Sum of the Lerch Zeta-Function over Nontrivial Zeros of the Dirichlet L-Function

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Dedicated to the memory of Professor Wolfgang Schwarz

Abstract For $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, λ rational, we consider the sum of values of the Lerch zeta-function $L(\lambda, \alpha, s)$ taken at the nontrivial zeros of the Dirichlet L-function $L(s, \chi)$, where χ mod Q, $Q \geq 1$, is a primitive Dirichlet character.

Keywords Dirichlet L-function • Value-distribution

2010 Mathematics subject classification: Primary 11M06; Secondary 11M26

1 Introduction

Let $s = \sigma + it$ denote a complex variable. We use the notation $e(x) = \exp(2\pi ix)$. By $\{x\}$, (a, b), and $[a, b]$ we denote the fractional part of the real number x, the greatest common divisor of integers a, b, and the least common multiple of integers a, b, respectively. In this paper T always tends to plus infinity and ε is any positive number.

The Lerch zeta-function is defined by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s} \quad (\sigma > 1),$$

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J. Sander et al. (eds.), *From Arithmetic to Zeta-Functions*,
DOI 10.1007/978-3-319-28203-9_10

where $0 < \lambda \leq 1$ and $0 < \alpha \leq 1$. The Dirichlet L-function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo some positive integer Q. For $\chi \bmod 1$ we get the Riemann zeta-function $L(s, \chi) = \zeta(s)$. The yet unsolved generalized Riemann hypothesis (GRH) states that inside the critical strip $0 < \sigma < 1$ every Dirichlet L-function has zeros only on the critical line $\sigma = \frac{1}{2}$. Zeros in the critical strip are called nontrivial and we denote them by $\rho_\chi = \beta_\chi + i\gamma_\chi$. In view of the functional equation (see formula (2.4) below) the nontrivial zeros are symmetrically distributed with respect to the critical line. A Dirichlet character $\chi \bmod Q$ is said to be primitive if it is not induced by any other character of modulus strictly less than Q. The unique principal character modulo Q is denoted by χ_0. The character $\chi_0 \bmod 1$ is the only one principal and primitive character. For $T > 0$, let $N(T, \chi)$ denote the number of the nontrivial zeros with $0 \leq \gamma_\chi \leq T$. For the primitive character $\chi \bmod Q$, we have (Montgomery and Vaughan [17, Corollary 14.7])

$$N(T, \chi) = \frac{T}{2\pi} \log \frac{QT}{2\pi e} + O(\log QT),$$

where $T \geq 4$.

For special values of α and λ the Lerch zeta-function reduces to the Riemann zeta-function $L(1, 1, s) = \zeta(s)$, $L(1, 1/2, s) = (2^s - 1)\zeta(s)$, $L(1/2, 1, s) = (1 - 2^{1-s})\zeta(s)$, the Dirichlet L-function $L(1/2, 1/2, s) = 2^s L(s, \psi)$, where $\psi \bmod 4$ is an odd Dirichlet character. If we fix $\lambda = 1$ we get the Hurwitz zeta-function $L(1, \alpha, s) = \zeta(s, \alpha)$ and if we fix $\alpha = 1$ we get the periodic zeta-function $e(\lambda)L(\lambda, 1, s) = F(s, \lambda)$. Nontrivial zeros $\rho(\lambda, \alpha) = \beta(\lambda, \alpha) + i\gamma(\lambda, \alpha)$ of the Lerch zeta-function $L(\lambda, \alpha, s)$ are located in the strip $-1 < \sigma < 1 + \alpha$ ([9]). In view of the formula [8, Corollary 2]

$$\sum_{|\gamma(\lambda,\alpha)| \leq T} \left(\beta(\lambda, \alpha) - \frac{1}{2} \right) = \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda(1 - \{\lambda\})}} + O(\log T),$$

we see that the nontrivial zeros are not always symmetrically distributed with respect to the critical line. It is pleasant to recall that the paper [8] was written when the first author was visiting W. Schwarz at Frankfurt university.

Let

$$\Lambda(x) = \begin{cases} \log p & \text{if } x = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

be the von Mangoldt function. For an integer n and a character χ mod Q the Gauss sum is defined by

$$G(n,\chi)=\sum_{a=1}^{Q}\chi(a)\exp\left(2\pi i\frac{an}{Q}\right).$$

If $(n,Q)=1$, then for a primitive character χ mod Q, we have $|G(n,\chi)|=\sqrt{Q}$ and, for the principle character χ_0 mod Q, it is known that $G(n,\chi_0)=\mu(n)$, where $\mu(n)$ is the Möbius function. We shall prove the following result:

Theorem 1.1 *Let χ mod Q, $Q\geq 1$, be a primitive Dirichlet character.*
We have, for $0<\lambda=\frac{k}{q}\leq 1$, $(k,q)=1$, and $0<\alpha<1$,

$$\sum_{0<\gamma_\chi\leq T} L\left(\lambda,\alpha,\rho_\chi\right)=-\left(\varLambda\left(\frac{1}{\alpha}\right)\chi\left(\frac{1}{\alpha}\right)+\delta(Q,q)e\left(-\alpha\lambda\right)\frac{\mu(q)}{\phi(q)}L(1-\alpha,\lambda,1)\right)\frac{T}{2\pi}$$
$$+O\left(T\exp(-c\log^{\frac{1}{4}-\epsilon}T)\right), \tag{1.1}$$

where $\delta(Q,q)=1$ if $Q|q$, $\delta(Q,q)=0$ otherwise, and c is a positive absolute constant.
Further, for $0<\lambda=\frac{k}{q}<1$, $(k,q)=1$,

$$\sum_{0<\gamma_\chi\leq T} L(\lambda,1,\rho_\chi)=\frac{T}{2\pi}\log\frac{TQ}{2\pi e}-\delta(Q,q)e(-\lambda)\frac{\mu(q)}{\phi(q)}\frac{T}{2\pi}\log\frac{T}{2\pi e}+C(\chi,\lambda)\frac{T}{2\pi}$$
$$+O\left(T\exp(-c\log^{\frac{1}{4}-\epsilon}T)\right), \tag{1.2}$$

where the constant $C(\chi,\lambda)$ is defined by Eq. (2.17) below.
Let A be a positive constant. Both asymptotic formulas of this theorem are valid uniformly for $Q\ll\log^A T$.

Next we discuss the asymptotic formula (1.1) of Theorem 1.1. From the proof of Theorem 1.2 in Nakamura [18] we know that, for $0<\lambda<1$ and $0<\alpha\leq 1$,

$$L(\lambda,\alpha,1)\neq 0.$$

The proof of the last formula is nice and short. By the integral representation (Lerch [16, formula (2)] or [15, formula (2.6)]) we have

$$\varGamma(\sigma)L(\lambda,\alpha,\sigma)=\int_0^\infty\frac{x^{\sigma-1}e^{(1-\alpha)x}}{e^x-e^{2\pi i\lambda}}dx=\int_0^\infty\frac{x^{\sigma-1}e^{(1-\alpha)x}(e^x-e^{-2\pi i\lambda})}{|e^x-e^{2\pi i\lambda}|^2}dx\quad(\sigma>0).$$

Now $\operatorname{Re}L(\lambda,\alpha,1)>0$ since $\operatorname{Re}(e^{(1-\alpha)x}(e^x-e^{2\pi i\lambda}))>0$ for $x>0$.

Formula (1.1) extends results obtained in Fujii [4], Steuding [10, 21], where the case $\lambda=1$, $0<\alpha<1$, and $Q=1$ was investigated. In [6] and [14] Formula (1.2) with $0 < \lambda = k/q < 1$, $(k, q) = 1$, $\alpha = 1$, and $Q = 1$ was considered, see also Steuding [22].

In the paper [11] the sum of $L(s, \chi)$ and in [7] the sum of $L(s, \chi)L(1 - s, \chi)$ over nontrivial zeros of another Dirichlet L-function were studied.

The next section is devoted to the proof of Theorem 1.1.

2 Proof of Theorem 1.1

We will use contour integration. The proof of the theorem relies on the method of Conrey et al. [2]. We divide the proof into the following Sects.: 2.1. Beginning of the proof: Functional equations; Sect. 2.2. Gonek's lemma; and Sect. 2.3. Perron's formula.

2.1 Beginning of the Proof: Functional Equations

Similarly as in [11], without loss of generality, we consider T satisfying the inequality

$$\min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log(QT)}. \tag{2.1}$$

For $q \geq 1$, χ mod q and $t \geq 0$ we have (Prachar [20, Theorem 3.3 in Chap. 7])

$$N_\chi(t + 1) - N_\chi(t) := \#\{\rho_\chi = \beta_\chi + i\gamma_\chi \, : \, t < \gamma_\chi \leq t + 1\} \ll \log Q(t + 2).$$

Thus

$$\sum_{0<\gamma_\chi \leq T} L(\lambda, \alpha, \rho_\chi) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s, \chi)L(\lambda, \alpha, s)ds + O(\log Q),$$

where the contour $\mathfrak{C}$ is a rectangle with vertices $a + ib$, $a + iT$, $1 - a + iT$, and $1 - a + ib$, where $a = 1 + 1/\log(QT)$, $2 \leq b \leq 3$, $\min_{\gamma_\chi} |b - \gamma_\chi| \gg 1/\log Q$. Further

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s, \chi)L(\lambda, \alpha, s)ds \\
&= \frac{1}{2\pi i} \left\{ \int_{a+ib}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{1-a+ib}^{a+ib} \right\} \frac{L'}{L}(s, \chi)L(\lambda, \alpha, s)ds \\
&=: \sum_{j=1}^{4} \mathcal{J}_j.
\end{aligned}$$

First we obtain

$$\begin{aligned}\mathcal{J}_1 =& \frac{1}{2\pi i}\int_{a+ib}^{a+iT} \frac{L'}{L}(s,\chi)L(\lambda,\alpha,s)ds\\ =& \frac{1}{2\pi}\int_{b}^{T} \frac{L'}{L}(a+it,\chi)L(\lambda,\alpha,a+it)dt\\ =& -\sum_{m=2}^{\infty}\sum_{n=0}^{\infty}\frac{\chi(m)\Lambda(m)e(\lambda n)}{(m(n+\alpha))^a}\frac{1}{2\pi}\int_b^T \frac{1}{(m(n+\alpha))^{it}}dt\\ =& -\Lambda\left(\frac{1}{\alpha}\right)\chi\left(\frac{1}{\alpha}\right)\frac{T}{2\pi}+O(\log^2(QT)).\end{aligned}$$

Second we consider $\mathcal{J}_2$. Formula (6) from [11] gives [recall that T satisfies (2.1)]

$$\frac{L'}{L}(\sigma+iT,\chi) \ll \log^2(QT) \qquad \text{for} \quad -1\le\sigma\le 2, T\ge 2. \tag{2.2}$$

By Corollary 4 from [5] we have

$$L(\lambda,\alpha,\sigma+iT) \ll T^{\frac{1}{2}}\log T \qquad \text{for} \quad -\frac{1}{\log(QT)}\le\sigma\le 1+\frac{1}{\log(QT)}. \tag{2.3}$$

Thus we get

$$\mathcal{J}_2 = \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT}\frac{L'}{L}(s,\chi)L(\lambda,\alpha,s)ds = O\left(T^{\frac{1}{2}}\log^3(QT)\right).$$

Similarly we see that

$$\mathcal{J}_4 = O(\log^3 Q).$$

A change of variables $s\mapsto 1-\overline{s}$ gives

$$\mathcal{J}_3 = -\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{L'}{L}(1-\overline{s},\chi)L(\lambda,\alpha,1-\overline{s})ds.$$

After conjugation we get

$$\overline{\mathcal{J}_3} = -\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{L'}{L}(1-s,\overline{\chi})\overline{L(\lambda,\alpha,1-\overline{s})}ds.$$

To evaluate this integral we will use functional equations. A Dirichlet L-function to a primitive character χ mod Q satisfies the functional equation (Apostol [1, Theorem 12.11])

$$L(1-s,\chi) = \Delta(1-s,\chi)L(s,\overline{\chi}), \tag{2.4}$$

where

$$\Delta(1-s,\chi) = \tau(\psi)\frac{1}{Q}\left(\frac{Q}{2\pi}\right)^s \Gamma(s)\left(\exp\left(-\frac{\pi i s}{2}\right) + \psi(-1)\exp\left(\frac{\pi i s}{2}\right)\right).$$

Taking the logarithmic derivative of the functional equation we get

$$\frac{L'}{L}(1-s,\chi) = \frac{\Delta'}{\Delta}(1-s,\chi) - \frac{L'}{L}(s,\overline{\chi}),$$

where

$$\frac{\Delta'}{\Delta}(s,\chi) = \frac{\Delta'}{\Delta}(1-s,\overline{\chi}) = -\log\frac{tQ}{2\pi} + O\left(\frac{1}{t}\right), \tag{2.5}$$

for $t > 1$. The Lerch zeta-function satisfies the functional equation (Lerch [16] or [15])

$$\begin{aligned} L(\lambda,\alpha,1-s) =& (2\pi)^{-s}\Gamma(s)\biggl(e\left(\frac{s}{4}-\alpha\lambda\right)L(1-\alpha,\lambda,s) \\ &+ e\left(-\frac{s}{4}+\alpha(1-\{\lambda\})\right)L(\alpha,1-\{\lambda\},s)\biggr). \end{aligned} \tag{2.6}$$

Therefore

$$\begin{aligned} &\frac{L'}{L}(1-s,\overline{\chi})\overline{L(\lambda,\alpha,1-\overline{s})} \\ &= \left(\frac{\Delta'}{\Delta}(s,\chi) - \frac{L'}{L}(s,\chi)\right)(2\pi)^{-s}\Gamma(s)\biggl(e\left(-\frac{s}{4}+\alpha\lambda\right)L(\alpha,\lambda,s) \\ &\quad + e\left(\frac{s}{4}-\alpha(1-\{\lambda\})\right)L(1-\alpha,1-\{\lambda\},s)\biggr). \end{aligned}$$

Then

$$\begin{aligned} \overline{\mathcal{J}_3} =& -e(\alpha\lambda)\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)(2\pi)^{-s}\Gamma(s)e\left(-\frac{s}{4}\right)L(\alpha,\lambda,s)ds \\ &- e(-\alpha(1-\{\lambda\}))\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)(2\pi)^{-s}\Gamma(s)e\left(\frac{s}{4}\right)L(1-\alpha,1-\{\lambda\},s)ds \end{aligned}$$

$$
\begin{aligned}
&+ e(\alpha\lambda)\frac{1}{2\pi i}\int_{a+ib}^{a+iT}(2\pi)^{-s}\Gamma(s)e\left(-\frac{s}{4}\right)\frac{L'}{L}(s,\chi)L(\alpha,\lambda,s)ds\\
&+ e(-\alpha(1-\{\lambda\}))\frac{1}{2\pi i}\int_{a+ib}^{a+iT}(2\pi)^{-s}\Gamma(s)e\left(\frac{s}{4}\right)\frac{L'}{L}(s,\chi)L(1-\alpha,1-\{\lambda\},s)ds\\
=:&\sum_{j=1}^{4}\mathcal{F}_j,
\end{aligned}
$$

say. In view of the bound (2.5), Stirling's formula, and by the bound [5, Corollary 2]

$$
L(\lambda,\alpha,a+it)\ll \log t, \tag{2.7}
$$

we have

$$
\mathcal{F}_2,\mathcal{F}_4\ll 1.
$$

Summarizing the so far obtained results we see that, for $0<\lambda\le 1$, $0<\alpha\le 1$, and $Q\ge 1$,

$$
\sum_{0<\gamma_\chi\le T}L(\lambda,\alpha,\rho_\chi)=-\Lambda\left(\frac{1}{\alpha}\right)\chi\left(\frac{1}{\alpha}\right)\frac{T}{2\pi}+\overline{\mathcal{F}_1}+\overline{\mathcal{F}_3}+O(T^{1/2+\epsilon}). \tag{2.8}
$$

2.2 Gonek's Lemma

For $\mathcal{F}_1$ and $\mathcal{F}_3$ we will use the following version of Gonek's Lemma (c.f. Gonek [12, Lemma 5] and [3, Lemma 2]).

Lemma 2.1 *Assume that* $\sum_{m\le x}|a_m|\ll x$ *and* $b_n\ll 1$. *Let* $1<c\le 1+1/\log\tau$ *and* $0<\delta<1$, *then*

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{c+i}^{c+i\tau}\Delta(1-s)\sum_{m=1}^{\infty}\frac{a_m}{m^s}\sum_{n=0}^{\infty}\frac{b_n}{(n+\alpha)^s}ds\\
&=\sum_{\substack{m\ge1,n\ge0\\ m(n+\alpha)\le\frac{\tau}{2\pi}}}a_mb_ne(-m(n+\alpha))+O\left(\tau^{\frac{1}{2}}(c-1)^{-2}\right)
\end{aligned}
$$

uniformly in $\alpha\in[\delta,1]$. *Here*

$$
\Delta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}
$$

is a factor from the functional equation $\zeta(1-s)=\Delta(1-s)\zeta(s)$.

Proof This is Lemma 5 in [10].

Let $\delta = \pm 1$. In order to apply Lemma 2.1 for our purpose note that

$$\frac{e\left(\delta \frac{s}{4}\right)}{2\cos\frac{\pi s}{2}} = \begin{cases} O\left(\exp\left(-\pi|t|\right)\right) & \text{if } \delta t \geq 0, \\ 1 + O\left(\exp\left(-\pi|t|\right)\right) & \text{otherwise.} \end{cases} \tag{2.9}$$

We return to the integral $\mathcal{F}_1$ and rewrite it as follows:

$$\mathcal{F}_1 = -e(\alpha\lambda)\int_b^T \frac{\Delta'}{\Delta}(a+i\tau,\chi)\, d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}(2\pi)^{-s}\Gamma(s)e\left(-\frac{s}{4}\right)L(\alpha,\lambda,s)ds\right).$$

In view of (2.9) and the bound from (2.7) Lemma 2.1 gives

$$\begin{aligned} &\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}(2\pi)^{-s}\Gamma(s)e\left(-\frac{s}{4}\right)L(\alpha,\lambda,s)ds \\ &= \sum_{0\leq n\leq\frac{\tau}{2\pi}-\lambda} e(\alpha n)e(-(n+\lambda)) + O(\tau^{\frac{1}{2}+\epsilon}) \\ &= e(-\lambda)\sum_{0\leq n\leq\frac{\tau}{2\pi}-\lambda} e(\alpha n) + O(\tau^{\frac{1}{2}+\epsilon}) = O(\tau^{\frac{1}{2}+\epsilon}). \end{aligned}$$

Then using Expression (2.5) we have, for $\alpha \neq 1$,

$$\mathcal{F}_1 = -e(\alpha\lambda)\int_b^T \frac{\Delta'}{\Delta}(a+i\tau,\chi)\, d\left(O(\tau^{\frac{1}{2}+\epsilon})\right) = O(T^{1/2+\epsilon})$$

and, for $\alpha = 1$,

$$\mathcal{F}_1 = \frac{T}{2\pi}\log\frac{TQ}{2\pi e} + O(T^{1/2+\epsilon}).$$

Next we consider the integral $\mathcal{F}_3$. Again by formula (2.9) and Lemma 2.1 we get

$$\mathcal{F}_3 = -\,e(\alpha\lambda)\sum_{\substack{m\geq 1, n\geq 0\\ m(n+\lambda)\leq\frac{T}{2\pi}}} \Lambda(m)\chi(m)e(\alpha n)e(-m(n+\lambda)) + O(T^{1/2+\epsilon}).$$

We split the last sum into two sums with $(m,q) = 1$ and $(m,q) > 1$ accordingly. For the case $(m,q) > 1$ we obtain, for $\alpha \neq 1$,

$$\sum_{\substack{m\geq 1, n\geq 0, (m,q)>1\\ m(n+\lambda)\leq\frac{T}{2\pi}}} \Lambda(m)\chi(m)e(\alpha n)e(-m\lambda) = O\left(\sum_{\substack{m\leq\frac{T}{2\pi}\\ (m,q)>1}} \Lambda(m)\right) = O(\log T)$$

and, for $\alpha = 1$,

$$\sum_{\substack{m\geq 1, n\geq 0, (m,q)>1 \\ m(n+\lambda)\leq \frac{T}{2\pi}}} \Lambda(m)\chi(m)e(\alpha n)e(-m\lambda) = \frac{T}{2\pi} \sum_{\substack{m\leq \frac{T}{2\pi} \\ (m,q)>1}} \Lambda(m)\chi(m)e(-m\lambda)\frac{1}{m} + O(\log T)$$

$$= \frac{T}{2\pi} \sum_{p|q} \log p \sum_{j=1}^{\infty} \frac{\chi(p^j)e(-p^j\lambda)}{p^j} + O(\log T).$$

Now we use that $\lambda = \frac{k}{q}$. If $(m, q) = 1$, then the orthogonality relation for Dirichlet characters gives

$$\Lambda(m)\chi(m)e\left(-m\frac{k}{q}\right) = \frac{1}{\phi(q)} \sum_{\psi \bmod q} \sum_{a=1}^{q} \overline{\psi}(a)e\left(-a\frac{k}{q}\right) \Lambda(m)\chi(m)\psi(m)$$

$$= \frac{1}{\phi(q)} \sum_{\psi \bmod q} G(-k, \overline{\psi})\Lambda(m)\chi(m)\psi(m),$$

where $G(-k, \overline{\psi})$ is the Gauss sum defined before Theorem 1.1. Then, for $0 < \alpha \leq 1$,

$$\mathcal{F}_3 = -\frac{e(\alpha\lambda)}{\phi(q)} \sum_{\psi \bmod q} G(-k, \overline{\psi}) \sum_{\substack{m\geq 1, n\geq 0 \\ m(n+\lambda)\leq \frac{T}{2\pi}}} \Lambda(m)\chi(m)\psi(m)e(\alpha n) \tag{2.10}$$

$$- (\alpha - \{\alpha\})e(\lambda)\frac{T}{2\pi} \sum_{p|q} \log p \sum_{j=1}^{\infty} \frac{\chi(p^j)e(-p^j\lambda)}{p^j} + O(T^{1/2+\epsilon}).$$

Again, we summarize obtained results. In view of formula (2.8) Sect. 2.2 gives, for $0 < \lambda \leq 1$, $0 < \alpha \leq 1$, and $Q \geq 1$,

$$\sum_{0<\gamma_\chi\leq T} L(\lambda, \alpha, \rho_\chi) = (\alpha - \{\alpha\})\frac{T}{2\pi} \log \frac{TQ}{2\pi e} - \Lambda\left(\frac{1}{\alpha}\right)\chi\left(\frac{1}{\alpha}\right)\frac{T}{2\pi} + \overline{\mathcal{F}_3} + O(T^{1/2+\varepsilon}). \tag{2.11}$$

2.3 Perron's Formula

In Titchmarsh [23, Lemma 3.12] Perron's formula for ordinary Dirichlet series is proved. Easy to check that the analogous lemma is true for Dirichlet series $\frac{L'}{L}(s, \chi\psi)L(\alpha, \lambda, s)$. Note that Perron [19] proved his formula (without error term) for general Dirichlet series. Therefore, for the second sum of the formula (2.10), Perron's formula gives

$$\sum_{\substack{m\ge 1,n\ge 0\\ m(n+\lambda)\le \frac{T}{2\pi}}} \Lambda(m)\chi(m)\psi(m)e(\alpha n) = -\frac{1}{2\pi i}\int_{a-iU}^{a+iU}\frac{L'}{L}(s,\chi\psi)L(\alpha,\lambda,s)\left(\frac{T}{2\pi}\right)^s\frac{ds}{s} + O\left(\frac{T\log^2 T}{U}\right).$$

In [20, Chap. 8, Theorem 6.2] considering $[q,Q] \ll \log^B T$, with B being any positive constant, we find

$$L(s,\chi\psi)\neq 0 \quad \text{for} \quad \sigma > 1-\frac{c}{\log^{\frac{3}{4}+\epsilon} T},$$

where c is an absolute positive constant. With regard to this zero-free region for $L(s,\chi\psi)$ let $b_1 = 1 - c/\log^{(\frac{3}{4}+\epsilon)} T$. Shifting the line of integration we get

$$\begin{aligned}
\sum_{\substack{m\ge 1,n\ge 0\\ m(n+\lambda)\le \frac{T}{2\pi}}} \Lambda(m)\chi(m)\psi(m)e(\alpha n) &= -\operatorname{Res}_{s=1}\frac{L'}{L}(s,\chi\psi)L(\alpha,\lambda,s)\left(\frac{T}{2\pi}\right)^s\frac{1}{s}\\
&+\frac{1}{2\pi i}\left\{\int_{a+iU}^{b_1+iU}+\int_{b_1+iU}^{b_1-iU}+\int_{b_1-iU}^{a-iU}\right\}\frac{L'}{L}(s,\chi\psi)L(\alpha,\lambda,s)\left(\frac{T}{2\pi}\right)^s\frac{ds}{s} \qquad (2.12)\\
&+O\left(\frac{T\log^2 T}{U}\right)\\
&=: -\operatorname{Res}_{s=1}\frac{L'}{L}(s,\chi\psi)L(\alpha,\lambda,s)\left(\frac{T}{2\pi}\right)^s\frac{1}{s}+\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+O\left(\frac{T\log^2 T}{U}\right).
\end{aligned}$$

Recall that the character χ is primitive. Thus if $\chi\psi$ is a principal character mod $[q,Q]$, then $Q|q$ and

$$\psi = \overline{\chi}\psi_0,$$

where ψ_0 is a principle character mod$q$. In view of this we introduce the notation

$$\delta(Q,q) = \begin{cases} 1 & \text{if } Q|q,\\ 0 & \text{otherwise.}\end{cases} \qquad (2.13)$$

The Dirichlet L-function attached to the principal character has a simple pole at $s=1$, otherwise it is an entire function. In view of $L(s,\psi_0) = \zeta(s)\prod_{p|q}(1-p^{-s})$

we have

$$\frac{L'}{L}(s,\psi_0) = -\frac{1}{s-1} + \gamma + \sum_{p|q} \frac{\log p}{p^s - 1} + O(s-1) \qquad (s \to 1),$$

here γ is the Euler–Mascheroni constant. For $0 < \alpha < 1$, the function $L(\alpha, \lambda, s)$ is entire and, for $\alpha = 1$, the function $L(\alpha, \lambda, s) = \zeta(s, \lambda)$ has a simple pole at $s = 1$. It is known (Ivić [13, formula (1.122)]) that

$$\zeta(s,\lambda) = \frac{1}{s-1} + \gamma(\lambda) + O(s-1) \qquad (s \to 1),$$

where

$$\gamma(\lambda) = \lim_{N\to\infty} \left(\sum_{m=0}^{N} \frac{1}{m+\lambda} - \log(N+\lambda) \right) = -\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}.$$

The last equality can be found in Wilton [24]. Note that $\gamma(1)$ is the Euler–Mascheroni constant. Thus, for $\alpha \neq 1$,

$$-\operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L(\alpha,\lambda,s) \left(\frac{T}{2\pi}\right)^s \frac{1}{s} = L(\alpha,\lambda,1)\frac{T}{2\pi} \tag{2.14}$$

and, for $\alpha = 1$,

$$-\operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L(\alpha,\lambda,s) \left(\frac{T}{2\pi}\right)^s \frac{1}{s} = \frac{T}{2\pi}\log\frac{T}{2\pi e} - \left(\frac{\Gamma'(\lambda)}{\Gamma(\lambda)} + \gamma + \sum_{p|q}\frac{\log p}{p-1}\right)\frac{T}{2\pi}. \tag{2.15}$$

If $\chi\psi$ is a nonprincipal character $\bmod[q,Q]$, then for $\alpha = 1$,

$$-\operatorname{Res}_{s=1} \frac{L'}{L}(s,\chi\psi) L(\alpha,\lambda,s) \left(\frac{T}{2\pi}\right)^s \frac{1}{s} = -\frac{L'}{L}(1,\chi\psi)\frac{T}{2\pi}. \tag{2.16}$$

Integrals $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ are analogous to the integrals which appear in Formula (9) of [11]. By a similar reasoning as in [11], using bounds (2.2), (2.3), and choosing $U = T^{1-b_1}$, we get that

$$\mathcal{K}_j \ll T^{1-\frac{c}{\log^{3/4+\varepsilon} T}},$$

for $j = 1, 2, 3$. In view of formulas (2.10)–(2.13) the residue (2.14) leads to Expression (1.1) of Theorem 1.1. Finally, residues (2.15) and (2.16) give (1.2) with the constant

$$
\begin{aligned}
C(\chi,\lambda) = {} & -\delta(Q,q)e(-\lambda)\frac{\mu(q)}{\phi(q)}\left(\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}+\gamma+\sum_{p|q}\frac{\log p}{p-1}\right)+ \\
& +\frac{e(-\lambda)}{\phi(q)}\sum_{\substack{\psi \bmod q\\ \chi\psi\neq\psi_0}} G(k,\psi)\frac{L'}{L}(1,\overline{\chi\psi}) \\
& -e(-\lambda)\sum_{p|q}\log p\sum_{j=1}^{\infty}\frac{\overline{\chi}(p^j)e(p^j\lambda)}{p^j}.
\end{aligned} \tag{2.17}
$$

This proves Theorem 1.1.

Acknowledgement The authors are supported by grant No. MIP-049/2014 from the Research Council of Lithuania.

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