Plane and Planarity Thresholds for Random Geometric Graphs

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Abstract. A random geometric graph, $G(n, r)$, is formed by choosing n points independently and uniformly at random in a unit square; two points are connected by a straight-line edge if they are at Euclidean distance at most r. For a given constant k, we show that $n^{\frac{-k}{2k-2}}$ is a distance threshold function for $G(n, r)$ to have a connected subgraph on k points. Based on that, we show that $n^{-2/3}$ is a distance threshold function for $G(n, r)$ to be plane, and $n^{-5/8}$ is a distance threshold function for $G(n, r)$ to be planar.

1 Introduction

Wireless networks are usually modeled as disk graphs in the plane. Given a set P of points in the plane and a positive parameter r, the *disk graph* is the geometric graph with vertex set P which has a straight-line edge between two points $p, q \in P$ if and only if $|pq| \leq r$, where $|pq|$ denotes the Euclidean distance between p and q. If $r = 1$, then the disk graph is referred to as *unit disk graph.* A *random geometric graph*, denoted by $G(n, r)$, is a geometric graph formed by choosing n points independently and uniformly at random in a unit square; two points are connected by a straight-line edge if and only if they are at Euclidean distance at most r, where $r = r(n)$ is a function of n and $r \to 0$ as $n \to \infty$.

We say that two line segments in the plane *cross* each other if they have a point in common that is interior to both edges. Two line segments are *non-crossing* if they do not cross. Note that two non-crossing line segments may share an endpoint. A geometric graph is said to be *plane* if its edges do not cross, and *non-plane*, otherwise. A graph is *planar* if and only if it does not contain K_5 (the complete graph on 5 vertices) or $K_{3,3}$ (the complete bipartite graph on six vertices partitioned into two parts each of size 3) as a minor. A *non-planar graph* is a graph which is not planar.

A graph property $\mathcal{P}$ is *increasing* if a graph G satisfies $\mathcal{P}$, then by adding edges to G, the property $\mathcal{P}$ remains valid in G. Similarly, $\mathcal{P}$ is *decreasing* if a graph G satisfies $\mathcal{P}$, then by removing edges from G, the property $\mathcal{P}$ remains valid in G. $\mathcal{P}$ is called a *monotone* property if $\mathcal{P}$ is either increasing or decreasing. Connectivity and "having a clique of size k" are increasing monotone properties,

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while planarity and "being plane" are decreasing monotone properties in $G(n,r)$, where the value of r increases.

By [13] any monotone property of a random geometric graphs has a threshold function. The thresholds in random geometric graphs are expressed by the distance r. In the sequel, the term w.h.p. (with high probability) is to be interpreted to mean that the probability tends to 1 as $n \to \infty$. For an increasing property $\mathcal{P}$, the threshold is a function $t(n)$ such that if $r = o(t(n))$ then w.h.p. $\mathcal{P}$ does not hold in $G(n,r)$, and if $r = \omega(t(n))$ then w.h.p. $\mathcal{P}$ holds in $G(n,r)$. Symmetrically, for a decreasing property $\mathcal{P}$, the threshold is a function $t(n)$ such that if $r = o(t(n))$ then w.h.p. $\mathcal{P}$ holds in $G(n,r)$, and if $r = \omega(t(n))$ then w.h.p. $\mathcal{P}$ does not hold in $G(n,r)$. Note that a threshold function may not be unique. It is well known that $\sqrt{\ln n/n}$ is a connectivity threshold for $G(n,r)$; see [14,19,20]. In this paper we investigate thresholds in random geometric graphs for having a connected subgraph of constant size, being plane, and being planar.

1.1 Related Work

Random graphs were first defined and formally studied by Gilbert in [10] and Erdös and Rényi [8]. It seems that the concept of a random geometric graph was first formally suggested by Gilbert in [11] and for that reason is also known as Gilbert's disk model. These classes of graphs are known to have numerous applications as a model for studying communication primitives (broadcasting, routing, etc.) and topology control (connectivity, coverage, etc.) in idealized wireless sensor networks as well as extensive utility in theoretical computer science and many fields of the mathematical sciences.

An instance of Erdös-Rényi graph [8] is obtained by taking n vertices and connecting any two with probability p, independently of all other pairs; the graph derived by this scheme is denoted by $G_{n,p}$. In $G_{n,p}$ the threshold is expressed by the edge existence probability p, while in $G(n,r)$ the threshold is expressed in terms of r. In both random graphs and random geometric graphs, property thresholds are of great interest [4,7,9,13,18]. Note that edge crossing configurations in $G(n,r)$ have a geometric nature, and as such, have no analogues in the context of the Erdös-Rényi model for random graphs. However, planarity, and having a clique of specific size are of interest in both $G_{n,p}$ and $G(n,r)$.

Bollobás and Thomason [5] showed that any monotone property in random graphs has a threshold function. See also a result of Friedgut and Kalai [9], and a result of Bourgain and Kalai [6]. In the Erdös-Rényi random graph $G_{n,p}$, the connectivity threshold is $p = \log n/n$ and the threshold for having a giant component is $p = 1/n$; see [1]. The planarity threshold for $G_{n,p}$ is $p = 1/n$; see [4,23].

A general reference on random geometric graphs is [22]. There is extensive literature on various aspects of random geometric graphs of which we mention the related work on coverage by [15,16] and a review on percolation, connectivity, coverage and colouring by [3]. As in random graphs, any monotone property in geometric random graphs has a threshold function [7,13,17,18].

Random geometric graphs have a connectivity threshold of $\sqrt{\ln n/n}$; see [14, 19, 20]. Gupta and Kumar [14] provided a connectivity threshold for points that are uniformly distributed in a disk. By a result of Penrose [21], in $G(n,r)$, any threshold function for having no isolated vertex (a vertex of degree zero) is also a connectivity threshold function. Panchapakesan and Manjunath [19] showed that $\sqrt{\ln n/n}$ is a threshold for being an isolated vertex in $G(n,r)$. This implies that $\sqrt{\ln n/n}$ is a connectivity threshold for $G(n,r)$. For $k \geq 2$, the details on the k-connectivity threshold in random geometric graphs can be found in [21, 22]. Connectivity of random geometric graphs for points on a line is studied by Godehardt and Jaworski [12]. Appel and Russo [2] considered the connectivity under the L_∞-norm.

1.2 Our Results

In this paper we investigate thresholds for some monotone properties in random geometric graphs. In Sect. 2 we show that for a constant k, the distance threshold for having a connected subgraph on k points is $n^{\frac{-k}{2k-2}}$. We show that the same threshold is valid for the existence of a clique of size k. Based on that, we prove the following thresholds for a random geometric graph to be plane or planar. In Sect. 3, we prove that $n^{-2/3}$ is a distance threshold for a random geometric graph to be plane. In Sect. 4, we prove that $n^{-5/8}$ is a distance threshold for a random geometric graph to be planar.

2 The Threshold for Having a Connected Subgraph on k Points

In this section, we look for the distance threshold for "existence of connected subgraphs of constant size"; this is an increasing property. For a given constant k, we show that $n^{\frac{-k}{2k-2}}$ is the threshold function for the existence of a connected subgraph on k points in $G(n,r)$. Specifically, we show that if $r = o(n^{\frac{-k}{2k-2}})$, then w.h.p. $G(n,r)$ has no connected subgraph on k points, and if $r = \omega(n^{\frac{-k}{2k-2}})$, then w.h.p. $G(n,r)$ has a connected subgraph on k points. We also show that the same threshold function holds for the existence of a clique of size k.

Theorem 1. *Let $k \geq 2$ be an integer constant. Then, $n^{\frac{-k}{2k-2}}$ is a distance threshold function for $G(n,r)$ to have a connected subgraph on k points.*

Proof. Let $P_1, \ldots, P_{\binom{n}{k}}$ be an enumeration of all subsets of k points in $G(n,r)$. Let $DG[P_i]$ be the subgraph of $G(n,r)$ that is induced by P_i. Let X_i be the random variable such that

$$X_i = \begin{cases} 1 & \text{if } DG[P_i] \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases}$$

Let the random variable X count the number of sets P_i for which $DG[P_i]$ is connected. It is clear that

$$X = \sum_{i=1}^{\binom{n}{k}} X_i. \quad (1)$$

Observe that $\mathrm{E}[X_i] = \Pr[X_i = 1]$. Since the random variables X_i have identical distributions, we have

$$\mathrm{E}[X] = \binom{n}{k} \mathrm{E}[X_1]. \quad (2)$$

We obtain an upper bound and a lower bound for $\Pr[X_i = 1]$. First, partition the unit square into squares of side equal to r. Let $\{s_1, \dots, s_{1/r^2}\}$ be the resulting set of squares. For a square s_t, let S_t be the $kr \times kr$ square which has s_t on its left bottom corner; see Fig. 1(a). S_t contains at most k^2 squares each of side length r (each S_t on the boundary of the unit square contains less than k^2 squares). Let $A_{i,t}$ be the event that all points in P_i are contained in S_t. Observe that if $DG[P_i]$ is connected then P_i lies in S_t for some $t \in \{1, \dots, 1/r^2\}$. Therefore,

$$\text{if } DG[P_i] \text{ is connected, then } (A_{i,1} \vee A_{i,2} \vee \cdots \vee A_{i,1/r^2}),$$

and hence we have

$$\Pr[X_i = 1] \le \sum_{t=1}^{1/r^2} \Pr[A_{i,t}] \le \sum_{t=1}^{1/r^2} (k^2 r^2)^k = k^{2k} r^{2k-2}. \quad (3)$$

Now, partition the unit square into squares with diagonal length equal to r. Each such square has side length equal to $r/\sqrt{2}$. Let $\{s_1, \dots, s_{2/r^2}\}$ be the resulting set of squares. Let $B_{i,t}$ be the event that all points of P_i are in s_t. Observe that if all points of P_i are in the same square, then $DG[P_i]$ is a complete graph and hence connected. Therefore,

$$\text{if } (B_{i,1} \vee B_{i,2} \vee \cdots \vee B_{i,2/r^2}), \text{ then } DG[P_i] \text{ is connected,}$$

and hence we have

$$\Pr[X_i = 1] \ge \sum_{t=1}^{2/r^2} \Pr[B_{i,t}] = \sum_{t=1}^{2/r^2} \left(\frac{r^2}{2}\right)^k = \frac{1}{2^{k-1}} r^{2k-2}. \quad (4)$$

Since $k \ge 2$ is a constant, Inequalities (3) and (4) and Eq. (2) imply that

$$\mathrm{E}[X_i] = \Theta(r^{2k-2}), \quad (5)$$

$$\mathrm{E}[X] = \Theta(n^k r^{2k-2}). \quad (6)$$

If $n \to \infty$ and $r = o(n^{\frac{-k}{2k-2}})$ we conclude that the following inequalities are valid

$$\begin{aligned} \Pr[X \ge 1] &\le \mathrm{E}[X] \text{ (by Markov's Inequality)} \\ &= \Theta(n^k r^{2k-2}) \text{ (by (6))} \\ &= o(1). \end{aligned} \quad (7)$$

Therefore, w.h.p. $G(n, r)$ has no connected subgraph on k points.

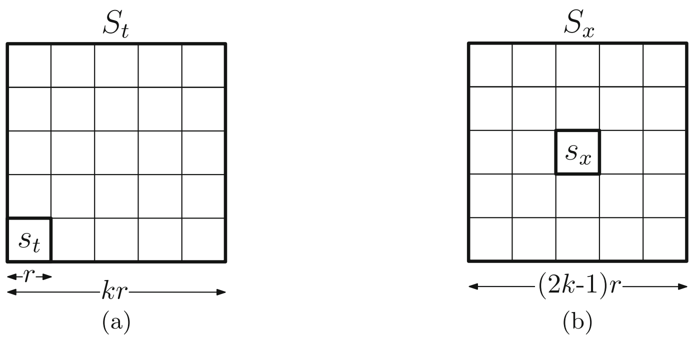

Fig. 1. (a) The square S_t has s_t on its left bottom corner. (b) The square S_x which is centered at s_x.

In the rest of the proof, we assume that $r = \omega(n^{\frac{-k}{2k-2}})$. In order to show that w.h.p. $G(n, r)$ has at least one connected subgraph on k vertices, we show, using the second moment method [1], that $\Pr[X = 0] \to 0$ as $n \to \infty$. Recall from Chebyshev's inequality that

$$\Pr[X = 0] \leq \frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2}. \tag{8}$$

Therefore, in order to show that $\Pr[X = 0] \to 0$, it suffices to show that

$$\frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2} \to 0. \tag{9}$$

In view of Identity (1) we have

$$\mathrm{Var}(X) = \sum_{1 \leq i,j \leq \binom{n}{k}} \mathrm{Cov}(X_i, X_j), \tag{10}$$

where $\mathrm{Cov}(X_i, X_j) = \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\mathrm{E}[X_j] \leq \mathrm{E}[X_i X_j]$. If $|P_i \cap P_j| = 0$ then $DG[P_i]$ and $DG[P_j]$ are disjoint. Thus, the random variables X_i and X_j are independent, and hence $\mathrm{Cov}(X_i, X_j) = 0$. It is enough to consider the cases when P_i and P_j are not disjoint. Assume $|P_i \cap P_j| = w$, where $w \in \{1, \ldots, k\}$. Thus, in view of Eq. (10), we have

$$\begin{aligned} \mathrm{Var}(X) &= \sum_{w=1}^{k} \sum_{|P_i \cap P_j| = w} \mathrm{Cov}(X_i, X_j) \\ &\leq \sum_{w=1}^{k} \sum_{|P_i \cap P_j| = w} \mathrm{E}[X_i X_j]. \end{aligned} \tag{11}$$

The computation of $\mathrm{E}[X_i, X_j]$ involves some geometric considerations which are being discussed in detail below. Since X_i and X_j are 0–1 random variables, $X_i X_j$ is a 0–1 random variable and

$$X_i X_j = \begin{cases} 1 & \text{if both } DG[P_i] \text{ and } DG[P_j] \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of the expected value we have

$$\begin{aligned} \mathrm{E}[X_i X_j] &= \Pr[X_j = 1 | X_i = 1] \Pr[X_i = 1] \\ &= \Pr[X_j = 1 | X_i = 1] \mathrm{E}[X_i]. \end{aligned} \tag{12}$$

By (5), $\mathrm{E}[X_i] = \Theta(r^{2k-2})$. It remains to compute $\Pr[X_j = 1|X_i = 1]$, i.e., the probability that $DG[P_j]$ is connected given that $DG[P_i]$ is connected. Consider the k-tuples P_i and P_j under the condition that $DG[P_i]$ is connected. Let x be a point in $P_i \cap P_j$. Partition the unit square into squares of side length equal to r. Let s_x be the square containing x. Let S_x be the $(2k-1)r \times (2k-1)r$ square centered at s_x. S_x contains at most $(2k-1)^2$ squares each of side length r (if S_x is on the boundary of the unit square then it contains less than $(2k-1)^2$ squares); see Fig. 1(b). The area of S_x is at most $(2kr)^2$, and hence the probability that a specific point of P_j is in S_t is at most $4k^2r^2$. Since P_i and P_j share w points, in order for $DG[P_j]$ to be connected, the remaining $k-w$ points of P_j must lie in S_x. Thus, the probability that $DG[P_j]$ is connected given that $DG[P_i]$ is connected is at most $(4k^2r^2)^{k-w} \leq c_w r^{2k-2w}$, for some constant $c_w > 0$. Thus, $\Pr[X_j = 1|X_i = 1] \leq c_w r^{2k-2w}$. In view of Eq. (12), we have

$$\mathrm{E}[X_i X_j] \leq c'_w \cdot r^{2k-2w} \cdot r^{2k-2} = c'_w r^{4k-2w-2}, \tag{13}$$

for some constant $c'_w > 0$.

Since P_i and P_j are k-tuples which share w points, $|P_i \cup P_j| = 2k - w$. There are $\binom{n}{2k-w}$ ways to choose $2k-w$ points for $P_i \cup P_j$. Since we choose w points for $P_i \cap P_j$, $k-w$ points for P_i alone, and $k-w$ points for P_j alone, there are $\binom{2k-w}{w,k-w,k-w}$ ways to split the $2k-w$ chosen points into P_i and P_j. Based on this and Inequality (13), Inequality (11) turns out to

$$\begin{aligned} \mathrm{Var}(X) &\leq \sum_{w=1}^{k} \sum_{|P_i \cap P_j| = w} \mathrm{E}[X_i X_j] \\ &\leq \sum_{w=1}^{k} \binom{n}{2k-w} \binom{2k-w}{w, k-w, k-w} c'_w r^{4k-2w-2} \\ &\leq \sum_{w=1}^{k} c''_w n^{2k-w} r^{4k-2w-2}. \end{aligned}$$

for some constants $c''_w > 0$. Consider (9) and note that by (6), $\mathrm{E}[X]^2 \geq c'' n^{2k} r^{4k-4}$, for some constant $c'' > 0$. Thus,

$$\begin{aligned} \frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2} &\leq \sum_{w=1}^{k} \frac{c''_w n^{2k-w} r^{4k-2w-2}}{c'' n^{2k} r^{4k-4}} = \sum_{w=1}^{k} \frac{c''_w}{c''} \cdot \frac{1}{n^w r^{2w-2}} \\ &= \frac{c''_1}{c''} \cdot \frac{1}{n^1 r^0} + \frac{c''_2}{c''} \cdot \frac{1}{n^2 r^2} + \cdots + \frac{c''_k}{c''} \cdot \frac{1}{n^k r^{2k-2}} \end{aligned} \tag{14}$$

Since $r = \omega(n^{\frac{-k}{2k-2}})$, all terms in (14) tend to zero. This proves the convergence in (9). Thus, $\Pr[X = 0] \to 0$ as $n \to \infty$. This implies that if $r = \omega(n^{\frac{-k}{2k-2}})$, then $G(n, r)$ has a connected subgraph on k vertices with high probability. ■

In the following theorem we show that if $k = O(1)$, then $n^{\frac{-k}{2k-2}}$ is also a threshold for $G(n, r)$ to have a clique of size k; this is an increasing property.

Theorem 2. *Let $k \geq 2$ be an integer constant. Then, $n^{\frac{-k}{2k-2}}$ is a distance threshold function for $G(n, r)$ to have a clique of size k.*

Proof. By Theorem 1, if $r = o(n^{\frac{-k}{2k-2}})$, then w.h.p. $G(n, r)$ has no connected subgraph on k vertices, and hence it has no clique of size k. This proves the first statement. We prove the second statement by adjusting the proof of Theorem 1, which is based on the second moment method. Assume $r = \omega(n^{\frac{-k}{2k-2}})$. Let $P_1, \dots, P_{\binom{n}{k}}$ be an enumeration of all subsets of k points. Let X_i be equal to 1 if $DG[P_i]$ is a clique, and 0 otherwise. Let $X = \sum X_i$.

Partition the unit square into a set $\{s_1, \dots, s_{1/r^2}\}$ of squares of side length r. Let S_t be the $2r \times 2r$ square which has s_t on its left bottom corner. If $DG[P_i]$ is a clique then P_i lies in S_t for some $t \in \{1, \dots, 1/r^2\}$. Therefore,

$$\Pr[X_i = 1] \leq 4^k r^{2k-2}.$$

Now, partition the unit square into a set $\{s_1, \dots, s_{2/r^2}\}$ of squares with diagonal length r. If all points of P_i fall in the square s_t, then $DG[P_i]$ is a clique. Thus,

$$\Pr[X_i = 1] \geq \frac{1}{2^{k-1}} r^{2k-2}.$$

Since $k \geq 2$ is a constant, we have

$$\mathrm{E}[X_i] = \Theta(r^{2k-2}),$$
$$\mathrm{E}[X] = \Theta(n^k r^{2k-2}).$$

In view of Chebyshev's inequality we need to show that $\frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2}$ tends to 0 as n goes to infinity. We bound $\mathrm{Var}(X)$ from above by Inequality (11). Consider the k-tuples P_i and P_j under the condition that $DG[P_i]$ is a clique. Let $|P_i \cap P_j| = w$, and let x be a point in $P_i \cap P_j$. Partition the unit square into squares of side length r. Let s_x be the square containing x. Let S_x be the $3r \times 3r$ square centered at s_x. In order for $DG[P_j]$ to be a clique, the remaining $k - w$ points of P_j must lie in S_x. Thus,

$$\mathrm{E}[X_i X_j] \leq c'_w r^{4k-2w-2},$$

for some constant $c'_w > 0$. By a similar argument as in the proof of Theorem 1, we can show that for some constants $c'', c''_w > 0$ the followings inequalities are valid:

$$\mathrm{Var}(X) \leq \sum_{w=1}^{k} c''_w n^{2k-w} r^{4k-2w-2},$$

$$\frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2} \leq \sum_{w=1}^{k} \frac{c''_w}{c''} \cdot \frac{1}{n^w r^{2w-2}}.$$

Since $r = \omega(n^{\frac{-k}{2k-2}})$, the last inequality tends to 0 as n goes to infinity. This completes the proof for the second statement. ■

As a direct consequence of Theorem 2, we have the following corollary.

Corollary 1. *n^{-1} is a threshold for $G(n, r)$ to have an edge, and $n^{-\frac{3}{4}}$ is a threshold for $G(n, r)$ to have a triangle.*

3 The Threshold for $G(n, r)$ to be Plane

In this section we investigate the threshold for a random geometric graph to be plane; this is a decreasing property. Recall that $G(n, r)$ is plane if no two of its edges cross. As a warm-up exercise we first prove a simple result which is based on the connectivity threshold for random geometric graphs, which is known to be $\sqrt{\ln n/n}$.

Theorem 3. If $r \geq \sqrt{\frac{c \ln n}{n}}$, with $c \geq 36$, then w.h.p. $G(n, r)$ is not plane.

proof In order to prove that w.h.p. $G(n, r)$ is not plane, we show that w.h.p. it has a pair of crossing edges. Partition the unit square into squares each with diagonal length r. Then subdivide each such square into nine sub-squares as depicted in Fig. 2. There are $\frac{18}{r^2}$ sub-squares, each of side length $\frac{r}{3\sqrt{2}}$. The probability that no point lies in a specific sub-square is $(1 - \frac{r^2}{18})^n$. Thus, the probability that there exists an empty sub-square is at most

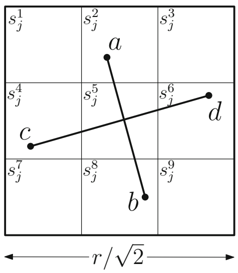

Fig. 2. An square of diameter r which is partitioned into nine sub-squares.

$$\frac{18}{r^2}\left(1 - \frac{r^2}{18}\right)^n \leq n\left(1 - \frac{c \ln n}{18n}\right)^n \leq n^{1-c/18} \leq \frac{1}{n},$$

when $c \geq 36$. Therefore, with probability at least $1 - \frac{1}{n}$ all sub-squares contain points. By choosing four points a, b, c, and d as depicted in Fig. 2, it is easy to see that the edges (a, b) and (c, d) cross. Thus, w.h.p. $G(n, r)$ has a pair of crossing edges, and hence w.h.p. it is not plane. ■

In fact, Theorem 3 ensures that w.h.p. there exists a pair of crossing edges in each of the squares. This implies that there are $\Omega\left(\frac{n}{\ln n}\right)$ disjoint pair of crossing edges, while for $G(n, r)$ to be not plane we need to show the existence of at least one pair of crossing edges. Thus, the value of r provided by the connectivity threshold seems rather weak. By a different approach, in the rest of this section we show that $n^{-\frac{2}{3}}$ is the correct threshold.

Lemma 1. *Let (a, b) and (c, d) be two crossing edges in $G(n, r)$, and let Q be the convex quadrilateral formed by a, b, c, and d. Then, two adjacent sides of Q are edges of $G(n, r)$.*

Proof. Refer to Fig. 3. At least one of the angles of Q, say $\angle cad$, is bigger than or equal to $\pi/2$. It follows that in the triangle $\triangle cad$ the side cd is the longest, i.e., $|cd| \geq \max\{|ac|, |ad|\}$. Since $|cd| \leq r$, both $|ac|$ and $|ad|$ are at most r. Thus, ac and ad—which are adjacent—are edges of $G(n, r)$. ■

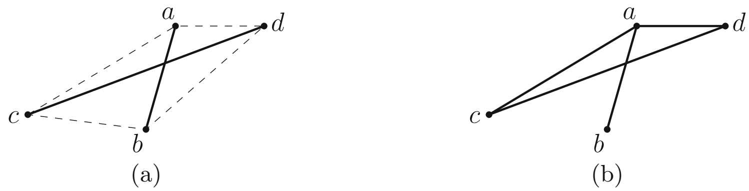

Fig. 3. (a) Illustration of Lemma 1. (b) Crossing edges (a, b) and (c, d) form an anchor.

In the proof of Lemma 1, a is connected to b, c, and d. So the distance between a to each of b, c, and d is at most r. Thus, we have the following corollary.

Corollary 2. *The endpoints of every two crossing edges in $G(n, r)$ are at distance at most $2r$ from each other. Moreover, there exists an endpoint which is within distance r from other endpoints.*

Based on the proof of Lemma 1, we define an *anchor* as a set $\{a, b, c, d\}$ of four points in $G(n, r)$ such that three of them form a triangle, say $\triangle cad$, and the fourth vertex, b, is connected to a by an edge which crosses cd; see Fig. 3(b). We call a as the *crown* of the anchor. The crown is within distance r from the other three points. Note that bc and bd may or may not be edges of $G(n, r)$. In view of Lemma 1, two crossing edges in $G(n, r)$ form an anchor. Conversely, every anchor in $G(n, r)$ introduces a pair of crossing edges.

Observation 1. *$G(n, r)$ is plane if and only if it has no anchor.*

Theorem 4. *$n^{-\frac{2}{3}}$ is a threshold for $G(n, r)$ to be plane.*

Proof. In order to show that $G(n, r)$ is plane, by Observation 1, it is enough to show that it has no anchors. Every anchor has four points and it is connected. By Theorem 1, if $r = o(n^{-\frac{2}{3}})$, then w.h.p. $G(n, r)$ has no connected subgraph on 4 points, and hence it has no anchors. This proves the first statement.

We prove the second statement by adjusting the proof of Theorem 1 for $k = 4$. Assume $r = \omega(n^{-\frac{2}{3}})$. Let $P_1, \dots, P_{\binom{n}{4}}$ be an enumeration of all subsets of 4 points. Let X_i be equal to 1 if $DG[P_i]$ contains an anchor, and 0 otherwise. Let $X = \sum X_i$. In view of Chebyshev's inequality we need to show that $\frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2}$ tends to 0 as n goes to infinity.

Partition the unit square into a set $\{s_1, \dots, s_{2/r^2}\}$ of squares with diagonal length r. Then, subdivide each square s_j, into nine sub-squares $s_j^1, \dots, s_j^9$ as depicted in Fig. 2. If each of $s_j^1, s_j^3, s_j^7, s_j^9$ or each of $s_j^2, s_j^4, s_j^6, s_j^8$ contains a point of P_i, then $DG[P_i]$ is a convex clique of size four and hence it contains an anchor. Thus,

$$\Pr[X_i = 1] \geq \frac{r^6}{2^3} \cdot \frac{2}{9^4}.$$

This implies that $\mathrm{E}[X_i] = \Omega(r^6)$, and hence $\mathrm{E}[X] = \Omega(n^4 r^6)$. Therefore,

$$\mathrm{E}[X]^2 \geq c'' n^8 r^{12},$$

for some constant $c'' > 0$. By a similar argument as in the proof of Theorem 1 we bound the variance of X from above by

$$\mathrm{Var}(X) \leq c_1'' n^7 r^{12} + c_2'' n^6 r^{10} + c_3'' n^5 r^8 + c_4'' n^4 r^6.$$

Since $r = \omega(n^{-\frac{2}{3}})$, $\frac{\mathrm{Var}(X)}{\mathrm{E}[X]^2}$ tends to 0 as n goes to infinity. That is, w.h.p. $G(n, r)$ has an anchor. By Observation 1, w.h.p. $G(n, r)$ is not plane. ■

As a direct consequence of the proof of Theorem 4, we have the following:

Corollary 3. *With high probability if a random geometric graph is not plane, then it has a clique of size four.*

Note that every anchor introduces a crossing and each crossing introduces an anchor. Since, every anchor is a connected graph and has four points, by (6) we have the following corollary.

Corollary 4. *The expected number of crossings in $G(n, r)$ is $\Theta(n^4 r^6)$.*

4 The Threshold for $G(n, r)$ to be Planar

In this section we investigate the threshold for the planarity of a random geometric graph; this is a decreasing property. By Kuratowski's theorem, a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or of $K_{3,3}$. Note that any plane random geometric graph is planar too; observe that the reverse statement may not be true. Thus, the threshold for planarity seems to be larger than the threshold of being plane. By a similar argument as in the proof of Theorem 3 we can show that if $r \geq \sqrt{c \ln n / n}$, then w.h.p. each square with diagonal length r contains K_5, and hence $G(n, r)$ is not planar.

Theorem 5. *$n^{-\frac{5}{8}}$ is a threshold for $G(n, r)$ to be planar.*

Proof. By Theorem 2, if $r = \omega(n^{-\frac{5}{8}})$, then w.h.p. $G(n, r)$ has a clique of size 5. Thus, w.h.p. $G(n, r)$ contains K_5 and hence it is not planar. This proves the second statement of the theorem.

If $r = o(n^{-\frac{5}{8}})$, then by Theorem 1, w.h.p. $G(n, r)$ has no connected subgraph on 5 points, and hence it has no K_5. Similarly, if $r = o(n^{-\frac{3}{5}})$, then w.h.p. $G(n, r)$ has no connected subgraph on 6 points, and hence it has no $K_{3,3}$.

Since $n^{-\frac{5}{8}} < n^{-\frac{3}{5}}$, it follows that if $r = o(n^{-\frac{5}{8}})$, then w.h.p. $G(n,r)$ has neither K_5 nor $K_{3,3}$ as a subgraph.

Note that, in order to prove that $G(n,r)$ is planar, we have to show that it does not contain any subdivision of either K_5 or $K_{3,3}$. Any subdivision of either K_5 or $K_{3,3}$ contains a connected subgraph on $k \geq 5$ vertices. Since $n^{-5/8} < n^{-k/(2k-2)}$ for all $k \geq 5$, in view of Theorem 1, we conclude that if $r = o(n^{-\frac{5}{8}})$, then w.h.p. $G(n,r)$ has no subdivision of K_5 and $K_{3,3}$, and hence $G(n,r)$ is planar. This proves the first statement of the theorem. ■

As a direct consequence of the proof of Theorem 5, we have the following:

Corollary 5. *With high probability if a random geometric graph does not contain a clique of size five, then it is planar.*

5 Conclusion and Further Results

We presented thresholds for random geometric graphs to have a connected subgraph of constant size, to be plane, and to be planar. A natural open problem is to extend Theorem 1 for connected subgraphs of k vertices where k is not necessarily a constant, and for connected subgraphs of k vertices which have diameter δ.

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