Guided by Schwarz' Functions: A Walk Through the Garden of Mahler's Transcendence Method

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Dedicated to the Memory of Professor Wolfgang Schwarz

Abstract In this paper, transcendence results and, more generally, results on the algebraic independence of functions and their values are proved via Mahler's analytic method. Here the key point is that the functions involved satisfy certain types of functional equations as $G_d(z^d) = G_d(z) - z/(1-z)$ in the case of $G_d(z) := \sum_{h\geq 0} z^{d^h}/(1-z^{d^h})$ for $d \in \{2,3,4,\ldots\}$. In 1967, these particular functions $G_d(z)$ were arithmetically studied by W. Schwarz using Thue–Siegel–Roth's approximation method.

Keywords Algebraic independence of functions • Mahler's method

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1 Introduction

In 1967, Schwarz [14] studied the arithmetic nature of the particular Lambert series

$$G_d(z) := \sum_{h=0}^{\infty} \frac{z^{d^h}}{1-z^{d^h}}$$

at certain rational points of the unit interval, where d is an integer parameter. Using Thue–Siegel–Roth's approximation theorem he proved: If $d \geq 3, t \geq 2$, and

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$0 < b < t^{1-5/(2d)}$, then $G_d(b/t)$ is transcendental. Under weaker conditions on d and b, he obtained irrationality, non-quadracity, etc. Note that, e.g., the transcendence of

$$\sum_{h=0}^{\infty} \frac{1}{2^{2^h}+1} = 2 - G_2\left(\frac{1}{2}\right),$$

the reciprocal sum of the Fermat numbers, was not covered by the results just quoted.

Shortly later, in the first volume of the Journal of Number Theory, Mahler [9] roughly reported on earlier work of his on the transcendence of values of functions satisfying functional equations like

$$G_d(z^d) = G_d(z) - \frac{z}{1-z} \tag{1.1}$$

(see [6–8]). In this note, Mahler suggested several directions, in which his work might possibly be extended.

It is the main aim of this article to give an impression of the development, which was initiated by Schwarz' note and Mahler's reaction. To this purpose, we consider coprime[1] $A, B \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ satisfying $A(0) = 0$ and put $H := \frac{A}{B}$. This rational function has coefficients in an algebraic number field (called K), and convergence radius Δ being $+\infty$ for constant B, and the smallest absolute value of all zeros of B otherwise. Clearly, the series

$$\mathcal{H}_d(z) := \sum_{h=0}^{\infty} H(z^{d^h})$$

with integer $d \geq 2$ (*to be always assumed from now on*) is in $K[[z]]$, converges in $|z| < \min(1, \Delta)$, and satisfies the functional equation

$$\mathcal{H}_d(z^d) = \mathcal{H}_d(z) - H(z) \tag{1.2}$$

"of Mahler-type." Taking $H(z) = z/(1-z)$ we get $\mathcal{H}_d = G_d$, the *Schwarz functions*, and on taking $H(z) = z$ we get $\mathcal{H}_d = F_d$, where the so-called Fredholm series $F_d(z) = \sum_{h \geq 0} z^{d^h}$ is the prototype of a Mahler function.

[1] Here and in the sequel, $\overline{\mathbb{Q}}$ denotes the field of all complex algebraic numbers.

2 Transcendence

From Mahler's transcendence criterion in [6] (see also [12, Theorem 1.2]) we deduce

Theorem 2.1 *$\mathcal{H}_d(\alpha)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < \min(1, \Delta)$ and $B(\alpha^{d^j}) \neq 0$ ($j = 0, 1, \ldots$) provided that $\mathcal{H}_d(z)$ is transcendental over $K(z)$.*

Note here that something analytical in this direction is necessary: Namely, on taking $H(z) = z/(1-z^2)$ we find $\mathcal{H}_2(z) = z/(1-z)$; thus, a transcendence result of the above type cannot hold for this $\mathcal{H}_2$.

In some good-natured cases, the transcendence of $\mathcal{H}_d(z)$ follows from classical results in function theory, e.g., if $H(z) = z$. But it is important to have rather general criteria for the transcendence of Mahler-type functions. One of those reads as follows:

Theorem 2.2 ([11]) *If $f \in \mathbb{C}[[z]]$ satisfies $f(z^d) = \varphi(z, f(z))$ or $f(z) = \psi(z, f(z^d))$ for some $\varphi, \psi \in \mathbb{C}(z, w)$, then f is either rational or transcendental.*

Thus, under favorable conditions, one has only to exclude rationality. Precisely in this direction, Coons [2, Theorem 2.2] deduced quite recently the following functional transcendence criterion from Theorem 2.2:

Theorem 2.3 *If $f \in \mathbb{C}[[z]]$ satisfies $f(z^d) = f(z) - \frac{A(z)}{B(z)}$ with $A, B \in \mathbb{C}[z] \setminus \{0\}$ and $\max(\deg A, \deg B) \leq d - 1$, then f is transcendental over $\mathbb{C}(z)$.*

Note that this implies the transcendence of all Schwarz functions $G_d(z)$. Note also that, in general, the bound $d - 1$ for the degrees of A, B is best possible for any d as one can see by taking

$$A(z) = \alpha z \frac{z^{d-1} - \alpha^{d-1}}{z - \alpha}, \quad B(z) = \alpha - z^d$$

with α some $(d-1)$th root of unity, where we are led to the rational function

$$\sum_{h=0}^{\infty} \frac{A}{B}(z^{d^h}) = \frac{z}{\alpha - z}$$

(see [3, Theorem 9] and [1, Lemma 2.10]) generalizing our example $\mathcal{H}_2(z)$ after Theorem 2.1.

In the rest of this section, we include a rather shortened

Proof of Theorem 2.3 W.l.o.g. we may assume A, B coprime and, moreover, $A(0) = 0, B(0) \neq 0$. By Theorem 2.2, it is enough to show $f \notin \mathbb{C}(z)$. Thus, let us assume, on the contrary, that f is rational, say $f = u/v$ with coprime $u, v \in \mathbb{C}[z] \setminus \{0\}$. Then the functional equation can be equivalently written as

$$u(z^d)v(z)B(z) = v(z^d)\big(u(z)B(z) - v(z)A(z)\big),$$

hence, by the coprimality of $u(z^d), v(z^d)$,

$$v(z^d) \mid v(z)B(z) \quad \text{and} \quad u(z^d) \cdot \frac{v(z)B(z)}{v(z^d)} = u(z)B(z) - v(z)A(z) \tag{2.1}$$

implying $(d-1)\deg v \leq \deg B$.

Now, $v = \text{const}$ would imply $B\,|A$, by the equation in (2.1), hence $B = \text{const}$ since A, B are supposed to be coprime. Thus, $u(z^d) = u(z) - (v(0)/B(0))A(z)$ implying $d \deg u \leq \max(\deg u, d-1)$ hence $u = \text{const}$, and then $A = 0$, a contradiction.

So the case $\deg B \leq d-2$ is excluded [see after (2.1)] implying $\deg B = d-1$, and this, in turn, leads to $\deg v \leq 1$, where we may immediately assume $\deg v = 1$ or $v(z) = \alpha z + \beta$ with $\alpha \neq 0$. The degree of the left-hand side of the equation in (2.1) equals $d \deg u$, whereas the degree of the right-hand side is $\leq \max(d-1+\deg u, d)$, and this consideration yields $\deg u \leq 1$ or $u(z) = az + b$. Therefore the functional equation from Theorem 2.3 reduces to

$$\frac{A(z)}{B(z)} = \frac{az+b}{\alpha z+\beta} - \frac{az^d+b}{\alpha z^d+\beta} = \frac{(b\alpha - a\beta)z(z^{d-1}-1)}{(\alpha z^d+\beta)(\alpha z+\beta)}$$

with $a\beta \neq b\alpha$ since $A \neq 0$, whence

$$A(z)\big(\alpha z(z^{d-1}-1) + (\alpha z+\beta)\big)(\alpha z+\beta) = (b\alpha - a\beta)z(z^{d-1}-1)B(z)$$

implying $\deg A + 1 = \deg B \;\Leftrightarrow\; \deg A = d-2$ and, moreover,

$$A(z)(\alpha z+\beta)^2 = C(z) \cdot z(z^{d-1}-1) \tag{2.2}$$

with some $C \in \mathbb{C}[z] \setminus \{0\}$ of degree 0, hence $C(z) = \text{const} \neq 0$. Thus, the right-hand side of (2.2) has only simple zeros but the left-hand side has multiple ones, and this contradiction concludes our proof. □

3 Hypertranscendence

An analytic function is called *hypertranscendental* if no finite collection of derivatives of the function is algebraically dependent over $\mathbb{C}(z)$. A hypertranscendence criterion for Mahler-type functions is the following:

Theorem 3.1 ([10, Theorem 3]) *Suppose that* $f \in \mathbb{C}[[z]]$ *has the following two properties:*

(i) *For some integer* $n \geq 1$, *let* $f, f', \dots, f^{(n-1)}$ *be algebraically dependent over* $\mathbb{C}(z)$.
(ii) *For some integer* $d \geq 2$, f *satisfies* $f(z^d) = u(z)f(z) + v(z)$, *where* $u, v \in \mathbb{C}(z), u \neq 0$. *If* $u(z) = s_M z^M + \dots$ *with an integer* M *and* $s_M \neq 0$, *put* $Q = [M/(d-1)]$.

Then there exists some $w \in \mathbb{C}(z)$ *satisfying*

$$w(z^d) = u(z)w(z) + v(z) \quad or \quad w(z^d) = u(z)w(z) + v(z) - \gamma \frac{u_1(z)z^{Qd}}{u_2(z)},$$

where $u_1(z) = u(z)/(s_M z^M), u_2 \in \mathbb{C}(z) \setminus \{0\}$ *fulfills the condition* $u_2(z^d) = u_2(z)/u_1(z)$, *and* $\gamma \in \mathbb{C}$ *is the constant term in the* z*-expansion of the quotient* $v(z)u_2(z)/(u_1(z)z^{Qd})$ *in case* $s_M = 1, M = Q(d-1)$, *but* $\gamma = 0$ *otherwise.*

From this criterion we deduce the following consequence:

Corollary 3.2 *If* $f \in \mathbb{C}[[z]] \setminus \mathbb{C}(z)$ *satisfies* $f(z^d) - f(z) \in \mathbb{C}(z)$, *then* f *is hypertranscendental.*

Proof Let f, as in the corollary, satisfy $f(z^d) - f(z) \in \mathbb{C}(z), = v(z) \in \mathbb{C}[[z]]$, say, hence $v(0) = 0$. Assume that (i) from Theorem 3.1 holds. Since (ii) holds also, with $u(z) = 1$ (hence $M = 0, s_M = 1, Q = 0$) and $v(z)$ as above, there exists some $w \in \mathbb{C}(z)$ satisfying

$$w(z^d) = w(z) + v(z) \tag{3.1}$$

(as soon as we have checked $\gamma = 0$: $u_1(z) = 1; u_2 \in \mathbb{C}(z) \setminus \{0\}$ fulfills $u_2(z^d) = u_2(z)$, or $U(z^d)V(z) = U(z)V(z^d)$ with $u_2 = U/V$, $U, V \in \mathbb{C}[z] \setminus \{0\}$ coprime, whence $u_2 = \text{const} \in \mathbb{C}^\times$, and thus $\gamma = 0$).

By (3.1), $w(z)$ has no pole at 0, hence it is in $\mathbb{C}[[z]]$. Since f satisfies also (3.1), $\varphi := f - w \in \mathbb{C}[[z]]$ satisfies $\varphi(z^d) = \varphi(z)$, whence $\varphi = \text{const}$, and we arrive at the contradiction $f = w + \text{const} \in \mathbb{C}(z)$. □

In particular, all $G_d(z)$ and $F_d(z)$ are hypertranscendental.

To see what this implies arithmetically, we quote the following algebraic independence criterion:

Theorem 3.3 ([12, Theorem 4.2.1]) *Let* K *be an algebraic number field. Suppose that* $f_1, \dots, f_n \in K[[z]]$ *converge in* $|z| < \rho$ *with some* $0 < \rho \leq 1$, *where they satisfy*

$$^\tau(f_1(z^d), \dots, f_n(z^d)) = \mathcal{A}(z) \cdot {}^\tau(f_1(z), \dots, f_n(z)) + {}^\tau(b_1(z), \dots, b_n(z)) \tag{3.2}$$

with $\mathcal{A} \in \text{Mat}_{n\times n}(K(z)), b_1, \dots, b_n \in K(z)$, *and* τ *indicating the matrix transpose. If* $\alpha \in \overline{\mathbb{Q}}^\times$ *with* $|\alpha| < \rho$ *is such that no* α^{d^j} $(j = 0, 1, \dots)$ *is a pole of* $b_1, \dots, b_n$ *and* *the entries of $\mathcal{A}$, then the following inequality holds for transcendence degrees*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(f_1(\alpha),\ldots,f_n(\alpha)) \geq \mathrm{trdeg}_{K(z)}K(z)(f_1(z),\ldots,f_n(z)). \tag{3.3}$$

We apply this with $f_i(z) := f^{(i-1)}(z), f$ being an irrational solution of

$$f(z^d) = f(z) - H(z), \quad H = \frac{A}{B}, \tag{3.4}$$

with A, B as in our introduction [compare (1.2)]. According to Corollary 3.2, f is hypertranscendental, whence the right-hand side of (3.3) equals n for any $n \geq 1$. By successive differentiations of (3.4), we recognize

$$\tilde{\mathcal{A}}(z) \cdot {}^{\tau}(f(z^d),\ldots,f^{(n-1)}(z^d)) = {}^{\tau}(f(z),\ldots,f^{(n-1)}(z)) - {}^{\tau}(H(z),\ldots,H^{(n-1)}(z))$$

with lower triangular $\tilde{\mathcal{A}}(z) \in \mathrm{Mat}_{n\times n}(\mathbb{Z}[z])$ having $1, dz^{d-1},\ldots,(dz^{d-1})^{n-1}$ on the main diagonal. From this, (3.2) can be easily checked with $\mathcal{A}(z) := \tilde{\mathcal{A}}(z)^{-1}, {}^{\tau}(b_1(z),\ldots,b_n(z)) = -\mathcal{A}(z)\cdot {}^{\tau}(H(z),\ldots,H^{(n-1)}(z))$. Clearly, $\mathcal{A}$ has no poles $\neq 0$, whereas non-zero poles of b_i can only come from zeros of B, and we have established the following result:

Corollary 3.4 *Suppose that f is an irrational solution of* (3.4)*, where $A, B \in \overline{\mathbb{Q}}[z]\setminus\{0\}$. Then, for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ with $|\alpha| < \min(1, \Delta)$ and $B(\alpha^{d^j}) \neq 0$ for $j = 0, 1, \ldots,$ the numbers $f(\alpha), f'(\alpha), f''(\alpha), \ldots$ are algebraically independent.*

In particular, for any d and $\alpha \in \overline{\mathbb{Q}}^{\times}$, $|\alpha| < 1$, the numbers $G_d(\alpha), G_d'(\alpha), G_d''(\alpha), \ldots$ are algebraically independent.

4 Algebraic Independence of the Values of Schwarz' Functions

The most typical question here is as follows. For a Mahler-type function f, analytic on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, say, one is interested in necessary and sufficient conditions on $\alpha_1,\ldots,\alpha_n \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ such that $f(\alpha_1),\ldots,f(\alpha_n)$ are algebraically independent. This problem was essentially solved for the Fredholm series F_d, $d \geq 2$, by Loxton and van der Poorten [5]. But in the case of Schwarz' functions G_d, no significant results seem to exist in the literature, at least to our knowledge. We want to make here a modest contribution to this problem.

To begin with, we conclude analogously to [12, pp. 106–107] as follows. Suppose $\alpha_1,\ldots,\alpha_n \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ to be multiplicatively independent, and consider the multivariable functions $f_j(z_1,\ldots,z_n) := G_d(z_j)$, $j = 1,\ldots,n$. Since $G_d(z)$ is transcendental over $\mathbb{C}(z)$, the functions $f_1(\underline{z}),\ldots,f_n(\underline{z})$ of the multivariable $\underline{z} := (z_1,\ldots,z_n)$ are algebraically independent over $\mathbb{C}(\underline{z})$. Moreover, $\Omega := \mathrm{diag}(d,\ldots,d) \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_{\geq 0})$ and $\underline{\alpha} := (\alpha_1,\ldots,\alpha_n)$ satisfy the properties (I) through (IV) from [12, pp. 33–34, 62], whence we obtain the algebraic independence of $G_d(\alpha_1), \ldots, G_d(\alpha_n)$ if, as assumed above, the numbers $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent. Note that the main feature here consists in the trick to deduce algebraic independence results for one single-valued function at different points from corresponding results on several multivariable functions at one point.

In a very particular subcase of multiplicatively dependent $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}^\times \cap \mathbb{D}$, we are in a position to give a characterization of the algebraic independence of $G_d(\alpha_1), \ldots, G_d(\alpha_n)$, namely if all α_j's are powers of some $\alpha \in \overline{\mathbb{Q}}^\times \cap \mathbb{D}$.

Theorem 4.1 *Let $m_1, \ldots, m_n$ be $n \geq 2$ positive integers, and let $\alpha \in \overline{\mathbb{Q}}^\times \cap \mathbb{D}$. Then $G_d(\alpha^{m_1}), \ldots, G_d(\alpha^{m_n})$ are algebraically independent if and only if*

$$\frac{m_j}{m_i} \notin d^{\mathbb{Z}} \tag{4.1}$$

holds for any pair (i, j) with $i \neq j$.

To prepare our proof below, we next quote the one-variable version of a main tool for algebraic independence of Mahler-type functions.

Theorem 4.2 *Let $f_1, \ldots, f_n \in \mathbb{C}[[z]]$ satisfy the functional equation*

$$^\tau(f_1(z), \ldots, f_n(z)) = \mathcal{A} \cdot {}^\tau(f_1(z^d), \ldots, f_n(z^d)) + {}^\tau(b_1(z), \ldots, b_n(z)) \tag{4.2}$$

with $\mathcal{A} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and $b_1, \ldots, b_n \in \mathbb{C}(z)$. If $f_1, \ldots, f_n$ are algebraically dependent over $\mathbb{C}(z)$, then there exist $c_1, \ldots, c_n \in \mathbb{C}$, not all 0, such that $c_1 f_1 + \ldots + c_n f_n \in \mathbb{C}(z)$.

Proof This is a particular case of [12, Theorem 3.2.2], a forerunner of which was [4, Theorem 2].

Proof of Theorem 4.1 We consider the functions

$$f_j(z) := G_d(z^{m_j}) \quad (j = 1, \ldots, n) \tag{4.3}$$

satisfying, by (1.1),

$$f_j(z^d) = f_j(z) + \frac{z^{m_j}}{z^{m_j} - 1} \quad (j = 1, \ldots, n) \tag{4.4}$$

which is a system of functional equations of type (4.2). Iterating (4.4) we find

$$f_j(z^{d^\ell}) = f_j(z) + \sum_{\lambda=0}^{\ell-1} \frac{z^{d^\lambda m_j}}{z^{d^\lambda m_j} - 1}$$

for any integer $\ell \geq 0$, empty sums being 0, by convention. Assuming w.l.o.g. $m_2/m_1 = d^\ell$, this equation yields $G_d(\alpha^{m_2}) - G_d(\alpha^{m_1}) \in \overline{\mathbb{Q}}$, by (4.3), where the explicit value of this difference can be written down. Thus, the validity of (4.1) is necessary for the algebraic independence of $G_d(\alpha^{m_1}), \ldots, G_d(\alpha^{m_n})$.

That conditions (4.1) are sufficient for the algebraic independence of the numbers just mentioned can be deduced from Theorem 3.3 as follows: Applying this theorem with $K = \mathbb{Q}, \rho = 1$ and (4.4) as system (3.2), we obtain $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(f_1(\alpha), \ldots, f_n(\alpha)) \geq n$, by (3.3), hence our assertion, if we can show that our $f_1(z), \ldots, f_n(z)$ are algebraically independent over $\mathbb{Q}(z)$.

To prove even their algebraic independence over $\mathbb{C}(z)$, we use Theorem 4.2 which tells us the following. If $f_1, \ldots, f_n$ were algebraically dependent over $\mathbb{C}(z)$, then there exists a $\underline{c} := (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{\underline{0}\}$ such that $r(z) := c_1 f_1(z) + \ldots + c_n f_n(z)$ is a rational function satisfying the functional equation

$$r(z^d) = r(z) + \sum_{j=1}^{n} c_j \frac{z^{m_j}}{z^{m_j} - 1}, \tag{4.5}$$

by (4.4).

To get our desired contradiction more conveniently, we next transform, for our above $\underline{c} \neq \underline{0}$, Eq. (4.5) in $r, m_1, \ldots, m_n$ into an equivalent one in $s, k_1, \ldots, k_n$. To this purpose we write, for $j = 1, \ldots, n$, $m_j = d^{t(j)} k_j$ with integers $t(j) \geq 0$ and $k_j > 0$ such that $d \nmid k_j$. Then condition (4.1) is equivalent to the distinctness of $k_1, \ldots, k_n$. Moreover, with $\underline{c}$ and r as in (4.5), we define the rational function s by

$$s(z) := r(z) - \sum_{j=1}^{n} c_j \sum_{\tau=0}^{t(j)-1} \frac{z^{d^\tau k_j}}{z^{d^\tau k_j} - 1}$$

which satisfies

$$s(z^d) = s(z) + \sum_{j=1}^{n} c_j \frac{z^{k_j}}{z^{k_j} - 1}. \tag{4.6}$$

Therefore, to reach our contradiction, it suffices to establish the following auxiliary result the proof of which we defer to the last section. □

Lemma 4.3 *If $\underline{c} := (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{\underline{0}\}$, and $k_1, \ldots, k_n$ are distinct positive integers not divisible by d, then the functional equation (4.6) has no rational solution s.*

This may be the right place to ask an

Open Question *Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}^\times \cap \mathbb{D}$. Are the following two statements equivalent?*

(i) *$G_d(\alpha_1), \ldots, G_d(\alpha_n)$ are algebraically independent.*
(ii) *$\alpha_j \neq \alpha_i^{d^\ell}$ holds for any triple (i, j, ℓ) with $i \neq j$ and $\ell \geq 0$.*

Of course, the implication (i) ⇒ (ii) is easily seen. In the particular case $\alpha_j = \alpha^{m_j}$ treated in Theorem 4.1, the reversed implication is also valid.

5 Proof of Lemma 4.3

Step 1: *Partial fraction decomposition*. Assume on the contrary, that, for a $(c_1, \ldots, c_n) \neq \underline{0}$, Eq. (4.6) with distinct k_j not divisible by d has a rational solution s. Since all polynomials $z^{k_j} - 1$ divide $z^L - 1$, $L := \operatorname{lcm}(k_1, \ldots, k_n)$, it follows from [13, Lemma 1] that s must be of the shape

$$s(z) = \frac{a(z)}{z^L - 1}$$

with some $a \in \mathbb{C}[z]$, $a \neq 0$, by (4.6) and the distinctness of the k_j. Note also $\deg a \leq L$.

For integers $k > 0$, we write $\zeta_k := e^{2\pi i/k}$ and have

$$\frac{z^k}{z^k - 1} = 1 + \frac{1}{k} \sum_{j=0}^{k-1} \frac{\zeta_k^j}{z - \zeta_k^j} = 1 + \frac{1}{k} \sum_{\delta | k} f_\delta(z),$$

the second sum being over all positive divisors δ of k, where we put

$$f_\delta(z) := \sum_{\substack{j=0 \\ (j,\delta)=1}}^{\delta-1} \frac{\zeta_\delta^j}{z - \zeta_\delta^j}.$$

Here the poles of f_δ are exactly the primitive δth roots of unity, i.e., the roots of the δth cyclotomic polynomial.

In the same way, we may write down the partial fraction decomposition of s as

$$s(z) = S + \sum_{\delta | L} s_\delta(z) \quad \text{with} \quad s_\delta(z) := \sum_{\substack{j=0 \\ (j,\delta)=1}}^{\delta-1} \frac{s_{\delta,j}}{z - \zeta_\delta^j},$$

where S and the $s_{\delta,j}$'s are complex constants.

By the preceding notations, our functional equation (4.6) assumes the form

$$\sum_{\delta | L} \left(s_\delta(z^d) - s_\delta(z) \right) = \sum_{j=1}^{n} \frac{c_j}{k_j} \sum_{\delta | k_j} f_\delta(z), \tag{5.1}$$

where we already used $c_1 + \ldots + c_n = 0$, a result coming from the fact that both sides of (5.1) tend to 0 as $z \to \infty$. W.l.o.g., let us assume $c_1 \cdot \ldots \cdot c_m \neq 0$ but $c_{m+1} = \ldots = c_n = 0$ for $2 \leq m \leq n$, and furthermore $k_1 > \ldots > k_m$. Next, consider the set of all positive integers δ dividing at least one of $k_1, \ldots, k_m$, where none of these δ's is divisible by d since we assumed $d \nmid k_j$ for $j = 1, \ldots, n$. Clearly k_1 is the maximal element of this set and cannot occur among the divisors of $k_2, \dots, k_m$. Thus, all poles of f_{k_1} remain poles of the right-hand side of (5.1), and we may summarize the result of Step 1 as follows: Denoting by N the greatest positive integer such that the left-hand side

$$\sum_{\delta|L} \left(s_\delta(z^d) - s_\delta(z)\right) \tag{5.2}$$

of (5.1) and f_N have at least one pole in common, then $N = k_1$ holds, whence $d \nmid N$.

Step 2: *Study of $s_\delta(z^d)$ and final contradiction*. From the above definition of s_δ for positive divisors δ of L we obtain

$$s_\delta(z^d) = \sum_{\substack{j=0\\(j,\delta)=1}}^{\delta-1} \frac{s_{\delta,j}}{z^d - \zeta_\delta^j} = \sum_{\substack{j=0\\(j,\delta)=1}}^{\delta-1} s_{\delta,j} \sum_{\kappa=0}^{d-1} \frac{1}{d\zeta_{d\delta}^{(j+\kappa\delta)(d-1)}(z - \zeta_{d\delta}^{j+\kappa\delta})}\,. \tag{5.3}$$

Suppose, from now on, $p_1^{\nu(1)} \cdot \ldots \cdot p_\omega^{\nu(\omega)}$ to be the canonical factorization of d. Assume that $p_1, \dots, p_\sigma$ are not divisors of δ but $p_{\sigma+1}, \dots, p_\omega$ are, where we have to consider the cases $\sigma = 0, \dots, \omega$. Then we have the following equivalence:

$$(j,\delta) = 1 \iff (j + \kappa\delta, \delta) = (j + \kappa\delta, d\delta/\prod\nolimits_{i=1}^{\sigma} p_i^{\nu(i)}) = 1$$

with the usual convention here and later that empty products equal 1. Now, any positive divisor D of $p_1 \cdot \ldots \cdot p_\sigma$ is relatively prime to δ, whence there are precisely $\frac{d}{D}$ numbers $\kappa \in \{0, \dots, d-1\}$ satisfying $D|(j + \kappa\delta)$. Thus, by the well-known inclusion–exclusion principle, we can say that, for fixed coprime j, δ, the number of $\kappa \in \{0, \dots, d-1\}$ such that $j + \kappa\delta$ is prime to $p_1 \cdot \ldots \cdot p_\sigma$ (or equivalently to $\prod_{i=1}^{\sigma} p_i^{\nu(i)}$) equals $d\prod_{i=1}^{\sigma}(1 - 1/p_i)$. Therefore we can note that, for fixed coprime j, δ, there are exactly $d\prod_{i=1}^{\sigma}(1 - 1/p_i)$ values $\kappa \in \{0, \dots, d-1\}$ such that $(j + \kappa\delta, d\delta) = 1$ holds. Hence we conclude

$$s_\delta(z^d) = \sum_{\substack{j=0\\(j,d\delta)=1}}^{d\delta-1} \frac{s_{\delta,j}}{d\zeta_{d\delta}^{j(d-1)}(z - \zeta_{d\delta}^j)} + \Sigma_\delta(z) \tag{5.4}$$

from the double sum in (5.3), where, strictly speaking, we should write $s_{\delta,j-[j/\delta]\delta}$ in the numerator instead of simply $s_{\delta,j}$. The rational function Σ_δ in (5.4) vanishes identically in case $\sigma = 0$, whereas, in the cases $1 \le \sigma \le \omega$, it may have poles at certain primitive ρth roots of unity but with $\rho < d\delta$ only. Since $s_\delta \ne 0$ is equivalent to the fact that not all $s_{\delta,j}, j \in \{0, \dots, d\delta - 1\}$ and prime to $d\delta$, vanish, we conclude from (5.4) that, in this case of δ, the difference $s_\delta(z^d) - s_\delta(z)$ has poles at $(d\delta)$th roots of unity. Thus, the N defined before (5.2) must be of the form $d\delta$, whence $d|N$ holds, and we have obtained a contradiction proving Lemma 4.3.

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