Sums of Two Squares of Sums of Two Squares

Rebecca Ulrike Jakob

Dedicated to Wolfgang Schwarz

Abstract This article determines the order of magnitude of integers not exceeding x that can be written as sums of two squares of integers that are themselves sums of two squares. The tools include Selberg's sieve and contour integration in the spirit of the Selberg-Delange method.

Keywords Arithmetic progression • Square • Sum • Sums of two squares

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1 Introduction

In about 1900 Landau calculated the cardinality of sums of two squares in a long interval $[1, x]$. With $\mathcal{S} = \{n \in \mathbb{N} \mid n = a^2 + b^2\}$ he proved in [3] the asymptotic formula

$$\sum_{\substack{n \le x \\ n \in \mathcal{S}}} 1 = C \frac{x}{\sqrt{\log x}} + \mathcal{O}\left(\frac{x}{(\log x)^{3/2}}\right), \tag{1.1}$$

where $C := \frac{1}{\sqrt{2}} \prod_{p \equiv 3(4)} (1 - p^{-2})^{-1/2}$.

It is possible to diversify the summation condition in a lot of ways. In this paper we want to consider the situation where each of the two squares is again a sum of two squares.

R.U. Jakob (✉)
Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, Germany
e-mail: r.jakob@stud.uni-goettingen.de

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We define some notation. Let $\tilde{\mathcal{S}} = \{n \in \mathbb{N} \mid p|n \Rightarrow p \not\equiv 3(4)\} \subset \mathcal{S}$, and let

$$R(n) = \#\{a, b \in \mathcal{S} \mid n = a^2 + b^2\},$$
$$\tilde{R}(n) = \#\{a, b \in \tilde{\mathcal{S}} \mid n = a^2 + b^2\}.$$

Finally we write

$$S_i(x) = \sum_{n \le x} R(n)^i$$

for $i = 0, 1, 2$ with the convention $0^0 = 0$, and analogously

$$\tilde{S}_i(x) = \sum_{n \le x} \tilde{R}(n)^i.$$

Our aim is to estimate these quantities from above and below. We start with the following which can be obtained from (1.1) by partial summation.

Theorem 1.1 *For $x \ge 3$ we have the asymptotic formula*

$$S_1(x) = \frac{\pi}{4} \prod_{p \equiv 3(4)} (1 - p^{-2})^{-1} \frac{x}{\log x} + \mathcal{O}\Big(\frac{x}{(\log x)^2}\Big).$$

An analogous asymptotic formula (with a different constant) can be proved for $\tilde{S}_1(x)$.

Our next aim is an upper bound for $\tilde{S}_2(x)$. As a preparation we define the function

$$\varphi_s(n) := \prod_{p|n} \big(1 - p^{-s}\big). \tag{1.2}$$

Then we prove the following

Theorem 1.2 *Let $q_j, r_j \in \mathbb{Z}$ for $j = 1, \ldots, k$ with $\mathcal{Q} = \prod_j q_j \prod_{i \ne j} |q_i r_j - q_j r_i| \ne 0$. Then*

$$\#\{n \le x \mid q_j n + r_j \in \tilde{\mathcal{S}}\ \forall j = 1, \ldots, k\}$$
$$\le \frac{x}{(\log x)^{k/2}} \frac{\Gamma(k/2+1)2^k}{F^k(1)} \frac{1}{\varphi_1^k(\mathcal{Q})} \left(1 + \mathcal{O}\bigg(\Big(\frac{\log \mathcal{Q}}{\varphi_1^k(\mathcal{Q})} + \varphi_{1-\delta'}^{-2k}(\mathcal{Q})\Big)\frac{1}{\log x}\bigg)\right),$$

for a constant $\delta' > 0$ and $F^k(1)$ given by the convergent Euler product

$$\prod_p \Big(1 - \frac{\chi_{-4}(p)}{p}\Big)^{k/2} \prod_{p \equiv 3(4)} \Big(1 + \sum_{l=2}^{k} \tbinom{k}{l} p^{-l}\Big)\big(1 - p^{-2}\big)^{-k/2}.$$

This furnishes an upper bound for numbers such that k arithmetic progressions are not divisible by a prime congruent to 3 modulo 4.

To prove this theorem we use Selberg's sieve which requires a calculation that is reminiscent of the Selberg-Delange method. Having this bound available, we will prove the following theorem.

Theorem 1.3 *For $x \geq 3$ we have the upper bound*

$$\tilde{S}_2(x) \ll \frac{x}{\log x}. \tag{1.3}$$

We see that $\tilde{S}_1(x)$ and $\tilde{S}_2(x)$ have the same order of magnitude, thus with the Cauchy-Schwarz inequality we get a good lower bound for $\tilde{S}_0(x)$. Furthermore we have the inequality $\tilde{S}_0(x) \leq S_0(x)$, so we get a lower bound for $S_0(x)$. This gives us our final result.

Theorem 1.4 *For $x \geq 3$ we have*

$$\frac{x}{\log x} \ll S_0(x) \leq C_o \frac{x}{\log x}\Big(1 + \mathcal{O}\Big(\frac{1}{\log x}\Big)\Big)$$

with $C_o = \frac{\pi}{4} \prod_{p \equiv 3(4)} (1 - p^{-2})^{-1}$.

In other words, the number of sums of two squares of sums of two squares is of order of magnitude $\frac{x}{\log x}$.

2 Useful Lemmas

First we will prove three lemmas which are useful to prove the theorems.

Lemma 2.1 *For fixed $n, m \in \mathbb{N}$ we get*

$$-\int_0^{\sqrt{x}} \frac{t}{\sqrt{\log t}^m} \frac{d}{dt}\Big(\frac{\sqrt{x - t^2}}{\sqrt{\log(x - t^2)}^n}\Big) dt = \frac{\sqrt{2}^m \pi}{4} \frac{x}{\sqrt{\log x}^{n+m}}\Big(1 + \mathcal{O}\Big(\frac{1}{\log x}\Big)\Big).$$

Proof Let $X = \frac{\sqrt{x}}{(\log x)^{(m+n+2)/2}}$. We consider the two integrals over $[0, X]$ and $[X, \sqrt{x}]$ separately, and denote them by $I_e(x)$ and $I_m(x)$, respectively.

In the integrand of $I_e(x)$ the derivative is bounded by

$$\frac{d}{dt}\Big(\frac{\sqrt{x - t^2}}{\sqrt{\log(x - t^2)}^n}\Big) \ll \frac{X}{\sqrt{x - X^2}\sqrt{\log(x - X^2)}^n} \ll \frac{X}{\sqrt{x - X^2}}.$$

Using this and the estimate $\frac{t}{\sqrt{\log t}^m} \ll X$ we obtain

$$I_e(x) \ll \int_0^X \frac{X^2}{\sqrt{x - X^2}} dt \ll \frac{X^3}{\sqrt{x}} \ll \frac{x}{(\log x)^{(m+n+2)}}.$$

For the integrand of $I_m(x)$ we use Taylor's formula around $t = \sqrt{x}$

$$\frac{1}{\sqrt{\log t}^m} = \frac{\sqrt{2}^m}{\sqrt{\log x}^m} + \mathcal{O}\Big(\frac{t - \sqrt{x}}{\sqrt{x}\sqrt{\log x}^{m+2}}\Big).$$

For the integral over the error term we compute the derivative term and use trivial bounds for the factors

$$\int_X^{\sqrt{x}} \frac{(\sqrt{x} - t)t^2}{\sqrt{x - t^2}\sqrt{\log(x - t^2)}^n} |n - \log(x - t^2)^{-1}| dt \frac{1}{\sqrt{x}\sqrt{\log x}^{m+2}} \ll \frac{x}{\sqrt{\log x}^{m+n+2}}.$$

In the remaining integral for the main term $\big(\frac{2}{\log x}\big)^{m/2}$ of Taylor's formula we expand the lower bound to zero. With partial integration and the substitution $y = x - t^2$ we get an integral similar to the one above, whence

$$\int_X^{\sqrt{x}} t \frac{d}{dt}\Big(\frac{\sqrt{x - t^2}}{\sqrt{\log(x - t^2)}^n}\Big) dt = \int_0^x \frac{\sqrt{y}}{2\sqrt{x - y}\sqrt{\log y}^n} dy + \mathcal{O}\Big(\frac{x}{\sqrt{\log x}^{n+2}}\Big).$$

We need to compute the integral. Again we split it into two parts $[0, Z]$ and $[Z, x]$ with $Z = \frac{x}{(\log x)^{n/2+1}}$ and repeat the estimates to get

$$\int_0^x \frac{\sqrt{y}}{2\sqrt{x - y}\sqrt{\log y}^n} dy = \frac{1}{\sqrt{\log x}^n} \int_0^x \frac{\sqrt{y}}{2\sqrt{x - y}} dy \Big(1 + \mathcal{O}\Big(\frac{1}{\log x}\Big)\Big).$$

The remaining integral $\int_0^x \frac{\sqrt{y}}{\sqrt{x-y}} dy$ can be computed exactly with the result $\frac{x\pi}{2}$ and our claim follows.

Lemma 2.2 *The map*

$$\{(k, l, m, n) \in \mathbb{N}^4 \mid kl = mn\} \longleftrightarrow \{(s, t, u, v) \in \mathbb{N}^4 \mid (s, u) = 1\},$$

$$(k, l, m, n) \longmapsto \Big(\frac{k}{(k, m)}, (k, m), \frac{m}{(k, m)}, \frac{n(k, m)}{k}\Big),$$

$$(st, uv, tu, sv) \longleftarrow (s, t, u, v)$$

is a bijection.

The proof is straightforward.

Lemma 2.3 *Let $k \in \mathbb{N}$ and $s > 1/2$. Then for $x \geq 3$ we have*

$$\sum_{n \leq x} \frac{1}{n} \varphi_s^{-k}(n) \ll \log x.$$

Proof The case $s = 1$ and $k = 1$ was proved by Landau (c.f. [2]). Note that $n\varphi_1(n)$ is Euler's phi function.

The expression $\frac{1}{\varphi_s(n)}$ can be rewritten as

$$\frac{1}{\varphi_s(n)} = \prod_{p|n} \big(1 - \frac{1}{p^s}\big)^{-1} = \prod_{p|n} \big(1 + \frac{1}{p^s - 1}\big) = \sum_{d|n} \frac{\mu^2(d)}{d^s \varphi_s(d)}$$

where μ is the Möbius function. We insert this in our summation to get

$$\sum_{n \leq x} \frac{1}{n} \varphi_s^{-k}(n) = \sum_{n \leq x} \frac{1}{n} \Big(\sum_{d|n} \frac{\mu^2(d)}{d^s \varphi_s(d)} \Big)^k$$

$$= \sum_{\substack{d_j \leq x \\ j=1,\dots,k}} \frac{\mu^2(d_1) \cdots \mu^2(d_k)}{d_1^s \varphi_s(d_1) \cdots d_k^s \varphi_s(d_k)} \frac{1}{\mathrm{lcm}_i(d_i)} \sum_{m \leq \frac{x}{\mathrm{lcm}_i(d_i)}} \frac{1}{m}$$

$$\ll \sum_{\substack{d_j \leq x \\ j=1,\dots,k}} \Big| \frac{\mu^2(d_1) \cdots \mu^2(d_k)}{d_1^s \varphi_s(d_1) \cdots d_k^s \varphi_s(d_k)} \Big| \frac{1}{\mathrm{lcm}_i(d_i)} \cdot \log x.$$

We complete the remaining sum into an infinite sum and see

$$\sum_{\substack{d_j = 1 \\ j=1,\dots k}}^{\infty} \Big| \frac{\mu^2(d_1) \cdots \mu^2(d_k)}{d_1^s \varphi_s(d_1) \cdots d_k^s \varphi_s(d_k)} \Big| \frac{1}{\mathrm{lcm}_i(d_i)} \ll \prod_p \Big(1 + \frac{1}{p(p^s - 1)}\Big)^k.$$

This product converges absolutely for $s > 1/2$.

3 Numbers in $\tilde{\mathcal{S}}$ in Arithmetic Progressions

In this section we prove Theorem 1.2. For simplification we assume $k \geq 2$, since in the case $k = 1$ we can apply the proof of (1.1) with a different constant. In addition, if (q_j, r_j) has a divisor which is congruent to 3 (mod 4) there are no solutions, hence the inequality is true. If (q_j, r_j) has a prime divisor $p \equiv 1 \pmod 4$, we can simply divide it out without changing the condition $q_j n + r_j \in \tilde{\mathcal{S}}$. Hence we assume without loss of generality that $(q_j, r_j) = 1$. We use Selberg's sieve, c.f. [1, Theorem 6.4, p. 10]. Let

$$\mathcal{P} = \prod_{\substack{p \equiv 3(4) \\ k < p \leq x}} p \qquad \text{and}$$

$$Q(t) = (q_1 t + r_1) \dots (q_k t + r_k).$$

Then the cardinality of $\{n \leq x \mid (Q(n), \mathcal{P}) = 1\}$ is an upper bound for the quantity we want to bound.

Consider the function $\rho(d) = \#\{a \pmod d \mid Q(a) \equiv 0 \pmod d\}$. We see that $\rho(p) \leq \min\{k, p\}$ for $p \mid \mathcal{P}$. In particular it is $\rho(p) = k$ if $(p, \mathcal{Q}) = 1$. By the Chinese remainder theorem, ρ is multiplicative. Hence we get

$$\sum_{\substack{n \leq x \\ d \mid Q(n)}} 1 = \frac{x}{d} \rho(d) + \mathcal{O}(\tau_{k-1}(d)),$$

where τ_k is the k-th iterated divisor function $\sum_{a_1 \cdots a_k = n} 1$.

We define the multiplicative function g by $g(p) = \frac{\rho(p)}{p - \rho(p)}$ for primes $p \mid \mathcal{P}$, $g(p) = 0$ for $p \nmid \mathcal{P}$ and $g(p^\nu) = 0$ for all p and all $\nu \geq 2$. For a parameter z to be chosen later let $Z = \sum_{d \leq z} g(d)$. Therefore by Selberg's sieve

$$\sum_{\substack{n \leq x \\ (Q(n), \mathcal{P}) = 1}} 1 \leq \frac{x}{Z} + \mathcal{O}\Bigg(\sum_{\substack{d \leq z^2 \\ p \mid \mathcal{P}}} \tau_3(d) \tau_{k-1}(d) \Bigg). \tag{3.1}$$

Hence we need to find a lower bound for Z to get the information we want. Writing

$$Z' = \sum_{d \leq z} \mu^2(d) \prod_{\substack{p \mid d \\ p \mid \mathcal{Q} \\ p \equiv 3(4)}} g(p) \prod_{\substack{p \mid d \\ p \nmid \mathcal{Q} \\ p \equiv 3(4)}} \frac{\tau_{k-1}(p)}{p},$$

it is easy to see that $Z \geq Z'$.

For the sum Z' we use Perron's formula to find a lower bound. It will be useful to use a refined version due to Liu and Ye [4]. Consider the Dirichlet series of Z', given by the Euler product

$$\begin{aligned} D(s) := & \prod_{\substack{p \mid \mathcal{Q} \\ p \equiv 3(4) \\ p > k}} \left(1 + \frac{g(p)}{p^s}\right) \left(1 + \frac{\tau_{k-1}(p)}{p^{s+1}}\right)^{-1} \prod_{p \equiv 3(4)} \left(1 + \frac{\tau_{k-1}(p)}{p^{s+1}}\right) \\ &= D'(s) \cdot \zeta^{k/2}(s+1) L^{-k/2}(s+1, \chi_{-4}) H(s), \end{aligned}$$

where

$$D'(s) = \prod_{\substack{p\mid Q\\ p\equiv 3(4)\\ p>k}} \big(1+\frac{g(p)}{p^s}\big)\big(1+\frac{\tau_{k-1}(p)}{p^{s+1}}\big)^{-1}.$$

The function $H(s)$ is holomorphic and nonzero in $\operatorname{Re} s > -\frac{1}{2}$. The finite product $D'(s)$ also is a holomorphic function and bounded by

$$\varphi_{1+\operatorname{Re} s}^{2k}(\mathcal{Q}) \ll \left|D'(s)\right| \ll \frac{1}{\varphi_{1+\operatorname{Re} s}^{2k}(\mathcal{Q})},$$

where φ_s is as in (1.2). Define $F^2(s+1) = \zeta(s+1)L^{-1}(s+1,\chi_{-4})H^{2/k}(s)s$, then F is holomorphic in the zero-free region of the L-function as in [6, II §5.4] and we get $D(s) = F^k(s+1)s^{-k/2}D'(s)$.

Let $2 \le T < z$ be a parameter which we choose later. For the error terms in Perron's formula we need to bound the quantity

$$\sum_{|z-n|\le\frac{z}{\sqrt{T}}} g(n) \ll \sum_{|z-n|\le\frac{z}{\sqrt{T}}} \mu^2(n)\tau_{k-1}(n) \prod_{\substack{p\mid n\\ k<p,p\mid\mathcal{Q}}} \frac{1}{p-k} \prod_{\substack{p\mid n\\ p\nmid\mathcal{Q}}} \frac{1}{p}$$
$$\ll \varphi_1^{-k}(\mathcal{Q}) \sum_{|z-n|\le\frac{z}{\sqrt{T}}} \mu^2(n)\frac{\tau_{k-1}(n)}{n} \ll \varphi_1^{-k}(\mathcal{Q})\frac{(\log(z))^{k-2}}{\sqrt{T}}.$$

The last bound follows by [5, Theorem 2]. The second part of the error term is given by

$$\frac{z^b B(b)}{\sqrt{T}} \ll \frac{z^b}{\sqrt{T}}\sum_n \frac{|g(n)|}{n^b} \ll \frac{\zeta^k(1+\frac{1}{\log z})}{\sqrt{T}} \ll \frac{(\log z)^k}{\sqrt{T}},$$

where we choose $b = \frac{1}{\log z}$.

By applying Perron's formula in the version of [4, Corollary 2.2] we get

$$Z' = \frac{1}{2\pi i}\int_{\frac{1}{\log z}-iT}^{\frac{1}{\log z}+iT} \frac{F^k(s+1)}{s^{k/2}}D'(s)z^s\frac{ds}{s} + \mathcal{O}\Big(\varphi_1^{-k}(\mathcal{Q})\frac{(\log(z))^{k-2}}{\sqrt{T}} + \frac{(\log z)^k}{\sqrt{T}}\Big). \tag{3.2}$$

Now we integrate over the same contour as in Tenenbaum [6, II §5.4] but shifted to zero, so that the order of the singularity at 0 is $\frac{k}{2}+1$.

We split the contour into various pieces

$$\begin{aligned}
\gamma_1, \gamma_7 &= \Big[c \mp iT, -\frac{\delta}{2\log(2+T)} \mp iT\Big],\\
\gamma_2, \gamma_6 &= -\frac{\delta}{2\log(2+|t|)} + it, \text{for } t \in [-T, 0], t \in [0, T],\\
\gamma_4 &= \partial B_r(0),\\
\gamma_3 &= \Big[-\frac{\delta}{2\log 2}, -r\Big],\\
\gamma_5 &= \Big[-r, -\frac{\delta}{2\log 2}\Big],
\end{aligned}$$

with $r = \frac{1}{\log z}$ and $\delta > 0$ is chosen so small that

$$\frac{1}{L(s, \chi_{-4})} \ll \log|t|, \qquad \zeta(s) \ll \log|t|,$$

for $\sigma > 1 - \frac{\delta}{2\log(2+|t|)}$ (see [6, p. 262]).

For the parts γ_i with $i = 1, 2, 6, 7$ an upper bound for $D'(s)$ is given by the function $\varphi_{1-\delta'}^{-2k}(\mathcal{Q})$ with $\delta' = \frac{\delta}{2\log 2}$. We get for γ_1 and analogously for γ_7 the bound

$$\begin{aligned}
\int_{\gamma_1} \frac{F^k(s+1)}{s^{k/2}} D'(s) z^s \frac{ds}{s} &\ll \frac{(\log T)^k}{\varphi_{1-\delta'}^{2k}(\mathcal{Q})} \int_{-\frac{\delta}{2\log(2+T)}}^{\frac{1}{\log z}} z^\sigma \frac{d\sigma}{T}\\
&\ll \varphi_{1-\delta'}^{-2k}(\mathcal{Q}) \frac{(\log T)^k}{T} \int_{-\frac{\delta}{2\log(2+T)}}^{\frac{1}{\log z}} z^\sigma \, d\sigma \ll \varphi_{1-\delta'}^{-2k}(\mathcal{Q}) \frac{(\log T)^k}{T}.
\end{aligned} \tag{3.3}$$

For the parts γ_2 and γ_6 the bound is given by

$$\begin{aligned}
&\int_{\gamma_2} \frac{F^k(s+1)}{s^{k/2}} D'(s) z^s \frac{ds}{s}\\
&\ll \varphi_{1-\delta'}^{-2k}(\mathcal{Q}) (\log T)^k z^{-\delta/(2\log T)} \int_0^T \Big|\frac{\delta}{2\log T} - it\Big|^{-1} dt\\
&\ll \varphi_{1-\delta'}^{-2k}(\mathcal{Q}) z^{-\delta/(2\log T)} \frac{2(\log T)^{1+k}}{\delta}.
\end{aligned} \tag{3.4}$$

We choose $T = \exp(\frac{\delta}{2}\sqrt{\log z})$ to optimize the error term composed from (3.2), (3.3) and (3.4).

By writing $\Upsilon' = \gamma_3\gamma_4\gamma_5$ for the contribution not yet considered we get with the bounds from (3.2), (3.3) and (3.4) the asymptotic formula

$$Z' = \frac{1}{2\pi i}\int_{\Upsilon'} \frac{F^k(s+1)}{s^{k/2}} D'(s) z^s \frac{ds}{s} + \mathcal{O}\Big(\big(\varphi_{1-\delta'}^{-2k}(\mathcal{Q}) + \varphi_1^{-k}(\mathcal{Q})\big)\frac{(\log z)^k}{\exp(\delta\sqrt{\log z})}\Big). \tag{3.5}$$

The remaining integral does not have an explicit solution, but by using the Taylor expansion

$$F^k(s+1)D'(s) = F^k(1)D'(0) + \mathcal{O}\Big(\varphi_1^{-k}(\mathcal{Q})\log\mathcal{Q}\cdot|s|\Big)$$

we obtain

$$\Big|\frac{1}{2\pi i}\int_{\Upsilon'} \frac{F^k(s+1)}{s^{k/2}} D'(s) z^s \frac{ds}{s}\Big|$$

$$\geq \varphi_1^k(\mathcal{Q})F^k(1)\Big|\frac{1}{2\pi i}\int_{\Upsilon'} s^{-k/2} z^s \frac{ds}{s}\Big| + \mathcal{O}\Big(\varphi_1^k(\mathcal{Q})\log\mathcal{Q}\Big|\int_{\Upsilon'} \frac{|s|}{s^{k/2}} z^s \frac{ds}{s}\Big|\Big) = I_h(z) + I_e(z).$$

The product $D'(0)$ is bounded from below by $\varphi_1^k(\mathcal{Q})$. Now we change $I_h(z)$ by the substitution $s = \frac{w}{\log z}$ into the form of Theorem 5.2 from [6, II §5.2]. Let Υ be the resulting contour from Υ'. Applying Corollary 2.1 of [6, II §5.2], for Hankel's contour we get

$$\begin{aligned} I_h(z) &\geq F^k(1)\varphi_1^k(\mathcal{Q})(\log z)^{k/2}\Big|\frac{1}{2\pi i}\int_{\Upsilon} e^w w^{-(k/2+1)}dw\Big| \\ &= F^k(1)\varphi_1^k(\mathcal{Q})(\log z)^{k/2}\Big(\frac{1}{\Gamma(k/2+1)} + \mathcal{O}(z^{-\nu})\Big), \end{aligned}$$

for some $\nu > 0$.

To compute the integral $I_e(z)$ we use trivial bounds on the circle and on the lines separately to get $I_e \ll \varphi_1^{-k}(\mathcal{Q})\log\mathcal{Q}(\log z)^{k/2-1}$. Combining the obtained results we get a lower bound for Z' and hence for Z. The error terms from Perron's formula and from the other pieces of the contour (3.5) combine with $I_e(x)$ to give

$$\begin{aligned} &\varphi_1^{-k}(\mathcal{Q})\log\mathcal{Q}(\log z)^{k/2-1} + \big(\varphi_{1-\delta'}^{-2k}(\mathcal{Q}) + \varphi_1^{-k}(\mathcal{Q})\big)\frac{(\log z)^k}{\exp(\delta\sqrt{\log z})} \\ &\ll \big(\varphi_1^{-k}(\mathcal{Q})\log\mathcal{Q} + \varphi_{1-\delta'}^{-2k}(\mathcal{Q})\big)(\log z)^{k/2-1}. \end{aligned} \tag{3.6}$$

The error terms in Eqs. (3.1) and (3.6) are optimized by choosing $z = \frac{x^{1/2}}{(\log x)^d}$ with $d = \frac{1}{2}(\frac{k}{2} + 1 + 3^k)$ and we get

$$\begin{aligned}&\#\{n \le x \mid q_j n + r_j \in \tilde{\mathcal{S}}\ \forall j = 1, \ldots, k\}\\ &\le \frac{x}{(\log x)^{k/2}} \frac{\Gamma(k/2+1)2^{k/2}}{F^k(1)} \frac{1}{\varphi_1^k(\mathcal{Q})}\Big(1 + \mathcal{O}\Big(\Big(\frac{\log \mathcal{Q}}{\varphi_1^k(\mathcal{Q})} + \varphi_{1-\delta'}^{-2k}(\mathcal{Q})\Big)\frac{1}{\log x}\Big)\Big).\end{aligned}$$

4 Proof of Theorem 1.1

In this section we find an asymptotic formula for $S_1(x)$.

Let $r_0(n)$ be the characteristic function on the numbers that can be written as a sum of two squares. By rewriting the sum in terms of $r_0(n)$ and (1.1) we get

$$S_1(x) = \sum_{a \le \sqrt{x}} r_0(a) C \frac{\sqrt{2}\sqrt{x-a^2}}{\sqrt{\log(x-a^2)}}\Big(1 + \mathcal{O}\big((\log(x-a^2))^{-1}\big)\Big).$$

By applying partial summation and once more (1.1) we get the integrals

$$\begin{aligned}S_1(x) =& -C^2\sqrt{2}\int_0^{\sqrt{x}} \frac{t}{\sqrt{\log t}} \frac{d}{dt}\Big(\frac{\sqrt{x-t^2}}{\sqrt{\log(x-t^2)}}\Big)dt\\ &+ \sum_{n \le \sqrt{x}} r_0(n) \lim_{a \to \sqrt{x}} \frac{\sqrt{x-a^2}}{\sqrt{\log(x-a^2)}}\Big(1 + \frac{1}{\log(x-a^2)}\Big)\\ &+ \mathcal{O}\Bigg(\int_0^{\sqrt{x}} \frac{t}{\sqrt{\log t}^3} \frac{d}{dt}\Big(\frac{\sqrt{x-t^2}}{\sqrt{\log(x-t^2)}}\Big)dt + \int_0^{\sqrt{x}} \frac{t}{\sqrt{\log t}} \frac{d}{dt}\Big(\frac{\sqrt{x-t^2}}{\sqrt{\log(x-t^2)}^3}\Big)dt\Bigg)\\ &=: M(x) + 0 + E(x).\end{aligned}$$

The boundary expression vanishes since $\lim_{a \to \sqrt{x}} \frac{\sqrt{x-a^2}}{\sqrt{\log(x-a^2)}}\big(1 + (\log(x-a^2))^{-1}\big) = 0$. With Lemma 2.1 at our disposal the computation of $M(x)$ and $E(x)$ is simple.

For $M(x)$ we have $m = n = 1$ and hence

$$M(x) = C^2\sqrt{2}\frac{\sqrt{2}\pi}{4} \frac{x}{\log x} + \mathcal{O}\Big(\frac{x}{(\log x)^2}\Big).$$

The error term $E(x)$ has two integrals, where in the first one we have $m = 3$, $n = 1$ and in the second one $m = 1$, $n = 3$. Since Lemma 2.1 is symmetric in n and m it yields $E(x) \ll \frac{x}{(\log x)^2}$.

Combining these two terms we conclude that

$$S_1(x) = C^2 \frac{\pi}{2} \frac{x}{\log x} + \mathcal{O}\Big(\frac{x}{(\log x)^2}\Big). \tag{4.1}$$

The asymptotic formula for $\tilde{S}_1(x)$ is almost the same as $S_1(x)$, and for $x \geq 3$ we get

$$\tilde{S}_1(x) = \frac{\pi}{4} \prod_{p \equiv 3(4)} (1 - p^{-2}) \frac{x}{\log x} + \mathcal{O}\Big(\frac{x}{(\log x)^2}\Big).$$

5 Proof of Theorem 1.3

The upper bound of $\tilde{S}_2(x)$ requires a bit more work. Opening the square we have

$$\tilde{S}_2(x) = \sum_{\substack{a,b,c,d \in \tilde{\mathcal{S}} \\ a^2+b^2=c^2+d^2 \leq x}} 1.$$

We need to understand the summation condition $a^2 + b^2 = c^2 + d^2$. The diagonal term is $\tilde{S}_1(x)$ which we have bound above. Hence we can assume that $\{a, b\} \neq \{c, d\}$ and the condition $a^2 + b^2 = c^2 + d^2$ is equivalent to $(a+d)(a-d) = (c+b)(c-b)$. Using Lemma 2.2 this can be rewritten with $s, t, u, v \in \mathbb{N}$ with $(s, u) = 1$ as

$$a + d = st, \quad a - d = uv, \quad c + b = tu, \quad c - b = sv.$$

Without loss of generality we can assume that $a > b$ and $c > d$, then the condition $a, b, c, d \in \tilde{\mathcal{S}}$ changes to $st + uv$, $st - uv$, $ut + sv$, $ut - sv \in \tilde{\mathcal{S}}$. It follows that

$$\tilde{S}_2(x) \leq \sum_{\substack{st,uv,ut,sv \leq 2\sqrt{x} \\ st \pm uv, ut \pm sv \in \tilde{\mathcal{S}}}} 1.$$

Now we want to apply Theorem 1.2. We fix u, t, v and think of s as our variable. In accordance with Theorem 1.2, we define $\mathcal{Q} = tv(t - v)(t + v)(t^2 + v^2)$.

In the case $t = v$ our conditions imply that $t(s \pm u) > 0$ and $t(u \pm s) > 0$ which is obviously impossible.

In the following let $t \neq v$, so that $\mathcal{Q} \neq 0$. Isolating the sum over s we get

$$\sum_{\substack{st,uv,ut,sv \leq 2\sqrt{x} \\ st \pm uv, ut \pm sv \in \tilde{\mathcal{S}}}} 1 = \sum_{tu,uv \leq 2\sqrt{x}} \sum_{\substack{s \leq \min\{\frac{ut}{v}, \frac{2\sqrt{x}}{t}, \frac{2\sqrt{x}}{v}\} \\ st \pm uv, ut \pm sv \in \tilde{\mathcal{S}}}} 1.$$

We use Theorem 1.2 to calculate the inner sum, and obtain

$$\sum_{\substack{s\le\min\{\frac{ut}{v},\frac{2\sqrt{x}}{t},\frac{2\sqrt{x}}{v}\}\\ st\pm uv,ut\pm sv\in\tilde{S}}} 1 \ll \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^2}\frac{1}{\varphi_1^4(\mathcal{Q})}\Big(1+\Big(\frac{\log\mathcal{Q}}{\varphi_1^4(\mathcal{Q})}+\varphi_{1-\delta'}^{-8}(\mathcal{Q})\Big)\frac{1}{\log\frac{ut}{v}}\Big),$$

with δ' defined in Sect. 3. For the sum over t, u, v we get the condition $v < t$, since $\frac{uv}{t} < \frac{ut}{v}$. Furthermore in the term $\varphi_1^4(\mathcal{Q})$ we split the product $\mathcal{Q}$. To this end a short look at the definition shows that $\varphi_s(nm) \ge \varphi_s(n)\varphi_s(m)$ and with the Hölder inequality it follows that

$$\begin{aligned}
&\sum_{\substack{uv<ut\\ ut<2\sqrt{x}}} \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^2}\Big(\frac{1}{\varphi_1(t)}\frac{1}{\varphi_1(v)}\frac{1}{\varphi_1(t-v)}\frac{1}{\varphi_1(t+v)}\frac{1}{\varphi_1(t^2+v^2)}\Big)^4\\
&\ll\Big(\sum_{\substack{uv<ut\\ ut<2\sqrt{x}}}\frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^2}\frac{1}{\varphi_1^{20}(v)}\Big)^{4/5}\Big(\sum_{\substack{u<2\sqrt{x}\\ n<\frac{2x}{u^2}}}\frac{u}{(\log u)^2}\frac{r(n)}{\varphi_1^{20}(n)}\Big)^{1/5},
\end{aligned} \tag{5.1}$$

where $r(n)$ counts the number of solutions $a^2 + b^2 = n$.

First we compute the sum over v and use $(\log\frac{ut}{v})^{-2} < (\log u)^{-2}$. Using Lemma 2.3 to compute the remaining sum over v we get

$$\sum_{ut\le 2\sqrt{x}}\sum_{v\le t}\frac{ut}{v}\frac{1}{(\log u)^2}\frac{1}{\varphi_1^4(v)} \ll \sum_{ut\le 2\sqrt{x}}\frac{u}{(\log u)^2}t\log t.$$

With partial integration we get for the sum over $t\log t$ the upper bound

$$\sum_{t\le\frac{2\sqrt{x}}{u}} t\log t \ll \frac{x}{u^2}\Big(\log\frac{2\sqrt{x}}{u}-1\Big).$$

There remains the sum over u for which once again we use partial summation. We obtain

$$\sum_{u\le 2\sqrt{x}}\frac{1}{u(\log u)^2}\log\frac{2\sqrt{x}}{u} \ll \frac{1}{\log x}.$$

Hence the first factor of (5.1) is computed.

Using $r(n) = \sum_{d|n}\chi_{-4}(d)$ the second factor of (5.1) can be transformed similarly as in Lemma 2.3 and it follows that

$$\sum_{n\le\frac{2x}{u^2}}\frac{r(n)}{\varphi_1^{20}(n)} \ll \frac{x}{u^2}.$$

And the sum over u can be computed as above to get

$$\sum_{\substack{u<2\sqrt{x}\\ n<2\frac{x}{u^2}}} \frac{u}{(\log u)^2}\frac{r(n)}{\varphi_1^{20}(n)} \ll \frac{x}{\log x}.$$

Combining these two results we get

$$\sum_{\substack{uv<ut\\ ut<2\sqrt{x}}} \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^2}\frac{1}{\varphi_1^4(\mathcal{Q})} \ll \frac{x}{\log x}.$$

For the sums over $\frac{\log\mathcal{Q}}{\varphi_1^4(\mathcal{Q})}$ and $\varphi_{1-\delta'}^{-8}(\mathcal{Q})$ the argument is the same and it follows in the same way that

$$\sum_{uv<ut<2\sqrt{x}} \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^3}\frac{\log\mathcal{Q}}{\varphi_1^8(\mathcal{Q})} \ll \sum_{uv<ut<2\sqrt{x}} \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^3}\frac{\log t}{\varphi_1^8(\mathcal{Q})} \ll \frac{x}{\log x}$$

and

$$\sum_{uv<ut<2\sqrt{x}} \frac{ut}{v}\frac{1}{(\log\frac{ut}{v})^3}\varphi_{1-\delta'}^{-8}(\mathcal{Q})\varphi_1^{-4}(\mathcal{Q}) \ll \frac{x}{(\log x)^2}.$$

Combining these two results we get the final bound

$$\tilde{S}_2(x) \ll \frac{x}{\log x}.$$

6 Final Result

Theorem 1.4 is now a consequence of the Cauchy-Schwarz inequality. We have

$$\tilde{S}_1(x) \le \Big(\sum_{n\le x}\tilde{R}(n)^0\Big)^{1/2}\Big(\sum_{n\le x}\tilde{R}(n)^2\Big)^{1/2} = \tilde{S}_0(x)^{1/2}\tilde{S}_2(x)^{1/2}.$$

Thus

$$S_0(x) \ge \tilde{S}_0(x) \ge \frac{\tilde{S}_1^2(x)}{\tilde{S}_2(x)}.$$

By replacing $\tilde{S}_2(x)$ with (1.3) and $\tilde{S}_1(x)$ with (4.1) we get the lower bound stated in Theorem 1.4.

On the other hand we have $S_0(x) \le S_1(x)$, which yields the upper bound.

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