Mathematical Culture and Mathematics Education in Hungary in the XXth Century

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Hungarian mathematics and mathematical teaching traditions are often considered as corresponding to a special “Hungarian mathematical culture”, focused mainly on problem-solving and on heuristic methods. However, a detailed characterisation of “Hungarian methods” has been lacking, as well as a coherent historical description of the development of a “Hungarian school”.[1] This paper attempts to contribute to the treatment of this subject.

One of the key moments in the history of Hungarian mathematics education is the reform prepared by the team of Tamás Varga during the 1960s and 1970s, which was officially introduced in 1978.[2] Varga himself participated actively in the international discussions of the period,[3] and was deeply inspired by the different

[1] There exist mostly some commemorations on single mathematicians, by colleagues see e.g. the recent book about Rényi (2013). An interesting attempt, in English, to give a panorama on Hungarian mathematical life is that of (Hersh and John-Steiner 1993). A brief history of mathematics education can be found in Császár (2005) and in Frank (2012). About contemporary mathematics education, the comparative researches of Paul Andrews give some caracterisation (e.g. Andrews 2003; Andrews and Hatch 2001).

[2] The Hungarian mathematics education community remembers the reform led by Varga as a decisive moment which has significant influence on mathematics education until today: this is attested for example by the numerous commemorations on the yearly conference “Varga Tamás Napok” as well as by Szendrei (2007). For details on the reform movement in English, see (Halmos and Varga 1978).

[3] For example, he edited with the Belgian W. Servais a UNESCO book following a UNESCO conference about mathematics education, organised in Budapest in 1962 (Servais and Varga 1971). He was also vice-president of the CIEAEM. (See Szendrei 2007).

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experiences and reforms of the “new math” movement.[4] At the same time, even if this is poorly documented, his work seems also to bear some important characteristics of a local tradition. In this paper, I focus on this internal influence: I show that a coherent, distinct conception of mathematics and its teaching developed in Hungary, in a community of mathematicians, educators and other thinkers in the mid-20th century, which could have influenced not only the later reform of Varga, but also the later philosophy of Imre Lakatos, among others.[5]

In a first, introductory part, I will briefly explain the circumstances of the formation of high quality Hungarian mathematics in the late 19th- early 20th century, touching on the socio-economic and cultural context as well as the educational reforms of the period. Without being able to give a general, coherent analysis of Hungarian mathematical culture of this period, I will insist that several characteristics of the later developed tradition can be discovered already in the formative period of 20th-century Hungarian mathematical life.

In the second part, I will present the community in question, a mid-20th century group constituted by first-rate Hungarian mathematicians (László Kalmár, Rózsa Péter, Alfréd Rényi), mathematics educators (János Surányi, Tamás Varga), the later historian of mathematics Árpád Szabó and the later philosopher of science Imre Lakatos among others: most of them were related in the 1940 to an interdisciplinary community thinking about education, around the Calvinist pastor and educator Sándor Karácsony. From the 1950’s they met regularly in the mathematical research institut founded by Rényi. I will briefly present the people in question and I will describe the possible ways of influence among them.

In the third, main part of the paper, through an analysis of their diverse writings, I will attempt to describe some main characteristics of the conception of the nature of mathematics and its teaching represented by this community. (I also include George Pólya in this analysis, even if he was not in close contact with the community around Karácsony, as he spent most of his life outside of Hungary; but he studied in his home country and had great effect on the reform movements of the 1960s and ’70s.) I will show that the members of this community emphasise the developing nature of mathematics, and the role of problems in this development; they attribute a great importance to intuition and experiences in mathematics. For them, mathematical activity is basically dialogical; teaching mathematics serves first of all to educate students to think; they caution against excessive use of formal language; and they put the accent on the creative nature of mathematics, and its relations with arts and playfulness.

[4]E.g. in Varga (1975) he details who inspired him. Incidentally, these influences seem to arrive mostly from the western countries and less from the “Eastern bloc”. This phenomenon doesn’t seems to be unique for the case of Hungary: we plan to develop a research project with C. Radtka and S. Lawrence among others to study the ways of circulation between Eastern and Western Europe in the new math reform period.

[5]This work was the basis of a chapter of my thesis (Gosztonyi 2015a) where I compare the Hungarian reform led by Tamás Varga with the French “mathématiques modernes” reform of the same period, in order to understand the influence of the international discourses on one hand and the local traditions on the other hand on the characteristics of these reforms.

1 The Emergence of the XXth Century's Hungarian Mathematical Culture[6]

Up to the last few decades of the 19th century, there was hardly any organized mathematical research in Hungary. Until then, the only significant mathematicians were Farkas and János Bolyai, father and son, but they worked in relative isolation, and their importance was only realized by the very end of the century. (By then, János Bolyai became quite a cult figure, inspiring the mathematical life forming around that time.) But from the early 20th century, a striking number of internationally significant mathematicians were educated in Hungary, developing a state-of-the-art mathematical culture.

How was this fast and spectacular development possible? As far as I know, there is no definite answer to this question yet; however, there are many factors that are often mentioned as playing a possible role in it.

1.1 The Socio-Economic and Cultural Context

First of all, it is worth mentioning that this sudden improvement of mathematics is not isolated from a larger socio-economic and cultural context in Hungary: following the Austro-Hungarian Compromise and the establishment of the Austro-Hungarian Monarchy in 1867, a broad and fast social and economic development began in the country, especially in Budapest, the capital newly created by the fusion of three cities. The rapid industrialization created a need for improvement in science. Hungarian society, which was essentially feudal until then, started to change, and a new middle class formed, partly from the decaying landed gentry of feudal origins, and partly from the small, mainly German and Jewish bourgeoisie already present in Hungary.

It is important to mention that the legal emancipation of the Jewish population took place in 1867 followed by its significant assimilation into Hungarian society. Culture and science provided the opportunity for Jewish youth to rise socially, and the cultural energies released by the emancipation enriched Hungarian culture—especially in the field of sciences and mathematics. In the early 20th century, the overwhelming majority of Hungarian mathematicians were of Jewish origin. (Following World War I and the Treaty of Trianon, this process of assimilation was interrupted by the increasing anti-Semitism and the *numerus clausus* law limiting the admission of Jewish students to university. Many young talents of Jewish origin went abroad to study during these years—and later gained international success.)

During this exceptionally fast social and economic development, thriving cultural life arose in *fin-de-siècle* Budapest. Thus, a vivid, very open and inspiring atmosphere gave birth to high level workshops on literature, arts, natural and social sciences etc.

[6]The following summary is mostly based on the works of (Hersh and John-Steiner 1993; Frank 2012; Németh and Pukánszky 1996; Császár 2006; Kántor-Varga 2006).

1.2 Educational Reforms and the Improvement of Mathematics Education

Several educational reforms also took place in the era of the Austro-Hungarian Monarchy. The first law on general education, making six-year primary school obligatory and tuition-free for the poor, was already accepted in 1868. Secondary education was also reformed and improved several times, and the public training of teachers was also begun by the founding of the Institute for Teacher Training and the "Mintagimnázium," the "Practicing High School" where trainee teachers could get teaching experience.

Regarding the improvement of mathematics education in secondary schools, significant Hungarian mathematicians of the period such as Manó Beke and Gyula Kőnig played a decisive role by participating in the reform of syllabuses and by writing textbooks. Beke also joined the international movement for improving mathematics education, directed by Felix Klein, around the turn of the century.[7]

Higher education in mathematics was accessible at the Faculty of Philosophy of the University of Budapest and at the Technical University. The University of Kolozsvár, founded in 1872, had an independent Faculty of Sciences and Mathematics. (Since 1921, this University is located in Szeged.)

The physicist Loránd Eötvös, who was a significant organizer of Hungarian scientific life, as well as Minister of Culture and Education for a short period, founded the Eötvös Collegium[8] in 1895 explicitly modelled after the École Normale Supérieure in Paris. This institute became an important centre for training scholars skilled in both research and teaching. It was also thanks to him that the "Mathematikai és Physikai Társulat" [Society of Mathematics and Physics] was created in 1891. Among other things, this society organized a mathematical competition for students (later adopting Eötvös's name) and published the *Mathematikai és Physikai Lapok* [Mathematical and Physical Journal] and the *Középiskolai Matematikai Lapok* [High School Journal in Mathematics], or KÖMAL for short. The latter journal did not only publish articles on mathematics, but also proposed problems for secondary school students, made the best solutions public, and at the end of the year, printed the photographs of the best problem solvers. KÖMAL and the tradition of mathematical competitions are still today essential in training gifted students in Hungary: they help to reveal talents early, motivate the students and improve their problem solving skills.[9] They provide the basis, since the end of the 19th century, of a mathematical culture oriented to problem-solving.

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[7]It would be interesting to study the influence of this international reform movement, and more particularily of Felix Klein on the Hungarian mathematics education, as some interesting similarity can be found in their principles: e.g. the experimental nature of mathematics or the role attributed to intuition. (See e.g. Gispert and Schubring 2011).

[8]The Collegium gained its name after József Eötvös, the father of Loránd: lawyer, novelist and also Minister of Culture and Education, who was responsible for the 1868 law of education.

[9]KÖMAL is partly available in English; students may also participate in English in the competition by correspondence. See http://www.komal.hu/info/bemutatkozas.e.shtml for the English version.

1.3 The Role of Lipót Fejér

Regarding the success of mathematics in Hungary in the 20th century, George Pólya said the following:

> Why did Hungary produce so many mathematicians of our time? Many people have asked this question which, I think, nobody can fully answer. There were, however, two factors whose influence on Hungarian mathematics is manifest and undeniable, and one of these was Leopold Fejér, his work, his personality. The other factor was the combination of a competitive examination in mathematics with a periodical. (Pólya 1961, p. 501)

I already mentioned these competitions—now a few words about Leopold Fejér (1880–1959). Fejér was not only a significant scholar in mathematical analysis, but also the first one in Hungary to have a coherent mathematical school organized around his person. In the first half of the 20th century, he lectured to practically everyone who learned mathematics at the University of Budapest. He was the supervisor of mathematicians such as Marcel Riesz, George Pólya, László Kalmár, Pál Turán, Pál Erdős etc., but even his other students remembered him as a charismatic and influential teacher (Hersh and John-Steiner 1993, p. 18). Without being able, at the moment, to analyse in detail the role of Fejér in the development of Hungarian mathematical culture and teaching traditions, I quote some reminiscences of his disciples that suggest that several characteristics of these traditions were already present in Fejér's teaching practice, and he might have influenced his disciples' thinking.

> Fejér gave very short, very beautiful lectures. They lasted less than an hour. You sat there for a long time before he came. When he came in, he would be in a sort of frenzy. He was very ugly-looking when you first examined him, but he had a very lively face with a lot of expression. *The lecture was thought out in very great detail, with dramatic denouement. He seemed to relive the birth of the theorem; we were present at the creation*. He made his famous contemporaries equally vivid; they rose from the pages of the textbooks. *That made mathematics appear as a social as well as an intellectual activity.* (Ágnes Berger, quoted by Hersh and John-Steiner 1993, p. 18. Emphasis added.)

We will see in the third part that presenting the mathematics as a social activity, using dramatic forms and presenting mathematical knowledge in the processus of its creation are recurrent caracteristics of the tradition in question.

> When Fejér stumbled upon an article which was written in such a mystical style that it was impossible to understand, he said: oh, these young people, these young people, their ambition is that if someone read their paper, they should think what a genius the author was, discovering such things the reader would have never even thought about. On the other hand – he went on –, *if I write a paper, my ambition is that the reader should think: - What's the big deal? Even I could have done that.* That is why all of Fejér's papers were so easy to understand and enjoyable to read. (László Kalmár in Szabó 2005, p. 457. My translation and highlights.)

This idea, "let the reader or the student think that, '*even he or she could have done that*,'" returns several times in the writings of Kalmár and Rózsa Péter as well as in *How to solve it?* by Pólya or in the work of Varga. In the quoted article by Pólya, he emphasises the care Fejér took to give to his papers an intuitive clarity

(Pólya 1961, p. 505.): this effort can be identified in the style of Kalmár, Péter, Rényi, or even Varga. Moreover, Pólya stresses the role of aesthetic considerations behind this effort of Fejér, and in more general, the parallel between Fejér's artistic and mathematical interests: this is again a recurrent characteristic of later traditions, and is especially explicit in the work of Péter and Varga.

Finally, to illustrate the general admiration for Fejér's person and the possible power of his influence, I quote the novelist Géza Ottlik, also former student of Fejér in mathematics:

> It's quite impossible to describe to an outsider what Lipót Fejér was like. A giant he was. His sheer being was an otherwordly kind of consolation. For those who didn't know him, there is some aspect of the world that they do not know and will never know. (Ottlik 2004, p. 268. My translation.)

2 Mathematicians in the Circle of Karácsony

This part of the paper is focused on a group of mathematicians, mathematics educators and other thinkers who were involved with a circle of many different intellectuals discussing education in the 1940s, and who were later more or less directly implicated in the development of mathematics education in Hungary.

The leader of this interdisciplinary circle was Sándor Karácsony, a Calvinist pastor and a unique mind, a bit of a psychologist, a bit of a philosopher, but primarily a very influential character in pedagogy.[10] His teachings already gained attention between the two World Wars, and during the 1940s, a group discussing different questions about education formed around him. This included some first-rate mathematicians like László Kalmár, Rózsa Péter along with János Surányi and Tamás Varga,[11] who later became the leading figure of the reform movements on mathematics education in the '60s and '70s. We do not know much yet about how Karácsony personally affected mathematics education; however, some references[12] make it clear that he played an important role in the development of the thinking of the above mentioned mathematicians—especially of Kalmár and Varga—about education.

The participation of these mathematicians in Karácsony's circle confirms however the existence of vivid reflexions about questions of mathematics education, already in the 1940s. Kalmár's article from 1942, published in a book of

[10]For details about Karácsony see (Kontra 1992).

[11]Kontra (1992) mentions the participation of Kalmár, Péter, Varga. About the participation of János Surányi, his son, László Surányi has given some information.

[12]In his recent article, Szabó attempts to demonstrate how the "social philosophy" of Karácsony and the role he dedicated to the pictures and visuality in his works influenced Kalmár's thinking about mathematical research and mathematics education (Máté 2006; Gurka 2001; Szabó 2013). Karácsony appears several times in the correspondence of Kalmár and Péter, Kalmár and Varga: these references don't give details about the nature of Karácsony's influence, but confirm his importance for Kalmár and Varga.

Karácsony's circle, contains already several principles developed later by other members of the circle, as we will see below. The reflexions could continue later in the mathematics research institute founded by Alfréd Rényi,[13] where all these people met regularly.

The classical philologist, and later historian of mathematics Árpád Szabó (1913–2001) and the philosopher of science and mathematics Imre Lakatos (1922–1974) were also both in contact with Sándor Karácsony, the former being his colleague, and the latter being his student in Debrecen in the '40s. Recent papers[14] have suggested that they were both influenced by the mathematicians in Karácsony's circle, especially by Kalmár. In the '50s, they both worked for some time at Rényi's research institute where they also could have been in contact with the mentioned mathematicians. At the institute, Lakatos was given the task of translating *How to solve it*? by George Pólya (1887–1985): because of this translation, Pólya could have a great influence on the development of Hungarian mathematics education (Pólya is commonly regarded as one of the 'fathers' of the Hungarian reform, Varga himself refers him regularly), as well as on Lakatos himself, who considered Pólya one of his masters.

In general, we can say that in this community, philosophical reflexions about the nature of mathematics developed in close connection to questions of mathematics education, and of education more generally.

Before passing on to the analysis of their writings, I briefly present the mathematicians in question, and I refer to their (direct or indirect) contributions in mathematics education.

László Kalmár (1905–1976) worked mainly on logic and mathematical linguistics, but he was well-versed in many fields of mathematics. He pioneered computer science in Hungary. His work was not directly focused on public education, but his colleagues and disciples remember him as having important influence on their thinking about mathematics and also about its teaching. He was a passionate and influential lecturer at university, and also keen on explaining mathematics to his colleagues and amateur friends. (Even when he was still a student—for example, Rózsa Péter, his classmate and close friend, considered Kalmár as her mentor throughout all of her life). His famous long mathematical letters, published posthumously (Kalmár 1986), but broadly diffused even earlier, explain complex subjects or proofs very clearly, and his vivid explanatory style can be recognized in the works of Rózsa Péter, among others. Some of his examples and problems were taken over by Rózsa Péter[15] and later by Tamás Varga. As mentioned earlier, he also seems to have had an important influence on the philosophy of Lakatos (Máté 2006; Gurka 2001).

[13]The mathematics research institute of the Hungarian Academy of Science today bears Rényi's name: http://www.renyi.hu/.

[14]Máté (2006), Gurka (2001).

[15]See the preface of *Playing with infinity* (Péter 1961) where he appears as one of her most important references.

Rózsa Péter (1905–1977) is known primarily for her research on recursive functions, and for her significant efforts in popularizing mathematics and improving mathematics teaching. She gained experience in public education as well (since she did not get a position at university before the war). Her mathematics popularizing book, *Playing with infinity*, first published in 1945, was translated into a number of languages and is still regularly reprinted in Hungary. From 1949, she wrote a series of secondary school textbooks with Tibor Gallai and others (Gallai and Péter 1949): these books were significantly different from previous textbooks, and could influence the later educational reforms. Péter actively supported the reform movement of Varga, and her *Playing with infinity* is mentioned in Varga's teacher's handbook for some mathematical and didactical questions (C. Neményi et al. 1978).

Alfréd Rényi (1921–1970), a specialist of the theory of probability, also played a prominent role in organizing Hungarian mathematical life. He was the founder of the Institute of Mathematics of the Hungarian Academy of Sciences in 1950 and remained its director until the end of his life. He was not particularly engaged in public education, but he was interested in the topic, writing and talking about it several times, and his Institute of Mathematics gave rise to a special research group on the educational reforms. He also supported the reform movement of Tamás Varga politically. His mathematics popularizing works written in the style of Plato's dialogues and Pascal's letters *(Dialogues on Mathematics, Letters on Probability*) had great success (Rényi 1967, 1972). Varga later became a specialist in the teaching of probability—further research might study to what extent his work was inspired by Rényi.

János Surányi (1918–2006) did mathematical research in logic, combinatorics and number theory, and he also did important work in mathematics education, especially concerning gifted students. He was, among other things, the reviver of the KÖMAL journal after the Second World War, and the head of the group, in Rényi's research institute, preparing the first special mathematics class curricula in the Fazekas high school during the 1960s. (This group worked in close connection with Varga's team, preparing the primary school mathematics education reform.)

Tamás Varga (1919–1987), the later head of the reform of the 1970s, was a young mathematics teacher in the 1940s, and frequented the Karácsony circle together with some of his brothers. From the end of the 1940s, he took a role in the development of national curricula and of textbooks, and he developed a series of experiments in primary schools in the 1960s, which led to the so-called "complex mathematics education" reform program, introduced on the national level in 1978. His correspondence with Kalmár and his colleagues' testimonies attest his vivid contacts with the aforementioned mathematicians.[16]

Finally, in my analysis I also take into account the work of George Pólya, even if he was not in close contact with the community around Karácsony, living abroad from 1914. But he had studied in Hungary, in a similar context to the aforementioned

[16]These details are confirmed among others by Mária Halmos, one of his closest colleagues who was also a member of Surányi's aforementioned team in Rényi's research institute. See also (Máté 2006).

mathematicians (he was, for example, one of the students of Fejér), and later he exerted significant influence in Hungary, especially on Varga's work (as we have seen, *How to solve it* was translated into Hungarian by Lakatos, at the request of Rényi and Varga).

3 Conception of Mathematics and of Its Education According to the Mathematicians of the Karácsony Circle

On the next few pages, I will sketch the main principles found in common between these scholars concerning mathematics and mathematics education, by analysing their diverse writings: lectures about education, mathematics popularizing books, letters etc.

I will quote primarily from a lecture by Kalmár, titled *The Development of Mathematical Rigor from Intuition to Axiomatic Method*, which appeared in a collection of essays by the Karácsony circle in 1942, because in this text we can find almost all these main ideas later developed by different members of the group around Karácsony. I will compare Kalmár's text with quotations from works of the other authors mentioned.

3.1 Mathematics Is a Developing Science

The historical aspect of mathematics appears several times in the studied authors' writings. They present mathematics as a continuously changing, developing science and they suggest that students should be led through a similar process of evolution. However, it does not mean that the *real* history of mathematics should be studied or taught, rather a *rational reconstruction*, as we can see also in *Proofs and Refutations* by Lakatos (1976a).

Rényi for example chooses historical contexts for his works by writing them in the form of Platonic or Galilean dialogues (*Dialogues on Mathematics*) or Pascalian letters (*Letters on Probability*). As he explains in the introduction of the *Dialogues*, he wants to present his subjects "in statu nascendi", "in the very freshness of its becoming" (Rényi 1972, p. 55).

Kalmár, in the introduction of his lecture on *The Development of Mathematical Rigor…*, writes:

> **1.a)** I will not discuss the issues from a historical perspective, I leave this task to someone with a thorough knowledge of the history of mathematics. Instead, I will describe the road that individual mathematicians travel while constructing for themselves a rigorous system of mathematical concepts and theorems. And I will describe this path as I see it in hindsight. I realize that often I am no longer seeing the road that I actually traveled, but see instead the shortest path that I could have taken to get to where I am standing now. (Kalmár 2011, p. 270)

And in the last section of his paper, concerning the context of the education:

> **1.b)** Now, however far we have gotten in this developmental process, and whatever our opinion might be about the steps ahead, we must realize that if we want to introduce others to mathematics, we must help them so they, too, can follow along this path, for it is only through these levels that they can reach our position. (Idem p. 282)

> **1.c) […]** with a small amount of extra effort, we can always present things in such a way that we honestly reveal how we came to realize those things, or how we could have come to realize them, and we could wait until later to cast the theory in its final form. It is not at all a problem – in fact, it is a good thing – if our students eventually come away with the impression: this was no big deal, I could have arrived at it myself. It also fits better with the scientific perspective if we present the process of development rather than the axiomatic theory in its finished form; for it is not the latter that expresses the present state of science, but the fact that this is where the developmental path has led us. (Idem p. 287)

Rózsa Péter's *Playing with infinity* also guides her readers through a process of development of mathematics: she proposes problems and questions, shows different solutions or attempts at answers, which lead to further and further questions. The end of the last chapter is not only a good example of a development-centred approach, but it also illustrates the relation of this conception to the mathematical researches of Kalmár and Péter.

After presenting Church's and Gödel's theorems, Rózsa Péter writes:

> **1.d)** This is where I must stop writing. We have come up against the limits of present-day mathematical thinking. Our epoch is the epoch of increasing consciousness; in this field Mathematics has done its bit. It has made us conscious of the limits of its own capabilities.
>
> But have we come up against final obstacles? Up to the present there has always been a way out of all the *culs-de-sac* encountered in the history of mathematics. There is one point about Church's proof which we might do well to ponder over: it would be necessary to formulate quite precisely what the arguments are that mathematicians today can think of, if we wanted to employ the processes of Mathematics in connection with such a concept. The moment something is formulated, it is already circumscribed. Every fence encloses a narrow space. The undecidable problems that turn up manage to get through the fence.
>
> Future development is sure to enlarge the framework, even if we cannot as yet see how. The eternal lesson is that Mathematics is not something static, closed, but living and developing. Try as we may to constrain it into a closed form, it finds an outlet somewhere and escapes alive. (Péter 1961, pp. 264–265)

Both in Kalmár's and Péter's researches, the big negative results, such as Church's and Gödel's theorems, took central position (Máté 2008). In their interpretation, the consequence of these results is that mathematics can never be a perfectly founded, perfectly infallible, closed system. The new problems that emerge during the evolution of mathematics will also change its form, its language, and its methods of proof.

It seems that for both Kalmár and Péter, the problems are what drive the development of mathematics. In his works, Pólya emphasizes the importance of problem resolution, of inductive methods, and of heuristic (concerning also mathematical research and teaching). Varga's reform conception is often characterised with the expression: "get students to discover the mathematics": he focuses less on the transmission of mathematical knowledge, more on guiding students through processes of mathematical invention, processes lead by series of problems and tentative solutions. The primary school teacher's handbooks of his team give varied instructions on how to construct teaching processes through series of problems; and in the middle-school textbooks, in place of deductive description of mathematical knowledge, one finds fictive students' discussions of problems to introduce new chapters.[17]

3.2 The Importance of Intuition and of Gaining Experience

According to Kalmár, mathematical concepts arise from intuition. It is important to add that the Hungarian word used by Kalmár, "szemlélet", which is translated in English as "intuition", refers to vision.[18] Kalmár's explications have an important visual aspect: intuitive mathematical knowledge is presented in the form of mental pictures. This emphasis on vision may have stemmed from Karácsony who also spoke often about pictures and vision (Szabó 2013).

> **2.a)** The point of departure for our journey is the intuition. Everyone accepts that our geometrical concepts – like point, line, surface, direction, angle, length, area, volume, etc. – derive from the contents of intuition. If we consider things closely, we realize that the same holds for the concepts of arithmetic, too: five chalks, half an apple – these denote clear contents of intuition. But there is general agreement among experts that certain rather abstract concepts of mathematics have nothing whatsoever to do with intuition. Set theory is perhaps the most abstract branch of mathematics; […]; nonetheless, at the most rudimentary level of concept formation, we imagine sets intuitively, as though they were like sacks into which someone has put their elements. (Kalmár 2011, p. 270)

> **2.b)** As soon as we recognize, via logical steps, a property we could not read off the picture originally, we return to the picture, coloring it with the newly unveiled property. Thus the picture becomes more colorful and vivid, so we can read off it the new, hitherto hidden properties as well. For mathematicians, this development of intuition amply makes up for the fading effect of the abstraction process; they are even emboldened enough to carry out another round of abstraction on the newly re-colored concepts gotten through abstraction. (Idem p. 272)

Intuitive, vivid demonstration is essential to the mathematicians of Karácsony's circle; they consider it one of the most important points while writing textbooks or constructing teaching material. Kalmár's mathematical letters (addressed to colleagues or non-mathematician friends) are famous for their vivid explanatory style. The effect of this style is recognisable also in Péter's *Playing with infinity*: this book, while very easy to read, is an attempt to introduce her readers to the pleasure of mathematical research; precise mathematical proofs are often replaced with vivid analogies.

[17]For more detail about the role and appearance of the series of problems in the work of Rózsa Péter and Tamás Varga, see (Gosztonyi 2015b)—a paper published as part of the French "Series of problems" project in history of sciences (problemata.hypotheses.org).

[18]In this sense, the use of the Hungarian term seems to be similar to the German "Anschauung".

At the same time, it is clearly visible from Kalmár's text that intuition or the gaining of experience can be understood not only on the physical level, but also in a more abstract sense, such as in thought-experiments. Intuition develops dynamically by collecting experiences and processing notions. One can also experiment and gain experience—on an appropriate level of development—with prime numbers or equations, for example (Pólya's works present numerous examples).[19]

Varga's educational reform syllabus (Szebenyi 1978) provides numerous opportunities for gaining a diversity of experiences and deliberately delays the introduction of new mathematical notions and knowledge to let them emerge from students' experiences. The structure of the syllabus is based on a dialectic relation of different mathematical domains, in order to show a wide variety of examples of the emerging notions. Vision often has a prominent role in this process: Varga emphasised the variety of representational tools (see e.g. Varga 1972), and one of the key functions of geometry in Varga's syllabus is to furnish models and illustrative examples for other mathematical domains, like arithmetic, functions or combinatorics (Varga and Szendrei 1979, p. 135).

3.3 Dialogue

Kalmár explains the launch of the progression of mathematical rigor with a quite surprising idea:

> 3.a) The major incentive that prompts us to break away from intuition is, I think, the fact that humans, including mathematicians, are social creatures.

At this point, the influence of Sándor Karácsony can probably be recognized, who developed a kind of collective psychological theory (Szabó 2013). Kalmár continues as follows:

> They like to communicate to others what strikes them as interesting and notable. This is when they are in for the first round of disappointments. It turns out that what is obvious to me based on my intuition might inspire a puzzled look from others. […]
>
> The easiest way to handle this is by listing, before presenting a certain idea, the concepts and the properties of those concepts to which I will refer as evidently given by my intuition. Those to whom I am presenting my proof can examine these one by one, check them against their own intuition, to see whether they likewise find clear what these "basic concepts" mean, and whether they likewise find these "basic truths" evident. […] (Kalmár 2011, p. 272)

This passage, going on longer, is strikingly reminiscent of certain dialogues of Plato and of the practice of antique dialectical debate. The connection between Greek dialectics and mathematics was later pioneered by Árpád Szabó, and his

[19] Probably that is what Lakatos meant when he called mathematics a 'quasi-empirical science'. Lakatos invited Kalmár to take part in a conference on philosophy of mathematics in London in 1965, where he presented similar thoughts on the empirical nature of mathematics. Lakatos commented on his talk, which he later expanded in his article on mathematics as a quasi-emprical science (Kalmár 1967; Lakatos 1976b).

thought also had influence on Rényi, who wrote his mathematics popularizing works in the form of dialogues.[20] The fact that Lakatos wrote his *Proofs and Refutations* in the form of classroom dialogue may be related to the above.[21]

It seems that in these works, the dialogue form is in close connection with the idea of presenting development. For example, in the postscript of Rényi's "Dialogues on mathematics" we find:

> 3.b) The Socratic dialogue presents thoughts while they are being created and dramatizes ideas. By so doing it keeps the attention awake and facilitates understanding. (Rényi 1967, p. 90)

The dialogue form or dramatization is also a central idea in this group of scholars from the point of view of education. The teacher and the students are partners: they develop mathematics together. According to Pólya:

> 3.c) Moreover, when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. (5. Teacher and student in Pólya 1990, p. 5)

As we have seen, *Playing with infinity* by Rózsa Péter and her textbooks are also constructed according to similar principles. She based her works on series of problems, series of questions and answers, where each problem is a natural consequence of the preceding ones, and could have occurred to the reader or student.

Varga's teachers' handbooks contain numerous tasks and situations based on the dialogue between students or between the teacher and the students; progress is promoted by students' remarks, ideas or even by their mistakes (E.g. in C. Neményi et al. 1978). And, as we have seen, middle-school textbooks present fictional student dialogues. The related teacher's handbooks encourage provoking similar dialogues in the classroom.

3.4 Education for Thinking People

I should mention here that the mathematicians of Karácsony's circle draw an analogy between the mathematician and the student learning mathematics. We have seen with Kalmár and Rózsa Péter the intention to conduct the students through a similar process to that which the mathematician follows. Varga attempts to place the students in the research mathematician's situation to let them discover mathematical results as the mathematician who first discovered them would do. Pólya's works are written for the student, the teacher and the young mathematician alike (Pólya 1990, p. XL).

This seems to be the result of a non-evident choice: the aim of teaching mathematics—by this conception—is not to transmit "ready-made" knowledge to users (e.g. one could argue, that an engineer only needs the theorems and the recipes for

[20]Rényi himself refers to Szabó in the Postface of his *Dialogues*. (Rényi 1967, p. 91).

[21]About the relation of Szabó and Lakatos see (Máté 2006).

computation), not even to present mathematical structures as models for thinking (as we can see in the Bourbakist school); it is to provide an introduction to the process of mathematical discovery, it is a reinvention of mathematics, together with the students.

Of course, this does not mean that Hungarian mathematicians want to raise every student to be a mathematician; rather, it means that they regard mathematics as one of the most important grounds for human thought.

Two illustrative passages from Pólya's *How to solve it?*:

> **4.a)** […] mathematics, besides being a necessary avenue to engineering jobs and scientific knowledge, may be fun and may also open up a vista of mental activity on the highest level. (Preface to the Second Edition, Pólya 1990, p. XXXV)

> **4.b)** […] Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact. (5. Teacher and student, idem p. 5)

In *Playing with infinity*, Rózsa Péter describes a classroom experience starting with a curious question from a student. The role of the teacher consists then in guiding the collective research, helping to find suitable research questions and linked new problems to help the class to construct mathematical knowledge to answer the original question (Péter 1961, Chap. 4).

Concerning Varga's reform, he often called attention to the importance of patience when introducing new mathematical knowledge: the emphasis being not on the formal knowledge of a notion, operation or theorem, but on its profound understanding, which develops during a discovery process, based on numerous examples (as we have seen above). For this reason, his syllabus presents two levels of knowledge: a broader level has to be experienced with students, but only a narrower level has to be acquired in a given grade of education. Teachers' handbooks give advice on how to encourage students' autonomous ideas and how to guide their thinking processes.

3.5 Limited Use of Formal Language

The view of these mathematicians on the use of formal mathematical language is not independent either from the above.

Kalmár's 1942 article goes through the different levels of mathematical rigour from intuition to formal axiomatics. However, he rejects the possibility of mathematical creation based purely on formal axiomatics.

> **5.a)** [The formal axiomatic approach] is given in principle only; in reality, pursuing it for its own sake would be a game only, not mathematics. Its significance resides in its utility as a working principle, when it comes to examining various questions within Hilbert-style proof theory: whether arithmetic is free of contradiction, or whether all problems of arithmetic (or some other system) can be solved. (Kalmár 2011, p. 278)

According to him, mathematical creation arises always from intuition:

> **5.b)** There are no mathematicians, however abstract their subject matter might be, who do not initially think intuitively, "heuristically" during their research; subsequently, they would cast their results in axiomatic form, thereby camouflaging how they arrived at them. (Idem p. 282)

As we have seen, formalisation has a certain communicative function in Kalmár's view: it helps the mediation of ideas and results; it secures the conviction of a community. Actual mathematical language is also the result of an evolution, also an answer to a series of problems as the mathematical notions or theorems. In consequence, when students are introduced to the logic of mathematical development, this also involves the language of the mathematics.

The mathematicians of Karácsony's circle were strongly against any kind of excessive formalization, especially in public education. The word "formalism" is still used by many Hungarian mathematics teachers and pedagogues with a negative connotation. This does not mean that any formalization is refused, but that one should take care to introduce formal languages with good reason, after careful preparation and at a slow pace, after the students have understood the underlying concepts. Pólya for example, in the article about *Notation* of his *How to solve it?*, after discussing devices to introduce effective notations, remarks:

> **5.c)** Not only the most hopeless boys in the class but also quite intelligent students may have an aversion for algebra. There is always something arbitrary and artificial about notation; to learn a new notation is a burden for the memory. The intelligent student refuses to assume the burden if he does not see any compensation for it. The intelligent student is justified in his aversion for algebra if he is not given ample opportunity to convince himself by his own experience that *the language of mathematical symbols assist the mind*. To help him to such experience is an important task of the teacher, one of his most important tasks. (Pólya 1990, p. 140)

Péter Rózsa's *Playing with infinity*, written in a literary style and almost without formulas, offers a beautiful example of this endeavour. One of the rare places in the book where she introduces a formula is at the end of the mentioned classroom example, which she concludes in the following way:

> The above formula is only a symbol and means nothing by itself; everybody can substitute into it his own experiences. For one it might mean the counting of the diagonals of a polygon, for another the counting of the number of possibilities for choosing the leading pair among his pupils. The writing down of a formula is an expression of our joy that we can answer all these questions by means of one argument. (Péter 1961, p. 33)

So, a formula is not an a priori given mathematical object, but the result of a process, filled with meaning obtained from a diversity of experiences.

Varga's educational conception encourages students' personal notations and their comparison in concrete problem situations. He often introduces non-standard notations and representational tools that can be still illustrative and effective at a certain level of learning mathematics (e.g. in C. Neményi and Varga 1978).

3.6 Art, Playing, Creativity

The role of arts and playing in mathematics is a recurring theme, especially in the works of Tamás Varga and Rózsa Péter. According to them, playfulness is inseparable from the process of mathematical creation; and in playfulness, the nature of mathematics as an art manifests itself.

Rózsa Péter—who herself translated Rilke into Hungarian and used to write film reviews—fought regularly for the unity of the "two cultures": mathematics and arts.[22] In her opinion, the main common characteristic of these two domains is that both are *free creations of the human mind*. In her *Playing with infinity*, curiosity and pleasure of discovery often appear as the main driving forces of mathematical research.

Tamás Varga in 1946, as a 26-year-old teacher wrote to Kalmár:

> **6.a)** […] there are two subjects. Of course not arithmetic and geometry. But:
> 1) Computing the world
> 2) Playing with numbers (and figures, and objects… this is just as intertwined with the natural sciences as 1).
> 1) is the science side,
> 2) is the art side of the *me* and *him*.
> […] I always preferred the arts side. I noticed this, as I always only liked showing this kind of things to the first year students. (Szabó 2005, p. 403. My translation)

And in his last article in 1987, he wrote:

> **6.b)** Mathematics, from the lowest to the highest levels, is always based on experience: trial and error, conjectures and their checks, their rejection or confirmation. Still, it is a free creation of the human mind. It is a bridge between the two cultures. It is full of playfulness and aesthetics: it is also an art. (Varga 1987, p. 28. My translation)

In the context of arts, playing and mathematical research, both Rózsa Péter and Tamás Varga underline the role of personality, intellectual liberty and of affective elements. Therefore, they stress that playing, beauty, children's natural curiosity and creativity are not only consistent with, but even necessary elements of mathematics teaching. The learning of mathematics can only be effective if, in consequence of the foregoing, it is a joyful activity.

4 Summary

In this article, I attempted to confirm the existence of a Hungarian mathematical culture with specific traditions and specific conceptions of mathematics and its teaching. In the first part, I resumed the educational reforms and the social-economical-cultural context of the emergence of a Hungarian mathematical research culture about the turn of the 19th and the 20th century. In the second part, I

[22] E.g. Péter (2004) or the appendix of *Playing with Infinity* published in several Hungarian editions (e.g. in Péter 1969, pp. 257–267).

presented a distinct group of Hungarian mathematicians and I analysed their different writings, published mostly from the 1940s to the 1970s. This analysis revealed some main principles of a quite coherent conception: a conception that could influence general education in Hungary through the reform led by Tamás Varga. According to these mathematicians:

1. Mathematics is not static and everlasting, but is rather a constantly developing and changing creation of the human mind. Students should be accompanied through the same process.
2. The source of mathematics is intuition and experience. (Not constrained to actual physical observations.) Without it, neither mathematical creation, nor real understanding can be achieved, so it is important to develop intuition with the help of a handful of experiences in every level of education.
3. Mathematical activity is basically dialogical; it is a sequence of questions, problems and the attempts to answer them. Teaching mathematics is not a one-sided passing on of knowledge; it is more a joint activity of the student and the teacher. The teacher acts as an aid in the students' rediscovery of mathematics.
4. The aim of teaching mathematics is not to pass on recipes for computation for a user without reflection; it is to provide an introduction to the process of mathematical creation, and consequently educate thinking people.
5. Excessive formalism is discouraged; a formal language should be introduced only after proper preparations.
6. The process of mathematical creation is closely connected with play. In playfulness, the artistic nature of mathematics manifests itself.

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