The Joint Discrete Universality of Periodic Zeta-Functions

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To the memory of Professor Wolfgang Schwarz

Abstract In the paper, a joint discrete universality theorem on approximation of a pair of analytic functions by shifts of periodic zeta-functions and periodic Hurwitz zeta-functions is obtained. For the proof the linear independence over $\mathbb{Q}$ of a certain set is used.

Keywords Algebraically independent numbers • Joint universality • Linear independence • Periodic Hurwitz zeta-function • Periodic zeta-function • Weak convergence

2010 Mathematics subject classification: Primary 11M41; Secondary 32E30

1 Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be two periodic sequences of complex numbers with minimal periods $k \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively. The periodic zeta-function $\zeta(s; \mathfrak{a})$ and periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$ with parameter α, $0 < \alpha \leq 1$, are defined, for $\sigma > 1$, by the series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

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J. Sander et al. (eds.), *From Arithmetic to Zeta-Functions*,
DOI 10.1007/978-3-319-28203-9_15

Let $\zeta(s,\alpha)$ denote the classical Hurwitz zeta-function which is for $\sigma > 1$ given by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. In view of the periodicity of the sequences $\mathfrak{a}$ and $\mathfrak{b}$, we have that, for $\sigma > 1$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta\left(s, \frac{m}{k}\right)$$

and

$$\zeta(s,\alpha;\mathfrak{b}) = \frac{1}{l^s} \sum_{m=0}^{l-1} b_m \zeta\left(s, \frac{m+\alpha}{l}\right).$$

Thus, by the above remark on the function $\zeta(s,\alpha)$, these equalities give analytic continuation for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ to the whole complex plane, except for a hypothetical simple pole at $s = 1$ with residues

$$a = \frac{1}{k} \sum_{m=1}^{k} a_m \quad \text{and} \quad b = \frac{1}{l} \sum_{m=0}^{l-1} b_m,$$

respectively. If $a = 0$ or $b = 0$, then both functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ are entire. Clearly, $\zeta(s;\mathfrak{a})$ is a generalization of the Riemann zeta-function, and $\zeta(s,\alpha;\mathfrak{b})$ generalizes the Hurwitz zeta-function.

In [3], a joint universality theorem on the approximation of a pair of analytic functions by shifts $\zeta(s+i\tau;\mathfrak{a})$ and $\zeta(s+i\tau,\alpha;\mathfrak{b})$ has been obtained. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K, and by $H_0(K)$, $K \in \mathcal{K}$, the subclass of $H(K)$ of non-vanishing functions on K. We remind that a sequence of numbers a_n is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprime integers m, n. Now we state the main result of [3].

Theorem 1.1 *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and that the number α is transcendental. Let $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Theorem 1.1 is a generalization of the Mishou theorem proved in [9] for the Riemann zeta and Hurwitz zeta-functions. Theorem 1.1 is the so-called continuous universality theorem because analytic functions are approximated by the continuous shifts $\zeta(s + i\tau; \mathfrak{a})$ and $\zeta(s + i\tau, \alpha; \mathfrak{b})$, $\tau \in \mathbb{R}$. The aim of this paper is to prove a discrete version of Theorem 1.1 where analytic functions are approximated by discrete shifts $\zeta(s + ikh; \mathfrak{a})$ and $\zeta(s + ikh, \alpha; \mathfrak{b})$, or, more generally, by shifts $\zeta(s + ikh_1; \mathfrak{a})$ and $\zeta(s + ikh_2, \alpha; \mathfrak{b})$, $k \in \mathbb{N}_0$, with fixed positive numbers h, h_1, and h_2.

Denote by $\mathcal{P}$ the set of all prime numbers, and define the set

$$L(\mathcal{P}, \alpha, h, \pi) = \left\{(\log p : \ p \in \mathcal{P}), (\log(m + \alpha) : \ m \in \mathbb{N}_0), \frac{\pi}{h}\right\}.$$

Theorem 1.2 *Suppose that the set $L(\mathcal{P}, \alpha, h, \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and that the sequence $\mathfrak{a}$ is multiplicative. Let $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \ \sup_{s \in K_1} |\zeta(s + ikh; \mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

Now let

$$L(\mathcal{P}, \alpha, h_1, h_2, \pi) = \{(h_1 \log p : \ p \in \mathcal{P}), (h_2 \log(m + \alpha) : \ m \in \mathbb{N}_0), \pi\}.$$

We note that this set consists of all elements of two sequences in addition with the number π. Then we have the following generalization of Theorem 1.2:

Theorem 1.3 *Suppose that the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and that the sequence $\mathfrak{a}$ is multiplicative. Let $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \ \sup_{s \in K_1} |\zeta(s + ikh_1; \mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s \in K_2} |\zeta(s + ikh_2, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

Obviously, for $h_1 = h_2 = h$ Theorem 1.3 reduces to Theorem 1.2.

By the Nesterenko theorem [11], the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Therefore, the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ with $\alpha = \pi^{-1}$ and rational h_1, h_2 is linearly independent over $\mathbb{Q}$.

Professor Wolfgang Schwarz was a mathematician with a broad outlook. Among other problems of probabilistic and analytic number theory, he was also interested in universality of zeta-functions, and this is confirmed by the paper [7].

2 Two Lemmas

For the proof of Theorem 1.3, we will apply the probabilistic approach based on a limit theorem on weakly convergent probability measures in the space of analytic functions. In this section, we present two lemmas which play an important role in the proof of such a limit theorem.

Let $\gamma = \{s \in \mathbb{C} : \; |s| = 1\}$, and

$$\Omega_1 = \prod_{p \in \mathcal{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where γ_p for all $p \in \mathcal{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the classical Tikhonov theorem, with the product topology and pointwise multiplication, the tori Ω_1 and Ω_2 are compact topological abelian groups. Therefore, denoting by $\mathcal{B}(X)$ the Borel σ-field of the space X, we observe that on $(\Omega_j, \mathcal{B}(\Omega_j))$ the probability Haar measure m_{jH} can be defined, $j = 1, 2$. Denote by $\omega_1(p)$ the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_p, $p \in \mathcal{P}$, and by $\omega_2(m)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Then $\omega_1(p)$, $p \in \mathcal{P}$, and $\omega_2(m)$, $m \in \mathbb{N}_0$, are complex-valued random variables defined on the spaces $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ and $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, respectively.

We define one more set by $\Omega = \Omega_1 \times \Omega_2$. Then Ω is also a compact topological abelian group. This leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ with the probability Haar measure m_H. We denote by $\omega = (\omega_1, \omega_2)$ the elements of Ω, where $\omega_j \in \Omega_j, j = 1, 2..$

Next, for $A \in \mathcal{B}(\Omega)$, let

$$Q_N(A) \overset{def}{=} \frac{1}{N+1} \# \Big\{ 0 \le k \le N : \; \big((p^{-ikh_1} : \; p \in \mathcal{P}), \\ ((m+\alpha)^{-ikh_2} : \; m \in \mathbb{N}_0) \big) \in A \Big\}.$$

Lemma 2.1 *Suppose that the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof We apply the Fourier transform method, and use similar arguments as in the proof of Lemma 7 in [3]; recalling the proof of this lemma, we observe that the Fourier transform $g_N(\underline{k}, \underline{l})$, $\underline{k} = (k_p : \; k_p \in \mathbb{Z}, \; p \in \mathcal{P})$, $\underline{l} = (l_m : \; l_m \in \mathbb{Z}, \; m \in \mathbb{N}_0)$ of the measure Q_N is of the form

$$g_N(\underline{k}, \underline{l}) = \int_{\Omega} \left(\prod_{p \in \mathcal{P}} \omega_1^{k_p}(p) \prod_{m \in \mathbb{N}_0} \omega_2^{k_m}(m) \right) \mathrm{d}Q_N,$$

where only a finite number of integers k_p and l_m are distinct from zero. Thus, by the definition of Q_N,

$$\begin{aligned} g_N(\underline{k}, \underline{l}) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{p \in \mathcal{P}} p^{-ikk_p h_1} \prod_{m \in \mathbb{N}_0} (m+\alpha)^{-ikl_m h_2} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ -ikh_1 \sum_{p \in \mathcal{P}} k_p \log p - ikh_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) \right\}, \end{aligned} \tag{2.1}$$

where, as above, only a finite number of integers k_p and l_m are distinct from zero. Obviously,

$$g_N(\underline{0}, \underline{0}) = 1. \tag{2.2}$$

Before to consider the case $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$, we observe that, in this case,

$$\exp\left\{ -ih_1 \sum_{p \in \mathcal{P}} k_p \log p - ih_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) \right\} \neq 1. \tag{2.3}$$

Indeed, if inequality (2.3) is not true, then, for a certain $\hat{m} \in \mathbb{Z}$,

$$\exp\left\{ -ih_1 \sum_{p \in \mathcal{P}} k_p \log p - ih_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) \right\} = e^{2\pi i \hat{m}}.$$

Hence,

$$h_1 \sum_{p \in \mathcal{P}} k_p \log p + h_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) = \pi \hat{l}$$

with $\hat{l} \in \mathbb{Z}$. Since only a finite number of integers k_p and l_m are distinct from zero, this contradicts the linear independence of the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$. Thus, (2.3) is true. Therefore, in the case $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$, using the formula for the geometric series, we obtain

$$g_N(\underline{k}, \underline{l}) = \frac{1 - \exp\left\{ -i(N+1)\left(h_1 \sum_{p \in \mathcal{P}} k_p \log p + h_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) \right) \right\}}{(N+1)\left(1 - \exp\left\{ -i\left(h_1 \sum_{p \in \mathcal{P}} k_p \log p + h_2 \sum_{m \in \mathbb{N}_0} l_m \log(m+\alpha) \right) \right\} \right)}.$$

From this and (2.2), we find that

$$\lim_{N\to\infty} g_N(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if} \quad (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0 & \text{if} \quad (\underline{k},\underline{l}) \neq (\underline{0},\underline{0}). \end{cases}$$

Therefore, a continuity theorem for probability measures on compact groups proves the lemma.

The next lemma uses again the linear independence of the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ over $\mathbb{Q}$, and is devoted to ergodicity of a certain transformation on the space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Define the element $a_{\alpha,h_1,h_2} \in \Omega$ by the formula

$$a_{\alpha,h_1,h_2} = \left\{(p^{-ih_1} : p \in \mathcal{P}), ((m+\alpha)^{-ih_2} : m \in \mathbb{N}_0)\right\},$$

and let, for $\omega \in \Omega$,

$$\varphi_{\alpha,h_1,h_2}(\omega) = a_{\alpha,h_1,h_2}\omega.$$

Since the Haar measure m_H is invariant with respect to translations by points from Ω, we have that φ_{α,h_1,h_2} is a measurable measure preserving transformation on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. We remind that a set $A \in \mathcal{B}(\Omega)$ is called invariant with respect to the transformation φ_{α,h_1,h_2} if the sets A and $\varphi_{\alpha,h_1,h_2}(A)$ differ one from another at most by a set of zero m_H-measure, and that the transformation φ_{α,h_1,h_2} is ergodic if the σ-field of invariant sets consists of sets with m_H-measure 0 or 1.

Lemma 2.2 *Suppose that the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then the transformation φ_{α,h_1,h_2} is ergodic.*

Proof We know (see the proof of Lemma 11 of [3]) that characters χ of the group Ω are of the form

$$\chi(\omega) = \prod_{p\in\mathcal{P}} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0} \omega_2^{l_m}(m),$$

where only a finite number of integers k_p and l_m are distinct from zero. If χ is non-trivial character of Ω, then, from this and inequality (2.3), it follows that, for $(\underline{k},\underline{l}) \neq (\underline{0},\underline{0})$,

$$\chi(a_{\alpha,h_1,h_2}) = \exp\left\{-ih_1 \sum_{p\in\mathcal{P}} k_p \log p - ih_2 \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha)\right\} \neq 1. \tag{2.4}$$

Let A be an invariant set of the transformation φ_{α,h_1,h_2}. Denote by I_A the indicator function of A, and by $\hat{f}$ the Fourier transform of the function f. Then

$$\hat{I}_A(\chi) = \int_{\Omega} \chi(\omega) I_A(\omega) m_H(\mathrm{d}\omega) = \chi(a_{\alpha,h_1,h_2}) \hat{I}_A(\chi),$$

since the measure m_H is invariant, and, for almost all $\omega \in \Omega$,

$$I_A(a_{\alpha,h_1,h_2}\omega) = I_A(\omega).$$

This and (2.4) imply that

$$\hat{I}_A(\chi) = 0 \tag{2.5}$$

for every non-trivial character χ of Ω.

Now let χ_0 be the trivial character of Ω, i.e., $\chi_0(\omega) \equiv 1$. Suppose that $\hat{I}(\chi_0) = a$. Since

$$\int_{\Omega} \chi(\omega) m_H(\mathrm{d}\omega) = \begin{cases} 1 & \text{if} \quad \chi = \chi_0, \\ 0 & \text{if} \quad \chi \neq \chi_0, \end{cases}$$

we have in view of (2.5) that, for every character χ of the group Ω,

$$\hat{I}_A(\chi) = a \int_{\Omega} \chi(\omega) m_H(\mathrm{d}\omega) = a\hat{1}(\chi) = \hat{a}(\chi). \tag{2.6}$$

It is well known that the function $I_A(\omega)$ is uniquely determined by its Fourier transform $\hat{I}_A(\omega)$. Therefore, we have from (2.6) that $I_A(\omega) = a$ for almost all $\omega \in \Omega$. However, since $I_A(\omega)$ is the indicator of A, $a = 0$ or 1. Hence, $I_A(\omega) = 0$ for almost all $\omega \in \Omega$, or $I_A(\omega) = 1$ for almost all $\omega \in \Omega$. This shows that $m_H(A) = 0$ or $m_H(A) = 1$, i.e., the transformation φ_{α,h_1,h_2} is ergodic.

3 A Limit Theorem

Denote by $H(D)$ the space of functions which are analytic on D, equipped with the topology of uniform convergence on compacta, and, for $A \in \mathcal{B}(H^2(D))$, define

$$P_N(A) = \frac{1}{N+1} \# \left\{ 0 \le k \le N : \ \underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \in A \right\},$$

where $\underline{h} = (h_1, h_2)$ and

$$\underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) = \left(\zeta(s + ikh_1; \mathfrak{a}), \zeta(s + ikh_2, \alpha; \mathfrak{b}) \right).$$

Moreover, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element $\underline{\zeta}(s, \alpha, \omega; \mathfrak{a}, \mathfrak{b})$ by the formula, see [3],

$$\underline{\zeta}(s, \alpha, \omega; \mathfrak{a}, \mathfrak{b}) = (\zeta(s, \omega_1; \mathfrak{a}), \zeta(s, \alpha, \omega_2; \mathfrak{b})),$$

where

$$\zeta(s, \omega_1; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}$$

with

$$\omega_1(m) = \prod_{\substack{p^r | m \\ p^{r+1} \nmid m}} \omega_1^r(p),$$

and

$$\zeta(s, \alpha, \omega_2; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}.$$

We note that the series for $\zeta(s, \omega_1; \mathfrak{a})$ and $\zeta(s, \alpha, \omega_2; \mathfrak{b})$ converge uniformly on compact subsets of the strip D for almost all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, respectively, and thus they define $H(D)$-valued random elements on the probability spaces $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ and $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, respectively. Moreover, in view of multiplicativity of the sequence $\mathfrak{a}$, we have that for almost all $\omega_1 \in \Omega_1$ and $s \in D$,

$$\zeta(s, \omega_1; \mathfrak{a}) = \prod_{p \in \mathcal{P}} \left(1 + \sum_{k=1}^{\infty} \frac{a_{p^k} \omega_1^k(p)}{p^{ks}}\right).$$

Let $P_{\underline{\zeta}}$ be the distribution of random element $\underline{\zeta}(s, \alpha, \omega; \mathfrak{a}, \mathfrak{b})$, i.e., let $P_{\underline{\zeta}}$ be a probability measure given by

$$P_{\underline{\zeta}}(A) = m_H\left(\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega; \mathfrak{a}, \mathfrak{b}) \in A\right), \quad A \in \mathcal{B}(H^2(D)).$$

Now we are ready to state a limit theorem for P_N. Let

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 3.1 *Suppose that the set $L(\mathcal{P}, \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and that the sequence $\mathfrak{a}$ is multiplicative. Then P_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$. Moreover, the support of $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof The proof is similar to the one for Theorem 6 from [3], therefore, we do not give all details.

For fixed $\sigma_1 > \frac{1}{2}$ and $m, n \in \mathbb{N}$, let

$$v(m,n) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\},$$

and for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v(m,n,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}.$$

Define the functions

$$\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v(m,n)}{m^s}$$

and

$$\zeta_n(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v(m,n,\alpha)}{(m+\alpha)^s}.$$

Then it is known (see [2, 6]) that the latter series are absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, let

$$\underline{\zeta}_n(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta_n(s+ikh_1;\mathfrak{a}), \zeta_n(s+ikh_2,\alpha;\mathfrak{b}))$$

and

$$P_{N,n}(A) = \frac{1}{N+1}\#\left\{0 \le k \le N : \underline{\zeta}_n(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)).$$

Similarly, for $\omega = (\omega_1, \omega_2) \in \Omega$, define

$$\zeta_n(s,\omega_1;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m) v(m,n)}{m^s}$$

and

$$\zeta_n(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m) v(m,n,\alpha)}{(m+\alpha)^s},$$

the series also being absolutely convergent for $\sigma > \frac{1}{2}$. Let

$$\underline{\zeta}_n(s + ik\underline{h}, \alpha, \omega; \mathfrak{a}, \mathfrak{b}) = (\zeta_n(s + ikh_1, \omega_1; \mathfrak{a}), \zeta_n(s + ikh_2, \alpha, \omega_2; \mathfrak{b})),$$

and

$$P_{N,n,\omega}(A) = \frac{1}{N+1}\#\left\{0 \le k \le N : \underline{\zeta}_n(s + ik\underline{h}, \alpha, \omega; \mathfrak{a}, \mathfrak{b}) \in A\right\},$$
$$A \in \mathcal{B}(H^2(D)).$$

Now, using Lemma 2.1, Theorem 5.1 of [1] and the invariance of the Haar measure m_H, we easily find that, on $(H^2(D), \mathcal{B}(H^2(D)))$, there exists a probability measure P_n such that the measures $P_{N,n}$ and $P_{N,n,\omega}$ both converge weakly to P_n as $N \to \infty$. The latter assertion is an analogue of Theorem 8 from [3].

The next step of the proof of Theorem 3.1 consists of approximation. For this purpose, the metric of $H^2(D)$ is substantial. It is well known that there exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset D$ such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric on $H(D)$ which induces the topology of uniform convergence on compacta. Now if, for $\underline{g}_1 = (g_{11}, g_{12})$, $\underline{g}_2 = (g_{21}, g_{22}) \in H^2(D)$,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{j=1,2} \rho(g_{1j}, g_{2j}),$$

then $\underline{\rho}$ is a metric in $H^2(D)$ inducing its topology.

We also remind the Gallagher lemma, Lemma 1.4 of [10], which allows to deduce discrete mean square estimates from those of continuous type.

Lemma 3.2 *Let T_0 and $T \ge \delta > 0$ be real numbers, and let $\mathcal{T}$ be a finite set lying in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1,$$

and let $S(x)$ *be a complex-valued continuous function on* $[T_0, T + T_0]$ *having a continuous derivative on* $(T_0, T + T_0)$. *Then*

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta}\int_{T_0}^{T} |S(x)|^2 dx + \left(\int_{T_0}^{T} |S(x)|^2 dx \int_{T_0}^{T} |S'(x)|^2 dx\right)^{\frac{1}{2}}.$$

For $n \in \mathbb{N}$, let

$$l_n(s) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)n^s,$$

where $\Gamma(s)$ is the Euler gamma-function. Then an application of the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(s)a^{-s}\mathrm{d}s = e^{-a}, \quad a,\ b > 0,$$

leads to the representation

$$\zeta_n(s; \mathfrak{a}) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta(s + z; \mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z}.$$

Let $K \subset D$ be a compact subset. Then the above representation for $\zeta_n(s; \mathfrak{a})$ and an application of the residue theorem imply, for an arbitrary $h_1 > 0$, the estimate

$$\frac{1}{N}\sum_{k=0}^{N} \sup_{s \in K} |\zeta(s + ikh_1; \mathfrak{a}) - \zeta_n(s + ikh_1; \mathfrak{a})| \ll \int_{-\infty}^{\infty} |l_n(\hat{\sigma} + i\tau)| \left(\frac{1}{N}\sum_{k=0}^{N} |\zeta(\sigma + it + ikh_1 + i\tau; \mathfrak{a})|\right) \mathrm{d}t, \tag{3.1}$$

where $\hat{\sigma} < 0$, $\frac{1}{2} < \sigma < 1$ and t is bounded by a constant. Since, for a fixed σ, $\frac{1}{2} < \sigma < 1$, the estimate [6]

$$\int_0^T |\zeta(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t = O(T)$$

is valid, Lemma 3.2 implies the bound

$$\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma+\boldsymbol{it}+ikh_1+i\tau;\mathfrak{a})| \leq \left(\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma+\boldsymbol{it}+ikh_1+i\tau;\mathfrak{a})|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{h_1}\int_0^{Nh_1}|\zeta(\sigma+it+i\tau;\mathfrak{a})|^2dt+$$
$$\left(\int_0^{Nh_1}|\zeta(\sigma+it+i\tau;\mathfrak{a})|^2dt\int_0^{Nh_1}|\zeta'(\sigma+it+i\tau;\mathfrak{a})|^2dt\right)^{\frac{1}{2}} \ll 1+|\omega\tau|.$$

Therefore, in view of (3.1),

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh_1;\mathfrak{a})-\zeta_n(s+ikh_1;\mathfrak{a})|=0.$$

From this, by the definition of the metric ρ, we find that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\left(\zeta(s+ikh_1;\mathfrak{a}),\zeta_n(s+ikh_1;\mathfrak{a})\right)=0. \tag{3.2}$$

In [5], Theorem 4.1, it was shown that for an arbitrary $h_2>0$ and a compact subset $K\subset D$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh_2,\alpha;\mathfrak{b})-\zeta_n(s+ikh_2,\alpha;\mathfrak{b})|=0.$$

Thus,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\left(\zeta(s+ikh_2,\alpha;\mathfrak{b}),\zeta_n(s+ikh_2,\alpha;\mathfrak{b})\right)=0.$$

The latter equality together with (3.2) and the definition of the metric $\underline{\rho}$ shows that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\underline{\rho}\left(\underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}),\underline{\zeta}_n(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b})\right)=0. \tag{3.3}$$

Standard arguments show that for almost all $\omega_1\in\Omega_1$,

$$\int_0^T|\zeta(\sigma+it,\omega_1;\mathfrak{a})|^2\mathrm{d}t=O(T)$$

with a fixed σ, $\frac{1}{2} < \sigma < 1$. Therefore, by a reasoning as in the proof of (3.2) we find that, for almost all $\omega_1 \in \Omega_1$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(\zeta(s+ikh_1,\omega_1;\mathfrak{a}), \zeta_n(s+ikh_1,\omega_1;\mathfrak{a})\right) = 0. \tag{3.4}$$

The linear independence over $\mathbb{Q}$ of the set $L(\mathcal{P},\alpha,h_1,h_2,\pi)$ implies the linear independence of the set

$$\left\{ ((m+\alpha) : \ m \in \mathbb{N}_0), \frac{\pi}{h_2} \right\}.$$

Thus, repeating the proof of Lemma 2.6 of [4], we have that for almost all $\omega_2 \in \Omega_2$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(\zeta(s+ikh_2,\alpha,\omega_2;\mathfrak{b}), \zeta_n(s+ikh_2,\alpha,\omega_2;\mathfrak{b})\right) = 0.$$

This together with (3.4), for almost all $\omega \in \Omega$, implies the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\rho}\left(\underline{\zeta}(s+ik\underline{h},\alpha,\omega;\mathfrak{a},\mathfrak{b}), \underline{\zeta}_n(s+ik\underline{h},\alpha,\omega;\mathfrak{a},\mathfrak{b})\right) = 0, \tag{3.5}$$

where

$$\underline{\zeta}(s+ik\underline{h},\alpha,\omega;\mathfrak{a},\mathfrak{b}) = \left(\zeta(s+ikh_1,\omega_1;\mathfrak{a}), \zeta(s+ikh_2,\alpha,\omega_2;\mathfrak{b})\right).$$

For $\omega \in \Omega$, we define one more probability measure by

$$P_{N,\omega}(A) = \frac{1}{N+1} \# \left\{ 0 \le k \le N : \ \underline{\zeta}(s+ik\underline{h},\alpha,\omega;\mathfrak{a},\mathfrak{b}) \in A \right\}, \quad A \in \mathcal{B}(H^2(D)).$$

Then the weak convergence of $P_{N,n}$ and $P_{N,n,\omega}$ to the same measure P_n as $N \to \infty$, equalities (3.3) and (3.5), and Theorem 4.2 of [1] show that the measures P_N and $P_{N,\omega}$ both converge weakly to the same probability measure P on $(H^2(D), \mathcal{B}(H^2(D)))$ as $N \to \infty$.

It remains to prove that $P = P_{\underline{\zeta}}$. For this, Lemma 2.2 is applied. On the space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable θ by the formula

$$\theta(\omega) = \begin{cases} 1 & \text{if} \quad \underline{\zeta}(s,\alpha,\omega;\mathfrak{a},\mathfrak{b}) \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a fixed continuity set of the measure P. Then, for the expectation $\mathbb{E}\theta$, we have

$$\mathbb{E}\theta = \int_{\Omega} \theta \mathrm{d}m_H = P_{\underline{\zeta}}(A). \tag{3.6}$$

Lemma 2.2 together with the Birkhoff–Khintchine ergodicity theorem yields the equality

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \theta \left(\varphi^{k}_{\alpha,h_1,h_2}(\omega)\right) = \mathbb{E}\theta. \tag{3.7}$$

On the other hand, by the definitions of θ and φ_{α,h_1,h_2}, we have that

$$\frac{1}{N+1} \sum_{k=0}^{N} \theta \left(\varphi^{k}_{\alpha,h_1,h_2}(\omega)\right) = P_{N,\omega}(A). \tag{3.8}$$

Since $P_{N,\omega}$ converges weakly to P, and A is a continuity set of P, it follows that

$$\lim_{N\to\infty} P_{N,\omega}(A) = P(A).$$

However, (3.6)–(3.8) imply that, for every continuity set A of P,

$$P_{\underline{\zeta}}(A) = P(A).$$

Hence, $P_{\underline{\zeta}} = P$.

The limit measure $P_{\underline{\zeta}}$ does neither depend on h_1 and h_2 nor on the arithmetic of α. Therefore, by Lemma 12 of [3], the support of $P_{\underline{\zeta}}$ is the set $S \times H(D)$. The theorem is proved.

4 Proof of the Universality Theorem

A proof of Theorem 1.3 is based on Theorem 3.1 and the Mergelyan theorem on the approximation of analytic functions by polynomials, and is quite standard. We state the Mergelyan theorem as the following lemma:

Lemma 4.1 *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

The proof of this lemma can be found in [8] and [12].

We will also use an equivalent of the weak convergence of probability measures in terms of open sets.

Lemma 4.2 *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(X, \mathcal{B}(X))$. Then P_n converges weakly to P as $n \to \infty$ if, and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

The proof of this lemma is given in [1], Theorem 2.1.

Proof (Proof of Theorem 1.3) By Lemma 4.2, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \left| f_1(s) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}. \tag{4.1}$$

and

$$\sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{4.2}$$

Define the set

$$G = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} \left| g_1(s) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\}.$$

Then we observe that G is an open set, and, in view of the second assertion of Theorem 3.1, $(e^{p_1(s)}, p_2(s))$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, $P_{\underline{\zeta}}(G) > 0$. Thus, by first assertion of Theorem 3.1 and Lemma 4.2,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \underline{\zeta}(s + ik\underline{h}, \alpha, \omega; \mathfrak{a}, \mathfrak{b}) \in G \right\} \geq P_{\underline{\zeta}}(G) > 0.$$

Therefore, the definition of G implies that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K_1} \left| \zeta(s + ikh_1; \mathfrak{a}) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s + ikh_2, \alpha; \mathfrak{b}) - p_2(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (4.1) and (4.2) proves the theorem.

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