A Ternary Problem in Additive Prime Number Theory

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Abstract Estimates are obtained for the number of natural numbers below a parameter that do not have a representation as the sum of two squares of primes and a kth power of a prime. These improve earlier bounds in the order of magnitude. The method is then also applied to some related questions.

Keywords Circle method • Sums of prime powers

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1 Introduction

This collection of research articles is dedicated to the memory of Wolfgang Schwarz. Early in his career he was a practitioner of the Hardy–Littlewood circle method [14–16], and he followed later developments of the method itself and its range of applicability with great interest. His survey article [17] gives ample proof of this, and the collection of open problems contained therein helped this writer into the subject [1]. It therefore seems fitting for the occasion to examine here the current status of a problem that Schwarz discussed in his thesis. Published in two parts [14, 15] and devoted in general to the additive theory of prime numbers, it is in the second of his papers where the intent is to obtain proof that almost all numbers expected to be representable in a certain proposed form indeed have such representations.

Perhaps the most interesting of his results relates to representations of the natural number n in the form

$$n = p_2^2 + p_2^2 + p_3^k \qquad (1.1)$$

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in which $k \geq 2$ is a given natural number, and the variables p_j range over primes. One may expect such a representation to exist whenever the congruences

$$x_1^2 + x_2^2 + x_3^k \equiv n \bmod q \tag{1.2}$$

are soluble with $(x_j, q) = 1$ $(1 \leq j \leq 3)$, for all $q \in \mathbb{N}$, and we write $\mathscr{H}_k$ for the set of all $n \in \mathbb{N}$ where this is so. It is well known and easy to prove[1] that $\mathscr{H}_k$ is the union of finitely many residue classes. In particular, $\mathscr{H}_k$ has positive density among the natural numbers. Schwarz [15, Satz 3] proved that for any fixed $A > 0$ the number $E_k(N)$ of integers $n \in \mathscr{H}_k$ with $1 \leq n \leq N$ for which (1.1) has no solution in primes p_1, p_2, p_3 satisfies $E_k(N) \ll N(\log N)^{-A}$. Earlier Hua [7] had obtained this same conclusion for *some* positive A. After the fundamental innovations of Montgomery and Vaughan [13] in their treatment of the binary Goldbach problem, an estimate of the shape

$$E_k(N) \ll N^{1-\delta(k)} \tag{1.3}$$

should have been within reach for some positive $\delta(k)$, but a proof was only published in 1993 by Leung and Liu [9]. Their argument relied on the Deuring–Heilbronn phenomenon, as did the work of Montgomery and Vaughan, and hence little could be said about the dependence of δ on k at the time. Five years later Liu and Zhan [11] noticed that for representation problems with more than two variables, potential Siegel zeros impact the major arc analysis to a lesser extent than is the case in a binary situation. This interesting observation has been instrumental in many applications since. In particular, dramatic progress was possible with the quadratic case of the Waring–Goldbach problem. We only mention the latest bound

$$E_2(N) \ll N^{17/20+\varepsilon}$$

due to Harman and Kumchev [5]. For larger values of k, we have (1.3) with

$$\delta(3) = \frac{1}{21} - \varepsilon, \quad \delta(k) = \frac{1}{3k \cdot 2^{k-2}} - \varepsilon \quad (k \geq 4) \tag{1.4}$$

where ε is any positive number, as established by Lü [12] and Li [10, Theorem 1], respectively. These two papers are based on very similar ideas. An elementary bound for the number of solutions of the equation

$$p_1^2 + p_2^2 = p_3^2 + p_4^2$$

in primes p_j not exceeding $N^{1/2}$ is combined with an estimate of Weyl's type for the degree k Weyl sum in a circle method analysis of the problem at hand. When

[1] For details see, for example, the footnote on p. 41 of [3], or *inter alia* in [15].

$k \geq 3$, the best known Weyl bounds for trigonometric sums over prime powers are due to Kumchev [8], and their direct use produces the estimates (1.4). For larger values of k, Li's results can be improved by using Weyl bounds stemming from Vinogradov's mean value theorem. Wooley's groundbreaking work [19] on the latter can be combined with the strategy of Li to show that in (1.3) one may take $\delta(k) = c/k^3$, for some constant $c > 0$.

In this paper we follow a different route and treat the kth power in mean. This increases the admissible values for $\delta(k)$ by a factor k when k is large, and when $k = 3$ we also obtain a sizeable improvement over the work of Lü [12].

Theorem 1.1 *One has*

$$E_3(N) \ll N^{15/16+\varepsilon} \quad \text{and} \quad E_k(N) \ll N^{1-1/(2k)^2+\varepsilon} \quad (k \geq 4).$$

To access the strength of this result, it is perhaps worth pointing out that the circle method approach to problems of this type is currently limited by the square root cancellation barrier. For the problem under consideration, this corresponds to $\delta(k) = 1/k$ in (1.3). If one is prepared to give up the primality of p_3 in (1.1), then one can get reasonably close to this barrier. More precisely, let $\mathscr{G}_k$ be the set of all natural numbers n for which the congruences (1.2) are soluble with $(x_1x_2, q) = 1$, for all $q \in \mathbb{N}$. Let $E_k^*(N)$ denote the number of all $n \in \mathscr{G}_k$ with $1 \leq n \leq N$ for which the equation

$$p_1^2 + p_2^2 + x^k = n \tag{1.5}$$

has no solution in primes p_1, p_2 and natural numbers x.

Theorem 1.2 *Let* $k \geq 4$. *Then* $E_k^*(N) \ll N^{1-1/(97k)}$.

No effort has been made to optimize the constant 97 that occurs in the exponent of the bound for $E_k^*(N)$. Emphasis is on larger k, and consequently, we have not attempted to tune our approach so as to yield good bounds when k is of moderate size.

It is interesting to compare the conclusion of Theorem 1.2 with related estimates for the prominent Hardy–Littlewood problem concerning sums of a prime and a kth power. When $n \in \mathscr{G}_k$ we expect Eq. (1.5) to have about $n^{1/k}(\log n)^{-2}$ solutions, while whenever $n - x^k$ is irreducible as a polynomial over the rationals, the equation $n = p + x^k$ should have about $n^{1/k}(\log n)^{-1}$ solutions with p prime. Thus, one might be led to believe that the two problems are roughly of the same degree of difficulty. However, this does not seem to be the case, given our current understanding of exponential sums over primes. Indeed, if $D_k(N)$ denotes the number of all $n \leq N$ for which $n - x^k$ is irreducible over $\mathbb{Z}[x]$ but $n = p + x^k$ has no solution with p prime, then the bound

$$D_k(N) \ll N^{1-1/(25k)} \tag{1.6}$$

was obtained in collaboration with Perelli [4] almost 20 years ago, yet assuming that no Dirichlet L-function has a zero in the half-plane Re $s > \frac{1}{2}$. Unconditional bounds for $D_k(N)$ come nowhere close to (1.6). The significant difference between the Hardy–Littlewood problem and the representation problem (1.1) is that the major arc estimates of Kumchev [8] have an amplifying effect whenever three summands are present, but only then. Kumchev's bounds control Weyl sums related to $p_1^2 + p_2^2$ to a precision that is available for a single prime only if one is prepared to accept the use of unproven hypotheses.

In his thesis work, Schwarz also considered additive problems with more than three prime powers. For example, when $k \geq 2$, representations of the form

$$p_1^2 + p_2^4 + p_3^4 + p_4^k = n \tag{1.7}$$

or

$$p_1^2 + p_2^3 + p_3^6 + p_4^k = n \tag{1.8}$$

are considered. We focus our attention temporarily on Eq. (1.8), and on those integers n for which the congruences $x_1^2 + x_2^3 + x_3^6 + x_4^k \equiv n \bmod q$ have a solution with $(x_1x_2x_3x_4, q) = 1$, for all $q \in \mathbb{N}$. Let $\mathscr{F}_k$ denote the set of all such n. Then Schwarz shows [15, Satz 6] that when $n \leq N$ and $n \in \mathscr{F}_k$ but n has no representation in the form (1.8), with all p_j prime, then n is in a set with no more than $O(N(\log N)^{-A})$ members, again for any fixed $A > 0$. There is also a corresponding result for Eq. (1.7) [15, Satz 5]. Our methods apply to such problems as well, leading to analogues of Theorems 1.1 and 1.2 with little difficulty, the overall details being somewhat simpler thanks to the presence of a fourth summand. In fact, it is possible in the two problems (1.7) and (1.8) to establish an asymptotic formula for the number of solutions, with the exceptional set estimate still of strength comparable to the conclusion in Theorem 1.1. We content ourselves with a brief discussion of such a result for Eq. (1.8). In this context, define the counting function

$$\varrho(n) = \sum_{p_1^2+p_2^3+p_3^6+p_4^k=n} (\log p_1)(\log p_2)(\log p_3)\log p_4 \tag{1.9}$$

and the singular series

$$\mathfrak{s}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{1\leq x_j \leq q \\ 1 \leq j \leq 4 \\ (x_j,q)=1}} \varphi(q)^{-4} e(a(x_1^2 + x_2^3 + x_3^6 + x_4^k - n)/q). \tag{1.10}$$

Theorem 1.3 *Let $k \geq 2$ be a natural number, and let $A > 1$. Then the inequality*

$$\left|\varrho(n) - \Gamma(\tfrac{1}{2})\Gamma(\tfrac{4}{3})\Gamma(\tfrac{7}{6})\mathfrak{s}(n)n^{1/k}\right| > n(\log n)^{-A}$$

holds for no more than $O(N^{1-1/(8k^2)+\varepsilon})$ *of the natural numbers n not exceeding N. Further, there is a number* $C \geq 1$ *such that for* $n \in \mathscr{F}_k$ *one has*

$$\mathfrak{s}(n) \gg (\log\log n)^{-C}. \tag{1.11}$$

In particular, all but $O(N^{1-1/(8k^2)+\varepsilon})$ of the numbers $n \in \mathscr{F}_k$ with $1 \leq n \leq N$ have a representation in the form (1.8). It should be noted that $\mathscr{F}_k$ is a union of finitely many arithmetic progressions (see [15, Satz 6]), and is therefore a set of positive density.

Throughout this paper, we apply the convention that whenever the letter ε occurs in a statement, it is asserted that this statement holds for all positive real numbers ε. Constants implicit in Landau's and Vinogradov's well-known symbols may depend on ε. Eulers totient is $\varphi(n)$, and $\omega(n)$ is the number of distinct prime factors of n.

2 A Mean Value Estimate

We begin our deliberations by discussing a moment estimate for a certain Weyl sum that we now introduce. Let $k \geq 3$, and put

$$h_0(\alpha, X) = \sum_{1 \leq x \leq X} e(\alpha x^k). \tag{2.1}$$

Let $M = X^{1/k}$, and let $\mathscr{P}$ denote the set of primes p with $p \equiv -1 \bmod k$ and $M < p \leq 2M$. Then, for $t \in \mathbb{N}$, we define the sum $h_t(\alpha, X)$ by means of the recursion

$$h_t(\alpha, X) = \sum_{p \in \mathscr{P}} h_{t-1}(\alpha p^k, X/p). \tag{2.2}$$

By (2.1) and (2.2), it follows that one may also write

$$h_t(\alpha, X) = \sum_{x \leq X} w_t(x, X) e(\alpha x^k) \tag{2.3}$$

where $w_0(x, X)$ is the indicator function on $\{1, 2, \ldots, [X]\}$, and where for $t \geq 1$, the coefficients $w_t(x, X)$ satisfy the relation

$$w_t(x, X) = \sum_{\substack{p \mid x \\ p \in \mathscr{P}}} w_{t-1}(x/p, X/p). \tag{2.4}$$

The sum on the right-hand side of (2.4) has at most $k-1$ summands. By induction on t, it follows that $w_t(x,X)=0$ for $x>X$, and that for $x\le X$ one has

$$w_t(x,X)\le k^t. \tag{2.5}$$

For $t\ge 0$ and $s\ge 1$, we shall bound the mean value

$$I(X,s,t)=\int_0^1 |h_0(\alpha,X)|^2|h_t(\alpha,X)|^{2s}\,\mathrm{d}\alpha. \tag{2.6}$$

It will be convenient to write $\theta=1-1/k$.

Lemma 2.1 *For $t\ge 0$ and $s\ge 1$, one has*

$$I(X,s,t)\ll X^{2s+2-k+k(\theta^t+3\theta^s)}. \tag{2.7}$$

This estimate should be compared with Lemma 2.1 of Brüdern and Perelli [4] where a slightly superior estimate was obtained, yet with a Weyl sum that is not as flexible as our h_t. Indeed, the Weyl sum used in [4] arises through the same recursion process as our h_t, but one would start with the extra constraint in (2.1) that the x are restricted to primes. With the Riemann Hypothesis for Dirichlet L-functions in hand, this makes little difference, but in unconditional work our new Weyl sum is much easier to use. This will become transparent in Sect. 4 below.

We now commence the proof of the lemma. By (2.3) and (2.5), we have $h_t(\alpha)\ll X$, and consequently, by (2.6), one has the trivial bound $I(X,s,t)\ll X^{2s+2}$. When $t=0$, this already established (2.7), for all s. Further, when $1\le s\le k$, one has $\theta^s\ge\theta^k\ge 1/\mathrm{e}>1/3$, and (2.7) is again evident.

We may now suppose that $s>k$ and $t\ge 1$, and then proceed by induction on $s+t$. By orthogonality, (2.6), (2.2) and (2.3), one finds that $I(X,s,t)$ equals the number of solutions of the diophantine equation

$$x_1^k-x_2^k=\sum_{j=1}^{2s}(-1)^j p_j^k y_j^k, \tag{2.8}$$

each solution counted with multiplicity

$$\prod_{j=1}^{2s} w_{t-1}(y_j,X/p_j), \tag{2.9}$$

and the natural numbers x_l and the primes p_j subject to $1\le x_l\le X$ and $p_j\in\mathscr{P}$. It may be worth remarking that we have $p_jy_j\le X$ for all $1\le j\le 2s$ whenever the multiplicity (2.9) is non-zero.

Let I_0 denote the number of solutions of (2.8), counted with weight (2.9), where for all $1\le j\le 2s$ one has $p_j\mid x_1$ and $p_j\mid x_2$. For $l=1$ or 2, let I_l denote the number of solutions of (2.8) where $p_1 \nmid x_l$, counted with weight (2.9). Then, since for any solution counted by $I(X,s,t)$ but not by I_0, there is some j such that p_j does not divide one of x_1, x_2, we deduce by symmetry that

$$I(X,s,t) \le I_0 + 2s(I_1 + I_2). \tag{2.10}$$

We require an upper bound for I_0. For $\mathbf{p} = (p_1, \dots, p_{2s}) \in \mathscr{P}^{2s}$, let

$$G(\alpha, \mathbf{p}) = \sum_{\substack{1 \le x \le X \\ p_j | x \, (1 \le j \le 2s)}} e(\alpha x^k). \tag{2.11}$$

Then, by orthogonality and the triangle inequality, and then applying the elementary bound $|z_1 z_2 \dots z_{2s}| \ll |z_1|^{2s} + \dots + |z_{2s}|^{2s}$, we infer that

$$\begin{aligned} I_0 &\le \sum_{\mathbf{p} \in \mathscr{P}^{2s}} \int_0^1 |G(\alpha, \mathbf{p})|^2 |h_{t-1}(\alpha p_1^k, X/p_1) \dots h_{t-1}(\alpha p_{2s}^k, X/p_{2s})| \, \mathrm{d}\alpha \\ &\ll \sum_{\mathbf{p} \in \mathscr{P}^{2s}} \int_0^1 |G(\alpha, \mathbf{p})|^2 |h_{t-1}(\alpha p_1^k, X/p_1)|^{2s} \, \mathrm{d}\alpha. \end{aligned} \tag{2.12}$$

Note that at most $k-1$ different primes $p_j \in \mathscr{P}$ can divide a number x with $x \le X$. Hence, for the sum in (2.11) to be non-empty, at most $k-1$ of the entries of $\mathbf{p}$ can be distinct, and we may restrict the sum in (2.12) to such $\mathbf{p}$, of which there are no more than $O(M^{k-1})$ choices. For a fixed such choice, we deduce from orthogonality that the integral in (2.12) does not exceed the number of solutions of the diophantine equation

$$x_1^k - x_2^k = p_1^k (y_1^k - y_2^k + \dots + y_{2s-1}^k - y_{2s}^k), \tag{2.13}$$

with the variables x_1, x_2 constrained to $1 \le x_l \le X$ and $p_1 \mid x_l$ $(l = 1, 2)$, each solution counted with weight

$$\prod_{j=1}^{2s} w_{t-1}(y_j, X/p_1). \tag{2.14}$$

We put $x_l = p_1 z_l$ in (2.13) and use orthogonality again to conclude from (2.6) that

$$\int_0^1 |G(\alpha, \mathbf{p})|^2 |h_{t-1}(\alpha p_1^k, X/p_1)|^{2s} \, \mathrm{d}\alpha \le I(X/p_1, s, t-1). \tag{2.15}$$

But $p_1 \in \mathscr{P}$ so that $p_1 > M$, and then the induction hypothesis supplies the bound

$$I(X/p_1, s, t-1) \ll (X/M)^{2s+2-k+k(\theta^{t-1}+3\theta^s)}.$$

By (2.12) and (2.15), we then deduce that

$$I_0 \ll M^{k-1}(X/M)^{2s+2-k+k(\theta^{t-1}+3\theta^s)} \ll X^\mu$$

where

$$\begin{aligned}\mu &= \theta + \theta\big(2s+2-k+k(\theta^{t-1}+3\theta^s)\big)\\ &\le 2s+2-k+k(\theta^t+3\theta^s)+\theta-\frac{1}{k}(2s+2-k).\end{aligned}$$

Hence, since $s>k$, we see that

$$I_0 \ll X^{2s+2-k+k(\theta^t+3\theta^s)}, \tag{2.16}$$

as required.

We now proceed by considering cases. First suppose that $I_1+I_2 \le I_0$. Then, by (2.10), we conclude that $I(X,s,t) \ll I_0$, and (2.7) follows from (2.16).

This leaves the case where $I_0 < I_1 + I_2$. Write

$$H_p(\alpha) = \sum_{\substack{1\le x\le X\\ p\nmid x}} e(\alpha x^k).$$

Then, for $l=1$ and 2, we infer by orthogonality that

$$I_l = \sum_{p\in\mathscr{P}} \int_0^1 H_p(\alpha)h_{t-1}((-1)^{l+1}p^k\alpha, X/p)h_0(-\alpha)h_t((-1)^l\alpha)|h_t(\alpha)|^{2s-2}\,\mathrm{d}\alpha,$$

where now we write $h_t(\alpha) = h_t(\alpha,X)$ in the interest of brevity. By Hölder's inequality and (2.6), one finds that

$$I_l \le \sum_{p\in\mathscr{P}} J_p^{\frac{1}{2s}} K_p^{\frac{1}{2}-\frac{1}{2s}} I(X,s,t)^{\frac{1}{2}}, \tag{2.17}$$

where

$$J_p = \int_0^1 |H_p(\alpha)|^2 |h_{t-1}(p^k\alpha, X/p)|^{2s}\,\mathrm{d}\alpha$$

and

$$K_p = \int_0^1 |H_p(\alpha)|^2 |h_t(\alpha)|^{2s}\,\mathrm{d}\alpha.$$

Note that the right-hand side of (2.17) is independent of l. Furthermore, by orthogonality and a consideration of the underlying diophantine equations, one has $K_p \le I(X,s,t)$. In the case under consideration, (2.10) yields $I(X,s,t) \ll I_1 + I_2$, and (2.17) delivers

$$I(X,s,t)^{1/(2s)} \ll \sum_{p\in\mathscr{P}} J_p^{1/(2s)}. \tag{2.18}$$

Fix a prime $p_1 \in \mathscr{P}$. Then, by orthogonality one finds that J_{p_1} is equal to the number of solutions of Eq. (2.13) with $1 \le x_1 \le X$, $1 \le x_2 \le X$, $p_1 \nmid x_1x_2$, each solution counted with multiplicity (2.14). By (2.13), we have $x_1^k \equiv x_2^k \bmod p_1^k$, and since $p_1 \in \mathscr{P}$, it follows that $p_1^k \mid x_1 - x_2$. However, $|x_1 - x_2| < X = M^k < p_1^k$ so that we must have $x_1 = x_2$. Hence, by (2.5), (2.6) and orthogonality again, we conclude that $J_{p_1} \ll XI(X/p_1, s-1, t-1)$. Since $X/p_1 \le X^\theta$, we first deduce from the induction hypothesis and (2.7) that

$$J_{p_1} \ll X^{1+\theta(2s-k)+k(\theta^t+3\theta^s)},$$

and then infer from (2.18) the final bound

$$I(X,s,t) \ll M^{2s}X^{1+\theta(2s-k)+k(\theta^t+3\theta^s)} = X^{2s+2-k+k(\theta^t+3\theta^s)},$$

completing the induction.

3 The Basic Argument

In this section, we shall present the basic argument that underpins all proofs. Along the way, we shall establish Theorem 1.1. Let $k \ge 3$, and consider the Weyl sums

$$f(\alpha) = \sum_{\frac14 N < p^2 \le N} e(\alpha p^2)\log p, \quad g(\alpha) = \sum_{\frac14 N < p^k \le N} e(\alpha p^k)\log p. \tag{3.1}$$

Whenever $\mathscr{B} \subset [0,1]$ is measurable, we put

$$r_{\mathscr{B}}(n,N) = \int_{\mathscr{B}} f(\alpha)^2 g(\alpha) e(-\alpha n)\,\mathrm{d}\alpha. \tag{3.2}$$

By orthogonality,

$$r_{[0,1]}(n,N) = \sum_{\substack{p_1^2+p_2^2+p_3^k=n\\ \frac14 N<p_1^2,p_2^2,p_3^k\le N}} (\log p_1)(\log p_2)\log p_3.$$

For $N < n \le 2N$, $n \in \mathscr{H}_k$, one expects $r_{[0,1]}(n, N)$ to be of size $N^{1/k}$, and we proceed to establish this for almost all n. Let

$$Q = N^{1/(4k)}, \tag{3.3}$$

and write $\mathfrak{M}$ for the union of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \le QN^{-1}(\log N)^{-30k}\}$$

with $0 \le a \le q$, $(a, q) = 1$ and $1 \le q \le Q$. We also put $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

The evaluation of $r_{\mathfrak{M}}(n, N)$ we are fortunate to be able to borrow from the work of Li [10]. Let

$$\mathfrak{S}(n, Q) = \sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{x_1,x_2,x_3=1 \\ (x_1x_2x_3,q)=1}}^{q} \varphi(q)^{-3} e(a(x_1^2 + x_2^2 + x_3^k - n)/q),$$

$$\mathfrak{J}(n, N) = \sum_{\substack{m_1+m_2+m_3=n \\ \frac14 N < m_1,m_2,m_3 \le N}} (m_1m_2)^{-1/2} m_3^{(1/k)-1}. \tag{3.4}$$

Then[2] Proposition 2.1 of Li [10] asserts that whenever $N < n \le 2N$, one has

$$r_{\mathfrak{M}}(n, N) = \frac{1}{4k}\mathfrak{S}(n, Q)\mathfrak{J}(n, N) + O(N^{1/k}(\log N)^{-30k}),$$

and for these n, the lower bound $\mathfrak{J}(n, N) \gg N^{1/k}$ is immediate. Further, Lemma 3.1 of Li [10] yields $\mathfrak{S}(n, Q) \gg (\log N)^{-15k}$ for all but $O(N^{1-1/(8k)+\varepsilon})$ of the integers $n \in \mathscr{H}_k$ with $N < n \le 2N$. This establishes the following result:

Lemma 3.1 *For all but $O(N^{1-1/(9k)})$ of the integers $n \in \mathscr{H}_k$ with $N < n \le 2N$, one has $r_{\mathfrak{M}}(n, N) \gg N^{1/k}(\log N)^{-15k}$.*

Our treatment of the minor arcs depends on an important estimate of Kumchev [8] that we now describe in a language suitable for application within this paper. Let $1 \le Y \le N^{1/8}$, and let $\mathfrak{N}(Y)$ denote the union of the pairwise disjoint intervals

$$\mathfrak{N}_{q,a}(Y) = \{\alpha \in [0,1] : |q\alpha - a| \le Y/N\}$$

with $0 \le a \le q$, $(a, q) = 1$ and $1 \le q \le Y$. We write $\mathfrak{N} = \mathfrak{N}(N^{1/8})$ and $\mathfrak{n} = [0,1] \setminus \mathfrak{N}$. We define the function $\Upsilon : [0,1] \to [0,1]$ by putting $\Upsilon(\alpha) = 0$ for $\alpha \in \mathfrak{n}$,

[2] An oversight in [10] is corrected here. The variables p_1^2, p_2^2, p_3^k run over $(\frac12 N, N]$ in [10], but then $\mathfrak{J}(n, N) = 0$ for $n < \frac32 N$ which is not acceptable. If these variables run over $(\frac14 N, N]$ instead, as we have arranged matters here, then the proof of Proposition 2.1 in Li [10] becomes valid.

and when $\alpha \in \mathfrak{N} \cap \mathfrak{N}_{q,a}(N^{1/8})$ by writing

$$\Upsilon(\alpha) = (q + N|q\alpha - a|)^{-1}.$$

Lemma 3.2 *Uniformly for* $\alpha \in [0, 1]$, *one has*

$$|f(\alpha)|^2 \ll N^{7/8+\varepsilon} + N^{1+\varepsilon}\Upsilon(\alpha).$$

Proof Theorem 3 of Kumchev [8] provides an estimate slightly stronger than that claimed in Lemma 3.2, but for the sum

$$\sum_{\frac{1}{4}N<p^2\leq N} e(\alpha p^2).$$

Thus, Lemma 3.2 follows by partial summation.

Before embarking on the estimation of the minor arc integral, we collect a number of mean value estimates. The first of these is the inequality

$$\int_0^1 |f(\alpha)|^4 \, \mathrm{d}\alpha \ll N^{1+\varepsilon} \tag{3.5}$$

that follows from a consideration of the underlying diophantine equation and Hua's lemma [18, Lemma 2.5]. Further, when $k = 3$, we put $u = 4$, and when $k \geq 4$, we put $u = k^2$. Then, the upper bound

$$\int_0^1 |g(\alpha)|^{2u} \, \mathrm{d}\alpha \ll N^{(2u/k)-1+\varepsilon} \tag{3.6}$$

is again a consequence of Hua's lemma in the special case $k = 3$, and for larger k this bound follows from Wooley's estimates for Vinogradov's mean value (for example, [20, Corollary 1.2], where an even stronger bound is obtained) and a consideration of the underlying diophantine equation.

We initiate the minor arc analysis by applying Bessel's inequality to (3.2). Thus

$$\sum_{N<n\leq 2N} r_{\mathfrak{m}}(n, N)^2 \leq \int_{\mathfrak{m}} |f(\alpha)|^4 |g(\alpha)|^2 \, \mathrm{d}\alpha. \tag{3.7}$$

By Hölder's inequality,

$$\int_{\mathfrak{n}} |f(\alpha)|^4 |g(\alpha)|^2 \, \mathrm{d}\alpha \leq \sup_{\alpha\in\mathfrak{n}} |f(\alpha)|^{4/u} \Big(\int_0^1 |f(\alpha)|^4 \, \mathrm{d}\alpha \Big)^{1-1/u} \Big(\int_0^1 |g(\alpha)|^{2u} \, \mathrm{d}\alpha \Big)^{1/u}.$$

Hence, by (3.5), (3.6) and Lemma 3.2,

$$\int_{\mathfrak{n}} |f(\alpha)|^4 |g(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{7/(4u)+\varepsilon} N^{1-1/u} N^{(2/k)-1/u} \ll N^{1+(2/k)-\delta+\varepsilon}, \tag{3.8}$$

where $\delta = 1/(4u)$.

This leaves the set $\mathfrak{N} \cap \mathfrak{m}$ for treatment, and this set is covered by the union of sets $\mathfrak{K}(Y) = \mathfrak{N}(2Y) \setminus \mathfrak{N}(Y)$ as Y runs over $2^{-j}N^{1/8}$, with $Q(\log N)^{-30k} \le Y \le \frac{1}{2}N^{1/8}$. Note that $\Upsilon(\alpha) \le Y^{-1}$ for $\alpha \notin \mathfrak{N}(Y)$. Further, Lemma 2 of Brüdern [2] supplies the bound

$$\int_{\mathfrak{N}(2Y)} \Upsilon(\alpha) |g(\alpha)|^2 \,\mathrm{d}\alpha \ll Y N^{1/k+\varepsilon-1} + N^{2/k+\varepsilon-1}.$$

This implies that

$$\int_{\mathfrak{K}(Y)} \Upsilon(\alpha)^2 |g(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1/k+\varepsilon-1} + Y^{-1} N^{2/k+\varepsilon-1}.$$

For $\alpha \in \mathfrak{N}$, we deduce from Lemma 3.2 that $|f(\alpha)|^2 \ll N^{1+\varepsilon}\Upsilon(\alpha)$. Hence, on summing over Y, we infer that

$$\int_{\mathfrak{N}\cap\mathfrak{m}} |f(\alpha)|^4 |g(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1+7/(4k)+\varepsilon}. \tag{3.9}$$

Note that this bound is superior to the one in (3.8), so that we now deduce from (3.7) that

$$\sum_{N<n\le 2N} r_{\mathfrak{m}}(n,N)^2 \ll N^{1+2/k-\delta+\varepsilon}.$$

Consequently, the inequality $|r_{\mathfrak{m}}(n,N)| > N^{1/k}(\log N)^{-30k}$ can hold for no more than $O(N^{1-\delta+\varepsilon})$ of the integers $n \in (N, 2N]$. The conclusion of Theorem 1.1 now follows by combining this last observation with Lemma 3.1 and a dyadic splitting up argument.

4 A Variant of the Main Theme

We now tune the basic argument to deliver Theorem 1.2. Fix $k \ge 4$, and recall that $\theta = 1 - 1/k$. Then choose $t \in \mathbb{N}$ such that $\theta^{t-1} > \frac{1}{36} \ge \theta^t$. With this choice of t, we put $h(\alpha) = h_t(\alpha, N^{1/k})$. With $f(\alpha)$ as in (3.1), we define

$$r^*_{\mathscr{B}}(n,N) = \int_{\mathscr{B}} f(\alpha)^2 h(\alpha) e(-\alpha n) \,\mathrm{d}\alpha,$$

where $\mathscr{B} \subset [0, 1]$ denotes a measurable set. By orthogonality, (3.1) and (2.3), one has

$$r^*_{[0,1]}(n, N) = \sum_{\substack{p_1^2+p_2^2+x^k=n \\ \frac{1}{4}N<p_1^2,p_2^2\le N}} w_t(x; N^{1/k})(\log p_1)\log p_2. \tag{4.1}$$

Much as in the previous section, we expect that for most $n \in (N, 2N] \cap \mathscr{G}_k$ the count $r^*_{[0,1]}(n, N)$ comes close to $n^{1/k}$, and this can be established by invoking Lemma 2.1 in place of (3.6) within the minor arc work performed in the previous section. However, the use of the sums $h(\alpha)$ causes extra complication in the major arc analysis.

We overwrite previous usage by now putting

$$Q = N^{\theta^t/k}.$$

Let $\mathfrak{M}$ and $\mathfrak{m}$ be defined as in Sect. 3, but with this new value of Q. The arcs $\mathfrak{N}$, $\mathfrak{N}(X)$ and $\mathfrak{n}$ retain their meaning from Sect. 3. We now mimic the argument departing from (3.7). Put $s = 4k$ and then apply Hölder's inequality to see that

$$\int_{\mathfrak{n}} |f(\alpha)|^4 |g(\alpha)|^2 \,\mathrm{d}\alpha \le \sup_{\alpha\in\mathfrak{n}} |f(\alpha)|^{4/s} \Big(\int_0^1 |f(\alpha)|^4 \,\mathrm{d}\alpha \Big)^{1-1/s} \Big(\int_0^1 |h(\alpha)|^{2s} \,\mathrm{d}\alpha \Big)^{1/s}.$$

By Lemma 2.1, (3.5) and Lemma 3.2, we discern that

$$\int_{\mathfrak{n}} |f(\alpha)^4 h(\alpha)^2| \,\mathrm{d}\alpha \ll N^{7/(4s)+\varepsilon} N^{1-1/s} (N^{(2s/k)-1+\theta^t+3\theta^s})^{1/s} \ll N^{1+(2/k)+\lambda+\varepsilon}, \tag{4.2}$$

where

$$\lambda = \frac{1}{s}\Big(\theta^t + 3\theta^s - \frac{1}{4}\Big).$$

Recall the definition of t, and observe that $\theta^k \le \mathrm{e}^{-1}$ and $\mathrm{e}^4 > 54$. It is now readily confirmed that

$$\lambda \le \frac{1}{s}\Big(3\theta^s - \frac{2}{9}\Big) \le -\frac{1}{24k}. \tag{4.3}$$

Next, we consider the range $\mathfrak{N} \cap \mathfrak{m}$. Mimicry of the deduction of (3.9) yields

$$\int_{\mathfrak{N}\cap\mathfrak{m}} |f(\alpha)|^4 |h(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1+2/k+\varepsilon} Q^{-1}.$$

Now, since $k \geq 4$ and $\theta^t \geq \frac{1}{36}\theta > \frac{1}{48}$, we infer from (4.2) and (4.3) that

$$\int_{\mathfrak{m}} |f(\alpha)|^4 |h(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\frac{1}{48k}}.$$

By using Bessel's inequality as in (3.7), this estimate allows us to conclude as follows:

Lemma 4.1 *For all but $O(N^{1-1/(49k)})$ of the integers $n \in (N, 2N]$ one has $|r^*_{\mathfrak{m}}(n, N)| < N^{1/k}(\log N)^{-100k}$.*

For the major arcs, we have to prepare for an application of the methods of Liu and Zhan [11] with quite some effort. We begin with an explicit formula for the sums $h(\alpha)$. Put $P = N^{1/k}$. With t still as above, let $\mathscr{U}(N)$ denote the set of all tuples $\mathbf{p} = (p_1, \ldots, p_t)$ built from primes $p_j \equiv -1 \bmod k$ $(1 \leq j \leq t)$ that satisfy the inequalities

$$P^{1/k} < p_1 \leq 2P^{1/k}, \quad (P/(p_1 \ldots p_j))^{1/k} < p_{j+1} \leq 2(P/(p_1 \ldots p_j))^{1/k} \quad (1 \leq j < t). \tag{4.4}$$

For $\mathbf{p} \in \mathscr{U}(N)$ we write $u = u(\mathbf{p}) = p_1 p_2 \ldots p_t$. Then, repeated use of (2.2) yields

$$h(\alpha) = \sum_{\mathbf{p} \in \mathscr{U}(N)} \sum_{x \leq P/u} e(\alpha(ux)^k). \tag{4.5}$$

We summarize some estimates concerning $\mathscr{U}(N)$ in the next lemma.

Lemma 4.2 *For all $\mathbf{p} \in \mathscr{U}(N)$ one has $u(\mathbf{p}) \asymp P^{1-\theta^t}$. Further,*

$$\#\mathscr{U}(N) \asymp P^{1-\theta^t}(\log N)^{-t}.$$

Proof An inspection of (4.4) shows that there exist constants $0 < c'_{1,j} < c_{1,j} < c_{2,j} < c'_{2,j}$ with the property that whenever $c_{1,j}P^{\theta^{j-1}} < p_j < c_{2,j}P^{\theta^{j-1}}$ holds for all $1 \leq j \leq t$, then $\mathbf{p} \in \mathscr{U}(N)$, and whenever $\mathbf{p} \in \mathscr{U}(N)$, then $c'_{1,j}P^{\theta^{j-1}} < p_j < c'_{2,j}P^{\theta^{j-1}}$ for all j. The conclusions of Lemma 4.2 now follow from Chebychev's estimates.

We are ready to develop a major arc approximation for the sum $h(\alpha)$. When $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, we put

$$S(q, a) = \sum_{x=1}^{q} e(ax^k/q), \quad V(\beta, Y) = \int_0^Y e(\beta\gamma^k)\,\mathrm{d}\gamma.$$

Then, according to Theorem 4.1 of Vaughan [18], one has

$$\sum_{x \leq P/u} e\Big(\Big(\frac{a}{q} + \beta\Big)(ux)^k\Big) = q^{-1}S(q, au^k)V(\beta u^k, P/u) + O(q^{\varepsilon}(q + Nq|\beta|)^{1/2}).$$

A change of variable shows that $V(\beta u^k, P/u) = u^{-1}V(\beta, P)$. Hence, on taking $\alpha = (a/q) + \beta$ in (4.5), we deduce from Lemma 4.2 that

$$h(\alpha) = V(\beta, P) \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{S(q, au^k)}{qu} + O\big(P^{1-\theta^t} q^{\varepsilon}(q + Nq|\beta|)^{1/2})\big)$$

where again we wrote $u = u(\mathbf{p})$ in the interest of brevity. Next, consider the sum

$$v(\beta) = \frac{1}{k}\sum_{m\le N} m^{-\theta} e(\beta m).$$

Maclaurin's summation formula yields $V(\beta, P) = v(\beta) + O(1 + P|\beta|)$. Hence, if we define the function $h^*(\alpha)$ for $\alpha = (a/q) + \beta \in \mathfrak{M}$ with $q \le Q$, $(a, q) = 1$ and $|\beta| \le Q/N$ by

$$h^*(\alpha) = v(\beta) \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{S(q, au^k)}{qu}, \tag{4.6}$$

we find that uniformly for $\alpha \in \mathfrak{M}$ one has

$$h(\alpha) = h^*(\alpha) + O(P^{1-\frac{1}{2}\theta^t+\varepsilon}).$$

Let

$$r'(n, N) = \int_{\mathfrak{M}} h^*(\alpha) f(\alpha)^2 e(-\alpha n)\,\mathrm{d}\alpha.$$

Then, via Bessel's inequality, we deduce from the previous estimate that

$$\sum_{N<n\le 2N} \big(r^*_{\mathfrak{M}}(n, N) - r'(n, N)\big)^2 \le \int_{\mathfrak{M}} |(h(\alpha) - h^*(\alpha)) f(\alpha)^2|^2\,\mathrm{d}\alpha \ll P^{2-\theta^t+\varepsilon} N,$$

and we may conclude as follows:

Lemma 4.3 *For all but $O(N^{1-1/(49k)})$ of the integers $n \in (N, 2N]$ one has*

$$|r^*_{\mathfrak{M}}(n, N) - r'(n, N)| < N^{1/k}(\log N)^{-100k}.$$

We now evaluate $r'(n, N)$ further, replacing the exponential sum $f(\alpha)$ by its natural approximation. This is straightforward by the method proposed in [11]. In this process we receive help from the function $h^*(\alpha)$, as this decays in q. To see this, note that whenever $(a, q) = 1$, one finds from Theorem 4.2 of Vaughan [18] that $q^{-1}S(q, au^k) \ll q^{-1/k}(q, u^k)^{1/k}$. Consequently, by (4.6), Lemma 4.2 and [18, Lemma 2.8],

$$h^*(\alpha) \ll P(q + P^k|q\alpha - a|)^{-1/k} \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{(q, u)}{u} \ll P(\log N)^{-t}(q + P^k|q\alpha - a|)^{-1/k}.$$

In such a situation, the method of Liu and Zhan [11] is particularly easy to apply, and on following the recent exposition of Hoffman and Yu [6], for example, one identifies a leading term, featuring the singular integral (3.4) and a kind of singular series that we now introduce. Let

$$S^*(q, a) = \sum_{\substack{x=1\\(x,q)=1}}^{q} e(ax^2/q),$$

and then put

$$A_u(q, n) = \varphi(q)^{-2}q^{-1} \sum_{\substack{a=1\\(a,q)=1}}^{q} S^*(q, a)^2 S(q, au^k)e(-an/q) \tag{4.7}$$

to form the sum

$$\Theta(n, N) = \sum_{\mathbf{p}\in\mathscr{U}(N)} u(\mathbf{p})^{-1} \sum_{q\le Q} A_{u(\mathbf{p})}(q, n). \tag{4.8}$$

Equipped with this notation, we summarize the outcome of the Liu–Zhan method in the next lemma, but as we pointed out already, there is no need to present a detailed proof because the reader will have no difficulty in providing one along the lines of Hoffmann and Yu [6, Sect. 6].

Lemma 4.4 *Let $A > 1$. Then, for all $N < n \le 2N$, one has*

$$r'(n, N) = \Theta(n, N)\mathfrak{J}(n, N) + O(N^{1/k}(\log N)^{-A}).$$

Our next task is to disentangle the sum over $\mathbf{p}$ in (4.8), and realize the partial singular series

$$\mathfrak{S}^*(n, q) = \sum_{q\le Q} A_1(q, n) \tag{4.9}$$

as a factor. With this end in view, we examine $A_u(q, n)$ more closely. The number of incongruent solutions x_1, x_2, x_3 of the congruence $x_1^2 + x_2^2 + (ux_3)^k \equiv n \bmod q$ with $(x_1x_2, q) = 1$ is a multiplicative function of q, and by orthogonality, this function is given by

$$q^{-1}\sum_{a=1}^{q} S^*(q,a)^2 S(q,au^k)e(-an/q).$$

Hence, by Möbius's inversion formula, we read off from (4.7) that $A_u(q,n)$ is also multiplicative as a function of q. Thus, we may restrict attention to the case where $q = p^l$ is a power of a prime. In this instance, we apply a result of Hua [7], showing that $S^*(p^l,a) = 0$ holds whenever $p \nmid a$ and $l \geq 2$ when p is odd, and also for $p = 2$, $l \geq 4$. Hence, by (4.7), for all $u \in \mathbb{N}$ one has

$$A_u(p^l,n) = 0 \quad (l \geq 2, p \geq 3, \text{ or } l \geq 4, p = 2). \tag{4.10}$$

Now suppose that p is an odd prime. When $p \mid u$, we find from (4.7) that

$$A_u(p,n) = A_p(p,n) = (p-1)^{-2}\sum_{a=1}^{p-1} S^*(p,a)^2 e(-an/p).$$

We write

$$\tau_p = \sum_{x=1}^{p} e(x^2/p), \quad c_p(n) = \sum_{a=1}^{p-1} e(-an/p).$$

Then, by familiar properties of the quadratic Gauß sum, whenever $p \nmid a$ we have

$$S^*(p,a) = \sum_{x=1}^{p} e(ax^2/p) - 1 = \Big(\frac{a}{p}\Big)\tau_p - 1.$$

Hence

$$S^*(p,a)^2 = \tau_p^2 + 1 - 2\Big(\frac{a}{p}\Big)\tau_p. \tag{4.11}$$

and

$$A_p(p,n) = (p-1)^{-2}(\tau_p^2+1)c_p(n) - 2(p-1)^{-2}\tau_p\sum_{a=1}^{p-1}\Big(\frac{a}{p}\Big)e\Big(-\frac{an}{p}\Big).$$

The remaining sum on the right transforms into $(n \mid p)_L\bar{\tau}_p$, so that

$$A_p(p,n) = (p-1)^{-2}(\tau_p^2+1)c_p(n) - 2(p-1)^{-2}|\tau_p|^2\Big(\frac{n}{p}\Big).$$

From the classical identity $|\tau_p|^2 = p$ we now conclude that

$$|A_p(p,n)| \le 8p^{-1}(p,n). \tag{4.12}$$

Next consider odd primes p with $p \nmid u$. Then, a simple transformation shows that $S(p, au^k) = S(p,a)$, and hence that $A_u(p,n) = A_1(p,n)$. When $p \nmid a$ one also has (Vaughan [18, Lemma 4.3])

$$S(p,a) = \sum_{\chi} \bar{\chi}(a)\tau(\chi)$$

where χ runs over the $(k, p-1) - 1$ non-principal characters χ, mod p, for which χ^k is principal, and where

$$\tau(\chi) = \sum_{a=1}^{p} \chi(a)e(a/p).$$

By (4.11),

$$\begin{aligned} A_1(p,n) &= (p-1)^{-2}p^{-1}\sum_{a=1}^{p-1}\sum_{\chi}\bar{\chi}(a)\tau(\chi)\Big(\tau_p^2 + 1 - 2\Big(\frac{a}{p}\Big)\tau_p\Big)e\Big(-\frac{an}{p}\Big) \\ &= (p-1)^{-2}p^{-1}\sum_{\chi}\tau(\chi)(\tau_p^2+1)\sum_{a=1}^{p-1}\bar{\chi}(a)e\Big(-\frac{an}{p}\Big) \\ &\quad - 2(p-1)^{-2}p^{-1}\sum_{\chi}\tau(\chi)\tau_p\sum_{a=1}^{p-1}\bar{\chi}(a)\Big(\frac{a}{p}\Big)e\Big(-\frac{an}{p}\Big). \end{aligned}$$

When $p \mid n$, the first summand on the right vanishes, and so does the second unless $\chi = \bar{\chi}$ is the Legendre symbol. From the standard upper bound for Gauß sums, it now follows that $|A_1(p,n)| \le 2/(p-1)$ whenever $p \mid n$. If $p \nmid n$, then the sum over a in the first summand on the right-hand side of the preceding display again transforms into a Gauß sum, and so does the sum over a in the second summand unless again χ is the Legendre symbol, in which case the relevant sum becomes the Ramanujan sum $c_p(n)$. A short calculation now shows that whenever $p \nmid n$ one has

$$|A_1(p,n)| \le \frac{4k}{p-1} \le \frac{6k}{p}, \tag{4.13}$$

and we then see that this bound holds for all odd primes p.

Based on (4.12) and (4.13), we may now evaluate $\Theta(n,N)$ in terms of the singular series.

Lemma 4.5 *We have*

$$\Theta(n,N) = \mathfrak{S}^*(n,Q) \sum_{\mathbf{p}\in\mathscr{U}(N)} u(\mathbf{p})^{-1} + O(N^{-\theta^t/k}).$$

Proof By (4.8) and (4.9),

$$\Theta(n,N) - \mathfrak{S}^*(n,Q) \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{1}{u} = \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{1}{u} \sum_{q\le Q} \big(A_u(q,n) - A_1(q,n)\big). \tag{4.14}$$

By (4.10), we may restrict the sum over q to the set

$$\mathscr{Q} = \{q \le Q : 16 \nmid q,\ p^2 \nmid q \text{ for all odd } p\}.$$

Further, for each pair $\mathbf{p}, q$ with $\mathbf{p} \in \mathscr{U}(N)$, $q \in \mathscr{Q}$ and $(u(\mathbf{p}), q) = 1$, one has $A_u(q,n) = A_1(q,n)$, so that these pairs do not contribute to (4.14). Hence the sum in (4.14) may be restricted further, to those pairs $\mathbf{p}, q$ where $(p_1p_2\dots p_t, q) > 1$.

Let I denote a non-empty subset of $\{1,2,\dots,y\}$, and let

$$\mathscr{K}(I) = \{(\mathbf{p},q) \in \mathscr{U}(N)\times\mathscr{Q} : p_i \mid q \text{ for } i \in I,\ p_j \nmid q \text{ for } j \notin I\}.$$

The argument from the preceding paragraph shows that any pair $\mathbf{p}, q$ that makes a non-zero contribution to the sum on the right-hand side of (4.14) is in some $\mathscr{K}(I)$. Now suppose that $(\mathbf{p},q) \in \mathscr{K}(I)$. Then, at least when N is large, the $p_1,\dots,p_t$ are all distinct, and hence, the number

$$u' = \prod_{i\in I} p_i$$

is a divisor of q with $(q/u', u(\mathbf{p})) = 1$. It follows that

$$A_u(q,n) = A_u(u',n)A_u(q/u',n) = A_{u'}(u',n)A_1(q/u',n),$$

and we infer that

$$A_u(q,n) - A_1(q,n) = \big(A_{u'}(u',n) - A_1(u',n)\big)A_1(q/u',n).$$

Consequently, the contribution from $(\mathbf{p},q) \in \mathscr{K}(I)$ to the sum in (4.14) is no larger than

$$\sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{1}{u} \sum_{\substack{q\in\mathscr{Q}\\ q\equiv 0 \bmod u'}} \big|A_1(q/u',n)\big(A_{u'}(u',n) - A_1(u',n)\big)\big|.$$

We now write $v = q/u'$ and deduce from (4.12) and (4.13) that the expression in the preceding display is bounded by

$$\ll \sum_{\mathbf{p}\in\mathscr{U}(N)} \frac{1}{u}\Big(\sum_{v\le Q}\frac{(6k)^{\omega(v)}}{v}\Big)\frac{8^{\omega(u')}}{u'}(u',n) \ll (\log Q)^{6k}\sum_{\mathbf{p}\in\mathscr{U}(N)}\frac{(u',n)}{uu'}.$$

Here the sum over $\mathbf{p}$ factorizes. For $i\in I$ the corresponding factor does not exceed

$$\sum_{p_i\asymp N^{\theta^{i-1}/k}} \frac{(p_i,n)}{p_i^2} \ll N^{-\theta^{i-1}/k}, \tag{4.15}$$

while for $j\notin I$ the sum over p_j is certainly bounded. Since I is non-empty, we have at least one factor (4.15), producing the estimate $O(N^{-\theta^t/k})$ for the portion of (4.14) where $(\mathbf{p},q)\in\mathscr{K}(I)$. Summation over I completes the proof of Lemma 4.5.

We now need a lower bound for the singular series. Although there is no explicit reference at hand for the sum $\mathfrak{S}^*(n,Q)$, there is no difficulty in adjusting the arguments of [9, Sect. 6] to the present needs, and we obtain the following result, analogous to Lemma 3.1 of Li [10].

Lemma 4.6 *For all but $O(NQ^{\varepsilon-1/2})$ of the integers $n\in(N,2N]\cap\mathscr{G}_k$, one has $\mathfrak{S}^*(n,Q)\gg(\log n)^{-15k}$.*

We are ready to establish Theorem 1.2. Indeed, from Lemmas 4.2, 4.5 and 4.6, we find that $\Theta(n,N)\gg(\log N)^{-15k-t}$ holds for all but $O(N^{-1/(97k)})$ of the integers $n\in(N,2N]\cap\mathscr{G}_k$. By Lemma 4.4 and (3.4), we deduce that for these n we also have $r'(n,N)\gg N^{1/k}(\log N)^{-15k-t}$. The definition of t implies that $t\le 5k$, and hence, by Lemmas 4.1 and 4.3, we finally see that the lower bound

$$r^*_{[0,1]}(n,N)\gg N^{1/k}(\log N)^{-20k}$$

holds for all but $O(N^{-1/(97k)})$ of the integers $n\in(N,2N]\cap\mathscr{G}_k$. By (4.1) and a dyadic dissection argument, this confirms the conclusions recorded in Theorem 1.2.

5 The Quaternary Problem

We end with a short sketch of a proof of Theorem 1.3. Since we attempt to establish an asymptotic formula, we can no longer work with localized Weyl sums but have to use their brethren

$$F_l(\alpha)=\sum_{p^l\le N} e(\alpha p^l)\log p.$$

Then, by (1.9) and orthogonality, whenever $n \le N$ one has

$$\varrho(n) = \int_0^1 F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

We define major and minor arcs $\mathfrak{M}$ and $\mathfrak{m}$ as in Sect. 3, with Q defined by (3.3). When $\mathscr{B}$ is a measurable subset of $[0, 1]$, we put

$$\varrho_{\mathscr{B}}(n, N) = \int_{\mathscr{B}} F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

The major arc work will not detain us for long, as this is standard for the experienced worker in the area. First, one begins by applying the now standard methods of Liu and Zhan [11], and this leads to the preliminary asymptotic relation

$$\varrho_{\mathfrak{M}}(n, N) = \Gamma(\tfrac{1}{2})\Gamma(\tfrac{4}{3})\Gamma(\tfrac{7}{6})\mathfrak{s}(n, Q)n^{1/k} + O(N^{1/k}(\log N)^{-A})$$

that is valid for any fixed $A > 1$ and all $n \le N$, and in which the partial singular series is given via

$$\mathfrak{s}(n, Q) = \sum_{q \le Q} B(q, n)$$

and

$$B(q, n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{1 \le x_j \le q \\ 1 \le j \le 4 \\ (x_j,q)=1}} \varphi(q)^{-4} e(a(x_1^2 + x_2^3 + x_3^6 + x_4^k - n)/q).$$

The theory of the arithmetic function $B(q, n)$ is similar to that of $A(q, n)$ in the previous section. In particular, $B(q, n)$ is again multiplicative in q, and by an argument paralleling that leading to (4.10), (4.12) and (4.13), one finds that $B(p, n) \ll p^{-3/2}(p, n)^{1/2}$ while $B(p^l, n) = 0$ holds for all p, all $l \ge 2$. It follows easily that the sum (1.10) converges absolutely, and that $\mathfrak{s}(n) - \mathfrak{s}(n, Q) \ll n^{\varepsilon}Q^{\varepsilon-1/2}$. In particular, we see that

$$\varrho_{\mathfrak{M}}(n, N) = \Gamma(\tfrac{1}{2})\Gamma(\tfrac{4}{3})\Gamma(\tfrac{7}{6})\mathfrak{s}(n)n^{1/k} + O(N^{1/k}(\log N)^{-A}). \tag{5.1}$$

To obtain a lower bound for $\mathfrak{s}(n)$, we use multiplicativity to write the series as an Euler product which takes the shape

$$\mathfrak{s}(n) = \prod_p (1 + B(p, n)).$$

By orthogonality, one deduces that $1+B(p,n) = p(p-1)^{-4}M(p,n)$ where $M(p,n)$ is the number of incongruent solutions of the congruence $y_1^2+y_2^3+y_3^6+y_4^k \equiv n \bmod p$ with $p \nmid y_1y_2y_3y_4$. The congruence condition in Theorem 1.3 assures that $M(p,n) \geq 1$, and so the Euler factors of $\mathfrak{s}(n)$ are positive, and bounded below uniformly in n by $p(p-1)^{-4}$. Further, for $p \nmid n$, we noted earlier that $B(p,n) \ll p^{-3/2}$. Hence, we conclude that

$$\mathfrak{s}(n) \gg \prod_{p|n}(1 + B(p,n)).$$

For $p \mid n$, we mentioned already that there is a number $c > 0$ with $|B(p,n)| \leq cp^{-1}$. We apply this bound for $p \mid n$ with $p > 2c$, and use the uniform lower bound for $1 + B(p,n)$ for the smaller primes to confirm (1.11).

This leaves the minor arcs $\mathfrak{m}$. The main difficulty here is that Kumchev's exponential sum estimates refer to localized sums. We have to turn these into bounds for $F_l(\alpha)$, and this requires some care.

Lemma 5.1 *Uniformly for* $\alpha \in [0,1]$, *one has*

$$F_2(\alpha)^2 \ll N^{7/8+\varepsilon} + N^{1+\varepsilon}\Upsilon(\alpha), \quad F_3(\alpha)^2 \ll N^{13/21+\varepsilon} + N^{2/3+\varepsilon}\Upsilon(\alpha).$$

Proof Let $P = N^{1/2}$ and $\alpha \in [0,1]$. For $1 \leq R \leq P$, put

$$f(\alpha, R) = \sum_{R<p\leq 2R} e(\alpha p^2)\log p.$$

A dyadic dissection argument provides some $R = R(\alpha)$ with $P^{7/8} \leq R \leq \frac{1}{2}P$ and the property that

$$F_2(\alpha) = \sum_{P^{7/8}<p\leq P} e(\alpha p^2)\log p + O(P^{7/8}) \ll P^{7/8} + |f(\alpha,R)|\log N. \tag{5.2}$$

Fix a (small) $\delta > 0$. If $|f(\alpha,R)| \leq P^{7/8+\delta}$, then $F_2(\alpha) \ll P^{7/8+\delta}$. In the opposite case, we apply Dirichlet's theorem on diophantine approximation to find $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $1 \leq q \leq R^{3/2}$ and $|q\alpha - a| \leq R^{-3/2}$ before using Theorem 3 of Kumchev [8] in conjunction with partial summation. This yields

$$P^{7/8+\delta} < |f(\alpha,R)| \ll R^{7/8+\varepsilon} + R^{1+\varepsilon}(q + R^2|q\alpha - a|)^{-1/2}. \tag{5.3}$$

It follows that $q + R^2|q\alpha - a| \ll R^{2+\varepsilon}P^{-7/4-2\delta}$. For large N, this implies that $q \leq P^{1/4}$ and $|q\alpha - a| \leq P^{-7/4}$, and hence that $\alpha \in \mathfrak{N}$. The trivial bound

$$R^2(q + P^2|q\alpha - a|) \leq P^2(q + R^2|q\alpha - a|)$$

now suffices to conclude from (5.3) that $|f(\alpha,R)|^2 \ll N^{1+\varepsilon}\Upsilon(\alpha)$, and (5.2) then delivers $|F_2(\alpha)|^2 \ll N^{1+\varepsilon}\Upsilon(\alpha)$. We have now shown that whenever $\delta > 0$, one has

$$|F_2(\alpha)|^2 \ll N^{7/8+\delta} + N^{1+\varepsilon}\Upsilon(\alpha).$$

This establishes Lemma 5.1 for F_2, and the bound for F_3 follows by the same argument.

We are ready to run the basic argument from Sect. 3 for the minor arcs. One observes that

$$\int_0^1 |F_2(\alpha)F_3(\alpha)F_6(\alpha)|^2\,\mathrm{d}\alpha \ll N^{1+\varepsilon} \tag{5.4}$$

(see Schwarz [15], Lemma 1.2 and Korollar), and as in (3.6) we also have

$$\int_0^1 |F_k(\alpha)|^{2k^2}\,\mathrm{d}\alpha \ll N^{2k-1+\varepsilon}. \tag{5.5}$$

By Hölder's inequality, we find that the integral

$$\int_{\mathfrak{n}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)|^2\,\mathrm{d}\alpha$$

does not exceed

$$\sup_{\alpha\in\mathfrak{n}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)|^{2/k^2}\Big(\int_0^1 |F_2F_3F_6|^2\,\mathrm{d}\alpha\Big)^{1-1/k^2}\Big(\int_0^1 |F_k|^{2k^2}\,\mathrm{d}\alpha\Big)^{1/k^2}.$$

We now use (5.4), (5.5) and Lemma 5.1 for F_2 together with trivial bounds for F_3F_6 to conclude that

$$\int_{\mathfrak{n}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)|^2\,\mathrm{d}\alpha \ll N^{1+\frac{2}{k}+\frac{1}{8k^2}+\varepsilon}. \tag{5.6}$$

For the set $\mathfrak{N}\cap\mathfrak{m}$ the argument from Sect. 3 requires an upgrade. First we apply Lemma 2 of Brüdern [2] to confirm that

$$\int_{\mathfrak{N}} \Upsilon(\alpha)|F_6(\alpha)|^2\,\mathrm{d}\alpha \ll N^{\varepsilon-2/3},$$

and we then see via Lemma 5.1 that

$$\int_{\mathfrak{N}} |F_2(\alpha)F_6(\alpha)|^2\,\mathrm{d}\alpha \ll N^{1/3+\varepsilon}.$$

For $\alpha \in \mathfrak{N} \setminus \mathfrak{N}(N^{1/21})$, we find from Lemma 5.1 that $|F_3(\alpha)|^2 \ll N^{13/21+\varepsilon}$. The trivial bound for F_k is now enough to confirm the estimate

$$\int_{\mathfrak{N}\setminus\mathfrak{N}(N^{1/21}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{\frac{20}{21}+\frac{2}{k}+\varepsilon} \tag{5.7}$$

For $\alpha \in \mathfrak{N}(N^{1/21})$ Lemma 5.1 produces $|F_2(\alpha)F_3(\alpha)| \ll N^{5/3+\varepsilon}\Upsilon(\alpha)^2$, and then the argument leading to (3.9) delivers the bound

$$\int_{\mathfrak{N}(N^{1/21})\cap\mathfrak{m}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1+\frac{7}{4k}+\varepsilon}.$$

This estimate combines with (5.6), (5.7) and Bessel's inequality to

$$\sum_{n\le N} \varrho_{\mathfrak{m}}(n,N)^2 \le \int_{\mathfrak{m}} |F_2(\alpha)F_3(\alpha)F_6(\alpha)F_k(\alpha)|^2 \,\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\frac{1}{8k^2}+\varepsilon}.$$

It follows that for all but $O(N^{1-1/(8k^2)+2\varepsilon})$ of the integers $n \le N$ one has $\varrho_{\mathfrak{m}}(n,N) \ll N^{1/k-\varepsilon}$. This combines with (5.1) to confirm the conclusions in Theorem 1.3.

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