On Error Sum Functions for Approximations with Arithmetic Conditions

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Dedicated to the memory of Professor Wolfgang Schwarz

Abstract Let $\mathcal{E}_{k,l}(\alpha) = \sum_{q_m \equiv l \pmod{k}} |q_m\alpha - p_m|$ be error sum functions formed by convergents p_m/q_m $(m \geq 0)$ of a real number α satisfying the arithmetical condition $q_m \equiv l \pmod{k}$ with $0 \leq l < k$. The functions $\mathcal{E}_{k,l}$ are Riemann-integrable on $[0, 1]$, so that the integrals $\int_0^1 \mathcal{E}_{k,l}(\alpha)\, d\alpha$ exist as the arithmetical means of the functions $\mathcal{E}_{k,l}$ on $[0, 1]$. We express these integrals by multiple sums on rational terms and prove upper and lower bounds. In the case when l vanishes (i.e. k divides q_m) and when the smallest prime divisor p_1 of $k = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ satisfies $p_1 > k^{\varepsilon}$ for some positive real number ε, we have found an asymptotic expansion in terms of k, namely $\int_0^1 \mathcal{E}_{k,0}(\alpha)\, d\alpha = \zeta(2)\big(2\zeta(3)k^2\big)^{-1} + \mathcal{O}\big(3^t k^{-2-\varepsilon}\big)$. This result includes all integers k which are of the form $k = p^a$ for primes p and integers $a \geq 1$.

Keywords Approximation with arithmetical conditions • Continued fractions • Convergents • Error sum functions

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1 Introduction

There are many results in the literature concerned with rational approximations p/q to irrational numbers, where p and q are restricted by additional arithmetical conditions. An important result in this direction is due to Uchiyama [11].

Theorem A *For every real irrational number α and integers $s > 1$, $a \geq 0$, $b \geq 0$ such that a and b are not simultaneously divisible by s, there are infinitely many*

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integers p and q $\neq 0$ *satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{s^2}{4q^2}, \quad p \equiv a \pmod{s}, \quad q \equiv b \pmod{s}.$$

In [3] the author proved that the constant 1/4 in Uchiyama's paper cannot be improved. Let $\|\eta\|$ denote the distance of a real number η from the nearest integer. Then we deduce the following corollary from Theorem A:

Corollary A1 *Let* $f : \mathbb{N} \to \mathbb{R}_{>0}$ *be a function satisfying* $f(q) = o(q)$ *for positive integers q tending to infinity. Then, for every integers* $s > 0$, $a \geq 0$ *and every real irrational number* α *we have*

$$\liminf_{\substack{q>0 \\ q \equiv a \pmod{s}}} f(q)\|q\alpha\| = 0.$$

In particular cases stronger results are possible, e.g., for the number $e = \exp(1)$ by Theorem 1.3 in [4].

Theorem B *Let a and s be arbitrary positive integers. Then*

$$\liminf_{\substack{q>0 \\ q \equiv a \pmod{s}}} q\|qe\| = 0.$$

About 5 years later Komatsu [9, Theorem 4] showed that the result of Theorem B remains true for e replaced by every number $e^{1/k}$ ($k \in \mathbb{N}$).

Recently, the author [5] studied the so-called *error sum functions*. Let

$$\mathcal{E}(\alpha) := \sum_{m=0}^{\infty} |q_m\alpha - p_m|, \qquad \mathcal{E}^*(\alpha) := \sum_{m=0}^{\infty} (q_m\alpha - p_m),$$

where for $m \geq 0$ the fraction p_m/q_m is the m-th convergent of the real number α. The numbers p_m and q_m can be computed recursively from the continued fraction expansion of α. Various aspects of these functions have been investigated in [5–7], among them it is shown that $0 \leq \mathcal{E}(\alpha) \leq (1+\sqrt{5})/2$ and $0 \leq \mathcal{E}^*(\alpha) \leq 1$ for all real numbers α. Both, $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$, measure the average of error terms for diophantine approximations of α by rationals. Moreover, $\mathcal{E}(\alpha) \in \mathbb{Q}(\alpha)$ holds for real numbers of algebraic degree 1 and 2. For $e = \exp(1)$ we have the formula

$$\mathcal{E}(e) = 2e\int_0^1 \exp(-t^2)\,dt - e = 1.3418751\ldots,$$

which proves that $\mathcal{E}(e) \notin \mathbb{Q}(e)$. The function $\mathcal{E}(\alpha)$ is continuous for every real irrational point α, and discontinuous for all rational numbers α (see [6, Theorem 2]).

Therefore, the function $\mathcal{E}$ is Riemann-integrable on [0, 1]. It turns out [6, Theorem 5] that

$$\int_0^1 \mathcal{E}(\alpha)\,d\alpha \;=\; -\frac{5}{8}+\frac{3\zeta(2)\log 2}{2\zeta(3)} \;=\; 0.79778798\ldots,$$

where ζ denotes the Riemann Zeta function. This integral represents the arithmetical mean of the function $\mathcal{E}$ on [0, 1]. This result can be generalized. Let $n = 1, 2, 3, \ldots$ and

$$I_n \;:=\; \int_0^1 \sum_{m=0}^{\infty} \left|q_m\alpha - p_m\right|^n d\alpha\,.$$

It can be shown [6, Sec. 4] that

$$I_n \;=\; \frac{1}{n+1}\left(1-\frac{1}{2^{n+1}}-\frac{2\zeta(n+1,-1)}{\zeta(n+2)}\right),$$

where

$$\zeta(n+1,-1) \;:=\; \sum_{m_2>m_1>0}\frac{(-1)^{m_1}}{m_1 m_2^{n+1}} \;=\; \sum_{m_2=1}^{\infty}\frac{1}{m_2^{n+1}}\sum_{m_1=1}^{m_2-1}\frac{(-1)^{m_1}}{m_1}$$

is known as multiple Zeta function. Borwein et al. [2] expressed $\zeta(n+1,-1)$ in terms of $\log 2, \zeta(2), \zeta(3), \ldots, \zeta(n+2)$. Thus we obtain the following results:

Theorem C *Let $n \geq 1$ be an integer. Then we have*

$$\int_0^1 \sum_{m=0}^{\infty}\left|q_m\alpha-p_m\right|^n d\alpha = \frac{1}{n+1}\left(1-\frac{1}{2^{n+1}}\right)-1+\frac{4\log(2)\,\zeta(n+1)}{(n+1)\zeta(n+2)}\left(1-\frac{1}{2^{n+1}}\right)$$
$$+\frac{1}{(n+1)\zeta(n+2)}\sum_{k=1}^{n-1}\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{n-k}}\right)\zeta(k+1)\zeta(n+1-k)\,.$$

In particular, for $n = 2, 3, 4$ we have the identities

$$\int_0^1 \sum_{m=0}^{\infty}\left|q_m\alpha-p_m\right|^2 d\alpha = \frac{7\log 2\,\zeta(3)}{6\zeta(4)}+\frac{\zeta^2(2)}{12\zeta(4)}-\frac{17}{24} \;=\; 0{,}39813\ldots,$$

$$\int_0^1 \sum_{m=0}^{\infty}\left|q_m\alpha-p_m\right|^3 d\alpha = \frac{15\log 2\,\zeta(4)}{16\zeta(5)}+\frac{3\zeta(2)\zeta(3)}{16\zeta(5)}-\frac{49}{64} \;=\; 0{,}27019\ldots,$$

$$\int_0^1 \sum_{m=0}^{\infty}\left|q_m\alpha-p_m\right|^4 d\alpha = \frac{31\log 2\,\zeta(5)}{40\zeta(6)}+\frac{7\zeta(2)\zeta(4)}{40\zeta(6)}+\frac{9\zeta^2(3)}{80\zeta(6)}-\frac{129}{160} \;=\; 0{,}20731\ldots.$$

Taking $\zeta(2s) \in \mathbb{Q}(\pi)$ into account, it follows that

$$\begin{aligned}
I_1 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3)\big)\,,\\
I_2 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3)\big)\,,\\
I_3 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3), \zeta(5)\big)\,,\\
I_4 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3), \zeta(5)\big)\,,\\
I_5 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3), \zeta(5), \zeta(7)\big)\,,\\
I_6 &\in \mathbb{Q}\big(\pi, \log(2), \zeta(3), \zeta(5), \zeta(7)\big)\,;
\end{aligned}$$

in particular we know $I_1, \ldots, I_6 \in \mathbb{Q}\big(\pi, \log(2), \zeta(3), \zeta(5), \zeta(7)\big)$. This proves that $I_1, \ldots, I_6$ are algebraically dependent over $\mathbb{Q}$. But indeed a stronger result holds, which can be verified using a suitable computer algebra system.

Corollary C1 *The numbers I_1, I_2, I_3, I_4 are algebraically dependent over $\mathbb{Q}$. For $x_i = I_i$ $(i = 1, 2, 3, 4)$ the algebraic equation*

$$\begin{aligned}
0 = {} & 10240x_1x_3x_4 - 2976x_1^2x_2 - 1488x_1^2 - 5952x_1x_2 + 5120x_1x_3 + 7840x_1x_4\\
& -2592x_2x_3 + 6400x_3x_4 + 944x_1 - 4542x_2 + 1904x_3 + 4900x_4 + 179
\end{aligned}$$

holds.

The proof works by substituting the above expressions for I_1, I_2, I_3, I_4 into the equation given in the corollary, where additionally

$$\zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \qquad \zeta(6) = \frac{\pi^6}{945}$$

must be taken into account.

Note that

$$I_1, \ldots, I_{2n} \in \mathbb{Q}\big(\underbrace{\pi, \log(2), \zeta(3), \zeta(5), \ldots, \zeta(2n+1)}_{n+2}\big)\,.$$

This proves

Corollary C2 *For every integer $n \geq 3$ any $n+3$ numbers from the set $\{I_1, I_2, \ldots I_{2n}\}$ are algebraically dependent over $\mathbb{Q}$.*

In this paper we focus our interest on a generalized error sum function. Let α be a real number, and let $k \geq 1$ and $0 \leq l < k$ be integers. We define

$$\mathcal{E}_{k,l}(\alpha) := \sum_{\substack{m=0\\ q_m \equiv l \pmod{k}}}^{\infty} |q_m\alpha - p_m|\,,$$

in particular we set $\mathcal{E}_k(\alpha) = \mathcal{E}_{k,0}(\alpha)$. It is clear that $\mathcal{E}_1(\alpha) = \mathcal{E}(\alpha)$, and

$$\sum_{l=0}^{k-1} \mathcal{E}_{k,l}(\alpha) \;=\; \mathcal{E}(\alpha) \qquad (k \geq 1)\,.$$

For $k > 1$ the error sum function $\mathcal{E}_k(\alpha)$ can be transformed into a more striking form. Since k does not divide $q_0 = 1$, the term for $m = 0$ in $\mathcal{E}_k(\alpha)$ does not occur. Moreover, for the convergents p_m/q_m $(m \geq 1)$ of α satisfying $q_m \equiv 0 \pmod{k}$ we obtain the inequalities

$$|q_m\alpha - p_m| \;\leq\; \frac{1}{q_m} \;\leq\; \frac{1}{k} \;\leq\; \frac{1}{2}\,.$$

This proves

$$\mathcal{E}_k(\alpha) \;=\; \sum_{\substack{m\,=\,1\\ k|q_m}}^{\infty} \|q_m\alpha\| \qquad (k > 1)\,. \tag{1.1}$$

We continue to point out more basic properties of $\mathcal{E}_k(\alpha)$ for $k > 1$. Since q_m and q_{m+1} are coprime, at most every second term in $\mathcal{E}_k(\alpha)$ does not vanish. So we obtain the following upper bound for $\mathcal{E}_k(\alpha)$:

$$\mathcal{E}_k(\alpha) \;=\; \sum_{\substack{m\,=\,1\\ k|q_m}}^{\infty} |q_m\alpha - p_m| \;\leq\; \sum_{m=0}^{\infty} |q_{2m+1}\alpha - p_{2m+1}| \;=\; \frac{\mathcal{E}(\alpha) - \mathcal{E}^*(\alpha)}{2} \qquad (k > 1)\,.$$

The identities

$$\mathcal{E}_k(\alpha) \;=\; \frac{\mathcal{E}(\alpha) - \mathcal{E}^*(\alpha)}{2} \;=\; \alpha$$

with $k > 1$ hold for all numbers α given by their continued fraction expansion

$$\alpha \;=\; [0; k, 1, k, 1, k, 1, \dots] \;=\; [0; \overline{k, 1}] \;=\; \sqrt{\frac{1}{4} + \frac{1}{k}} - \frac{1}{2}\,,$$

since for $m \geq 0$ we have the congruence relations $q_{2m+1} \equiv 0 \pmod{k}$ and $q_{2m} \equiv 1 \pmod{k}$. Moreover,

$$\mathcal{E}_k(\alpha) \;=\; \sum_{m=0}^{\infty} |q_{2m+1}\alpha - p_{2m+1}| \;=\; \sum_{m=0}^{\infty} \left(\frac{k+2}{2} - \sqrt{\left(\frac{k+2}{2}\right)^2 - 1} \right)^{m+1} \;=\; \alpha\,.$$

There exist real irrational numbers α for which the series $\mathcal{E}_k(\alpha)$ consists of at most finitely many terms, contrary to the series $\mathcal{E}(\alpha)$. To prove the existence of such an irrational number, we define α recursively by its continued fraction expansion $\alpha = [0; a_1, a_2, \dots] = [0; 2, 1, 1, 2, 2, 4, 6, \dots]$ as follows. We have

$$q_1 = 2\,, \quad q_2 = 3\,, \quad q_3 = 5\,, \quad q_4 = 13\,, \quad q_5 = 31\,, \quad q_6 = 137\,, \quad q_7 = 853\,.$$

Now let us assume that for $m \geq 8$ the denominators q_{m-1} and q_{m-2} are primes. Then, by the Dirichlet prime number theorem, there are infinitely many positive integers a such that $q_m = aq_{m-1} + q_{m-2} \in \mathbb{P}$. The number a_m is uniquely defined by the smallest positive integer a satisfying this condition. Then, for *every* integer $k > 1$, the series $\mathcal{E}_k(\alpha)$ consists of at most one term. Furthermore, there are many situations in which $\mathcal{E}_k(\alpha)$ vanishes.

Proposition 1.1 *For every integer $k > 1$ there are uncountably many irrational numbers α such that $\mathcal{E}_k(\alpha) = 0$.*

To prove this proposition, let $k > 1$ be any integer. We define an irrational number α depending on k and on a sequence $(b_n)_{n\geq 2}$ of positive integers by

$$\alpha = [0; 1, kb_2, kb_3, kb_4, \dots] = [0; a_1, a_2, a_3, \dots]\,.$$

The denominators q_m of the convergents p_m/q_m of α satisfy the recurrence formula

$$q_0 = 1\,, \quad q_1 = a_1 = 1\,, \quad q_{m+2} = a_{m+2}q_{m+1} + q_m \quad (m = 0, 1, 2, \dots)\,.$$

Since $q_{m+2} \equiv q_m \pmod{k}$ for $m = 0, 1, 2, \dots$ it follows recursively that $1 \equiv q_0 \equiv q_2 \equiv q_4 \equiv \cdots \equiv q_{2m} \pmod{k}$ and, similarly, $1 \equiv q_1 \equiv q_3 \equiv q_5 \equiv \cdots \equiv q_{2m+1} \pmod{k}$ for $m = 0, 1, 2, \dots$. This proves that no denominator q_m is divisible by k. Hence, $\mathcal{E}_k(\alpha) = 0$. By Cantor's counting principle we have found uncountably many real numbers α satisfying $\mathcal{E}_k(\alpha) = 0$.

The main goal of this paper is to study the behaviour of the numbers $\int_0^1 \mathcal{E}_{k,l}(\alpha)\, d\alpha$ depending on k and l. For $l = 0$ and k restricted to those numbers having *no small prime divisors* we prove the asymptotic behaviour of these integrals for k tending to infinity (Theorem 2.1 and Corollaries 2.2–2.5). For integers k having *many small prime divisors* the numbers $\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha$ tend more quickly to zero than in the case $k = p^a$ for fixed $a \geq 1$ and primes p (Theorem 2.6 and Corollary 2.4). The integrals on the error sum functions $\mathcal{E}_{k,l}$ with $l > 0$ are treated in Theorem 2.7.

2 Statement of the Results

Let $\mu : \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function, and let $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for $s \geq 2$ be the Riemann Zeta function. By $J_3 : \mathbb{N} \to \mathbb{N}$ we denote Jordan's arithmetical function defined by $J_3(1) = 1$ and

$$J_3(n) = n^3 \prod_{p|n} \Big(1 - \frac{1}{p^3}\Big) \qquad (n > 1), \tag{2.1}$$

where p runs through all prime divisors of n. Moreover, for any integer n let $\mathcal{D}_n$ denote the set of all positive divisors of n. For every positive integer r we define the number T_r by

$$T_r := \sum_{n=r}^{\infty} \sum_{1 \leq m \leq n/r} \frac{(-1)^{m+1}}{mn^2}. \tag{2.2}$$

The identity from the following theorem can be considered as the main result of this paper, which contrasts with the property of the function $\mathcal{E}_k(\alpha)$ given by Proposition 1.1.

Theorem 2.1 *For every integer $k > 1$ we have*

$$\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha = \frac{1}{\zeta(3)} \sum_{r \in \mathcal{D}_k} \sum_{s \in \mathcal{D}_r} \frac{\mu(s)\mu(ks/r)T_r}{rJ_3(ks/r)}.$$

Corollary 2.2 *Let $k > 1$ be any integer having t prime divisors , where P denotes the smallest prime divisor of k. Then we have*

$$\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha = \frac{\zeta(2)}{2\zeta(3)k^2} + \mathcal{O}\Big(\frac{1}{k^3} + \frac{3^t}{k^2 P}\Big).$$

Corollary 2.3 *For all primes p we have*

$$\int_0^1 \mathcal{E}_p(\alpha)\, d\alpha = \frac{1}{p^3 - 1}\Big(\frac{p^2 T_p}{\zeta(3)} - \frac{3\zeta(2)\log 2}{2\zeta(3)} + \frac{1}{4}\Big) = \frac{\zeta(2)}{2\zeta(3)p^2} + \mathcal{O}\Big(\frac{1}{p^3}\Big)$$

and

$$\frac{2}{77p^2} < \int_0^1 \mathcal{E}_p(\alpha)\, d\alpha < \frac{97}{109p^2} \qquad (p \geq 3).$$

Corollary 2.4 *Let p be a prime and a be a positive integer. Set $k := p^a$. Then we have*

$$\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \;=\; \frac{\zeta(2)}{2\zeta(3)k^2} + \mathcal{O}\Big(\frac{1}{k^{2+1/a}}\Big).$$

Corollary 2.5 *Let $k > 1$ be an integer having at most t prime divisors. The smallest prime divisor P of k satisfies $P > k^{\varepsilon}$ for any $0 < \varepsilon < 1$. Then we have*

$$\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \;=\; \frac{\zeta(2)}{2\zeta(3)k^2} + \mathcal{O}\Big(\frac{3^t}{k^{2+\varepsilon}}\Big).$$

To state the results in the subsequent theorems we need Euler's totient φ.

Theorem 2.6 *For every integer $k \geq 3$ we have*

$$\frac{1}{k^2 \log\log k} \;\ll\; \frac{\varphi(k)}{4k^3} \;<\; \int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \;<\; \frac{\zeta(2)}{k^2}.$$

For the numbers $k = p_1p_2\cdots p_r$ given by the product on the first $r \geq 2$ primes $p_1 = 2, p_2 = 3, \ldots$ we have

$$\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \;\asymp\; \frac{1}{k^2 \log\log k}.$$

Theorem 2.6 shows that $\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \asymp k^{-2}$ does not hold for $k \in \mathbb{N}$ tending to infinity. In the following theorem we estimate the integral on the error sum function $\mathcal{E}_{k,l}(\alpha)$ for $l > 0$, where the case $l = 1$ is treated separately. By (a, b) we denote the greatest common divisor of two integers a and b.

Theorem 2.7 (i) *For every integer $k \geq 2$ and $l = 1$ we have*

$$\frac{5}{8} + \frac{\varphi(k+1)}{4(k+1)^3} < \int_0^1 \mathcal{E}_{k,1}(\alpha)\,d\alpha = \sum_{\substack{a=1 \\ a \equiv 1 \pmod{k}}}^{\infty} \; \sum_{\substack{b=0 \\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} - \frac{3}{8} \leq \frac{5}{8} + \frac{\zeta(2)}{k^2}.$$

(ii) *For integers $k \geq 3$ and $2 \leq l < k$ we have*

$$\frac{\varphi(l)}{4l^3} < \int_0^1 \mathcal{E}_{k,l}(\alpha)\,d\alpha = \sum_{\substack{a=1 \\ a \equiv l \pmod{k}}}^{\infty} \; \sum_{\substack{b=0 \\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} \leq \frac{1}{l^2} + \frac{\zeta(2)}{k^2}.$$

For two consecutive primes p_{r-1} and p_r ($r \geq 2$) it follows from *(ii)* in Theorem 2.7 by Bertrand's Postulate and Theorem 9 in [8] that

$$\int_0^1 \mathcal{E}_{p_r,p_{r-1}}(\alpha)\, d\alpha \asymp \frac{1}{p_r^2} \asymp \frac{1}{r^2 \log^2 r}.$$

3 Auxiliary Results

Lemma 3.1 *Let $k > 1$ be an integer, and let r be any positive divisor of k. Then we have the identity*

$$\sum_{\substack{d=1 \\ (d,k)=k/r}}^{\infty} \frac{\mu(d)}{d^3} = \frac{1}{\zeta(3)} \sum_{s\in\mathcal{D}_r} \frac{\mu(s)\mu(ks/r)}{J_3(ks/r)}.$$

Proof We obtain

$$S := \sum_{\substack{d=1 \\ (d,k)=k/r}}^{\infty} \frac{\mu(d)}{d^3} = \sum_{\substack{m=1 \\ (m,r)=1}}^{\infty} \frac{\mu(mk/r)}{(mk/r)^3} = \left(\frac{r}{k}\right)^3 \sum_{m=1}^{\infty} \sum_{\substack{s\geq 1 \\ s|(m,r)}} \frac{\mu(s)\mu(mk/r)}{m^3}$$

$$= \left(\frac{r}{k}\right)^3 \sum_{\substack{s\geq 1 \\ s|r}} \mu(s) \sum_{\substack{m=1 \\ m\equiv 0 \pmod s}}^{\infty} \frac{\mu(mk/r)}{m^3} = \left(\frac{r}{k}\right)^3 \sum_{s\in\mathcal{D}_r} \frac{\mu(s)}{s^3} \sum_{m=1}^{\infty} \frac{\mu(mks/r)}{m^3}.$$

For any positive integer t we have

$$\sum_{m=1}^{\infty} \frac{\mu(mt)}{m^3} = \mu(t) \sum_{\substack{m=1 \\ (m,t)=1}}^{\infty} \frac{\mu(m)}{m^3} = \frac{\mu(t)t^3}{\zeta(3)J_3(t)},$$

where the identity on the right-hand side can be obtained by using the method explained in [10]. Substituting the last expression into S by setting $t = ks/r$, we complete the proof of the desired identity from the lemma. □

Lemma 3.2 *For every positive integer r we have*

$$T_r = \sum_{n=1}^{\infty}\sum_{m=0}^{nr-1}\frac{1}{n(nr+m)^2}, \qquad T_1 = \frac{3}{2}\zeta(2)\log 2 - \frac{1}{4}\zeta(3),$$

$$T_r = \frac{\zeta(2)}{2r} + \mathcal{O}\Big(\frac{1}{r^2}\Big), \qquad \textit{and} \qquad \frac{1}{2r} < T_r < \begin{cases} \dfrac{1}{r-1} & \textit{if } r \geq 2, \\ 1 & \textit{if } r \geq 1. \end{cases}$$

In particular, we have $T_r < 2/r$ for all $r \geq 1$.

Proof Let $r \geq 1$ be an integer. To prove the alternative expression of T_r, we first observe that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{nr-1}\frac{1}{n(nr+m)^2} = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=nr}^{2nr-1}\frac{1}{k^2} = \sum_{k=r}^{\infty}\frac{1}{k^2}\sum_{n=\lfloor k/2r\rfloor+1}^{\lfloor k/r\rfloor}\frac{1}{n}, \tag{3.1}$$

where the last identity follows by interchanging the order of summation, and where $\lfloor\eta\rfloor$ denotes the floor function, i.e. the greatest integer not exceeding η. Next, let $\beta \geq 1$ be a real number, and

$$\delta := \begin{cases} 1, \text{ if } \lfloor\beta\rfloor \equiv 1 \pmod 2, \\ 0, \text{ if } \lfloor\beta\rfloor \equiv 0 \pmod 2. \end{cases}$$

Then we have

$$\sum_{n=1}^{\lfloor\beta\rfloor}\frac{1}{n} = \sum_{m=1}^{\lfloor\beta/2\rfloor+\delta}\frac{1}{2m-1} + \sum_{m=1}^{\lfloor\beta/2\rfloor}\frac{1}{2m},$$

which yields, equivalently,

$$\begin{aligned}\sum_{n=\lfloor\beta/2\rfloor+1}^{\lfloor\beta\rfloor}\frac{1}{n} &= \sum_{m=1}^{\lfloor\beta/2\rfloor+\delta}\frac{1}{2m-1} + \frac{1}{2}\sum_{m=1}^{\lfloor\beta/2\rfloor}\frac{1}{m} - \sum_{n=1}^{\lfloor\beta/2\rfloor}\frac{1}{n} \\ &= \sum_{m=1}^{\lfloor\beta/2\rfloor+\delta}\frac{1}{2m-1} - \frac{1}{2}\sum_{m=1}^{\lfloor\beta/2\rfloor}\frac{1}{m} \\ &= \sum_{m=1}^{\lfloor\beta\rfloor}\frac{(-1)^{m+1}}{m}.\end{aligned}$$

With $\beta = k/r$ for $k \geq r$ we conclude from (3.1) and (2.2) that

$$\sum_{n=1}^{\infty} \sum_{m=0}^{nr-1} \frac{1}{n(nr+m)^2} = \sum_{k=r}^{\infty} \frac{1}{k^2} \sum_{1 \leq m \leq k/r} \frac{(-1)^{m+1}}{m} = T_r \, .$$

For the asymptotic expansion of T_r we apply Euler's summation formula to the function $f(x) = 1/(x + nr - 1)^2$: Let $B(\eta) = \eta - [\eta] - 1/2$. Then,

$$\begin{aligned} \sum_{m=1}^{nr} \frac{1}{(x+nr-1)^2} &= \int_1^{nr} f(x)\,dx + \int_1^{nr} B(x) f'(x)\,dx - B(1)f(1) - B(nr)f(nr) \\ &= \frac{1}{nr} - \frac{1}{2nr-1} + \mathcal{O}\Big(\frac{1}{n^2 r^2}\Big), \end{aligned}$$

which yields

$$\begin{aligned} T_r &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{nr} \frac{1}{(x+nr-1)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2 r} - \sum_{n=1}^{\infty} \frac{1}{n(2nr-1)} + \mathcal{O}\Big(\sum_{n=1}^{\infty} \frac{1}{n^3 r^2}\Big) \\ &= \frac{\zeta(2)}{r} - \sum_{n=1}^{\infty} \Big(\frac{1}{2n^2 r} + \frac{1}{2n^2 r(2nr-1)}\Big) + \mathcal{O}\Big(\frac{1}{r^2}\Big) \\ &= \frac{\zeta(2)}{r} - \frac{\zeta(2)}{2r} + \mathcal{O}\Big(\sum_{n=1}^{\infty} \frac{1}{n^2 r(2nr-1)}\Big) + \mathcal{O}\Big(\frac{1}{r^2}\Big) \\ &= \frac{\zeta(2)}{2r} + \mathcal{O}\Big(\frac{1}{r^2}\Big). \end{aligned}$$

T_1 is a special case of the multivariate zeta function $\zeta(m, n)$, see [1, Sect. 2.6]:

$$\begin{aligned} T_1 &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(-1)^{m+1}}{mn^2} = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{mn^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \\ &= -\zeta(2,-1) + \frac{3}{4}\zeta(3) = \frac{3}{2}\zeta(2)\log 2 - \frac{1}{4}\zeta(3)\, . \end{aligned}$$

The bounds for T_r stated in the lemma follow from (2.2) by using the inequalities

$$\frac{1}{2} \leq \sum_{1\leq m\leq n/r} \frac{(-1)^{m+1}}{m} \leq 1 \qquad (n \geq r),$$

$$\sum_{m=r}^{\infty} \frac{1}{m^2} < \int_{r-1}^{\infty} \frac{dt}{t^2} = \frac{1}{r-1} \leq \frac{2}{r} \quad (r \geq 2), \qquad T_1 = 1.409757\cdots < 2,$$

and

$$\sum_{m=r}^{\infty} \frac{1}{m^2} > \int_{r}^{\infty} \frac{dt}{t^2} = \frac{1}{r} \quad (r \geq 1).$$

4 Proof of Theorem 2.1

Let $\chi_{k,l} : \mathbb{N} \to \{0,1\}$ be defined by

$$\chi_{k,l}(n) := \begin{cases} 1 \text{ if } n \equiv l \pmod{k}, \\ 0 \text{ otherwise}. \end{cases}$$

Note that $\chi_{k,l}(1) = 1$ holds if and only if $l = 1$. At the beginning of the proof of Theorem 2.1 we follow the lines in Sect. 4 in [6] and modify the arguments. Let m and $a_1, \ldots, a_m$ be positive integers. We define the rational numbers ξ_1, ξ_2 by their continued fraction expansion:

$$\xi_1 = [0; a_1, \ldots, a_{m-1}, a_m] \qquad \text{and} \qquad \xi_2 = [0; a_1, \ldots, a_{m-1}, a_m + 1].$$

We have $\xi_1 < \xi_2$ for even m and $\xi_2 < \xi_1$ for odd m. We define the interval I_m by $I_m = (\xi_1, \xi_2)$ for even m and $I_m = (\xi_2, \xi_1)$ otherwise. It is well known that the intervals I_m are disjoint for different positive integers $a_1, \ldots, a_m$, and that for any fixed m the union of all closed intervals $\overline{I}_m$ gives the interval $[0,1]$. For this decomposition of $[0,1]$ we express the integral as follows:

$$\begin{aligned}
\int_0^1 \mathcal{E}_{k,l}(\alpha)\,d\alpha &= \int_0^1 \sum_{m=0}^{\infty} (-1)^m \chi_{k,l}(q_m)(q_m\alpha - p_m)\,d\alpha \\
&= \frac{\chi_{k,l}(1)}{2} + \sum_{m=1}^{\infty} (-1)^m \int_0^1 \chi_{k,l}(q_m)(q_m\alpha - p_m)\,d\alpha \\
&= \frac{\chi_{k,l}(1)}{2} + \sum_{m=1}^{\infty} (-1)^m \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \int_{I_m} \chi_{k,l}(q_m)(q_m\alpha - p_m)\,d\alpha \\
&= \frac{\chi_{k,l}(1)}{2} + \sum_{m=1}^{\infty} \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \chi_{k,l}(q_m) \int_{\xi_1}^{\xi_2} (q_m\alpha - p_m)\,d\alpha .
\end{aligned}$$

Note that p_m and q_m depend on $a_1, \dots, a_m$. The continued fraction expansion of every point $\alpha \in I_m$ has the form $\alpha = [0; a_1, \dots, a_{m-1}, a_m, \dots]$. Hence the convergents p_ν / q_ν for $\nu \leq m$ depend on I_m, but not on $\alpha \in I_m$. Therefore we compute the above integral on $[\xi_1, \xi_2]$ by

$$\int_{\xi_1}^{\xi_2} (q_m \alpha - p_m)\, d\alpha = (\xi_2 - \xi_1)\, \frac{(\xi_2 + \xi_1) q_m - 2p_m}{2}\,.$$

Using

$$\xi_1 = \frac{p_m}{q_m} \quad \text{and} \quad \xi_2 = \frac{(a_m + 1)p_{m-1} + p_{m-2}}{(a_m + 1)q_{m-1} + q_{m-2}}$$

we compute the expressions

$$\xi_2 - \xi_1 = \frac{(-1)^m}{(q_m + q_{m-1}) q_m}$$

and

$$\xi_2 + \xi_1 = \frac{p_{m-1} q_m + q_{m-1} p_m + 2 p_m q_m}{(q_m + q_{m-1}) q_m}\,,$$

which give

$$\int_{\xi_1}^{\xi_2} (q_m \alpha - p_m)\, d\alpha = \frac{1}{2 q_m (q_m + q_{m-1})^2}\,,$$

and consequently

$$\int_0^1 \mathcal{E}_{k,l}(\alpha)\, d\alpha \;=\; \frac{\chi_{k,l}(1)}{2} + \sum_{m=1}^{\infty} \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \frac{\chi_{k,l}(q_m)}{2 q_m (q_m + q_{m-1})^2}\,. \tag{4.1}$$

For the denominators of two subsequent convergents of the continued fraction of $\alpha = \langle 0; a_1, \dots, a_m, \dots \rangle$ it is well known that $(q_m, q_{m-1}) = 1$. For fixed $q_m = a$ we count the solutions of $q_{m-1} = b$ with $(a, b) = 1$ and $0 \leq b \leq a - 1$ in the multiple sum on the left-hand side of (4.1). It is necessary to distinguish the cases $m \geq 2$ and $m = 1$.

Case 1: $m \geq 2$. First let $a_1 = 1$. Then,

$$\frac{q_{m-1}}{q_m} = \langle 0; a_m, \dots, a_2, 1 \rangle = \langle 0; a_m, \dots, a_2 + 1 \rangle\,.$$

For $a_1 \geq 2$ we have

$$\frac{q_{m-1}}{q_m} = \langle 0; a_m, \ldots, a_2, a_1 \rangle = \langle 0; a_m, \ldots, a_2, a_1 - 1, 1 \rangle .$$

Case 2: $m = 1$. For $a_1 = 1$ we have a unique representation of the fraction

$$\frac{q_{m-1}}{q_m} = \frac{q_0}{q_1} = \frac{1}{a_1} = \frac{1}{1} = \langle 0; 1 \rangle ,$$

since the integer part $a_0 = 0$ must not be changed. For $a_1 \geq 2$ there are again two representations:

$$\frac{q_{m-1}}{q_m} = \frac{q_0}{q_1} = \frac{1}{a_1} = \langle 0; a_1 \rangle = \langle 0; a_1 - 1, 1 \rangle .$$

Therefore it becomes clear that for any fixed $q_m = a$ every coprime integer b with $0 \leq b \leq a - 1$ occurs exactly two times in the multiple sum on the right-hand side of (4.1), except for $m = 1$ and $a_1 = 1$. For this exceptional case we separate the term

$$\frac{\chi_{k,l}(q_1)}{2q_1(q_1 + q_0)^2} = \frac{\chi_{k,l}(1)}{8}$$

from the multiple sum. Therefore we obtain

$$\begin{aligned}
&\int_0^1 \mathcal{E}_{k,l}(\alpha)\, d\alpha \\
&= \frac{\chi_{k,l}(1)}{2} + \sum_{m=2}^{\infty} \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \frac{\chi_{k,l}(q_m)}{2q_m(q_m + q_{m-1})^2} + \sum_{a_1=2}^{\infty} \frac{\chi_{k,l}(q_1)}{2q_1(q_1 + 1)^2} + \frac{\chi_{k,l}(1)}{8} \\
&= \frac{5\chi_{k,l}(1)}{8} + \sum_{a=1}^{\infty} \sum_{\substack{b=1 \\ (a,b)=1}}^{a-1} \frac{\chi_{k,l}(a)}{a(a+b)^2} \\
&= -\frac{3\chi_{k,l}(1)}{8} + \sum_{\substack{a=1 \\ a \equiv l \pmod{k}}}^{\infty} \sum_{\substack{b=0 \\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} . \qquad (4.2)
\end{aligned}$$

Note that for $b = 0$ the condition $(a, 0) = 1$ holds for $a = 1$ only. For the proof of Theorem 2.1 we now assume that $l = 0$, so that $\chi_{k,l}(1)$ vanishes. Then (4.2) simplifies to

$$\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha = \sum_{\substack{a=1 \\ k|a}}^{\infty} \sum_{\substack{b=0 \\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2}. \tag{4.3}$$

Next, we express the arithmetic condition $(a, b) = 1$ on a and b from the inner sum by the Möbius function. Then we proceed by interchanging the order of the resulting triple sum. Here, $[d, k]$ denotes the least common multiple of d and k.

$$\begin{aligned}
\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha &= \sum_{\substack{a=1 \\ k|a}}^{\infty} \sum_{b=0}^{a-1} \sum_{\substack{d>0 \\ d|(a,b)}} \frac{\mu(d)}{a(a+b)^2} = \sum_{d=1}^{\infty} \sum_{\substack{a=1 \\ [d,k]\,|a}}^{\infty} \sum_{\substack{b=0 \\ d|b}}^{a-1} \frac{\mu(d)}{a(a+b)^2} \\
&= \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{([d,k]n-1)/d} \frac{\mu(d)}{[d,k]n\big([d,k]n + dm\big)^2} \\
&= \sum_{r\in\mathcal{D}_k} \sum_{\substack{d=1 \\ (d,k)=k/r}}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{[d,k]n/d-1} \frac{\mu(d)}{[d,k]n\big([d,k]n + dm\big)^2}.
\end{aligned}$$

The condition $(d, k) = k/r$ implies that

$$[d, k] = \frac{dk}{(d, k)} = dr\,.$$

Hence the above multiple sum takes the form

$$\begin{aligned}
\int_0^1 \mathcal{E}_k(\alpha)\, d\alpha &= \sum_{r\in\mathcal{D}_k} \sum_{\substack{d=1 \\ (d,k)=k/r}}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{nr-1} \frac{\mu(d)}{nrd(nrd+md)^2} \\
&= \sum_{r\in\mathcal{D}_k} \frac{1}{r} \Bigg(\sum_{\substack{d=1 \\ (d,k)=k/r}}^{\infty} \frac{\mu(d)}{d^3} \Bigg) \Bigg(\sum_{n=1}^{\infty} \sum_{m=0}^{nr-1} \frac{1}{n(nr+m)^2} \Bigg).
\end{aligned}$$

Finally, we express the two terms in brackets by the identities given in Lemma 3.1 and Lemma 3.2, respectively. This completes the proof of the theorem. □

5 Proofs of Corollaries 2.2–2.5

Proof of Corollary 2.2 From the multiple sum in Theorem 2.1 we separate the term for $r = k$ and $s = 1$:

$$\begin{aligned}\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha &= \frac{T_k}{\zeta(3)k} + \frac{1}{\zeta(3)} \sum_{\substack{r\in\mathcal{D}_k \\ r\neq k \vee s\neq 1}} \sum_{s\in\mathcal{D}_r} \frac{\mu(s)\mu(ks/r)T_r}{rJ_3(ks/r)} \\ &= \frac{T_k}{\zeta(3)k} + \mathcal{O}\Big(\sum_{\substack{r\in\mathcal{D}_k \\ r\neq k \vee s\neq 1}} \sum_{s\in\mathcal{D}_r} \frac{|\mu(s)\mu(ks/r)|T_r}{rJ_3(ks/r)} \Big) \\ &= \frac{T_k}{\zeta(3)k} + \mathcal{O}\Big(\sum_{\substack{r\in\mathcal{D}_k \\ r\neq k \vee s\neq 1}} \sum_{s\in\mathcal{D}_r} |\mu(s)\mu(ks/r)|\,\frac{r}{k^3s^3} \Big). \end{aligned} \tag{5.1}$$

Here we have applied the inequalities $T_r \ll 1/r$ (Lemma 3.2) and

$$J_3(n) = n^3 \prod_{p|n}\Big(1-\frac{1}{p^3}\Big) > n^3 \prod_{p\in\mathbb{P}}\Big(1-\frac{1}{p^3}\Big) = \frac{n^3}{\zeta(3)} \qquad (n\geq 1) \tag{5.2}$$

(see [8, Theorem 280]). In order to estimate k^3s^3/r we discuss the following two cases. Recall that $r|k$ and $s|r$, and that the number P is the smallest prime divisor of k.

Case 1: $1 \leq r < k$ and $s = 1$.

$$\frac{k^3s^3}{r} = \frac{k^3}{r} \geq \frac{k^3}{k/P} = Pk^2\,.$$

Case 2: $1 \leq r \leq k$ and $s \geq P$.

$$\frac{k^3s^3}{r} \geq \frac{k^3s^3}{k} \geq P^3k^2 \geq Pk^2\,.$$

Using these bounds we estimate the error term in (5.1). This gives

$$\begin{aligned}\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha &= \frac{T_k}{\zeta(3)k} + \mathcal{O}\Big(\sum_{\substack{r\in\mathcal{D}_k \\ r\neq k \vee s\neq 1}} \sum_{s\in\mathcal{D}_r} |\mu(s)\mu(ks/r)|\,\frac{1}{Pk^2} \Big) \\ &= \frac{T_k}{\zeta(3)k} + \mathcal{O}\Big(\frac{1}{Pk^2} \sum_{r\in\mathcal{D}_k} \sum_{s\in\mathcal{D}_r} |\mu(s)\mu(ks/r)| \Big) \\ &= \frac{\zeta(2)}{2\zeta(3)k^2} + \mathcal{O}\Big(\frac{1}{k^3}\Big) + \mathcal{O}\Big(\frac{1}{Pk^2} \sum_{r\in\mathcal{D}_k} \sum_{s\in\mathcal{D}_r} |\mu(s)\mu(ks/r)| \Big), \end{aligned}$$

where we have applied the asymptotic formula for T_k from Lemma 3.2. To complete the proof of the corollary we finally prove the identity

$$q_k := \sum_{r\in\mathcal{D}_k}\sum_{s\in\mathcal{D}_r} |\mu(s)\mu(ks/r)| \;=\; 3^t \qquad (k = p_1^{a_1}\cdots p_t^{a_t})$$

by induction with respect to t. For $t = 1$ let $k = p^a$. We count three pairs $[r,s] \in \mathcal{D}_k \times \mathcal{D}_r$ such that $|\mu(s)\mu(ks/r)| = 1$ given by $[p^a, 1]$, $[p^a, p]$, and $[p^{a-1}, 1]$. Now we assume that $q_{k'} = 3^t$ holds for all integers k' having t prime divisors. Let $k = p_1^{a_1}\cdots p_t^{a_t}p_{t+1}^{a_{t+1}}$ and $k' = p_1^{a_1}\cdots p_t^{a_t}$. While $[r', s'] \in \mathcal{D}_{k'} \times \mathcal{D}_{r'}$ runs through all 3^t pairs which are counted for $q_{k'}$, we obtain q_k by counting the $3\cdot 3^t$ pairs $[r,s] \in \mathcal{D}_k \times \mathcal{D}_r$ given by $[r'p_{t+1}^{a_{t+1}}, s']$, $[r'p_{t+1}^{a_{t+1}}, s'p_{t+1}]$, and $[r'p_{t+1}^{a_{t+1}-1}, s']$. This completes the proof of Corollary 2.2. □

Proof of Corollary 2.3 For $k = p \in \mathbb{P}$ the multiple sum in Theorem 2.1 consists of three terms corresponding to $[r = 1, s = 1]$, $[r = p, s = 1]$, and $[r = p, s = p]$. Therefore, we obtain

$$\int_0^1 \mathcal{E}_p(\alpha)\,d\alpha \;=\; \frac{1}{\zeta(3)}\Big(-\frac{T_1}{J_3(p)} + \frac{T_p}{p} + \frac{T_p}{pJ_3(p)}\Big).$$

With $J_3(p) = p^3 - 1$ and

$$T_1 \;=\; \frac{3}{2}\zeta(2)\log 2 - \frac{1}{4}\zeta(3)$$

given in Lemma 3.2 we prove the identity stated in the corollary. Now, for $p \geq 3$ we have with Lemma 3.2:

$$\int_0^1 \mathcal{E}_p(\alpha)\,d\alpha \;<\; \frac{1}{p^3-1}\Big(\frac{p^2}{(p-1)\zeta(3)} - \frac{3\zeta(2)\log 2}{2\zeta(3)} + \frac{1}{4}\Big) \;<\; \frac{97}{109p^2},$$

and

$$\int_0^1 \mathcal{E}_p(\alpha)\,d\alpha \;>\; \frac{1}{p^3-1}\Big(\frac{p}{2\zeta(3)} - \frac{3\zeta(2)\log 2}{2\zeta(3)} + \frac{1}{4}\Big) \;>\; \frac{2}{77p^2}.$$

This proves the inequalities stated in Corollary 2.3. □

Proofs of Corollaries 2.4 and 2.5 We apply Corollary 2.2 with $k = p^a$, $t = 1$, and $P = p$. Then the error term in Corollary 2.2 takes the form

$$\mathcal{O}\Big(\frac{1}{k^3} + \frac{3^t}{k^2P}\Big) \;=\; \mathcal{O}\Big(\frac{1}{k^3} + \frac{1}{k^2p}\Big) \;=\; \mathcal{O}\Big(\frac{1}{k^3} + \frac{1}{k^2k^{1/a}}\Big) \;=\; \mathcal{O}\Big(\frac{1}{k^{2+1/a}}\Big).$$

Thus Corollary 2.4 is proven. Also Corollary 2.5 follows immediately from Corollary 2.2. □

6 Proofs of Theorems 2.6 and 2.7

Proof of Theorem 2.6 For the upper bound we estimate the right-hand side of the identity in (4.3). Let $k \geq 2$. Then

$$\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha = \sum_{m=1}^{\infty} \sum_{\substack{b=1 \\ (km,b)=1}}^{km-1} \frac{1}{km(km+b)^2} < \sum_{m=1}^{\infty} \sum_{b=1}^{km-1} \frac{1}{k^3 m^3} < \sum_{m=1}^{\infty} \frac{1}{k^2 m^2} = \frac{\zeta(2)}{k^2}.$$

On the other hand, the lower bound for the integral in Theorem 2.6 follows from the identity (4.3), too. Here we assume that $k \geq 3$.

$$\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha \geq \sum_{\substack{b=1 \\ (k,b)=1}}^{k-1} \frac{1}{k(k+b)^2} \geq \frac{\varphi(k)}{k(2k-1)^2} > \frac{\varphi(k)}{4k^3} \gg \frac{1}{k^2 \log\log k}. \tag{6.1}$$

The inequality on the right-hand side involving a lower bound of Euler's totient follows from Theorem 328 in [8].

Next, let $k = p_1 p_2 \cdots p_r$ for some positive integer $r \geq 2$. Then we have

$$\log k = \sum_{\substack{p \leq p_r \\ p \in \mathbb{P}}} \log p = \vartheta(p_r) \ll p_r$$

by Theorem 414 in [8]. Applying additionally Theorem 429 in [8], we obtain

$$\prod_{\substack{p|k \\ p \in \mathbb{P}}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p \leq p_r \\ p \in \mathbb{P}}} \left(1 - \frac{1}{p}\right) \ll \frac{1}{\log p_r} \ll \frac{1}{\log\log k}. \tag{6.2}$$

Moreover,

$$\begin{aligned}
\int_0^1 \mathcal{E}_k(\alpha)\,d\alpha &\leq \sum_{m=1}^{\infty} \sum_{\substack{b=1 \\ (km,b)=1}}^{km-1} \frac{1}{k^3 m^3} = \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{\varphi(km)}{m^3} \\
&= \frac{1}{k^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \prod_{\substack{p|km \\ p \in \mathbb{P}}} \left(1 - \frac{1}{p}\right) \leq \frac{\zeta(2)}{k^2} \prod_{\substack{p|k \\ p \in \mathbb{P}}} \left(1 - \frac{1}{p}\right) \\
&\ll \frac{1}{k^2 \log\log k} \qquad \text{(by (6.2))}.
\end{aligned}$$

Together with the lower bound (6.1) we complete the proof of the theorem. □

Proof of Theorem 2.7 By (4.2) we have already shown the identities in the theorem. So it remains to prove the inequalities. First, we prove the upper bounds.

$$\sum_{\substack{a=1\\ a\equiv l \pmod k}}^{\infty} \sum_{\substack{b=0\\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} - \frac{3\chi_{k,l}(1)}{8}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{b=0\\ (km+l,b)=1}}^{km+l-1} \frac{1}{(km+l)(km+l+b)^2} - \frac{3\chi_{k,l}(1)}{8}$$

$$\leq \sum_{m=0}^{\infty} \sum_{b=0}^{km+l-1} \frac{1}{(km+l)^3} - \frac{3\chi_{k,l}(1)}{8} = \frac{1}{l^2} + \sum_{m=1}^{\infty} \frac{1}{(km+l)^2} - \frac{3\chi_{k,l}(1)}{8}$$

$$\leq \frac{1}{l^2} + \sum_{m=1}^{\infty} \frac{1}{k^2m^2} - \frac{3\chi_{k,l}(1)}{8} = \frac{1}{l^2} + \frac{\zeta(2)}{k^2} - \frac{3\chi_{k,l}(1)}{8}$$

$$= \begin{cases} \dfrac{5}{8} + \dfrac{\zeta(2)}{k^2} & \text{if } l = 1\,, \\[2ex] \dfrac{1}{l^2} + \dfrac{\zeta(2)}{k^2} & \text{if } l > 1\,. \end{cases}$$

For the lower bounds we treat the cases $l = 1$ and $l > 1$ separately. First, let $l = 1$. Then

$$\sum_{\substack{a=1\\ a\equiv 1 \pmod k}}^{\infty} \sum_{\substack{b=0\\ (a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} - \frac{3\chi_{k,1}(1)}{8}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{b=0\\ (km+1,b)=1}}^{km} \frac{1}{(km+1)(km+1+b)^2} - \frac{3}{8}$$

$$= \frac{5}{8} + \sum_{m=1}^{\infty} \sum_{\substack{b=0\\ (km+1,b)=1}}^{km} \frac{1}{(km+1)(km+1+b)^2}$$

$$\geq \frac{5}{8} + \sum_{\substack{b=1\\ (k+1,b)=1}}^{k} \frac{1}{(k+1)(k+1+b)^2}$$

$$> \frac{5}{8} + \sum_{\substack{b=1\\ (k+1,b)=1}}^{k+1} \frac{1}{4(k+1)^3} = \frac{5}{8} + \frac{\varphi(k+1)}{4(k+1)^3}\,.$$

Next, let $l > 1$. Then

$$\sum_{\substack{a=1\\a\equiv l \pmod k}}^{\infty} \sum_{\substack{b=0\\(a,b)=1}}^{a-1} \frac{1}{a(a+b)^2} - \frac{3\chi_{k,1}(1)}{8}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{b=1\\(km+l,b)=1}}^{km+l-1} \frac{1}{(km+l)(km+l+b)^2}$$

$$\geq \sum_{\substack{b=1\\(l,b)=1}}^{l-1} \frac{1}{l(l+b)^2} > \sum_{\substack{b=1\\(l,b)=1}}^{l} \frac{1}{4l^3} = \frac{\varphi(l)}{4l^3}.$$

This completes the proof of the theorem. □

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