of a matrix

$$B_Z = \begin{pmatrix} u_2 & u_1 & tu_0 & 0 \\ 0 & tv_2 & v_1 & v_0 \end{pmatrix}$$

and such that $\mathbf{F}''$ restricts as

$$\mathbf{F}''|\tilde{P} = 2\mathcal{O}_{\tilde{P}}, \quad \mathbf{F}''|\tilde{D} = \mathcal{O}_{\tilde{D}}(-l_0) \oplus \mathcal{O}_{\tilde{D}}(-l_2), \quad \mathbf{F}''|S_i = \mathcal{O}_{S_i} \oplus \mathcal{I}_{q_0,S_i}(1),$$

where $q_i \in S_i \smallsetminus \tilde{D}$.

Finally, as in Sect. 3.2, there is an elementary transform

$$0 \to \mathbf{E}' \to \mathbf{F}'' \to \mathcal{O}_{S_0} \oplus \mathcal{O}_{S_0}$$

such that $\mathbf{E}'$ is locally free on W and such that $\mathbf{E} := \mathbf{E}'(S_0 + S_2)$ has the desired restrictions

$$\mathbf{E}|\tilde{P} = 2\mathcal{O}_{\tilde{P}}, \quad \mathbf{E}|\tilde{D} = 2\mathcal{O}_{\tilde{D}}$$

and such that $\mathbf{E}|S_i$ do have the Chern classes $c_1 = 0,\ c_2 = 1$. So $\mathbf{E}$ is a limit tree bundle on the tree of surfaces $W_0 = \tilde{P} \cup \tilde{D} \cup S_0 \cup S_2$.

4 Kirwan Blowup I

The 2×2-matrices with entries in V in (1) parametrize the sheaves in $M(2; 0, 2)$ and at the same time the conics of their jumping lines in the dual plane $P(V^*)$ by their determinants in S^2V. Since the isomorphisms of the left-hand term in (1) are not essential, only the subspaces $[A]$ spanned by the rows of A matter, so that the Grassmannian $G_2(k^2 \otimes V)$ is a parameter space of $M(2;0,2)$. The Plücker embedding

$$p : G_2(k^2 \otimes V) \subset P(\wedge^2(k^2 \otimes V)) = P(\wedge^2 k^2 \otimes SV \oplus S^2k^2 \otimes \wedge^2 V)$$

can be expressed in terms of the entries, using the standard basis of k^2, by

$$[\begin{pmatrix} x\, x' \\ y\, y' \end{pmatrix}] \quad \stackrel{p}{\mapsto} \quad [xy' - x'y;\ x \wedge y,\ x \wedge y' + x' \wedge y,\ x' \wedge y'].$$

One should note here that there is the relation

$$\begin{pmatrix} x\, x' \\ y\, y' \end{pmatrix} \wedge \begin{pmatrix} x \wedge y & x \wedge y' + x' \wedge y & x' \wedge y' & 0 \\ 0 & x \wedge y & x \wedge y' + x' \wedge y & x' \wedge y' \end{pmatrix} = 0. \tag{2}$$

There is an action of $\mathrm{SL}_2(k)$ on both sides of the Plücker embedding, induced by the natural action on k^2 and written as

$$[A]g = [Ag] \quad \text{and} \quad [q;\ \Phi]g = [q;\ \Phi S^2 g],$$

explicitly with

$$Ag = \begin{pmatrix} x\, x' \\ y\, y' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \Phi S^2 g = (\xi,\ \omega,\ \eta) \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & 2\beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix},$$

such that the Plücker embedding is equivariant. An element [A] in the Grassmannian is semistable if and only if $\det(A) \neq 0$, and it is stable if and only if $\det(A)$ is the equation of a nondegenerate quadric in the dual plane $P(V^*)$. Moreover, the morphism $[A] \to [\det(A)]$,

$$G_2(k^2 \otimes V)^{ss} \longrightarrow PS^2V \cong \mathbb{P}_5 \cong \bar{M}(2;0,2,0)$$

is a good GIT quotient; see [7].

For the construction of a compactification of $M^b(0,2)$ by tree bundles, we need to replace the Grassmannian by a parameter space with only stable points in order to avoid unnatural identifications in the boundary. This is done by the method of Kirwan [4] in two consecutive blowups.

4.1 The First Blowup

In the following we use the abbreviations $X = G_2(k^2 \otimes V)$ and $G = \mathrm{SL}_2(k)$. The group G has the fixed points $[\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)]$. According to [4], let then Z_G denote the subset

$$Z_G = \{[A] \in X |\ \text{the affine fiber of } p(A) \text{ fixed by } G\}.$$

It follows that

$$Z_G = \{[\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)]\} \cong P(V),$$

that it is a closed and smooth subvariety of X and that $Z_G = GZ_G$. The vanishing of the components of Φ characterizes the points of Z_G and these components define its ideal sheaf $\mathcal{I}_G$. Let then

$$\tilde{X} := Bl_{Z_G}(X)$$

be the blowup of X along Z_G. In this situation

$$\tilde{X} \subset X \times P(S^2k^2 \otimes \wedge^2 V)$$

is the closure of the graph of the map $\Phi : X \smallsetminus Z_G \to P(S^2k^2 \otimes \wedge^2 V)$, see [1], [2], [3] for classical descriptions of blowups. This blowup can geometrically be described as follows.

Lemma 1

(a) $\tilde{X}$ *is the subvariety of* $X \times P(S^2k^2 \otimes \wedge^2 V)$ *of points* $([A],\ [\xi,\ \omega,\ \eta])$ *satisfying*

(i) $(x \wedge y,\ x \wedge y' + x' \wedge y,\ x' \wedge y') \in k(\xi,\ \omega,\ \eta)$
(ii) $A \wedge \begin{pmatrix} \xi & \omega & \eta & 0 \\ 0 & \xi & \omega & \eta \end{pmatrix} = 0$

(b) The exceptional divisor E_G *in* $\tilde{X}$ *is the subvariety of pairs* $([A],\ [x \wedge u,\ x \wedge w,\ x \wedge v])$ *with* $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$.
(c) $\tilde{X}$ *is smooth and the projection* $\tilde{X} \to X$ *is G-equivariant.*

Sketch of Proof Because $\tilde{X}$ is the closure of graph, (i) follows immediately and also (ii) by formula (2). Let conversely $Y \subset X \times P(S^2k^2 \otimes \wedge^2 V)$ be defined by (i) and (ii). Then $\tilde{X} \subset Y$ and $\tilde{X} \smallsetminus E_G = Y \smallsetminus E_G$. One shows now that the fiber Y_p for a point $p \in Z_G$ coincides with the fiber $\tilde{X}_p = E_{G,p}$. Such a point has as its first component $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, and (ii) implies that its second component is of the form $[x \wedge u,\ x \wedge w,\ x \wedge v]$. Consider then the 1-parameter family $A(t) = \left(\begin{smallmatrix} x & -tv \\ tu & x+tw \end{smallmatrix}\right)$. For $t \neq 0$, $[A(t)] \in X \smallsetminus Z_G$, and its lift to $\tilde{X}$ has the limit $\tilde{A}$ with components $[A]$ and $[x \wedge u,\ x \wedge w,\ x \wedge v]$. This proves (a) and also (b) as a corollary. For (c) smoothness follows from that of X and Z_G, and the equivariance directly from (a).

It follows from (b) that E_G is the $\mathbb{P}_5$-bundle

$$\begin{array}{ccc} E_G & \xrightarrow{\approx} & P(S^2k^2 \otimes Q) \\ \downarrow & & \downarrow \\ Z_G & \xrightarrow{\approx} & P(V), \end{array}$$

where Q is the tautological quotient bundle of P(V).

Stability in $\tilde{X}$: By definition of $\tilde{X}$, there is the Plücker embedding:

$$\tilde{X} \subset P((\wedge^2 k^2 \otimes SV \oplus S^2k^2 \otimes \wedge^2 V) \otimes (S^2k^2 \otimes \wedge^2 V))$$

and by this the action on $\tilde{X}$ is induced by the obvious linear action of G on the ambient projective space.

Proposition 2 *Let* $\tilde{A} = ([A],\ [\xi,\ \omega,\ \eta])$ *be a point of* $\tilde{X}$. *Then:*

(i) $\tilde{A}$ *is semistable if and only if both of* $\det(A) = xy' - x'y$ *and* $\omega^2 - 4\xi\eta$ *are non-zero.*
(ii) *If* $\tilde{A} \notin E_G$, *then* $\tilde{A}$ *is stable if and only if* $\pi(\tilde{A}) = [A]$ *is stable.*
(iii) *If* $\tilde{A} \in E_G$, *then* $\tilde{A}$ *is stable if and only if* $\omega^2 - 4\xi\eta$ *is not a square in* $S^2(V/k.x)$

For the proof, notice first that the quadratic forms $\det(A) = xy' - x'y$ and $\omega^2 - 4\xi\eta$ of the components of $\tilde{A}$ are invariant under this action. Then the statements can be canonically verified by either looking for the points in the affine cone or by using the Mumford criterion for the action of 1-parameter subgroups. For the latter, the weights can be computed via the tensor products in the Plücker space.

Some elementary calculations with the explicit description of the group action show:

Lemma 3 *Let* $\tilde{A} = ([A],\ [\xi,\ \omega,\ \eta])$ *be a point of* $\tilde{X}$. *Then:*

(i) $\omega^2 - 4\xi\eta = 0$ *if and only if there is a* $g \in G$ *such that* $[\xi,\ \omega,\ \eta]S^2g = [\xi', 0, 0]$.
(ii) $\omega^2 - 4\xi\eta$ *is square if and only if there is a* $g \in G$ *such that* $[\xi,\ \omega,\ \eta]S^2g = [\xi',\ \omega',\ \eta']$ *with* $\xi' = 0$ *or* $\eta' = 0$.
(iii) $\omega^2 - 4\xi\eta$ *is a product if and only if there is a* $g \in G$ *such that* $[\xi,\ \omega,\ \eta]S^2g = [\xi', 0,\ \eta']$.

Let now $H_0^{ss} \subset H_1^{ss} \subset X^{ss}$ be the subvarieties of points $[A]$ for which $\det(A)$ is a square, respectively, a product in S^2V. These are the inverse images in X^{ss} of the double lines, respectively, pairs of lines in the space $P(S^2V)$ of conics in $P(V^*)$. Let $H_0 \subset H_1$ be their closures in X. By definition $Z_G \subset H_0^{ss}$. Since the matrices $[A] \in H_0^{ss}$ are of type $[\left(\begin{smallmatrix} x & 0 \\ z & x \end{smallmatrix}\right)]g,\ g \in G$, one finds that $H_0^{ss} \smallsetminus Z_G$ consists of all non-closed orbits whose closures meet Z_G, the orbits of the latter being its points. Then

$$P(V) \cong Z_G = H_0^{ss}//G \subset X^{ss}//G \cong P(S^2V)$$

is the Veronese embedding. Moreover, all the points $H_0^{ss} \smallsetminus Z_G$ become unstable in $\tilde{X}$; see Lemma 4.

Let $\tilde{H}_0 \subset \tilde{H}_1$ be the proper transforms of $H_0 \subset H_1$ in $\tilde{X}$. Then the following holds.

Lemma 4

(a) $\tilde{H}_0 \cap \tilde{X}^{ss} = \emptyset$ *and* $E_G \cap \tilde{H}_0 = E_G \smallsetminus E_G^{ss}$.
(b) $\tilde{H}_1 \cap \tilde{X}^{s} = \emptyset$ *and* $E_G \cap \tilde{H}_1 = E_G \smallsetminus E_G^{s}$.

Sketch of Proof A point in $H_0^{ss} \smallsetminus Z_G$ is equivalent to a point $[\left(\begin{smallmatrix} x & 0 \\ z & x \end{smallmatrix}\right)]$ and this has the second component $[x \wedge z, 0, 0]$ in $\tilde{X}$. By Remark 3 it is not semistable. Then also $\omega^2 - 4\xi\eta = 0$ for the limit points. To show that $E_G \smallsetminus E_G^{ss} \subset \tilde{H}_0$, we may assume that a point p in $E_G \smallsetminus E_G^{ss}$ has the components $[\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)]$, $[x \wedge z, 0, 0]$. As in the proof of Lemma 1, the family defined by $[\left(\begin{smallmatrix} x & 0 \\ tz & x \end{smallmatrix}\right)]$ shows that $p \in \tilde{H}_0$. This proves (a). The proof of (b) is analogous.

By the characterization of semistable points, the equivariant morphism $\pi : \tilde{X} \to X$ maps (semi-)stable points to (semi-)stable and gives rise to a morphism $\tilde{X}^{ss}//G \to X^{ss}//G \cong P(S^2V)$, which is an isomorphism over the complement of the Veronese surface Z_G/G. Because $\Sigma_2 := E_G^{ss}//G$ becomes the inverse image of Z_G/G and is a Cartier divisor, we obtain the

Proposition 5 $\widetilde{P(S^2V)} := \tilde{X}^{ss}//G$ *is the blowup of* $P(S^2V)$ *along the Veronese surface.*

4.2 Related Geometry of Conics

For any point $\tilde{A}$ in $\tilde{X}$, the quadratic form $\omega^2 - 4\xi\eta$ can be seen as an element of S^2V^* because of $\wedge^2 V \cong V^*$. One can then easily verify that for any nondegenerate $A = \left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)$ (for which $[x \wedge y,\ x \wedge y' + x' \wedge y,\ x' \wedge y')] = [\xi,\ \omega,\ \eta]$)), the quadradic form $\omega^2 - 4\xi\eta$ is the equation of the dual conic in $P(V)$ of the conic $\{\det(A) = 0\} \subset P(V^*)$ (of jumping lines of the corresponding vector bundle). Because $\tilde{X}^{ss}$ is defined by $\det(A) \neq 0$ and $\omega^2 - 4\xi\eta \neq 0$, we can define the universal family of conics

$$Q \subset \tilde{X}^{ss} \times P(V)$$

as the subvariety of pairs $(\tilde{A}, [v])$ with $(\omega^2 - 4\xi\eta)(v) = 0$. If $\tilde{A} \in E_G^{ss}$, i.e. $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, then $(\xi,\ \omega,\ \eta) = (x \wedge u,\ x \wedge w,\ x \wedge v)$ and the fiber $Q_{\tilde{A}}$ is a pair of lines through [x] in $P(V)$ or a double line.

Secondly, the related quadratic form $w^2 - 4uv \in S^2V/k.x$ without the factor x defines two points or a double point on the double line $\{x^2 = 0\}$ in $P(V^*)$.

Recalling that the space of **complete conics** in the plane $P(V^*)$ consists of conics, for which the double lines are enriched by two points or a double point, one finds that $\tilde{X}^{ss}$ parametrizes this space and that the quotient $\widetilde{P(S^2V)} := \tilde{X}^{ss}//G$ is the space of complete conics in $P(V^*)$. Moreover, because the forms $\det(A)$ and $\omega^2 - 4\xi\eta$ are invariant, the conic bundle Q descents to a conic bundle embedded in $\widetilde{P(S^2V)} \times P(V)$ and describes the duality for complete conics.

5 Kirwan Blowup II

It is easy to see that there are no semistable points in $\tilde{X}$ with a two-dimensional stabilizer by checking the types of points. But there are two-dimensional such stabilizers. For the Kirwan blowup, it is enough to consider only connected reductive ones. Again by checking the different types of points, one finds that the only such stabilizers are $R = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\} \cong k^*$ and its conjugates. According to [4] we consider for the center of the blowup of $\tilde{X}$ the subvariety Z_R of points $\tilde{A}$ in $\tilde{X}$ which are fixed by R, and such in addition, R acts trivially on the affine fiber of $\tilde{A}$ in $\wedge^2(k^2 \otimes V) \otimes (S^2k^2 \otimes \wedge^2 V)$. A direct computation shows that

$$Z_R \quad \text{is the set of points} \quad ([\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)],\ [0, \omega, 0]) \quad \text{in} \quad \tilde{X}.$$

Then $GZ_R \subset \tilde{X}^{ss} \smallsetminus \tilde{X}^s$ and is of dimension 6.

By definition $GZ_R \subset \tilde{H}_1^{ss}$, and GZ_R is the subset of points in $\tilde{H}_1^{ss}$ with closed orbits. The good quotient $GZ_R//G = \tilde{H}_1^{ss}//G$ is then the proper transform in $\widetilde{P(S^2V)}$ of the divisor Σ_1 of products in $P(S^2V)$.

Lemma 6

(1) The closure $\overline{GZ_R}$ is the subvariety of points $([A], [\xi,\ \omega,\ \eta])$ in $\tilde{X}$ for which $\xi,\ \omega,\ \eta$ are pairwise linearly dependent in $\wedge^2 V$.
(2) $\overline{GZ_R} \cap \tilde{X}^{ss} = GZ_R$.
(3) $\overline{GZ_R}$ is smooth.
(4) $\overline{GZ_R}$ and E_G intersect transversily in dimension 5.

Proof Let Y be the closed subvariety of $\tilde{X}$ defined by the condition in (1). Then $\overline{GZ_R} \subset Y$. When $y \in Y \cap \tilde{X}^{ss}$, then $y = ([A], [a\xi,\ b\xi,\ c\xi])$ with $b^2 - 4ac \neq 0$, and there is a group element g and some λ so that $\lambda(a, b, c) = (0, 1, 0)S^2g$, because y is supposed to be semistable. Then $yg^{-1} = ([B], [0, \xi, 0])$ and thus an element of Z_R. Now $Y \cap \tilde{X}^{ss} = GZ_R$. If y is unstable, there is a group element g so that $\lambda(a, b, c) = (1, 0, 0)S^2g$. Then $yg^{-1} = ([B], [\xi, 0, 0])$ and such points are limits of points in GZ_R: such matrices B can only be of type $[\left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)]$ or of type $[\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)]$. In the first case, $[\left(\begin{smallmatrix} x & 0 \\ y & ty \end{smallmatrix}\right)]$ is family, whose members are G-equivalent to points in Z_R for $t \neq 0$. In the second case, the members of the family $[\begin{pmatrix} x & t^2y \\ y & x \end{pmatrix}]$ for $t \neq 0$ are G-equivalent to $[\begin{pmatrix} x+ty & 0 \\ 0 & x-ty \end{pmatrix}]$ belonging also to Z_R. This proves $Y \subset \overline{GZ_R}$ and thus (1) and (2). The lengthy but elementary proof of (3) and (4) by use of local coordinates for the Grassmannian and its blowup is omitted here. □

Remark The set $\overline{GZ_R} \smallsetminus GZ_R$ consists entirely of the orbits of the unstable points $([\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}], [x \wedge y, 0, 0])$ and $([\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}], [\xi, 0, 0])$.

Lemma 7 $E_G \smallsetminus E_G^{ss} \subset E_G \cap \overline{GZ_R} \subset E_G \cap \tilde{H}_1 = E_G \smallsetminus E_G^s$ *and these sets are of dimensions 4, 5, and 6, respectively.*

Proof When a point $p \in E_G$ is unstable, it is in the orbit of a point $q = ([\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}], [\xi, 0, 0])$ and then $p \in \overline{GZ_R}$. Such points have a two-dimensional stabilizer G_q, and then $E_G \smallsetminus E_G^{ss}$ is parametrized by $P(Q) \times G/G_q$, where Q is the tautological quotient bundle on $P(V)$. Hence $E_G \smallsetminus E_G^{ss}$ is four-dimensional. The points in $E_G \cap \overline{GZ_R}$ are of type $([\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}], [a\xi, b\xi, c\xi,])$ with $\xi = x \wedge u$ and $u \in V/k.x$. Therefore there is a surjective morphism $P(Q) \times \mathbb{P}_2 \to E_G \cap \overline{GZ_R}$ which is generically injective. Hence $\dim(E_G \cap \overline{GZ_R}) = 5$. Finally $E_G \cap \tilde{H}_1$ is an intersection of hypersurfaces and so of dimension 6. □

The condition in Lemma 6 for points in $\overline{GZ_R}$ is equivalent to the vanishing of $\xi \wedge \omega,\ \xi \wedge \eta,\ \omega \wedge \eta$. Moreover, the homomorphism $(\xi,\ \omega,\ \eta) \mapsto (\xi \wedge \omega,\ \xi \wedge \eta,\ \omega \wedge \eta)$ describes the canonical wedge map

$$\mathrm{Hom}((S^2k^2)^*, \wedge^2 V) \to \mathrm{Hom}(\wedge^2(S^2k^2)^*, \wedge^2 \wedge^2 V),$$

and this is G-equivariant, explicitly described by

$$(\xi,\ \omega,\ \eta) \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix} \mapsto (\xi \wedge \omega,\ \xi \wedge \eta,\ \omega \wedge \eta) \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}. \tag{3}$$

So the map

$$\tilde{X} \smallsetminus \overline{GZ_R} \xrightarrow{\Phi} P(\wedge^2(S^2k^2) \otimes \wedge^2 \wedge^2 V) \cong P(k^3 \otimes V),$$

given by $p \to [\xi \wedge \omega,\ \xi \wedge \eta,\ \omega \wedge \eta]$ is well-defined and G-equivariant and the components of this map generate the ideal sheaf of $\overline{GZ_R}$

5.1 The Second Blowup

Can now be defined as the blowup of $\tilde{X}$ along $\overline{GZ_R}$:

$$Y := Bl_{\overline{GZ_R}}(\tilde{X}) \xrightarrow{\pi} \tilde{X}$$

It is simultaneously the **closure of the graph of** Φ. By the smoothness of the ingredients, Y is smooth. Moreover, $Y \subset \tilde{X} \times P(k^3 \otimes V)$ is acted on by G and the projection $Y \to \tilde{X}$ is G-equivariant according to formula (3). We let E_R denote the exceptional divisor.

Remark The condition for $\overline{GZ_R}$ says that the second components of its points are of type $[a\xi, b\xi, c\xi] = (a, b, c) \otimes \xi$ in $P(S^2k^2 \otimes \wedge^2 V)$. This means that $\overline{GZ_R}$ is the pullback of the Segre variety $S = P(S^2k^2) \times P(\wedge^2 V)$ in $P(S^2k^2 \otimes \wedge^2 V)$. It follows that also the blowup $Bl_{\overline{GZ_R}}(\tilde{X})$ is the pullback of the blowup of $P(S^2k^2 \otimes \wedge^2 V)$ along the Segre variety S.

5.2 Stability in Y

By definition Y is embedded in $\tilde{X} \times P(\wedge^2(S^2k^2) \otimes \wedge^2 \wedge^2 V)$. Combined with Segre embeddings, we have

$$Y \subset P((\wedge^2 k^2 \otimes S^2 V \oplus S^2k^2 \otimes \wedge^2 V) \otimes (S^2k^2 \otimes \wedge^2 V) \otimes (\wedge^2(S^2k^2) \otimes \wedge^2 \wedge^2 V)).$$

Then using the Mumford criterion, see [8] and [6], and considering the weights of 1-parameter subgroups, one can derive:

(i) Points in Y over points in GZ_R are stable.
(ii) Points in Y over stable points in $\tilde{X}$ are stable.
(iii) Points in Y over unstable points in $\tilde{X}$ are unstable.
(iv) Properly semistable points in $\tilde{X}^{ss} \smallsetminus GZ_R$ become unstable in Y.
(v) Every semistable point in Y is stable.

Remark One can as well show that the stabilizer of any semistable point in Y is finite.

The G-equivariant morphism π induces a surjective G-equivariant morphism $Y^s \longrightarrow \tilde{X}^{ss}$ and thus a surjective morphism of the good quotients

$$\tau : Y^s/G \longrightarrow \tilde{X}^{ss}//G = \widetilde{P(S^2V)}$$

with surjective restriction

$$\tilde{\Sigma}_1 := E_R^s/G \longrightarrow GZ_R//G = \Sigma_1,$$

whereas

$$Y^s/G \smallsetminus \tilde{\Sigma}_1 \xrightarrow{\approx} \tilde{X}^{ss}//G \smallsetminus \Sigma_1$$

must be an isomorphism because π is an isomorphism outside E_R. Moreover, because $Y^s \xrightarrow{\pi} \tilde{X}^{ss}$ is a blowup, also the induced morphism τ is a blowup along the divisor Σ_1. Hence the

Proposition 8 $\tau : Y^s/G \longrightarrow \tilde{X}^{ss}//G = \widetilde{P(S^2V)}$ *is an isomorphism.*

Remark While the second Kirwan blowup has no effect on the quotient, it describes $\widetilde{P(S^2V)}$ as a geometric quotient, so that non-isomorphic S-equivalent limit sheaves w.r.t. the parameter space Y^s are excluded. This is needed for the construction of families which include admissible tree bundles because S-equivalence for tree bundles is not defined.

6 Families Including Tree Bundles

In this section the construction of families of sheaves, including all admissible tree bundles for the tree compactification of $M^b(0,2)$, will be sketched in two steps. In step one we construct such a family over the base space $\tilde{X}^{ss}$.

Firstly we recall the presentation of the semi-universal family for the Gieseker-Maruyama space $M(2;0,2)$. Let $0 \to \mathcal{U} \to k^2 \otimes \mathcal{O}_X \to \mathcal{Q} \to 0$ be the tautological sequence on the Grassmannian $X = G_2(k^2 \otimes V)$. As in formula (1), there are two such equivalent presentations. The second is the exact sequence over $X^{ss} \times P$:

$$0 \to k^2 \otimes \mathcal{O}_X \boxtimes \mathcal{O}_P(-2) \longrightarrow \mathcal{Q} \boxtimes \mathcal{O}_P(-1) \to \mathcal{F} \to 0. \tag{4}$$

Recall from Lemma 3 that $H_1^{ss} \subset X^{ss}$ is the hypersurface of points $[A]$ for which $\det(A)$ decomposes, i.e. the inverse image of Σ_1, and that $H_0^{ss} \subset H_1^{ss}$ is the subvariety where $\det(A)$ is a square. Let now $S_1 \subset X^{ss} \times P$ be the subvariety of points $([A],[v])$ for which v divides $\det(A)$ and $S_0 \subset S_1$ where $\det(A) = v^2$. Then S_1 is seven-dimensional and 2:1 over $H_1^{ss} \smallsetminus H_0^{ss}$.

It follows that $\mathcal{F}$ is locally free on $X^{ss} \times P \smallsetminus S_1$ whose restriction to fibers over $X^{ss} \smallsetminus H_1$ is the vector bundles in $M^b(0,2)$, whereas the sheaves over points in H_1 become the semistable sheaves in the boundary of $M^b(0,2)$.

Notice however that the sheaf $\mathcal{F}$ restricted to $\{p\} \times P$ may be singular only in one of the points of S_1 over p; see the Notice before Sect. 3.1.

6.1 First Step

Let now $\tilde{X}^{ss} \times P \xrightarrow{\alpha} X^{ss} \times P$ be the map $\phi = \alpha \times \mathrm{id}$, where α is the blowup map of Sect. 4, and consider the lifted family $\mathbb{F} = \phi^*\mathcal{F}$. Then $\mathbb{F}$ is locally free over the inverse image of $X^{ss} \smallsetminus E_G^{ss} \cup \tilde{H}_1^{ss}$.

Analogously to S_0 and S_1, let then $\tilde{S}_0$ be the set of points $(p, [v]) \in \tilde{X}^{ss} \times P$ over E_G^{ss} where $\det(A) = v^2$, and let similarly $\tilde{S}_1 \subset \tilde{X}^{ss} \times P$ be the set of points over $\tilde{H}_1^{ss}$ where v is a factor of $\det(A)$. Then $\mathbb{F}$ is locally free outside $\tilde{S}_0 \cup \tilde{S}_1$, $\tilde{S}_0$ is mapped 1:1 to E_G^{ss}, and the map $\tilde{S}_1 \smallsetminus \tilde{S}_0 \to \tilde{H}_1^{ss} \smallsetminus E_G^{ss}$ is 2:1.

Consider now the blowup $Z \xrightarrow{\sigma_0} \tilde{X}^{ss} \times P$ along $\tilde{S}_0$ and let D denote the exceptional divisor. Let

$$\mathcal{F} := \sigma_0^* \mathbb{F}/torsion$$

be the torsion-free pullback on Z. Now the situation of the families $\mathcal{F}$ and $\mathbb{F}$ restricted to the open subset $\tilde{X}^{ss} \smallsetminus \tilde{H}_1^{ss}$ of the base is the higher-dimensional analog to that of the families over the curve C in Sect. 3.1, with $0 \in C$ replaced by the divisor $E_G^{ss} \subset \tilde{X}^{ss} \smallsetminus \tilde{H}_1^{ss}$.

Moreover, one can compare the two situations by considering a curve $C \subset \tilde{X}^{ss}$ transversal to E_G^{ss} in a point $p \notin \tilde{H}_1^{ss}$. Then the blowup Z_C of $C \times P$ in the point $(p, q) \in \tilde{S}_0$ can be identified with the restriction of Z to C. Moreover, by flatness, the sheaves $\mathcal{F}_C$ and $\mathbb{F}_C$ on Z_C from Sect. 3.1 can be identified with the restrictions of $\mathcal{F}$ and $\mathbb{F}$ to $Z|C$. Because $\mathcal{F}_C$ is locally free on Z_C, it follows that $\mathcal{F}$ is locally free in a neighborhood of the fiber Z_p of Z over p. Finally, because the fiber Z_p is the union of the blowup of P at q and the restriction D_p of exceptional divisor D, the sheaf $\mathcal{F}|Z_p$ is a tree bundle on Z_p. In order to obtain the correct Chern classes, we have to replace $\mathcal{F}$ by its twist $\mathcal{F}(D)$ as in Sect. 3.1, which is also compatible with the restriction. It has been shown:

Proposition 9 *With the notation above, the family $\mathcal{F}$ is a family of tree bundles over the restricted base variety $\tilde{X}^{ss} \smallsetminus \tilde{H}_1^{ss}$.*
If $p \in \tilde{X}^{ss} \smallsetminus \tilde{H}_1^{ss} \cup E_G^{ss}$, then $\mathcal{F}|Z_p$, $Z_p = P$, is a bundle in $M^b(0, 2)$.
If $p \in E_G^{ss} \smallsetminus \tilde{H}_1^{ss}$, then $\mathcal{F}|Z_p$, where $Z_p = \tilde{P} \cup D_p$, $D_p \cong \mathbb{P}_2$, is a tree bundle with $\mathcal{F}|\tilde{P} \cong 2\mathcal{O}_{\tilde{P}}$ and $\mathcal{F}|D_p \in M_{D_p}^b(0, 2)$.

For the fibers over points in $\tilde{H}_1^{ss}$, we have:

Lemma 10 *Let $\hat{S}_1$ be the proper transform of $\tilde{S}_1$ in Z. Then $\hat{S}_1 \to \tilde{H}_1^{ss}$ is 2:1.*

Remark For a point $p \in E_G^{ss} \cap \tilde{H}_1^{ss}$, the two points of $\hat{S}_1$ over p will be contained in the fiber $D_p \cong \mathbb{P}_2$ of the exceptional divisor D. By the previous, $\mathcal{F}$ is locally free on $Z \smallsetminus \hat{S}_1$.

Proof The method of proof is again by restriction to transversal curves: Let $q \in \tilde{S}_0 \cap \tilde{S}_1 \subset \tilde{X}^{ss} \times P$ and $p \in E_G^{ss} \cap \tilde{H}_1^{ss}$ its image. Then p has the components $\left[\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right]$ and $[a\xi, b\xi, c\xi]$ with $b^2 - 4ac \neq 0$ and $\xi = x \wedge y$ for some $y \in V$. Then

$$p(t) := (\left[\begin{smallmatrix} x & 0 \\ 0 & x+ty \end{smallmatrix}\right], [a\xi, b\xi, c\xi])$$

is a 1-parameter family in $GZ_R \subset \tilde{H}_1^{ss}$ defining a normal direction to E_G^{ss} at p. Let C denote the image of $p(t)$ for small t. For $t \neq 0$ the points [x] and [x+ty] define then sections of $\tilde{S}_1|C \smallsetminus \{0\}$, which fill this subset. Because q is the only point in $\tilde{S}_0$ over p, $q \in \tilde{S}_1|C$, the closure of $\tilde{S}_1|C \smallsetminus \{0\}$. Let now $S_C := \tilde{S}_1|C \subset C \times P$ and consider the blowups

$$Bl_q(C \times P) \subset Bl_{\tilde{S}_0}(\tilde{X}^{ss}) = Z$$

as the proper transform. Then the restriction $\hat{S}_1|C$ of the proper transform $\hat{S}_1$ can be identified with the proper transform of S_C in $Bl_q(C \times P)$. This situation corresponds to the figure in Sect. 3.1 with the two sections [x] and [x+ty] added. Then the proper transforms of these linear sections do not meet on the exceptional divisor D_p. Hence also $\hat{S}_1 \cap D_p$ consists of two different points. □

6.2 Second Step

By the above, $\mathcal{F}$ is locally free on $Z \smallsetminus \hat{S}_1$, and one could try to construct the tree bundles over $\tilde{H}_1^{ss}$ by directly blowing up Z along $\hat{S}_1$ and modifying the lifted sheaf. However, over points $p \in \tilde{H}_1^{ss} \smallsetminus GZ_R$, the sheaf $\mathcal{F}|Z_p$ has only one singular point and is not stable; see the remark at the beginning of this Sect. 6. Secondly, $\tilde{H}_1^{ss} \smallsetminus GZ_R$ consists only of non-closed orbits. On the other hand, the orbits in GZ_R are closed, and for $p \in GZ_R$, the two points of $\hat{S}_1$ are the singular points of $\mathcal{F}|Z_p$.

Now this insufficiency can be eliminated by using the second Kirwan blowup $Y \to \tilde{X}$ and pulling the pair $(Z, \mathcal{F})$ back to Y^s. After this the points of $\tilde{H}_1^{ss} \smallsetminus GZ_R$ become unstable and can be neglected, and $\hat{S}_1|GZ_R$ is the reasonable locus to be blown up. Therefore, let

$$\begin{array}{ccc} Z_Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y^s & \longrightarrow & \tilde{X}^{ss}, \end{array}$$

be the pullback of Z and let $\mathcal{F}_Y$ be the lift of $\mathcal{F}$ to Z_Y. The situation of the pair $(Z_Y, \mathcal{F}_Y)$ is now the relative version of the situation in Sect. 3.2 before using a double cover.

6.2.1 Properties of $(Z_Y, \mathcal{F}_Y)$:

Let E_R^s denote the exceptional divisor of Y^s over GZ_R, (see Sect. 5), let $E_{G,Y}$ denote the proper transform of E_G, and let D_Y be the pullback of D in Z. Then $\mathcal{F}_Y$ is singular exactly along the pullback $S_{1,Y}$ of $\hat{S}_1$, and $S_{1,Y}$ is 2:1 over E_R^s everywhere by Lemma 10. For points p in $E_R^s \smallsetminus E_{G,Y}$, the two points of $S_{1,Y}$ over p will be in the fiber $Z_{Y,p} \cong P$, but for points p in $E_R^s \cap E_{G,Y}$, the two points of $S_{1,Y}$ over p will be in the fiber $D_{Y,p}$ of D_Y.

Remark The variety Z_Y may also be obtained as the blowup of the variety $S_{0,Y} \subset Y^s \times P$ over $E_{G,Y}$, defined as $\tilde{S}_0$ over E_G.

In order to construct a family of tree bundles in this new relative situation, Z_Y has to be blown up along $S_{1,Y}$ as in the case Sect. 3.2. Then the torsion-free pullback of $\mathcal{F}_Y$ would give a family of tree bundles parametrized along E_R^s. But as in Sect. 3.2, these tree bundles would not be admissible as defined in Sect. 2.2. In analogy to Sect. 3.2, one would have to use a double cover of Y^s which is branched exactly over E_R^s in order to construct admissible tree bundles. However, such a double cover may not exist globally. But one could consider such local covers $U \to Y^s$ over affine open parts. Then we have Cartesian diagrams

$$\begin{array}{ccccc} W_U & \xrightarrow{\tau} & Z_U & \xrightarrow{g} & Z_Y \\ & & \downarrow & & \downarrow \\ & & U \times P & \longrightarrow & Y^s \times P \\ & & \downarrow & & \downarrow \\ & & U & \xrightarrow{f} & Y^s, \end{array}$$

where τ is the blowup of Z_U along the subvariety $S_U = g^* S_{1,Y}$. This is the subvariety where $\mathcal{F}_U := g^*\mathcal{F}_Y$ is not locally free. By the previous, it is 2:1 over the branch locus $B := f^* E_R^s \subset U$. Consider then the sheaf

$$\mathcal{E} := \tau^* \mathcal{F}_U / torsion.$$

One can show as in the curve case that $\mathcal{E}$ is flat over U.

Now one can argue as in 6.1 using curves C which are transversal to B : There is an elementary transform $\mathcal{E}'$ of $\mathcal{E}$ on W_U with transformation support over B which is locally free on W_U. Then $\mathcal{E}'$ is a family of tree bundles, whose fibers over points in $U \smallsetminus B$ are the same as for points in $Y^s \smallsetminus E_R^s$ or in $\tilde{X}^{ss} \smallsetminus GZ_R$. After twisting with the exceptional divisor in W_U, we may finally assume that $\mathcal{E}'$ is a family of admissible tree bundles with prescribed Chern classes. Hence the

Proposition 11 *For any 2:1 cover $U \xrightarrow{f} Y^s$ of an affine open subset of Y^s, branched exactly along E_R^s, the following holds:*

(i) *For points p in $U \smallsetminus f^*\tilde{E}_G \cup B$, the bundle $\mathcal{E}'_p$ is a member of $M^b_{W_{U,p}}(0,2)$, where $W_{U,p} \cong \mathbb{P}_2$.*
(ii) *For points p in $f^*\tilde{E}_G \smallsetminus B$, the bundle $\mathcal{E}'_p$ is of the type described in Sect. 3.1.*

(iii) *For points p in $B \smallsetminus f^*\tilde{E}_G$, the bundle $\mathcal{E}'_p$ is of the type described in Sect. 3.2.*
(iv) *For points p in $B \cap f^*\tilde{E}_G$, the bundle $\mathcal{E}'_p$ is of the type described in Sect. 3.3.*

The families of tree bundles so constructed may not descend to a global family over the Kirwan blowup $\tilde{M}_2 \cong \widetilde{P(S^2V)}$ of $M(2;0,2) \cong P(S^2V)$ because the automorphism groups of the tree bundles include automorphisms of the supporting surfaces; see Sect. 2.2. However, delicately, their isomorphism classes are determined precisely by the points of $\tilde{M}_2$:

Proposition 12 *The set of points of $\tilde{M}_2$ is the set isomorphism classes of the tree bundles constructed above. In particular, let as above $\Sigma_2 \subset \widetilde{P(S^2V)}$ be the blowup of the Veronese surface in $P(S^2V)$ and $\Sigma_1 \subset \widetilde{P(S^2V)}$ the proper transform of the subvariety of decomposable conics. Then:*

(i) $\tilde{M}_2 \smallsetminus \Sigma_1 \cup \Sigma_2 = M^b(0,2)$ *is the set the isomorphism classes of the (stable) bundles in $M(2;0,2)$.*
(ii) *The set $\Sigma_2 \smallsetminus \Sigma_1$ is the set of isomorphism classes of limit tree bundles of type 1 described in Sect. 3.1.*
iii) *The set $\Sigma_1 \smallsetminus \Sigma_2$ is the set of isomorphism classes of limit tree bundles of type 2 described in Sect. 3.2.*
(iv) *The set $\Sigma_1 \cap \Sigma_2$ is the set of isomorphism classes of limit tree bundles of type 3 described in Sect. 3.3.*

Proof There is nothing to proof for (i). For the proof of (ii), recall that $\Sigma_2 \smallsetminus \Sigma_1$ is the geometric quotient of the open part $E_G^s \subset \tilde{X}^s$ of the exceptional divisor E_G whose points are of type

$$p = ([\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}], [\xi,\ \omega,\ \eta]),$$

where $\omega^2 - 4\xi\eta$ decomposes into two different factors and $\xi,\ \omega,\ \eta \in x \wedge V$. By 3 we may assume that $\omega = 0$. The two factors $\xi,\ \eta$ determine two lines in P=P(V) through $[x]$; see Sect. 4.2. Now the fiber Z_p is a union $\tilde{P}(x) \cup D_p$, where $\tilde{P}(x)$ is the blowup of P at $[x]$ and $D_p \cong \mathbb{P}_2$. Then the two lines in P determine two points $q_1,\ q_2$ on the exceptional line $\ell_p = \tilde{P}(x) \cap D_p$. Let now $\mathcal{F}$ on Z be the sheaf constructed in 6.1. By Proposition 9 $\mathcal{F}|Z_p$ has the restrictions $\mathcal{F}|\tilde{P}(x) \cong 2\mathcal{O}_{\tilde{P}(x)}$ and $\mathcal{F}|D_p \in M^b_{D_p}(0,2)$. So $\mathcal{F}|D$ corresponds to its smooth conic of jumping lines in the dual plane D_p^* or to the dual conic $\Gamma_p \subset D_p$ of the latter. □

Claim The conic Γ_p meets the line ℓ_p in the two points $q_1,\ q_2$.

In addition, there is the following elementary.

Lemma 13 *Let ℓ be the line through two points $a_1,\ a_2 \in \mathbb{P}_2$ and let* $\mathrm{Aut}_\ell(\mathbb{P}_2)$ *be the subgroup of the group of automorphisms of $\mathbb{P}_2$ which fixes the points of ℓ. Then $Aut_\ell(\mathbb{P}_2)$ acts transitively on the set of nondegenerate conics through $a_1,\ a_2$.*

If the claim is verified, the lemma implies that the isomorphism class of $\mathcal{F}|D_p$ and then also of $\mathcal{F}|Z_p$ only depends on the two points $q_1,\ q_2$, which are determined by the point p. Then the isomorphism class of $\mathcal{F}|Z_p$ also depends only on the image $[p]$ of p in the quotient $\tilde{M}_2$, which proves (ii).

In order to prove the claim, we use again 1-parameter degenerations with limit point p which are transversal to E_G^s.

For that we may assume that

$$p = ([\begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix}], [e_0 \wedge e_1,\ 0,\ e_0 \wedge e_2]),$$

where e_0, e_1, e_2 form a basis of V and that the first component of the 1-parameter family is given by

$$A(t) = \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix} + t \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}.$$

with t in a neighborhood C of $0 \in \mathbb{A}^1(k)$. This is a smooth curve in X^{ss}, and its lift to $\tilde{X}^{ss}$ is transversal to E_G^s and has second component

$$[e_0 \wedge y + t\xi,\ e_0 \wedge (y' - x) + t\omega,\ -e_0 \wedge x' + t\eta],$$

where $(\xi,\ \omega,\ \eta) = (x \wedge y,\ x \wedge y' + x' \wedge y,\ x' \wedge y')$. Because p is supposed to be the limit at $t = 0$, we may assume, up to a scalar factor, that the components of the vectors satisfy

$$y_1 = 1,\ y_2 = 0,\ y_1' = x_1,\ y_2' = x_2,\ x_1' = 0,\ x_2' = -1$$

In addition we replace the basis $e_1 \wedge e_2,\ -e_0 \wedge e_2,\ e_0 \wedge e_1$ of $\wedge^2 V$ by the basis z_0, z_1, z_2 of V^*, dual to the basis e_0, e_1, e_2 of V. Then the second component of $p(t)$ reads

$$[z_2 - tx_2z_0 + t\xi', tz_0 + t\omega', -z_1 + x_1z_0 + t\eta'],$$

where $\xi',\ \omega',\ \eta' \in \operatorname{Span}(z_1, z_2)$.

Let now Z_C be the restriction of Z to C. Then Z_C can be considered the blowup of $C \times P$ at $(0, [e_0])$ as a proper transform, and $\mathcal{F}|D_p$ can be computed as in 3.1, as well as its conic $\Gamma_p \subset D_p$. As $\mathcal{F}|Z_C$ is the torsion-free pullback of the sheaf on $C \times P$ defined by $A(t)$, its family of conics becomes the proper transform of the family

$$Q = \{(tz_0 + t\omega')^2 - 4(z_2 - tx_2z_0 + t\xi')(-z_1 + tx_1z_0 + t\eta') = 0\},$$

whose fibers for $t \neq 0$ are the conics of $\mathcal{F}|\{t\} \times P$, c.f. Sect. 4.2. This proper transform is obtained by substituting the forms tz_0, z_1, z_2 by u_0, u_1, u_2, which are the coordinate forms of $D_p \cong \mathbb{P}_2$; see 3.1. So the proper transform $\tilde{Q}$ of Q is defined by the equation

$$(u_0 + t\omega')^2 - 4(u_2 - x_2u_0 + t\xi')(-z_1 + x_1u_0 + t\eta'),$$

where now $\xi', \omega', \eta' \in \operatorname{Span}(u_1, u_2)$. For $t = 0$, the conic Γ_p of $\mathcal{F}|D_p$ has the equation $u_0^2 - 4(u_2 - x_2u_0)(-u_1 + x_1u_0)$. Because the line ℓ_p and $P([\tilde{e}_0])$ are given by $u_0 = 0$, Γ_p meets ℓ_p in the two points q_1, q_2 with equation u_1u_2. This proves the claim and thus (ii) of Proposition 12.

For the proof of (iii), let a point in $\Sigma_1 \setminus \Sigma_2$ be the image of a point $p \in E_R^s \subset Y^s$. We may assume that its image $\bar{p} \in GZ_R$ under the second blowup has the components

$$\left(\begin{smallmatrix} x_1 & 0 \\ 0 & x_2 \end{smallmatrix}\right), [0,\ x_1 \wedge x_2,\ 0].$$

Under an auxiliary blowup W_U as in 6.2.1, the fiber $W_{U,p}$ is isomorphic to $\tilde{P}(x_1, x_2) \cup D_1 \cup D_2$, where $\tilde{P}(x_1, x_2)$ is the blowup of P at x_1, x_2, $D_i \cong \mathbb{P}_2$, containing the exceptional lines ℓ_i of $\tilde{P}(x_1, x_2)$. Moreover, $W_{U,p}$ is determined by the data of the point p or its image in $\Sigma_1 \setminus \Sigma_2$ up to isomorphism. By Proposition 11, (iii), the tree bundle $\mathcal{E}'_p = \mathcal{E}'|W_{U,p}$ is trivial on $\tilde{P}(x_1, x_2)$ and restricts to bundles $\mathcal{E}'_i$ on D_i with Chern classes $c_1 = 0, c_2 = 2$. By the following Lemma 14 the isomorphism class of each $\mathcal{E}'_i$ corresponds uniquely to a point $q_i \in D_i \setminus \ell_i$. Since the group $\operatorname{Aut}_{\ell_i}(D_i)$ acts transitively on $D_i \setminus \ell_i$ (see Lemma 13), these isomorphism classes are uniquely determined by $W_{U,p}$ and finally determined by the point $[p] \in \Sigma_1 \setminus \Sigma_2$, because the automorphisms of $W_{U,p}$ must be identities on $\tilde{P}(x_1, x_2)$.

Lemma 14 *Let $\ell \subset \mathbb{P}_2 = P$ be a line. Then the moduli space $M_\ell(0,1)$ of isomorphism classes of rank 2 vector bundles on $\mathbb{P}_2$ which are trivial on ℓ with Chern classes $c_1 = 0, c_2 = 1$ can be identified with the set $\mathbb{P}_2 \setminus \ell$.*

Proof of the Lemma Let ℓ have the equation z_0 and let $a = [a_0, a_1, a_2] \in \mathbb{P}_2 \setminus \ell$. Let

$$B = \begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ a_0 & a_1 & a_2 & z_0 \end{pmatrix},$$

and define $\mathcal{E}(a)$ as cokernel in the sequence

$$0 \to \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-1) \xrightarrow{B} 3\mathcal{O}_P(-1) \oplus \mathcal{O}_P \to \mathcal{E}(a) \to 0.$$

Then the class of $\mathcal{E}(a)$ belongs to $M_\ell(0,1)$. Conversely, given any $\mathcal{E}$ in $M_\ell(0,1)$, it is well known that $\mathcal{E}$ is an elementary transform of the twisted tangent bundle $\mathcal{T}_P(-2)$ with exact extension sequence

$$0 \to \mathcal{T}_P(-2) \to \mathcal{E} \to \mathcal{O}_\ell \to 0.$$

From that we get a resolution matrix B of $\mathcal{E}$ as above. In that, (a_0, a_1, a_2) represents the extension class and $[a_0, a_1, a_2]$ the isomorphism class of $\mathcal{E}$.

This completes the proof of (iii). The proof of (iv) is analogous to that of (iii). In this case a point $p \in B \cap f^*\tilde{E}_G$ or $p \in E_R^s \cap \tilde{E}_G$ over a point in in $\Sigma_1 \cap \Sigma_2$ can be supposed to have as components

$$[\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)], [a\xi,\ b\xi,\ c\xi], [u, w, v]$$

with $\xi \in x \wedge V$ and $b^2 - ac \neq 0$. Then $W_{U,p}$ as a fiber of the blowup is isomorphic to $\tilde{P}([x]) \cup \tilde{D}_0(p_1, p_2) \cup D_1 \cup D_2$, where $\tilde{D}_0(p_1, p_2)$ is the blowup at two points of a plane D_0 which contains the exceptional line ℓ_0 of $\tilde{P}([x])$ and where D_i are again planes containing the two exceptional lines ℓ_i of $\tilde{D}_0(p_1, p_2)$. Then $W_{U,p}$ depends only on the geometry and the point p up to isomorphism. Now the tree bundle $\mathcal{E}'$ on W_U of Proposition 11 is trivial on $\tilde{P}([x])$ and $\tilde{D}_0(p_1, p_2)$, whereas $\mathcal{E}'|D_i$ has Chern classes $c_1 = 0, c_2 = 1$. It follows again from 14 that the isomorphism classes of $\mathcal{E}'|D_i$ are unique, and then that $\mathcal{E}'|W_{U,p}$ is uniquely determined because the automorphisms of $W_{U,p}$ must be identities on the components $\tilde{P}([x])$ and $\tilde{D}_0(p_1, p_2)$. This proves (iv) of the Proposition 12. □

6.3 The Stack

By the above construction of families of tree bundles, a global family of such bundles could not be obtained. Instead, we have families of tree bundles on local 2:1 covers of the parameter space Y^s. These are forming an obvious moduli stack over the category of such open covers. It is plausible to claim that this is a Deligne-Mumford stack which is corepresented by $Y^s/G \cong \widetilde{P(S^2V)}$.

In [5] global families of limit tree bundles of stable rank 2 vector bundles on surfaces have been constructed by other abstract procedures, which led to algebraic spaces as moduli spaces. The question of their relation to the above stack being open at present.

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