Quantum Correlations: Challenging the Tsirelson Bound

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Abstract. A detailed study of quantum correlations reveals that reconstructions based on physical principles often fail to reproduce the quantum bound in the general case of N-partite correlations. We read here an indication that the notion of system, implicitly assumed in the operational approaches, becomes problematic. Our approach addresses this issue using algebraic coding theory. If the observer is defined by a limit on string complexity, information dynamics leads to an emergent continuous model in the critical regime. Restricting it to a family of binary codes describing 'bipartite systems,' we find strong evidence of an upper bound on bipartite correlations equal to 2.82537, which is measurably lower than the Tsirelson bound.

1 Mathematics Guides Understanding

A large swath of new research in the foundations of quantum theory addresses the problem of correlations between distant parties. It had been known since the "second quantum revolution" started by John Bell [3] that the amount of correlations is a decisive quantity distinguishing between local (classical) and nonlocal physical theories. However, it was only noticed quite recently that the amount of correlations or, more broadly, quantum bounds in Bell inequalities are, by themselves, a formidable puzzle through which, once we are able to understand it better, we may get an entirely new understanding of quantum theory.

Quantum logical reconstructions have reached their peak when George Mackey and, later, Constantin Piron gave examples of mathematical derivations of the Hilbert space formalism from an orthomodular lattice with additional assumptions [18,26]. After a decline of two decades, operational reconstructions of quantum theory took over from quantum logic around the turn of the century. Finite-dimensional Hilbert spaces came into focus of these reconstructions as a result of the development of quantum information. To derive the Hilbert space, one posits several physical principles that are given a mathematical formulation with the operation framework [5,15,22]. Such reconstructions contain an important insight: an assumption of continuity is a necessary, but not a sufficient, ingredient of quantum mechanical axiomatic systems. Differently worded continuity assumptions exist in every reconstruction [16,32]: a prominent representative is the existence of a continuous reversible transformation between any

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two pure states of the system [8]. On their own, however, these assumptions are insufficient for reconstructing quantum theory, as demonstrated by C^*-algebraic approaches [7]. Moreover, quantum theory has emerged from the reconstruction program [13], not only as a description of individual systems with continuous state spaces, but also requiring an extra axiom about how such systems compose [29]. This insight must be complemented with a quantitative bound on the amount of correlations given by the Bell inequalities and explored in postquantum models [2,30,31]. There exists a fundamental fact about nature: the amount of correlations between distant subsystems is limited by a non-classical bound, e.g., the Tsirelson bound for bipartite correlations [6]. All mathematical alternatives to the Hilbert space formalism must strive to predict its empirical value.

Reaching the Tsirelson bound of the CHSH inequality is not yes sufficient for a complete reconstruction of quantum mechanics. Various attempts to characterize this boundary have led, in particular, to the understanding that no bipartite principle on how two systems compose can fully characterize the quantum bound [12]. Further, many principles developed in the reconstruction program [4,11,17,25] were shown to be satisfied by a set of correlations which is larger than the quantum set [24]. This line of research, then, though stemming from the reconstructions, comes to invalidate most, if not all, of the desiderata on which the latter are built [23].

In our view, this points to a staggering evolution: the notion of system may not be an essential building block of quantum theory. Like other formerly unquestionable fundamental concepts in physics, a 'system' is now amenable to analysis and derivation. One is trying to imagine a quantum theory without systems. In Sect. 2 we introduce a framework based on the algebraic coding theory. It provides a general model of communication and deals mathematically with errors or ambiguity. The use of coding theory is enabled by the definition of observer in information-theoretic terms introduced in Sect. 3. It involves a limit on the complexity of strings, which (to use a common-language expression) the observer can 'store and handle'. Strings contain all descriptions of states allowed by quantum theory but also much more: they need not refer to systems or be interpretable in terms of preparations or measurements. Using the work of Manin and Marcolli, we show that symbolic dynamics on such strings leads to an emergent continuous model in the critical regime (Sect. 4). Restricting this model to a subfamily of 'quantum' binary codes describing 'bipartite systems' (Sect. 5), we find strong evidence of an upper bound on bipartite correlations equal to 2.82537. The difference between this number and the Tsirelson bound $2\sqrt{2}$ can be tested. The Hilbert space formalism, then, emerges from this mathematical approach as an effective description of a fundamental discrete theory of 'quantum' languages in the critical regime, somewhat similarly to the description of phase transitions by the effective Landau theory.

2 Codes

Communication implies that messages are encoded and transmitted in suitable codes using an alphabet shared between the parties involved in communication.

An alphabet is a finite set A of cardinality $q \geq 2$. A code is a subset $C \subset A^n$ consisting of some of the words of length $n \geq 1$. A language is an ensemble of codes of different lengths using the same alphabet. As an example, take binary codes of length n based on a two-letter alphabet, say, $\{0, 1\}$. Strings of zeros and ones of arbitrary length belong to a language formed by binary codes with different values of n.

In full generality, nothing can be stipulated about message semantics, material support of the encoding and decoding operations or their practical efficiency. One can observe, however, that decoding a message is less prone to error if the number of words in the code is small. On the other hand, reducing the number of code words requires the words to be longer. The number

$$R = \frac{\log_q \#C}{n} \tag{1}$$

is called the (transmission) rate of code C.

One can associate a fractal to any code in the following way [20,21]. Define a rarified interval $(0,1)_q = [0,1] \setminus \{m/q^n | m, n \in \mathbb{Z}\}$. Points $x = (x_1, \ldots, x_n) \in (0,1)_q^n$ can be identified with $(\infty \times n)$ matrices whose k-th column is the q-ary decomposition of x_k. For a code C define $S_C \subset (0,1)_q^n$ as the set of all matrices with rows in C. It is a Sierpinski fractal and its Hausdorff dimension is R. The closure of S_C inside the cube $[0,1]^n$ includes the rational points with q-ary digits. This new fractal $\hat{S}_C$ is a metric space in the induced topology from $[0,1]^n$. Now consider a family of codes C_r of $\#C_r = q^{k_r}$ words of length n_r, with rates:

$$\frac{k_r}{n_r} \nearrow R. \tag{2}$$

They define a fractal $S_R = \bigcup_r S_{C_r}$ of Hausdorff dimension $\dim_H(S_R) = R$.

3 Bounded Complexity

Any observer's memory is limited in size. While their material constitution may be radically different, different observers with the same memory size should demonstrate similar performance in handling information. This intuition serves as a motivation for the following information-theoretic definition of observer.

Definition. An observer is a subset of strings of bounded complexity, i.e., strings compressible below a certain threshold.

This limit can be viewed as the length of observer's memory. If a string has high complexity, it cannot describe an observer with a memory smaller that the minimal length required to store it; but it remains admissible for an observer with a larger memory.

The definition of observer requires a notion of string complexity independent of the observer's material organization. Kolmogorov complexity is a suitable candidate, for it strives to grasp complexity in a machine-independent way. For a set of strings that are code words of code C with rate R, the lower Kolmogorov complexity satisfies [20]:

$$\sup_{x \in \hat{S}_C} \kappa(x) = R. \tag{3}$$

For all words $x \in S_R$ in a language formed by codes C_r, the lower Kolmogorov complexity is bounded by $\kappa(x) \leq R$. Hence the closure $\hat{S}_R$ of the fractal S_R is a metric space that describes the handling of words of bounded Kolmogorov complexity. It is a 'minimal' geometric structure corresponding to the notion of observer.

4 Critical Language Dynamics

A change in the observer's information can be modeled via dynamical evolution on the fractal set S_R. In quantum theory, new information enters when a projective or a POVM measurement produce a new string in the observer's memory. Taking inspiration from Manin and Marcolli [20], we represent this process as a statistical mechanical system evolving on the set of all possible strings in codes C_r. A change in the observer's information corresponds to a change in the 'occupation numbers' λ_a of words $a \in \bigcup_r C_r$. The evolution of λ_a is described via Hamiltonian dynamics on the Fock space:

$$H_{stat} \epsilon_{a_1 \ldots a_m} = (\lambda_{a_1} + \ldots + \lambda_{a_m}) \epsilon_{a_1 \ldots a_m}, \tag{4}$$

with the Keane 'ergodicity' condition:

$$\sum_{a \in \cup_r C_r} e^{-R\lambda_a} = 1, \tag{5}$$

where vectors $\epsilon_{a_1 \ldots a_m}$ belong to the Fock space representation $l^2(W(\bigcup_r C_r))$ of the set W of all words in the codes C_r. To be precise, the Fock space is a representation of the algebra defined below in (7); at this stage it is justified by the completeness of W, which by construction includes all possible observer information states. If the observer's information remains within W, then the Keane condition gives a meaning to the weights λ_a as normalized logarithms of inverse probabilities that a is stored in the observer's memory. This evolution has a partition function:

$$Z(\beta) = \frac{1}{1 - \sum\limits_{a \in \cup_r C_r} e^{-\beta \lambda_a}}. \tag{6}$$

Manin and Marcolli show that at the critical temperature (equivalently, string complexity) $\beta = R$, the behaviour of this system is given by a KMS state on an algebra respecting unitarity [20]. This algebra is built out of a geometric object, namely the fractal $\hat{S}_C$, as follows. Consider characteristic functions $\chi_{\hat{S}_C(w)}$, where $w = w_1 \ldots w_m$ runs over finite words composed of $w_i \in C$ and $\hat{S}_C(w)$ denotes the subset of infinite words $x \in \hat{S}_C$ that begin with w. These functions can be identified with the range projections

$$P_w = T_w T_w^* = T_{w_1} \dots T_{w_m} T_{w_m}^* \dots T_{w_1}^*. \tag{7}$$

At the low temperature $\beta > R$ there exists a unique type I_∞ KMS-state ϕ_R on the statistical system of codes, which is a Toeplitz-Cuntz algebra with time evolution:

$$\sigma_t(T_w) = q^{itn} T_w. \tag{8}$$

The partition function is:

$$Z_C(\beta) = (1 - q^{(R-\beta)n})^{-1}. \tag{9}$$

However, the isometries in the algebra do not add up to unity. Only at the critical temperature $\beta = R$, where a phase transition occurs for all codes C_r, is there a unique KMS state on the Cuntz algebra, i.e., an algebra such that, importantly for our argument, isometries add up to unity: $\sum_a T_a T_a^* = 1$.

Critical behavior of the original discrete linguistic model is described at $\beta = R$ by a field theory on the metric space $\hat{S}_R$, which obeys unitarity. By construction, this fractal also has scaling symmetry. This yields a field theory respecting scaling and unitarity. While there has been some discussion of models that are scale invariant but not conformal, we assume that, in agreement with Polyakov's general conjecture [28], this field theory is conformal. The field it describes is clearly an emergent phenomenon, for its underlying dynamics is given in terms of codes. However, within the conformal field theory this field, now a basic object, is to be considered fundamental. Due to the properties of continuity, unitarity and to the geometric character of its state space, the conformal model becomes a tentative candidate for a reconstruction of quantum theory.

5 Amount of Correlations

Since quaternionic quantum mechanics or, in some limited cases, real-number quantum mechanics can be represented in the Hilbert space [1], one should expect that continuity and unitarity alone do not single out quantum theory. In other words, the conformal model of Sect. 4 likely contains more than a description of 'quantum' languages. In this section, we do not seek to provide a necessary and sufficient condition that selects only code words generated by quantum theory. Rather, we pick out a particular example, namely a class of models corresponding to the critical regime of binary codes describing measurements on bipartite quantum systems in the usual 3-dimensional Euclidean space.

First we define an informational analog of 'bipartite.' In quantum theory, subsystems that are entangled can be materially different but they are described by the same number of entangled degrees of freedom. Their informational content is represented by strings of identical complexity. For example, measurements in a

CHSH-type experiment produce binary strings of results for a choice of $\sigma_x, \sigma_y, \sigma_z$ measurements. The no-signalling condition implies that the probability of 0 on Alice's side is independent of Bob's settings, and vice versa. Hence the strings resemble Bernoulli distributions with a Kolmogorov complexity equal to the binary entropy of the probability of 0, plus a correction due to the existence of non-zero mutual information between Alice's and Bob's outputs. Since both sides enter symmetrically in the CHSH inequality, this correction to Kolmogorov complexity is *a priori* the same on Alice's and Bob's sides. We use this argument to replace Eq. (4) with a class of Hamiltonians assumed to describe a 'bipartite system' in the framework of codes.

The Kolmogorov order is an arrangement of words $a_i \in \bigcup_r C_r$ in the increasing order of complexity [19]. It is not computable and it differs radically from any numbering of a_i based on the Hamming distance in the codes C_r. Words that are adjacent in the Kolmogorov order have similar complexity. We now select an Ising-type Hamiltonian:

$$H_2 = -\sum_{ij} a_i \times a_j, \tag{10}$$

as a dynamical model on the language that describes bipartite quantum systems. The sum is taken over N neighbors in the Kolmogorov order, i.e. all strings of identical complexity. The result of multiplication on binary words is a new word with letters isomorphic to multiplication results in a two-element group $\{\pm 1\}$. Hence, for a two-letter alphabet $\{a, b\}$,

$$a \times a = b \times b = b, \quad a \times b = b \times a = a. \tag{11}$$

A binary language with $N = 6$ using H_2 gives rises to information dynamics which is, on the one hand, equivalent to information dynamics of a bipartite quantum system and, on the other hand, equivalent to the dynamics of a 3-dim Ising model. This is because a class of Hamiltonians with $N = 6$ has the same number of terms as in three spatial dimensions, although the codes that belong to this class are uncomputable due to the properties of Kolmogorov complexity. Plainly, one cannot tell which binary codes give rise to the $N = 6$ situation nor should one expect that Hamiltonians H_{stat} and H_2 belong to the same universality class. However, the equivalence of (10) with a 3-dim Ising model suggests that, just like the Ising model itself, the Hamiltonian H_2 also exhibits critical behaviour described by a conformal field theory.

As it is usually the case in statistical mechanics, the critical regime can be studied without knowing the details of the dynamics. Correlations of order 2 in this regime are described by the lowest-dimensional even primary scalar $\epsilon = \sigma \times \sigma$ in the conformal field theory. This field is symmetric; hence it provides a good candidate to describe the symmetry of bipartite correlations in the CHSH inequality under the switch between Alice and Bob. Following the above intuition, we assume that it provides a description of 'bipartiteness' within the conformal model. The operator dimension of ϵ is

$$\Delta_\epsilon = 3 - \frac{1}{\nu}, \tag{12}$$

where ν is a well-known critical exponent describing the correlation length [9].

The 3-dim Ising equivalence has its limitations since the true metric space of code evolution is not flat space but the fractal $\hat{S}_C$. Still, it provides significant evidence that H_2 has a critical regime. Further, the exponential character of the mapping that links the fractal embedded in the unit cube with flat Euclidean space hints at the existence of a connection between the critical behaviours of the Ising model and the code. The correlation length in the fractal representation of a language describes a logarithmic distance in $\hat{S}_C$ from which words are brought in groups of equal complexity by the Kolmogorov reordering. If H_2 exhibits a critical behaviour similar to that of H_{stat}, then correlations in the critical regime at string complexity $\beta = R$ come from the entire fractal. The Ising analogy with the scaling of the correlator of the lowest primary even field suggests a power law for the amount of correlations on the words of equal complexity:

$$\langle \epsilon(a)\epsilon(0)\rangle \sim a^{-2\Delta_\epsilon}. \quad (13)$$

We conjecture that, due to the exponential mapping between spaces, the corresponding correlations in the fractal are limited by the logarithm of the RHS of (13). Their maximum strength $2\Delta_\epsilon$ can be computed based on the value $\nu = 0.62999(5)$ in [10]:

$$2\Delta_\epsilon = 2.82537(2). \quad (14)$$

An attempt to test the difference between this value and the Tsirelson bound is currently in progress [14,27].

6 Conclusion

Historically, quantum logical reconstructions of quantum theory drive home the importance of the assumptions of continuity and composition rule. These are two pillars of quantum theory. A detailed study of quantum correlations reveals, however, that reconstructions based on physical principles often fail to lead to the quantum bound in the general case of N-partite correlations. We take this seriously as an indication that the notion of system, implicitly assumed in the operational approaches, becomes problematic. Using a model based on codes, we suggest an approach free of the observer-system distinction. Although our model is highly speculative, we believe that it demonstrates the interest to explore quantum theory via novel mathematical formalisms.

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