Chapter 19
On One Boundary-Value Problem with Two Nonlocal Conditions for a Parabolic Equation

Olga Danilkina

Abstract This work is concerned with a boundary-value problem for a parabolic equation with nonlocal integral conditions of the second kind. Existence and uniqueness of a generalized solution are proved.

19.1 Introduction

In recent years, nonlocal problems for PDEs have received a great deal of attention as a convenient way of description of different physical phenomena. These problems arise in a wide variety of applications, including heat conduction, processes in liquid plasma, dynamics of ground waters, thermo-elasticity and some technological processes.

In this paper, our main interest lies in the field of nonlocal problems with integral conditions that generalizes the discrete case. We mention the first papers in this area [6, 14] devoted to problems for parabolic equations. Then these results were extended [2, 7–11, 13, 16, 26, 28, 29]. For papers related to nonlocal problems for other evolution equations, we refer the reader to [1, 3–5, 12, 17, 19–25, 27].

In [25], the author studied two problems for the hyperbolic equation

$$u_{tt} - u_{xx} + c(x,t)u = f(x,t)$$

with the initial condition

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

and two types of nonlocal conditions.

O. Danilkina (✉)
Department of Mathematics, The University of Dodoma, Tanzania
e-mail: olga.danilkina@gmail.com; olga.danilkina@udom.ac.tz

G.A. Anastassiou, O. Duman (eds.), *Computational Analysis*, Springer Proceedings in Mathematics & Statistics 155, DOI 10.1007/978-3-319-28443-9_19

Case 1 :

$$\int_0^l K_i(x)u(x,t)\,dx = 0, \quad i = 1, 2.$$

Case 2 :

$$u_x(0,t) - \int_0^l K_1(x,t)u(x,t)\,dx = 0,$$

$$u_x(l,t) - \int_0^l K_2(x,t)u(x,t)\,dx = 0.$$

Motivated by the ideas of Pulkina [25], in this paper we extend the results of Pulkina [25] to a special class of boundary-value problems with nonlocal integral conditions for parabolic equations. The proof of the main result is based on the method of energy estimates and the Faedo–Galerkin approximations.

19.2 Preliminaries

In the cylinder $Q_T = \{(x,t)\colon x \in (0,l),\ t \in (0,T)\}$ we consider the problem for the equation

$$u_t = u_{xx} + c(x,t)u \tag{19.1}$$

with the initial condition

$$u(x,0) = \varphi(x) \tag{19.2}$$

and the nonlocal conditions

$$u_x(0,t) = \int_0^l K_1(x,t)u_x(x,t)\,dx + \int_0^l M_1(x,t)u(x,t)\,dx, \tag{19.3}$$

$$u_x(l,t) = \int_0^l K_2(x,t)u_x(x,t)\,dx + \int_0^l M_2(x,t)u(x,t)\,dx. \tag{19.4}$$

In this paper, we shall assume that the following assumptions are satisfied.

(*A*1) $c(x,t) \in C(\overline{Q}_T)$, $\varphi(x) \in C^1[0,l]$;
(*A*2) $K_1(x,t),\ K_2(x,t),\ M_1(x,t),\ M_2(x,t) \in C^1(\overline{Q}_T)$.

We note that presence of partial derivatives on the right-hand side of the nonlocal conditions (19.3), (19.4) can cause difficulties in constructing of a priori estimates. Therefore, to avoid this we integrate by parts in (19.3), (19.4) and obtain

$$u_x(0,t) = K_1(l,t)u(l,t) - K_1(0,t)u(0,t) + \int_0^l R_1(x,t)u(x,t)\,dx, \tag{19.5}$$

$$u_x(l,t) = K_2(l,t)u(l,t) - K_2(0,t)u(0,t) + \int_0^l R_2(x,t)u(x,t)\,dx, \tag{19.6}$$

where $R_1(x,t) = M_1(x,t) - (K_1(x,t))_x$, $R_2(x,t) = M_2(x,t) - (K_2(x,t))_x$.

Let $W_2^{1,0}(Q_T)$ be the usual Sobolev space. We define the space $V_2(Q_T)$ which consists of elements of $W_2^{1,0}(Q_T)$ with the norm

$$|u|^2 = ess \sup_{0\le t\le T} \int_0^l u^2(x,t)\,dt + \int_{Q_T} u_x^2(x,t)\,dx\,dt.$$

Definition 19.1. A function $u(x,t) \in V_2(Q_T)$ is said to be a generalized solution to the problem (19.1), (19.2), (19.5), (19.6) provided for any function $\eta(x,t) \in W_2^1(Q_T)$, $\eta(x,T) = 0$, the following integral identity holds:

$$\begin{aligned}
&\int_{Q_T} (-u\eta_t + u_x\eta_x - cu\eta)\ dx\,dt \\
&\quad = \int_0^l \varphi(x)\eta(x,0)\,dx + \int_0^T (K_1(0,t)\eta(0,t) - K_2(0,t)\eta(l,t))\,u(0,t)\,dt \\
&\qquad + \int_0^T (K_2(l,t)\eta(l,t) - K_1(l,t)\eta(0,t))\,u(l,t)\,dt \\
&\qquad + \int_{Q_T} (R_1(x,t)\eta(l,t) - R_2(0,t)\eta(0,t))\,u(x,t)\,dx\,dt.
\end{aligned} \tag{19.7}$$

Lemma 19.2. *Let a function $u(x,t)$ be a solution to the problem* (19.1)*,* (19.2)*,* (19.5)*,* (19.6)*. Then the following identity holds:*

$$\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + \int_{Q_T} u_x^2\,dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_{Q_T} cu^2\,dx\,dt$$
$$+ \int_0^\tau K_2(l,t)u^2(l,t)\,dt - \int_0^\tau K_1(l,t)u(0,t)u(l,t)\,dt$$
$$+ \int_0^\tau K_1(0,t)u^2(0,t)\,dt - \int_0^\tau K_2(0,t)u(0,t)u(l,t)\,dt$$
$$+ \int_{Q_T} R_1(x,t)u(x,t)\,dx\,u(l,t)\,dt$$
$$- \int_{Q_T} R_2(x,t)u(x,t)\,dx\,u(0,t)\,dt$$

for a.e. $\tau \in [0,T]$.

Proof. Let a function $u(x,t) \in W_2^1(Q_\tau)$ and satisfy the integral identity (19.7) for all functions $\eta(x,t) \in W_2^1(Q_T),\ \eta(x,T) = 0$. For an arbitrary $\tau \in [0,T]$, we take

$$\eta(x,t) = \begin{cases} u(x,t), & 0 < t < \tau, \\ 0, & \tau \le t < T. \end{cases}$$

After integration by parts in (19.7) we obtain

$$\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + \int_{Q_\tau} u_x^2\,dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_{Q_\tau} cu^2\,dx\,dt$$
$$+ \int_0^\tau K_2(l,t)u^2(l,t)\,dt - \int_0^\tau K_1(l,t)u(0,t)u(l,t)\,dt$$
$$+ \int_0^\tau K_1(0,t)u^2(0,t)\,dt - \int_0^\tau K_2(0,t)u(0,t)u(l,t)\,dt$$
$$+ \int_{Q_\tau} R_1(x,t)u(x,t)\,dx\,u(l,t)\,dt$$
$$- \int_{Q_\tau} R_2(x,t)u(x,t)\,dx\,u(0,t)\,dt. \qquad (19.8)$$

We shall prove that a function $u(x,t) \in V_2(Q_T)$ also satisfies (19.8). To this aim consider a sequence $v^m(x,t) \in W_2^1(Q_T)$ which satisfies the identity (19.7) and hence, (19.8), that is

$$\begin{aligned}\frac{1}{2}\int_0^l (v^m)^2(x,\tau)\,dx + \int_{Q_\tau} (v^2)_x^2\,dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_{Q_\tau} c(v^m)^2\,dx\,dt \\ + \int_0^\tau K_2(l,t)(v^m)^2(l,t)\,dt \\ - \int_0^\tau K_1(l,t)v^m(0,t)v^m(l,t)\,dt \\ + \int_0^\tau K_1(0,t)(v^m)^2(0,t)\,dt \\ - \int_0^\tau K_2(0,t)v^m(0,t)v^m(l,t)\,dt \\ + \int_{Q_\tau} R_1(x,t)v^m(x,t)\,dx\,v^m(l,t)\,dt \\ - \int_{Q_\tau} R_2(x,t)v^m(x,t)\,dx\,v^m(0,t)\,dt.\end{aligned} \tag{19.9}$$

Note that $W_2^1(Q_\tau)$ is dense in $V_2^{1,0}(Q_T)$ [15] and hence, in $V_2(Q_T)$. Therefore, there exists a function $u^* \in V_2(Q_T)$ such that $|v^m - u^*|_{Q_T} \to 0$ as $m \to \infty$:

$$ess \sup_{0\le t\le T} \int_0^l (v^m - u^*)^2\,dx + \int_{Q_T} (v^m - u^*)_x^2\,dx\,dt \to 0.$$

It implies that $v^m(x,t) \to u^*$ strongly in $L_2(0,l)$ and $v_x^m(x,t) \to u_x^*$ strongly in $L_2(Q_T)$. We also note that $v^m \to u^*$ in $L_2(Q_T)$. Our next aim is to estimate terms on the right-hand side of (19.9). The assumptions $(A1)$, $(A2)$ imply that there exist positive numbers c_1, k_2 such that $|c(x,t)| \le c_1$, $|K_2(x,t)| \le k_2$. Applying ε-inequality [18]

$$v^2\big|_{x=0,\,x=l} \le \int_0^l \left(\varepsilon v_x^2 + C(\varepsilon)v^2\right)\,dx$$

we obtain

$$\left|\int_0^\tau K_2(l,t)(v^m(l,t))^2\,dt\right| \le k_2\int_0^\tau (v^m(l,t))^2\,dt$$
$$\le k_2\varepsilon||v_x^m||^2 + k_2C_\varepsilon||v^m||^2$$
$$\le H_1\left(||v_x^m||^2 + ||v^m||^2\right), \tag{19.10}$$

where $H_1 = \max\{k_2\varepsilon,\ k_2C_\varepsilon\}$. Similarly, we derive the estimates

$$\left|\int_0^\tau K_1(0,t)(v^m(0,t))^2\,dt\right| \le H_2\left(||v_x^m||^2 + ||v^m||^2\right), \tag{19.11}$$

$$\left|\int_0^\tau (K_1(l,t)+K_2(0,t))v^m(0,t)v^m(l,t)\,dt\right| \le H_3\left(||v_x^m||^2 + ||v^m||^2\right). \tag{19.12}$$

To obtain an estimate for the term

$$\int_{Q_\tau} R_1(x,t)v^m(x,t)\,dx\,v^m(l,t)\,dt,$$

we use Young's inequality and the Cauchy–Schwartz inequality and then

$$\left|\int_{Q_\tau} R_1(x,t)v^m(x,t)\,dx\,v^m(l,t)\,dt\right|$$
$$\le \frac{l}{2}\int_0^\tau (v^m(l,t))^2\,dt + \frac{r_1}{2}\int_{Q_\tau}(v^m(x,t))^2\,dx\ dt.$$

Similarly,

$$\left|\int_{Q_\tau} R_1(x,t)v^m(x,t)\,dx\,v^m(l,t)\,dt\right| \le H_4\left(||v_x^m||^2 + ||v^m||^2\right) \tag{19.13}$$

and

$$\left|\int_{Q_\tau} R_2(x,t)v^m(x,t)\,dx\,v^m(0,t)\,dt\right| \le H_5\left(||v_x^m||^2 + ||v^m||^2\right), \tag{19.14}$$

where the constants H_i, $i = 2, 3, 4, 5$ do not depend on m. Furthermore, we note that

$$\left| \int\limits_{Q_\tau} c(v^m)^2 \, dx \, dt \right| \le c_1 ||v^m||^2. \tag{19.15}$$

Therefore, using strong convergence $v^m(x,t) \to u^*$ and the estimates (19.10)–(19.15) we pass to the limit as $m \to \infty$ in and obtain (19.8) for $u(x,t) \in V_2(Q_T)$.

Lemma 19.3. *Let a function $u(x,t)$ be a solution to the problem* (19.1)*,* (19.2)*,* (19.5)*,* (19.6)*. Then there exists $H > 0$ such that $|u|_{Q_T} \le H$.*

Proof. By Lemma 19.2, the solution $u(x,t)$ satisfies the integral identity (19.8). We shall estimate the right-hand side of (19.8). Note that for $\varepsilon_1, \varepsilon_2 > 0$

$$u^2(0,t) \le \int\limits_0^l \left(\varepsilon_1 u_x^2 + C(\varepsilon_1) u^2\right) dx, \ u^2(l,t)$$

$$\le \int\limits_0^l \left(\varepsilon_2 u_x^2 + C(\varepsilon_2) u^2\right) dx$$

and hence,

$$|u(0,t)u(l,t)| \le \frac{1}{2} \int\limits_0^l \left((\varepsilon_1 + \varepsilon_2) u_x^2 + (C(\varepsilon_1) + C(\varepsilon_2)) u^2\right) dx.$$

Therefore,

$$\left| \int\limits_0^\tau K_2(l,t) u^2(l,t) \, dt \right| \le k_2 \varepsilon \int\limits_0^\tau \int\limits_0^l u_x^2 \, dx \, dt + k_2 C_\varepsilon \int\limits_0^\tau \int\limits_0^l u^2 \, dx \, dt, \tag{19.16}$$

$$\left| \int\limits_0^\tau (K_1(l,t) + K_2(0,t)) \, u(0,t) u(l,t) \right|$$

$$\le \frac{(k_1 + k_2)}{2} (\varepsilon_1 + \varepsilon_2) \int\limits_{Q_\tau} u_x^2 \, dx \, dt +$$

$$+ \frac{(k_1 + k_2)}{2} (C_{\varepsilon_1} + C_{\varepsilon_2}) \int\limits_{Q_\tau} u^2 \, dx \, dt.$$

Moreover,

$$\left|\int\limits_{Q_\tau} R_1(x,t)u(x,t)\,dx\,u(l,t)\,dt\right| \le \frac{1}{2}(r_1 + lC_{\varepsilon_2})\int\limits_{Q_\tau}(u(x,t))^2\,dx\;dt + \frac{\varepsilon_2 l}{2}\int\limits_{Q_\tau}(u_x(x,t))^2\,dx\,dt$$

and

$$\left|\int\limits_{Q_\tau} R_2(x,t)u(x,t)\,dx\,u(0,t)\,dt\right| \le \frac{1}{2}(r_2 + lC_{\varepsilon_1})\int\limits_{Q_\tau}(u(x,t))^2\,dx\;dt + \frac{\varepsilon_1 l}{2}\int\limits_{Q_\tau}(u_x(x,t))^2\,dx\,dt. \tag{19.17}$$

From the estimates (19.16)–(19.17) and the integral identity (19.8) it follows that

$$\int\limits_0^l u^2(x,\tau)\,dx + \int\limits_{Q_\tau} u_x^2\,dx\,dt \le \frac{1}{2}\int\limits_0^l \varphi^2(x)\,dx + P\int\limits_{Q_\tau} u^2\,dx\,dt.$$

In particular,

$$\int\limits_0^l u^2(x,\tau)\,dx \le \frac{1}{2}\int\limits_0^l \varphi^2(x)\,dx + P\int\limits_{Q_\tau} u^2\,dx\,dt. \tag{19.18}$$

By Gronwall's lemma we conclude that

$$\int\limits_{Q_\tau} u^2(x,t)\,dx\,dt \le H_1\int\limits_0^l \varphi^2(x)\,dx, \tag{19.19}$$

and hence,

$$\int\limits_{Q_\tau} u_x^2(x,t)\,dx\,dt \le H_2 \int\limits_0^l \varphi^2(x)\,dx. \tag{19.20}$$

Therefore, from (19.19) and (19.20) we obtain

$$|u|_{Q_T} \le H.$$

19.3 The Main Result

In this section we shall prove existence and uniqueness theorem for the problem (19.1), (19.2), (19.5), (19.6).

Theorem 19.4. *Let the conditions* $(A1)$–$(A2)$ *hold and*

$(A3)$ $K_1(\xi_1,0),\ K_2(\xi_i,0)=0,\ i=1,2,\ \xi_1=0,\ \xi_2=l,\ K_1(l,t)=K_2(0,t)$,
$(A4)$ $R_1^2+R_2^2\le\frac{1}{2}$.

Then there exists a unique generalized solution to the problem (19.1)*,* (19.2)*,* (19.5)*,* (19.6).

Proof. The proof of the theorem is organized as follows. First, to prove the existence part we construct a sequence of Faedo–Galerkin approximations and show its convergence to the solution of the problem. Second, we prove uniqueness of the generalized solution. Let a system of functions $\{\varphi_i(x)\}\in C^1[0,l]$ be complete in W_2^1 and

$$\left(\varphi_i,\ \varphi_j\right)_{L_2(0,l)} = \begin{cases} 1,\ i=j,\\ 0,\ i\ne j.\end{cases}$$

We define for each $N\in\mathbb{N}$ the approximate solution in the following form

$$u^N(x,t)=\sum_{k=1}^N c_k^N(t)\varphi_k(x),$$

where the functions $c_k(t)$ are unknown for the moment. We shall consider $c_k(t)$ which are solutions to the Cauchy problem

$$\int\limits_0^l u_t^N\varphi_i\,dx+\int\limits_0^l u_x^N\varphi_i'\,dx-\int\limits_0^l c(x,t)u^N\varphi_i\,dx$$
$$=K_1(0,t)u^N(0,t)\varphi_i(0)-K_1(l,t)u^N(l,t)\varphi_i(0)$$

$$+\int_0^l R_1(x,t)u^N\,dx\varphi_i(0)-\int_0^l R_2(x,t)u^N\,dx\varphi_i(l)$$

$$-K_2(0,t)u^N(0,t)\varphi_i(l)+K_2(l,t)u^N(l,t)\varphi_i(l), \tag{19.21}$$

$$c_i^N(0)=(\varphi,\ \varphi_i)\,, \tag{19.22}$$

$i=\overline{1,\,N}$. We write the Cauchy problem (19.21)–(19.22) such that

$$\frac{d}{dt}c_i^N(t)+\sum_{k=1}^{N}c_k^N(t)A_{k,i}(t)=0,\ \ i=\overline{1,\,N}, \tag{19.23}$$

where

$$A_{k,i}(t)=\int_0^l \varphi_k'(x)\varphi_i'(x)\,dx-\int_0^l c(x,t)\varphi_k(x)\varphi_i(x)\,dx$$
$$-\varphi_i(0)\left(K_1(l,t)\varphi_k(l)-K_1(0,t)\varphi_k(0)+\int_0^l R_1(x,t)\varphi_k(x)\,dx\right)$$
$$+\varphi_i(l)\left(K_2(l,t)\varphi_k(l)-K_2(0,t)\varphi_k(0)+\int_0^l R_2(x,t)\varphi_k(x)\,dx\right).$$

We estimate the coefficients $A_{k,i}$ as follows:

$$|A_{k,i}(t)|\le\frac{1}{2}\int_0^l \varphi_k'^2(x)\,dx+\frac{1}{2}\int_0^l \varphi_i'^2(x)\,dx$$
$$+\frac{\overline{c}}{2}\int_0^l \varphi_k^2(x)\,dx+\frac{\overline{c}}{2}\int_0^l \varphi_i^2(x)\,dx$$
$$+|\varphi_i(0)|\left(k_1\left(\varphi_k(l)|+|\varphi_k(0)|\right)+\frac{r_1}{2}+\frac{1}{2}\int_0^l \varphi_k^2(x)\,dx\right)$$
$$+\ |\varphi_i(0)|\left(k_2\left(\varphi_k(l)|+|\varphi_k(0)|\right)+\frac{r_2}{2}+\frac{1}{2}\int_0^l \varphi_k^2(x)\,dx\right).$$

The assumptions $(A1)$–$(A2)$ imply that $A_{k,i}$ are bounded. Therefore, the Cauchy problem has a unique solution $c_k^N \in C^1(0,T)$ and all approximations $u^N(x,t)$ are defined. The next aim is to show that the sequence $\{u^N(x,t)\}$ converges to the solution to the problem (19.1), (19.2), (19.5), (19.6). To this aim we multiply each (19.21) by $c_i^N(t)$, sum it up from $i = 0$ to $i = N$ and integrate the result with respect to t from 0 to $t_1 < T$. Thus we obtain

$$\frac{1}{2}\int_0^l \left(u^N(x,t_1)\right)^2 dx + \int_{Q_{t_1}} \left(u_x^N\right)^2 dx\,dt = \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \int_{Q_{t_1}} c(u^N)^2\,dx\,dt$$
$$+ \int_0^{t_1} K_2(l,t)(u^N(l,t))^2\,dt$$
$$- \int_0^{t_1} K_1(l,t)u^N(0,t)u^N(l,t)\,dt$$
$$+ \int_0^{t_1} K_1(0,t)(u^N(0,t))^2\,dt$$
$$- \int_0^{t_1} K_2(0,t)u^N(0,t)u^N(l,t)\,dt$$
$$+ \int_{Q_{t_1}} R_1(x,t)u^N(x,t)\,dx\,u^N(l,t)\,dt$$
$$- \int_{Q_{t_1}} R_2(x,t)u^N(x,t)\,dx\,u^N(0,t)\,dt.$$

Therefore, from Lemmas 19.2 and 19.3 it follows that $\left|u^N\right|_{Q_T} \le \text{Const}$. It implies that there exists a subsequence of $\{u^N(x,t)\}$ which converges weakly in $L_2(0,l)$ and uniformly with respect to $t \in [0,T]$ to some function $u(x,t)$ [18]. We shall prove that this function $u(x,t)$ satisfies the integral identity (19.7) from the definition of a generalized solution to the problem (19.1), (19.2), (19.5), (19.6). To this end, we multiply each (19.21) by a smooth function $d_i(t)$, $d_i(T) = 0$, sum it up from $i = 1$ to $i = N$, integrate with respect to t from 0 to T and denote $\Phi^{N'} = \sum_{i=1}^{N'} d_i(t)\varphi_i(x)$.

As a result we obtain

$$\int_0^T \Big(-(u^N, \Phi_t^{N'}) + (u_x^N, \Phi_x^{N'}) - (cu^N, \Phi^{N'})\Big)$$

$$= \int_0^l \varphi(x)\Phi^{N'}(x,0)\,dx + \int_0^T \int_0^l R_1(x,t)u^N(x,t)\,dx\,\Phi^{N'}(0,t)\,dt$$

$$- \int_0^T \int_0^l R_2(x,t)u^N(x,t)\,dx\,\Phi^{N'}(l,t)\,dt - \int_0^T K_2(0,t)u^N(0,t)\Phi^{N'}(l,t)\,dt$$

$$+ \int_0^T K_2(l,t)u^N(l,t)\Phi^{N'}(l,t)\,dt - \int_0^T K_1(l,t)u^N(l,t)\Phi^{N'}(0,t)\,dt$$

$$- \int_0^T K_1(0,t)u^N(l,t)\Phi^{N'}(0,t)\,dt. \tag{19.24}$$

Since $\Phi^{N'}$, $\Phi_t^{N'}$, $\Phi_x^{N'} \in L_2(Q_T)$, the subsequence $\{u^{N_m}(x,t)\}$ converges weakly in $L_2(Q_T)$, so it is possible to pass to the limit in (19.24) as $m \to \infty$ for any fixed $\Phi^{N'}$. Thus, for any $u(x,t) \in V_2(Q_T)$ the following identity holds

$$\int_0^T \Big(-(u, \Phi_t^{N'}) + (u_x, \Phi_x^{N'}) - (cu, \Phi^{N'})\Big)$$

$$= \int_0^l \varphi(x)\Phi^{N'}(x,0)\,dx + \int_0^T \int_0^l R_1(x,t)u(x,t)\,dx\,\Phi^{N'}(0,t)\,dt$$

$$- \int_0^T \int_0^l R_2(x,t)u(x,t)\,dx\,\Phi^{N'}(l,t)\,dt - \int_0^T K_2(0,t)u(0,t)\Phi^{N'}(l,t)\,dt$$

$$+ \int_0^T K_2(l,t)u(l,t)\Phi^{N'}(l,t)\,dt - \int_0^T K_1(l,t)u(l,t)\Phi^{N'}(0,t)\,dt$$

$$+ \int_0^T K_1(0,t)u(l,t)\Phi^{N'}(0,t)\,dt. \tag{19.25}$$

Denote $\Phi = \bigcup_{N'=1}^{\infty} \Phi^{N'}$. The set Φ is dense in $W_2^1(Q_T)$ and hence, there exists a function $\Phi(x,t) \in W_2^1(Q_T)$ that is the limit of the sequence $\Phi^{N'}$. Finally, we conclude that the relation (19.25) holds for all functions $\Phi(x,t) \in W_2^1(Q_T)$ and therefore, there exists the solution $u(x,t) \in V_2(Q_T)$ to the problem (19.1), (19.2), (19.5), (19.6) in sense of Definition 19.1. Assume that there exist two different generalized solutions $u_1(x,t)$, $u_2(x,t) \in V_2(Q_T)$ to the problem (19.1), (19.2), (19.5), (19.6). Then

$$u = u_1 - u_2 \in V_2(Q_T)$$

satisfies the following identity

$$\begin{aligned}
&\int\limits_{Q_T} (-u\eta_t + u_x\eta_x - cu\eta)\, dx\, dt \\
&\quad = \int\limits_{Q_T} (R_1(x,t)\eta(l,t) - R_2(0,t)\eta(0,t))\, u(x,t)\, dx\, dt \\
&\qquad + \int\limits_0^T (K_1(0,t)\eta(0,t) - K_2(0,t)\eta(l,t))\, u(0,t)\, dt \\
&\qquad + \int\limits_0^T (K_2(l,t)\eta(l,t) - K_1(l,t)\eta(0,t))\, u(l,t)\, dt. \qquad (19.26)
\end{aligned}$$

We take

$$\eta(x,t) = \begin{cases} 0, & b \le t \le T, \\ \int\limits_b^t u(x,\tau)\, d\tau, & 0 \le t \le b, \end{cases}$$

where $b \in [0,T]$ is arbitrary. Note that

$$\eta(x,t) \in W_2^1(Q_T), \eta(x,T) = 0$$

and since $\eta_{xt} = u_x$, so $\eta_{xt} \in L_2(Q_T)$. We substitute $\eta(x,t)$ into (19.26) and express u, u_x in terms of η. Then (19.26) becomes

$$\begin{aligned}
&\int_{Q_b} \left(-\eta_t^2 + \eta_{tx}\eta_x - c\eta\eta_t\right)\, dx\, dt \\
&\quad = \int_{Q_b} (R_1(x,t)\eta(l,t) - R_2(0,t)\eta(0,t))\, \eta_t(x,t)\, dx\, dt \\
&\qquad + \int_0^b (K_1(0,t)\eta(0,t) - K_2(0,t)\eta(l,t))\, \eta_t(0,t)\, dt \\
&\qquad + \int_0^T (K_2(l,t)\eta(l,t) - K_1(l,t)\eta(0,t))\, \eta_t(l,t)\, dt. \qquad (19.27)
\end{aligned}$$

Integrating by parts in (19.27) we obtain

$$\frac{1}{2}\int_0^l \eta_x^2(x,0)\,dx + \int_{Q_b} \eta_t^2\,dx\,dt$$
$$= \frac{1}{2}\left(-\int_0^l c(x,0)\eta^2(x,0)\,dx + \int_{Q_b} c_t\eta^2\,dt\right)$$
$$+ \int_{Q_b} \left(-R_1(x,t)\eta(l,t) + R_2(0,t)\eta(0,t)\right)\eta_t(x,t)\,dx\,dt$$
$$+ \frac{1}{2}K_1(0,0)\eta^2(0,0) + \frac{1}{2}\int_0^b (K_1(0,t))_t\eta^2(0,t)\,dt$$
$$+ \frac{1}{2}K_2(l,0)\eta^2(l,0) + \frac{1}{2}\int_0^b (K_2(l,t))_t\eta^2(l,t)\,dt$$
$$- K_2(0,0)\eta(l,0)\eta(0,0) - \int_0^b K_2(0,t)\eta_t(l,t)\eta(0,t)\,dt$$
$$+ \int_0^b K_1(l,t)\eta(l,t)_t\eta(0,t)\,dt - \int_0^b (K_2(0,t))_t\eta(l,t)\eta(0,t)\,dt. \quad (19.28)$$

Under the assumptions $(A1)$–$(A2)$ from (19.28) we obtain the following estimate

$$\int_0^l \eta_x^2(x,0)\,dx + \int_{Q_b} \eta_t^2\,dx\,dt \le (c_1 b + P_1 b^2)\int_{Q_b} \eta_t^2\,dx\,dt + P_2\int_{Q_b} \eta_x^2\,dx\,dt,$$

where

$$P_1 = 2 + c_2 + \frac{k_3 + 2k_4}{l},\ P_2 = 2(2l^2 + k_3 l + 3k_4 l).$$

Since $b > 0$ is an arbitrary, so let b be such that $1 - c_1 b - Cb^2 > 0$ and in particular,

$$1 - c_1 b - Cb^2 \ge \frac{1}{2}.$$

That is,

$$b \in [0, \nu]), \ \nu = \frac{-c_1 + \sqrt{c_1^2 + 2C}}{2C}.$$

Then for all $b \in [0, \nu]$

$$\int\limits_0^l \eta_x^2(x,0)\,dx + \frac{1}{2}\int\limits_{Q_b} \eta_t^2\,dx\,dt \le P_2 \int\limits_{Q_b} \eta_x^2\,dx\,dt. \tag{19.29}$$

Define the function $u(x,t) = \int\limits_0^t u(x,\tau)\,d\tau$. Then $\eta(x,t) = y(x,t) - y(x,b)$ for $t \in [0, b]$ and (19.29) can be represented as

$$\int\limits_0^l y_x^2(x,b)\,dx + \frac{1}{2}\int\limits_{Q_b} y_t^2\,dx\,dt \le P_2 \int\limits_{Q_b} (y(x,t) - y(x,b))_x^2\,dx\,dt,$$

which implies that

$$\begin{aligned}\int\limits_0^l y_x^2(x,b)\,dx &\le P_2 \int\limits_{Q_b} (y(x,t) - y(x,b))_x^2\,dx\,dt \\ &\le 2P_2 \int\limits_{Q_b} y_x^2(x,t)\,dx\,dt + 2P_2 b \int_0^l y_x^2(x,b)\,dx.\end{aligned}$$

In particular, for $b \le \dfrac{1}{4P_2}$ we obtain

$$\int\limits_0^l y_x^2(x,b)\,dx \le 4P_2 \int\limits_{Q_b} y_x^2(x,t)\,dx\,dt. \tag{19.30}$$

The estimate (19.30) is valid for all $b \in [0, b_1]$, where $b_1 = \min\left\{\dfrac{1}{4P_2}, \nu\right\}$. From (19.30), Gronwall's lemma and the condition $y_x(x,0) = 0$ we obtain that $y_x^2(x,b) = 0$ for all $b \in [0, b_1]$. And hence, $\eta_x(x,t) = 0$, $t \in [0, b_1]$. Then from (19.30) it follows that $\eta_t(x,t) = u(x,t) = 0$, $t \in [0, b_1]$. Likewise, we repeat the above arguments and obtain $u(x,t) = 0$ for all $t \in [b_1, 2b_1]$ and so on. Finally, we conclude that $u(x,t) = 0$ in Q_T that in turn implies uniqueness of the solution to the problem (19.1), (19.2), (19.5), (19.6).

References

1. A.M. Abdrakhmanov, Solvability of a boundary-value problem with an integral boundary condition of the second kind for equations of odd order. Math. Notes **88**(1), 151–159 (2010)
2. A. Ashyralyev, A. Sarsenbi, Well-posedness of a parabolic equation with nonlocal boundary condition. Bound. Value Probl. **2015**, 38 (2015)
3. G. Avalishvili, M. Avalishvili, D. Gordeziani, On integral nonlocal boundary value problems for some partial differential equations. Bull. Georgian Natl. Acad. Sci. **5**(1), 31–37 (2011)
4. A. Bouziani, Solution forte d'un problem mixte avec condition non locales pour une classe d'equations hyperboliques. Bull. Cl. Sci. Acad. R. Belg. **8**, 53–70 (1997)
5. A. Bouziani, Initial-boundary value problems for a class of pseudoparabolic equations with integral boundary conditions. J. Math. Anal. Appl. **291**, 371–386 (2004)
6. J.R. Cannon, The solution of the heat equation subject to the specification of energy. Quart. Appl. Math. **21**(2), 155–160 (1963)
7. J.R. Cannon, Y. Lin, An inverse problem of finding a parameter in a semi-linear heat equation. J. Math. Anal. Appl. **145**, 470–484 (1990)
8. J.R. Cannon, Y. Lin, S. Wang, Determination of a control parameter in a parabolic partial differential equation. J. Austral. Math. Soc. Ser. B **33**, 149–163 (1991)
9. O. Danilkina, On a nonlocal problem with the second kind integral condition for a parabolic equation. Int. J. Part. Differ. Equ. Appl. **2**(4), 62–67 (2014)
10. O. Danilkina, On a certain nonlocal problem for a heat equation. Glob. J. Math. Anal. **2**(4), 235–242 (2014)
11. O. Danilkina, On one non-classical problem for a multidimensional parabolic equation. J. Concr. Appl. Math. **4**(13), 266–278 (2015)
12. D.G. Gordeziani, G.A. Avalishvili, Solutions of nonlocal problems for one-dimensional oscillations of the medium. Mat. Modelir. **12**(1), 94–103 (2000)
13. B. Jumarhon, S. McKee, On the heat equation with nonlinear and nonlocal boundary conditions. J. Math. Anal. Appl. **190**, 806–820 (1995)
14. L.I. Kamynin, On a boundary problem in the theory of heat conduction with a non-classical boundary conditions. Zh. Vychisl. Math. Math. Fiz. **4**(6), 1006–1024 (1964)
15. A.N. Kolmogorov, S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Dover Publications, New York, 1999)
16. A.I. Kozhanov, Parabolic equations with an unknown time-dependent coefficient. Comput. Math. Math. Phys. **12**, 2085–2101 (2005)
17. A.I. Kozhanov, L.S. Pulkina, On the solvability of boundary value problems with a nonlocal boundary condition of integral form for multidimensional hyperbolic equations. Differ. Equ. **42**(9), 1233–1246 (2006)
18. O.A. Ladyzhenskaya, *Boundary-Value Problems of Mathematical Physics* (Nauka, Moscow, 1973)
19. A. Merad, A. Bouziani, Numerical solutions of the hyperbolic equation with purely integral condition by using Laplace transform method. Palest. J. Math. **4**(1), 30–36 (2015)
20. S. Mesloub, A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. J. Math. Anal. Appl. **316**, 189–209 (2006)
21. L.S. Pulkina, A mixed problem with integral condition for the hyperbolic equation. Math. Notes **74**(3), 411–421 (2003)
22. L.S. Pulkina, A nonlocal problem with integral conditions for a hyperbolic equation. Differ. Equ. **40**(7), 947–953 (2004)
23. L.S. Pulkina, Initial-boundary value problem with a nonlocal boundary condition for a multidimensional hyperbolic equation. Differ. Equ. **44**(8), 1119–1125 (2008)
24. L.S. Pulkina, A nonlocal problem with integral conditions for the hyperbolic equation. Nanosyst.: Phys. Chem. Math. **2**(4), 61–70 (2011)
25. L.S. Pulkina, Boundary value problems for a hyperbolic equation with nonlocal conditions of the I and II kind. Russ. Math. (Iz. VUZ) **56**(4), 62–69 (2012)

26. J.M. Rassias, E.T. Karimov, Boundary-value problems with non-local initial condition for parabolic equations with parameter. Eur. J. Pure Appl. Math. **3**(6), 948–957 (2010)
27. A.A. Samarskii, On certain problems of the modern theory of differential equations. Differ. Equ. **16**(11), 1221–1228 (1980)
28. R.V. Shamin, Nonlocal parabolic problems with the support of nonlocal terms inside a domain. Funct. Differ. Equ. **10**(1–2), 307–314 (2003)
29. M. Tadi, M. Radenkovic, A numerical method for 1-D parabolic equation with nonlocal boundary conditions. Int. J. Comput. Math. **2014**, Article ID 923693, 9 (2014)