Dynamical Systems and Uniform Distribution of Sequences

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Dedicated to the memory of Professor Wolfgang Schwarz

Abstract We give a survey on classical and recent applications of dynamical systems to number theoretic problems. In particular, we focus on normal numbers, also including computational aspects. The main result is a sufficient condition for establishing multidimensional van der Corput sets. This condition is applied to various examples.

Keywords Dynamical systems • Uniform distribution • van der Corput set

2010 Mathematics subject classification: Primary 11J71; Secondary 11K38

1 Dynamical Systems in Number Theory

In the last decades dynamical systems became very important for the development of modern number theory. The present paper focuses on Furstenberg's refinements of Poincaré's recurrence theorem and applications of these ideas to Diophantine problems.

A (measure-theoretic) dynamical system is formally given as a quadruple $(X, \mathfrak{B}, \mu, T)$, where $(X, \mathfrak{B}, \mu)$ is a probability space with σ-algebra $\mathfrak{B}$ of measurable sets and μ a probability measure; $T\colon X \to X$ is a measure-preserving

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J. Sander et al. (eds.), *From Arithmetic to Zeta-Functions*,
DOI 10.1007/978-3-319-28203-9_17

transformation on this space, i.e. $\mu(T^{-1}A) = \mu(A)$ for all measurable sets $A \in \mathfrak{B}$. In the theory of dynamical systems, properties of the iterations of the transformation T are of particular interest. For this purpose we only consider invertible transformations and call such dynamical systems invertible.

The first property, we consider, originates from Poincaré's famous recurrence theorem (see Theorem 1.4 of [32] or Theorem 2.11 of [13]) saying that starting from a set A of positive measure $\mu(A) > 0$ and iterating T yields infinitely many returns to A. More generally, we call a subset $\mathcal{R} \subset \mathbb{N}$ of the positive integers a set of recurrence if for all invertible dynamical systems and all measurable sets A of positive measure $\mu(A) > 0$ there exists $n \in \mathcal{R}$ such that $\mu(A \cap T^{-n}A) > 0$. Then Poincaré's recurrence theorem means that $\mathbb{N}$ is a set of recurrence.

A second important theorem for dynamical systems is Birkhoff's ergodic theorem (see Theorem 1.14 of [32] or Theorem 2.30 of [13]). We call T ergodic if the only invariant sets under T are sets of measure 0 or of measure 1, i.e. $T^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$. Then Birkhoff's ergodic theorem connects average in time with average in space, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x) = \int_X f(x) d\mu(x)$$

for all $f \in L^1(X, \mu)$ and μ-almost all $x \in X$.

Let us explain an important application of this theorem to number theory. For $q \geq 2$ a positive integer, consider $T\colon [0,1) \to [0,1)$ defined by $T(x) = \{qx\}$, where $\{t\} = t - \lfloor t \rfloor$ denotes the fractional part of t. If $x \in \mathbb{R}$ is given by its q-ary digit expansion $x = \lfloor x \rfloor + \sum_{j=1}^{\infty} a_j(x) q^{-j}$, then the digits $a_j(x)$ can be computed by iterating this transformation T: $a_j(x) = d$ if $T^{j-1}x \in \left[\frac{d}{q}, \frac{d+1}{q}\right)$ with $d \in \{0, 1, \ldots, q-1\}$. Moreover, since $a_j(Tx) = a_{j+1}(x)$ for $j \geq 1$ the transformation T can be seen as a left shift of the expansion.

Now we call a real number x simply normal in base q if

$$\lim_{\mathbb{N}\to\infty} \frac{1}{N} \#\{j \leq N\colon a_j(x) = d\} = \frac{1}{q}$$

for all $d = 0, \ldots, q-1$, i.e., all digits d appear asymptotically with equal frequencies $1/q$. A number x is called q-normal if it is simply normal with respect to all bases $q, q^2, q^3, \ldots$. This is equivalent to the fact that the sequence $(\{q^n x\})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 (for short: u.d. mod 1), which also means that all blocks $d_1, d_2, \ldots, d_L$ of subsequent digits appear in the expansion of x asymptotically with the same frequency q^{-L} (cf. [8, 12, 16]). For completeness, let us give here one possible definition of u.d. sequences (x_n): a sequence of real numbers x_n is called u.d. mod 1 if for all continuous functions $f : [0,1] \to \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x) dx. \tag{1.1}$$

Note, that by Weyl's criterion the class of continuous functions can be replaced by trigonometric functions $e(hx) = e^{2\pi ihx}$, $h \in \mathbb{N}$ or by characteristic functions $1_I(x)$ of intervals $I = [a, b)$. Applying Birkhoff's ergodic theorem shows that Lebesgue almost all real numbers are q-normal in any base $q \geq 2$. Defining a real number to be absolutely normal if it is q-normal for all bases $q \geq 2$, this immediately yields that almost all real numbers are absolutely normal.

In particular, this shows the existence of absolutely normal numbers. However, it is a different story to find constructions of (absolutely) normal numbers. It is a well-known difficult open problem to show that important numbers like $\sqrt{2}$, ln 2, e, π etc. are simply normal with respect to some given base $q \geq 2$. A much easier task is to give constructions of q-normal numbers for fixed base q. Champernowne [9] proved that

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\ldots$$

is normal to base 10 and later this type of constructions was analyzed in detail. So, for instance, for arbitrary base $q \geq 2$

$$0.\langle\lfloor g(1)\rfloor\rangle_q\ \langle\lfloor g(2)\rfloor\rangle_q \ldots$$

is q-normal, where $g(x)$ is a non-constant polynomial with real coefficients and the q-normal number is constructed by concatenating the q-ary digit expansions $\langle\lfloor g(n)\rfloor\rangle_q$ of the integer parts of the values $g(n)$ for $n = 1, 2, \ldots$. These constructions were extended to more general classes of functions g (replacing the polynomials) (see [11, 18, 19, 22, 23, 29]) and the concatenation of $\langle[g(p)]\rangle_q$ along prime numbers instead of the positive integers (see [10, 17, 18, 24]).

All such constructions depend on the choice of the base number $q \geq 2$, and thus they are not suitable for constructing absolutely normal numbers. A first attempt to construct absolutely normal numbers is due to Sierpinski [30]. However, Turing [31] observed that Sierpinski's "construction" does not yield a computable number, thus it is not based on a recursive algorithm. Furthermore, Turing gave an algorithm for a construction of an absolutely normal number. This algorithm is very slow and, in particular, not polynomially in time. It is very remarkable that Becher et al. [2] established a polynomial time algorithm for the construction of absolutely normal numbers. However, there remain various questions concerning the analysis of these algorithms. The discrepancy of the corresponding sequences is not studied and the order of convergence of the expansion is very slow and should be investigated in detail. Furthermore, digital expansions with respect to linear recurring base sequences seem appropriate to be included in the study of absolute normality from a computational point of view.

Let us now return to Poincaré's recurrence theorem which shortly states that the set $\mathbb{N}$ of positive integers is a recurrence set. In the 1960s various stronger concepts were introduced:

1. $\mathcal{R} \subseteq \mathbb{N}$ is called a nice recurrence set if for all invertible dynamical systems and all measurable sets A of positive measure $\mu(A) > 0$ and all $\varepsilon > 0$, there exist infinitely many $n \in \mathcal{R}$ such that

$$\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon.$$

2. $\mathcal{H} \subseteq \mathbb{N}$ is called a van der Corput set (for short: vdC set) if the following implications holds:

$$(x_{n+h} - x_n)_{n\in\mathbb{N}} \text{ is u.d. mod 1 for all } h \in \mathcal{H} \Longrightarrow (x_n)_{n\in\mathbb{N}} \text{ is u.d. mod 1.}$$

Clearly, any nice recurrence set is a recurrence set. By van der Corput's difference theorem (see [12, 16]) the set $\mathcal{H} = \mathbb{N}$ of positive integers is a vdC set. Kamae and Mendès-France [15] proved that any vdC set is a nice recurrence set. Ruzsa [25] conjectured that any recurrence set is also a vdC set. An important tool in the analysis of recurrence sets is their equivalence with intersective (or difference) sets established by Bertrand-Mathis [5]. We call a set $\mathcal{I}$ intersective if for each subset $E \subseteq \mathbb{N}$ of positive (upper) density, there exists $n \in \mathcal{I}$ such that $n = x - y$ for some $x, y \in E$. Here the upper density of E is defined as usual by

$$\overline{d}(E) = \limsup_{N\to\infty} \frac{\#(E \cap [1, N])}{N}.$$

Bourgain [7] gave an example of an intersective set which is not a vdC set, hence contradicting the above mentioned conjecture of Ruzsa.

Furstenberg [14] proved that the values $g(n)$ of a polynomial $g \in \mathbb{Z}[x]$ with $g(0) = 0$ form an intersective set and later it was shown by Kamae and Mendès-France [15] that this is a vdC set, too. It is also known that for fixed $h \in \mathbb{Z}$ the set of shifted primes $\{p \pm h : p \text{ prime}\}$ is a vdC set if and only if $h = \pm 1$. [20, Corollary 10]. This leads to interesting applications to additive number theory, for instance to new proofs and variants of theorems of Sárközy [26–28]. A general result concerning intersective sets related to polynomials along primes is due to Nair [21].

In the present paper we want to extend the concept of recurrence sets, nice recurrence sets, and vdC sets to subsets of $\mathbb{Z}^k$, following the program of Bergelson and Lesigne [3] and our earlier paper [4]. In Sect. 2 we summarize basic facts concerning these concepts, including general relations between them and counter examples. Section 3 is devoted to a sufficient condition for establishing the vdC property. In the final Sect. 4 we collect various examples and give some new applications.

2 van der Corput Sets

In this section we provide various equivalent definitions of van der Corput sets in $\mathbb{Z}^k$. In particular, we give four different definitions, which are k-dimensional variants of the one-dimensional definitions, whose equivalence is due to Ruzsa [25]. These generalizations were established by Bergelson and Lesigne [3]. Then we present a set, which is not a vdC set in order to give some insight into the structure of vdC sets. Finally, we define the higher-dimensional variant of nice recurrence sets.

2.1 Characterization via Uniform Distribution

Similar to above we first define a van der Corput set (vdC set for short) in $\mathbb{Z}^k$ via uniform distribution.

Definition 2.1 *A subset $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$ is a vdC set if any family $(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ of real numbers is u.d. mod 1 provided that it has the property that for all $\mathbf{h} \in \mathcal{H}$ the family $(x_{\mathbf{n}+\mathbf{h}} - x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ is u.d. mod 1.*

Here the property of u.d. mod 1 for the multi-indexed family $(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ is defined via a natural extension of 1.1:

$$\lim_{N_1,N_2,\ldots,N_k\to+\infty} \frac{1}{N_1N_2\cdots N_k} \sum_{0\le\mathbf{n}<(N_1,N_2,\ldots,N_k)} f(x_{\mathbf{n}}) = \int_0^1 f(x)dx \tag{2.1}$$

for all continuous functions $f : [0,1] \to \mathbb{R}$. Here in the limit $N_1, N_2, \ldots, N_k$ are tending to infinity independently and $<$ is defined componentwise.

Using the k-dimensional variant of van der Corput's inequality we could equivalently define a vdC set as follows:

Definition 2.2 *A subset $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$ is a van der Corput set if for any family $(u_{\mathbf{n}})_{n\in\mathbb{Z}^k}$ of complex numbers of modulus 1 such that*

$$\forall \mathbf{h} \in \mathcal{H}, \quad \lim_{N_1,N_2,\ldots,N_k\to+\infty} \frac{1}{N_1N_2\cdots N_k} \sum_{0\le\mathbf{n}<(N_1,N_2,\ldots,N_k)} u_{\mathbf{n}+\mathbf{h}}\overline{u_{\mathbf{n}}} = 0$$

the relation

$$\lim_{N_1,N_2,\ldots,N_k\to+\infty} \frac{1}{N_1N_2\cdots N_k} \sum_{0\le\mathbf{n}<(N_1,N_2,\ldots,N_k)} u_{\mathbf{n}} = 0$$

holds.

2.2 Trigonometric Polynomials and Spectral Characterization

The first two definitions are not very useful for proving or disproving that a set $\mathcal{H}$ is a vdC set. Similar to the one-dimensional case the following spectral characterization involving trigonometric polynomials is a better tool.

Theorem 2.3 ([3, Proposition 1.18]) *A subset $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$ is a van der Corput set if and only if for all $\varepsilon > 0$, there exists a real trigonometric polynomial P on the k-torus $\mathbb{T}^k$ whose spectrum is contained in $\mathcal{H}$ and which satisfies $P(0) = 1$, $P \geq -\varepsilon$.*

The set of polynomials fulfilling the last theorem for a given ε forms a convex set. Moreover the conditions may be interpreted as some infimum. Therefore we might expect some dual problem, which is actually provided by the following theorem. For details see Bergelson and Lesigne [3] or Montgomery [20].

Theorem 2.4 ([3, Theorem 1.8]) *Let $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$. Then $\mathcal{H}$ is a van der Corput set if and only if for any positive measure σ on the k-torus $\mathbb{T}^k$ such that, for all $\mathbf{h} \in \mathcal{H}$, $\hat{\sigma}(\mathbf{h}) = 0$, this implies $\sigma(\{(0, 0, \ldots, 0)\}) = 0$.*

2.3 Examples

The structure of vdC sets is better understood by first giving a counter example. The following lemma shows to be very useful in the construction of counter examples.

Lemma 2.5 *Let $\mathcal{H} \subset \mathbb{N}$. If there exists $q \in \mathbb{N}$ such that the set $\mathcal{H} \cap q\mathbb{N}$ is finite, then the set $\mathcal{H}$ is not a vdC set.*

Proof The proof is a combination of the following two observations of Ruzsa [25] (see Theorem 2 and Corollary 3 of [20]):

1. Let $m \in \mathbb{N}$. The sets $\{1, \ldots, m\}$ and $\{n \in \mathbb{N} : m \nmid n\}$ are both not vdC sets.
2. Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \subset \mathbb{N}$. If $\mathcal{H}$ is a vdC set, then $\mathcal{H}_1$ or $\mathcal{H}_2$ also has to be a vdC set.

Suppose there exists a $q \in \mathbb{N}$ such that $\mathcal{H} \cap q\mathbb{N}$ is finite. Then we may split $\mathcal{H}$ into the sets $\mathcal{H} \cap q\mathbb{N}$ and $\mathcal{H} \setminus q\mathbb{N}$. The first one is finite and the second one contains no multiples of q. Therefore both are not vdC sets and hence $\mathcal{H}$ is not a vdC set.

The first counter example deals with arithmetic progressions.

Lemma 2.6 *Let $a, b \in \mathbb{N}$. If the set $\{an + b : n \in \mathbb{N}\}$ is a vdC set, then $a \mid b$.*

Proof Let $b \in \mathbb{N}$ and $\mathcal{H} = \{an + b : n \in \mathbb{N}\}$ be a vdC set. Then by Lemma 2.5 we must have

$$an + b \equiv b \equiv 0 \bmod a \quad \text{infinitely often.}$$

This implies that $a \mid b$.

The sufficiency (and also the necessity) of the requirement $a \mid b$ follows from the following result of Kamae and Mendès-France [15] (cf. Corollary 9 of [20]).

Lemma 2.7 *Let $P(z) \in \mathbb{Z}[z]$ and suppose that $P(z) \to +\infty$ as $z \to +\infty$. Then $\mathcal{H} = \{P(n) > 0 : n \in \mathbb{N}\}$ is a vdC set if and only if for every positive integer q the congruence $P(z) \equiv 0 \pmod q$ has a root.*

Now we want to establish a similar result for sets of the form $\{ap + b : p \text{ prime}\}$. In this case the following result is due to Bergelson and Lesigne [3] which is a generalization of the case $f(x) = x$ due to Kamae and Mendès-France [15].

Lemma 2.8 ([3, Proposition 1.22]) *Let f be a (non zero) polynomial with integer coefficients and zero constant term. Then the sets $\{f(p-1) : p \in \mathbb{P}\}$ and $\{f(p+1) : p \in \mathbb{P}\}$ are vdC sets in $\mathbb{Z}$.*

We show the converse direction.

Lemma 2.9 *Let a and b be nonzero integers. Then the set $\{ap + b : p \in \mathbb{P}\}$ is a vdC set if and only if $|a| = |b|$, i.e., $ap + b = a(p \pm 1)$.*

Proof It is clear from Lemma 2.8 that $\{ap + b : p \in \mathbb{P}\}$ is a vdC set if $|a| = |b|$.

On the contrary a combination of Lemma 2.5 and Lemma 2.6 yields that $a \mid b$. Now we consider the sequence modulo b. Then by Lemma 2.5 we get that

$$ap + b \equiv ap \equiv 0 \bmod b \quad \text{infinitely often.}$$

Since $(p, b) > 1$ only holds for finitely many primes p we must have $b \mid a$. Combining these two requirements yields $|a| = |b|$.

3 A Sufficient Condition

In this section we want to formulate a general sufficient condition which provides us with a tool to show for plenty of different examples that they generate a vdC set. This is a generalization of the conditions of Kamae and Mendès-France [15] and Bergelson and Lesigne [3]. Before stating the condition we need an auxiliary lemma.

Lemma 3.1 ([3, Corollary 1.15]) *Let d and e be positive integers, and let L be a linear transformation from $\mathbb{Z}^d$ into $\mathbb{Z}^e$ (represented by an $e \times d$ matrix with integer entries). Then the following assertions hold:*

1. *If D is a vdC set in $\mathbb{Z}^d$ and if $0 \notin L(D)$, then $L(D)$ is a vdC set in $\mathbb{Z}^e$.*
2. *Let $D \subseteq \mathbb{Z}^d$. If the linear map L is one-to-one, and if $L(D)$ is a vdC set in $\mathbb{Z}^e$, then D is a vdC set in $\mathbb{Z}^d$.*

Our main tool is the following general result. Applications are given in the next section.

Proposition 3.2 *Let* $g_1, \dots, g_k\colon \mathbb{N} \to \mathbb{Z}$ *be arithmetic functions. Suppose that* $g_{i_1}, \dots, g_{i_m}$ *is a basis of the* $\mathbb{Q}$*-vector space* $\operatorname{span}(g_1, \dots, g_k)$*. For each* $q \in \mathbb{N}$*, we introduce*

$$D_q := \left\{(g_{i_1}(n), \dots, g_{i_m}(n)\colon n \in \mathbb{N} \text{ and } q! \mid g_{i_j}(n) \text{ for all } j = 1, \dots, m\right\}.$$

Suppose further that, for every q*, there exists a sequence* $(h_n^{(q)})_{n\in\mathbb{N}}$ *in* D_q *such that, for all* $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m \setminus \mathbb{Q}^m$*, the sequence* $(h_n^{(q)} \cdot \mathbf{x})_{n\in\mathbb{N}}$ *is uniformly distributed mod* 1*. Then*

$$\tilde{D} := \{(g_1(n), \dots, g_k(n))\colon n \in \mathbb{N}\} \in \mathbb{Z}^k$$

is a vdC set.

Proof We first show that the set

$$D := \{(g_{i_1}(n), \dots, g_{i_m}(n))\colon n \in \mathbb{N}\}$$

is a vdC set in $\mathbb{Z}^m$. For $q, N \in \mathbb{N}$ we define a family of trigonometric polynomials

$$P_{q,N} := \frac{1}{N} \sum_{n=1}^{N} e\left(h_n^{(q)} \cdot \mathbf{x}\right).$$

By hypothesis, $\lim_{N\to\infty} P_{q,N}(x) = 0$ for $x \notin \mathbb{Q}^m$. For fixed q there exists a subsequence $(P_{q,N'})$ which converges pointwise to a function g_q. Since $g_q(x) = 1$ (for $x \in \mathbb{Q}^m$ and q sufficiently large) and $g_q(x) = 0$ (for $x \notin \mathbb{Q}^m$), the sequence (g_q) is pointwise convergent to the indicator function of $\mathbb{Q}^m$. For a positive measure σ on the m-dimensional torus with vanishing Fourier transform $\hat{\sigma}$ on D, we have $\int P_{q,N} \mathrm{d}\sigma = 0$ for all q, N. Thus $\sigma(\mathbb{Q}^m) = 0$ follows from the dominating convergence theorem, obviously $\sigma(\{0, 0, \dots, 0\}) = 0$, and thus D is a vdC set.

In order to prove that $\tilde{D}$ is a vdC set we apply Lemma 3.1 twice. Since $g_{i_1}, \dots, g_{i_m}$ is a base of $\operatorname{span}(g_1, \dots, g_k)$, we can write each g_j as a linear combination (with rational coefficients) of $g_{i_1}, \dots, g_{i_m}$. Multiplying with the common denominator of the coefficients yields

$$a_j g_j = b_{j,1} g_{i_1} + \cdots + b_{j,m} g_{i_m}$$

for $j = 1, \dots, k$ and certain $a_j, b_{j,\ell} \in \mathbb{Z}$. Considering the transformation $L\colon \mathbb{Z}^m \to \mathbb{Z}^k$ given by the matrix $(b_{j,\ell})$ and applying part (1) of Lemma 3.1 shows that

$$\{(a_1 g_1(n), \dots, a_k g_k(n))\colon n \in \mathbb{N}\}$$

is a vdC set for certain integers $a_1, \dots, a_k$.

Now consider the transformation $\tilde{L}: \mathbb{Z}^k \to \mathbb{Z}^k$ given by the $k \times k$ diagonal matrix with entries $a_1, \ldots, a_k$ in the diagonal. Then by part (2) of Lemma 3.1 also $\tilde{D}$ is a vdC set and the proposition is proved.

4 Various Examples and Applications to Additive Problems

In this section we consider multidimensional variants of prime powers, entire functions and $x^\alpha \log^\beta x$ sequences.

4.1 Prime Powers

In a recent paper the authors together with Bergelson, Kolesnik, and Son [4] consider sets of the form

$$\{(\alpha_1(p_n \pm 1)^{\theta_1}, \ldots, \alpha_k(p_n \pm 1)^{\theta_k}): n \in \mathbb{N}\},$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and $p_n \in \mathcal{P}$ runs over all prime numbers. These sets are vdC, however, we missed the treatment of a special case in the proof. In particular, if for some $i \neq j$ the exponents satisfy $\theta_i = \theta_j =: \theta$, then the vector (p_n^θ, p_n^θ) is not uniformly distributed mod 1.

Here we close this gap.

Theorem 4.1 *If α_i are positive integers and β_i are positive and non-integers, then*

$$D_1 = \{\big((p-1)^{\alpha_1}, \cdots, (p-1)^{\alpha_k}, [(p-1)^{\beta_1}], \cdots, [(p-1)^{\beta_\ell}]\big) \mid p \in \mathcal{P}\},$$

and

$$D_2 = \{\big((p+1)^{\alpha_1}, \cdots, (p+1)^{\alpha_k}, [(p+1)^{\beta_1}], \cdots, [(p+1)^{\beta_\ell}]\big) \mid p \in \mathcal{P}\}$$

are vdC sets in $\mathbb{Z}^{k+\ell}$.

Proof Since x^{θ_1} and x^{θ_2} are $\mathbb{Q}$-linear dependent for all $x \in \mathbb{Z}$ if and only if $\theta_1 = \theta_2$, an application of Proposition 3.2 yields that it suffices to consider the case where all exponents are different. However, this follows by the same arguments as in the proof of Theorem 4.1 in [4].

4.2 Entire Functions

In this section we consider entire functions of bounded logarithmic order. We fix a transcendental entire function f and denote by $S(r) := \max_{|z|\leq r} |f(z)|$. Then we call λ the logarithmic order of f if

$$\limsup_{r\to\infty} \frac{\log S(r)}{\log r} = \lambda.$$

The central tool is the following result of Baker [1].

Theorem 4.2 ([1, Theorem 2]) *Let f be a transcendental entire function of logarithmic order $1 < \lambda < \frac{4}{3}$. Then the sequence*

$$(f(p_n))_{n\geq 1}$$

is uniformly distributed mod 1.

Our second example of a class of vdC sets is the following.

Theorem 4.3 *Let $f_1, \ldots, f_k$ be entire functions with distinct logarithmic orders $1 < \lambda_1, \lambda_2, \ldots, \lambda_k < \frac{4}{3}$, respectively. Then the set*

$$D := \{(\lfloor f_1(p_n)\rfloor, \ldots, \lfloor f_k(p_n)\rfloor) : n \in \mathbb{N}\}$$

is a vdC set.

Proof We enumerate $D = (\mathbf{d}_n)_{n\geq 1}$, where

$$\mathbf{d}_n := (\lfloor f_1(p_n)\rfloor, \ldots, \lfloor f_k(p_n)\rfloor).$$

First we show that for every $q \in \mathbb{N}$ the set

$$D^{(q)} := \{(d_1, \ldots, d_k) \in D : q \mid d_i\}$$

has positive relative density in D. We note that if $0 \leq \left\{\frac{f_i(p_n)}{q}\right\} < \frac{1}{q}$ for $1 \leq i \leq k$, then $\mathbf{d}_n \in D^{(q)}$. By Theorem 4.2 the sequence

$$\left(\left(\frac{f_1(p_n)}{q}, \ldots, \frac{f_k(p_n)}{q}\right)\right)_{n\geq 1}$$

is uniformly distributed and thus $D^{(q)}$ has positive density in D.

For each $q \in \mathbb{N}$ we enumerate the elements of $D^{(q!)} = (\mathbf{d}_n^{(q!)})_{n\geq 1}$, such that $\left|\mathbf{d}_n^{(q!)}\right|$ is increasing. Since the logarithmic orders are distinct we immediately get that the functions f_i are $\mathbb{Q}$-linearly independent. Thus by Proposition 3.2 it is sufficient to show that for all $q \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k \setminus \mathbb{Q}^k$ the sequence $(\mathbf{d}_n^{(q!)} \cdot \mathbf{x})_{n \geq 1}$ is u.d. mod 1.

Using the orthogonality relations for additive characters we get for any nonzero integer h that

$$\frac{1}{\left|\{n \leq N \colon \mathbf{d}_n \in D^{(q!)}\}\right|} \sum_{n \leq N} e\left(h\left(d_n^{(q!)} \cdot \mathbf{x}\right)\right)$$

$$= \frac{1}{\left|\{n \leq N \colon \mathbf{d}_n \in D^{(q!)}\}\right|} \frac{1}{(q!)^k} \sum_{j_1=1}^{q!} \cdots \sum_{j_k=1}^{q!} \frac{1}{N} \sum_{n \leq N} e\left(d_n \cdot \left(h\mathbf{x} + \left(\frac{j_1}{q!}, \ldots, \frac{j_k}{q!}\right)\right)\right).$$

The innermost sum is of the form

$$\sum_{n \leq N} e(g(p_n)),$$

with $g(x) = \sum_{i=1}^k \alpha_i \lfloor f_i(x) \rfloor$ for a certain $(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \setminus \mathbb{Q}^k$.

By relabeling the terms we may suppose that there exists an ℓ such that $\alpha_1, \ldots, \alpha_\ell \notin \mathbb{Q}$ and $\alpha_{\ell+1}, \ldots, \alpha_k \in \mathbb{Q}$. Furthermore we may write $\alpha_j = \frac{a_j}{q}$ for $\ell + 1 \leq j \leq m$. Then

$$e(g(p_n)) = e\left(\sum_{i=1}^k \alpha_k \lfloor f_i(p_n) \rfloor\right) = \prod_{j=1}^{\ell} s_j(\alpha_j f_j(p_n), f_j(p_n)) \prod_{j=\ell+1}^{k} t_j(\lfloor f_j(p_n) \rfloor),$$

where $s_j(x, y) = e(x - \{y\}\alpha_j)$ $(1 \leq j \leq \ell)$ and $t_j(z) = e\left(a_j \frac{z}{q}\right)$ $(\ell + 1 \leq j \leq k)$.

Since $s_j(x, y)$ is Riemann-integrable on $\mathbb{T}^2$ for $j = 1, \ldots, \ell$ and $t_j(z)$ is continuous on $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, the function $\prod_{j=1}^{\ell} s_j \prod_{j=\ell+1}^{k} t_j$ is Riemann-integrable on $\mathbb{T}^{2\ell} \times \mathbb{Z}_q^{k-\ell}$.

Now an application of Theorem 4.2 yields that for any $u \in \mathbb{N}$ the sequence

$$\left(\alpha_1 f_1(p_n), f_1(p_n), \ldots, \alpha_\ell f_\ell(p_n), f_\ell(p_n), \frac{f_{\ell+1}(p_n)}{u}, \ldots, \frac{f_k(p_n)}{u}\right)_{n \geq 1}$$

is u.d. in $\mathbb{T}^{2\ell} \times \mathbb{T}^{k-\ell}$. Since $\lfloor x \rfloor \equiv a \pmod{q}$ is equivalent to $\frac{x}{q} \in [\frac{a}{q}, \frac{a+1}{q}]$, we deduce that

$$(\alpha_1 f_1(p_n), f_1(p_n), \ldots, \alpha_\ell f_\ell(p_n), f_\ell(p_n), \lfloor f_{\ell+1}(p_n) \rfloor, \ldots, \lfloor f_k(p_n) \rfloor)_{n \geq 1}$$

is u.d. in $\mathbb{T}^{2\ell} \times \mathbb{Z}_q^{k-\ell}$, and Weyl's criterion implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} e\left(\sum_{i=1}^{k} \alpha_i \lfloor f_i(p_n)\rfloor\right) = 0,$$

proving the theorem.

4.3 Functions of the Form $x^\alpha \log^\beta x$

In the one-dimensional case Boshernitzan et al. [6] showed, among other result, that these sets are vdC sets. Our aim is to show an extended version for the k-dimensional case. Therefore we use the following general criterion, which is a combination of Fejer's theorem and van der Corput's difference theorem.

Theorem 4.4 ([16, Theorem 3.5]) *Let $f(x)$ be a function defined for $x > 1$ that is k-times differentiable for $x > x_0$. If $f^{(k)}(x)$ tends monotonically to 0 as $x \to \infty$ and if $\lim_{x\to\infty} x\left|f^{(k)}(x)\right| = \infty$, then the sequence $(f(n))_{n\ge1}$ is u.d. mod 1.*

Applying this theorem we get the following

Corollary 4.5 *Let $\alpha \neq 0$ and*

- *either $\sigma > 0$ not an integer and $\tau \in \mathbb{R}$ arbitrary*
- *or $\sigma > 0$ an integer and $\tau \in \mathbb{R} \setminus [0, 1]$.*

Then the sequence $(\alpha n^\sigma \log^\tau n)_{n\ge2}$ is u.d. mod 1.

Our third example is the following class of vdC sets.

Theorem 4.6 *Let $\alpha_1, \dots, \alpha_k > 0$ and $\beta_1, \dots, \beta_k \in \mathbb{R}$, such that $\beta_i \notin [0, 1]$ whenever $\alpha_i \in \mathbb{Z}$ for $i = 1, \dots, k$. Then the set*

$$D := \{(\lfloor n^{\alpha_1} \log^{\beta_1} n\rfloor, \dots, \lfloor n^{\alpha_k} \log^{\beta_k} n\rfloor) : n \in \mathbb{N}\}$$

is a vdC set.

Proof Following the same arguments as is the proof of Theorem 4.3 and replacing the uniform distribution result for entire functions (Theorem 4.3) by the corresponding result for $n^\alpha \log^\beta n$ sequences (Corollary 4.5) yields the proof.

Acknowledgements This research work was done when the first author was a visiting professor at the Department of Analysis and Number Theory at Graz University of Technology. The author thanks the institution for its hospitality. For the realization of the present paper the first author received support from the Conseil Régional de Lorraine.

The second author acknowledges support of the project F 5510-N26 within the special research area "Quasi Monte-Carlo Methods and applications" founded by the Austrian Science Fund.

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