

Sums of Two Squares and a Power

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Dedicated to the memory of Wolfgang Schwarz, with admiration for his broad interests, inside and outside mathematics.

Abstract We extend results of Jagy and Kaplansky and the present authors and show that for all $k \geq 3$ there are infinitely many positive integers n , which cannot be written as $x^2 + y^2 + z^k = n$ for positive integers x, y, z , where for $k \not\equiv 0 \pmod{4}$ a congruence condition is imposed on z . These examples are of interest as there is no congruence obstruction itself for the representation of these n . This way we provide a new family of counterexamples to the Hasse principle or strong approximation.

Keywords Hasse principle • Strong approximation • Ternary additive problems • Waring type problems

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1 Introduction

This paper is dedicated to the memory of Wolfgang Schwarz, who was the PhD advisor of the second named author. In particular Wolfgang Schwarz's books "Einführung in Siebmethoden der analytischen Zahlentheorie" and "Arithmetical functions" were very useful for the second author's own studies.

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Looking at Schwarz's own PhD thesis, see [11, 12], which is on sums of prime powers, i.e. on the Goldbach–Waring problem, one finds a great number of results, one of those being the following (Theorem 3 of [12]):

For fixed $k \geq 1$ let $S_k(N)$ be the set of positive integers n , with

$$\begin{aligned} 3 &\leq n \leq N, \\ n &\not\equiv 0 \pmod{2}, \quad n \not\equiv 2 \pmod{3}, \quad \text{for odd } k \\ n &\equiv 3 \pmod{24}, && \text{for even } k, \\ n &\not\equiv 0 \pmod{5}, && \text{for } k \equiv 2 \pmod{4} \\ n &\not\equiv 0, 2 \pmod{5}, && \text{for } k \equiv 0 \pmod{4} \\ n &\not\equiv 1 \pmod{p}, && \text{for each } p \equiv 3 \pmod{4} \text{ with } (p-1) \mid k. \end{aligned}$$

Then the number of integers $n \in S_k(N)$ not of the form

$$n = p_1^2 + p_2^2 + p_3^k,$$

is, for all $B > 0$, at most

$$O_B \left(\frac{N}{(\log N)^B} \right).$$

This improved on a result of Hua [8, Theorem 1], who proved this with $B = \frac{k}{k+2}$. As we had worked earlier on solutions of $x^2 + y^2 + z^k = n$, it is due to this connection that we have chosen to contribute the present note to the volume in Memory of Wolfgang Schwarz.

As it turns out, also one of the first named author's PhD advisors worked on this kind of problem in his PhD thesis: without restricting the variables to primes, one should be able to obtain stronger results, and indeed, improving on earlier work pioneered by Davenport and Heilbronn [3] and further developed by many other authors, Brüdern [1] has shown that there are at most $O(N^{1-\frac{1}{k}+\epsilon})$ integers $n \leq N$ with no solutions of

$$n = x^2 + y^2 + z^k, \tag{1.1}$$

where n is not in a residue class excluded by congruence obstructions. For a survey of results on sums of mixed powers, see also [2] and [14].

It was generally expected that for all sufficiently large n the Hasse principle for Eq. (1.1) holds true, i.e. for all such n satisfying the necessary congruence conditions there would exist a solution of (1.1) in positive integers, see, for example, Chap. 8 in [13]. However, in 1995 Jagy and Kaplansky [9] shattered this belief by proving that for $k = 9$ and some positive constant c there are at least $c \frac{N^{1/3}}{\log N}$ positive integers $n \leq N$ that are not sums of two squares and one k th power. In fact, their method works for any odd composite number k , but not for the other cases of k . In [4] we proved that a similar restriction holds for $k = 4$. That approach actually generalizes

to all k divisible by four (see Theorem 3.1), and by slightly modifying it we can not only get a bigger set of exceptional n but we can also handle k not divisible by four; to be more specific, we prove that (1.1) does not satisfy 'strong approximation': For $k \equiv 2 \pmod{4}$, $k \geq 6$ and sufficiently large N we show that there are asymptotically $\gg N^{1/2}/(\log N)^{1/2}$ positive integers $n \leq N$ for which Eq. (1.1) has no solution with z fixed into a certain residue class, though there are no congruence obstructions (see Theorem 3.2). For odd $k \geq 3$ we show that there are asymptotically at least $\frac{kN^{1/k}}{2\varphi(k)\log N}$ such exceptional positive integers $n \leq N$ (see Theorem 2.1).

Let us further mention that Hooley [7] investigated sums of three squares and a k th power, Friedlander and Wooley [5] sums of two squares and three biquadrates, and Wooley [15] sums of squares and a 'micro square', in connection with a conjecture of Linnik. In this connection we would like to add a seemingly forgotten old reference: Theorem 7 of Rieger [10] states that the number of integers $n \leq N$ which can be written as $n = x^2 + y^2 + z^k$, where $z \leq F(N)$, and F is a function tending monotonically to infinity, with $F(n) \leq \sqrt{\log N}$, is $\gg_{k,F} \frac{NF(N)}{\sqrt{\log N}}$, in other words, as good as it can be.

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2 Two Squares and an Odd k th Power

Theorem 2.1 *Let $k \geq 3$ be odd. Let p be a prime with $p \equiv 1 \pmod{4k}$. Then there are no integers x, y, z , positive or negative, with $x^2 + y^2 + z^k = p^k$ and $z \equiv 2k \pmod{4k}$.*

Proof Assume there are solutions, then $x^2 + y^2 = (p - z)(p^{k-1} + p^{k-2}z + \dots + pz^{k-2} + z^{k-1})$. If $z \equiv 2k \pmod{4k}$, then $p - z \equiv 2k + 1 \pmod{4k}$. Since k is odd, $2k + 1 \equiv 3 \pmod{4}$. Hence $p - z$ must contain a prime divisor $q \equiv 3 \pmod{4}$ with odd multiplicity. Note that $\gcd(q, k) = 1$, as otherwise $q|k$ and $0 \equiv p - z \equiv 2k + 1 \equiv 1 \pmod{q}$ gives a contradiction.

Recall that by the general classification of integers which are sums of two squares the integer $x^2 + y^2$ contains prime factors $q \equiv 3 \pmod{4}$ with even multiplicity only. Therefore both $p - z$ and $p^{k-1} + p^{k-2}z + \dots + pz^{k-2} + z^{k-1}$ are divisible by q . With $p \equiv z \pmod{q}$ it follows that

$$p^{k-1} + p^{k-2}z + \dots + pz^{k-2} + z^{k-1} \equiv kz^{k-1} \equiv 0 \pmod{q}.$$

This implies that $q \mid z$ and hence $q \mid p$, which is impossible, as $q = p$ would contradict $p \equiv 1 \pmod{4}$.

Also note that there are no congruence obstructions that would imply that in $x^2 + y^2 + z^k = p^k$ there are no solutions with $z \equiv 2k \pmod{4k}$.

To see this first observe that for a fixed odd prime q one can choose an integer $z \equiv 2k \pmod{4k}$ such that q is coprime to $p^k - z^k$; similarly, for $q = 8$ just choose

$z = 2k$. For this fixed z the congruence $x^2 + y^2 + z^k \equiv p^k \pmod q$ has a nonsingular solution in x and y which by Hensel's lemma can be lifted to a q -adic or 2-adic solution, respectively.

By the prime number theorem in arithmetic progressions, the number of such examples, $p^k \leq N$ with $p \equiv 1 \pmod{4k}$, is asymptotically

$$\frac{1}{\varphi(4k)} \int_2^{N^{1/k}} \frac{dt}{\log t} \sim \frac{k}{2\varphi(k)} \frac{N^{1/k}}{\log N}.$$

3 Two Squares and an Even k th Power

3.1 Two Squares and a k th Power, $k \equiv 0 \pmod 4$

Theorem 3.1 *Suppose that $4 \mid k$ and let p be a prime with $p \equiv 7 \pmod 8$. Let $n \equiv 1 \pmod 8$ be either 1 or consist of prime factors congruent to 1 mod 4 only, and assume that $n < p$. Then there are no positive integers x, y, z with $x^2 + y^2 + z^k = (np)^2$.*

Proof Let $k = 2t$, where t is even. Assume there are solutions, then $x^2 + y^2 = (np - z^t)(np + z^t)$. If z is even, then $np - z^t \equiv 3 \pmod 4$. If z is odd, then $np - z^t \equiv 6 \pmod 8$. In both cases $np - z^t$ must contain a prime divisor $q \equiv 3 \pmod 4$ with odd multiplicity. Therefore, as in the proof of Theorem 2.1, we conclude that both $np - z^t$ and $np + z^t$ are divisible by q . Hence their sum $2np$ and their difference $-2z^t$ are also divisible by q . Since $2n \not\equiv 0 \pmod q$, and since p is prime: $p = q$, and since $z \neq 0$: q divides z . But this gives a contradiction:

$$x^2 + y^2 + z^k > q^k \geq q^4 > (nq)^2 = (np)^2.$$

Let us give an estimate of the number of integers $np \leq N$, with $n \equiv 1 \pmod 8$ consisting of prime factors 1 mod 4 only, and $n < p$.

Recall that by a theorem of Landau the number of integers $n \leq N$ consisting of prime factors 1 mod 4 only is of order of magnitude $\frac{N}{(\log N)^{1/2}}$, and about one half of these numbers satisfy the congruence restriction $n \equiv 1 \pmod 8$.

Let $f : \mathbb{N} \rightarrow \{0, 1\}$ be the characteristic function of these integers n , i.e. we put $f(n) = 1$, if $n \equiv 1 \pmod 8$, and all prime factors of n are 1 mod 4; otherwise, we put $f(n) = 0$. Now

$$\sum_{np \leq N, n < p} f(n) = \sum_{n \leq N/p, n < p} f(n) \gg \sum_{N^{1/2} \leq p \leq N^{3/4}} \frac{N/p}{(\log(N/p))^{1/2}} \gg \frac{N}{(\log N)^{1/2}},$$

where we used that

$$\sum_{N^{1/2} \leq p \leq N^{3/4}} \frac{1}{p} = \log \log N^{3/4} - \log \log N^{1/2} + o(1) = \log(3/2) + o(1) \gg 1.$$

(In view of Landau's theorem this order is the right order of magnitude.) Hence the number of exceptional $(np)^2 \leq N$ provided by Theorem 3.1 is $\gg \frac{N^{1/2}}{(\log N)^{1/2}}$.

Note that as for Theorem 2.1 one can check that there are no congruence obstructions for the representation of $(np)^2$.

3.2 Two Squares and a k th Power, $k \equiv 2 \pmod{4}$

Theorem 3.2 *Suppose that $k \equiv 2 \pmod{4}$, $k \geq 6$ and let p be a prime with $p \equiv 7 \pmod{8}$. Let $n < p$ be an integer either 1 or consisting of prime factors congruent to 1 mod 4 only, and $n \equiv 1 \pmod{8}$. Then there are no positive integers x, y, z , where $2 \mid z$, with $x^2 + y^2 + z^k = (np)^2$.*

Proof The proof is almost verbatim as above.

Let us remark that as above one shows that the number of exceptional $(np)^2 \leq N$ provided by Theorem 3.2 is $\gg \frac{N^{1/2}}{(\log N)^{1/2}}$. Further note that in a similar way as for Theorem 2.1 one observes that there are no congruence obstructions for the requested representation of $(np)^2$.

4 Afterthought

A major part of the paper was actually written around 2007/2008. We had shown earlier versions of this paper to several colleagues, hoping that someone would write a more detailed explanation based on tools from arithmetic geometry such as the Brauer–Manin obstruction. Indeed, in this way the question has come to Fabian Gundlach [6] who was very recently able to give a detailed and general account.

As Gundlach refers to our work as an unpublished manuscript, and as our proofs use a much less sophisticated language, it seems desirable to have this paper in final form. The main part of this paper is a slightly improved version, compared to the manuscript Gundlach referred to. In particular, the version cited by Gundlach [6] had in Theorem 2.1 the same statement and proof with p^{2k} rather than p^k . Also in Theorems 3.1 and 3.2 we now have an additional factor n , thanks to an observation of J.C. Schlage-Puchta. In other words, the current version gives slightly stronger results.

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