Chapter 11
Multigrid Iterations

Abstract Multigrid methods belong to the class of fastest linear iterations, since their convergence rate is bounded independently of the step size h. Furthermore, their applicability does not require symmetry or positive definiteness. Books devoted to multigrid are Hackbusch [183], Wesseling [395], Trottenberg–Oosterlee–Schuller [367], Shaidurov [338], and Vassilevski [378]; see also [205, pp. 1–312].

The 'smoothing step' and the 'coarse-grid correction' together with the involved restrictions and prolongations are typical ingredients of the multigrid iteration. They are introduced in Section 11.1 for the Poisson model problem. The two-grid iteration explained in Section 11.2 is the first step towards the multigrid method. The iteration matrix is provided in §11.2.3. First numerical examples are presented in §11.2.4.

Before a more general proof of convergence is presented, Section 11.3 investigates the one-dimensional model problem. The proof demonstrates the complementary roles of the smoothing part and the coarse-grid correction. Moreover, the dependence of the convergence rate on the number of smoothing steps is determined.

The multigrid iteration is defined in Section 11.4. Its computational work is discussed and numerical examples are presented. The iteration matrix is described in §11.4.4.

The nested iteration presented in Section 11.5 is a typical technique combined with the multigrid iteration. In principle, it can be combined with any iteration, provided that a hierarchy of discretisations is given. Besides a reduction of the computational work, the nested iteration technique allows us to adjust the iteration error to the discretisation error.

A general convergence analysis of the W-cycle is presented in Section 11.6. Stronger statements are possible in the positive definite case which is studied in Section 11.7. Here, also the V-cycle convergence is proved. As long as lower order terms are responsible for the nonsymmetric structure, the symmetric convergence results can be transferred as shown in §11.7.6. This includes the case of the V-cycle.

Possible combinations with semi-iterative methods are discussed in Section 11.8. Concluding comments are given in Section 11.9.

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11.1 Introduction

Multigrid iterations consist of two complementary parts: the *smoothing step* and the *coarse-grid correction*. Below we explain both steps in the case of the Poisson model problem.

11.1.1 Smoothing

Let A be the matrix of the Poisson model problem with step size h. As the simplest example we choose Richardson's iteration:

$$x^{m+1} = \Phi_\Theta^{\mathrm{Rich}}(x^m, b) = x^m - \Theta(Ax^m - b) \qquad \text{with} \tag{11.1a}$$

$$\Theta = \tfrac{1}{8}h^2 \approx 1/\lambda_{\max}(A) = 1/\rho(A) \qquad \text{(cf. (3.1c))}. \tag{11.1b}$$

$\rho(A) = \lambda_{\max}(A)$ is the eigenvalue corresponding to the eigenfunction

$$e^{\alpha\beta}(x,y) = 2h\sin(\alpha\pi x)\sin(\beta\pi y) \quad (1 \le \alpha, \beta \le N-1, (x,y) \in \Omega_h) \tag{11.2a}$$

of the highest frequency $\alpha = \beta = N-1$ (cf. (3.2)). The convergence rate $\rho(M_\Theta^{\mathrm{Rich}}) = 1 - \Theta\lambda_{\min} \approx 1 - \lambda_{\min}/\lambda_{\max} \le 1 - \mathcal{O}(h^2)$ is attained by the lowest frequency $\alpha = \beta = 1$, i.e., when the error $e^m = x^m - x$ is a multiple of the eigenfunction $e^{1,1}$.

All $x \in X := \mathbb{R}^I$ can be represented by the orthonormal eigenvector basis (11.2a):

$$x = \sum_{\alpha,\beta=1}^{N-1} \xi_{\alpha\beta}\, e^{\alpha\beta} \qquad \text{with } \xi_{\alpha\beta} := \left\langle x, e^{\alpha\beta} \right\rangle. \tag{11.2b}$$

Since high frequencies α, β correspond to *strong oscillations* of the sine functions (11.2a), we define

$$X_{\mathrm{osc}} := \mathrm{span}\left\{ e^{\alpha\beta} :\ 1 \le \alpha, \beta \le N-1,\ \max\{\alpha,\beta\} > \tfrac{N}{2} \right\} \tag{11.2c}$$

as a subspace of the *oscillatory components*. Note that at least one of the indices α, β lies in the *high-frequency* part $(N/2, N)$ of the frequency interval $[1, N-1]$. If we are able to generate an approximation x^0, whose error lies in the subspace X_{osc},

$$e^0 := x^0 - x \in X_{\mathrm{osc}} \qquad (e^0 \text{ is the error, not an eigenvector!}), \tag{11.2d}$$

the simple Richardson iteration yields fast convergence.

Lemma 11.1. *Assume the Poisson model case with (11.2d). Then all succeeding errors e^m also belong to X_{osc} and satisfy the error estimate*

$$\|e^m\|_2 \le \tfrac{3}{4}\|e^{m+1}\|_2, \tag{11.3}$$

i.e., restricted to X_{osc}, the convergence rate is h-independent.

Proof. Since the vectors (11.2a) are orthonormal (cf. Lemma 3.2), we have

$$\|x\|_2^2 = \sum_{\alpha,\beta=1}^{N-1} |\xi_{\alpha\beta}|^2 \qquad \text{for } x \text{ in (11.2b).}$$

Because of $Me^{\alpha\beta} = (1 - \Theta\,\lambda_{\alpha\beta})\, e^{\alpha\beta}$, applying the iteration matrix $M = I - \Theta A$ to the error e^m with coefficients $\xi_{\alpha\beta}$ yields e^{m+1} satisfying

$$\begin{aligned} \|e^{m+1}\|_2^2 \;&\le\; \sum |1 - \Theta\lambda_{\alpha\beta}|^2\, |\xi_{\alpha\beta}|^2 \;\le\; \max |1 - \Theta\lambda_{\alpha\beta}|^2 \sum |\xi_{\alpha\beta}|^2 \\ &=\; \max |1 - \Theta\lambda_{\alpha\beta}|^2\, \|e^m\|_2^2 \,, \end{aligned}$$

with $\lambda_{\alpha\beta} = 4h^{-2}\left[\sin^2(\alpha\pi h/2) + \sin^2(\beta\pi h/2)\right]$ (cf. (3.1a)). The maximum has to be taken over all α, β appearing in (11.2c). By symmetry, we may restrict the frequencies to $0 < \alpha < N$ and $N/2 < \beta < N$. For these α, β,

$$2h^{-2} = 4h^{-2}\sin^2(\pi/4) < \lambda_{\alpha\beta} \le \lambda_{N-1,N-1} < 8h^{-2}$$

holds; hence, $|1 - \Theta\lambda_{\alpha\beta}| = |1 - \frac{h^2}{8}\lambda_{\alpha\beta}| < 1 - \frac{h^2}{8}2h^{-2} = \frac{3}{4}$ proves the desired inequality (11.3). □

The statement of the lemma is not of direct practical use because the assumption (11.2d) cannot be established in practice (at least not with less work than for solving $Ax = b$ exactly). However, we can conclude the following estimate involving the *smooth subspace* $X_{\rm sm} := X_{\rm osc}^{\perp} = \operatorname{span}\{e^{\alpha\beta} :\ 1 \le \alpha, \beta \le N/2\}$.

Conclusion 11.2. *Split the starting error e^0 into*

$$e^0 = e^0_{\rm osc} + e^0_{\rm sm}, \quad e^0_{\rm osc} \in X_{\rm osc}, \quad e^0_{\rm sm} \in X_{\rm sm} := X_{\rm osc}^{\perp}.$$

Then, after m steps of Richardson's iteration (11.1a,b), we have

$$\begin{aligned} e^m &= e^m_{\rm osc} + e^m_{\rm sm} \qquad \textit{with} \\ &\quad e^m_{\rm osc} = M^m e^0_{\rm osc} \in X_{\rm osc}, \quad e^m_{\rm sm} = M^m e^0_{\rm sm} \in X_{\rm sm}, \\ &\quad \|e^m_{\rm osc}\|_2 \le \left(\tfrac{3}{4}\right)^m \|e^0_{\rm osc}\|_2, \end{aligned}$$

while $e^m_{\rm sm}$ converges only very slowly to 0. Since $e^m_{\rm osc}$ decreases faster than $e^m_{\rm sm}$, the smooth part of e^m has increased, and one may regard e^m as 'smoother' than e^0. The smoothness of e^m can be measured by the ratio $\|e^m_{\rm sm}\|_2/\|e^m_{\rm osc}\|_2$.

For illustration purposes, we present the numerical results for the system

$$Ax = b \qquad \text{with} \quad A = h^{-2}\,\operatorname{tridiag}\{-1, 2, -1\} \tag{11.4a}$$

of $n = N - 1 = \frac{1}{h} - 1$ equations corresponding to the one-dimensional Poisson boundary value problem

$$-u''(x) = f(x) \quad \text{for } 0 < x < 1, \qquad u(0) = u_0, \quad u(1) = u_1. \tag{11.4b}$$

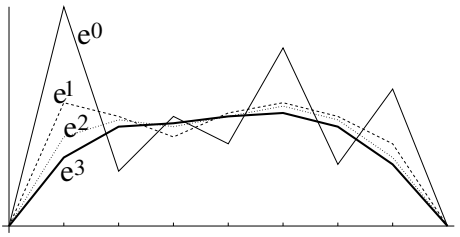

Fig. 11.1 Errors $e^m \in X_{\rm osc}$ of example (11.4a).

Figure 11.1 shows the (piecewise linearly connected) initial values e_i^0 $(0 \le i \le N = 8)$ and the errors e^m of the first three Richardson iterates. The errors e^m are insignificantly smaller than e^0 but clearly smoother.

We call iterative methods such as the Richardson iteration (11.1a,b) *smoothing iterations* and use the symbol $\mathcal{S}$ instead of Φ.

Exercise 11.3. The choice of Θ in (11.1b) is not the optimal value. Determine Θ such that the bound in (11.3) becomes minimal.

In the following, an iteration Ψ with a complementary property is desired: Ψ should effectively reduce the smooth components in

$$X_{\rm sm} = X_{\rm osc}^{\perp} = \operatorname{span}\left\{e^{\alpha\beta} : 1 \le \alpha, \beta \le N/2\right\}.$$

Ψ is not required to have good convergence with respect to the subspace $X_{\rm osc}$. Then the product method $\Psi \circ \mathcal{S}$ would have the property that for both subspaces $X_{\rm osc}$ and $X_{\rm sm}$ one of the factors in $\Psi \circ \mathcal{S}$ yields fast convergence.

Unfortunately, none of the methods mentioned so far has this property. To construct such an iteration Ψ, we adhere to the concept that smooth grid functions can be well approximated by using a coarser grid. After introducing coarser grids in §§11.1.2–11.1.4, we shall return to the construction of the *coarse-grid correction* Ψ in §11.1.5.

11.1.2 Hierarchy of Systems of Equations

For the following considerations, we have to embed the problem $Ax = b$ into a family of systems. In the model case, for all step sizes $h = 1/N$, we obtain a system $Ax = b$ depending on N or h, respectively. Let

$$h_0 > h_1 > \ldots > h_{\ell-1} > h_\ell > \ldots \qquad \text{with } \lim_{\ell\to\infty} h_\ell = 0$$

be a sequence of step sizes, which may be generated, e.g., by

$$h_\ell := h_0/2^\ell \qquad (\ell \ge 0). \tag{11.5a}$$

The index ℓ is the level-number. $\ell = 0$ corresponds to the coarsest grid. In the model case, for which the grid $\Omega_\ell := \Omega_{h_\ell}$ is contained in the unit square, the step size

$$h_0 = 1/2 \tag{11.5b}$$

is the coarsest one. Then $\Omega_0 = \Omega_{h_0}$ contains only one interior grid point.

Each step size h_ℓ (i.e., each level ℓ) corresponds to a system

$$A_\ell x_\ell = b_\ell \qquad (\ell = 0, 1, 2, \ldots) \tag{11.6a}$$

of order n_ℓ, which in the model case amounts to

$$n_\ell = (N_\ell - 1)^2 = (1/h_\ell - 1)^2. \tag{11.6b}$$

The family of systems (11.6a) for $\ell = 0, 1, 2, \ldots$ represents the hierarchy of systems of equations. The actual problem $Ax = b$ to be solved corresponds to a particular level $\ell = \ell_{\max}$. For solving $A_\ell x_\ell = b_\ell$ at $\ell = \ell_{\max}$, we shall use all lower levels $\ell < \ell_{\max}$.

Remark 11.4. Concerning the construction of the family $\{A_\ell, b_\ell\}_{\ell=0,1,\ldots}$ of discretisations, we mention two quite different approaches:

(A) A maximal level $\ell_{\max}$ and the corresponding system $A_{\ell_{\max}} x_{\ell_{\max}} = b_{\ell_{\max}}$ are given. Then auxiliary problems $A_\ell x_\ell = b_\ell$ for $\ell < \ell_{\max}$ are created in some way.

(B) The discretisation starts with $A_0 x_0 = b_0$. Local (or global) grid refinement is used to construct $A_\ell x_\ell = b_\ell$ for $\ell = 1, 2, \ldots$ until the discretisation error is sufficiently small.

11.1.3 Prolongation

The vectors x_ℓ and b_ℓ in (11.6a) are elements of the vector space

$$X_\ell = \mathbb{R}^{n_\ell}. \tag{11.7}$$

To connect different levels $\ell = 0, 1, 2, \ldots, \ell_{\max}$, we introduce the prolongation

$$p : X_{\ell-1} \to X_\ell \qquad (\ell \geq 1), \tag{11.8}$$

which is assumed to be a linear and injective mapping (more precisely; a family of mappings[1] for all $\ell \geq 1$) from the coarse grid into the fine one.

In the one-dimensional case (11.4a), the vector x_ℓ can be regarded as a grid function defined on $\Omega_\ell = \{\mu h_\ell : 0 < \mu < N_\ell = 1/h_\ell\}$. The vector x_ℓ is rewritten as u_ℓ if it is understood as a grid function on Ω_ℓ with values

$$u_\ell(\mu h_\ell) = x_{\ell,\mu} \qquad (1 \leq \mu \leq N_\ell - 1),$$

i.e., for all step sizes the arguments of u_ℓ belong to the interval $\Omega = (0, 1)$ and are restricted to the nodal points in Ω_ℓ. For ease of notation, we include the boundary values

$$u_\ell(0) = u_\ell(1) = 0. \tag{11.9}$$

[1] A more precise notation would be $p_{\ell,\ell-1}$ indicating the involved levels. However, the context will uniquely determine the levels.

An obvious proposal for the prolongation p is piecewise linear interpolation between the grid points of $\Omega_{\ell-1}$:

$$(pu_{\ell-1})(\xi) := u_{\ell-1}(\xi) \qquad \text{for } \xi \in \Omega_{\ell-1} \subset \Omega_\ell, \tag{11.10a}$$

$$(pu_{\ell-1})(\xi) := \frac{1}{2}\left[u_{\ell-1}(\xi + h_\ell) + u_{\ell-1}(\xi - h_\ell)\right] \qquad \text{for } \xi \in \Omega_\ell \backslash \Omega_{\ell-1}, \tag{11.10b}$$

where definition (11.9) is used at $\xi = h_\ell$ and $\xi = 1 - h_\ell$. A shorter characterisation of the prolongation p is the symbol (11.10c):

$$p = \left[\tfrac{1}{2} \quad 1 \quad \tfrac{1}{2}\right]. \tag{11.10c}$$

The stencil in (11.10c) indicates that the unit vector $x_{\ell-1} = (\ldots, 0, 1, 0, \ldots)^\mathsf{T}$ is mapped into $x_\ell = p x_{\ell-1} = (\ldots, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \ldots)^\mathsf{T}$.

For the *two*-dimensional Poisson equation, the vector x_ℓ is represented by the grid function u_ℓ:

$$u_\ell(\xi, \eta) = x_{\ell,ij} \qquad \text{for } 1 \le i, j \le N_\ell - 1,\ (\xi, \eta) = (ih_\ell, jh_\ell) \in \Omega_\ell,$$

where the boundary values are defined by

$$u_\ell(\xi, \eta) := 0 \qquad \text{for } \xi = 0 \text{ or } \xi = 1 \text{ or } \eta = 0 \text{ or } \eta = 1.$$

The two-dimensional generalisation of the piecewise linear interpolation (11.10a,b) (bilinear interpolation) reads as follows:

$$\begin{aligned}
(pu_{\ell-1})(\xi, \eta) &:= u_{\ell-1}(\xi, \eta) \qquad \text{for } (\xi, \eta) \in \Omega_{\ell-1} \subset \Omega_\ell, \\
(pu_{\ell-1})(\xi, \eta) &:= \frac{1}{2}\left[u_{\ell-1}(\xi + h_\ell, \eta) + u_{\ell-1}(\xi - h_\ell, \eta)\right] \\
&\qquad \text{for } \xi/h_\ell \text{ odd},\ \eta/h_\ell \text{ even}, \\
(pu_{\ell-1})(\xi, \eta) &:= \frac{1}{2}\left[u_{\ell-1}(\xi, \eta + h_\ell) + u_{\ell-1}(\xi, \eta - h_\ell)\right] \\
&\qquad \text{for } \xi/h_\ell \text{ even},\ \eta/h_\ell \text{ odd}, \\
(pu_{\ell-1})(\xi, \eta) &:= \frac{1}{4}\left[\begin{array}{r} u_{\ell-1}(x + h_\ell, h + h_\ell) + u_{\ell-1}(x - h_\ell, h - h_\ell) \\ +u_{\ell-1}(x - h_\ell, h + h_\ell) + u_{\ell-1}(x + h_\ell, h - h_\ell) \end{array}\right] \\
&\qquad \text{for } \xi/h_\ell \text{ and } \eta/h_\ell \text{ odd}.
\end{aligned}$$

The abbreviation of p defined above is the star

$$p = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \qquad (\textit{nine-point prolongation}), \tag{11.11}$$

since the application of p to a unit vector yields the values indicated in (11.11) extended by zero is the remaining grid.

In general, the stencil

$$p = \begin{bmatrix} \pi_{-1,1} & \pi_{0,1} & \pi_{1,1} \\ \pi_{-1,0} & \pi_{0,0} & \pi_{1,0} \\ \pi_{-1,-1} & \pi_{0,-1} & \pi_{1,-1} \end{bmatrix} \tag{11.12}$$

describes the following mapping, where the summation is taken over all i, j with $(\xi - ih_\ell, \eta - jh_\ell) \in \Omega_{\ell-1}$:

$$(pu_{\ell-1})(\xi,\eta) := \sum_{i,j} \pi_{ij} u_{\ell-1}(\xi - ih_\ell, \eta - jh_\ell) \quad \text{for } (\xi,\eta) \in \Omega_\ell.$$

Other linear interpolations as well as prolongations of higher order are discussed by Hackbusch [183, §3.4]. A so-called *matrix-dependent prolongation* is defined by (11.12) with the coefficients

$$\pi_{00} := 1, \quad \pi_{\pm1,0} := -\frac{\sum_j \alpha_{\mp1,j}}{\sum_j \alpha_{0,j}}, \quad \pi_{0,\pm1} := -\frac{\sum_i \alpha_{i,\mp1}}{\sum_i \alpha_{i,0}}, \tag{11.13a}$$

$$(A_\ell\, p\, u_{\ell-1})(\xi,\eta) = 0 \qquad \text{for } \xi/h_\ell \text{ and } \eta/h_\ell \text{ odd}, \tag{11.13b}$$

where $\alpha_{i,j}$ are the coefficients of A_ℓ according to (1.13a,b). Condition (11.13b) determines $\pi_{\pm1,\pm1}$ (cf. Hackbusch [183, §10.3] and de Zeeuw [104]).

11.1.4 Restriction

The restriction r is a linear and surjective mapping

$$r : X_\ell \to X_{\ell-1} \qquad (\ell \geq 1)\,,$$

which maps fine-grid functions into coarse-grid functions. If $\Omega_{\ell-1} \subset \Omega_\ell$ holds as in the model case, the simplest choice is the trivial restriction

$$(r_{\text{triv}} u_\ell)(\xi,\eta) = u_\ell(\xi,\eta) \qquad \text{for } (\xi,\eta) \in \Omega_{\ell-1}\,.$$

However, because of certain disadvantages, we advise against its use (cf. Hackbusch [183, §3.5]). Instead, we define $(r\,u_\ell)(\xi,\eta)$ as the weighted mean of the neighbouring values. The stencil

$$r = \begin{bmatrix} \rho_{-1,1} & \rho_{0,1} & \rho_{1,1} \\ \rho_{-1,0} & \rho_{0,0} & \rho_{1,0} \\ \rho_{-1,-1} & \rho_{0,-1} & \rho_{1,-1} \end{bmatrix} \tag{11.14}$$

characterises the restriction

$$(ru_\ell)(\xi,\eta) = \sum_{i,j=-1}^{1} \rho_{ij}\, u_\ell(\xi + ih_\ell, \eta + jh_\ell) \qquad \text{for } (\xi,\eta) \in \Omega_{\ell-1}.$$

The nine-point prolongation (11.11) corresponds to the nine-point restriction

$$r = \frac{1}{4}\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}, \tag{11.15}$$

which can be considered as the adjoint to (11.11), where the definition of adjoint mappings is based on the scalar products

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\ell \qquad \text{with} \quad \langle u_\ell, v_\ell \rangle_\ell = h_\ell^d \sum\nolimits_{\alpha \in I} u_{\ell,\alpha}\, \overline{v_{\ell,\alpha}} \tag{11.16}$$

for X_ℓ. d is the dimension of the grid $\Omega_\ell \subset \mathbb{R}^d$. The adjoint mapping is denoted by p^*. Since p can also considered as a matrix, the transposed matrix p^T is defined. Because of the different weighting factors h_ℓ^d in (11.16), p^* and p^T differ by a factor as stated in the next exercise.

Exercise 11.5. Assume the two-dimensional case $d = 2$ and prove that the mapping adjoint to p defined in (11.12) is r in (11.14) with $\rho_{ij} = \pi_{ij}/4$. Prove for general d that $p^* = 2^{-d} p^\mathsf{T}$.

Having fixed the prolongation, we can always choose the adjoint mapping

$$r := p^* \tag{11.17}$$

as a restriction. For example, we can define a matrix-dependent restriction by (11.13a,b) and (11.17).

11.1.5 Coarse-Grid Correction

Let $\bar{x}_\ell$ be the result of a few steps of the smoothing iteration (11.1a,b). The corresponding error $\bar{e}_\ell := \bar{x}_\ell - x_\ell$ is the exact correction; i.e., the solution can be obtained by

$$x_\ell = \bar{x}_\ell - \bar{e}_\ell \,.$$

Since $A_\ell \bar{e}_\ell = A_\ell(\bar{x}_\ell - x_\ell) = A_\ell \bar{x}_\ell - A_\ell x_\ell = A_\ell \bar{x}_\ell - b_\ell$, the correction $\bar{e}_\ell$ satisfies the equation

$$A_\ell\, \bar{e}_\ell = d_\ell \qquad \text{with the } \textit{defect } \; d_\ell := A_\ell \bar{x}_\ell - b_\ell \,. \tag{11.18a}$$

According to considerations in §11.1.1, $\bar{e}_\ell$ is smooth. Therefore, it should be possible to approximate $\bar{e}_\ell$ by using the coarse grid: $\bar{e}_\ell \approx p e_{\ell-1}$. As ansatz for $e_{\ell-1}$, we take the coarse-grid equation corresponding to (11.18a):

$$A_{\ell-1} e_{\ell-1} = d_{\ell-1} \qquad \text{with } \; d_{\ell-1} := r d_\ell. \tag{11.18b}$$

Assume that we are able to solve the coarse-grid equation (11.18b) exactly:

$$e_{\ell-1} = A_{\ell-1}^{-1} d_{\ell-1}\,. \tag{11.18c}$$

Its image $pe_{\ell-1}$ under the prolongation p should approximate the solution $\bar{e}_\ell$ of (11.18a), so that the coarse-grid correction is completed by

$$x_\ell^{\mathrm{new}} := \bar{x}_\ell - p\, e_{\ell-1}. \tag{11.18d}$$

In compact form, the coarse-grid correction (11.18a–d) reads as follows:

$$\bar{x}_\ell \longmapsto x_\ell^{\mathrm{new}} := \bar{x}_\ell - pA_{\ell-1}^{-1} r(A_\ell \bar{x}_\ell - b_\ell).$$

Renaming $\bar{x}_\ell$ and x_ℓ^{new} by x_ℓ^m and x_ℓ^{m+1}, the mapping above defines an iterative method which we call the *coarse-grid correction*:

$$\Phi_\ell^{\mathrm{CGC}}(x_\ell, b_\ell) := x_\ell - pA_{\ell-1}^{-1} r(A_\ell x_\ell - b_\ell). \tag{11.19}$$

Remark 11.6. The iteration matrix M_ℓ^{CGC} and the matrix N_ℓ^{CGC} of the second normal form of the coarse-grid correction are

$$M_\ell^{\mathrm{CGC}} = I - pA_{\ell-1}^{-1} r A_\ell\,, \qquad N_\ell^{\mathrm{CGC}} = pA_{\ell-1}^{-1} r\,.$$

Φ_ℓ^{CGC} (as such without smoothing) is not an interesting iteration as stated next.

Remark 11.7. The coarse-grid correction Φ_ℓ^{CGC} is consistent, but not convergent.

Proof. The consistency is a consequence of the second normal form. $n_\ell > n_{\ell-1}$ implies $\dim X_\ell > \dim X_{\ell-1}$; hence, the kernel of the restriction r is nontrivial. Let $0 \neq x \in \ker(r)$. Since $M_\ell^{\mathrm{CGC}}\eta = \eta$ for $\eta := A_\ell^{-1}x$, the matrix M_ℓ^{CGC} has an eigenvalue $\lambda = 1$, so that $\rho(M_\ell^{\mathrm{CGC}}) \geq 1$ indicates divergence. □

For systems obtained from Galerkin discretisation (cf. Proposition E.16 and Hackbusch [183, Note 3.6.6]), the so-called *Galerkin product* representation of $A_{\ell-1}$ is valid:

$$A_{\ell-1} = r\, A_\ell\, p. \tag{11.20}$$

Remark 11.8. Given $A = A_{\ell_{\max}}$, one can use (11.20) as a recursive definition of the coarse-grid matrices A_ℓ for $\ell = \ell_{\max} - 1, \ldots, 0$, provided that suitable mappings r and p are available (see case (A) in Remark 11.4). If one uses the definition (11.17) of r, only the prolongations p have to be defined.

Lemma 11.9. *Assume (11.20). Then $\Phi_\ell^{\mathrm{CGC}}(\hat{x}_\ell, b_\ell) = \hat{x}_\ell$ holds for all vectors $\hat{x}_\ell$ with $\hat{e}_\ell = \hat{x}_\ell - x_\ell \in \mathrm{range}(p)$; i.e., Φ_ℓ^{CGC} is a projection (cf. Definition 5.12).*

Proof. Use $M_\ell^{\mathrm{CGC}} p = p - pA_{\ell-1}^{-1} r A_\ell p = p - pA_{\ell-1}^{-1} A_{\ell-1} = p - p = 0$. □

The next exercise shows that the coarse-grid equation (11.18b) is a reasonable ansatz for $e_{\ell-1}$.

Exercise 11.10. Let A_ℓ be positive definite. The best approximation of $\bar{e}_\ell \in X_\ell$ with respect to the A_ℓ norm $\|x_\ell\|_A := \langle A_\ell x_\ell, x_\ell \rangle_\ell^{1/2}$ is $pe_{\ell-1}$, where $p = r^*$ according to (11.17) and $e_{\ell-1}$ is the solution of (11.18b) with the Galerkin matrix (11.20).

11.2 Two-Grid Method

11.2.1 Algorithm

The smoothing iteration $\mathcal{S}_\ell$ is defined in §11.1.1 and the coarse-grid correction Φ_ℓ^{CGC} is constructed in §11.1.5. The two-grid iteration is the product iteration

$$\Phi_\ell^{\mathrm{TGM}} := \Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^\nu \qquad (\ell \geq 1,\ \nu \geq 1)$$

(cf. §5.4), where ν is the number of smoothing steps. In algorithmic notation, the iteration Φ_ℓ^{TGM} takes the form

(11.21)

function $\Phi_\ell^{\mathrm{TGM}}(x_\ell, b_\ell)$;
begin for $i := 1$ **to** ν **do** $x_\ell := \mathcal{S}_\ell(x_\ell, b_\ell)$; (11.21a)
$d_{\ell-1} := r(A_\ell x_\ell - b_\ell)$; (11.21b)
$e_{\ell-1} := A_{\ell-1}^{-1} d_{\ell-1}$; (11.21c)
$x_\ell := x_\ell - p e_{\ell-1}$; (11.21d)
$\Phi_\ell^{\mathrm{TGM}} := x_\ell$ (11.21e)
end;

11.2.2 Modifications

As stated in Proposition 5.25b, $\Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^\nu$ has the same convergence behaviour as

$$\Phi_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} := \mathcal{S}_\ell^{\nu_2} \circ \Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^{\nu_1} \qquad \text{with } \nu = \nu_1 + \nu_2. \tag{11.22a}$$

In this case, ν_1 pre- and ν_2 post-smoothing steps are applied. Algorithm (11.21) is the special case of iteration (11.22a) with $\nu_1 = \nu$ and $\nu_2 = 0$. In the sequel, we use the more general version (11.22a).

One may also use different iterations $\mathcal{S}_\ell$ and $\hat{\mathcal{S}}_\ell$ as pre- and post-smoothers:

$$\hat{\mathcal{S}}_\ell^{\nu_2} \circ \Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^{\nu_1}. \tag{11.22b}$$

A semi-iterative smoothing instead of (11.21a) will be discussed in §11.8.1.

11.2.3 Iteration Matrix

Lemma 11.11. *Let $\mathcal{S}_\ell$ be a consistent iteration with iteration matrix S_ℓ. Then $\Phi_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}$ is a consistent iteration with the iteration matrix*

$$M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} = S_\ell^{\nu_2} \left(I - p A_{\ell-1}^{-1} r A_\ell\right) S_\ell^{\nu_1}. \tag{11.23}$$

Proof. According to Proposition 5.25b, M_ℓ^{TGM} is the product of the iteration matrices of $\hat{\mathcal{S}}_\ell^{\nu_2}$, Φ_ℓ^{CGC}, $\mathcal{S}_\ell^{\nu_1}$. Equation (11.23) follows from Remark 11.6. □

Since $\mathcal{S}_\ell$ is consistent, the matrix $N_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}$ of the second normal form is implicitly determined by

$$M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} = I - N_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} A_\ell \,.$$

As known from (5.12b), the second normal form matrices do not have a simple representation for product iterations. The same statement holds for $W_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}$.

11.2.4 Numerical Examples

As an example we choose the two-dimensional Poisson model problem with the step sizes h_ℓ in (11.5a,b). $A_{\ell_{\max}}$ and the auxiliary matrices A_ℓ ($\ell < \ell_{\max}$) are defined by the five-point discretisation (1.4a). The two-grid parameters are $\nu_1 = 2$, $\nu_2 = 0$. The smoothing iteration is the chequer-board variant of the Gauss–Seidel iteration (cf. (1.20)). The error norms $\|e_\ell^m\|_2 = \|x_\ell^m - x_\ell\|_2$ at level $\ell = 5$ with $h_5 = 1/64$ are shown in Table 11.1. Table 11.2 contains the reduction factors $\|x_\ell^m - x_\ell\|_2 / \|x_\ell^{m-1} - x_\ell\|_2$. The last row in Table 11.2 shows the averaged convergence factors $\rho_\ell := (\|e_\ell^8\|_2 / \|e_\ell^0\|_2)^{1/8}$. In contrast to the foregoing iterative methods, the convergence factors hardly depend on the step size. Furthermore, the convergence rate of about 0.06 is very favourable.

m	$\|x_\ell^m - x_\ell\|_2$
0	2.935_{10}-02
1	1.210_{10}-03
2	6.206_{10}-05
3	3.378_{10}-06
4	1.939_{10}-07
5	1.152_{10}-08
6	7.058_{10}-10
7	4.432_{10}-11
8	7.188_{10}-12

Table 11.1 Iteration errors for $h_5 = \frac{1}{64}$.

Since the two-grid method depends on the parameters ν_1 and ν_2, their influence on the convergence is be investigated. As mentioned in §11.2.2, the convergence rate depends only on $\nu = \nu_1 + \nu_2$. Therefore, we may choose $\nu_1 = \nu$ and $\nu_2 = 0$ without loss of generality. The convergence factors ρ_ℓ for $h_3 = 1/16$ determined as above are shown in Table 11.3. As expected, convergence improves with increasing ν. In the last row, $\rho_3(\nu)$ is compared with the function $C/(C+\nu)$ for $C = 0.135$. It suggests the asymptotic behaviour $\rho_\ell(\nu) \approx \mathcal{O}(1/\nu)$.

m	$h_\ell = \frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
1	0.10391	0.10420	0.07778	0.05465	0.03807	0.02661
2	0.06210	0.05549	0.04730	0.04336	0.04121	0.04009
3	0.06248	0.05738	0.05409	0.05238	0.05132	0.05077
4	0.06250	0.05851	0.05804	0.05565	0.05445	0.05375
5	0.06250	0.05963	0.06191	0.05866	0.05741	0.05665
6	0.06250	0.06061	0.06512	0.06089	0.05937	0.05835
7	0.06250	0.06143	0.06768	0.06292	0.06124	0.05996
8	0.06250	0.06208	0.06954	0.06463	0.06285	0.06132
ρ_ℓ	0.06654	0.06360	0.06203	0.05626	0.05248	0.04939

Table 11.2 Error ratios $\|x^m - x_\ell\|_2 / \|x^{m-1} - x_\ell\|_2$ and averaged convergence factors ρ_ℓ for the two-grid method with $\nu_1 = 2$ and $\nu_2 = 0$.

ν	1	2	3	4	5	6	10
$\rho_3(\nu)$	0.222	0.062	0.04	0.03	0.023	0.0196	0.0133
$\frac{0.135}{0.135+\nu}$	0.119	0.063	0.043	0.033	0.026	0.022	0.0133

Table 11.3 Convergence factors for different smoothing numbers ν.

11.3 Analysis for a One-Dimensional Example

In principle, one can analyse the two-grid convergence for the two-dimensional Poisson model problem (cf. Hackbusch [183, §8.1.1]); however, it is not sufficiently transparent for an introductory consideration. Therefore, we consider the tridiagonal equation (11.4a):

$$Ax = b \qquad \text{with } A = h^{-2} \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \tag{11.24}$$

discretising the one-dimensional Poisson equation (11.4b). It should be emphasised that tridiagonal matrices are easy to solve directly. Analysis of iterative methods for these tridiagonal equations is of interest only because of the fact that the convergence properties also carry over to the general case of two or more spatial dimensions. Furthermore, this chapter serves as a demonstration of how model problems can be investigated by the help of Fourier analysis.

11.3.1 Fourier Analysis

We abbreviate the quantities at levels ℓ and $\ell - 1$ by

$$N = N_\ell, \quad N' = N_{\ell-1}, \quad h = h_\ell = 1/N, \quad h' = h_{\ell-1} = 2h.$$

The vector $x = (x_k)_{1 \le k \le N-1}$ is formally extended by the components

$$x_0 = x_N = 0. \tag{11.25a}$$

The vectors (grid functions) e^α with the components

$$e^\alpha_k = \sqrt{2h}\, \sin(\alpha k \pi h) \qquad (0 \le k \le N) \tag{11.25b}$$

satisfy condition (11.25a) for all frequencies $\alpha \in \mathbb{Z}$. According to Exercise 3.3, $\{e^\alpha : 1 \le \alpha \le N-1\}$ forms an orthonormal basis. Therefore, the matrix Q built by e^α as columns is unitary: $Q^{\mathsf{H}} Q = I$ (cf. Definition A.27):

$$Q := \left[e^1, e^{N-1},\ e^2, e^{N-2},\ \ldots, e^\alpha, e^{N-\alpha},\ \ldots, e^{\frac{N}{2}-1}, e^{\frac{N}{2}+1},\ e^{\frac{N}{2}} \right]. \tag{11.25c}$$

The reason for the special ordering of the columns will be seen next. $M := M_\ell^{\mathrm{TGM}}$ denotes the iteration matrix of the two-grid method. Since multiplying by Q or

$Q^{\mathsf{H}} = Q^{-1}$ does not change the spectral norm and spectral radius (cf. Lemma B.18), we conclude that

$$\|\hat{M}\|_2 = \|M\|_2\,, \quad \rho(\hat{M}) = \rho(M) \quad \text{for } \hat{M} := Q^{-1}MQ. \tag{11.25d}$$

$\hat{M}$ is the Fourier-transformed iteration matrix. We shall show in §11.3.2 that $\hat{M}$ has a block-diagonal structure:

$$\begin{aligned} &\hat{M} = \text{blockdiag}\,\{M_1, M_2, \ldots, M_{N'-1}, M_{N'}\} \quad \text{with} \\ &M_\alpha:\ 2\times 2 \text{ matrices for } 1 \le \alpha \le N'-1, \quad M_{N'}:\ 1\times 1 \text{ matrix.} \end{aligned} \tag{11.25e}$$

Applying (A.10) to $\hat{M}$ and $\hat{M}^{\mathsf{H}}\hat{M}$, we obtain the next statement.

Lemma 11.12. *Matrices of the form (11.25e) satisfy*

$$\|\hat{M}\|_2 = \max_{1\le\alpha\le N'} \|M_\alpha\|_2\,, \qquad \rho(\hat{M}) = \max_{1\le\alpha\le N'} \rho(M_\alpha).$$

We choose the Richardson iteration with $\Theta = \frac{h^2}{4} \approx \frac{1}{\rho(A_\ell)}$ as the smoothing iteration. For proving the block structure (11.25e), we transform the iteration matrix

$$M = (I - pA_{\ell-1}^{-1} r A_\ell) S_\ell^\nu \qquad \text{with } S_\ell = I - \Theta A_\ell,\ \Theta = h^2/4\,.$$

The Fourier transform applied to the matrices A_ℓ, S_ℓ yields

$$\hat{A}_\ell := Q^{-1} A_\ell\, Q, \qquad \hat{S}_\ell := Q^{-1} S_\ell\, Q\,. \tag{11.26a}$$

Next, we need a Fourier map $Q' : X_{\ell-1} \to X_{\ell-1}$ defined in the coarse grid space. It is defined by

$$Q' = \left[e'^{\,1}, e'^{\,2}, \ldots, e'^{\,N'-1}\right]$$

with the orthonormal columns

$$e'^{\,\alpha}_k = \sqrt{4h}\,\sin(2\alpha k\pi h) \qquad (0 \le k \le N'). \tag{11.26b}$$

The vectors $e'^{\,\alpha} \in X_{\ell-1}$ are obtained from e^α in (11.25b) by replacing $h = h_\ell$ with $h' = h_{\ell-1}$. Now p, $A_{\ell-1}$, and r can be transformed into

$$\hat{p} = Q^{-1} p\, Q', \quad \hat{A}_{\ell-1} := Q'^{-1} A_{\ell-1}\, Q', \quad \hat{r} := Q'^{-1}\, r\, Q.$$

One verifies that $\hat{M}$ in (11.25d) takes the form

$$\hat{M} = (I - \hat{p}\hat{A}_{\ell-1}^{-1}\hat{r}\hat{A}_\ell)\,\hat{S}_\ell^\nu$$

(check that $\hat{p}\hat{A}_{\ell-1}^{-1}\hat{r}\hat{A}_\ell = Q^{-1}pA_{\ell-1}^{-1}rA_\ell Q$).

11.3.2 Transformed Quantities

Now we prove that all factors $\hat{p},\ \hat{A}_{\ell-1}^{-1},\ \hat{r},\ \hat{A}_\ell,\ \hat{S}_\ell^\nu$ are block-diagonal as stated in (11.25e). According to §3.1, $A_\ell e^\alpha = \lambda_\alpha e^\alpha$ holds with $\lambda_\alpha = 4h^{-2}\sin^2(\alpha\pi h/2)$. We introduce

$$s_\alpha^2 = \sin^2(\alpha\pi h/2), \qquad c_\alpha^2 = \cos^2(\alpha\pi h/2).$$

Noting that $\lambda_{N-\alpha} = s_{N-\alpha}^2 = c_\alpha^2$, we obtain

$$\hat{A}_\ell := Q^{-1}A_\ell Q = \mathrm{blockdiag}\{A_1,\dots,A_{N'}\} \quad \text{with the blocks} \tag{11.27a}$$

$$A_\alpha = 4h^{-2}\begin{bmatrix} s_\alpha^2 & 0 \\ 0 & c_\alpha^2 \end{bmatrix} \quad \text{for } 1\le\alpha\le N'-1, \qquad A_{N'} = 2h^{-2}. \tag{11.27b}$$

Since $S_\ell = I - \frac{1}{4}h^2 A_\ell$ and $s_\alpha^2 + c_\alpha^2 = 1$, equations (11.27a,b) yield the result

$$\hat{S}_\ell = Q^{-1}S_\ell Q = \mathrm{blockdiag}\{S_1,\dots,S_{N'}\} \text{ with the blocks} \tag{11.27c}$$

$$S_\alpha = \begin{bmatrix} c_\alpha^2 & 0 \\ 0 & s_\alpha^2 \end{bmatrix} \quad \text{for } 1 \le \alpha \le N'-1, \qquad S_{N'} = \frac{1}{2}\,. \tag{11.27d}$$

Because of $A_{\ell-1}e'^{\,\alpha} = \lambda'_\alpha e'^{\,\alpha}$ with $\lambda'_\alpha = 4h'^{-2}\sin^2(\alpha\pi h'/2) = h^{-2}\sin^2(\alpha\pi h)$ and using $\sin^2(\alpha\pi h) = 4s_\alpha^2 c_\alpha^2$, we obtain the diagonal matrix

$$\hat{A}_{\ell-1} := Q'^{-1}A_{\ell-1}Q' = \mathrm{diag}\{A'_1,\dots,A'_{N'}\} \text{ with } A'_\alpha = \frac{4}{h^2}s_\alpha^2 c_\alpha^2\,. \tag{11.27e}$$

Next, we transform p and r. Let p be defined by (11.10a–c). For r, we choose the adjoint mapping $r = p^*$:

$$r = \tfrac{1}{2}\left[\tfrac{1}{2}\ 1\ \tfrac{1}{2}\right], \text{ i.e., } (ru_\ell)(\xi) = \tfrac{1}{4}u_\ell(\xi-h) + \tfrac{1}{2}u_\ell(\xi) + \tfrac{1}{4}u_\ell(\xi+h). \tag{11.27f}$$

r and $\hat{r}$ are matrices of the format $(N'-1)\times(N-1) = (N'-1)\times(2N'-1)$. The representation

$$\hat{r} := [\,\mathrm{blockdiag}\{r_1,\dots,r_{N'-1}\},0\,] \quad \text{with } r_\alpha = \sqrt{\tfrac{1}{2}}\left[c_\alpha^2, -s_\alpha^2\right] \tag{11.27g}$$

means that the last column of $\hat{r} := Q'^{-1}r\,Q$ vanishes (this follows from $re^{N'} = 0$) and that the remaining part of the format $(N'-1)\times(2N'-1)$ consists of $N'-1$ blocks r_α of size 1×2. For the proof of (11.27g), it must be shown that

$$re^\alpha = c_\alpha^2 e'^\alpha/\sqrt{2}, \quad re^{N-\alpha} = -s_\alpha^2 e'^\alpha/\sqrt{2} \qquad \text{for } 1\le\alpha\le N'-1\,.$$

The restriction (11.27f) yields

$$\begin{aligned} r\sin(\alpha x\pi) &= \left[\sin(\alpha(x-h)\pi) + 2\sin(\alpha x\pi) + \sin(\alpha(x+h)\pi)\right]/4 \\ &= \left[1+\cos(\alpha h\pi)\right]\sin(\alpha x\pi)/2 = \cos(\alpha h\pi/2)^2\sin(\alpha x\pi) = c_\alpha^2\sin(\alpha x\pi) \end{aligned}$$

for all frequencies α. The different scaling of the vectors e^α, e'^α explains the additional factor in $re^\alpha = c_\alpha^2 e'^\alpha/\sqrt{2}$. Since this identity holds for all $\alpha \in \mathbb{Z}$, we may replace α by $N-\alpha$: $re^{N-\alpha} = c_{N-\alpha}^2 e'^{N-\alpha}/\sqrt{2}$. For $0 \le k \le N'$, the equality $\sin(2\alpha k\pi h) = -\sin(2(N-\alpha)k\pi h)$ leads to $e'^{N-\alpha} = -e'^\alpha$ (cf. definition (11.26b)). Finally, $c_{N-\alpha}^2 = s_\alpha^2$ proves $re^{N-\alpha} = -s_\alpha^2 e'^\alpha/\sqrt{2}$.

p in (11.10a–c) and r in (11.27f) are connected by $r = p^*$. From $p^* = \frac{1}{2}\,p^{\mathsf{T}}$, we derives the representation

$$\hat{p} = Q^{-1} p\, Q' = Q^{-1}(2r)^{\mathsf{T}} Q' = Q^{\mathsf{T}}(2r)^{\mathsf{T}} Q' = 2\left[Q'^{\mathsf{T}} r Q\right]^{\mathsf{T}} = 2\,\hat{r}^{\mathsf{T}}.$$

Therefore, the result (11.27g) for $\hat{r}$ proves

$$\hat{p} = Q^{-1} p\, Q' = \begin{bmatrix} \text{blockdiag}\,\{p_1, \ldots, p_{N'-1}\} \\ 0 \end{bmatrix} \tag{11.27h}$$

with $p_\alpha = \sqrt{2}\begin{bmatrix} c_\alpha^2 \\ -s_\alpha^2 \end{bmatrix}$.

11.3.3 Convergence Results

Since all factors in (11.26a) have a block-diagonal structure, this carries over to $\hat{M}$ and proves the structure (11.25e). For the 2×2 blocks M_α $(1 \le \alpha \le N'-1)$ and the 1×1 block $M_{N'}$, the statements (11.27b, d, e, g, h) yield

$$M_\alpha = (I - p_\alpha A_\alpha'^{-1} r_\alpha A_\alpha) S_\alpha^\nu \quad (1 \le \alpha \le N'-1), \quad M_{N'} = 2^{-\nu}.$$

Inserting the representations of p_α, A'_α, r_α, A_α, S_α^ν, we obtain

$$\begin{aligned} M_\alpha &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} c_\alpha^2 \\ -s_\alpha^2 \end{bmatrix} \frac{h^2}{4 s_\alpha^2 c_\alpha^2} \begin{bmatrix} c_\alpha^2 & -s_\alpha^2 \end{bmatrix} 4h^{-2} \begin{bmatrix} s_\alpha^2 & 0 \\ 0 & c_\alpha^2 \end{bmatrix} \right) \begin{bmatrix} c_\alpha^2 & 0 \\ 0 & s_\alpha^2 \end{bmatrix}^\nu \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} c_\alpha^2 & -c_\alpha^2 \\ -s_\alpha^2 & s_\alpha^2 \end{bmatrix} \right) \begin{bmatrix} c_\alpha^2 & 0 \\ 0 & s_\alpha^2 \end{bmatrix}^\nu = \begin{bmatrix} s_\alpha^2 & c_\alpha^2 \\ s_\alpha^2 & c_\alpha^2 \end{bmatrix} \begin{bmatrix} c_\alpha^2 & 0 \\ 0 & s_\alpha^2 \end{bmatrix}^\nu. \end{aligned} \tag{11.28}$$

The block M_α describes the application of M to the two functions $e^\alpha, e^{N-\alpha}$ (the respective columns of the matrix Q, cf. (11.25c)). Since $\alpha < N' < N-\alpha$, e^α corresponds to a smooth grid function and $e^{N-\alpha}$ to an oscillatory one. Obviously, the inequalities $0 < \alpha < N' < N-\alpha < N$ lead to

$$0 < s_\alpha^2 < \frac{1}{2} < c_\alpha^2 < 1. \tag{11.29}$$

The two 2×2 matrices in (11.28) characterise the coarse-grid correction and the smoothing iteration, respectively. Let the error have a representation $\sum_{\alpha=1}^{N-1} \xi_\alpha e^\alpha$ as in (11.2b). The entries $c_\alpha^2 > s_\alpha^2$ express the fact that the smooth e^α-components

converge more slowly than the nonsmooth $e^{N-\alpha}$-components. The first matrix reflects the complementary behaviour of the coarse-grid correction: The smooth components (s_α^2 in the first column) are better reduced than the oscillatory ones (c_α^2 in the second column).

Exercise 11.13. Prove that $\rho(M_\alpha)=\rho_\nu(s_\alpha^2)$ and $\|M_\alpha\|_2=\zeta_\nu(s_\alpha^2)$ with

$$\rho_\nu(\xi) := \xi(1-\xi)^\nu + (1-\xi)\xi^\nu, \tag{11.30a}$$

$$\zeta_\nu(\xi) := \sqrt{2\left[\xi^2(1-\xi)^{2\nu} + (1-\xi)^2\xi^{2\nu}\right]}. \tag{11.30b}$$

Combining (11.25d), Lemma 11.12, and Exercise 11.13 yields

$$\rho(M) = \max\{\rho_\nu(s_\alpha^2) : 1 \le \alpha \le N'\},\ \|M_\alpha\|_2 = \max\{\zeta_\nu(s_\alpha^2) : 1 \le \alpha \le N'\}.$$

Since the values of s_α^2 for $1 \le \alpha \le N'$ are between 0 and $\frac{1}{2}$ (cf. (11.29)), the following estimates are valid:

$$\rho(M) \le \rho_\nu := \max\{\rho_\nu(\xi) : 0 \le \xi \le 1/2\}, \tag{11.31a}$$

$$\|M\|_2 \le \zeta_\nu := \max\{\zeta_\nu(\xi) : 0 \le \xi \le 1/2\}. \tag{11.31b}$$

The bounds ρ_ν and ζ_ν of the convergence rate and contraction numbers depend on the smoothing number ν; however, they do not depend on the step size h. Since ρ_ν and ζ_ν decrease monotonically with increasing ν and $\rho_1 = \zeta_1 = \frac{1}{2} < 1$, the convergence of the two-grid method for the one-dimensional model problem (11.24) is proved. A more detailed discussion of the functions $\rho_\nu(\xi)$ and $\zeta_\nu(\xi)$ and their maxima in $[0, \frac{1}{2}]$ yields the following.

ν	ρ_ν	ζ_ν
1	1/2	1/2
2	1/4	1/4
3	1/8	0.150
4	0.0832	0.1159
5	0.0671	0.0947
10	0.0350	0.0496

Table 11.4 ρ_ν and ζ_ν.

Theorem 11.14. *Let the two-grid method for solving the system (11.24) be characterised by Richardson's iteration with $\Theta = h^2/4$ (identical to the Jacobi iteration damped by $\frac{1}{2}$) as smoother, by the piecewise linear prolongation p, and the adjoint restriction (11.27f). Then the two-grid method with $\nu \ge 1$ smoothing steps converges with the rate ρ_ν in (11.31a), which is h-independent. The contraction number (with respect to the Euclidean norm) is bounded by ζ_ν in (11.31b). For increasing ν, these bounds have the asymptotic behaviour*

$$\rho_\nu = \frac{1}{\mathrm{e}\,\nu} + \mathcal{O}\big(\nu^{-2}\big), \qquad \zeta_\nu = \frac{\sqrt{2}}{\mathrm{e}\,\nu} + \mathcal{O}\big(\nu^{-2}\big).$$

Some values of ρ_ν, ζ_ν are listed in Table 11.4. Obviously, the quantities $\rho(M)$, $\|M\|_2$ converge with a decreasing step size parameter to their bounds ρ_ν and ζ_ν; hence, the given estimates are strict. In §11.6 we will derive a convergence rate for general problems that also behaves like $\mathcal{O}(1/\nu)$. Theorem 11.14 demonstrates that such results about the asymptotic behaviour for large ν are not too pessimistic.

11.4 Multigrid Iteration

11.4.1 Algorithm

The two-grid method is not yet suited for practical applications because one still has to solve one system per iteration at level $\ell-1$. The problem to be solved in (11.21c) has the form

$$A_{\ell-1}\, e_{\ell-1} = d_{\ell-1}; \tag{11.32}$$

hence it is of the same structure as the original problem $A_\ell x_\ell = b_\ell$. Instead of solving the system (11.32) exactly, one may approximate the solution iteratively. The iteration of choice is again the two-grid method, now applied to levels $\ell-1$ and $\ell-2$ instead of ℓ and $\ell-1$. Then new auxiliary problems $A_{\ell-2}e_{\ell-2} = d_{\ell-2}$ arise, for which again the two-grid method (now at level $\ell-2$) can be applied until equations $A_0 e_0 = d_0$ arise at the coarsest grid. The corresponding recursive method is the multigrid iteration $\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}$, which has the following algorithmic form:

(11.33)

procedure $\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}(x_\ell, b_\ell)$;
if $\ell = 0$ **then** $\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)} := A_0^{-1} b_0$ **else** (11.33a)
begin for $i := 1$ **to** ν_1 **do** $x_\ell := \mathcal{S}_\ell(x_\ell, b_\ell)$; (11.33b)
$d_{\ell-1} := r(A_\ell x_\ell - b_\ell)$; (11.33c)
$e_{\ell-1}^{(0)} := 0$; (11.33d$_1$)
for $i := 1$ **to** γ **do** $e_{\ell-1}^{(i)} := \Phi_{\ell-1}^{\mathrm{MGM}(\nu_1,\nu_2)}(e_{\ell-1}^{(i-1)}, d_{\ell-1})$; (11.33d$_2$)
$x_\ell := x_\ell - p e_{\ell-1}^{(\gamma)}$; (11.33e)
for $i := 1$ **to** ν_2 **do** $x_\ell := \hat{\mathcal{S}}_\ell(x_\ell, b_\ell)$; (11.33f)
$\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)} := x_\ell$ (11.33g)
end;

The pre- and post-smoothing steps are the same as in (11.22b). Obviously, the recursive calls terminate after ℓ steps when level $\ell = 0$ is reached. Hence, the algorithm is well-defined.

ν_1 and ν_2 denote again the number of pre- and post-smoothing steps. A natural assumption is $\nu := \nu_1 + \nu_2 > 0$. For the iterative solution of the coarse-grid equation (11.32), γ steps of the iteration $\Phi_{\ell-1}^{\mathrm{MGM}(\nu_1,\nu_2)}$ are applied to the starting value (11.33d$_1$). We shall see that $\gamma = 2$ is sufficient. Therefore, only the cases $\gamma = 1$ and $\gamma = 2$ are of practical interest. The multigrid iteration with $\gamma = 1$ has the name 'V-cycle', whereas the iteration with $\gamma = 2$ is called the 'W-cycle' (concerning the reason for these names, see Hackbusch [183, §2.5]).

The exact solution of linear equations is not completely avoided in the multigrid algorithm (11.33). In (11.33a), the system $A_0 x_0 = b_0$ corresponding to the coarsest grid has to be solved. Since the coarsest grid has the smallest number of grid points, the solution should not lead to practical difficulties. In the model case, according to (11.5b), $h_0 = \frac{1}{2}$ would be a possible choice of the coarsest grid size. In this case, $A_0 x_0 = b_0$ represents a single scalar equation.

Formally, the multigrid method is the product of the smoothing iteration and coarse-grid correction, where the latter almost corresponds to a composed method with a secondary iteration as described in §5.5. But different from §5.5, the auxiliary problem, which has to be approximated by the secondary iteration, does not belong to the same space X_ℓ but to the lower-dimensional space $X_{\ell-1}$.

11.4.2 Numerical Examples

The model problem with the step size $h = h_5 = 1/64$ is taken as an example. Table 11.5 shows the Euclidean norm $\|e^m\|_2$ of the errors and the reduction factors $\rho_{m+1,m} = \|e^m\|_2/\|e^{m-1}\|_2$. The parameters are $\nu_1 = 2$, $\nu_2 = 0$, $h_0 = \frac{1}{2}$. All matrices A_ℓ are defined by the five-point discretisation, p is the nine-point prolongation, and r the nine-point restriction. We choose the chequer-board Gauss–Seidel method as the smoothing iteration. The comparison of the results for $\gamma = 1$ (V-cycle) and $\gamma = 2$ (W-cycle) in Table 11.5 with the two-grid results (corresponding formally to $\gamma = \infty$; the values are copied from Table 11.2) show that $\gamma = 2$ yields almost the same fast convergence as the two-grid method, whereas the V-cycle results are less favourable.

	$\gamma = 1$ (V-cycle)		$\gamma = 2$ (W-cycle)		$\gamma = \infty$ $(two-grid\ algorithm)$
m	$\|e^m\|_2$	$\rho_{m+1,m}$	$\|e^m\|_2$	$\rho_{m+1,m}$	$\rho_{m+1,m}$
1	1.3274_{10}-1	0.1727	2.9984_{10}-02	0.03902	0.03807
2	2.2223_{10}-2	0.1674	1.3219_{10}-03	0.04408	0.04121
3	3.7656_{10}-3	0.1694	6.9050_{10}-05	0.05223	0.05132
4	6.4110_{10}-4	0.1702	3.7824_{10}-06	0.05477	0.05445
5	1.0941_{10}-4	0.1706	2.1584_{10}-07	0.05706	0.05741
6	1.8701_{10}-5	0.1709	1.2689_{10}-08	0.05879	0.05937
7	3.1996_{10}-6	0.1710	7.6788_{10}-10	0.06051	0.06124

Table 11.5 Multigrid iteration for the Poisson model problem with step size $h = 1/64$.

m	$\rho_{m+1,m}$
1	0.03025
2	0.04722
3	0.05308
4	0.05510
5	0.05694
6	0.05835
7	0.05970
8	0.06092
9	0.06206
10	0.06312

	pointwise Gauss–Seidel		row Gauss–Seidel
m	$\gamma = 1$	$\gamma = 2$	$\gamma = 1$
1	0.1584	0.0275	0.0465
2	0.2602	0.0955	0.0999
3	0.3351	0.2734	0.0952
4	0.3479	0.3003	0.1319
5	0.3360	0.2945	0.1267
6	0.3142	0.3062	0.1471
7	0.2920	0.3200	0.1304
8	0.2720	0.3348	0.1487
9	0.2553	0.3257	0.1328

Table 11.6 Multigrid convergence rates $\rho_{m+1,m}$ for Eq. (11.34) with $c = 4$ (left) and $c = 100$ (right), $h = 1/64$ and Gauss–Seidel smoothing.

In order to demonstrate that the multigrid iteration not only works well for positive definite problems, the next example is the nonsymmetric differential equation (convection-diffusion equation):

$$-\Delta u + cu_x = f$$

in $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary values (1.1b), discretised by

$$A_\ell = h_\ell^{-2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} + \frac{1}{2} c h_\ell^{-1} \begin{bmatrix} & 0 & \\ -1 & 0 & 1 \\ & 0 & \end{bmatrix}. \tag{11.34}$$

First, we choose $c = 4$ (for this value, all A_ℓ are M-matrices). $f = u = 0$ are taken as the right-hand side and exact solution, while $x(1-x+y)$ serves as the starting value. The other parameters are the same as in Table 11.5. The W-cycle ($\gamma = 2$) shows a convergence rate of ≈ 0.06 (cf. Tab. 11.6) and hardly differs from the corresponding rate of the Poisson model case.

As soon as the coefficient c becomes substantially larger, e.g., $c = 100$, a stability problem arises. Discretisation (11.34) yields an M-matrix for $h_5 = 1/64$, but not for larger h. A remedy is the matrix-dependent prolongation (11.13a,b) and the corresponding restriction, together with the Galerkin product (11.20) for $\ell < 5$ (cf. Hackbusch [183, §10.4]). Table 11.6 shows the convergence rates $\rho_{m+1,m}$ for $\gamma = 1$ and $\gamma = 2$. Different from the model case, the results for $\gamma = 2$ are hardly better than those for $\gamma = 1$. Furthermore, the rate ≈ 0.3 is not so favourable. The rate can be improved to ≈ 0.14 by row-wise Gauss–Seidel smoothing instead of the chequer-board Gauss–Seidel iteration (Table 11.6, right column).

In §§10.3.5–10.4 (cf. Tables 10.4, 10.6) the indefinite problem with the matrix

$$A_\ell := h_\ell^{-2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} - \begin{bmatrix} & 0 & \\ 0 & 50 & 0 \\ & 0 & \end{bmatrix} \tag{11.35}$$

is solved. As we shall see in §11.6.2, for indefinite problems the choice of the coarsest step size is restricted. Here $h_0 = \frac{1}{2}$ is too coarse, but $h = \frac{1}{4}$ is possible. However, better results can be obtained with $h = \frac{1}{8}$ as the coarsest step size. Table 11.7 shows the results for $h_5 = \frac{1}{64}$, $h_0 = \frac{1}{8}$, $\gamma = 2$, nine-point prolongation, and nine-point restriction. $\nu_1 = 2$ chequer-board Gauss–Seidel steps are applied ($\nu_2 = 0$) as smoothing. The convergence rate (here 0.442) improves with decreasing grid size. Vice versa, the worse rate 0.613 results for $h = 1/16$.

m	$\|e^m\|_2$	$\rho_{m+1,m}$
1	1.301_{10}-1	0.169309
2	5.607_{10}-2	0.430985
3	2.480_{10}-2	0.442381
4	1.097_{10}-2	0.442503
5	4.857_{10}-3	0.442505

Table 11.7 Results for the indefinite problem (11.35).

It is not necessary to choose h_0 sufficiently small if one uses the Kaczmarz iteration for smoothing (cf. §5.6.3), which is also convergent for the indefinite matrix (11.35). However, for the parameters $h_0 = \frac{1}{2}$, $\nu_1 = 2$, $\nu_2 = 0$, $\gamma = 2$, one obtains the rather unfavourable convergence rate 0.833 (for $h = \frac{1}{16}$ even 0.917).

11.4.3 Computational Work

To judge the convergence rates in §11.4.2, we have to take into account the amount of work per iteration (cf. §2.3). Because of the recursive structure, the amount of work is not quite obvious. The operations appearing in (11.33) are $\mathcal{S}_\ell$ in (11.33b,f), $r(A_\ell x_\ell - b_\ell)$ in (11.33c), and $x_\ell - pe_{\ell-1}$ in (11.33e). We denote the corresponding work by

$$C_S\, n_\ell \quad \text{operations for} \quad x_\ell \longmapsto \mathcal{S}_\ell(x_\ell, b_\ell) \text{ or } \hat{\mathcal{S}}_\ell(x_\ell, b_\ell), \tag{11.36a}$$

$$C_D\, n_\ell \quad \text{operations for} \quad x_\ell \longmapsto r(A_\ell x_\ell - b_\ell), \tag{11.36b}$$

$$C_C\, n_\ell \quad \text{operations for} \quad x_\ell \longmapsto x_\ell - pe_{\ell-1}\,. \tag{11.36c}$$

Proportionality to dimension n_ℓ is a consequence of the sparsity of the matrix A_ℓ (cf. (2.28)). For standard approaches to fully populated matrices, n_ℓ would have to be replaced by n_ℓ^2 in (11.36a,b) (but see §D or Hackbusch [198, §10]).

The dimensions n_ℓ should increase with increasing level-number ℓ at least by a fixed factor C_h:

$$n_{\ell-1} \le n_\ell / C_h \qquad \text{for } \ell \ge 1\,. \tag{11.37}$$

Otherwise, the difficulty would arise that the auxiliary problems $A_{\ell-1}e_{\ell-1} = d_{\ell-1}$ are of a similar dimension as $A_\ell x_\ell = b_\ell$.

Remark 11.15. For the standard choice $h_\ell = h_{\ell-1}/2$ and the spatial dimension d: $\Omega \subset \mathbb{R}^d$, inequality (11.37) holds with $C_h = 2^d$. In the model case, $d = 2$ is valid.

Theorem 11.16. *Assume (11.36a–c) and (11.37). Let γ in (11.33d$_2$) satisfy*

$$\gamma < C_h\,. \tag{11.38}$$

Then the work of the multigrid iteration is proportional to n_ℓ:

$$\begin{aligned} &\mathrm{Work}(\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}) \le C(\nu_1+\nu_2)\cdot n_\ell \qquad \textit{with} \\ &C(\nu) = \frac{\nu C_S + C_D + C_C}{1-\gamma/C_h} + \mathcal{O}\left((\gamma/C_h)^\ell\right)\,. \end{aligned} \tag{11.39}$$

Proof. Let $C_\ell\, n_\ell$ be the work for one $\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}$ step. From the representation (11.33), we conclude that $C_\ell n_\ell \le (\nu C_S + C_D + C_C) n_\ell + \gamma C_{\ell-1} n_{\ell-1}$. Inequality (11.37) yields $C_\ell \le (\nu C_S + C_D + C_C) + \vartheta C_{\ell-1}$ with $\vartheta := \gamma/C_h$ and results in the geometrical sum

$$C_\ell \;\le\; (\nu C_S + C_D + C_C)(1 + \vartheta + \ldots + \vartheta^{\ell-1}) + \gamma C_0 / n_\ell\,,$$

where C_0 denotes the work for (11.33a) (independent of h_ℓ). Since $\gamma^\ell/n_\ell \le \vartheta^\ell/n_1$, (11.39) follows. □

Remark 11.17. In the two-dimensional case $d = 2$, (11.38) is satisfied for the interesting values $\gamma = 1, 2$ because of $C_h = 4$ (cf. Remark 11.15). The following constants are obtained for (11.39):

$$C_V(\nu) = \frac{4}{3}(\nu C_S + C_D + C_C) + \mathcal{O}\left((\gamma/C_h)^\ell\right) \qquad \text{for } \gamma = 1, \tag{11.40a}$$

$$C_W(\nu) = 2\,(\nu C_S + C_D + C_C) + \mathcal{O}\left((\gamma/C_h)^\ell\right) \qquad \text{for } \gamma = 2. \tag{11.40b}$$

Since $\gamma/C_h < 1$ (cf. (11.38)), formulae (11.39) and (11.40a,b) show that for increasing ℓ the work for solving $A_0 x_0 = b_0$ in (11.33a) requires a vanishing portion of the total work.

Exercise 11.18. Although the one-dimensional case (11.24) is not of practical interest, one may apply the multigrid algorithm. Then (11.38) is not satisfied because $C_h = 2$ holds for the W-cycle ($\gamma = 2$). Prove that the work is equal to $\mathcal{O}(\ell\, n_\ell) = \mathcal{O}(n_\ell \log n_\ell)$.

For the standard multigrid parameters as used before, the work amounts to

$$C_S = 2(C_A - 1) \qquad \text{for the Gauss–Seidel iteration, cf. (3.20b),} \tag{11.41a}$$

$$C_D = 2C_A + 11/4 \qquad \text{for } r = \text{nine-point restriction (11.15),} \tag{11.41b}$$

$$C_C = 3/2 \qquad \text{for } p = \text{nine-point prolongation (11.11).} \tag{11.41c}$$

The constants C_S and C_D improve for the Poisson model case ($C_A = 5$, since multiplications by coefficients 1 can be omitted):

$$C_S = 5 \qquad \text{for the Gauss–Seidel iteration, cf. (3.21),} \tag{11.41a$'$}$$

$$C_D = 5 + 10/4 \qquad \text{for } r = \text{nine-point restriction (11.15),} \tag{11.41b$'$}$$

If the chequer-board Gauss–Seidel method is used, some operations can be saved when applying r and p (cf. Hackbusch [183, Note 4.3.4]). Inserting formulae (11.41a–11.41b$'$), the numbers (11.40a,b) become

$$C_V(\nu) = \frac{8}{3}(\nu + 1)C_A + \frac{17 - 8\nu}{3} + \mathcal{O}(1/4^\ell) \quad \text{for } \gamma = 1, \tag{11.41d}$$

$$C_V(\nu) = 12 + \frac{20}{3}\nu + \mathcal{O}(1/4^\ell) \quad \text{(Poisson model case, } \gamma = 1), \tag{11.41d$'$}$$

$$C_W(\nu) = 4(\nu + 1)C_A + \frac{17 - 8\nu}{3} + \mathcal{O}(1/2^\ell) \quad \text{for } \gamma = 2, \tag{11.41e}$$

$$C_W(\nu) = 18 + 10\nu + \mathcal{O}(1/2^\ell) \quad \text{(Poisson model case, } \gamma = 2). \tag{11.41e$'$}$$

The corresponding effective work of the V- and W-cycle for the Poisson model problem with $\nu = 2$ is $C_{V[W]}(2)/\left|C_A \log(\rho)\right|$. Using the convergence rates ρ in Table 11.5, we obtain

$$\mathrm{Eff}(\Phi_\ell^{\mathrm{MGM}(2,0)}) = -C_V(2)/\left[5\log(0.171)\right] \approx 2.89 \quad \text{for } \gamma = 1,$$

$$\mathrm{Eff}(\Phi_\ell^{\mathrm{MGM}(2,0)}) = -C_W(2)\,/\left[5\log(0.06)\right] \approx 2.7 \quad \text{for } \gamma = 2.$$

Together with the convergence rate, the effective work is also h-independent. One should compare the numbers $\mathrm{Eff}(\Phi_\ell^{\mathrm{MGM}(2,0)})$ with the competing values in

Remark 8.49 (for $h = 1/32$). The numbers (11.41d′,e′) can also be interpreted as follows. One V-cycle step costs as much as ≈ 5 Gauss–Seidel iteration steps, one W-cycle step corresponds to 7.6 Gauss–Seidel steps.

Finally, we address the question how many smoothing steps should be performed. The numerical results in §11.4.2 have shown good agreement with the two-grid results. According to Table 11.3, these rates behave as $\approx C_\rho/(1+\nu)$, where $\nu = \nu_1+\nu_2$. For simplification, assume that $C_C+C_D = C_S$. Then the multigrid work behaves like $\mathrm{Work}(\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}) \approx (1+\nu)C$. With an increasing number ν of smoothing steps, the convergence improves, however, the work also increases. We have to minimise the effective work

$$-\frac{(1+\nu)C/C_A}{\log\left(C_\rho/(1+\nu)\right)} = C'\frac{1+\nu}{\log(1+\nu)-\log\left(C_\rho\right)}\,.$$

The minimum is taken for $\nu^* = C_\rho \mathrm{e} - 1$ and has the value $\mathrm{e}C_\rho C/C_A$. This shows at least asymptotically that the faster the multigrid iteration (i.e., the smaller C_ρ), the smaller the number of smoothing steps should be. If, vice versa, it turns out that many smoothing steps are necessary, the multigrid method is not favourable and one should look for a better suited smoothing iteration.

11.4.4 Iteration Matrix

Since the iteration is defined recursively, the multigrid iteration matrix is also determined recursively.

Theorem 11.19. *Let S_ℓ, $\hat{S}_\ell$ be the iteration matrices of the respective consistent pre- and post-smoothing iterations $\mathcal{S}_\ell$ and $\hat{\mathcal{S}}_\ell$. Then the multigrid iteration $\Phi_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}$ is also consistent. Its iteration matrix $M_\ell^{\mathrm{MGM}} = M_\ell^{\mathrm{MGM}(\nu_1,\nu_2)}$ is defined by*

$$M_0^{\mathrm{MGM}} = 0, \qquad M_1^{\mathrm{MGM}} = M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}, \tag{11.42a}$$

$$M_\ell^{\mathrm{MGM}} = M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} + \hat{S}_\ell^{\nu_2} p \left(M_{\ell-1}^{\mathrm{MGM}}\right)^\gamma A_{\ell-1}^{-1} r A_\ell S_\ell^{\nu_1} \quad \textit{for } \ell \geq 1. \tag{11.42b}$$

Proof. Obviously, the coarse-grid correction (11.33c–e) is consistent. Hence, Proposition 5.25a shows that Φ_ℓ^{MGM} is consistent. For $\ell = 0$, Φ_0^{MGM} describes the exact solution, i.e., $M_0^{\mathrm{MGM}} = 0$. For $\ell = 1$, the multi- and two-grid methods coincide. This proves (11.42a). The iteration matrix of the coarse-grid correction (11.33c–e) is

$$M_\ell^{\mathrm{CGC}} = I - p\left[I - \left(M_{\ell-1}^{\mathrm{MGM}}\right)^\gamma\right] A_{\ell-1}^{-1} r A_\ell,$$

because we have $e_{\ell-1}^{(\gamma)} = I - p\left[I - \left(M_{\ell-1}^{\mathrm{MGM}}\right)^\gamma\right] A_{\ell-1}^{-1} r A_\ell e_\ell$ in (11.33e), as can be shown similarly as in the proof of (5.21b). (5.12a) and (11.23) prove (11.42b).□

11.5 Nested Iteration

Contrary to the name 'nested iteration', the following scheme is not an iteration, but a finite technique which can be applied to any iterative method, provided that there is a hierarchy of problems

$$A_\ell\, x_\ell = b_\ell \qquad (\ell = 0, 1, 2, \ldots).$$

The latter requirement is the same as for constructing the multigrid method. Therefore, it is natural to combine the multigrid iteration with the nested iteration. This will be done in §11.5.4. First, we discuss the nested iteration independently of the multigrid method. Concerning the construction of discretisations we refer to Remark 11.4. The concept of the nested iteration is of particular help for nonlinear problems, for which the choice of sufficiently close starting values is essential (cf. §11.9.5).

11.5.1 Discretisation Error and Relative Discretisation Error

We recall Remark 2.34. As long as x_ℓ is only considered as an approximation to a continuous solution of a differential equation, it makes no sense to compute x_ℓ more precisely than indicated by the discretisation error. The nested iteration provides a convenient way to obtain this goal.

For standard discretisations, the consistency order κ is known; i.e., the dependence on the step size is given by

$$\text{discretisation error:} \quad \delta_\ell \le C_{\mathrm{de}}\, h_\ell^\kappa,$$

but the constant C_{de} is usually unknown (in principle, it may be described by the derivatives of the solution, but these are unknown). One may use error estimators, to bound δ_ℓ (cf. Verfürth [380]) and to stop the iteration as soon as the iteration error is below δ_ℓ. Since δ_ℓ is the difference[2] between the exact solution and x_ℓ, the triangle inequality yields an estimate $\mathcal{O}(h_\ell^\kappa + h_{\ell-1}^\kappa)$ of the difference of x_ℓ and $x_{\ell-1}$ by $\delta_\ell + \delta_{\ell+1}$. Provided that $h_\ell/h_{\ell-1}$ is uniformly bounded, the previous estimate yields the following bound of the *relative discretisation error*:

$$\|x_\ell - \tilde{p} x_{\ell-1}\| \le C_1 h_\ell^\kappa \quad (\kappa > 0,\ x_\ell, x_{\ell-1} \text{ solutions to (11.6a)}). \tag{11.43}$$

Here, $\tilde{p} : X_{\ell-1} \to X_\ell$ is a suitable prolongation, which may not necessarily coincide with p in §11.1.3 and (11.33e). In the following, only the exponent κ in (11.43) must be known, not the constant C_1.

[2] The precise notation of the difference might be $x - Px_\ell$ or $Rx - x_\ell$ (x and x_ℓ belong to different spaces; P and R are prolongations and restrictions between these spaces).

11.5.2 Algorithm

Obviously, the result x_ℓ^m of an iteration is more desirable, the better the starting iterate x_ℓ^0 is. So far, we did not study the choice of a starting iterate.[3] Inequality (11.43) suggests using approximations to $x_{\ell-1}$ as the starting iterate of the iteration at level ℓ. The algorithm as proposed by Kronsjø–Dahlquist [248] reads as follows:

$$\boxed{\begin{array}{l}\tilde{x}_0 := \text{suitable approximation of the solution of } A_0 x_0 = b_0;\\ \textbf{for } \ell := 1 \textbf{ to } \ell_{\max} \textbf{ do}\\ \textbf{begin } \tilde{x}_\ell := \tilde{p}\,\tilde{x}_{\ell-1};\\ \qquad \textbf{for } i := 1 \textbf{ to } m_\ell \textbf{ do } \tilde{x}_\ell := \Phi_\ell(\tilde{x}_\ell, b_\ell)\\ \textbf{end};\end{array}} \tag{11.44}$$

Here, Φ_ℓ is any convergent and consistent linear iteration.[4] The number m_ℓ of iterations is still to be determined. Theorem 11.20 will propose an appropriate choice.

Note that (11.44) is not an iteration in the proper sense, but a *finite* process. Furthermore, it produces approximate solutions $\tilde{x}_\ell$ for *all* levels $1 \le \ell \le \ell_{\max}$.

11.5.3 Error Analysis

First, we analyse the case of non-optimal linear iterations; i.e., the contraction number behave as

$$\|M_\ell^\Phi\| \le 1 - c_\ell^\Phi h_\ell^\tau \quad \text{with } \tau > 0 \text{ for all } \ell \ge 1 \tag{11.45a}$$

(cf. (2.32c)). Here, M_ℓ^Φ is the iteration matrix of Φ_ℓ. An inequality opposite to condition (11.37), $n_{\ell-1} \le n_\ell / C_h$, is

$$n_\ell \le \overline{C}_h\, n_{\ell-1}.$$

By $n_\ell / n_{\ell-1} \approx (h_{\ell-1}/h_\ell)^d$, the latter inequality also gives an estimate of $h_{\ell-1}/h_\ell$ appearing above. Together with the norm of $\tilde{p}$, we obtain an estimate of the form

$$\|\tilde{p}\|(h_{\ell-1}/h_\ell)^\kappa \le C_2 \qquad (\,\tilde{p} : X_{\ell-1} \to X_\ell\,) \tag{11.45b}$$

with κ as in (11.43). The inequalities (11.43) and (11.45a,b) must use the same family of norms in X_ℓ.

[3] If one has to solve $A_\ell^{(\nu)} x_\ell^{(\nu)} = b_\ell^{(\nu)}$ ($\nu = 1, 2$) for similar data $(A_\ell^{(\nu)}, b_\ell^{(\nu)})$, one may take the solution of $A_\ell^{(1)} x_\ell^{(1)} = b_\ell^{(1)}$ as starting value for solving $A_\ell^{(2)} x_\ell^{(2)} = b_\ell^{(2)}$.

[4] The m_ℓ-fold application of Φ_ℓ may be replaced by a semi-iteration, or acceleration methods may be used (cf. §10).

Theorem 11.20. *Assume (11.43) and (11.45a,b). Fix some constant* $K > 0$ *and choose*

$$m_\ell \geq \frac{1 + \log(C_2 + 1/K)}{c_\ell^\Phi} \, h_\ell^{-\tau}.$$

Then the nested iteration (11.44) with (11.49) produces results $\tilde{x}_\ell$ *for all levels* $0 \leq \ell \leq \ell_{\max}$ *satisfying the error estimates*

$$\|\tilde{x}_\ell - x_\ell\| \leq K C_1 h_\ell^\kappa \tag{11.46}$$

provided that the starting iterate $\tilde{x}_0$ *satisfies inequality (11.46) for* $\ell = 0$.

Proof. By assumption, (11.46) holds for $\ell = 0$. Assume (11.46) for all level numbers $\leq \ell - 1$. The starting iterate $x_\ell^0 := \tilde{p}\,\tilde{x}_{\ell-1}$ has an error that can be bounded by

$$\begin{aligned}
\|x_\ell^0 - x_\ell\| &\leq \|\tilde{p}\,x_{\ell-1} - x_\ell\| + \|\tilde{p}\,(\tilde{x}_{\ell-1} - x_{\ell-1})\| \\
&\leq \|\tilde{p}\,x_{\ell-1} - x_\ell\| + \|\tilde{p}\,\|\,\|\tilde{x}_{\ell-1} - x_{\ell-1}\| \\
&\leq C_1 h_\ell^\kappa + \|\tilde{p}\,\| K C_1 h_{\ell-1}^\kappa \\
&\leq C_1 h_\ell^\kappa \, [1 + \|\tilde{p}\,\|(h_{\ell-1}/h_\ell)^\kappa K] \\
&\leq C_1 h_\ell^\kappa \, [1 + C_2 K)] \, .
\end{aligned}$$

m_ℓ iteration steps reduce the error to $\|x_\ell^{m_\ell} - x_\ell\| \leq \left(1 - c_\ell^\Phi h_\ell^\tau\right)^{m_\ell} \|x_\ell^0 - x_\ell\|$. The general inequality $1 + \xi \leq \exp(\xi)$ for all $\xi \in \mathbb{R}$ yields

$$\begin{aligned}
\left(1 - c_\ell^\Phi h_\ell^\tau\right)^{m_\ell} &\leq \exp(1 - m_\ell c_\ell^\Phi h_\ell^\tau) \leq \exp\left(1 - \left(1 + \log(C_2 + \frac{1}{K})\right)\right) \\
&= 1/\Big(C_2 + \frac{1}{K}\Big) = \frac{K}{1 + C_2 K}.
\end{aligned}$$

By the previous estimate of $\|x_\ell^0 - x_\ell\|$, (11.46) holds for ℓ. □

Choose K somewhat smaller than one. Since $C_1 h_\ell^\kappa$ is the relative discretisation error, we obtain approximations $\tilde{x}_\ell$ with an iteration error similar in size:

$$\|\tilde{x}_\ell - x_\ell\| \leq K \times \text{relative discretisation error.} \tag{11.47}$$

Note that this statement holds, although the size of C_1 involved in the relative discretisation error $C_1 h_\ell^\kappa$ does not enter the algorithm.

The cost of the nested iteration is dominated by the work $\mathcal{O}(n_\ell h_\ell^{-\tau})$ at the maximal level $\ell = \ell_{\max}$. However, the standard approach using the starting value $x_{\ell_{\max}}^0 = 0$ requires $\mathcal{O}(n_{\ell_{\max}} h_{\ell_{\max}}^{-\tau} \kappa \log(1/h_{\ell_{\max}}))$ operations (cf. (2.31b)).

The analysis of the *cascade algorithm* in Bornemann–Deuflhard [56] demonstrates that the choice of the norm $\|\cdot\|$ is essential.

11.5.4 Application to Optimal Iterations

Now we assume that the iteration (as, e.g., the multigrid method; cf. Kronsjø [247]) has an h-independent contraction number:

$$\|M_\ell^\Phi\| \le \zeta < 1 \quad \text{for all } \ell \ge 1, \quad M_\ell^\Phi\text{: iteration matrix of } \Phi_\ell\,. \tag{11.48}$$

Here the numbers m_ℓ in (11.44) can be chosen independently of ℓ:

$$m_\ell = m \qquad (\ell \ge 1)\,. \tag{11.49}$$

In Remark 11.22 we shall see that even the smallest possible number $m = 1$ is of practical interest.

Theorem 11.21. *Assume (11.43), (11.48), and (11.45b). The iteration number $m_\ell = m$ (cf. (11.49)) should be sufficiently large so that*

$$C_2\,\zeta^m < 1. \tag{11.50}$$

Then the nested iteration (11.44) with (11.49) produces results $\tilde{x}_\ell$ for all levels $0 \le \ell \le \ell_{\max}$ satisfying the error estimates

$$\|\tilde{x}_\ell - x_\ell\| \le C_3(\zeta^m) C_1 h_\ell^\kappa \quad \textit{with} \quad C_3(\zeta^m) := \zeta^m/(1 - C_2\zeta^m), \tag{11.51}$$

provided that the starting iterate $\tilde{x}^0$ satisfies inequality (11.51) for $\ell = 0$.

Proof. We repeat the induction proof of Theorem 11.20. Assume (11.51) for levels $\le \ell - 1$. The starting iterate $x_\ell^0 := \tilde{p}\,\tilde{x}_{\ell-1}$ has an error that can be bounded by

$$\begin{aligned}
\|x_\ell^0 - x_\ell\| &\le \|\tilde{p}\,x_{\ell-1} - x_\ell\| + \|\tilde{p}\,(\tilde{x}_{\ell-1} - x_{\ell-1})\| \\
&\le \|\tilde{p}\,x_{\ell-1} - x_\ell\| + \|\tilde{p}\,\|\,\|\tilde{x}_{\ell-1} - x_{\ell-1}\| \\
&\le C_1 h_\ell^\kappa + \|\tilde{p}\,\| C_3(\zeta^m) C_1 h_{\ell-1}^\kappa \\
&\le C_1 h_\ell^\kappa\,[1 + \|\tilde{p}\,\|(h_{\ell-1}/h_\ell)^\kappa C_3(\zeta^m)] \\
&\le C_1 h_\ell^\kappa\,[1 + C_2 C_3(\zeta^m)]\,.
\end{aligned}$$

After m iteration steps, the error is reduced to $\|x_\ell^m - x_\ell\| \le \zeta^m \|x_\ell^0 - x_\ell\| \le C_1\,h_\ell^\kappa\,\{\zeta^m\,[\,1 + C_2\,C_3\,(\zeta^m)]\}$ because of (11.48). Definition of $C_3(\cdot)$ in (11.51) shows that $\{\ldots\} = C_3(\zeta^m)$ and proves (11.51) for ℓ. □

Again the iteration error $\|\tilde{x}_\ell - x_\ell\|$ coincides up to a factor $C_3(\zeta^m)$ with the relative discretisation error $C_1 h_\ell^\kappa$, i.e., (11.47) holds with $K := C_3(\zeta^m)$.

Remark 11.22. The standard choice $h_\ell = h_{\ell-1}/2$ and the inequality $\|\tilde{p}\,\| \le 1$, which is valid for standard interpolations, yield the constant $C_2 = 2^\kappa$ in (11.45b). The consistency order of the model case is $\kappa = 2$, from which $C_2 = 4$. Therefore, the factor $C_3(\zeta^m)$ is equal to

$$C_3(\zeta^m) = \zeta^m/\,(1 - 4\zeta^m)\,.$$

For multigrid methods with convergence rates $\le \zeta = 0.2$ (see the results in §11.4.2), condition (11.50) is satisfied for only one iteration step (i.e., $m = 1$) and produces the value $C_3(0.2) = 1$.

11.5.5 Amount of Computational Work

Let Cn_ℓ be the work required by one step of the iteration Φ_ℓ at level ℓ and assume (11.37): $n_{\ell-1} \le n_\ell / C_h$. The work for $\tilde{x}_{\ell-1} \mapsto \tilde{p}\tilde{x}_{\ell-1}$ is considered negligible. The total work amounting to $C_{\text{nested}} n_\ell \le mC(n_1 + n_2 + \ldots + n_\ell)$ can be estimated, using the geometrical sum $n_1 + \ldots + n_\ell \le n_\ell \sum_k C_h^{-k} \le C_h n_\ell/(C_h - 1)$, by

$$C_{\text{nested}} \le m\, C\, C_h \,/\, (C_h - 1).$$

For the standard case $C_h = 2^d = 4$ (cf. Remark 11.15), we obtain the result

$$\text{Work}_{(11.44)} \le \frac{4m}{3} \text{Work}(\Phi_{\ell_{\max}}). \tag{11.52}$$

If we try to achieve an accuracy of $\varepsilon = Ch^\kappa$ at level $\ell = \ell_{\max}$ with the starting iterate $\tilde{x}_\ell := 0$ by iterating with Φ_ℓ, the work would be proportional to $\mathcal{O}(|\log \varepsilon|) = \mathcal{O}(|\log h_\ell|)$ (cf. (2.31b)). According to Remark 11.22, $m = 1$ is a realistic choice. Inequality (11.52) shows that sufficient accuracy for all levels $0 \le \ell \le \ell_{\max}$ can be attained with the 4/3-fold work of a single $\Phi_{\ell_{\max}}$ step.

Together with the numbers in (11.41d′, e′) and Table 11.5 (with $\nu_1 = 2$, $\nu_2 = 0$, $m = 1$), we obtain the following results:

$$\begin{aligned} &\text{the V-cycle } (\gamma = 1) \text{ requires } 34 n_{\ell_{\max}} \text{ operations to produce} \\ &\qquad \|\tilde{x}_\ell - x_\ell\| \le 0.53\, C_1 h_\ell^\kappa \quad \text{for } 0 \le \ell \le \ell_{\max}, \end{aligned} \tag{11.53a}$$

$$\begin{aligned} &\text{the W-cycle } (\gamma = 2) \text{ requires } 51 n_{\ell_{\max}} \text{ operations to produce} \\ &\qquad \|\tilde{x}_\ell - x_\ell\| \le 0.08\, C_1 h_\ell^\kappa \quad \text{for } 0 \le \ell \le \ell_{\max}. \end{aligned} \tag{11.53b}$$

The work given in (11.53b) corresponds to about 10 steps of the Gauss–Seidel iteration at level $\ell_{\max}$.

Since the nested iteration (11.44) is a finite process and not an iteration, considerations in §2.3 are not applicable. How many operations are necessary, depends on the desired accuracy.

11.5.6 Numerical Examples

First, the nested iteration is applied to the differential equation

$$-\Delta u = f := -\Delta(\mathrm{e}^{x+y^2}) \tag{11.54a}$$

with boundary values $\varphi = e^{x+y^2}$. The negative Laplacian $-\Delta$ is discretised at all levels by the standard five-point star. $\tilde{p}$ is cubic interpolation. Let x_ℓ^* be the restriction of the exact solution e^{x+y^2} of (11.54a) to the grid Ω_ℓ. Note that x_ℓ^* does not coincide with the discrete solution x_ℓ of the system $A_\ell x_\ell = b_\ell$

corresponding to (11.54a). In Table 11.8, the results $\tilde{x}_\ell$ of the nested iteration are compared with x_ℓ^* because this is the error most interesting in practice. The maximum norm $\|\tilde{x}_\ell - x_\ell\|_\infty$ of these errors is given for cases $m = 1$ and $m = 2$ (m in (11.49)). For comparison, the last column shows the discretisation error $\|x_\ell - x_\ell^*\|_\infty$, which formally corresponds to $m = \infty$. The multigrid iteration used for solving (11.44) has the same parameters as the W-cycle ($\gamma = 2$) in Table 11.5. The data in Table 11.8 demonstrate that the choice $m = 1$ is sufficient. $m = 2$ doubles the work but cannot improve the total error $\|\tilde{x}_\ell - x_\ell^*\|_\infty$ substantially.

ℓ	h_ℓ	$m = 1$	$m = 2$	$m = \infty$
0	1/2	7.9944658_{10}-2	7.9944658_{10}-2	7.9944658_{10}-2
1	1/4	3.9908756_{10}-2	2.9215605_{10}-2	2.8969488_{10}-2
2	1/8	1.5788721_{10}-2	8.1023136_{10}-3	8.0307789_{10}-3
3	1/16	3.2919346_{10}-3	2.0768391_{10}-3	2.0729855_{10}-3
4	1/32	5.7591549_{10}-4	5.2253758_{10}-4	5.2247399_{10}-4
5	1/64	1.3291689_{10}-4	1.3093946_{10}-4	1.3093956_{10}-4

Table 11.8 Errors $\|\tilde{x}_\ell - x_\ell^*\|_\infty$ of the nested iteration for (11.54a).

Analogous data are given in Table 11.9 for the differential equation

$$-\Delta u = f := -\Delta(y \sin(10x)) \tag{11.54b}$$

with a solution $y \sin(10x)$ which is oscillatory in the x direction. By the nonsmooth behaviour of the solution, the discretisation error (last column) for problem (11.54b) is nearly one digit worse than for (11.54a). Therefore, the additional error $\mathcal{O}(h_\ell^2)$ of linear interpolation $\tilde{p}$, which is used instead of the cubic one, is of minor consequence. Also for this example, it does not pay to perform $m = 2$ iterations per level.

ℓ	h_ℓ	$m = 1$	$m = 2$	$m = \infty$
0	1/2	2.8249099_{10}-0	2.8249099_{10}-0	2.8249099_{10}-0
1	1/4	5.0876212_{10}-1	4.6124302_{10}-1	4.7880033_{10}-1
2	1/8	9.5881341_{10}-2	1.0330948_{10}-1	1.0308770_{10}-1
3	1/16	2.7648979_{10}-2	2.6636710_{10}-2	2.6689213_{10}-2
4	1/32	6.8798570_{10}-3	6.6486368_{10}-3	6.6506993_{10}-3
5	1/64	1.6998365_{10}-3	1.6716069_{10}-3	1.6714014_{10}-3

Table 11.9 Errors $\|\tilde{x}_\ell - x_\ell^*\|_\infty$ of the nested iteration for (11.54b).

11.5.7 Comments

Additional variants for the nested iteration (e.g., combinations with extrapolation techniques) are discussed in Hackbusch [183, §5.4, §9.3.4, §16.4] and [191, §5.6.5].

Although nonlinear systems are not the subject of this book, we remark that the nested iteration is of even greater importance for nonlinear systems of equations. In the linear case, it helps to save computer time. For nonlinear iterations, however, the availability of sufficiently good starting iterates often decides on convergence (to the desired solution) or divergence. The nested iteration with its starting value $\tilde{x}_\ell := \tilde{p}\,\tilde{x}_{\ell-1}$ is a suitable technique for generating such starting iterates.

A description and analysis of the nonlinear multigrid method and the corresponding nested iteration can be found in §11.9.5.

11.6 Convergence Analysis

11.6.1 Summary

The convergence proof of multigrid methods differs from the convergence proofs of other iterations because here the relationship between the equations $A_\ell\, x_\ell = b_\ell$ and $A_{\ell-1} x_{\ell-1} = b_{\ell-1}$ plays an important role.

As sufficient criteria, we introduce and discuss two conditions in §§11.6.2–11.6.3: the smoothing and approximation property. The smoothing property is of algebraic nature, whereas the proof of the approximation property involves the continuous problem, whose discretisation is described by $A_\ell x_\ell = b_\ell$. Together, the smoothing and approximation properties yield the convergence statement for the two-grid iteration (§11.6.4). For $\gamma \geq 2$, multigrid convergence can be concluded directly from the two-grid convergence (§11.6.5).

For positive definite matrices A_ℓ, the multigrid method can be designed as a positive definite iteration. In this case, we shall achieve even better convergence results, including the V-cycle ($\gamma = 1$; see §11.7). These results are generalised in Theorem 11.61 to the nonsymmetric case.

The analysis represented below is strongly simplified compared with that of Hackbusch [194], since presently we base our considerations mostly on the Euclidean and spectral norm. Other norms are mentioned in §11.6.6 and §11.7.2.

In contrast to what has been said above, there are multigrid methods for which convergence proofs can be performed by purely algebraic considerations. These variants will be discussed in §12.9.

11.6.2 Smoothing Property

In §11.1.1 we called a grid function $x_\ell = \sum \xi_{\alpha\beta}\, e^{\alpha\beta}$ (cf. (11.2b)) *smooth* if the coefficients $\xi_{\alpha\beta}$ of high frequencies α, β (corresponding to the large eigenvalues $\lambda_{\alpha\beta}$ in (3.1a)) are small. Quantitatively, one may measure the smoothness by $\|A_\ell x_\ell\|_2 = (\sum \lambda_{\alpha\beta} \xi_{\alpha\beta}|^2)^{1/2}$. If the smoothing step (11.21a) really leads to a smoothing of the errors $e_\ell = x_\ell^0 - x_\ell$, the error $S_\ell^\nu e_\ell$ produced by the smoothing step must have a better smoothing measure $\|A_\ell S_\ell^\nu e_\ell\|_2$ than e_ℓ. Therefore, the smoothing ability is characterised by the spectral norm $\|A_\ell S_\ell^\nu\|_2$. Before defining the smoothing property, we analyse $\|A_\ell S_\ell^\nu\|_2$ for Richardson's iteration with positive definite A_ℓ:

$$\mathcal{S}_\ell(x_\ell, b_\ell) := x_\ell - \Theta(A_\ell x_\ell - b_\ell) \tag{11.55a}$$

$$\text{with} \quad \Theta = \Theta_\ell = 1/\rho(A_\ell) = 1/\|A_\ell\|_2\,. \tag{11.55b}$$

We have $\|A_\ell S_\ell^\nu\|_2 = \|A_\ell (I - \Theta A_\ell)^\nu\|_2 = \|X(I-X)^\nu\|_2/\Theta$ with $X := \Theta A_\ell$. The following lemma applies to the matrix polynomial $X(I-X)^\nu$.

Lemma 11.23. *(a) For all matrices X with $0 \le X \le I$, the inequality*

$$\|X(I-X)^\nu\|_2 \le \eta_0(\nu) \qquad (\nu \ge 0)$$

holds, where the function $\eta_0(\nu)$ is defined by

$$\eta_0(\nu) := \nu^\nu / (\nu+1)^{\nu+1}. \tag{11.56}$$

(b) The asymptotic behaviour of $\eta_0(\nu)$ for $\nu \to \infty$ is

$$\eta_0(\nu) = \frac{1}{\mathrm{e}\,\nu} + \mathcal{O}(\nu^{-2}).$$

Proof. Set $f(\xi) := \xi(1-\xi)^\nu$. According to Lemma A.11a, we have

$$\|X(I-X)^\nu\|_2 = \rho(X(I-X)^\nu) = \max\{|f(\xi)| : \xi \in \sigma(X)\}.$$

By $f(\xi) \le f(1/(\nu+1)) = \eta_0(\nu)$ for all $\xi \in [0,1] \supset \sigma(X)$, part (a) is proved. The discussion of the function $\eta_0(\nu)$ yields statement (b). □

Remark 11.24. For $A_\ell > 0$, Richardson's method (11.55a,b) leads to

$$\|A_\ell S_\ell^\nu\|_2 \le \eta_0(\nu)\,\|A_\ell\|_2 \qquad \text{for all } \nu \ge 0,\ \ell \ge 0. \tag{11.57}$$

Note that the factor $\eta_0(\nu)$ is independent of h_ℓ and ℓ. The smoothing property, which we are going to define, is an estimate with a form similar to (11.57). Instead of $\eta_0(\nu)$, we may take an arbitrary zero sequence $\eta(\nu) \to 0$. Furthermore, it is neither necessary nor desirable to require an inequality as (11.57) for *all* $\nu \ge 0$.

Definition 11.25 (smoothing property). An iteration $\mathcal{S}_\ell$ $(\ell \ge 0)$ with iteration matrix S_ℓ satisfies the smoothing property if there are functions $\eta(\nu)$ and $\bar\nu(h)$ independent of ℓ with

$$\|A_\ell S_\ell^\nu\|_2 \le \eta(\nu)\|A_\ell\|_2 \quad \text{for all } 0 \le \nu < \bar\nu(h_\ell),\ \ell \ge 1, \tag{11.58a}$$

$$\lim_{\nu\to\infty} \eta(\nu) = 0, \tag{11.58b}$$

$$\lim_{h\to 0} \bar\nu(h) = \infty \qquad \text{or} \qquad \bar\nu(h) = \infty. \tag{11.58c}$$

The equality $\bar\nu(h) = \infty$ in (11.58c) expresses the fact that (11.58a) holds for *all* ν. This happens only for convergent iterations $\mathcal{S}_\ell$, as shown below.

Remark 11.26. The conditions (11.58a,b) together with $\bar\nu(h) = \infty$ imply convergence of $\mathcal{S}_\ell$.

Proof. $\rho(S_\ell^\nu) \le \|S_\ell^\nu\|_2 \le \|A_\ell^{-1}\|_2 \|A_\ell S_\ell^\nu\|_2 \le \eta(\nu)\,\mathrm{cond}_2(A_\ell) < 1$ for sufficiently large ν follows from $\eta(\nu) \to 0$ and implies $\rho(S_\ell) < 1$. □

From Remark 11.24, we conclude the next theorem.

Theorem 11.27. *For $A_\ell > 0$, the Richardson iteration (11.55a,b) satisfies the smoothing property (11.58a–c) with $\eta(\nu) := \eta_0(\nu)$ and $\bar\nu(h) = \infty$.*

The reason for the more general condition (11.58c) instead of $\bar{\nu}(h) = \infty$ is that the smoothing property can also be formulated for non-convergent iterations. Examples of divergent iterations are the Gauss–Seidel iteration for the indefinite problem (11.35), as well as Richardson's iteration in the next remark.

Remark 11.28. Assume that the indefinite matrix $A_\ell = A_\ell^{\mathsf{H}}$ has the spectrum $\sigma(A_\ell) \subset [-\alpha_\ell, \beta_\ell]$ with $0 \le \alpha_\ell \le \beta_\ell$ and $\lim_{\ell\to\infty} \alpha_\ell/\beta_\ell = 0$. Although the Richardson iteration with $\Theta = 1/\beta_\ell$ is divergent, it satisfies the smoothing property.

Proof. The damping factor is $\Theta = 1/\beta_\ell$. As in the proof for Lemma 11.23, we have $\|A_\ell(I - \Theta A_\ell)^\nu\|_2 \le \max\{\eta_0(\nu), (\alpha_\ell/\beta_\ell)(1+\alpha_\ell/\beta_\ell)^\nu\}\,\|A_\ell\|_2$. Define $\bar{\nu}(h_\ell)$ by $\bar{\nu}(h_\ell) := \beta_\ell/\alpha_\ell \to \infty$. For $\nu < \bar{\nu} := \bar{\nu}(h_\ell)$, the inequalities

$$(\alpha_\ell/\beta_\ell)(1+\alpha_\ell/\beta_\ell)^\nu \le (\alpha_\ell/\beta_\ell)\exp\{\nu\alpha_\ell/\beta_\ell\} = \tfrac{1}{\nu}\left(\tfrac{\nu}{\bar{\nu}}\exp\tfrac{\nu}{\bar{\nu}}\right) \le \tfrac{\mathrm{e}}{\nu}$$

follow. Hence, (11.58a–c) is satisfied by $\eta(\nu) := \max\{\eta_0(\nu), \mathrm{e}/\nu\} = \mathrm{e}/\nu$. □

The assumptions of Remark 11.28 are fulfilled for discretisation of the Helmholtz equation $-\Delta u - cu = f$ $(c > 0)$, because $\mathcal{O}(\alpha_\ell/\beta_\ell) = \mathcal{O}(h_\ell^2)$.

The following theorem can be considered as a perturbation lemma. It shows that the smoothing property remains valid under the perturbation of the matrix A'_ℓ into $A_\ell = A'_\ell + A''_\ell$, where A_ℓ may be indefinite and nonsymmetric.

Theorem 11.29. *Let $A_\ell = A'_\ell + A''_\ell$ and $\mathcal{S}_\ell = \mathcal{S}_\ell(\cdot,\cdot,A_\ell)$ and $\mathcal{S}'_\ell = \mathcal{S}'_\ell(\cdot,\cdot,A'_\ell)$ be the smoothing iterations corresponding to A_ℓ and A'_ℓ, respectively. Their iteration matrices are denoted by S_ℓ and S'_ℓ with $S''_\ell := S_\ell - S'_\ell$. Assume that*

$$A'_\ell \text{ and } S'_\ell \text{ satisfy the smoothing property with } \eta'(\nu),\ \bar{\nu}'(h), \tag{11.59a}$$

$$\|S'_\ell\|_2 \le C'_S \qquad \text{for all } \ell \ge 1, \tag{11.59b}$$

$$\lim_{\ell\to\infty} \|S''_\ell\|_2 = 0, \tag{11.59c}$$

$$\lim_{\ell\to\infty} \|A''_\ell\|_2 / \|A'_\ell\|_2 = 0. \tag{11.59d}$$

Then the iteration $\mathcal{S}_\ell = \mathcal{S}_\ell(\cdot,\cdot,A_\ell)$ for A_ℓ also satisfies the smoothing property. The corresponding bound $\eta(\nu)$ can be chosen, e.g., as $\eta(\nu) := 2\eta'(\nu)$.

Proof. $C_S := C'_S + \max\{\|S''_\ell\|_2 : \ell \ge 1\}$ satisfies $\|S_\ell\|_2 \le C_S$ for all $\ell \ge 1$. Without loss of generality, we may suppose that $C_S \ge 1$. S_ℓ^ν can be split into $S_\ell'^\nu + S_\ell''^{(\nu)}$ with

$$\begin{aligned}
\|S_\ell''^{(\nu)}\|_2 &= \|S_\ell^\nu - S_\ell'^\nu\|_2 \\
&= \left\| \sum_{\mu=0}^{\nu-1} S_\ell^\mu (S_\ell - S'_\ell) S_\ell'^{\nu-1-\mu} \right\|_2 = \left\| \sum_{\mu=0}^{\nu-1} S_\ell^\mu S''_\ell S_\ell'^{\nu-1-\mu} \right\|_2 \\
&\le \left(\sum_{\mu=0}^{\nu-1} C_S^\mu C_S'^{\nu-1-\mu} \right) \|S''_\ell\|_2 \le \nu C_S^{\nu-1} \|S''_\ell\|_2 \underset{(11.59c)}{\to} 0
\end{aligned} \tag{11.59e}$$

for $\ell \to \infty$. For $1 \le \nu \le \bar\nu'(h_\ell)$, we have

$$\begin{aligned}\|A_\ell S_\ell^\nu\|_2 &\le \|A'_\ell S'^\nu_\ell\|_2 + \|A''_\ell\|_2 \|S^\nu_\ell\|_2 + \|A'_\ell\|_2 \|S''^{(\nu)}_\ell\|_2 \qquad (11.59f)\\ &\le \eta'(\nu)\|A'_\ell\|_2 + C_S^\nu \|A''_\ell\|_2 + \nu C_S^{\nu-1}\|S''_\ell\|_2\|A'_\ell\|_2 \\ &= \eta'(\nu)\|A_\ell\|_2 \left\{ \frac{\|A'_\ell\|_2}{\|A_\ell\|_2} + C_S^\nu \frac{\|A''_\ell\|_2}{\|A_\ell\|_2} + \nu C_S^{\nu-1} \frac{\|A'_\ell\|_2}{\|A_\ell\|_2}\|S''_\ell\|_2 \right\}.\end{aligned}$$

By $\|A''_\ell\|_2/\|A'_\ell\|_2 \to 0$, $\|S''_\ell\|_2 \to 0$, $\|A'_\ell\|_2/\|A_\ell\|_2 \to 1$, $\|A''_\ell\|_2/\|A_\ell\|_2 \to 0$, the expression $\{\ldots\}$ converges to 1 for $\ell \to \infty$ (i.e., for $h = h_\ell \to 0$) while ν is fixed. This proves that $\bar\nu''(h) \to \infty$ $(h \to 0)$ for

$$\bar\nu''(h) := \sup\left\{ \nu > 0 : \frac{\|A'_\ell\|_2}{\|A_\ell\|_2}\big(1 + \nu C_S^{\nu-1}\|S''_\ell\|_2\big) + C_S^\nu \frac{\|A''_\ell\|_2}{\|A_\ell\|_2} \le 2 \text{ for } h_\ell \le h \right\}.$$

We define $\eta(\nu) := 2\eta'(\nu)$ and $\bar\nu(\eta) := \min\{\bar\nu'(h), \bar\nu''(h)\}$. For $\nu \le \bar\nu(h)$, inequality (11.59f) proves the smoothing property $\|A_\ell S_\ell^\nu\|_2 \le \eta(\nu)\|A_\ell\|_2$. □

Usually, discretisations of elliptic differential equations satisfy the following conditions:

$$\begin{aligned}&\text{There is an } h\text{-independent constant } c_0 \text{ such that}\\ &A'_\ell := \tfrac{1}{2}(A_\ell + A_\ell^{\mathsf H}) + c_0 I \text{ is positive definite,} && (11.60a)\\ &\underline{C} h_\ell^{-2m} \le \|A'_\ell\|_2 \le \bar C h_\ell^{-2m} \qquad (2m\text{: order of the differential eq.}), && (11.60b)\\ &\|A''_\ell\|_2 \le C h_\ell^{1-2m} \qquad \text{for } A''_\ell := A_\ell - A'_\ell = \frac{1}{2}(A_\ell - A_\ell^{\mathsf H}) - c_0 I && (11.60c)\end{aligned}$$

(cf. Hackbusch [183, 201]). To apply Theorem 11.29, one proves the smoothing property for the positive definite matrix A'_ℓ and transfers this property to A_ℓ by Theorem 11.29. Condition (11.59d) follows from (11.60b,c) by $\|A''_\ell\|_2/\|A'_\ell\|_2 \le \mathcal{O}(h_\ell) \to 0$. Since $S''_\ell = -\Theta A''_\ell = -A''_\ell/\|A'_\ell\|_2$ in the case of the Richardson iteration, (11.59d) also implies (11.59c). (11.59b) is always satisfied by $C_S = 2$, because $S'_\ell = I - A'_\ell/\|A'_\ell\|_2$ (even $C_S = 1$ if $A'_\ell \ge 0$).

The smoothing property can be proved not only for the Richardson method but also for the damped (block-)Jacobi iteration, the 2-cyclic Gauss–Seidel iteration (in particular, the chequer-board Gauss–Seidel method for five-point formulae), and the Kaczmarz iteration. Furthermore, symmetric iterations like the symmetric Gauss–Seidel method, SSOR, and the ILU iteration (cf. deZeeuw[105]) belong to this class. The symmetric case will be considered in §11.7.3. The smoothing property does not hold, e.g., for the undamped Jacobi method or the SOR method with $\omega \ge \omega_{\rm opt}$. For the smoothing analysis of the iterations mentioned above, see Hackbusch [183, §6.2].

The proof of Lemma 11.23 is based on the properties of the spectral norm for normal matrices. Correspondingly, statements for general matrices are proved via perturbation arguments. Nevertheless, it is possible to obtain the smoothing property

for general matrices directly. Even other norms than the spectral norm are possible. The following result by Reusken appeared in a report of 1991 and later in [323, 322].

Theorem 11.30. *Let $\|\cdot\|$ be a matrix norm corresponding to a vector norm. Let $S_\ell = I - W_\ell^{-1} A_\ell$ be the iteration matrix of the smoothing iteration and assume that*

$$\|I - 2W_\ell^{-1} A_\ell\| \le 1\,, \tag{11.61a}$$

$$\|W_\ell\| \le C\,\|A_\ell\| \tag{11.61b}$$

with a constant C independent of ℓ. Then the smoothing property (11.61c) holds:

$$\|A_\ell S_\ell^\nu\| \le C\sqrt{2/(\pi\nu)}\,\|A_\ell\| \qquad \text{for all } \nu \ge 1. \tag{11.61c}$$

The matrix in (11.61a) is the iteration matrix of $\mathcal{S}_{\vartheta=2,\ell}$, the extrapolated version of $\mathcal{S}_\ell$ with $\vartheta = 2$. Inequality (11.61a) does not imply convergence of $\mathcal{S}_{\vartheta=2,\ell}$, but characterises the weak contractivity or nonexpansivity (cf. Definition 7.3). The proof of the theorem is based on the following lemma.

Lemma 11.31. *Let the matrix B satisfy $\|B\| \le 1$ with respect to a matrix norm corresponding to a vector norm. Then*[5]

$$\|(I-B)(I+B)^\nu\| \;\le\; 2\binom{\nu}{\lfloor \nu/2 \rfloor} \;\le\; 2^{\nu+1}\sqrt{2/(\pi\nu)}.$$

Proof. Note that

$$\begin{aligned}(I-B)(I+B)^\nu &= (I-B)\sum_{\mu=0}^{\nu}\tbinom{\nu}{\mu}B^\mu = I + \sum_{\mu=1}^{\nu}\tbinom{\nu}{\mu}B^\mu - \sum_{\mu=0}^{\nu-1}\tbinom{\nu}{\mu}B^{\mu+1} - B^{\nu+1}\\ &= \bigl(I - B^{\nu+1}\bigr) + \sum_{\mu=1}^{\nu}\Bigl[\tbinom{\nu}{\mu} - \tbinom{\nu}{\mu-1}\Bigr]B^\mu.\end{aligned}$$

By $\|B^\mu\| \le 1$, $\binom{\nu}{\mu-\alpha} = \binom{\nu}{\alpha}$ and $\binom{\nu}{\mu} \ge \binom{\nu}{\mu-1}$ for $\mu \le \lfloor \nu/2 \rfloor$ we obtain

$$\begin{aligned}\|(I-B)(I+B)^\nu\| &\le 2 + 2\sum_{\mu=1}^{\lfloor \nu/2 \rfloor}\left|\binom{\nu}{\mu} - \binom{\nu}{\mu-1}\right|\\ &= 2 + 2\sum_{\mu=1}^{\lfloor \nu/2 \rfloor}\left\{\binom{\nu}{\mu} - \binom{\nu}{\mu-1}\right\} = 2 + 2\binom{\nu}{\lfloor \nu/2 \rfloor} - 2\binom{\nu}{0} = 2\binom{\nu}{\lfloor \nu/2 \rfloor}.\end{aligned}$$

The sequence $a_k := \binom{2k}{k}\sqrt{k}\,/\,2^{2k}$ is monotonically increasing and tends to $\lim a_k = \frac{1}{\sqrt{\pi}}$. The identity $\binom{\nu}{\lfloor \nu/2 \rfloor} = a_{\nu/2}\,2^\nu/\sqrt{\nu/2}$ for even powers ν leads to the desired estimate $a_k \le 1/\sqrt{\pi}$. For odd ν use $\binom{\nu}{\lfloor \nu/2 \rfloor} = \frac{1}{2}\binom{\nu+1}{(\nu+1)/2}$. □

Proof of Theorem 11.30. $(I - B)(I + B)^\nu = 2^{\nu+1} W_\ell^{-1} A_\ell\, S_\ell^\nu$ holds with $B := I - 2W_\ell^{-1} A_\ell$; hence,

$$\|A_\ell S_\ell^\nu\| = 2^{-\nu-1}\|W_\ell(I-B)(I+B)^\nu\| \le 2^{-\nu-1}\|W_\ell\|\|(I-B)(I+B)^\nu\|.$$

Assumption (11.61b) and Lemma 11.31 yield the statement. □

[5] $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}$ is the rounding down to the next integer.

Example 11.32. (a) Let $C_i > 0$ $(1 \le i \le 4)$ be positive constants independent of ℓ with

$$\begin{aligned} C_1 I \;&\le\; \tfrac{1}{2}(A_\ell + A_\ell^{\mathsf{H}}) \;\le\; C_2 h_\ell^{-2} I, \\ \|\tfrac{1}{2}(A_\ell - A_\ell^{\mathsf{H}})\|_2 \;&\le\; C_3 h_\ell^{-1} I, \\ \|A_\ell\|_2 \;&\ge\; C_4 h_\ell^{-2} I. \end{aligned}$$

Set $\Theta = \Theta_\ell := h_\ell^2 C_1/(C_1C_2 + C_3^2)$ and $C := (C_1C_2 + vC_3^2)/(C_1C_4)$. Then the Richardson iteration damped by Θ_ℓ satisfies the smoothing property (11.61c) with the constant C above.

(b) Let $\mathcal{S}_\ell$ be the Jacobi or Gauss–Seidel iteration damped by $\vartheta = \frac{1}{2}$. Furthermore, A_ℓ is assumed to be weakly diagonally dominant. Then the smoothing property (11.61c) holds with $C = 2$ with respect to the row-sum norm $\|\cdot\|_\infty$.

Proof. (i) Theorem 3.30 proves (11.61a). (11.61b) follows with $C = 1/\Theta$.

(ii) Since $\vartheta = 1/2$, inequality (11.61a) is the estimation of the nondamped Jacobi or Gauss–Seidel iterations. Weak diagonal dominance implies (11.61a). From $\|D_\ell\|_\infty \le \|D_\ell - E_\ell\|_\infty \le \|A_\ell\|_\infty$ for $A = D - E - F$ (cf. (3.11a–d)), we conclude (11.61b) with $C = 1/\vartheta$. □

11.6.3 Approximation Property

11.6.3.1 Formulation

For the coarse-grid correction, the fine-grid solution e_ℓ of $A_\ell e_\ell = d_\ell$ is replaced by $p\, e_{\ell-1}$ obtained from $A_{\ell-1}e_{\ell-1} = d_{\ell-1} := r d_\ell$. Therefore, $p\, e_{\ell-1} \approx e_\ell$, i.e., $pA_{\ell-1}^{-1} r d_\ell \approx A_\ell^{-1} d_\ell$ should be valid. We quantify this requirement by

$$\|pA_{\ell-1}^{-1} r d_\ell - A_\ell^{-1} d_\ell\|_2 \le C_A \|d_\ell\|_2 / \|A_\ell\|_2 \quad \text{for } \ell \ge 1,\ d_\ell \in X_\ell.$$

This inequality can be rewritten by the matrix norm (spectral norm) as the *approximation property*

$$\|A_\ell^{-1} - pA_{\ell-1}^{-1} r\|_2 \le C_A / \|A_\ell\|_2 \qquad \text{for all } \ell \ge 1. \tag{11.63}$$

In general, proofs of the approximation property (11.63) are not of algebraic nature but use (at least indirectly) properties of the underlying boundary value problem. One possible route to the proof is as follows. Assume that $A_{\ell-1} = r A_\ell p$ holds according to (11.20). For an arbitrary restriction $r' : X_\ell \to X_{\ell-1}$, the following factorisation holds:

$$A_\ell^{-1} - pA_{\ell-1}^{-1} r = (I - pA_{\ell-1}^{-1} r A_\ell) A_\ell^{-1} = (I - pA_{\ell-1}^{-1} r A_\ell)(I - p\, r') A_\ell^{-1}.$$

Under suitable conditions,[6] the solution $v_\ell := A_\ell^{-1} f_\ell$ is sufficiently smooth, so that the interpolation error

$$d_\ell = (I - pr')v_\ell = v_\ell - pr'v_\ell$$

can be estimated by $\|d_\ell\|_2 \le C\|f_\ell\|_2/\|A_\ell\|_2$. The same tools can be used to show that $\|I - pA_{\ell-1}^{-1} rA_\ell\| \le \text{const}$. Together, one obtains the approximation property (11.63). In case $A_{\ell-1}$ is not the Galerkin product, see Hackbusch [183, Criteria 6.3.35 and 6.3.38].

The easiest proof of the approximation property can be given for Galerkin discretisations. The discretisation, together with the prolongations and restrictions, is defined in §§11.6.3.2–11.6.3.3. The crucial part of the proof of the approximation property is given in §11.6.3.4.

11.6.3.2 Galerkin Discretisation

The boundary value problem is described in the variational form (E.5). Instead of a single finite-dimensional subspace $V_n \subset V$ we consider a hierarchy of subspaces

$$V_0 \subset V_1 \subset \ldots \subset V_{\ell-1} \subset V_\ell \subset \ldots \subset V,$$

where V_ℓ replaces the notation V_{n_ℓ} ($n_\ell = \dim(V_\ell)$) used in §E.2. Similarly, all mappings $P_n = P_{n_\ell}, \ldots$ used in §§E.2–E.6 are now denoted by $P_\ell : V_\ell \to V, \ldots$

11.6.3.3 Canonical Prolongation and Restriction

Section E.6 discusses the relation of Galerkin discretisations using two subspaces $V_{n'} \subset V_n$, now denoted by $V_{\ell-1} \subset V_\ell$. According to Proposition E.15, there are mappings $p : X_{\ell-1} \to X_\ell$ and $r : X_\ell \to X_{\ell-1}$ with

$$P_\ell\, p = P_{\ell-1}, \qquad r = p^*, \qquad rR_\ell = R_{\ell-1}. \tag{11.64}$$

Since p and r are the natural choice (see the diagram in (E.19)), they are called the *canonical prolongation* and the *canonical restriction*.

Remark 11.33. (a) Using the representation $p = \hat{R}_\ell P_{\ell-1}$ in (E.18) and the bounds in (E.10a,b) and (E.11c), we get the uniform estimates

$$\|p\|_{X_\ell \leftarrow X_{\ell-1}} \;=\; \|r\|_{X_{\ell-1} \leftarrow X_\ell} \;\le\; \underline{C}_P \bar{C}_P \qquad \text{for all } \ell \ge 1.$$

(b) The matrices A_ℓ and $A_{\ell-1}$ are connected by (11.20):

$$A_{\ell-1} = r\, A_\ell\, p \qquad \text{for all } \ell \ge 1.$$

[6] In the case of difference schemes, the theory of *discrete regularity* can be used; cf. Hackbusch [180, 181], [183, §6.3.2.1], [201, §9.2], and Jovanovič–Süli [229].

11.6.3.4 Proof of the Approximation Property

Based on the $2m$-regularity (E.13b), the error estimate (E.14) is proved in §E.5:

$$\|E_\ell\|_{U\leftarrow U} \le C_E h_\ell^m \qquad \text{for all } \ell \ge 1, \tag{11.65a}$$

where $E_\ell := A^{-1} - P_\ell A_\ell^{-1} R_\ell$. $2m$ is the order of the differential operator. The inverse estimate, together with the boundedness of the bilinear form, yields (E.12c):

$$\|A_\ell\|_2 \le C_K h_\ell^{-2m} \qquad \text{for all } \ell \ge 1. \tag{11.65b}$$

The inequality

$$\|\hat{R}_\ell\|_{X_\ell\leftarrow U} = \|\hat{P}_\ell\|_{U\leftarrow X_\ell} \le \underline{C}_P \tag{11.65c}$$

is mentioned in (E.11c). A last condition for the approximation property is almost identical to the inequality $n_\ell \le \overline{C}_h\, n_{\ell-1}$ used in §11.5.3:

$$h_{\ell-1} \le C_h h_\ell \qquad \text{for all } \ell \ge 1. \tag{11.65d}$$

Usually, (11.65d) holds with $C_h = 2$.

Theorem 11.34. *Let A_ℓ be the matrices (E.7b) of the Galerkin discretisation. Choose the canonical p and r. Assume (11.65a–d). Then the approximation property (11.63) holds.*

Proof. Use inequality (11.65a) for ℓ and $\ell - 1$:

$$\begin{aligned}&\|P_\ell A_\ell^{-1} R_\ell - P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1}\|_{U\leftarrow U}\\ &= \|E_{\ell-1} - E_\ell\|_{U\leftarrow U} \le C_E \left(h_\ell^{2m} + h_{\ell-1}^{2m}\right).\end{aligned}$$

(11.65d) implies $h_{\ell-1}^{2m} \le C_h^{2m} h_\ell^{2m}$. From

$$h_\ell^{2m} \le C_K / \|A_\ell\|_2 \text{ (cf. (11.65b))} \quad \text{and} \quad P_\ell = P_{\ell-1} p,\ R_\ell = r R_\ell^{-1} \text{ (cf. (11.64))},$$

we conclude that

$$\|P_\ell (A_\ell^{-1} - p A_{\ell-1}^{-1} r) R_\ell\|_{U\leftarrow U} \le C' / \|A_\ell\|_2$$

with $C' := C_E C_K (1 + C_h^{2m})$. Multiplying $P_\ell (A_\ell^{-1} - p A_{\ell-1}^{-1} r) R_\ell$ by $\hat{R}_\ell$ from the left and by $\hat{P}_\ell$ from the right and using (E.11b,c), we obtain

$$\begin{aligned}\|A_\ell^{-1} - p A_{\ell-1}^{-1} r\|_2 &\le \|\hat{R}_\ell\|_{X_\ell\leftarrow U} \|P_\ell (A_\ell^{-1} - p A_{\ell-1}^{-1} r) R_\ell\|_{U\leftarrow U} \|\hat{P}_\ell\|_{U\leftarrow X_\ell}\\ &\le C' \underline{C}_P^2 / \|A_\ell\|_2,\end{aligned}$$

which is the approximation property with $C_A := C' \underline{C}_P^2$. □

11.6.4 Convergence of the Two-Grid Iteration

As mentioned in §11.2.2, $\rho(M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}) = \rho(M_\ell^{\mathrm{TGM}(\nu,0)})$ holds for $\nu = \nu_1 + \nu_2$, so that we may restrict our considerations to $\nu = \nu_1 > 0$, $\nu_2 = 0$. This choice is optimal for statements concerning the contraction number $\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2$ with respect to the spectral norm. The following Theorems 11.35 and 11.36 correspond to the cases $\bar\nu(h) = \infty$ and $\bar\nu(h) < \infty$, respectively.

Theorem 11.35. *Assume the smoothing and approximation properties (11.58a–c), (11.63) with $\bar\nu(h) = \infty$. For given $0 < \zeta < 1$, there exists a lower bound $\underline{\nu}$ such that*

$$\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2 \le C_A \eta(\nu) \le \zeta \qquad \text{for all } \nu \ge \underline{\nu},\ \ell \ge 1. \tag{11.66}$$

Here, C_A and $\eta(\nu)$ are the quantities in (11.63) and (11.58a,b). By $\zeta < 1$, inequality (11.66) implies convergence of the two-grid iteration. Note that the contraction bound $C_A\,\eta(\nu)$ is independent of h_ℓ.

Proof. The two-grid iteration matrix can be factorised as follows:

$$M_\ell^{\mathrm{TGM}(\nu,0)} = (I - pA_{\ell-1}^{-1} r A_\ell) S_\ell^\nu = \left[A_\ell^{-1} - pA_{\ell-1}^{-1} r\right] [A_\ell S_\ell^\nu]$$

(cf. Lemma 11.11). Estimating both factors by (11.58a) and (11.63), we obtain the inequality (11.66). □

Theorem 11.36. *Assume the smoothing and approximation properties (11.58a–c), (11.63), including the case $\bar\nu(h) < \infty$. For all $0 < \zeta < 1$, there exist bounds $\bar h > 0$ and $\underline{\nu}$ such that (11.66) holds for all $\nu \in [\underline{\nu}, \bar\nu(h))$ and all h_ℓ with $h_\ell \le \bar h$, where the interval $[\underline{\nu}, \bar\nu(h))$ is not empty (i.e., $\underline{\nu} < \bar\nu(h)$).*

Proof. Choose $\underline{\nu}$ as in Theorem 11.35. Because of $\bar\nu(h) \to \infty$ ($h \to 0$), $\bar h$ can be chosen such that $\bar\nu(h_\ell) > \underline{\nu}$ for all $h_\ell \le \bar h$. □

11.6.5 Convergence of the Multigrid Iteration

In Theorem 11.19, the representation $M_\ell^{\mathrm{MGM}(\nu,0)} = M_\ell^{\mathrm{TGM}(\nu,0)} - \ldots$ of the multigrid iteration matrix is shown. We are exploiting the fact that the perturbation '…' is sufficiently small; hence, two-grid convergence implies multigrid convergence. Besides the smoothing and approximation properties, we require additional conditions, which are easy to satisfy. The first one is

$$\|S_\ell^\nu\|_2 \le C_S \qquad \text{for all } \ell \ge 1,\ 0 < \nu < \bar\nu := \min_{\ell \ge 1} \bar\nu(h_\ell) \tag{11.67a}$$

with $\bar\nu(h_\ell)$ defined in (11.58c).

Exercise 11.37. Assume $S_\ell := S'_\ell + S''_\ell$. Let (11.67a) hold for S'^{ν}_ℓ and assume (11.59c): $\lim_{\ell\to\infty} \|S''_\ell\|_2 = 0$. Prove (11.67a) for S_ℓ (similar to Theorem 11.29).

Exercise 11.38. Prove the inequalities

$$\underline{C}_p^{-1} \|x_{\ell-1}\|_2 \le \|p x_{\ell-1}\|_2 \le \bar{C}_p \|x_{\ell-1}\|_2 \quad (x_{\ell-1} \in X_{\ell-1},\ \ell \ge 1) \tag{11.67b}$$

for the canonical choice (11.64) by using (E.9) with $\underline{C}_p = \bar{C}_p := \underline{C}_P \bar{C}_P$.

The identity $pA_{\ell-1}^{-1} r A_\ell S_\ell^\nu = S_\ell^\nu - \left[A_\ell^{-1} - pA_{\ell-1}^{-1} r\right] A_\ell S_\ell^\nu = S_\ell^\nu - M_\ell^{\mathrm{TGM}(\nu,0)}$ implies the next statement.

Lemma 11.39. *Let (11.67a,b) be valid. Then*

$$\|A_{\ell-1}^{-1} r A_\ell S_\ell^\nu\|_2 \le \underline{C}_p (C_S + \|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2). \tag{11.67c}$$

Let $\nu = \nu_1 > 0$ and $\nu_2 = 0$ be the numbers of smoothing steps as in §11.6.4. Using (11.67b, c), we can estimate the multigrid iteration matrix in (11.42a,b) by

$$\|M_\ell^{\mathrm{MGM}(\nu,0)}\|_2 \le \|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2 + C^* \|M_{\ell-1}^{\mathrm{MGM}(\nu,0)}\|_2^\gamma \quad \text{for } \ell \ge 1 \tag{11.68a}$$
$$\text{with } C^* := \underline{C}_p \bar{C}_p (C_S + 1).$$

Here, ν is assumed to be chosen large enough so that $\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2 \le 1$ according to Theorem 11.35 or 11.36. Together with $M_0^{\mathrm{MGM}} = 0$ (cf. (11.42a)), inequality (11.68a) leads to the recursive inequalities (11.68c) for the quantities ζ_ℓ:

$$\zeta_\ell := \|M_\ell^{\mathrm{MGM}(\nu,0)}\|_2 \qquad (\ell \ge 0), \tag{11.68b}$$
$$\zeta_0 := 0, \qquad \zeta_\ell \le \zeta + C^* (\zeta_{\ell-1})^\gamma \quad \text{for } \ell \ge 1. \tag{11.68c}$$

ζ is the ℓ-independent bound for the two-grid convergence, whose existence is stated by Theorem 11.35 or 11.36:

$$\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2 \le \zeta\,. \tag{11.68d}$$

Analysing the fixed-point equation $x = \zeta + C^* x^\gamma$, we obtain the next result.

Lemma 11.40. *Assume $\gamma \ge 2$, $C^*\gamma > 1$, and $\zeta \le \frac{\gamma-1}{\gamma} / \sqrt[\gamma-1]{C^*\gamma}$. Then all solutions of the inequalities (11.68c) are bounded by*

$$\zeta_\ell \le \zeta^* \le \frac{\gamma}{\gamma-1}\zeta < 1 \qquad \textit{for all } \ell \ge 0. \tag{11.69}$$

Exercise 11.41. For the most interesting case of $\gamma = 2$, prove that

$$\zeta^* = 2\zeta \,/\, \left(1 - \sqrt{1 - 4C^*\zeta}\right) \qquad \text{for } \zeta^* \text{ in (11.69)}.$$

Since, by (11.68b), ζ_ℓ are the contraction number bounds, we obtain the desired convergence result.

Theorem 11.42 (multigrid convergence). *Assume the smoothing and the approximation properties (11.58a–c) and (11.63), the conditions (11.67a,b), and, in addition,* $\gamma \geq 2$. *As in Theorems 11.35 and 11.36, for every* $0 < \zeta' < 1$ *there are* $\underline{\nu}$ *and* $\bar{h} > 0$ *such that*

$$\|M_\ell^{\mathrm{MGM}(\nu,0)}\|_2 \leq \zeta' < 1 \quad \textit{for } \underline{\nu} \leq \nu < \bar{\nu} := \min_{\ell \geq 1} \bar{\nu}(h_\ell),$$

provided that $h_1 \leq \bar{h}$. *Here,* $\underline{\nu} < \bar{\nu}$ *holds. In the case of* $\bar{\nu}(h) = \infty$, *one may set* $\bar{h} := \infty$ *(i.e., the choice of the grid size is not restricted).*

Proof. Choose $\zeta := \frac{\gamma-1}{\gamma}\zeta'$ small enough, so that ζ fulfils the assumptions of Lemma 11.40. According to Theorem 11.36, $\underline{\nu}$ and $\bar{h}$ have to be chosen in such a way that (11.68d) holds: $\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_2 \leq \zeta$ for $\underline{\nu} \leq \nu < \bar{\nu}$. Lemmata 11.39 and 11.40 give $\zeta_\ell = \|M_\ell^{\mathrm{MGM}(\nu,0)}\|_2 \leq \frac{\gamma}{\gamma-1}\zeta \leq \zeta'$. □

11.6.6 Case of Weaker Regularity

The proof of the approximation property uses the $2m$-regularity (cf. (E.13b)) which, in the case of the Poisson equation, is $A^{-1} = -\Delta^{-1} : U = L_2(\Omega) = H^0(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$. This assumption is true for the unit square $\Omega = (0,1) \times (0,1)$ as for any convex domain, but it does not hold, e.g., for domains with re-entrant corners. In the general case, one obtains only statements of the form

$$A^{-1} : H^{-\sigma m}(\Omega) \to H^{(2-\sigma)m}(\Omega) \cap H_0^m(\Omega) \qquad \text{for some } \sigma \in (0,1)$$

(cf. Hackbusch [193, §9.1]). A similar statement may be assumed for A^*. If $\sigma < 1$, the approximation property (11.63) cannot be proved but has to be formulated by the help of other norms.

Let $|\cdot|_t$ for $-1 \leq t \leq 1$ be a discrete analogue of the Sobolev norm $H^{tm}(\Omega)$. We define $U_\ell := (X_\ell, |\cdot|_\sigma)$ and $F_\ell := (X_\ell, |\cdot|_{-\sigma})$. Then

$$\|A_\ell^{-1} - pA_{\ell-1}^{-1}r\|_{U_\ell \leftarrow F_\ell} \leq (C_A / \|A_\ell\|_2)^{1-\sigma} \tag{11.70}$$

can be shown (cf. Hackbusch [183, §6.3.1.3]; cf. (E.15)). For the notation of the norm on the left-hand side, compare with (B.11). For $A_\ell > 0$, the norms can be defined by

$$\|x_\ell\|_{U_\ell} = |x_\ell|_\sigma := \|A_\ell^{\sigma/2} x_\ell\|_2, \quad \|f_\ell\|_{F_\ell} = |f_\ell|_{-\sigma} := \|A_\ell^{-\sigma/2} f_\ell\|_2. \tag{11.71}$$

In the general case, replace the matrix A_ℓ in (11.71) by the positive definite part $A'_\ell := \frac{1}{2}(A_\ell + A_\ell^{\mathsf{H}}) + c_0 I$ (cf. (11.60a)).

Part (11.58a) of the smoothing property (11.58a–c) has to be adapted to the new norms. Inequality (11.58a) becomes

$$\|A_\ell S_\ell^\nu\|_{F_\ell \leftarrow U_\ell} \leq \eta(\nu) \|A_\ell\|_2^{1-\sigma} \qquad \text{for } 0 \leq \nu \leq \bar{\nu}(h_\ell). \tag{11.72}$$

Exercise 11.43. Let $\mathcal{S}_\ell$ be the Richardson iteration (11.55a,b) and assume $A_\ell > 0$. Using the norms in (11.71), prove for all $\nu \geq 0$ that

$$\|A_\ell S_\ell^\nu\|_{F_\ell \leftarrow U_\ell} = \|A_\ell^{1-\sigma}(I - \Theta A_\ell)^\nu\|_2 \leq \left[\eta_0(\tfrac{\nu}{1-\sigma})\,\|A_\ell\|_2\right]^{1-\sigma}. \tag{11.73}$$

The two-grid contraction number with respect to $\|\cdot\|_{U_\ell}$ can be concluded from the product of (11.70) and (11.72):

$$\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_{U_\ell \leftarrow U_\ell} \leq \eta(\nu)\, C_A^{1-\sigma}. \tag{11.74}$$

Similar to §11.6.5, we obtain a corresponding convergence result for the multigrid iteration.

Consider the bound $\eta(\nu) = \left[\eta_0(\frac{\nu}{1-\sigma})\, C_A\right]^{1-\sigma}$ in (11.73). For the standard case discussed in §§11.6.2–11.6.5, we had $\sigma = 0$ and the bound $\eta(\nu)$ in (11.74) behaved as $\mathcal{O}(\frac{1}{\nu})$. For $0 < \sigma < 1$, the contraction number behaves as $\mathcal{O}(1/\nu^{1-\sigma})$. The value $\sigma = 1$ is not sufficient because $\eta(\nu)$ fails to fulfil (11.58b).

11.7 Symmetric Multigrid Methods

The multigrid analysis above addresses the general (nonsymmetric) case in order to emphasise that multigrid iterations are not restricted to symmetric or even only positive definite problems. However, the symmetric case admits some stronger statements that are covered in this chapter.

11.7.1 Symmetric and Positive Definite Multigrid Algorithms

We consider the two-grid algorithm (11.22b) and the multigrid iteration (11.33). The required symmetry conditions are

$$r = p^*, \quad \nu_1 = \nu_2 = \frac{\nu}{2}, \quad \hat{\mathcal{S}}_\ell = \mathcal{S}_\ell^* \text{ for all } \ell \geq 0 \tag{11.75a}$$

(cf. (11.17)). The second condition requires the post-smoothing $\hat{\mathcal{S}}_\ell$ to be adjoint to the pre-smoothing $\mathcal{S}_\ell$. Occasionally, we need the Galerkin product property:

$$A_{\ell-1} = r\, A_\ell\, p\,. \tag{11.75b}$$

The Galerkin product (11.75b), together with $r = p^*$, ensures that $A_{\ell_{\max}} = A_{\ell_{\max}}^{\mathsf{H}}$ implies $A_\ell = A_\ell^{\mathsf{H}}$ for all $0 \leq \ell < \ell_{\max}$. Otherwise, this property must be required explicitly:

$$A_{\ell_{\max}} = A_{\ell_{\max}}^{\mathsf{H}} \quad \Longrightarrow \quad A_\ell = A_\ell^{\mathsf{H}} \text{ for all } 0 \leq \ell < \ell_{\max}, \tag{11.75c}$$

Lemma 11.44. *Let (11.75a,c) be valid. Then the two- and multigrid iterations* $\Phi_\ell^{\mathrm{TGM}(\frac{\nu}{2},\frac{\nu}{2})}$ *and* $\Phi_\ell^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})}$ *are symmetric:* $\Phi_\ell^{\mathrm{TGM}(\frac{\nu}{2},\frac{\nu}{2})}, \Phi_\ell^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})} \in \mathcal{L}_{\mathrm{sym}}$.

Proof. (i) We have to prove that the Hermitian symmetry of $A = A_{\ell_{\max}}$ implies the symmetry of the matrix $N_{\ell_{\max}} = N_{\ell_{\max}}^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})}$ of the second normal form (cf. (5.4)).

(ii) Assume $A_{\ell_{\max}} = A_{\ell_{\max}}^{\mathsf{H}}$ and by (11.75c) that $A_\ell = A_\ell^{\mathsf{H}}$ holds for all levels. First, we prove $\Phi_\ell^{\mathrm{TGM}(\nu/2,\nu/2)} \in \mathcal{L}_{\mathrm{sym}}$. Note that $\Phi_\ell^{\mathrm{CGC}} \in \mathcal{L}_{\mathrm{sym}}$, since $N_\ell^{\mathrm{CGC}} = pA_{\ell-1}^{-1}r$ in Remark 11.6 is symmetric. By Corollary 5.30, the two-grid iteration $\Phi_\ell^{\mathrm{TGM}(\frac{\nu}{2},\frac{\nu}{2})} = (\mathcal{S}_\ell^*)^\nu \circ \Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^\nu = (\mathcal{S}_\ell^\nu)^* \circ \Phi_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^\nu$ is also symmetric.

(iii) Next, we use the definition (11.42a,b) for an induction on ℓ. Assume that $\Phi_{\ell-1}^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})} \in \mathcal{L}_{\mathrm{sym}}$. Then $\left(\Phi_{\ell-1}^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})}\right)^\gamma$ is also symmetric and, by Criterion 5.5, the matrix $M_{\ell-1}^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})} A_{\ell-1}^{-1}$ is symmetric. The steps (11.33c–e) define an coarse-grid correction $\hat{\Phi}_\ell^{\mathrm{CGC}}$ with the iteration matrix $M_\ell^{\mathrm{CGC}} := p\left(M_{\ell-1}^{\mathrm{MGM}}\right)^\gamma A_{\ell-1}^{-1} r A_\ell$. Obviously, $M_\ell^{\mathrm{CGC}} A_\ell^{-1} = p\left[\left(M_{\ell-1}^{\mathrm{MGM}}\right)^\gamma A_{\ell-1}^{-1}\right] r \in \mathcal{L}_{\mathrm{sym}}$ holds, and Criterion 5.5 proves the symmetry of $\hat{\Phi}_\ell^{\mathrm{CGC}}$. As in part (ii), the symmetry of $\Phi_\ell^{\mathrm{MGM}(\nu/2,\nu/2)}$ follows from the representation $\Phi_\ell^{\mathrm{MGM}(\nu/2,\nu/2)} = (\mathcal{S}_\ell^\nu)^* \circ \hat{\Phi}_\ell^{\mathrm{CGC}} \circ \mathcal{S}_\ell^\nu$. □

The positive definiteness of $\Phi_\ell^{\mathrm{TGM}(\frac{\nu}{2},\frac{\nu}{2})}$ and $\Phi_\ell^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})}$ is considered next.

Lemma 11.45. *Assume (11.75a) and* $A_\ell > 0$. *Set* $M_\ell := M_\ell^{\mathrm{MGM}(\nu/2,\nu/2)}$ *and* $W_\ell := W_\ell^{\mathrm{MGM}(\nu/2,\nu/2)}$. *The following statements also hold for the two-grid case.*
(a) Assume that the iteration $\Phi_\ell^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})}$ *converges. Then it is positive definite, i.e.,* $\Phi_\ell^{\mathrm{MGM}(\frac{\nu}{2},\frac{\nu}{2})} \in \mathcal{L}_{\mathrm{pos}}$, *and converges monotonically with respect the energy norm* $\|\cdot\|_{A_\ell}$. *The transformed iteration matrices* $A_\ell^{1/2} M_\ell A_\ell^{-1/2}$ *are Hermitian. The matrix* W_ℓ *of the third normal form is positive definite and fulfils*

$$(1-\rho_\ell)W_\ell \le A_\ell \le (1+\rho_\ell)W_\ell \quad \text{with } \rho_\ell = \rho(M_\ell) = \|A_\ell^{\frac{1}{2}} M_\ell A_\ell^{-\frac{1}{2}}\|_2. \tag{11.76a}$$

If, according to Theorems 11.35 or 11.42, $\rho_\ell \le \rho < 1$ *is* h_ℓ*-independent, then the condition (11.76b) is also* h_ℓ*-independent:*

$$\kappa(W_\ell^{-1} A_\ell) \le \frac{1+\rho_\ell}{1-\rho_\ell} \le \frac{1+\rho}{1-\rho}. \tag{11.76b}$$

(b) In the case of (11.75b), the inequalities (11.76a,b) can be improved:

$$(1-\rho_\ell)W_\ell \le A_\ell \le W_\ell, \quad \kappa(W_\ell^{-1} A_\ell) \le 1/(1-\rho_\ell) \le 1/(1-\rho).$$

Proof. The representations (11.23) and (11.42a,b) show that $A_\ell M_\ell A_\ell^{-1} = M_\ell^{\mathsf{H}}$, because $A_\ell \hat{S}_\ell A_\ell^{-1} = S_\ell^{\mathsf{H}}$ according to (5.2b). This proves part (a). Part (b) is obtained from Theorem 3.34c. Part (c) is based on the property $A_\ell^{1/2} M_\ell A_\ell^{-1/2} \ge 0$, which will be proved in (11.87b). □

11.7.2 Two-Grid Convergence for $\nu_1 > 0$, $\nu_2 > 0$

The case $\nu_1 = \nu > 0$ and $\nu_2 = 0$ is treated in §11.6. The technique used there can also be applied to the general case $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu := \nu_1 + \nu_2 > 0$, and especially to $\nu_1 = \nu_2 = \nu/2$.

Exercise 11.46. Assume (11.75a) without the condition $\nu_1 = \nu_2$. Prove

$$\Phi_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} = \left(\Phi_\ell^{\mathrm{TGM}(\nu_2,\nu_1)}\right)^*, \tag{11.77a}$$

$$\Phi_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} = \Phi_\ell^{\mathrm{TGM}(0,\nu_2)} \circ \Phi_\ell^{\mathrm{TGM}(\nu_1,0)} \text{ in the case of (11.75b).} \tag{11.77b}$$

Under assumption (11.75b), the statements (11.77c,d) for the two-grid iteration matrices $M_\ell(\nu_1,\nu_2) := M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}$ follow from (11.77a,b):

$$M_\ell(\nu_1,\nu_2) = M_\ell(0,\nu_2)M_\ell(\nu_1,0) = A_\ell^{-1}M_\ell(\nu_2,0)^{\mathsf{H}}A_\ell M_\ell(\nu_1,0), \tag{11.77c}$$

$$A_\ell^{\frac{1}{2}}M_\ell(\nu_1,\nu_2)A_\ell^{-\frac{1}{2}} = \left(A_\ell^{\frac{1}{2}}M_\ell(\nu_2,0)A_\ell^{-\frac{1}{2}}\right)^{\mathsf{H}}\left(A_\ell^{\frac{1}{2}}M_\ell(\nu_1,0)A_\ell^{-\frac{1}{2}}\right). \tag{11.77d}$$

For estimating $A_\ell^{1/2}\, M_\ell(\nu,0)\, A_\ell^{-1/2}$, we may use the approximation property (11.78a) and the smoothing property (11.78b):

$$\|A_\ell^{1/2}(A_\ell^{-1} - pA_{\ell-1}^{-1}r)\|_2 \leq \sqrt{C_A/\,\|A_\ell\|_2}\,, \tag{11.78a}$$

$$\|A_\ell S_\ell^\nu A_\ell^{-1/2}\|_2 \leq \sqrt{\eta(2\nu)\,\|A_\ell\|_2}\,, \tag{11.78b}$$

which correspond to (11.70) and (11.72) for the energy norm $\|\cdot\|_{U_\ell} = \|\cdot\|_{A_\ell}$ and the Euclidean norm $\|\cdot\|_{F_\ell} = \|\cdot\|_2$. Under assumption (11.75b), inequality (11.78a) is equivalent to the approximation property (11.63). In the case of Richardson's iteration (11.55a,b), inequality (11.78b) holds because of

$$\|A_\ell S_\ell^\nu A_\ell^{-1/2}\|_2 = \|A_\ell^{1/2} S_\ell^\nu\|_2^2 = \|A_\ell S_\ell^{2\nu}\|_2 \leq \eta_0(2\nu)\|A_\ell\|_2$$

with $\eta(2\nu) = \eta_0(2\nu)$ (η_0 in (11.56)). The inequalities (11.78a,b) yield the estimate

$$\|M_\ell^{\mathrm{TGM}(\nu,0)}\|_{A_\ell} = \|A_\ell^{1/2} M_\ell^{\mathrm{TGM}(\nu,0)} A_\ell^{-1/2}\|_2 \leq \sqrt{\eta(2\nu)C_A}.$$

Using (11.77d), we finally prove the following convergence theorem.

Theorem 11.47. *Assume (11.75a,b) without $\nu_1 = \nu_2$. The smoothing and approximation properties (11.78a,b) imply*

$$\begin{aligned} \|M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}\|_{A_\ell} &\leq C_A\sqrt{\eta(2\nu_1)\eta(2\nu_2)}, \\ \|M_\ell^{\mathrm{TGM}(\nu/2,\nu/2)}\|_{A_\ell} &\leq C_A\,\eta(\nu). \end{aligned}$$

Two-grid convergence follows as in Theorems 11.35–11.36.

As in §11.6.5, multigrid convergence can be concluded from the two-grid convergence. However, it has to be emphasised that the proof technique in §11.6.5 requires $\gamma \geq 2$ and therefore excludes the V-cycle ($\gamma = 1$).

11.7.3 Smoothing Property in the Symmetric Case

In particular, condition (11.75b): $\hat{S}_\ell = S_\ell^*$ is satisfied if $\hat{\mathcal{S}}_\ell = \mathcal{S}_\ell$ is a symmetric smoothing iteration. For symmetric iterations, the proof of the smoothing property is rather easy.

Lemma 11.48. *Let $S_\ell = I - W_\ell^{-1}A_\ell$ be the iteration matrix of a positive definite iteration $\mathcal{S}_\ell$ and assume that $\gamma W_\ell \le A_\ell \le \Gamma W_\ell$ for all $\ell \ge 0$ with $0 \le \gamma \le \Gamma < 2$. Then*

$$\|A_\ell S_\ell^\nu\|_2 \le \|W_\ell\|_2 \, \max\{\eta_0(\nu), \Gamma\,|1-\Gamma|^\nu\}$$

implies the smoothing property (11.58a–c) with $\bar{\nu}(h) = \infty$ if there is some C_W with

$$\|W_\ell\|_2 \le C_W \|A_\ell\|_2 \qquad \textit{for all } \ell > 0. \tag{11.79}$$

Proof. Define $Y := W_\ell^{-1/2} R_\ell W_\ell^{-1/2}$ with R_ℓ in $A_\ell = W_\ell - R_\ell$ and note that

$$\|A_\ell S_\ell^\nu\|_2 = \|W_\ell^{1/2}(I-Y)Y^\nu W_\ell^{1/2}\|_2 \le \|W_\ell^{1/2}\|_2^2 \, \|(I-Y)Y^\nu\|_2.$$

The first factor is equal to $\|W_\ell\|_2$, the second can be estimated as in Lemma 11.23 by $\max\{\eta_0(\nu), \Gamma\,|1-\Gamma|^\nu\}$ because of $(1-\Gamma)I \le Y \le (1-\gamma)I$. □

The following variant of the estimate is due to Wittum [403]. The estimate is helpful if good bounds for $\|R_\ell\|_2$ are known.

Lemma 11.49. *In addition to the assumptions of Lemma 11.48, assume $\nu \ge 2$. Define $R_\ell := W_\ell - A_\ell$. Then*

$$\|A_\ell S_\ell^\nu\|_2 \le \|S_\ell\|_2 \, \|R_\ell\|_2 \max\{\eta_0(\nu-2), \Gamma\,|1-\Gamma|^{\nu-2}\}.$$

Proof. Define Y as above, estimate $\|A_\ell S_\ell^\nu\|_2$ by $\|W_\ell^{1/2} Y\|_2^2 \|(I-Y)Y^{\nu-2}\|_2$ and use $\|W_\ell^{\frac{1}{2}} Y\|_2^2 = \|W_\ell^{-\frac{1}{2}} R_\ell^2 W_\ell^{-\frac{1}{2}}\|_2 = \rho(W_\ell^{-\frac{1}{2}} R_\ell^2 W_\ell^{-\frac{1}{2}}) = \rho(W_\ell^{-1} R_\ell^2) \le \|W_\ell^{-1} R_\ell\|_2 \|R_\ell\|_2 = \|S_\ell\|_2 \|R_\ell\|_2$. □

Exercise 11.50. Prove under the same assumption as in Lemma 11.48 that

$$\|A_\ell S_\ell^\nu A_\ell^{-1}\|_2 \le \sqrt{\|W_\ell\|_2 \max\{\eta_0(2\nu), \Gamma\,|1-\Gamma|^{2\nu}\}}. \tag{11.80}$$

Inequality (11.80) can be regarded as a modification of (11.78b).

The condition $\Gamma < 2$ in $0 < \gamma W_\ell \le A_\ell \le \Gamma W_\ell$ coincides with the convergence condition in Theorem 3.34b. However, $\gamma = 0$ is sufficient for the smoothing property, although the convergence rate $\rho(S_\ell)$ becomes worse the smaller γ is. Since damping corresponds to the replacement of W_ℓ by $\vartheta^{-1} W_\ell$, we obtain the following.

Remark 11.51. After a possibly necessary damping, all positive definite iterations satisfy the assumption $\gamma W_\ell \le A_\ell \le \Gamma W_\ell$ with $0 \le \gamma \le \Gamma < 2$.

11.7.4 Strengthened Two-Grid Convergence Estimates

To simplify the following considerations, the smoothing iteration $\mathcal{S}_\ell$ is assumed to satisfy the inequality $\gamma W_\ell \le A_\ell \le \Gamma W_\ell$ with $0 \le \gamma \le \Gamma \le 1$. As mentioned in Remark 11.51, this assumption can always be achieved by suitable damping. However, the following statements also hold in a somewhat modified form for $0 \le \gamma \le \Gamma < 2$. The assumptions are

$$\hat{\mathcal{S}}_\ell = \mathcal{S}_\ell, \qquad \hat{S}_\ell = S_\ell = I - W_\ell^{-1} A_\ell, \qquad 0 < A_\ell \le W_\ell. \tag{11.81}$$

The approximation property is required in the form (11.70) with $\|\cdot\|_{U_\ell} := \|\cdot\|_{W_\ell}$, $\|\cdot\|_{F_\ell} := \|\cdot\|_{W_\ell^{-1}}$ (see the following Remark 11.57):

$$\|W_\ell^{1/2}(A_\ell^{-1} - pA_{\ell-1}^{-1}r)W_\ell^{1/2}\|_2 \le C_A \qquad \text{for all } \ell \ge 1. \tag{11.82}$$

Lemma 11.52. *Assume (11.75a,b). The approximation property (11.82) is equivalent to the following inequality:*

$$0 \le A_\ell^{-1} - pA_{\ell-1}^{-1}r \le C_A W_\ell^{-1} \qquad \textit{for all } \ell \ge 1. \tag{11.83}$$

Proof. (C.3f) yields $-C_A I \le W_\ell^{\frac{1}{2}}(A_\ell^{-1} - pA_{\ell-1}^{-1}r)W_\ell^{\frac{1}{2}} \le C_A I$. Multiplying by $W_\ell^{-\frac{1}{2}}$ from both sides yields the bounds $\pm C_A W_\ell^{-1}$ for $A_\ell^{-1} - pA_{\ell-1}^{-1}r$. The lower bound $-C_A I$ can be replaced by 0, as can be concluded from Lemma 11.53. □

We postpone the proof of the modified approximation property (11.82) until Remark 11.57. Now we transform all quantities into a form better suited to symmetry:

$$\check{p} := A_\ell^{1/2} p A_{\ell-1}^{-1/2}, \quad \check{r} := \check{p}^* = A_{\ell-1}^{-1/2} r A_\ell^{1/2}, \quad Q_\ell := I - \check{p}\check{r},$$
$$X_\ell := A_\ell^{1/2} W_\ell^{-1} A_\ell^{1/2}, \qquad \check{S}_\ell := A_\ell^{1/2} S_\ell A_\ell^{-1/2} = I - X_\ell.$$

Since (11.75b) can be rewritten as $\check{r}\check{p} = I$, the following lemma can be concluded.

Lemma 11.53. *Under the assumption (11.75a,b), $Q_\ell = I - \check{p}\check{r}$ is an orthogonal projection: $Q_\ell = Q_\ell^{\mathsf{H}}$. As any orthogonal projection, it fulfils*

$$0 \le Q_\ell \le I \qquad \textit{for all } \ell \ge 1. \tag{11.84a}$$

$Q_\ell \ge 0$ also implies $0 \le A_\ell^{-1/2} Q_\ell A_\ell^{-1/2} = A_\ell^{-1} - pA_{\ell-1}^{-1}r$, so that the proof of the first inequality in (11.83) is completed. Multiplying (11.83) by $A_\ell^{1/2}$ from both sides yields the next lemma.

Lemma 11.54. *Assume (11.75a,b). The statements (11.82) or (11.83) are equivalent to*

$$0 \le Q_\ell \le C_A X_\ell \qquad \textit{for all } \ell \ge 1. \tag{11.84b}$$

According to (11.23), the transformed two-grid iteration matrix is

$$\check{M}_\ell(\nu_1,\nu_2) := A_\ell^{1/2} M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} A_\ell^{-1/2} = \check{S}_\ell^{\nu_2} Q_\ell \check{S}_\ell^{\nu_1}. \tag{11.85}$$

In contrast to Theorems 11.35 to 11.55, it is now possible to prove convergence for *all* $\nu > 0$.

Theorem 11.55 (two-grid convergence). *Assume (11.75a,b), (11.81), and the approximation property (11.82). Then the two-grid iteration converges monotonically with respect to the energy norm* $\|\cdot\|_{A_\ell}$*:*

$$\begin{aligned}\rho(M_\ell^{\mathrm{TGM}(\nu/2,\nu/2)}) &= \|M_\ell^{\mathrm{TGM}(\nu/2,\nu/2)}\|_{A_\ell} \\ &= \left\|\check{M}_\ell\left(\frac{\nu}{2},\frac{\nu}{2}\right)\right\|_2 \le \left\{\begin{array}{ll} C_A\eta_0(\nu) & \text{if } C_A \le 1+\nu \\ (1-1/C_A)^\nu & \text{if } C_A > 1+\nu \end{array}\right\} < 1.\end{aligned} \tag{11.86}$$

Proof. It remains to show the inequality '$\le$' in (11.86). The inequality (11.87a) following from (11.84a,b) can be inserted into (11.85) and yields (11.87b):

$$0 \le Q_\ell \le \alpha C_A X_\ell + (1-\alpha) I \qquad \text{for all } 0 \le \alpha \le 1, \tag{11.87a}$$

$$0 \le \check{M}_\ell \le \check{S}_\ell^{\nu/2} \left[\alpha C_A X_\ell + (1-\alpha) I\right] \check{S}_\ell^{\nu/2} \quad \text{for all } 0 \le \alpha \le 1. \tag{11.87b}$$

Since $\check{S}_\ell = I - X_\ell$, the right-hand side of (11.87b) is a polynomial $f(X_\ell;\alpha)$ with

$$f(\xi;\alpha) := (1-\xi)^\nu (1 - \alpha + \alpha\, C_A\, \xi). \tag{11.87c}$$

For all $0 \le \alpha \le 1$, inequality $0 \le X_\ell \le I$ (cf. (11.81)) implies the estimate

$$\|\check{M}_\ell\|_2 \le \|f(X_\ell;\alpha)\|_2 \le m(\alpha) := \max\{f(\xi;\alpha) : 0 \le \xi \le 1\}.$$

In particular, for $\alpha = 1$, we obtain the bound $C_A\eta_0(\nu)$. If $1 + \nu < C_A$, the value $\alpha^* := \frac{\nu}{C_A - 1}$ belongs to $[0,1]$ and yields the better bound $m(\alpha^*) = (1 - \frac{1}{C_A})^\nu$. □

Exercise 11.56. Prove the statements of Lemmata 11.52, 11.54 and Theorem 11.55 under the assumption $r A_\ell\, p \le A_{\ell-1}$ instead of (11.75b).

It remains to discuss the approximation property (11.82).

Remark 11.57. Assume the approximation property in the original form (11.63): $\|A_\ell^{-1} - p A_{\ell-1}^{-1} r\|_2 \le C_A' / \|A_\ell\|_2$. Furthermore, let (11.79) be valid: $\|W_\ell\|_2 \le C_W \|A_\ell\|_2$. Then (11.82) is satisfied by $C_A := C_A' C_W$.

Exercise 11.58. Assume (11.75a,b) without $\nu_1 = \nu_2$, as well as (11.81) and $\nu = \nu_1 + \nu_2 > 0$. Prove that the two-grid iteration $\Phi_\ell^{\mathrm{TGM}(\nu_1,\nu_2)}$ converges monotonically with respect to the energy norm $\|\cdot\|_{A_\ell}$. What is the h_ℓ-independent contraction number? Hint: First, use (11.77d) to estimate $\|\check{M}_\ell(\frac{\nu}{2},0)\|_2$ and thereafter apply (11.77d) to $\|\check{M}_\ell(\nu_1,\nu_2)\|_2$.

In the proof of Theorem 11.55, the smoothing property is also used indirectly; however, now it is formulated by the polynomial (11.87c) for arbitrary $0 \le \alpha \le 1$ instead of $\alpha = 1$.

11.7.5 V-Cycle Convergence

We apply the technique of §11.7.4 to the multigrid method. Since Theorem 11.42 excludes the V-cycle ($\gamma = 1$), we concentrate on this case. For another proof, see Braess–Hackbusch [65].

Theorem 11.59. *Under the same assumptions (11.75a,b), (11.81), (11.82) as in Theorem 11.55, the V-cycle ($\gamma = 1$) converges monotonically with respect to the energy norm $\|\cdot\|_{A_\ell}$ with the rate*

$$\rho\left(M_\ell^V(\tfrac{\nu}{2},\tfrac{\nu}{2})\right) = \left\|M_\ell^V(\tfrac{\nu}{2},\tfrac{\nu}{2})\right\|_{A_\ell} \le \frac{C_A}{C_A+\nu}$$

Proof. For $\gamma = 1$, abbreviate M_ℓ^{MGM} by M_ℓ^V. The recursive equations (11.42a,b) become

$$M_0^V(\nu_1,\nu_2)=0,\ M_\ell^V(\nu_1,\nu_2) = M_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} + S_\ell^\nu p M_{\ell-1}^V(\nu_1,\nu_2) A_{\ell-1}^{-1} r A_\ell S_\ell^\nu .$$

Transformation into the symmetric form yields

$$\begin{aligned}\check{M}_\ell^V &:= A_\ell^{1/2} M_\ell^V(\nu_1,\nu_2) A_\ell^{-1/2} = \check{M}_\ell^{\mathrm{TGM}(\nu_1,\nu_2)} + \check{S}_\ell^{\nu_2}\check{p}\check{M}_{\ell-1}^V\check{r}\check{S}_\ell^{\nu_1} \qquad (11.88)\\ &\overset{(11.85)}{=} \check{S}_\ell^{\nu_2}\left\{I-\check{p}\left[I-\check{M}_{\ell-1}^V\right]\check{r}\right\}\check{S}_\ell^{\nu_1} \quad \text{for } \ell\ge 1, \qquad \check{M}_0^V = 0.\end{aligned}$$

In the following, choose $\check{M}_\ell^V := \check{M}_\ell^V(\frac{\nu}{2},\frac{\nu}{2})$, i.e., $\nu_1 = \nu_2 = \frac{\nu}{2}$. Using (11.88), we obtain

$$\check{M}_\ell^V \ge 0$$

by induction: $\check{M}_0^V = 0$ and $I - \check{p}\big[I-\check{M}_{\ell-1}^V\big]\check{r} \ge I - \check{p}\check{r} = Q_\ell \ge 0$. Hence, the statements (11.89a) and (11.89b) are equivalent:

$$\|M_\ell^V(\tfrac{\nu}{2},\tfrac{\nu}{2})\|_{A_\ell} = \|\check{M}_\ell^V\|_2 \le \zeta_\ell \qquad (\check{M}_\ell^V := \check{M}_\ell^V(\tfrac{\nu}{2},\tfrac{\nu}{2})), \qquad (11.89a)$$

$$0 \le \check{M}_\ell^V \le \zeta_\ell I\,. \qquad (11.89b)$$

The induction hypothesis is $0 \le \check{M}_{\ell-1}^V \le \zeta_{\ell-1} I$ with $\zeta_{\ell-1} := \frac{C_A}{C_A+\nu}$. Inserting this inequality into (11.88), we arrive at

$$\begin{aligned}0 \le \check{M}_\ell^V &\le \check{S}_\ell^{\nu/2}\left\{I-(1-\zeta_{\ell-1})\check{p}\check{r}\right\}\check{S}_\ell^{\nu/2} = \check{S}_\ell^{\nu/2}\left\{(1-\zeta_{\ell-1})Q_\ell + \zeta_{\ell-1} I\right\}\check{S}_\ell^{\nu/2}\\ &\underset{(11.87a)}{\le} \check{S}_\ell^{\nu/2}\big\{(1-\zeta_{\ell-1})\left[\alpha C_A X_\ell + (1-\alpha)I\right] + \zeta_{\ell-1} I\ \big\}\check{S}_\ell^{\nu/2}\end{aligned}$$

for all $0 \le \alpha \le 1$. For $\alpha \in [0,1]$, the variable $\beta := (1-\zeta_{\ell-1})(1-\alpha) + \zeta_{\ell-1}$ varies in $[\zeta_{\ell-1}, 1]$. Substitution of α by β yields

$$0 \le \check{M}_\ell^V \le \check{S}_\ell^{\nu/2}\left\{(1-\beta)C_A X_\ell + \beta I\right\}\check{S}_\ell^{\nu/2} \qquad \text{for all } \zeta_{\ell-1} \le \beta \le 1.$$

The right-hand side is the polynomial $f(\xi;\beta) := (1-\xi)^\nu\left[\beta + (1-\beta)C_A\xi\right]$ for $\xi = X_\ell$ and can be estimated by

$$\|f(X_\ell;\beta)\|_2 \le m(\beta) := \max\{|f(\xi;\beta)| : 0 \le \xi \le 1\} \quad \text{(cf. (11.87c,d))}.$$

For $\beta = \zeta_{\ell-1} = C_A/(C_A+\nu)$, one finds $m(\beta) = f(0;\beta) = \beta = C_A/(C_A+\nu)$. Hence, (11.89b) holds with $\zeta_\ell = C_A/(C_A+\nu)$. □

Exercise 11.60. (a) Under the same assumptions, prove

$$\check{M}_\ell^V(0,\nu_2)\,\check{M}_\ell^V(\nu_1,0) = \check{M}_\ell^V(\nu_1,\nu_2)$$

and discuss convergence for $\nu = \nu_1 + \nu_2 > 0$.
(b) Prove the statement of Theorem 11.59 under the weaker condition $rA_\ell\, p \le A_{\ell-1}$ instead of (11.75b).

The condition $A_\ell \le W_\ell$ in (11.81) can be generalised to $A_\ell \le \sqrt{2}\, W_\ell$ (cf. Wittum [402, Proposition 4.2.4]).

Obviously, monotone and h_ℓ-independent convergence can also be shown for the W-cycle (more generally, for $\gamma \ge 2$). For this case (assuming $C_A \ge 1$), one finds, e.g., the estimate

$$\left\|M_\ell^W(\nu/2,\nu/2)\right\|_{A_\ell} \le \sqrt{C_A}\,/\left(\sqrt{C_A}+\nu\right).$$

In the case of weaker regularity (cf. §11.6.6) and for $\gamma = 2$ (W-cycle), one can still prove $\|M_\ell^W(\nu/2,\nu/2)\|_{A_\ell} \le \mathcal{O}(\nu^{\sigma-1}) < 1$ for all $\nu > 0$.

V-cycle convergence without full regularity assumptions is proved by Brenner [79]. See also §12.9.3.

11.7.6 Unsymmetric Multigrid Convergence for all $\nu > 0$

The analysis in §11.6 shows multigrid convergence for sufficiently large $\nu \ge \underline{\nu}$. In the symmetric case, §11.7.5 ensures convergence for all $\nu > 0$ and arbitrarily coarse h_0. In the general case, we still obtain convergence for all $\nu = \nu_1 + \nu_2 > 0$; however, h_0 must be sufficiently small: $h_0 \le \bar{h}$. The proof technique is the same as for Theorem 11.29.

Theorem 11.61. *Let the matrices A_ℓ ($\ell \ge 0$) be split into $A_\ell = A'_\ell + A''_\ell$ such that $A'_\ell > 0$. Let S_ℓ and S'_ℓ be the iteration matrices of the corresponding smoothing iterations $\mathcal{S}_\ell$ and $\mathcal{S}'_\ell$. For A''_ℓ and $S''_\ell := S_\ell - S'_\ell$, assume*

$$\|A_\ell'^{-1/2} A''_\ell A_\ell'^{-1/2}\|_2 \le C_1 h_\ell^\kappa, \quad \|A_\ell'^{1/2} S''_\ell A_\ell'^{-1/2}\|_2 \le C_2 h_\ell^\kappa \tag{11.90a}$$

with $\kappa > 0$. Assume that the following norms are bounded by 1:

$$\|A_\ell'^{1/2} S'_\ell A_\ell'^{-1/2}\|_2,\ \|A_\ell'^{1/2} p A_{\ell-1}'^{-1/2}\|_2,\ \|A_{\ell-1}'^{-1/2} r A_\ell'^{1/2}\|_2 \le 1 \tag{11.90b}$$

for all $\ell \ge 1$ and that the two- or multigrid method for A'_ℓ (with fixed parameters γ, ν_1, ν_2) converges monotonically with respect to the energy norm $\|\cdot\|_{A'_\ell}$ with the

contraction number ζ'. Further, let

$$\sup\{h_\ell/h_{\ell-1} : \ell \geq 1\} < 1 \quad \textit{and} \quad \varepsilon \in (0, 1-\zeta')$$

be valid. Then the two- and multigrid iterations for A_ℓ also converge monotonically with respect to the energy norm $\|\cdot\|_{A'_\ell}$ with the contraction number $\zeta = \zeta' + \varepsilon < 1$, provided that $h_0 \leq \bar{h}$ holds with sufficiently small $\bar{h}$.

Proof. First, the two-grid case is considered. The transformed iteration matrix $A_\ell'^{1/2} M'_\ell A_\ell'^{-1/2}$ (of the iteration for $A'_\ell x'_\ell = b'_\ell$) is the product

$$\begin{aligned}&\left[A_\ell'^{1/2} S'_\ell A_\ell'^{-1/2}\right]^{\nu_2} \times \left[A_\ell'^{1/2}(A_\ell'^{-1} - pA_{\ell-1}'^{-1} r) A_\ell'^{1/2}\right] \\ &\times \left[A_\ell'^{-1/2} A'_\ell A_\ell'^{-1/2}\right] \times \left[A_\ell'^{1/2} S'_\ell A_\ell'^{-1/2}\right]^{\nu_1}.\end{aligned}$$

Because of (11.90a), perturbations of S'_ℓ and A'_ℓ in the 1st, 3rd, and 4th factor by S''_ℓ and A''_ℓ, respectively, enlarge the spectral norm only by $\mathcal{O}(h_\ell^\kappa)$. A similar statement holds for the second factor because

$$\begin{aligned}&\left[A_\ell'^{1/2}(A_\ell^{-1} - pA_{\ell-1}^{-1} r) A_\ell'^{1/2}\right] - \left[A_\ell'^{1/2}(A_\ell'^{-1} - pA_{\ell-1}'^{-1} r) A_\ell'^{1/2}\right] \\ &= A_\ell'^{-1/2} A''_\ell A_\ell^{-1} A_\ell'^{1/2} + A_\ell'^{1/2} pA_{\ell-1}^{-1} A''_{\ell-1} A_{\ell-1}'^{-1} r A_\ell'^{1/2}.\end{aligned}$$

Let M_ℓ be the two-grid iteration matrix associated with the matrix A_ℓ. The assertion follows from $\left| \|M_\ell\|_{A'_\ell} - \|M'_\ell\|_{A'_\ell} \right| \leq C h_\ell^\kappa \leq C\bar{h}^\kappa$ for the choice $\bar{h} := (\varepsilon/C)^{1/\kappa}$. In the multigrid case, the following recursive estimate holds:

$$\left| \|M_\ell\|_{A'_\ell} - \|M'_\ell\|_{A'_\ell} \right| \leq C_0 h_\ell^\kappa + \left| \|M_{\ell-1}\|_{A'_{\ell-1}} - \left\|M'_{\ell-1}\right\|_{A'_{\ell-1}} \right|,$$

which by $h_\ell/h_{\ell-1} \leq C_h < 1$ leads to $\left| \|M_\ell\|_{A'_\ell} - \|M'_\ell\|_{A'_\ell} \right| \leq C h_0^\kappa \leq C\bar{h}^\kappa \leq \varepsilon$. □

Remark 11.62. The conditions in (11.90b) are satisfied if

$$S'_\ell = I - W_\ell'^{-1} A'_\ell \text{ with } 2W'_\ell \geq A'_\ell, \quad r = p^*, \quad rA'_\ell p \leq A'_{\ell-1}$$

(cf. (11.81), Exercises 11.56, and 11.60b). (11.75b) is sufficient for $rA'_\ell p \leq A'_{\ell-1}$.

The statement of Theorem 11.61 is not yet uniform with respect to $\nu = \nu_1 + \nu_2$. In particular, $\bar{h}$ might depend on ν. A ν-independent $\bar{h}$ can be obtained as follows: Theorem 11.42 (modified according to §11.7.2 to the energy norm $\|\cdot\|_{A'_\ell}$) shows convergence for $\nu \geq \underline{\nu}$ as long as $h_0 \leq \bar{h}_0$. For the finitely many $\nu = 1, \ldots, \underline{\nu} - 1$, we conclude convergence for $h_0 \leq \bar{h}_\nu$ from Theorem 11.61 with suitable $\bar{h}_\nu$. For $h_0 \leq \bar{h} := \min\{\bar{h}_\nu : 0 \leq \nu \leq \underline{\nu} - 1\}$, we obtain convergence for all $\nu > 0$.

For related results, see Mandel [269] and Bramble–Pasciak–Xu [75].

11.8 Combination of Multigrid Methods with Semi-Iterations

11.8.1 Semi-Iterative Smoothers

So far, only the ν-fold application of a smoothing iteration $\mathcal{S}_\ell$ has been considered as a smoothing step (11.33b,f). An alternative is a semi-iterative smoothing, where S_ℓ^ν is replaced by a polynomial $P_\nu(S_\ell)$ of degree ν with $P_\nu(1) = 1$. However, one should not choose the polynomials that were found to be optimal in §8 because those minimise $\rho(P_\nu(S_\ell))$. Using $\mathcal{S}_\ell$ for smoothing, we do not primarily want to make the error small but smooth. The smoothing property (11.58a) leads us to the following optimisation problem:

$$\text{minimise } \|A_\ell\, P_\nu(S_\ell)\|_2 \text{ over } P_\nu \in \mathcal{P}_\nu \quad \text{with} \quad P_\nu(1) = 1. \tag{11.91}$$

The semi-iterative Richardson method with $A_\ell > 0$ and $\sigma_M := [\,0, \|A_\ell\|_2\,]$ yields the optimisation problem

$$\min_{P_\nu \in \mathcal{P}_\nu,\, P_\nu(1)=1} \; \max_{0 \le \xi \le \|A_\ell\|_2} \left| \xi\, P_\nu\!\left(1 - \frac{\xi}{\|A_\ell\|_2}\right)\right| \tag{11.92}$$

(analogous to (8.23)). The solution reads as follows.

Theorem 11.63. *Let $A_\ell > 0$. The minimiser of (11.92) is a polynomial P_ν derived from the Chebyshev polynomial $T_{\nu+1}$ (cf. Lemma 8.23) by*

$$\tau\, P_\nu(1-\tau) = \eta(\nu)\, T_{\nu+1}\Big(\tau - (1-\tau)\cos\tfrac{\pi}{2\nu+2}\Big) \tag{11.93}$$

$$\text{with } \eta(\nu) = \frac{1}{\nu+1}\,\frac{\sin\left(\pi/\left(2\nu+2\right)\right)}{1+\cos\left(\pi/\left(2\nu+2\right)\right)} \le \frac{2\left(\sqrt{2}-1\right)}{\left(\nu+1\right)^2} \qquad (\nu \ge 1).$$

P_ν is the product $P_\nu(1-\tau) = \prod_{\mu=1}^{\nu}(1-\omega_\mu\tau)$ with

$$\omega_\mu = \left(1+\cos\frac{\pi}{2\nu+2}\right) \Big/ \left(\cos\frac{\pi}{2\nu+2} - \cos\frac{(2\mu+1)\pi}{2\nu+2}\right).$$

The expression (11.92) to be minimised takes the value

$$\|A_\ell P_\nu(I - A_\ell/\,\|A_\ell\|_2)\|_2 \le \eta(\nu)\,\|A_\ell\|_2\,.$$

Proof. (i) Evaluation of $T_{\nu+1}(\ldots)$ at $\tau = 0$ yields $T_{\nu+1}(-\cos\frac{\pi}{2\nu+2})$. Note that $-\cos\frac{\pi}{2\nu+2} = \cos(\pi - \frac{\pi}{2\nu+2}) = \cos(\frac{2\nu+1}{2\nu+2}\pi)$ and therefore

$$T_{\nu+1}(-\cos\tfrac{\pi}{2\nu+2}) = \cos\big((\nu+1)\tfrac{2\nu+1}{2\nu+2}\pi\big) = \cos\big((\nu+\tfrac{1}{2})\pi\big) = 0.$$

This justifies the factor τ on the left-hand side of (11.93).

(ii) The factor $\eta(\nu)$ is chosen such that $P_\nu(1) = \frac{d}{d\tau}\tau\, P_\nu(1-\tau)|_{\tau=0} = 1$ ensures the side condition.

(iii) Since the right-hand side in (11.93) takes the *equi-oscillating* values $\pm\eta(\nu)$ in $[0,1]$, it is the minimiser of (11.92). □

We add some comments to the results of Theorem 11.63.

(i) The semi-iterative smoothing achieves an order improvement. While the smoothing factor $\eta(\nu)$ of the stationary Richardson method behaves like $\mathcal{O}(\frac{1}{\nu+1})$, the order becomes $\mathcal{O}(1/(\nu+1)^2)$ in the semi-iterative case.

(ii) The application of the Chebyshev method requires knowledge of the interval $\sigma_M = [a,b]$ containing the spectrum of S_ℓ. Especially, the estimation of $b = 1 - \lambda_\ell / \|A_\ell\|_2$ with $\lambda_\ell = \lambda_{\min}(A_\ell)$ is of decisive importance. An overestimation of the upper bound $\Lambda_\ell = \|A_\ell\|_2$ in $\lambda_\ell I \le A_\ell \le \Lambda_\ell I$ is less sensitive (since $\Lambda_\ell/\lambda_\ell$ is the essential quantity). A different situation arises in Theorem 11.63, where we estimate the spectrum of A_ℓ simply by $0 \le A_\ell \le \Lambda_\ell I$, $\Lambda_\ell = \|A_\ell\|_2$, i.e., the lower bound λ_ℓ is trivially chosen as $a := 0$. The replacement of $0 \le A_\ell$ by $\lambda_\ell I \le A_\ell$ with $\lambda_\ell = \lambda_{\min}(A_\ell)$ would yield only an imperceptible improvement.

(iii) The statements from (ii) clarify the fact that the spectral condition number $\kappa(W_\ell^{-1}A_\ell)$ is not the essential quantity for smoothing.

(iv) The product representation $\prod(1-\omega_\mu\tau)$ seems to disregard the warnings in §8.3.4 concerning instabilities. The contradiction is solved by the fact that the number ν of smoothing steps should be relatively small according to the discussion in §11.4.3. Choosing, e.g., $\nu \le 4$, stability problems cannot arise.

For a general positive definite smoothing iteration with $S_\ell = I - W_\ell^{-1}A_\ell$, we obtain analogous results for minimising $\|W_\ell^{-1/2} A_\ell \, P_\nu(S_\ell) W_\ell^{-1/2}\|_2$, where the norms are chosen as for the approximation property (11.82). Corresponding to the smoothing property (11.78b), the minimisation of

$$\|W_\ell^{-1/2} A_\ell \, P_\nu(S_\ell) A_\ell^{1/2}\|_2 = \|Y^{1/2} P_\nu(I-Y)\|_2 \quad \text{with } Y := A_\ell^{1/2} W_\ell^{-1} A_\ell^{1/2}$$

is also of interest. The corresponding optimal polynomial can be found in Hackbusch [183, Proposition 6.2.35]. The bound $\mathcal{O}(1/\sqrt{\nu})$ in (11.78b) improves to $\mathcal{O}(1/(2\nu+1))$. The ADI parameters (cf. §8.5.3 and Hackbusch [183, §3.3.4 and Lemma 6.2.36]) have also to be chosen differently for optimising the smoothing effect.

The conjugate gradient method is only conditionally applicable. The standard CG method minimises $\|P_\nu(S_\ell)e_\ell\|_{A_\ell} = \|A_\ell^{1/2} P_\nu(S_\ell) e_\ell\|_2$, where e_ℓ is the error before smoothing and P_ν the corresponding optimal polynomial (cf. Proposition 10.11). However, since not the energy norm but the residual $\|A_\ell P_\nu(S_\ell) e_\ell\|_2$ has to be minimised, the method of the conjugate residuals (cf. §10.3) or the conjugate gradient method for the 'squared' equation $A_\ell^{\mathsf{H}} A_\ell x_\ell = A_\ell^{\mathsf{H}} b_\ell$ is better suited. These remarks apply to the pre-smoothing part only. The conjugate gradient methods do not seem to make much sense for the post-smoother. In any case, an nonsymmetric multigrid iteration results. See also Bank–Douglas [27].

The smoothing property of conjugate gradient methods has also been mentioned by Il'in [226].

11.8.2 Damped Coarse-Grid Corrections

The treatment of nonlinear equations suggests damping the coarse-grid correction as known from gradient methods, in order to obtain a descent method (cf. Hackbusch–Reusken [204]). It turns out that in the linear case, it is also possible to improve convergence. In particular, the V-cycle convergence can be accelerated (cf. Reusken [321], Braess [62]). The optimally damped coarse-grid correction step reads

$$x_\ell^{\text{new}} := x_\ell - \lambda p_\ell \quad \text{with } \lambda := \frac{\langle d_\ell, p_\ell \rangle_\ell}{\langle d_\ell, A_\ell p_\ell \rangle_\ell}\,, \quad \begin{cases} d_\ell := A_\ell x_\ell - b_\ell, \\ p_\ell := p\,\tilde{e}_{\ell-1}, \end{cases} \tag{11.94}$$

where $\tilde{e}_{\ell-1}$ is the approximation of the solution of the coarse-grid equation $A_{\ell-1}e_{\ell-1} = d_{\ell-1} := rd_\ell$.

Exercise 11.64. Let $A_\ell > 0$ and $\tilde{e}_{\ell-1}$ as above. Prove that

(a) $\lambda = 1$ is optimal for the two-grid method.

(b) If $r = p^*$ and $A_{\ell-1} = rA_\ell p$, λ in (11.94) can be written in the form

$$\lambda = \langle d_{\ell-1}, \tilde{e}_{\ell-1} \rangle_{\ell-1} \,/\, \langle A_{\ell-1}\tilde{e}_{\ell-1}, \tilde{e}_{\ell-1} \rangle_{\ell-1}\,.$$

Another possibility is the damping of the complete multigrid iteration. In the symmetric case, $\check{M}_\ell^{\text{MGM}} \geq 0$ holds (cf. (11.89b)) and implies that $\sigma(M_\ell^{\text{MGM}}) \subset [\,0, \rho(M_\ell^{\text{MGM}})\,]$. Extrapolation with $\Theta := 2/(2 - \rho\,(M_\ell^{\text{MGM}})) \approx 1 + \frac{1}{2}\rho\,(M_\ell^{\text{MGM}})$ leads to the nearly halved convergence rate $\rho(M_\ell^{\text{MGM}})/(2 - \rho(M_\ell^{\text{MGM}}))$.

11.8.3 Multigrid as Basic Iteration of the CG Method

As shown in §11.7.1, the multigrid method for a positive definite matrix A_ℓ can be designed as a positive definite iteration. The convergence statement $\sigma(M_\ell^{\text{MGM}}) \subset [\,0, \rho_\ell\,]$ with $\rho_\ell := \rho(M_\ell^{\text{MGM}}) < 1$ corresponds to the inequalities

$$\gamma W_\ell^{\text{MGM}} \leq A_\ell \leq W_\ell^{\text{MGM}} \qquad \text{with } \gamma := 1 - \rho_\ell$$

for the matrix W_ℓ^{MGM} of the third normal form of the multigrid iteration (cf. Theorem 6.10). Applying the CG method to Φ_ℓ^{MGM}, after m steps we obtain an improvement by $2\,[(\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)]^m$, where κ is the condition $\kappa = \frac{\Gamma}{\gamma} = 1/(1-\rho_\ell)$. A simple rewriting yields

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m = 2\rho_\ell^m / \left(1 + \sqrt{1-\rho_\ell}\right)^{2m} \approx 2\left(\frac{\rho_\ell}{4} + \mathcal{O}\big((\rho_\ell)^2\big)\right)^m.$$

Since ρ_ℓ may be assumed to be small (cf. §11.4.2), the convergence rate ρ_ℓ of the multigrid method can be accelerated by using the conjugate gradient method to $\rho_\ell/4$ (cf. Braess [61], Kettler [235]).

However, the use of the conjugate gradient method is of practical interest only if the multigrid convergence rate is relatively unfavourable (e.g., $\rho_\ell > 0.4$). The reason for this are considerations in §11.5.4. In the case of a good convergence rate, the nested iteration (11.44) requires only very few multigrid steps per level. For fast multigrid methods, the iteration number $m = 1$ has proved in §11.5.6 to be sufficient. For this value, the gradient and conjugate gradient methods still coincide. Only for $m \geq 2$ does the conjugate gradient method deserve attention.

11.9 Further Comments

11.9.1 Multigrid Method of the Second Kind

Discretisation of Fredholm's integral equations of the second kind leads to fixed-point equations of the form $x_\ell = K_\ell\, x_\ell + b_\ell$, i.e.,

$$A_\ell x_\ell = b_\ell \qquad \text{with } A_\ell = I - K_\ell.$$

Here, K_ℓ does not characterise a difference operator as in the case of discretised differential equations, but the discrete counterpart of an integral operator $(\mathcal{K}u)(x) = \int_D k(x,y)\mathrm{d}y$. Therefore, the *Picard iteration* $x_\ell^{m+1} := K_\ell\, x_\ell^m + b_\ell$ has a substantially better smoothing property. This implies that the multigrid method (with $\gamma = 2$) has a convergence rate $\rho_\ell = \mathcal{O}(h_\ell^\kappa)$ with a positive exponent κ. Hence, different from the situation considered before, convergence is better the larger the dimension of the problems is. Since the absolute value of ρ_ℓ may be of the size of 10^{-3} to 10^{-6}, the multigrid method is close to a direct solver. The work of the method is still proportional to the work of one Picard iteration.

The application is not restricted to discrete integral equations. In the example in §5.5.1, the equation $Ax = b$ was preconditioned by B, where B as well as A were discretisations of the differential equation. If both differential equations share the same principal part (i.e., if the terms of the highest order of differentiation coincide), the equation $A'x = b' := B^{-1}b$ with $A' := B^{-1}A$ leads to the fast multigrid convergence mentioned above. The application of the multigrid method of the second kind to $A'x = b'$ requires performing the Picard iteration $x^{m+1} = Kx^m + b' = x^m - B^{-1}(Ax - b)$. For example, B could be the five-point formula of the Poisson model problem, whereas A discretises the equation $-\Delta u + c_1 u_x + c_2 u_y + cu = f$.

An exact description and analysis of the multigrid method of the second kind as well as many examples of application can be found in Hackbusch [183, §16], [191], [184], [177], [179], [188].

Since the discrete integral operator K_ℓ is a fully populated matrix, the naive use of the Picard iteration leads to squared complexity. To obtain almost linear complexity the technique of hierarchical matrices can be applied (cf. Appendix D).

11.9.2 Robust Methods

To make our presentation brief, other smoothers than Richardson and chequer-board Gauss–Seidel are mentioned only marginally. In applications to more complicated systems, the problem of robustness arises. Are the convergence rates known for the Poisson model problem uniformly valid in a larger class of problems? The simplest case is an equation $A(\varepsilon)x = b$ depending on one parameter $\varepsilon \in (0, \infty)$. If the convergence rates not only are of the form $\rho(\varepsilon) \leq 1 - C(\varepsilon)h^\tau$ but hold with a constant $C(\varepsilon) = C$ uniformly in $\varepsilon \in (0, \infty)$): $\rho(\varepsilon) \leq 1 - Ch^\tau$, then this iteration is called *robust* with respect to the class of problems. Robust multigrid methods have to satisfy $\rho(\varepsilon) \leq \zeta < 1$ (ζ h_ℓ- and ε-independent; cf. Hackbusch [183, §10]).

Good experiences concerning robustness—at least for two spatial dimensions—are observed for ILU smoothers as introduced by Wesseling [394], [393] (cf. Kettler [235], Wittum [401]). Robustness holds for CG methods applied to the modified ILU iteration ($\omega = -1$), as well as for multigrid methods using point- or blockwise ILU iterations as smoother (then with $\omega = 0$ or even $\omega = 1$; cf. Wittum [403], Kettler [235], Oertel–Stüben [295]).

Another approach is the frequency decomposition multigrid method (cf. Hackbusch [186, 190]) which uses not only one but several coarse-grid corrections with different coarse-grid equations. The prolongations from the different coarse grids into the fine grid are constructed in such a way that the corrections cover different frequency intervals.

Constructing the coarse-grid equation at level $\ell - 1$ requires more data than given by the system $A_\ell x_\ell = b_\ell$ for $\ell = \ell_{\max}$. This fact may lead to difficulties when the multigrid iteration is wanted as a black-box solver. Therefore, it is remarkable that there are variants, the so-called *algebraic multigrid methods*, in which the coarse-grid matrix $A_{\ell-1}$ is only constructed by the entries of the matrix A_ℓ (cf. Stüben [359, 360], MacLachlan–Oosterlee [268], Xu–Zikatonov [410]).

11.9.3 History of the Multigrid Method

The first two-grid method was described by Brakhage [69] in 1960. More precisely, it was a two-grid method of the second kind because it was applied to problems mentioned in §11.9.1. In 1961, Fedorenko [131] described a two- and in 1964 a multigrid method for the Poisson model problem (cf. Fedorenko [132]). In 1966, Bakhvalov [23] proved the typical convergence properties for a more complicated situation. Additional early publications were due to Astrachancev [8] (1971), Hackbusch [176] (1976), Bank–Dupont (a report from 1977 was split into [28, 29]), Brandt [77] (1977), and Nicolaides [290] (1977). Further details concerning these and other papers by Frederickson, Wesseling, Hemker, and Braess are mentioned in Hackbusch [183, §2.6.5]. An extensive multigrid bibliography up to 1987 can be found in the proceedings [278].

The progress of multigrid algorithms and theory can be traced in the proceedings of the European Multigrid Conferences: [205, Cologne 1981], [206, Cologne 1985], [207, Bonn 1990], [216, Amsterdam 1993], [210, Stuttgart 1996], [108, Gent 1999]. The proceedings of the later EMG conferences in Hohenwart (2002), Scheveningen (2005), Bad Herrenalb (2008), Ischia (2010), Schwetzingen (2012), and Leuven (2014) can be found in special issues, e.g., of the journal Comput. Vis. Sci. Further proceedings in this field are [68, 182, 185, 209, 306].

11.9.4 Frequency Filtering Decompositions

An essential characteristic of the multigrid method, besides the use of a coarser grid, is the product form $\Phi_\ell^{\mathrm{CGC}} \circ S_\ell^\nu$ of two iterations which are active in different frequency intervals. Although many methods can be used for smoothing, the question remains as to whether there exists an alternative to $\Phi_\ell^{\mathrm{CGC}} \circ S_\ell^\nu$. It would be desirable to have a method filtering out the coarse frequencies and needing no hierarchy of grids. Such a method is proposed by Wittum [404, 405] and is based on a sequence of partial steps Φ_ν reducing certain frequency intervals.

First, we describe the standard *blockwise* ILU decomposition. Suppose that A has the block-tridiagonal structure

$$A = A^{\mathsf{H}} = \mathrm{blocktridiag}\{L_i, D_i, L_i^{\mathsf{H}} : i = 1, \ldots, N-1\} \tag{11.95a}$$

(cf. (1.8), (A.9)), which, e.g., holds for five- or nine-point formulae. As in (1.2), $N - 1 = h^{-1} - 1$ is the number of inner grid points per row. The exact LU decomposition is $A = LD^{-1}L^{\mathsf{H}}$ with

$$L := \mathrm{blocktridiag}\{L_i, T_i, 0\}, \qquad D := \mathrm{blockdiag}\{T_i\}, \tag{11.95b}$$

$$T_1 := D_1, \quad T_i := D_i - L_i T_{i-1}^{-1} L_i^{\mathsf{H}} \qquad (2 \le i \le N-1). \tag{11.95c}$$

Even if the blocks D_i are tridiagonal (cf. (1.8)), the matrices T_i are not sparse. The usual block-ILU decomposition is obtained from (11.95c) by replacing the full inverse of T_i^{-1} with the tridiagonal part $\mathrm{tridiag}\{T_{i-1}^{-1}\}$ of the exact inverse.

Another approach goes back to Axelsson–Polman [15]. Let $t^{(1)}$ and $t^{(2)}$ be two *test vectors*. The matrices T_i are defined in the next lemma.

Lemma 11.65. *Assume that* $t^{(1)}, t^{(2)} \in \mathbb{R}^{N-1}$ *satisfy*

$$\det\left((t_{i+j}^{(k)})_{j=0,1}^{k=1,2}\right) \neq 0 \qquad \textit{for all } 1 \le i \le N-2.$$

The vectors $c^{(1)}, c^{(2)} \in \mathbb{R}^{N-1}$ *may be arbitrary. Then there is a unique symmetric tridiagonal matrix* T *satisfying the equations* $Tt^{(k)} = c^{(k)}$ *for* $k = 1, 2$.

Hence, we can uniquely define symmetric tridiagonal matrices T_i by

$$T_1 := D_1, \qquad T_i\, t^{(k)} = (D_i - L_i T_{i-1}^{-1} L_i^{\mathsf{H}})\, t^{(k)} \quad (k = 1, 2) \tag{11.96}$$

for $i = 2, \ldots, N-1$. The matrices T_i inserted into (11.95b) yield a new incomplete blockwise triangular decomposition $A = LD^{-1}L^{\mathsf{H}} - C$. Definition (11.96) means

that T is exact with respect to the *test subspace* $\operatorname{span}\{t^{(1)}, t^{(2)}\}$. The corresponding iteration is

$$\Phi(x, b; t^{(1)}, t^{(2)}) := x - L^{-\mathsf{H}} D L^{-1}(Ax - b)$$

with L, D defined in (11.95b) and (11.96).

Wittum [404, 405] proposes taking the sine functions e^ν in (11.25b) with different frequencies ν as test vectors:

$$\Phi_\nu := \Phi(\cdot, \cdot\,; e^\nu, e^{\nu+1}) \qquad \text{with } \nu \in [1, N-2].$$

Choosing a factor $\alpha > 1$, which, e.g., may be chosen as $\alpha = 2$, a geometrical sequence of frequencies is selected (for $\lfloor \ldots \rfloor$ see Footnote 5 on page 297):

$$\nu_1 := 1, \quad \nu_{i+1} := \max\{\nu_i + 2, \lfloor \alpha \nu_i \rfloor\} \quad \text{as long as } \nu_{i+1} \le N - 2, \qquad (11.97)$$

Let k be the number of frequencies selected in (11.97). Obviously, this number is equal to $k = \mathcal{O}(\log N) = \mathcal{O}(|\log h|) = \mathcal{O}(\log n)$. The iteration of the frequency filtering decomposition is defined by the product:

$$\Phi_\alpha^{\mathrm{ffd}} := \Phi_{\nu_k} \circ \ldots \circ \Phi_{\nu_2} \circ \Phi_{\nu_1} \qquad (\alpha > 1 \text{ with } \nu_i \text{ in (11.97)}).$$

The work of one iteration $\Phi_\alpha^{\mathrm{ffd}}$ amounts to $\mathcal{O}(n \log n)$. The numerical results (cf. Wittum [404, 405]) demonstrate the very fast convergence of this iteration. Its efficacy can even exceed that of the standard multigrid methods.

The convergence is analysed for the case of a nine-point formula $A > 0$ with constant coefficients $D_i = D_{i+1}$, $L_i = L_{i+1} = L_i^{\mathsf{H}}$ (cf. Wittum [405]). The first step of the proof concerns the monotone convergence of Φ_ν with respect to the energy norm for all ν. However, the more characteristic step is a *neighbourhood property*. According to its definition, Φ_ν eliminates error components in $\operatorname{span}\{e^\nu, e^{\nu+1}\}$. It is essential that Φ_ν yields a uniform and h-independent contraction number for all frequencies in the interval $\nu \le \mu \le \alpha\nu$, i.e., that Φ_ν also acts efficiently in a certain neighbourhood of the *gauge frequency* ν.

The idea of frequency filtering decompositions can also be generalised to nonsymmetric or even nonlinear problems (cf. Wittum [405], Wagner [385, 386]). See also Weiler–Wittum [390], Wagner–Wittum [387], and Buzdin–Wittum [91].

Table 11.10 shows the iteration error $\|e^m\|_2 = \|x^m - x\|_2$ of the frequency filtering decomposition method $\Phi_\alpha^{\mathrm{ffd}}$ for $\alpha = 2$ applied to the Poisson model problem. The number k of partial steps ranges from 3 to 6. After 2 to 3 steps, machine precision is reached. One observes that with decreasing h, the convergence speed is bounded from above and therefore h-independently bounded.

	$h = 1/8, k = 3$		$h = 1/16, k = 4$		$h = 1/32, k = 5$		$h = 1/64, k = 6$	
m	$\|e^m\|_2$	$\rho_{m+1,m}$	$\|e^m\|_2$	$\rho_{m+1,m}$	$\|e^m\|_2$	$\rho_{m+1,m}$	$\|e^m\|_2$	$\rho_{m+1,m}$
0	6.3_{10}-01		7.0_{10}-01		7.4_{10}-01		7.6_{10}-01	
1	2.6_{10}-07	4.1_{10}-7	1.8_{10}-06	2.6_{10}-6	1.4_{10}-05	1.9_{10}-5	5.6_{10}-05	7.3_{10}-5
2	1.0_{10}-12	3.9_{10}-6	2.8_{10}-06	1.5_{10}-5	1.5_{10}-09	1.0_{10}-5	2.2_{10}-08	3.9_{10}-4
3	4.1_{10}-13	(4.0_{10}-1)	6.7_{10}-13	(2.3_{10}-2)	1.2_{10}-12	8.2_{10}-4	9.3_{10}-12	4.1_{10}-4

Table 11.10 Iteration of the frequency filtering decomposition for the Poisson model problem.

11.9.5 Nonlinear Systems

Although this monograph is devoted to systems of linear equations, the solution of nonlinear systems is of great importance. There are two principle approaches. In §11.9.5.1 we consider the Newton method, while in §11.9.5.2 proper nonlinear iterations are described.

The nonlinear system is of the form[7]

$$\mathcal{A}(x) = 0, \tag{11.98}$$

where the function $\mathcal{A} : \mathbb{K}^I \to \mathbb{K}^I$ is assumed to be continuously differentiable. We denote the derivative by

$$A(x) := \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{A}(x) \in \mathbb{K}^{I\times I}.$$

Let $x^* \in \mathbb{K}^I$ be the solution of (11.98) and define

$$A := A(x^*). \tag{11.99}$$

We require A to be regular. Then x^* is the unique solution in a neighbourhood $\mathcal{X}$ of x^* and $A(x)$ is regular for all $x \in \mathcal{X}$. If the problem (11.98) is derived by discretising a nonlinear partial differential equation, we expect the same sparse structure of the matrices $A(x)$ as usual.

11.9.5.1 Newton's Method

The Newton method is the standard technique to transfer the solution of a nonlinear system into a sequence of linear problems. Starting with $x^0 \in \mathcal{X}$, the exact Newton method yields the sequence

$$x^{m+1} := x^m - A(x^m)^{-1}\mathcal{A}(x^m). \tag{11.100}$$

If the neighbourhood $\mathcal{X}$ is small enough, the described sequence converges quadratically to x^* (cf. Quarteroni–Sacco–Saleri [314, §7.1]). Having in mind large-scale problems, the linear system

$$A(x^m)\,\delta = \mathcal{A}(x^m)$$

for the correction $\delta = x^m - x^{m+1}$ should not be computed directly. Instead any of the linear iterations described in this book can be applied to solve for δ.

Here the following comments apply:

- The derivative $A(x^m)$ has to be computed either analytically or by numerical differentiation. Since this may be costly, often $A(x^m)$ is replaced with an

[7] In the nonlinear case, without loss of generality, the right-hand side can be defined by zero.

approximation. For instance, only $A(x^0)$ is computed and the later $A(x^m)$ are replaced by $A(x^0)$.

- If the iteration for the linear problem requires a larger amount of work for initialisation, this cost is required for each step of the Newton method. This is another reason for replacing $A(x^m)$ by a fixed matrix $\tilde{A}$.
- If, as above, $A(x^m)$ is replaced by some $\tilde{A}$, quadratic convergence is lost and the convergence of
$$x^{m+1} := x^m - \tilde{A}^{-1}\mathcal{A}(x^m)$$
depends on $A(x^m) - \tilde{A}$.
- In the case of the true matrix $A(x^m)$, the stopping criterion for the iteration applied to $A(x^m)\,\delta = \mathcal{A}(x^m)$ should produce approximations for δ_m for δ such that the error $\delta_m - \delta$ is comparable with the error of $x^{m+1} - x^*$. A too accurate solution of δ in the beginning does not pay, whereas a too rough approximation for later m prevents quadratic convergence.
- Since $A(x)$ is continuous and regular for $x \in \mathcal{X}$, the matrices of the family
$$\{A(x^m) : m \in \mathbb{N}_0\}$$
are spectrally equivalent. Therefore, in principle, the same preconditioner can be used for all linear systems that arise.

The usual convergence behaviour of (11.100) shows two phases. In a pre-asymptotic first phase only linear convergence is observed (say for $0 \le m < m_0$). Later, for $m \ge m_0$, proper quadratic convergence occurs and only a few additional steps are needed. Above, the neighbourhood $\mathcal{X}$ is chosen so that iteration (11.100) converges. Proper quadratic behaviour requires iterates in an even smaller neighbourhood $\mathcal{X}_{\mathrm{quad}}$. It would be desirable to find a starting value in $\mathcal{X}_{\mathrm{quad}}$ instead of $\mathcal{X}$.

A good strategy for this purpose is the (nonlinear) nested iteration. This requires defining nonlinear systems at all discretisation levels $\ell = 0, \ldots, \ell_{\max}$, where the system at level $\ell_{\max}$ coincides with the original system (11.98):

$$\mathcal{A}_\ell(x_\ell) = b_\ell \qquad (0 \le \ell \le \ell_{\max})\,, \tag{11.101}$$

where $b_{\ell_{\max}} = 0$. The nonlinear nested iteration takes the following form:

$$\begin{array}{l}
\widetilde{x}_0 := \text{somehow computed approximation of } \mathcal{A}_0(x_0^*) = 0;\\
\textbf{for } \ell := 1 \textbf{ to } \ell_{\max} \textbf{ do}\\
\textbf{begin } \tilde{x}_\ell := \tilde{p}\,\tilde{x}_{\ell-1};\ \tilde{b}_{\ell-1} := \mathcal{A}_{\ell-1}(\tilde{x}_{\ell-1});\\
\qquad \text{apply an iterative solver starting with } \widetilde{x}_\ell \text{ delivering a new value } \widetilde{x}_\ell\\
\textbf{end};
\end{array} \tag{11.102}$$

The data $\tilde{b}_\ell$ $(0 \le \ell \le \ell_{\max} - 1)$ will be used later. Although $\tilde{x}_\ell$ is only an approximation, it is the exact solution of $\mathcal{A}_\ell(\tilde{x}_\ell) = \tilde{b}_\ell$.

11.9.5.2 Nonlinear Iterations

Φ in Definition 2.1 can be generalised to the nonlinear problem (11.98) by a nonlinear mapping

$$x^{m+1} = \Phi(x^m, \mathcal{A}).$$

For instance, the nonlinear analogue of the Richardson iteration (3.4) is

$$x^{m+1} = \Phi^{\mathrm{Rich}}_{\mathrm{nonl}}(x^m, \mathcal{A}) := x^m - \Theta\mathcal{A}(x^m).$$

Rewriting x^m by $x^* + e^m$ and assuming a small error e^m, we obtain the Taylor expansion

$$\mathcal{A}(x^m) = \mathcal{A}(x^*) + Ae^m + o(e^m) = Ae^m + o(e^m)$$

with A in (11.99) and therefore

$$x^{m+1} = x^m - \Theta Ae^m + o(e^m) \approx x^m - \Theta\left(Ax^m - b\right) = \Phi^{\mathrm{Rich}}(x^m, b, A)$$

with $b := Ax^*$. This proves that

$$\Phi^{\mathrm{Rich}}_{\mathrm{nonl}}(x, \mathcal{A}) \to \Phi^{\mathrm{Rich}}(x, b, A) \qquad \text{as } x \to x^*.$$

The nonlinear analogue of the Gauss–Seidel method replaces each step in (3.9) by solving the i-th equation in the system $\mathcal{A}(x) = 0$ with respect to the $x[i]$. The scalar nonlinear equations that arise can be solved, e.g., by Newton's method. In the same way, the nonlinear Jacobi iteration and the nonlinear SOR can be performed (cf. Törnig [365, §§8.2–8.4]).

More involved algebraic linear iterations as the ILU iteration are hard to transfer into a nonlinear counterpart since it requires the (incomplete) decomposition of the derivative A.

The linear iteration $\Phi_{\mathrm{lin}}(x, b, A) = x - N\left(Ax - b\right)$ has the obvious nonlinear counterpart $\Phi_{\mathrm{nonl}}(x, \mathcal{A}) := x - N\mathcal{A}(x)$. In all these cases, the asymptotic convergence speed of the nonlinear iteration Φ_{nonl} coincides with the convergence speed of the linear iteration Φ_{lin} applied to the linearised system $Ax - b$ with A in (11.99).

11.9.5.3 Nonlinear Two- and Multigrid Iteration

The multigrid iteration has a very natural generalisation to nonlinear systems. The underlying reason is that the method requires not the derivative $A(x) = \mathrm{d}\mathcal{A}(x)/\mathrm{d}u$ but only a directional derivative.

Instead of (11.98), we consider the family (11.101) of systems at all levels ℓ. We start with the two-grid iteration involving the levels ℓ and $\ell - 1$. We assume that the nested iteration is already used for levels below ℓ, so that a good starting value for x_ℓ, the approximate solution $\tilde{x}_{\ell-1}$ at level $\ell - 1$ and its defect $\tilde{b}_{\ell-1}$ are known.

The real number s used in lines 3 and 5 will be explained below.

function $\Phi_\ell^{\mathrm{NTGM}}(x_\ell, b_\ell)$; {solution of $\mathcal{A}_\ell(x_\ell) = b_\ell$ desired}
begin $x_\ell := \mathcal{S}_\ell^{\nu_1}(x_\ell, b_\ell)$; {pre-smoothing}
$d_{\ell-1} := r\,(\mathcal{A}_\ell(x_\ell) - b_\ell)\,;\ d_{\ell-1} := \tilde{b}_{\ell-1} + s \cdot d_{\ell-1}$;
$\xi_{\ell-1} := \mathcal{A}_{\ell-1}^{-1}(d_{\ell-1})$; {coarse-grid solve}
$x_\ell := x_\ell - p\,(\xi_{\ell-1} - \tilde{x}_{\ell-1})\,/s$; {coarse-grid correction}
$\Phi_\ell^{\mathrm{NTGM}} := \mathcal{S}_\ell^{\nu_2}(x_\ell, b_\ell)$; {post-smoothing}
end;

The pre- and post-smoothing iterations $\mathcal{S}_\ell$ may, e.g., be the nonlinear Richardson or Jacobi iteration. $\mathcal{S}_\ell^\nu$ denotes the ν-fold application.

Let x_ℓ^* be the solution of $\mathcal{A}_\ell(x_\ell^*) = 0$. We recall the neighbourhood $\mathcal{X}_\ell$ of x_ℓ^*, in which x_ℓ^* is the unique solution. Hence, the function $\mathcal{A}_\ell : \mathcal{X}_\ell \to \mathcal{Y}_\ell := \mathcal{A}_\ell(\mathcal{X}_\ell)$ is bijective. This allows us to define the inverse function $\mathcal{A}_\ell^{-1}$ on $\mathcal{Y}_\ell$. The function $\Phi_\ell^{\mathrm{NTGM}}$ uses $\mathcal{A}_{\ell-1}^{-1}$ for solving a coarse-grid equation $\mathcal{A}_{\ell-1}\xi_{\ell-1} = d_{\ell-1}$. This requires that $d_{\ell-1} \in \mathcal{Y}_{\ell-1}$. Since $\mathcal{Y}_{\ell-1}$ is a neighbourhood of zero, $d_{\ell-1}$ must be small enough. Since, by definition, $\tilde{x}_{\ell-1}$ is a good approximation of $x_{\ell-1}^*$, the defect $\tilde{b}_{\ell-1}$ is small enough. Choosing the number s small enough, $d_{\ell-1} = \tilde{b}_{\ell-1} - s \cdot d_{\ell-1}$ also belongs to $\mathcal{Y}_{\ell-1}$.

To understand the correction $x_\ell := x_\ell + p\,(\xi_{\ell-1} - \tilde{x}_{\ell-1})\,/s$, rewrite the bracket as

$$\xi_{\ell-1} - \tilde{x}_{\ell-1} = \mathcal{A}_{\ell-1}^{-1}(d_{\ell-1}) - \mathcal{A}_{\ell-1}^{-1}(\tilde{b}_{\ell-1}) \approx \left(\tfrac{\mathrm{d}}{\mathrm{d}y}\mathcal{A}_{\ell-1}^{-1}\right)\left(d_{\ell-1} - \tilde{b}_{\ell-1}\right).$$

The derivative of the inverse function $\mathcal{A}_{\ell-1}^{-1}(y)$ is

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\mathcal{A}_{\ell-1}(x)\right)^{-1} = A_{\ell-1}^{-1} \quad \text{for} \quad x = \mathcal{A}_{\ell-1}^{-1}(y).$$

Together with $d_{\ell-1} - \tilde{b}_{\ell-1} = s \cdot d_{\ell-1}$, we obtain $\xi_{\ell-1} - \tilde{x}_{\ell-1} = sA_{\ell-1}^{-1}d_{\ell-1}$ and the correction step yields asymptotically $x_\ell - p\,(\xi_{\ell-1} - \tilde{x}_{\ell-1})\,/s \approx x_\ell - pA_{\ell-1}^{-1}d_{\ell-1}$ with the restricted defect $d_{\ell-1} = r\,(\mathcal{A}_\ell(x_\ell) - b_\ell)$. This is the same expression as in (11.21b–d) and proves that the nonlinear two-grid iteration has an asymptotic convergence speed which coincides with the convergence speed of the linear two-grid iteration applied to the linearised system.

The recursive application of $\Phi_\ell^{\mathrm{NTGM}}$ yields the nonlinear multigrid iteration. Note that the application of $\Phi_\ell^{\mathrm{NTGM}}$ is interwoven with the nonlinear nested iteration (11.102) in which the solver is the m-fold application of $\Phi_\ell^{\mathrm{NTGM}}$. This implies that, when $\Phi_\ell^{\mathrm{NTGM}}$ is called, the quantities $\tilde{x}_k$ and $\tilde{b}_k$ are known for all lower levels $k < \ell$. In addition, we need a nonlinear function $\tilde{\Phi}_0(x_0, b_0)$ returning a good approximation of $\mathcal{A}_0^{-1}(b_0)$. This may be a Newton method. The number γ has the same meaning as in the linear case: $\gamma = 1$ is the V-cycle, $\gamma = 2$ is the W-cycle. The numbers $s = s(d_{\ell-1})$ play the same role as in the two-grid iteration. It can be chosen such that $s \cdot d_{\ell-1}$ is of the same size as $\tilde{b}_{\ell-1}$.

function $\Phi_\ell^{\mathrm{NMGM}}(x_\ell, b_\ell)$; {solution of $\mathcal{A}_\ell(x_\ell) = b_\ell$ desired}
begin $x_\ell := \mathcal{S}_\ell^{\nu_1}(x_\ell, b_\ell)$; {pre-smoothing}
$d_{\ell-1} := r\ (\mathcal{A}_\ell(x_\ell) - b_\ell)$; $d_{\ell-1} := \tilde{b}_{\ell-1} + s \cdot d_{\ell-1}$;
$\xi_{\ell-1} := \tilde{x}_{\ell-1}$; **for** $i := 1$ **to** γ **do** $\xi_{\ell-1} := \Phi_{\ell-1}^{\mathrm{NMGM}}(\xi_{\ell-1}, d_{\ell-1})$;
$x_\ell := x_\ell - p\ (\xi_{\ell-1} - \tilde{x}_{\ell-1})\,/s$; {coarse-grid correction}
$\Phi_\ell^{\mathrm{NMGM}} := \mathcal{S}_\ell^{\nu_2}(x_\ell, b_\ell)$; {post-smoothing}
end;

Again the nonlinear multigrid iteration has an asymptotic convergence speed which coincides with the convergence speed of the linear multigrid iteration applied to the linearised system. Details about the convergence proof can be found in [194, §9.5].

If one applies $\Phi_\ell^{\mathrm{NMGM}}$ to the linear problem $\mathcal{A}_\ell(x_\ell) = A_\ell x_\ell - b_\ell$, the auxiliary data $(\tilde{x}_{\ell-1}, \tilde{b}_{\ell-1})$ can be chosen as $(0, 0)$ and the algorithm coincides with the linear multigrid iteration.

There are different nonlinear multigrid versions using other reference data $(\tilde{x}_{\ell-1}, \tilde{b}_{\ell-1})$ and other factors s. A comparison with numerical examples is given in Hackbusch [189].