Randomized Block Data

8

This eighth chapter of *Permutation Statistical Methods* introduces a generalized Minkowski distance function and establishes the foundation for a set of Multivariate Randomized Block Permutation (MRBP) procedures for univariate and multivariate randomized-block data. MRBP procedures were introduced by Mielke and Iyer in 1982 and constitute a class of permutation methods for one or more response measurements among two or more treatments on the same or matched objects [299]. The MRBP procedures presented here provide a synthesizing foundation for a variety of statistical tests and measures that are further developed in Chaps. 9–11 for interval-, ordinal-, and nominal-level response measurements, respectively.

8.1 Multivariate Block Permutation Procedures

Suppose that a number of observed fields are compared to corresponding fields generated by one or more numerical models. Let the observed phenomena and the one or more numerical model predictions of these phenomena be termed "blocks," i.e., the first block might represent the observed phenomena and the remaining $b - 1$ blocks represent additional blocks, such as numerical model predictions of the phenomena for a total of $b \geq 2$ blocks. Also, let $r \geq 1$ denote the number of commensurate response measurements from each phenomenon and let $g \geq 2$ denote the number of phenomena, here called "treatments."

The terms representing "blocks" and "treatments" vary among disciplines. Oftentimes when $g = 2$ treatments and the same objects are represented in each treatment, the design is called a "before-and-after" or "subject-is-own-control" design. When $g = 2$ treatments and matched, but different, objects are represented in each treatment, the design is often called a "matched pairs" design. When $g > 2$ treatments and the same objects are represented in each treatment, the design is sometimes called a "repeated measures" design, and in this case the treatments are often labeled as "trials." Finally, in psychology randomized-block designs are known

K.J. Berry et al., *Permutation Statistical Methods*,
DOI 10.1007/978-3-319-28770-6_8

as "within-subjects" designs to distinguish them from completely randomized or "between-subjects" designs.

Let $x'_{ij} = (x_{1ij}, x_{2ij}, \ldots, x_{rij})$ denote a transposed vector of r response measurement scores associated with the ith treatment and jth block. Then the MRBP test statistic is given by

$$\delta = \left[g \binom{b}{2} \right]^{-1} \sum_{i=1}^{g} \sum_{j<k} \Delta(x_{ij}, x_{ik}) \ , \tag{8.1}$$

where $\sum_{j<k}$ denotes the sum over all j and k such that $1 \leq j < k \leq b$ and $\Delta(x, y)$ is a symmetric distance-function value of two points $x' = (x_1, x_2, \ldots, x_r)$ and $y' = (y_1, y_2, \ldots, y_r)$ in an r-dimensional Euclidean space. The generalized Minkowski distance function considered here is given by

$$\Delta(x, y) = \left(\sum_{i=1}^{r} \left| x_i - y_i \right|^p \right)^{v/p} \ , \tag{8.2}$$

where $p \geq 1$ and $v > 0$. Thus, $p = v = 2$ yields squared Euclidean distance, which is not a metric, and $p = 2$ and $v = 1$ yields ordinary Euclidean distance, which is a metric.[1]

The null hypothesis (H_0) states that the distribution of δ assigns an equal probability to each of the

$$M = \left(g! \right)^b$$

possible allocations of the r-dimensional response measurement scores to the g treatment positions within each of the b blocks. Consequently, the collection of r response measurement scores within each block yields g r-dimensional exchangeable random variables under the null hypothesis. The probability value associated with an observed value of δ, say δ_o, is the probability under the null hypothesis (H_0) of observing a value of δ as extreme or more extreme than δ_o. Thus, an exact probability value for δ_o may be expressed as

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} \ .$$

[1]Recall that a distance function is a metric if it satisfies three properties given by (1) $\Delta(x, y) \geq 0$ and $\Delta(x, x) = 0$, i.e., the distance is positive between two different points and is equal to zero from any point to itself; (2) the distance is symmetric: $\Delta(x, y) = \Delta(y, x)$, i.e., the distance between points x and y is the same in either direction; and (3) the triangle inequality is satisfied: $\Delta(x, y) \leq \Delta(x, z) + \Delta(z, y)$, i.e., the distance between any two points is the shortest distance along any path.

When M is very large, an approximate probability value for δ may be obtained from a resampling procedure, where

$$P\big(\delta \leq \delta_{\text{o}}|H_0\big) = \frac{\text{number of } \delta \text{ values} \leq \delta_{\text{o}}}{L} \ ,$$

and L denotes the number of randomly sampled test statistic values. Typically, L is set to a large value to ensure accuracy, e.g., $L = 1{,}000{,}000$. When M is very large and P is exceedingly small, a resampling-approximation permutation procedure may produce no δ values equal to or less than δ_{o}, even with $L = 1{,}000{,}000$, yielding an approximate resampling probability value of $P = 0.00$. In such cases, moment-approximation permutation procedures based on fitting the first three exact moments of the discrete permutation distribution to a Pearson type III distribution provide approximate probability values, as detailed in Chap. 1, Sect. 1.2.2 [284, 299].

As with MRPP, discussed in Chap. 2, a chance-corrected measure of agreement among all b blocks for all g treatments constitutes a universal measure of effect size for all randomized-block designs and is given by

$$\Re = 1 - \frac{\delta}{\mu_\delta} \ , \tag{8.3}$$

where μ_δ is the arithmetic average of the M δ values calculated on all possible arrangements of the observed response measurement scores given by

$$\mu_\delta = \frac{1}{M} \sum_{i=1}^{M} \delta_i \ . \tag{8.4}$$

Because μ_δ is a constant under H_0, the permutation distributions of δ and $\Re$ are equivalent, viz.,

$$P\left(\delta \leq \delta_{\text{o}}|H_0\right) = P\left(\Re \geq \Re_{\text{o}}|H_0\right) \ ,$$

where

$$\Re_{\text{o}} = 1 - \frac{\delta_{\text{o}}}{\mu_\delta}$$

and δ_{o} and $\Re_{\text{o}}$ denote the observed values of δ and $\Re$, respectively.

As with the chance-corrected within-group agreement measure presented in Chap. 2, the values of $\Re$ range from negative values to $\Re = +1$ when perfect agreement is achieved, the expected value of $\Re$ is zero under the null hypothesis, and agreement or disagreement is implied by $\Re > 0$ and $\Re < 0$, respectively. While probability values are highly dependent on sample size, this sample-size dependence does not hold for the chance-corrected within-block agreement measure, $\Re$.

8.1.1 Randomized-Block Designs and Alignment

For certain response patterns involving randomized-block designs, the observed test statistic δ_o, as defined in Eq. (8.1) on p. 422, is unable to detect treatment differences [299, pp. 1434–1435]. Such situations occur when the magnitude of the block differences exceeds the magnitude of the treatment differences. For a simple example, consider the univariate response measurement scores listed in Fig. 8.1 with $b = 2$ blocks and $g = 3$ treatments. If $p = 2$ and $v = 1$ in Eq. (8.2) on p. 422, then $\delta_o = 4$ and the random variable δ is also equal to 4 for all permutations of values within blocks; thus, the probability of $\delta_o = 4$ is 1. It is therefore impossible to detect treatment differences.

This problem is rectified by aligning the response measurement scores within each block, a technique initially described by Hodges and Lehmann in 1962 [178]. Alignment is accomplished for the example data in Fig. 8.1 by replacing x_{ij} with $x_{ij} - x_j^*$, where x_j^* is the median of $(x_{ij}, \ldots, x_{gj})$ for $j = 1, \ldots, b$.[2] The observed statistic δ_o is then computed on the aligned data. The median values for Blocks 1 and 2 in Fig. 8.1 are 2 and 6, respectively. If the median value is subtracted from the values in each block, the aligned data are then $y_{111} = 1 - 2 = -1$, $y_{112} = 2 - 2 = 0$, and $y_{113} = 3 - 2 = +1$ for Block 1, and $y_{121} = 5 - 6 = -1$, $y_{122} = 6 - 6 = 0$, and $y_{123} = 7 - 6 = +1$ for Block 2. The median-aligned data are given in Fig. 8.2.

After alignment $\delta_o = 0$ while the random variable δ assumes the values 0.00, 0.67, and 1.33 with respective probability values of 0.1667, 0.3333, and 0.5000, under the null hypothesis (the probability of δ_o is 1/6 after alignment). Note that if $v = 2$ and $r = 1$, the inferential results based on the random variable δ remain unaffected by the alignment.

Fig. 8.1 Example of unaligned data with $g = 3$ treatments, $b = 2$ blocks, and $r = 1$ response measurement

	Treatment		
Block	1	2	3
1	1	2	3
2	5	6	7

Fig. 8.2 Example of aligned data with $g = 3$ treatments, $b = 2$ blocks, and $r = 1$ response measurement

	Treatment		
Block	1	2	3
1	−1	0	+1
2	−1	0	+1

[2] In their 1982 article introducing MRBP, Mielke and Iyer initially suggested using the arithmetic mean instead of the median [299, p. 1435].

8.1.2 Example Univariate MRBP Analysis with $v = 2$

To illustrate an MRBP analysis with univariate response measurement scores and $v = 2$, consider a test of difference between $g = 2$ treatments, where a single response measurement has been obtained from each of $b = 4$ subjects, such as in a matched-pairs experimental design. For this example, there is $r = 1$ response measurement for each subject and $g = 2$ treatments for each of $b = 4$ blocks. The numbers of blocks, treatments, and response measurements are deliberately kept small to simplify the example analysis. The treatments and univariate response measurement scores are listed in Fig. 8.3.

Thus, following Eq. (8.2) on p. 422 for the univariate response measurement scores listed in Fig. 8.3 with $g = 2$, $r = 1$, $b = 4$, $p = 2$, and $v = 2$, the generalized Minkowski distance function yields

$$\Delta(1,2) = \Big[\big| (255 - 171) - (294 - 202) \big|^2 \Big]^{2/2} = 64.00 ,$$

$$\Delta(1,3) = \Big[\big| (255 - 171) - (259 - 247) \big|^2 \Big]^{2/2} = 5{,}184.00 ,$$

$$\Delta(1,4) = \Big[\big| (255 - 171) - (263 - 182) \big|^2 \Big]^{2/2} = 9.00 ,$$

$$\Delta(2,3) = \Big[\big| (294 - 202) - (259 - 247) \big|^2 \Big]^{2/2} = 6{,}400.00 ,$$

$$\Delta(2,4) = \Big[\big| (294 - 202) - (263 - 182) \big|^2 \Big]^{2/2} = 121.00 ,$$

and

$$\Delta(3,4) = \Big[\big| (259 - 247) - (263 - 182) \big|^2 \Big]^{2/2} = 4{,}761.00 .$$

When $r = 1$ and $g = 2$, Eq. (8.1) on p. 422 reduces to

$$\delta = \binom{b}{2}^{-1} \sum_{j<k} \Delta(x_j - x_k) . \tag{8.5}$$

Fig. 8.3 Example univariate data with $g = 2$ treatments, $b = 4$ blocks, and $r = 1$ response measurement

	Treatment	
Block	1	2
1	255	171
2	294	202
3	259	247
4	263	182

Then,

$$\delta = \binom{b}{2}^{-1} \Big[\Delta(1,2) + \Delta(1,3) + \Delta(1,4) + \Delta(2,3) + \Delta(2,4) + \Delta(3,4)\Big]$$

and the observed value of the MRBP test statistic with $v = 2$ is

$$\delta_{\mathrm{o}} = \binom{4}{2}^{-1} \big(64.00 + 5{,}184.00 + 9.00 + 6{,}400.00 + 121.00 + 4{,}761.00\big)$$

$$= \frac{1}{6}\big(16{,}539.00\big) = 2{,}756.50 \,.$$

Let δ_1 denote the MRBP test statistic for a matched-pairs t test with b blocks, $g = 2$ treatments, and $v = 2$, and let δ_2 denote the MRPP test statistic for a two-sample t test with $g = 2$ treatments, $v = 2$, $n_1 = n_2$, $C_1 = (n_1 - 1)/(N - g)$, and $C_2 = (n_2 - 1)/(N - g)$, where n_1 and n_2 denote the number of objects in treatments 1 and 2, respectively, and $N = n_1 + n_2$. Then the relationship between δ_1 and δ_2 is given by

$$\delta_1 = 2\left(\delta_2 - r_{12}\sqrt{\xi_1 \xi_2}\right) \,, \tag{8.6}$$

where ξ_i, $i = 1, 2$, are the average distance-function values for treatments 1 and 2, respectively, and r_{12} is the Pearson product-moment correlation coefficient calculated on the response measurement scores in treatments 1 and 2. See Chap. 2, Sect. 2.2 for detailed descriptions of ξ_i, $i = 1, 2$, and δ. For the interval-level response measurement scores listed in Fig. 8.3, the sample variances for treatments 1 and 2 are $s_1^2 = 316.9167$ and $s_2^2 = 1{,}125.6667$, $\xi_1 = 2s_1^2 = 2(316.9167) = 633.8333$, $\xi_2 = 2s_2^2 = 2(1{,}125.6667) = 2{,}251.3333$, $r_{12} = +0.0539$, $C_1 = (n_1 - 1)/(N - g) = (4 - 1)/(8 - 1)$, $C_2 = (n_2 - 1)/(N - g) = (4 - 1)/(8 - 2)$, and

$$\delta_2 = \sum_{i=1}^{g} C_i \xi_i = \frac{4-1}{8-2}(633.8333) + \frac{4-1}{8-2}(2{,}251.3333) = 1{,}442.5833 \,.$$

Then, following Eq. (8.6),

$$\delta_1 = 2\left(\delta_2 - r_{12}\sqrt{\xi_1 \xi_2}\right)$$

$$= 2\Big[1{,}442.5833 - 0.0539\sqrt{(633.8333)(2{,}251.3333)}\Big]$$

$$= 2{,}756.50 \,.$$

The

$$M = \left(g!\right)^{b} = \left(2!\right)^{4} = 16$$

possible, equally-likely arrangements of the observed univariate response measurement scores described in Fig. 8.3 on p. 425 are listed in Table 8.1 and are ordered by the δ values from lowest to highest.

The observed MRBP test statistic, $\delta_o = 2{,}756.50$, obtained from the original arrangement of the $N = 8$ univariate response measurement scores in Treatments 1 and 2,

$$\{255, 294, 259, 263\} \quad \{171, 202, 247, 182\}\,,$$

(Order 1 in Table 8.1) is unusual since 14 of the 16 δ values exceed the observed value of $\delta_o = 2{,}756.50$ and only two values of δ are equal to or less than the observed value.

If all arrangements of the $N = 8$ observed univariate response measurement scores listed in Fig. 8.3 occur with equal chance, the exact probability value of $\delta_o = 2{,}756.50$ computed on the $M = 16$ possible arrangements of the observed response measurement scores with $b = 4$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} = \frac{2}{16} = 0.1250\,.$$

Table 8.1 Permutations of the observed univariate response measurement scores listed in Fig. 8.3 with values for δ based on $v = 2$ ordered from lowest to highest

Order	Treatment 1	Treatment 2	δ
1	{255, 294, 259, 263}	{171, 202, 247, 182}	2,756.50
2	{171, 202, 247, 182}	{255, 294, 259, 263}	2,756.50
3	{255, 294, 247, 263}	{171, 202, 259, 182}	4,812.50
4	{171, 202, 259, 182}	{255, 294, 247, 263}	4,812.50
5	{171, 202, 247, 263}	{255, 294, 259, 182}	12,908.50
6	{255, 294, 259, 182}	{171, 202, 247, 263}	12,908.50
7	{171, 294, 259, 263}	{255, 202, 247, 182}	13,116.50
8	{255, 202, 247, 182}	{171, 294, 259, 263}	13,116.50
9	{255, 202, 259, 263}	{171, 294, 247, 182}	13,612.50
10	{171, 294, 247, 182}	{255, 202, 259, 263}	13,612.50
11	{171, 202, 259, 263}	{255, 294, 247, 182}	13,668.50
12	{255, 294, 247, 182}	{171, 202, 259, 263}	13,668.50
13	{171, 294, 247, 263}	{255, 202, 259, 182}	13,828.50
14	{255, 202, 259, 182}	{171, 294, 247, 263}	13,828.50
15	{255, 202, 247, 263}	{171, 294, 259, 182}	14,196.50
16	{171, 294, 259, 182}	{255, 202, 247, 263}	14,196.50

For comparison, a conventional matched-pairs t test calculated on the $b = 4$ pairs of response measurement scores listed in Fig. 8.3 yields an observed test statistic of $t_o = +3.6229$. Assuming independence and normality, t is approximately distributed as Student's t under the null hypothesis with $b - 1 = 4 - 1 = 3$ degrees of freedom. Under the null hypothesis, the observed value of $t_o = +3.6229$ yields an approximate two-sided probability value of $P = 0.0362$. Note the large difference between the conventional approximate probability value of $P = 0.0362$ and the exact permutation probability value of $P = 0.1250$. Such discrepancies are common when the number of blocks is small, as in this case with $b = 4$.

The total of the $M = 16$ δ values listed in Table 8.1 is 177,800. Thus, following Eq. (8.4) on p. 423, the exact average value of the $M = 16$ δ values listed in Table 8.1 is

$$\mu_\delta = \frac{1}{M}\sum_{i=1}^{M}\delta_i = \frac{1}{16}(177{,}800) = 11{,}112.50 .$$

Following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_o = 1 - \frac{\delta_o}{\mu_\delta} = 1 - \frac{2{,}756.50}{11{,}112.50} = +0.7519 ,$$

indicating approximately 75 % within-block agreement above that expected by chance.[3]

8.1.3 Example Univariate MRBP Analysis with $v = 1$

Because permutation statistical tests are data-dependent, distribution-free, and nonparametric, they require no distributional assumptions and make no estimates of population parameters. Consequently, it is not necessary to set $v = 2$, squaring the response-measurement differences between objects. As with MRPP in Chap. 2, a distance function based on $v = 1$, employing ordinary Euclidean distance between response measurement scores, is an attractive alternative to $v = 2$ as it is a metric, satisfies the triangle inequality, is robust to extreme values, provides an easy-to-understand Euclidean distance between objects, and ensures that the data and analysis spaces are congruent.

[3]The astute reader will have noted that the values of the generalized chance-corrected measure of agreement, $\Re$, are, in general, markedly greater in Chap. 8 than in Chaps. 2–7. Because Chaps. 8–11 analyze randomized-block data, there is less variability to be explained due to the matching of objects or subjects and, therefore, more agreement (less disagreement) between treatments than with the completely randomized designs analyzed in Chaps. 2–7.

To illustrate the computation of the MRBP test statistic with $v = 1$, consider the same finite sample of response measurement scores obtained from the $b = 4$ subjects listed in Fig. 8.3. For these data, there is $r = 1$ response measurement for each subject and $g = 2$ treatments for each of $b = 4$ blocks.

Following Eq. (8.2) on p. 422 for the data listed in Fig. 8.3 with $g = 2$, $r = 1$, $b = 4$, $p = 1$, and $v = 1$, the generalized Minkowski distance function yields

$$\Delta(1,2) = \Big[\big|(255-171)-(294-202)\big|^2\Big]^{1/2} = 8.00\ ,$$

$$\Delta(1,3) = \Big[\big|(255-171)-(259-247)\big|^2\Big]^{1/2} = 72.00\ ,$$

$$\Delta(1,4) = \Big[\big|(255-171)-(263-182)\big|^2\Big]^{1/2} = 3.00\ ,$$

$$\Delta(2,3) = \Big[\big|(294-202)-(259-247)\big|^2\Big]^{1/2} = 80.00\ ,$$

$$\Delta(2,4) = \Big[\big|(294-202)-(263-182)\big|^2\Big]^{1/2} = 11.00\ ,$$

and

$$\Delta(3,4) = \Big[\big|(259-247)-(263-182)\big|^2\Big]^{1/2} = 69.00\ .$$

Then, following Eq. (8.5) on p. 425,

$$\delta = \binom{b}{2}^{-1}\Big[\Delta(1,2)+\Delta(1,3)+\Delta(1,4)+\Delta(2,3)+\Delta(2,4)+\Delta(3,4)\Big]$$

and the observed value of the MRBP test statistic with $v = 1$ is

$$\delta_{\text{o}} = \binom{4}{2}^{-1}\big(8.00+72.00+3.00+80.00+11.00+69.00\big)$$

$$= \frac{1}{6}\big(243.00\big) = 40.50\ .$$

The

$$M = \big(g!\big)^{b} = \big(2!\big)^{4} = 16$$

Table 8.2 Permutations of the observed univariate response measurement scores listed in Fig. 8.3 with values for δ based on $v = 1$ ordered from lowest to highest

Order	Treatment 1	Treatment 2	δ
1	{255, 294, 259, 263}	{171, 202, 247, 182}	40.50
2	{171, 202, 247, 182}	{255, 294, 259, 263}	40.50
3	{255, 294, 247, 263}	{171, 202, 259, 182}	52.50
4	{171, 202, 259, 182}	{255, 294, 247, 263}	52.50
5	{171, 202, 247, 263}	{255, 294, 259, 182}	98.50
6	{255, 294, 259, 182}	{171, 202, 247, 263}	98.50
7	{171, 294, 259, 263}	{255, 202, 247, 182}	99.50
8	{255, 202, 247, 182}	{171, 294, 259, 263}	99.50
9	{255, 202, 259, 263}	{171, 294, 247, 182}	99.50
10	{171, 294, 247, 182}	{255, 202, 259, 263}	99.50
11	{171, 202, 259, 263}	{255, 294, 247, 182}	102.50
12	{255, 294, 247, 182}	{171, 202, 259, 263}	102.50
13	{171, 294, 247, 263}	{255, 202, 259, 182}	103.50
14	{255, 202, 259, 182}	{171, 294, 247, 263}	103.50
15	{255, 202, 247, 263}	{171, 294, 259, 182}	103.50
16	{171, 294, 259, 182}	{255, 202, 247, 263}	103.50

possible, equally-likely arrangements of the observed response measurement scores described in Fig. 8.3 are listed in Table 8.2 and are ordered by the δ values from lowest to highest.

The observed MRBP test statistic, $\delta_o = 40.50$, obtained from the original arrangement of the $N = 8$ univariate response measurement scores in Treatments 1 and 2,

$$\{255, 294, 259, 263\} \quad \{171, 202, 247, 182\} ,$$

(Order 1 in Table 8.2) is unusual since 14 of the 16 δ values exceed the observed value of $\delta_o = 40.50$ and only two values of δ are equal to or less than the observed value.

If all arrangements of the $N = 8$ observed univariate response measurement scores listed in Fig. 8.3 occur with equal chance, the exact probability value of $\delta_o = 40.50$ computed on the $M = 16$ possible arrangements of the observed response measurement scores with $b = 4$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} = \frac{2}{16} = 0.1250 .$$

The fact that both $v = 2$ and $v = 1$ yield the same probability value of $P = 0.1250$ is simply an artifact of the small data set given in Fig. 8.3 and is not, in general, to be expected. No comparison is made with Student's matched-pairs t test as Student's t test is undefined for $v = 1$.

The total of the $M = 16$ δ values listed in Table 8.2 is 1,400. Thus, following Eq. (8.4) on p. 423, the exact average value of the $M = 16$ δ values listed in Table 8.2 is

$$\mu_\delta = \frac{1}{M} \sum_{i=1}^{M} \delta_i = \frac{1}{16}(1{,}400) = 87.50 \,.$$

Following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_{\text{o}} = 1 - \frac{\delta_{\text{o}}}{\mu_\delta} = 1 - \frac{40.50}{87.50} = +0.5371 \,,$$

indicating approximately 54 % within-block agreement above that expected by chance.

8.1.4 Example Bivariate MRBP Analysis with $v = 2$

In this example, bivariate response measurement scores are used for simplicity to demonstrate a multivariate MRBP analysis. Consider a test of difference between $g = 2$ treatments, where bivariate response measurement scores have been obtained from each of $b = 4$ subjects, such as in a matched-pairs experimental design. For this example, there are $r = 2$ response measurement scores for each subject and $g = 2$ treatments for each of $b = 4$ blocks. The number of blocks, treatments, and response measurement scores are deliberately kept small to simplify the example analysis. The treatments and response measurement scores are listed in Fig. 8.4.

Following Eq. (8.2) on p. 422 for the Treatment 1 response measurement scores listed in Fig. 8.4 with $b = 4$ blocks, $r = 2$ response measurements, $p = 2$, and $v = 2$, the generalized Minkowski distance function yields

$$\Delta(1,2) = \Big[|73-59|^2 + |64-57|^2\Big]^{2/2} = \quad 245.00 \,,$$

$$\Delta(1,3) = \Big[|73-46|^2 + |64-35|^2\Big]^{2/2} = 1{,}570.00 \,,$$

Fig. 8.4 Example bivariate response measurement scores with $g = 2$ treatments, $b = 4$ blocks, and $r = 2$ response measurements

	Treatment	
Block	1	2
1	(73, 64)	(23, 47)
2	(59, 57)	(21, 43)
3	(46, 35)	(19, 31)
4	(23, 11)	(16, 28)

$$\Delta(1,4) = \left[|73-23|^2 + |64-11|^2 \right]^{2/2} = 5{,}309.00 \;,$$
$$\Delta(2,3) = \left[|59-46|^2 + |57-35|^2 \right]^{2/2} = 653.00 \;,$$
$$\Delta(2,4) = \left[|59-23|^2 + |57-11|^2 \right]^{2/2} = 3{,}412.00 \;,$$

and

$$\Delta(3,4) = \left[|46-23|^2 + |35-11|^2 \right]^{2/2} = 1{,}105.00 \;,$$

and for the Treatment 2 response measurement scores listed in Fig. 8.4, the generalized Minkowski distance function yields

$$\Delta(1,2) = \left[|23-21|^2 + |47-43|^2 \right]^{2/2} = 20.00 \;,$$
$$\Delta(1,3) = \left[|23-19|^2 + |47-31|^2 \right]^{2/2} = 272.00 \;,$$
$$\Delta(1,4) = \left[|23-16|^2 + |47-28|^2 \right]^{2/2} = 410.00 \;,$$
$$\Delta(2,3) = \left[|21-19|^2 + |43-31|^2 \right]^{2/2} = 148.00 \;,$$
$$\Delta(2,4) = \left[|21-16|^2 + |43-28|^2 \right]^{2/2} = 250.00 \;,$$

and

$$\Delta(3,4) = \left[|19-16|^2 + |31-28|^2 \right]^{2/2} = 18.00 \;.$$

Then, following Eq. (8.1) on p. 422,

$$\delta = \left[g \binom{b}{2} \right]^{-1} \left[\Delta(1,2) + \Delta(1,3) + \cdots + \Delta(2,4) + \Delta(3,4) \right]$$

and the observed value of the MRBP test statistic with $v = 2$ is

$$\delta_{\text{o}} = \left[2 \binom{4}{2} \right]^{-1} \left(245.00 + 1{,}570.00 + \cdots + 250.00 + 18.00 \right)$$
$$= \frac{1}{12} \left(13{,}412.00 \right) = 1{,}117.6667 \;.$$

In permutation analyses of randomized-block designs it is not always necessary to enumerate all

$$M = \left(g!\right)^{b}$$

possible, equally-likely arrangements of the observed data. It is obvious from a close inspection of Tables 8.1 and 8.2 that half of the arrangements are redundant, yielding duplicate δ values. Considerable savings in computing time can be achieved by eliminating the redundancy and computing only the

$$M = \left(g!\right)^{b-1} = \left(2!\right)^{4-1} = 8$$

non-redundant arrangements of the observed data.[4]

The $M = 8$ non-redundant, equally-likely arrangements of the observed response measurement scores described in Fig. 8.4 are listed in Table 8.3 and are ordered by the δ values from lowest to highest.

The observed MRBP test statistic, $\delta_{\mathrm{o}} = 1{,}117.6667$, obtained from the original arrangement of the $N = 8$ bivariate response measurement scores in Treatments 1 and 2,

$$\{(73, 64)(59, 57)(46, 35)(23, 11)\} \quad \{(23, 47)(21, 43)(19, 31)(16, 28)\}\,,$$

(Order 1 in Table 8.3) is unusual since seven of the eight δ values exceed the observed δ_{o} value of 1,117.6667 and only one δ value is equal to or less than the observed value.

If all non-redundant arrangements of the $N = 8$ observed bivariate response measurement scores listed in Fig. 8.4 occur with equal chance, the exact probability value of $\delta_{\mathrm{o}} = 1{,}117.6667$ computed on the $M = 8$ arrangements of the observed

Table 8.3 Permutations of the observed bivariate data listed in Fig. 8.4 with values for δ based on $v = 2$ ordered from lowest to highest

Order	Treatment 1	Treatment 2	δ
1	{(73, 64)(59, 57)(46, 35)(23, 11)}	{(23, 47)(21, 43)(19, 31)(16, 28)}	1,117.6667
2	{(73, 64)(59, 57)(46, 35)(16, 28)}	{(23, 47)(21, 43)(19, 31)(23, 11)}	1,152.6667
3	{(73, 64)(59, 57)(19, 31)(16, 28)}	{(23, 47)(21, 43)(46, 35)(23, 11)}	1,549.1667
4	{(73, 64)(59, 57)(19, 31)(23, 11)}	{(23, 47)(21, 43)(46, 35)(16, 28)}	1,554.5000
5	{(73, 64)(21, 43)(46, 35)(23, 11)}	{(23, 47)(59, 57)(19, 31)(16, 28)}	1,659.0000
6	{(73, 64)(21, 43)(46, 35)(16, 28)}	{(23, 47)(59, 57)(19, 31)(23, 11)}	1,684.6667
7	{(73, 64)(21, 43)(19, 31)(16, 28)}	{(23, 47)(59, 57)(46, 35)(23, 11)}	1,720.5000
8	{(73, 64)(21, 43)(19, 31)(23, 11)}	{(23, 47)(59, 57)(46, 35)(16, 28)}	1,735.1667

[4]This was a simplification used as far back as 1933 by Eden and Yates in their randomized-block analysis of Yeoman II wheat shoots [103].

response measurement scores with $b = 4$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_{\rm o}|H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_{\rm o}}{M} = \frac{1}{8} = 0.1250 \,.$$

The total of the $M = 8$ δ values listed in Table 8.3 is 12,173.3333. Thus, following Eq. (8.4) on p. 423, the exact average value of the $M = 8$ δ values listed in Table 8.3 is

$$\mu_\delta = \frac{1}{M}\sum_{i=1}^{M}\delta_i = \frac{1}{8}\,(12{,}173.3333) = 1{,}521.6667 \,.$$

Following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_{\rm o} = 1 - \frac{\delta_{\rm o}}{\mu_\delta} = 1 - \frac{1{,}117.6667}{1{,}521.6667} = +0.2655 \,,$$

indicating approximately 27 % within-block agreement above that expected by chance.

8.1.5 Example Bivariate MRBP Analysis with $v = 1$

As explained in previous examples, there is no need to square differences when employing permutation tests. To illustrate the computation of MRBP with bivariate response measurement scores and $v = 1$, employing ordinary Euclidean distance instead of squared Euclidean distance between response measurement scores, consider the same finite sample of $b = 4$ subjects listed in Fig. 8.4.

Following Eq. (8.2) on p. 422 for the Treatment 1 response measurement scores listed in Fig. 8.4 with $g = 2$ treatments, $b = 4$ blocks, $r = 2$ response measurements, $p = 2$, and $v = 1$, the generalized Minkowski distance function yields

$$\Delta(1,2) = \left[|73-59|^2 + |64-57|^2\right]^{1/2} = 15.6525 \,,$$

$$\Delta(1,3) = \left[|73-46|^2 + |64-35|^2\right]^{1/2} = 39.6232 \,,$$

$$\Delta(1,4) = \left[|73-23|^2 + |64-11|^2\right]^{1/2} = 72.8629 \,,$$

$$\Delta(2,3) = \left[|59-46|^2 + |57-35|^2\right]^{1/2} = 25.5539 \,,$$

$$\Delta(2,4) = \left[|59-23|^2 + |57-11|^2\right]^{1/2} = 58.4123 \,,$$

and

$$\Delta(3,4) = \left[|46-23|^2 + |35-11|^2\right]^{1/2} = 33.2415\ ,$$

and for the Treatment 2 response measurement scores listed in Fig. 8.4, the generalized Minkowski distance function yields

$$\Delta(1,2) = \left[|23-21|^2 + |47-43|^2\right]^{1/2} = \ \ 4.4721\ ,$$

$$\Delta(1,3) = \left[|23-19|^2 + |47-31|^2\right]^{1/2} = 16.4924\ ,$$

$$\Delta(1,4) = \left[|23-16|^2 + |47-28|^2\right]^{1/2} = 20.2485\ ,$$

$$\Delta(2,3) = \left[|21-19|^2 + |43-31|^2\right]^{1/2} = 12.1655\ ,$$

$$\Delta(2,4) = \left[|21-16|^2 + |43-28|^2\right]^{1/2} = 15.8114\ ,$$

and

$$\Delta(3,4) = \left[|19-16|^2 + |31-28|^2\right]^{1/2} = \ \ 4.2426\ .$$

Then, following Eq. (8.1) on p. 422,

$$\delta = \left[g\binom{b}{2}\right]^{-1}\left[\Delta(1,2) + \Delta(1,3) + \cdots + \Delta(2,4) + \Delta(3,4)\right]$$

and the observed value of the MRBP test statistic with $v = 1$ is

$$\delta_{\mathrm{o}} = \left[2\binom{4}{2}\right]^{-1}\left(15.6525 + 39.6232 + \cdots + 15.8114 + 4.2426\right)$$
$$= \frac{1}{12}\left(318.7788\right) = 26.5649\ .$$

The

$$M = \left(g!\right)^{b-1} = \left(2!\right)^{4-1} = 8$$

Table 8.4 Permutations of the observed bivariate data listed in Fig. 8.4 with values for δ based on $v = 1$ ordered from lowest to highest

Order	Treatment 1	Treatment 2	δ
1	{(73, 64)(59, 57)(46, 35)(23, 11)}	{(23, 47)(21, 43)(19, 31)(16, 28)}	26.5649
2	{(73, 64)(59, 57)(46, 35)(16, 28)}	{(23, 47)(21, 43)(19, 31)(23, 11)}	29.3755
3	{(73, 64)(59, 57)(19, 31)(23, 11)}	{(23, 47)(21, 43)(46, 35)(16, 28)}	33.4871
4	{(73, 64)(59, 57)(19, 31)(16, 28)}	{(23, 47)(21, 43)(46, 35)(23, 11)}	34.0114
5	{(73, 64)(21, 43)(19, 31)(16, 28)}	{(23, 47)(59, 57)(46, 35)(23, 11)}	36.2929
6	{(73, 64)(21, 43)(46, 35)(23, 11)}	{(23, 47)(59, 57)(19, 31)(16, 28)}	36.5032
7	{(73, 64)(21, 43)(19, 31)(23, 11)}	{(23, 47)(59, 57)(46, 35)(16, 28)}	37.3859
8	{(73, 64)(21, 43)(46, 35)(16, 28)}	{(23, 47)(59, 57)(19, 31)(23, 11)}	37.6965

non-redundant, equally-likely arrangements of the observed response measurement scores described in Fig. 8.4 are listed in Table 8.4 and are ordered by the δ values from lowest to highest.

The observed MRBP test statistic, $\delta_o = 26.5649$, obtained from the original arrangement of the $N = 8$ bivariate response measurement scores in Treatments 1 and 2,

$$\{(73, 64)(59, 57)(46, 35)(23, 11)\} \quad \{(23, 47)(21, 43)(19, 31)(16, 28)\} ,$$

(Order 1 in Table 8.4) is unusual since seven of the eight δ values exceed the observed value of $\delta_o = 26.5649$ and only one δ value is equal to or less than the observed value.

If all non-redundant arrangements of the $N = 8$ observed bivariate response measurement scores listed in Fig. 8.4 occur with equal chance, the exact probability value of $\delta_o = 26.5649$ computed on the $M = 8$ arrangements of the observed response measurement scores with $b = 4$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} = \frac{1}{8} = 0.1250 .$$

The total of the $M = 8$ δ values listed in Table 8.4 is 271.3176. Thus, following Eq. (8.4) on p. 423, the exact average value of the $M = 8$ δ values listed in Table 8.4 is

$$\mu_\delta = \frac{1}{M} \sum_{i=1}^{M} \delta_i = \frac{1}{8} (271.3176) = 33.9147 .$$

Following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_o = 1 - \frac{\delta_o}{\mu_\delta} = 1 - \frac{26.5649}{33.9147} = +0.2167 ,$$

indicating approximately 22 % within-block agreement above that expected by chance.

8.2 MRBP and Pearson's Product-Moment Correlation

It is not readily apparent that the MRBP test statistic, given by

$$\delta = \left[g \binom{b}{2} \right]^{-1} \sum_{i=1}^{g} \sum_{j<k} \Delta(x_{ij}, x_{ik}) \ ,$$

and the ordinary Pearson product-moment correlation coefficient are closely related when $v = 2$. Let R denote the Pearson product-moment correlation coefficient between two interval-level variables, $(x_{11}, \ldots, x_{g1})$ and $(x_{12}, \ldots, x_{g2})$, given by

$$R = \frac{\operatorname{cov}(x_1, x_2)}{s_1 s_2} \ ,$$

where the covariance of variables x_1 and x_2 is given by

$$\operatorname{cov}(x_1, x_2) = \frac{1}{g-1} \sum_{i=1}^{g} (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) \ ,$$

and the means and standard deviations are given by

$$\bar{x}_j = \frac{1}{g} \sum_{i=1}^{g} x_{ij} \quad \text{and} \quad s_j = \frac{1}{g-1} \sum_{i=1}^{g} \left(x_{ij} - \bar{x}_j \right)^2 \ ,$$

respectively, for $j = 1, 2$.

If $v = 2$, $b = 2$, and $r = 1$, then the functional relationships between R and δ are given by

$$R = \frac{\mu_\delta - \delta}{2S_1S_2} \quad \text{and} \quad \delta = \mu_\delta - 2RS_1S_2 \ ,$$

where

$$R = \frac{1}{gS_1S_2} \sum_{i=1}^{g} (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) \ ,$$

$$\mu_\delta = S_1^2 + S_2^2 + (\bar{x}_1 - \bar{x}_2)^2 \ ,$$

$$\bar{x}_j = \frac{1}{g} \sum_{i=1}^{g} x_{ij} \ , \quad \text{and} \quad S_j^2 = \frac{1}{g} \sum_{i=1}^{g} \left(x_{ij} - \bar{x}_j \right)^2 \tag{8.7}$$

for $j = 1, 2$.[5] Thus, R and δ are equivalent under the null hypothesis because $\bar{x}_1$, $\bar{x}_2$, S_1, and S_2 are invariant relative to the M possible permutations of the response measurement scores.

Because R and δ are equivalent under the null hypothesis, the permutation distributions of R and δ are also equivalent when $v = 2$, viz.,

$$P(R \geq R_o | H_0) = P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} ,$$

where $M = g!$ and R_o and δ_o denote the observed values of R and δ, respectively. Finally, the functional relationships between R and $\Re$ are given by

$$R = \frac{\Re \mu_\delta}{2S_1S_2} \quad \text{and} \quad \Re = \frac{2RS_1S_2}{\mu_\delta} .$$

8.2.1 Example MRBP Correlation Analysis

To illustrate the relationship between δ and Pearson's R, consider the univariate response measurement scores listed in Fig. 8.5 with $g = 7$ objects, $b = 2$ blocks, $r = 1$ response measurement, and $v = 2$, employing squared Euclidean distance between response measurement scores to correspond to the Pearson product-moment correlation coefficient. For the univariate response measurement scores listed in Fig. 8.5, $\bar{x}_1 = 2.00$, $\bar{x}_2 = 5.00$, $s_1 = 1.00$, $s_2 = 2.00$,

$$\text{cov}(x_1, x_2) = \frac{1}{7-1}(9.00) = 1.50 ,$$

Fig. 8.5 Example data with $g = 7$ objects, $b = 2$ blocks, and $r = 1$ response measurement

Object	x_1	x_2
1	3	8
2	3	6
3	3	5
4	2	6
5	1	5
6	1	3
7	1	2

[5]Note that the summation for S_j^2 in Eq. (8.7) is divided by g and not by $g - 1$, as degrees of freedom are irrelevant to permutation methods.

and the observed Pearson product-moment correlation coefficient is

$$R_o = \frac{\text{cov}(x_1 x_2)}{s_1 s_2} = \frac{1.50}{(1.00)(2.00)} = +0.75 \; .$$

Equivalently, for the response measurement scores listed in Fig. 8.5, $S_1 = 0.9258$, $S_2 = 1.8516$, $\delta_o = 10.7143$, $\mu_\delta = 13.2857$, and

$$R_o = \frac{\mu_\delta - \delta_o}{2S_1S_2} = \frac{13.2857 - 10.7143}{(2)(0.9258)(1.8516)} = +0.75 \; .$$

Since there are only $M = 7! = 5{,}040$ possible, equally-likely arrangements of the observed response measurement scores listed in Fig. 8.5, an exact solution is feasible. If all arrangements of the $N = 14$ observed response measurement scores listed in Fig. 8.5 occur with equal chance, the exact probability value of $\delta_o = 10.7143$ (or $R_o = +0.75$) computed on the $M = 5{,}040$ possible arrangements of the observed response measurement scores with $b = 2$ blocks preserved for each arrangement is

$$P(\delta \le \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \le \delta_o}{M} = \frac{216}{5{,}040} = 0.0429 \; .$$

For comparison, a conventional test of significance for R is given by

$$t = \left[\frac{(g-2)R^2}{1-R^2} \right]^{1/2}$$

and the observed value of t for $R_o = +0.75$ is

$$t_o = \left[\frac{(7-2)(+0.75)^2}{1-(+0.75)^2} \right]^{1/2} = +2.5355 \; .$$

Assuming independence and normality, t is approximately distributed as Student's t under the null hypothesis with $g - 2 = 7 - 2 = 5$ degrees of freedom. Under the null hypothesis, the observed value of $t_o = +2.5355$ yields an approximate two-sided probability value of $P = 0.0522$.

Also, for the $N = 7$ univariate response measurement scores listed in Fig. 8.5, the exact expected value of the $M = 5{,}040$ δ values is $\mu_\delta = 13.2857$ and following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_o = 1 - \frac{\delta_o}{\mu_\delta} = 1 - \frac{10.7143}{13.2857} = +0.1935 \; ,$$

indicating approximately 19 % within-block agreement above that expected by chance.

Finally, the relationships between the observed values of R_{o} and $\Re_{\mathrm{o}}$ are

$$R_{\mathrm{o}} = \frac{\Re_{\mathrm{o}}\mu_{\delta}}{2S_1S_2} = \frac{(+0.1935)(13.2857)}{2(0.9258)(1.8516)} = +0.75$$

and

$$\Re_{\mathrm{o}} = \frac{2R_{\mathrm{o}}S_1S_2}{\mu_{\delta}} = \frac{2(+0.75)(0.9258)(1.8516)}{13.2857} = +0.1935\ .$$

Analysis with $v = 1$

Although the Pearson product-moment correlation coefficient is not defined for $v = 1$, it is still possible to analyze the data with δ and $\Re$ based on ordinary Euclidean distances between response measurement scores. For the univariate response measurement scores listed in Fig. 8.5 with $g = 7$, $b = 2$, $r = 1$, and $v = 1$, employing ordinary Euclidean distance between response measurement scores, the observed value of the MRBP test statistic is $\delta_{\mathrm{o}} = 3.00$, the exact expected value of the $M = 5{,}040$ δ values is $\mu_{\delta} = 3.1224$, and the observed chance-corrected measure of effect size is

$$\Re_{\mathrm{o}} = 1 - \frac{\delta_{\mathrm{o}}}{\mu_{\delta}} = 1 - \frac{3.00}{3.1224} = +0.0392\ ,$$

indicating approximately chance within-block agreement. If all arrangements of the $N = 14$ observed response measurement scores listed in Fig. 8.5 occur with equal chance, the exact probability value of $\delta_{\mathrm{o}} = 3.00$ computed on the $M = 5{,}040$ possible arrangements of the observed response measurement scores with $b = 2$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_{\mathrm{o}}|H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_{\mathrm{o}}}{M} = \frac{216}{5{,}040} = 0.0429\ ,$$

which is the same as the probability value obtained with $v = 2$.

8.2.2 Permutations of g Response Measurements

If $(x_{11}, \ldots, x_{g1})$ is one of the $g!$ permutations of the observed response measurement scores and $v = 2$, then $\mu_{\delta} = 2S_1S_2$ and the Pearson product-moment correlation coefficient, R, is equivalent to the chance-corrected within-block measure of effect size, $\Re$, i.e., $R = \Re$, where

$$R = \frac{\mathrm{cov}(x_1, x_2)}{s_1 s_2} \quad \text{and} \quad \Re = 1 - \frac{\delta}{\mu_{\delta}}$$

Fig. 8.6 Example data set with $g = 10$ objects, $b = 2$ blocks, and $r = 1$ response measurement

Object	x_1	x_2
1	21	32
2	27	21
3	32	27
4	35	35
5	43	64
6	47	43
7	50	50
8	58	47
9	64	69
10	69	58

[297, pp. 132–133]. To illustrate the equivalence of Pearson's R and $\Re$ when $(x_{12}, \ldots, x_{g2})$ is a permutation of $(x_{11}, \ldots, x_{g1})$, consider the small data set listed in Fig. 8.6 with $g = 10$ objects, $r = 1$ response measurement, and $b = 2$ blocks. For these data the 10 response measurement scores listed under x_2 in Fig. 8.6 constitute a permutation of the 10 response measurement scores listed under x_1.

For the response measurement scores listed in Fig. 8.6, $\bar{x}_1 = \bar{x}_2 = 44.60$, $s_1 = s_2 = 16.0083$,

$$\mathrm{cov}(x_1, x_2) = \frac{1}{10-1}(1{,}853.4000) = 205.9333 ,$$

and the observed Pearson product-moment correlation coefficient is

$$R_{\mathrm{o}} = \frac{\mathrm{cov}(x_1, x_2)}{s_1 s_2} = \frac{205.9333}{(16.0083)(16.0083)} = +0.8036 .$$

Equivalently, for the response measurement scores listed in Fig. 8.6, the observed value of the MRBP test statistic with $v = 2$ is $\delta_{\mathrm{o}} = 90.60$, the exact expected value of the M δ values is $\mu_\delta = 461.2800$ and, following Eq. (8.3) on p. 423, the observed chance-corrected measure of effect size is

$$\Re_{\mathrm{o}} = 1 - \frac{\delta_{\mathrm{o}}}{\mu_\delta} = 1 - \frac{90.60}{461.2800} = +0.8036 ,$$

indicating approximately 80 % within-block agreement above that expected by chance.

Since there are $M = 10! = 3{,}628{,}800$ possible, equally-likely arrangements of the observed response measurement scores listed in Fig. 8.6, calculation of an exact probability value is prohibitive and an approximate resampling probability value is more practical. For the univariate response measurement scores listed in Fig. 8.6, the approximate resampling probability value of $\delta_{\mathrm{o}} = 90.60$ computed on $L = 1{,}000{,}000$ random arrangements of the observed response measurement scores is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{L} = \frac{3{,}217}{1{,}000{,}000} = 0.0032 \; .$$

While an exact solution may not be practical, it is not unrealistic, given current computer capabilities. For the response measurement scores listed in Fig. 8.6, the exact probability value of $\delta_o = 90.60$ computed on the $M = 3{,}628{,}800$ possible arrangements of the observed response measurement scores is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} = \frac{11{,}780}{3{,}628{,}800} = 0.0032 \; .$$

Analysis with $v = 1$

Although the Pearson product-moment correlation coefficient is not defined for $v = 1$, it is still possible to analyze the response measurement scores listed in Fig. 8.6 with δ and $\Re$ based on ordinary Euclidean distances between response measurement scores. For the response measurement scores listed in Fig. 8.6 with $g = 10$, $b = 2$, $r = 1$, and $v = 1$, the observed value of δ with $v = 1$ is $\delta_o = 7.40$, the exact expected value of the $M = 3{,}628{,}800$ δ values is $\mu_\delta = 17.40$ and the observed chance-corrected measure of effect size is

$$\Re_o = 1 - \frac{\delta_o}{\mu_\delta} = 1 - \frac{7.40}{17.40} = +0.5747 \; ,$$

indicating approximately 57 % within-block agreement above that expected by chance. If all M possible arrangements of the $N = 20$ observed response measurement scores listed in Fig. 8.6 occur with equal chance, the approximate resampling probability value of $\delta_o = 7.40$ computed on the $L = 1{,}000{,}000$ random arrangements of the observed response measurement scores with $b = 2$ blocks preserved for each arrangement is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{L} = \frac{5{,}192}{1{,}000{,}000} = 0.0052 \; .$$

For comparison, the exact probability value of $\delta_o = 7.40$ computed on the $M = 3{,}628{,}800$ possible arrangements of the observed response measurement scores is

$$P(\delta \leq \delta_o | H_0) = \frac{\text{number of } \delta \text{ values} \leq \delta_o}{M} = \frac{18{,}669}{3{,}628{,}800} = 0.0051 \; .$$

Finally, it should be noted that if $g = 2$, $r = 1$, $v = 2$, $x_{1j} = -x_{2j} = x_j$, and $|x_j| > 0$ for $j = 1, \ldots, b$, then the test based on δ is equivalent to the permutation version of either the matched-pairs or one-sample t test. When $v = 1$, $\Re$ possesses certain advantages over R; viz., $\Re$ is a measure of chance-corrected agreement rather than a measure of linearity, and second, $\Re$ is much more robust than R since it is based on ordinary Euclidean distances rather than squared Euclidean distances.

8.3 Coda

Chapter 8 provided the foundation for Multivariate Randomized Block Permutation (MRBP) procedures, with special emphasis on the generalized Minkowski distance function, $\Delta(x, y)$, as defined in Eq. (8.2) on p. 422; δ, the weighted mean of the specified distance-function values as defined in Eq. (8.1) on p. 422; and $\Re$, the chance-corrected within-block coefficient of agreement, as defined in Eq. (8.3) on p. 423. Chapters 9, 10, and 11 provide applications of MRBP to randomized-block data at the interval, ordinal, and nominal levels of measurement, respectively.

Chapter 9

Chapter 9 establishes the relationships between the MRBP test statistics, δ and $\Re$, and selected conventional tests and measures designed for the analysis of randomized-block data at the interval level of measurement. Considered in Chap. 9 are Student's t test for matched pairs, Hotelling's multivariate T^2 test for matched pairs, randomized-block analysis of variance, randomized-block multivariate analysis of variance, and Pearson's product-moment correlation coefficient.