Why Linear? An Illustration Using a Geometric Model of Quantum Interaction

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Abstract. Quantum mechanics is a linear theory. This is a strange fact. Why should nature be so convenient? Perhaps linearity is simply a consequence of small perturbations against a relatively uniform background. This can be formalized within the mathematical notions of *weak solution* and *linearization*. The true picture is unlikely to be so and some more appropriate framework may be required to deal with theories where this uniformity is lost, such as for example quantum gravity. A geometric model of quantum interaction provides a useful illustration. In this case the equations are quadratic, but both their weak form and linearization leads to a striking analogue of the Schrödinger equation.

1 Introduction

The formalism of quantum mechanics (QM) proceeds as follows. To a physical system one associates a separable Hilbert space over the complex numbers. States of the system correspond to rays in this space. An observable corresponds to a Hermitian operator whose eigenvalues are the only possible quantities that may be observed. Once an observation is made, the system collapses onto the corresponding eigenstate via a projection operator. Changes of state (not involving observation) are given by unitary transformations, which, when applied in a continuous setting lead to the (time dependent) Schrödinger equation. It is remarkable that nature should proceed in this way. Or is there perhaps a deeper underlying phenomenon, by which non-linear laws are perceived to be linear by dint of the contextual nature of the process? We generally make observations with respect to a relatively stable macroscopic background. But then the entire process consisting of the measuring apparatus, the environment both near and far and the system to be measured, may well not obey this convenient law.

The mathematics of partial differential equations exploits the notion of *linearization*. This is particularly useful for showing local existence, the Einstein equations being an important example in general realtivity. If φ is a function of several variables and $P(\varphi) = 0$ is a partial differential equation, where P is some operator that involves higher order derivatives; then one may calculate the

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expression $\frac{d}{d\epsilon}P(\varphi + \epsilon\psi)|_{\epsilon=0}$ to obtain a linear equation in ψ whose coefficients depend on φ. From a physical point of view, it is natural to consider φ as a fixed background with ψ a local perturbation and as such we would expect a local expression of the law $P \equiv 0$ to be given by some linear approximation.

An alternative approach is to study the weak form of the equations, whereby, using a suitable Hilbert space, we couple $P(\varphi)$ with some arbitray function ξ to form the inner product $\langle P(\varphi), \xi\rangle$, where derivatives are shifted across onto ξ. By choosing ξ appropriately, as we show below, one may also be led to a linear equation.

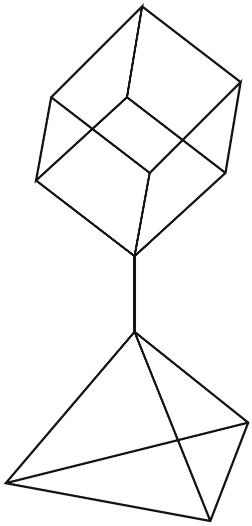
Quantum interaction describes macroscopic phenomema through quantum models, so we should be prepared to encompass non-linear laws into the theory, while retaining the essential quantum features of choice (uncertainty of measurement) and state collapse. Linear approximations should then arise when the process is occuring within a stable "flat" environement, such as our own ingrained concept of space. The term *flat* here should be interpreted as the vanishing of some intrinsic curvature. In the combinatorial setting of the model we discuss below, notions of curvature are developed in [5].

The author proposes a model of quantum interaction based on our perception of 3D objects. An approximation of an object is given by a combinatorial graph and its possible perceived states by a complex valued function on the vertices which satisfies a quadratic difference equation. This equation contains a real-valued parameter which we view as its spectrum: only certain values of the parameter allow for a non-trivial solution. Depending on the graph, these values may be discrete or continuous. For a given element of the spectrum, there is a linear freedom in the solution, so we may apply some normalization. A similar formalism to quantum mechanics can now be applied. Until an observation is made, the graph is not in any state; once an observation is made (correlation between systems takes place), it collapses into a geometric state. However, the quadratic nature of the equations means linear superposition of solutions is generally forbidden. In this article, we deduce a weak form of the equations, as well as apply linearization, to obtain a linear version on some fixed background. The resulting equation is remarkably similar to the Schrödinger equation.

2 A Geometric Model

The geometric model has been discussed in detail in a number of references, see [1–4], so we give an abridged version here. Our impression from the illustration below is of a framework in Euclidean 3-space. We see a tetrahedron attached to a cube by a rigid bar, rather than just lines on a piece of paper.

In fact there are potentially four apparent configurations depending upon whether the edges that cross are perceived to be above or below each other, with a consistency requirement for the two pairs of crossing edges of the cube. Our eyes may rove across the picture and flip between the different 3D–realizations, although it is quite hard to simultaneously concentrate on both the tetrahedron and the cube. Once we have fixed on a configuration, it is likely to persist until we make a significant mental effort to change it. However, let us remind ourselves that the physical reality consists of a collection of vertices joined by edges drawn on a flat piece of paper. So how do we get from the drawing to a 3D-realization?

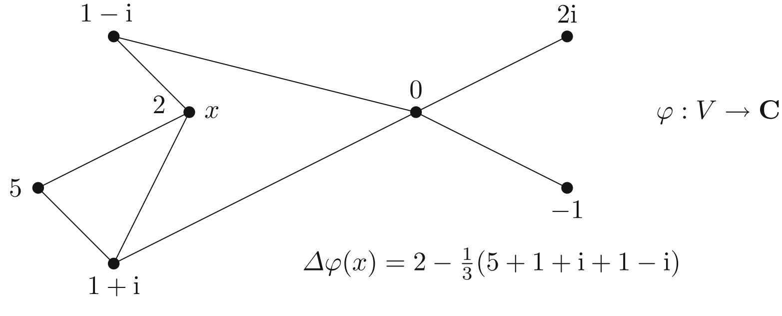


Let $\Gamma = (V, E)$ be a finite simple graph which may or may not be connected, with vertex set V and edge set E. Simple means there are no loops or multiple edges. For vertices x and y, we use the notation $x \sim y$ to indicate that x and y are joined by an edge and we let $n(x)$ denote the degree of vertex x, that is, the number of edges incident with x.

The Laplacian of a function $\varphi : V \to \mathbf{C}$ is defined by the mean-value property:

$$\Delta\varphi(x) := \frac{1}{n(x)} \sum_{y \sim x} (\varphi(x) - \varphi(y))$$

where $n(x)$ is the degree of vertex $x \in V$ (the number of edges incident with x). Although this appears to be a first order operator, formally, as we explain in Sect. 3, it arises as the negative of the coderivative of the derivative, which is its usual definition in smooth geometry.



The symmetric square of the derivative of φ is defined by (cf. Sect. 3):

$$(\nabla\varphi)^2(x) := \frac{1}{n(x)} \sum_{y\sim x} \big(\varphi(y) - \varphi(x)\big)^2.$$

Given a real-valued function $\gamma : V \to \mathbf{R}$, consider the equation

$$\frac{\gamma(x)}{n(x)} \Big(\sum_{y\sim x} \big(\varphi(y) - \varphi(x)\big)\Big)^2 = \sum_{y\sim x} \big(\varphi(x) - \varphi(y)\big)^2, \tag{1}$$

at each vertex $x \in V$, where $\varphi : V \to \mathbf{C}$ is a complex-valued function. In more economical form, this can be expressed as

$$\gamma(x)(\Delta\varphi(x))^2 = (\nabla\varphi)^2(x).$$

Clearly $\varphi =$ constant is a particular solution which we call trivial. Any real-valued φ will also give solutions, so we also impose the requirement that $\gamma < 1$, which, by the Cauchy-Schwarz inequality, means that any non-constant solution φ cannot be real on any vertex and all of its neighbours. The equation may or may not admit non-trivial solutions, depending on the function γ.

By a *framework* in Euclidean space, we mean a graph that is realized as a subset of Euclidean space with edges straight line segments joining the vertices. The framework is called *invariant* if, for a particular γ, it satisfies (1) with φ the restriction to the vertices of some orthogonal projection to the complex plane *independently of any similarity transformation of the framework.*

The underlying framework of a regular polytope is always invariant. For example, the restriction of any orthogonal projection φ to the underlying frameworks of the convex regular polytopes in Euclidean 3-space as well as the 4D 600-cell satisfies (1) with γ constant as given in the following table:

Polytope	γ
Tetrahedron	3/4
Cube	0
Octahedron	1/2
Icosahedron	$\frac{2-\sqrt{5}}{3-\sqrt{5}} < 0$
Dodecahedron	$\frac{3(1-\sqrt{5})}{2(3-\sqrt{5})} < 0$
600-cell	$\frac{5(1-2\sqrt{5})}{3} < 0$

There are many other examples of invariant frameworks [3]. The figure depicted above also satisfies (1) invariantly provided the bar joining the tetrahedron with the cube extends through the centre of mass of each of these objects.

It is important to note that the invariance property means that however we scale, translate, rotate or reflect the framework (a similarity transformation), it still satisfies (1) with γ unchanged. Scale invariance and translation invariance corresponds to the freedom $\varphi \mapsto \lambda\varphi + \mu$ ($\lambda, \mu \in \mathbf{C}$) of a solution to (1), whereas reflection invariance corresponds to the freedom of taking the complex conjugate: $\varphi \mapsto \overline{\varphi}$. Rotation invariance, when it occurs, is more subtle and requires an algebraic analysis of the equations.

For an invariant framework, the geometry is inherent in the underlying combinatorial structure and the function γ and does not depend on any embedding in Euclidean space. To see this, one should be able to reconstruct the framework from the given information, which is always locally possible provided $\gamma < 1$ [3]. A global reconstruction requires additional information such as an edge colouring of the underlying graph, see [5].

For a given graph, we would like to know what are the admissible functions $\gamma : V \to \mathbf{R}$ for which (1) has a solution. Define the *geometric spectrum* of Γ to be the collection of equivalence classes of such functions:

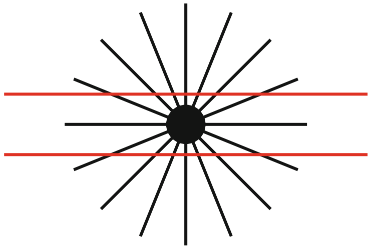
$$\Sigma = \{\gamma : V \to [-\infty, 1) \subset \mathbf{R} : \exists \text{ non} - \text{const } \varphi : V \to \mathbf{C} \text{ satisfying } (1)\},$$

where two functions are identified when they determine a common solution φ and agree on the compliment of the set $\{x \in V : \Delta\varphi(x) = (\nabla\varphi)^2(x) = 0\}$. We allow γ to take on the value $-\infty$ at points where the Laplacian vanishes. It is our contention that the geometric spectrum provides an analogue of the spectrum of a Hermitian operator and that the corresponding family of solutions to (1) together with their liftings into Euclidean 3-space (*geometric states*), an analogue of eigenstate in QM.

This point of view is put forward in the article [4] from QI-2014, where striking similarities are noted between the geometric states of the Necker cube and a spin 1/2 particle, particularly in respect of decoherence.

In [1,2], a model is proposed in which graphs can interact by falling into compatible geometric states to form a new state. This would seem to mirror the way in which perception translates physical data into "meaningful" images: our brain-system correlates with a configuration of lines on a piece of paper to produce compatible 3D-geometric states. However, to date, no reasonable model has been constructed to describe evolution of a geometric model, in a similar way to which unitary evolution models change in QM. We take the first steps here by first deriving the weak form of the Eq. (1) and then by applying the technique of linearization in a discrete setting. This requires a novel approach involving *symmetric* complex calculus, rather than the usual *Hermitian* forms used in QM. However, Hermitian forms still come into play when solutions are normalized.

Background is an essential factor in how we perceive objects as optical illusions demonstrate. The parallel lines below appear to bend apart due to the suggestion of a spherical background.



We propose that this is a phenomenon that can be formalized within the mathematical framework below[1]. Specifically, in order to interpret our geometric model, we introduce a fixed background with which the geometric state interacts. This can now be introduced into the equations via either *the weak form* or *linearization*. Both approaches lead to a linear Schrödinger-type equation that may now provide a reasonable framework of quantum interaction for the geometric model.

3 The Linearized and Weak Forms of the Equations

Let $\Gamma = (V, E)$ be a finite simple graph with vertex set V and edge set E, where elements e of E are expressed as unordered couples $e = xy$ for $x, y \in V$. For $x \in V$ define the *tangent space to Γ at x* to be the set of oriented edges with base point x together with an element $\overrightarrow{0}$: $T_x\Gamma = \{\overrightarrow{xy} : y \sim x\} \cup \{\overrightarrow{0}\}$. Define the *tangent bundle to Γ* to be the union: $T\Gamma = \cup_{x \in V} T_x\Gamma$. Then a 1-*form* on Γ is a map $\omega : T\Gamma \to \mathbf{C}$ such that $\omega(\overrightarrow{xy}) = -\omega(\overrightarrow{yx})$ and $\omega(\overrightarrow{0}) = 0$. To a function $\varphi : V \to \mathbf{C}$, we can naturally associate a 1-form, the *derivate* $\mathrm{d}\varphi$, by $\mathrm{d}\varphi(\overrightarrow{xy}) = \varphi(y) - \varphi(x)$.

For two 1-forms ω, η, define their *pointwise symmetric product at $x \in V$* by

$$\langle \omega, \eta \rangle_x = \sum_{y \sim x} \omega(\overrightarrow{xy})\eta(\overrightarrow{xy}),$$

and their *(global) symmetric product* by

$$(\omega, \eta) = \sum_{e \in E} \omega(e)\eta(e) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega(\overrightarrow{xy})\eta(\overrightarrow{xy}).$$

Note that in the first sum the 1-forms act on unoriented edges so that only their product is well-defined; the factor of one half occurs in the second sum, since there, unoriented edges are counted twice.

For functions $\varphi, \psi : V \to \mathbf{C}$, define their *(global) symmetric product* by

$$(\varphi, \psi) = \sum_{x \in V} n(x)\varphi(x)\psi(x),$$

[1] Another viable explanation for these kinds of optical illusions could be contextual emergence, see for example [7].

where $n(x)$ is the degree of vertex x.

The above definitions are the complex symmetric analogues of standard L^2 products that arise in functional analytic theory on a graph; in the latter situation they are replaced by Hermitian products rather than symmetric products [8].

Given a function $\xi : V \to \mathbf{C}$ and a 1-form $\omega : T\Gamma \to \mathbf{C}$, we can define a new 1-form $\xi\omega$ by

$$(\xi\omega)(\overrightarrow{xy}) = \frac{1}{2}(\xi(x) + \xi(y))\omega(\overrightarrow{xy}).$$

Then it is easily checked that

$$\mathrm{d}(\varphi\psi) = \varphi \mathrm{d}\psi + \psi \mathrm{d}\varphi.$$

Given a 1-form ω, define its *co-deriviative* $\mathrm{d}^*\omega$ to be the function which at each vertex $x \in V$ is given by

$$\mathrm{d}^*\omega(x) = -\frac{1}{n(x)} \sum_{y \sim x} \omega(\overrightarrow{xy}).$$

Then for a function $\varphi : V \to \mathbf{C}$,

$$\mathrm{d}^*\mathrm{d}\varphi = -\frac{1}{n(x)} \sum_{y \sim x} \mathrm{d}\varphi(\overrightarrow{xy}) = -\frac{1}{n(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x)) = -\Delta\varphi.$$

Lemma 1. *Let $\varphi, \psi : V \to \mathbf{C}$ be functions and $\omega : T\Gamma \to \mathbf{C}$ a 1-form. Then the following formulae hold:*

(i) $(\mathrm{d}\varphi, \omega) = (\varphi, \mathrm{d}^*\omega)$ *;*
(ii) $(\Delta\varphi, \psi) = -(\mathrm{d}\varphi, \mathrm{d}\psi)$ *;*
(iii) $(\Delta\varphi, \psi) = (\varphi, \Delta\psi)$ *;*
(iv) $\mathrm{d}^*(\varphi\omega)(x) = \varphi(x)\mathrm{d}^*\omega(x) - \frac{1}{2n(x)}\langle \mathrm{d}\varphi, \omega\rangle_x$ *for each $x \in V$.*

Proof. To prove (i), we notice that for $x \sim y$, the sum

$$(\mathrm{d}\varphi, \omega) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} (\varphi(y) - \varphi(x))\omega(\overrightarrow{xy})$$

contributes $(\varphi(y) - \varphi(x))\omega(\overrightarrow{xy}) = -\varphi(x)\omega(\overrightarrow{xy}) - \varphi(y)\omega(\overrightarrow{yx})$ (the term being symmetric in x and y), which equates to the corresponding terms in the sum

$$(\varphi, \mathrm{d}^*\omega) = -\sum_{x \in V} \varphi(x) \sum_{y \sim x} \omega(\overrightarrow{xy}).$$

The identity (ii) now follows from the fact that $\mathrm{d}^*\mathrm{d}\varphi = -\Delta\varphi$. Identity (iii) follows from (ii), by symmetry. Finally

$$\mathrm{d}^*(\varphi\omega)(x) = -\frac{1}{n(x)} \sum_{y \sim x} (\varphi\omega)(\overrightarrow{xy})$$

$$= -\frac{1}{2n(x)} \sum_{y \sim x} (\varphi(x) + \varphi(y))\omega(\overrightarrow{xy})$$
$$= \varphi(x)\mathrm{d}^*\omega(x) - \frac{1}{2n(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x))\omega(\overrightarrow{xy})$$
$$= \varphi(x)\mathrm{d}^*\omega(x) - \frac{1}{2n(x)} \langle \mathrm{d}\varphi, \omega \rangle_x,$$

which gives (iv).

With the above notation and formulae established, we can give a weak form of Eq. (1).

Proposition 1. *The Eq.* (1) *holds if and only if*

$$(\Delta\varphi, \xi\gamma\Delta\varphi) - 2(\mathrm{d}\varphi, n\xi\mathrm{d}\varphi) = 0, \tag{2}$$

for any function $\xi : V \to \mathbf{C}$.

Proof. Let $\xi : V \to \mathbf{C}$. Then if (1) holds, we have:

$$\sum_{x \in V} n(x)\xi(x) \left(\gamma(x)(\Delta\varphi(x))^2 - \mathrm{d}\varphi(x)^2\right) = 0,$$

equivalently

$$(\Delta\varphi, \xi\gamma\Delta\varphi) - (\mathrm{d}\varphi^2, \xi) = 0,$$

where we recall that $\mathrm{d}\varphi^2(x) = \sum_{y \sim x}(\varphi(y) - \varphi(x))^2 = \langle \mathrm{d}\varphi, \mathrm{d}\varphi \rangle_x$. But we claim that $(\mathrm{d}\varphi^2, \xi) = 2(\mathrm{d}\varphi, \xi\mathrm{d}\varphi)$. Indeed,

$$(\mathrm{d}\varphi^2, \xi) = \sum_{x \in V} n(x)\xi(x) \left(\sum_{y \sim x} (\varphi(y) - \varphi(x))^2 \right),$$

which, for each $x \sim y$, contributes the term $(n(x)\xi(x) + n(y)\xi(y))(\varphi(y) - \varphi(x))^2$. But precisely this term occurs within

$$2(\mathrm{d}\varphi, n\xi\mathrm{d}\varphi) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} (n(x)\xi(x) + n(y)\xi(y))(\varphi(y) - \varphi(x))^2.$$

(on noting the symmetry of the expression to be summed on the right-hand side).

Conversely, if we fix a vertex $x \in V$ and consider the function $\xi : V \to \mathbf{C}$ given by $\xi(x) = 1$ and $\xi(y) = 0$ for all $y \neq x$, then, recalling the factor of $1/2$ in the inner product of forms, (2) gives $\gamma(x)\Delta\varphi(x)^2 - \mathrm{d}\varphi^2(x) = 0$, so that (1) holds.

On taking the function ξ to be identically equal to 1, we obtain the following consequence.

Corollary 1. *Let* $\varphi : V \to \mathbf{C}$ *be a solution to Eq.* (1) *with* γ *constant. Then*

$$\left(\gamma\Delta(\Delta\varphi) + 2n\Delta\varphi + \frac{1}{n}\langle \mathrm{d}n, \mathrm{d}\varphi \rangle, \varphi \right) = 0.$$

4 The Schrödinger Equation

Corollary 1 suggests heuristic arguments as to why we might consider a pair (Γ, φ) consisting of a connected graph endowed with a solution φ to (1) with γ constant as an analogue of a quantum particle with mass inversely proportional to $|\gamma|$; in the case when $\gamma = 0$, we view the pair as representing a massless particle[2].

In the first instance, we do not admit any fixed background with respect to which we can define parameters of equations: the particle creates its own background, which is reflected in the form $(\mathcal{P}(\varphi), \varphi)$, where $\mathcal{P}$ is some (discrete) differential operator. Furthermore, in a geometric realization of a solution to (1) as an invariant framework in Euclidean space, the function φ is taken to be an orthogonal projection to the complex plane. Thus if we now fix the background with which φ interacts as say a fine mesh in Euclidean space with vertices placed on a regular lattice, then this in no way affects φ: the right-hand term of $(\mathcal{P}(\varphi), \varphi)$ it is still orthogonal projection, now defined on the vertices of the mesh, where the inner product should now be taken by summing over the vertices of the mesh that approximate the geometric realization of the original graph.

Let us therefore take the left-hand side of the inner product of Corollary 1 as a measure of change in our geometric model of quantum interaction. To do this, suppose that the centre of mass of the framework is located at the origin in Euclidean 3-space and that, as in quantum mechanics, a solution φ to (1) is normalized so as to have norm 1 with respect to some appropriate Hermitian inner product, so that, for example:

$$||\varphi|| := \left(\sum_{x \in V} |\varphi(x)|^2\right)^{1/2} = 1.$$

Now define a discrete step by step evolution $\{\varphi_t\}$ $(t = 0, 1, 2, \ldots)$ by the equation

$$\frac{\partial \varphi_t}{\partial t} := \varphi_{t+1} - \varphi_t = \gamma \Delta(\Delta \varphi_t) + 2n\Delta\varphi_t + \frac{1}{n}\langle \mathrm{d}n, \mathrm{d}\varphi_t \rangle, \qquad \varphi_0 = \varphi. \tag{3}$$

This should be compared with the time-dependent Schrödinger equation on a smooth fixed background with potential U [13]:

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U\psi.$$

for some function $\psi(x, t)$ which represents the probability density of finding a particle at position x. When ψ is identified with $\Delta\varphi$ the similarity is striking. It should be noted that at least for the regular polytopes, as the number of vertices increases and the polytope increasingly approximates a smooth object, the parameter γ becomes negative, reinforcing the idea that if we pursue the particle analogy, γ be inversely proportional to the mass.

[2] This latter case is precisely the model proposed in [6] to describe massless particles via twistor theoretic methods in a combinatorial setting.

The relation of the step by step process (3) to solutions to (1) needs to be further explored, however, we may investigate how the equation behaves for the regular polytopes. In fact, remarkably, provided the centre of mass is located at the origin in Euclidean space, we find that $\Delta\varphi = c\varphi$ for some constant c depending on the polytope. In particular, if we normalize at each step so that $||\varphi_t|| = 1$, then, since n is also constant, so is φ_t and the solution remains invariant. Thus the regular polytopes are stable solutions to (3). Linearization leads to a remarkably similar result.

In order to define the linearized equation, we consider a family $\{\varphi_t\}$ of functions such that $\varphi_0 = \varphi$ solves (1) with γ independent of t. On writing $\xi(x) = \frac{\partial\varphi(x)}{\partial t}|_{t=0}$, we obtain the equation linear in both ξ and φ:

$$\gamma(x)\Delta\varphi(x)\Delta\xi(x) = \langle \mathrm{d}\varphi(x), \mathrm{d}\xi(x)\rangle_x . \tag{4}$$

Note that $\xi = \lambda\varphi + \mu$ solves the linearized equation ($\lambda, \mu \in \mathbf{C}$ constant), reflecting the normalisation freedom $\varphi \mapsto \lambda\varphi + \mu$. If we multiply (4) by $n(x)$ and sum over $x \in V$, we obtain

$$0 = (\gamma\Delta\varphi, \Delta\xi) - \sum_{v\in V} n(x)\langle \mathrm{d}\varphi(x), \mathrm{d}\xi(x)\rangle_x .$$

It is not immediately obvious how to deal with the latter term of this equation. However, if we suppose that $n(x)$ is constant and apply Lemma 1, we obtain

$$\begin{aligned}(\Delta(\gamma\Delta\varphi), \xi) - 2n(\mathrm{d}\varphi, \mathrm{d}\xi) &= (\Delta(\gamma\Delta\varphi), \xi) - 2n(\mathrm{d}^*\mathrm{d}\varphi, \xi) \\ &= (\Delta(\gamma\Delta\varphi) + 2n\Delta\varphi, \xi) = 0.\end{aligned}$$

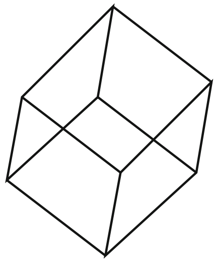
In particular, taking ξ to be 1 at a given vertex x and zero elsewhere, we deduce that at each $x \in V$, we have

$$\Delta(\gamma\Delta\varphi) + 2n\Delta\varphi = 0.$$

But, with both $\gamma(x)$ and $n(x)$ constant, this is precisely the time-independent version of (3).

5 Quantum Interaction

In order to pursue a quantum formalism, we can introduce a normalized linear superposition of states of the type $\lambda X + \mu Y$, where both X and Y are the lifts into Euclidean space of geometric states of (1). If we take standard projection $(x, y, z) \mapsto x + \mathrm{i}y$ from Euclidean space to the complex plane. Then the two corresponding states of the Necker cube have z coordinate differing by a sign on the relevant vertices. Factoring out an overall phase factor, we can suppose the coefficients of x and y are 1 to obtain the expression $(x, y, e^{\mathrm{i}\theta} z)$ as a representation of the possible superpositions at a particular vertex–see [4] for a more thorough account. Only when $e^{\mathrm{i}\theta} = \pm 1$ do we obtain an invariant framework corresponding to the geometric state.



Indeed, it would seem impossible to visualize an arbitrary superposition, but only the two invariant realizations, which provides a convincing argument for a quantum model of geometric interaction. Change of a system should now come about by some unitary-type transformation governed by the equations of the last section.

In deriving this model, the author has in mind a process of *reconstruction* of the geometric world from basic information-theoretic principles, as has been pursued in the domain of quantum mechanics by L. Hardy and A. Grinbaum [11,12]. As a first step, we need to create a graph. Given a set V of vertices, we can associate to each unordered couple $\{x, y\}$ in V $(x \neq y)$ a bit of information $q(x, y)$ which takes the value 0 or 1 with a certain probability. A basic state now corresponds to a graph, where we connect x and y with an edge whenever $q(x, y)$ takes the value 1. Geometric states, may be considered as potential information implicit in the graph [1,4]. Change of the system only makes sense relative to some other system, for example, a background state as introduced in the last section. The realization of geometric states and their potential step by step evolution could then be a driver of change of the underlying combinatori al structure.

Geometric information is only one kind of information implicit in a basic combinatorial system and the realization of other types of information could also drive change. The underlying principle we evoke is the way demands and responses in economic, linguistic, evolutionary and other systems drive change. These ideas are speculative and require further investigation to provide a workable theory.

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