Chapter 4
Controlling Oscillations in Nonlinear Systems with Delayed Output Feedback

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Abstract We discuss the problem of controlling oscillations in weakly nonlinear systems by delayed feedback. In classical control theory, the objective of the control action is typically to drive the system to a stable equilibrium. Here we also study the possibility of driving the system to a stable limit cycle having a prescribed amplitude and frequency, as well as suppressing unwanted oscillations, using partial state information in the feedback. The presence of the delay in the output feedback turns out to play a crucial role in achieving these goals.

4.1 Introduction

Controlling the behavior of dynamical systems is a problem of practical importance in many applications. In classical control theory, the basic goal is usually stated as a *regulator problem*, namely, to obtain an asymptotically stable equilibrium solution which attracts all nearby initial conditions. A more sophisticated aim in oscillation control can be defined as obtaining a stable periodic solution with desired properties, such as oscillation at a given amplitude or frequency. We call this goal the *oscillator problem*. This chapter deals with the oscillator problem under delayed output feedback.

Feedback delays are an inevitable feature of many natural and man-made control mechanisms. While they are often seen as an undesired characteristic that can destabilize the system or complicate the analysis, positive uses of delays have also been studied. These go back to the 1950s [1], followed by other works in later years [2–5], where delays were used to enhance the system performance in various ways. Most of the analytical studies have so far focused on linear systems and stability. In the present chapter we consider feedback laws to control the amplitude and stability of oscillations in nonlinear systems. Moreover, we consider the problem from an *output*

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feedback point of view, where only partial information about the state of the system is available for the feedback control.

The main results we present can be rigorously derived in full generality for weak nonlinearities, such as for systems near a Hopf bifurcation, which is an important mechanism for generating oscillations in nonlinear systems. The analysis then starts by projecting the dynamics onto a center manifold and proceeds by investigating the resulting two-dimensional system. This general approach for oscillation control can be found in [6–8]. For the purpose of simplicity, here we will assume that such a reduction step has already been done and a two-dimensional system has been obtained. Therefore, we will study systems described by equations of the form

$$\ddot{x} + \omega^2 x + \varepsilon g(x, \dot{x}, \varepsilon) = \varepsilon f(x(t - \tau)). \tag{4.1}$$

Here $x \in \mathbb{R}$, and ω and $\varepsilon \ll 1$ are positive parameters. The left hand side of (4.1) describes the dynamics of the system after projection onto a two-dimensional center manifold corresponding to a pair of imaginary eigenvalues $\pm i\omega$, whereas the right hand side represents a feedback of position that is delayed by $\tau \geq 0$. The feedback is, at the moment, scaled by the parameter ε so that it has a comparable magnitude with the nonlinearity g; however, we will relax this assumption in Sect. 4.5 when we study frequency control.

The form of left hand side of (4.1) is quite general and includes several paradigmatic systems as special cases, for instance the van der Pol (with $g(x, \dot{x}, \varepsilon) = (x^2 - 1)\dot{x}$) and the Duffing oscillators (with $g(x, \dot{x}, \varepsilon) = \alpha x + \beta x^3 + \gamma \dot{x}$). Equations of the form (4.1) also come up in various biological and industrial settings, for example in the production of proteins [9, 10], orientation control in the fly [11, 12], neuromuscular regulation of movement and posture [10, 13, 14], acousto-optical bistability [15], metal cutting [16], vibration absorption [17], and control of the inverted pendulum [18]. Feedback loops with only partial state information is typical in many biological control mechanisms. Furthermore, the classical control-theoretic approach of using an observer to reconstruct the full state is not an option in natural systems. Hence, it is an interesting and challenging goal to discover the theoretical basis for control under partial and delayed information.

The regulator and oscillator problems under delayed feedback have been studied for nonlinear equations of type (4.1) in several previous works. Some of the most relevant ones for the purposes of this chapter using similar techniques can be listed as follows. Controlling the amplitude of oscillations was investigated in [19] for the van der Pol oscillator and later in [20] for more general oscillators (4.1). Controlling the frequency of oscillations is studied in [6]. Suppressing oscillations in networks has been treated in [21]. A general study for controlling systems near Hopf bifurcation using distributed delays is given in [7], and for networks of oscillators in [21].

In the following we will analyze (4.1) and show that the goal of the regulator problem (stabilizing the zero solution) can be achieved by a linear delayed feedback of the variable x, and the goal of the oscillator problem (obtaining a stable limit cycle at a given amplitude and/or modifying its frequency) can be achieved by a nonlinear feedback function. The conclusion holds for general nonlinearities g and using only the feedback of the position x. On the other hand, the presence of a positive delay in (4.1) turns out to be essential in attaining most of these goals.

4.2 Averaging Theory and Periodic Solutions

For notation, we will use $\|\cdot\|$ for the usual Euclidean norm and $D_i g$ for the partial derivative of the function g with respect to its ith argument. Without loss of generality, it will be assumed throughout that $\omega = 1$ in (4.1), which can always be achieved by a rescaling of the time $t \mapsto \omega t$. Furthermore, it will be assumed that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are C^2, $f(0) = 0$, and $g(0, 0, \varepsilon) = 0$ for all ε.

The main tool for the analysis of (4.1) will be averaging theory for delay differential equations. Consider amplitude-phase variables (r, θ) defined by the transformation

$$\begin{aligned} x(t) &= \ \ r(t)\cos(t + \theta(t)) \\ \dot{x}(t) &= -r(t)\sin(t + \theta(t)). \end{aligned} \tag{4.2}$$

In these new coordinates, (4.1) takes the form

$$\begin{aligned} \dot{r} &= \varepsilon \sin(t + \theta)(g - f) \\ \dot{\theta} &= \varepsilon \frac{1}{r} \cos(t + \theta)(g - f), \end{aligned} \tag{4.3}$$

where the arguments of f and g are expressed in terms of r and θ, i.e.,

$$\begin{aligned} g &= g\,(r(t)\cos(t + \theta(t)), -r(t)\sin(t + \theta(t)), \varepsilon) \\ f &= f\,(r(t - \tau)\cos(t - \tau + \theta(t - \tau))). \end{aligned} \tag{4.4}$$

When $\varepsilon = 0$, the solutions of (4.3) are constants, which correspond by (4.2) to the usual harmonic oscillations. Thus, (4.3) can be viewed as a time-dependent perturbation of a simple harmonic oscillator in the amplitude-phase variables, which can be analyzed by the method of averaging for small ε.

Letting $y = (r, \theta) \in \mathbb{R}^2$, the system (4.3) and (4.4) is a delay differential equation describing the relation between the instantaneous derivative $\dot{y}(t)$ and the present and past values of $y(t)$. A solution $y(t)$ of (4.3) describes a trajectory in the infinite-dimensional state space $\mathcal{C} := C([-\tau, 0], \mathbb{R}^2)$, namely, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to $\mathbb{R}^2$, equipped with the supremum norm, $\|f\| = \sup_{x\in[-\tau,0]} f(x)$. A point y_t on a trajectory is a piece of the solution function over an interval of length τ, defined by $y_t(s) = y(t + s)$, $s \in [-\tau, 0]$. In this notation, (4.3) can be written as

$$\dot{y}(t) = \varepsilon h(t, y_t, \varepsilon) \tag{4.5}$$

where h is periodic in t with period $T = 2\pi$. The averaged equation corresponding to (4.5) is defined as

$$\dot{z}(t) = \varepsilon \bar{h}(z_t) \tag{4.6}$$

where

$$\bar{h}(\varphi) := \frac{1}{T} \int_0^T h(t, \varphi, 0)\, dt. \tag{4.7}$$

In (4.6) z_t is understood as a *constant* element of $\mathcal{C}$. One can intuitively understand this by noting that y is slowly changing by (4.5), so that $y_t(s) \equiv y(t) + \mathcal{O}(\varepsilon)$ for $s \in [-\tau, 0]$, i.e., y is almost constant over an interval of length τ. Thus, (4.6) is an ordinary differential equation. In this way, averaging reduces the infinite-dimensional system (4.5) to a finite dimensional one, (4.6). Furthermore, by the averaging theorem, hyperbolic equilibrium points of (4.6) correspond to hyperbolic periodic solutions of (4.5), with the same stability type [22].

We now return to our main equation (4.1) and its equivalent formulation (4.3) to apply averaging theory. We average the equation for r given in (4.3) in the sense of (4.7) to obtain

$$\begin{aligned} \dot{r} = \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \sin(t+\theta)\, g(r\cos(t+\theta), -r\sin(t+\theta), 0)\, dt - \\ \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \sin(t+\theta) f(r\cos(t-\tau+\theta))\, dt. \end{aligned} \tag{4.8}$$

Here, in accordance with (4.6), r and θ are treated as constants over one period. With the change of variables $u = -(t+\theta)$, and using the fact that the integrand is 2π-periodic in u, the first integral in (4.8) becomes

$$\varepsilon \frac{1}{2\pi} \int_{-\theta}^{-\theta-2\pi} (\sin u\,) g(r\cos u, r\sin u, 0)\, du.$$

Similarly, with $u = t - \tau + \theta$, the second integral in (4.8) can be written as

$$\begin{aligned} &- \varepsilon \frac{1}{2\pi} \int_{\theta-\tau}^{2\pi+\theta-\tau} \sin(u+\tau) f(r\cos u)\, du \\ &= -\varepsilon \frac{\sin\tau}{2\pi} \int_0^{2\pi} f(r\cos u)\cos u\, du - \varepsilon \frac{\cos\tau}{2\pi} \int_0^{2\pi} f(r\cos u)\sin u\, du \end{aligned} \tag{4.9}$$

where we have used the fact that the second integral in (4.9) is zero. Combining, we see that the averaged equation for r has the form

$$\dot{r} = -\varepsilon(F(r) + G(r)), \tag{4.10}$$

where

$$F(r) = \frac{\sin\tau}{2\pi}\int_0^{2\pi} f(r\cos t)\cos t\,dt, \tag{4.11}$$

$$G(r) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos t, r\sin t, 0)\sin t\,dt. \tag{4.12}$$

By the averaging theorem for delay differential equations [22] and the transformation (4.2), positive hyperbolic equilibria R of (4.10) yield hyperbolic periodic solutions of (4.1) of the form $x(t) \approx R\cos t$, with the same stability type. In other words, if $R > 0$ is such that $F(R) + G(R) = 0$ and $F'(R) + G'(R) \neq 0$, then (4.1) has a periodic solution which is orbitally asymptotically stable if $F'(R) + G'(R) > 0$, and unstable if $F'(R) + G'(R) < 0$, as long as $\varepsilon > 0$ is sufficiently small. In this way, studying nontrivial hyperbolic periodic solutions of (4.1) is reduced to investigating positive and hyperbolic equilibrium points of (4.10).

The stability argument extends to $R = 0$ and can be used to deduce the stability of the zero solution of (4.1). In fact, this can be done directly without resorting to averaging, but it is interesting to relate the conditions to the averaged quantities (4.11) and (4.12). Thus, linearization of (4.1) about the zero solution gives the characteristic equation

$$\Delta(\lambda,\varepsilon) := \lambda^2 + 1 + \varepsilon(D_1 g(0,0,\varepsilon) + \lambda D_2 g(0,0,\varepsilon)) - \varepsilon f'(0)e^{-\lambda\tau} = 0. \tag{4.13}$$

When $\varepsilon = 0$, there are two roots on the imaginary axis: $\lambda = \pm i$. By the implicit function theorem, the roots depend smoothly on ε in a neighborhood of $\varepsilon = 0$, and implicit differentiation of (4.45) gives

$$\mathrm{Re}[\lambda'(\varepsilon)|_{\varepsilon=0}] = -\frac{1}{2}(f'(0)\sin\tau + D_2 g(0,0,0)) \tag{4.14}$$

$$= -\big(F'(0) + G'(0)\big). \tag{4.15}$$

Hence, the roots λ move into the left (respectively, right) complex half-plane if $F'(0) + G'(0)$ is positive (resp., negative), and remain there for all sufficiently small $\varepsilon > 0$, indicating that the zero solution of (4.1) is asymptotically stable if $F'(0) + G'(0) > 0$ and unstable if $F'(0) + G'(0) < 0$. Thus, the stability of equilibrium solutions (regulator problem) and periodic orbits (oscillator problem) can be conveniently expressed within the same framework.

Remark 1 For calculations it is worthwhile to note that F and G defined in (4.11) and (4.12) are both odd functions of r; i.e.

$$F(-r) = -F(r) \quad \text{and} \quad G(-r) = -G(r) \quad \text{for all } r \in \mathbb{R}. \tag{4.16}$$

For details, see [20].

4.3 Linear Feedback

Classical control theory has been extensively developed for linear systems or their linearizations at suitable operating points. Hence, it is natural to first consider a linear feedback law, namely the case when f has the form

$$f(x) = k_1 x, \tag{4.17}$$

for some feedback gain $k_1 \in \mathbb{R}$. Then by (4.11),

$$F(r) = \frac{1}{2} r k_1 \sin \tau. \tag{4.18}$$

We first consider the regulator problem of stabilizing of the zero solution. From (4.14),

$$\mathrm{Re}[\lambda'(\varepsilon)|_{\varepsilon=0}] = -\frac{1}{2}(k_1 \sin \tau + D_2 g(0,0,0)).$$

We thus immediately obtain that, for small $\varepsilon > 0$, the zero solution of (4.1) is asymptotically stable if $k_1 \sin \tau > -D_2 g(0,0,0)$, and unstable if $k_1 \sin \tau < -D_2 g(0,0,0)$.

For periodic solutions, we seek positive fixed points R of the averaged equation (4.10), i.e., of

$$\dot{r} = -\varepsilon \left(\frac{1}{2} r k_1 \sin \tau + G(r) \right), \tag{4.19}$$

which gives

$$k_1 \sin \tau = -2 \frac{G(R)}{R}. \tag{4.20}$$

We define the function

$$\bar{G}(r) := \frac{G(r)}{r} \tag{4.21}$$

and note that

$$\bar{G}'(r) = \frac{r G'(r) - G(r)}{r^2} = \frac{1}{r}(G'(r) - \bar{G}(r)). \tag{4.22}$$

Combining (4.18), (4.20), and (4.21), we have

$$F'(R) + G'(R) = R \bar{G}'(R).$$

Therefore, a positive solution R of (4.20) is a fixed point of the averaged equation (4.10) and its stability is determined only by the sign of $\bar{G}'(R)$. By the averaging theorem, such points correspond to periodic solutions of the original equation (4.1) with amplitude R, which are orbitally asymptotically stable if $\bar{G}'(R) > 0$, and unstable if $\bar{G}'(R) < 0$, for sufficiently small $\varepsilon > 0$.

Now the important observation is that, for any desired amplitude $R > 0$, it is possible to find a feedback gain k_1 such that (4.20) is satisfied, provided $\sin\tau \neq 0$. Hence, delayed linear feedback can be effective in modifying the amplitude of periodic solutions. The condition $\sin\tau \neq 0$ shows that a nonzero delay in the feedback is essential for this task. However, the *stability* of these periodic solutions depends only on the function $\bar{G}$, and hence on the nonlinearity g. So, linear feedback is helpful in solving the oscillator problem only to the extent allowed by the nonlinearity g (for an example see [19]). On the other hand, linear feedback is effective in the regulator problem since it can stabilize the zero solution. We illustrate with examples.

Example 2 Consider the celebrated van der Pol oscillator under delayed feedback

$$\ddot{x}(t) + \varepsilon(x^2 - 1)\dot{x} + 1 = \varepsilon k_1 x(t - \tau). \tag{4.23}$$

It is well known that the uncontrolled system ($k_1 = 0$) has an attracting limit cycle solution $x(t) \approx 2\cos t$ for small ε whereas the origin is unstable. We will show that we can modify the amplitude of limit cycle oscillations or make the origin stable by an appropriate choice of feedback gain k_1. Now, (4.23) has the form (4.1) with $g(x, \dot{x}, \varepsilon) = (x^2 - 1)\dot{x}$ and $f(x) = k_1 x$. The averaged quantities are

$$G(r) = \frac{1}{2} r \left(\frac{r^2}{4} - 1 \right) \tag{4.24}$$

and F as in (4.18); so the averaged equation for (4.23) is

$$\dot{r} = -\varepsilon \frac{r}{2} \left(\frac{r^2}{4} - 1 + k_1 \sin\tau \right). \tag{4.25}$$

This equation has a fixed point at zero, and another one at $r = R = 2\sqrt{1 - k_1 \sin\tau}$ if $k_1 \sin\tau < 1$. We have $\bar{G}'(r) = r/4$, which is clearly positive for all $r > 0$; so the fixed point R is stable whenever it exists. Therefore, for $0 < \varepsilon \ll 1$, (4.23) can have a stable periodic solution with amplitude approximately $R = 2\sqrt{1 - k_1 \sin\tau}$. In the absence of feedback, i.e., when $k_1 = 0$, we recover the familiar periodic solution $x(t) \approx 2\cos t$ of (4.23), but we also see that we can set the amplitude arbitrarily by changing k_1. Moreover, by choosing $k_1 \sin\tau > -D_2 g(0, 0, 0) = 1$, the limit cycle oscillations can be destroyed and the origin can be made stable. Both situations are depicted in Fig. 4.1.

Example 3 We make a small modification to the van der Pol oscillator of Example 2 and consider the nonlinearity g with reversed sign, i.e., $g = -(x^2 - 1)\dot{x}$, again with linear delayed feedback:

$$\ddot{x}(t) - \varepsilon(x^2 - 1)\dot{x} + 1 = \varepsilon k_1 x(t - \tau). \tag{4.26}$$

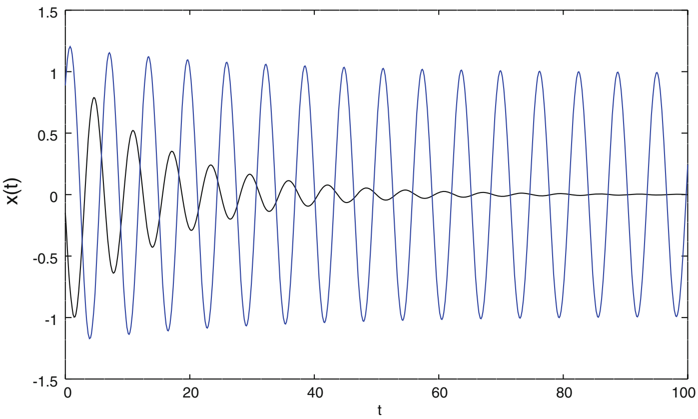

Fig. 4.1 Van der Pol oscillator under delayed feedback. Choosing a feedback gain of $k_1 = 0.75$ reduces the amplitude of limit cycle oscillations to 1 (*blue curve*), whereas increasing the gain to $k_1 = 2$ destroys the limit cycle and stabilizes the equilibrium point (*black curve*). *Parameter values* $\tau = \pi/2$ and $\varepsilon = 0.1$; random initial conditions

Now the averaged equation becomes

$$\dot{r} = -\varepsilon\frac{r}{2}\left(1 - \frac{r^2}{4} + k_1 \sin\tau\right), \tag{4.27}$$

which has a positive fixed point at $R = 2\sqrt{1 + k_1 \sin\tau}$ if $k_1 \sin\tau > -1$. As before, the amplitude of periodic solutions can be changed by appropriate choice of k_1 and τ. However, these solutions are all unstable because $\bar{G}'(R) = -R/4 < 0$. Thus, in this case the nonlinearity g does not allow the linear feedback to set up stable limit cycle oscillations at any amplitude R. (Note that the origin is locally stable as long as $k_1 \sin\tau > -D_2 g(0, 0, 0) = -1$.) In the next section we shall show how to overcome this limitation by adding a nonlinear term to the feedback function.

4.4 Nonlinear Feedback

As we have seen in Sect. 4.3, linear feedback is sufficient for the regulator problem but in general not for the oscillator problem. Therefore, we now turn to nonlinear feedback schemes. We show that, by adding a cubic term to the feedback function, the possibility of controlling oscillations through delayed feedback is greatly improved.

We consider a feedback function of the form

$$f(x) = k_1 x + k_3 x^3. \tag{4.28}$$

We will show that the coefficients k_i can be chosen so that the averaged equation (4.10) has a stable equilibrium point at a desired value R.

The averaged function F corresponding to (4.28) is also a cubic polynomial,

$$F(r) = q_1 r + q_3 r^3, \tag{4.29}$$

with

$$q_1 = \frac{1}{2} k_1 \sin \tau \quad \text{and} \quad q_3 = \frac{3}{8} k_3 \sin \tau. \tag{4.30}$$

Consequently,

$$F(r) + G(r) = r(q_1 + q_3 r^2 + \bar{G}(r)), \tag{4.31}$$

where the function $\bar{G}$ is defined in (4.21). Now let $R > 0$ be given. We will choose q_1 in a suitable manner, to be described shortly, and define q_3 in terms of q_1 as

$$q_3 = \frac{-q_1 - \bar{G}(R)}{R^2}. \tag{4.32}$$

With this choice of q_3, it follows from (4.31) that $F(R) + G(R) = 0$; so, R is an equilibrium point of the averaged equation (4.10). We will choose q_1 to ensure that R is a stable equilibrium, i.e., $F'(R) + G'(R) > 0$. From (4.31),

$$F'(R) + G'(R) = R(2Rq_3 + \bar{G}'(R)), \tag{4.33}$$

which is positive provided

$$q_3 > -\frac{\bar{G}'(R)}{2R}. \tag{4.34}$$

Using (4.34) in (4.32), the condition on q_1 is found as

$$q_1 < \frac{1}{2} R\bar{G}'(R) - \bar{G}(R). \tag{4.35}$$

From conditions (4.34) and (4.35), the feedback coefficients of (4.28) can then be calculated, using (4.30), as $k_1 = 2q_1(\sin \tau)^{-1}$ and $k_3 = 8q_3(3 \sin \tau)^{-1}$, whenever $\sin \tau \neq 0$. We thus have a procedure for feedback design to create a stable periodic solution with a prescribed amplitude R: First choose k_1 and/or τ so that

$$k_1 \sin \tau < R\bar{G}'(R) - 2\bar{G}(R). \tag{4.36}$$

Subsequently, calculate k_3 through the formula

$$k_3 = -\frac{8}{3R^2} \left(\frac{1}{2} k_1 + \frac{\bar{G}(R)}{\sin \tau} \right). \tag{4.37}$$

Then, for all sufficiently small $\varepsilon > 0$, the nonlinear feedback (4.28) ensures that the system (4.1) has an asymptotically orbitally stable periodic solution whose amplitude is $R + \mathcal{O}(\varepsilon)$. By a similar reasoning it can be seen that, by reversing the inequality in (4.36), one obtains an unstable periodic solution with amplitude $R + \mathcal{O}(\varepsilon)$.

Example 4 We consider the modified van der Pol equation of Example 3, this time with a nonlinear feedback:

$$\ddot{x}(t) - \varepsilon(x^2 - 1)\dot{x} + 1 = \varepsilon k_1 x(t - \tau) + \varepsilon k_3 x^3(t - \tau). \tag{4.38}$$

The averaged equation is

$$\dot{r} = -\varepsilon \frac{r}{2}\left(1 - \frac{r^2}{4} + k_1 \sin\tau + \frac{3r^2}{4} k_3 \sin\tau\right), \tag{4.39}$$

which has the positive fixed point

$$r = R = 2\sqrt{\frac{1 + k_1 \sin\tau}{1 - 3k_3 \sin\tau}} \tag{4.40}$$

whenever the radicand is positive. Furthermore,

$$R\bar{G}'(R) - 2\bar{G}(R) = -\frac{R^2}{4} - 2\left(\frac{1}{2} - \frac{R^2}{8}\right) = -1;$$

so, choosing $k_1 \sin\tau < -1$ satisfies (4.36) and ensures that the fixed point R is stable. Formula (4.37) then determines the remaining coefficient k_3. For instance, if it is desired to create stable oscillations at an amplitude of $R = 3$ with $\tau = \pi/2$, we can choose, e.g., $k_1 = -2$, and find $k_3 = 13/27$ from (4.40). Figure 4.2 shows the resulting limit cycle.

Remark 5 One may wonder why we have chosen to add a cubic term in (4.28). We note that if f is an even function, then (4.11) gives $F(-r) = F(r)$, so that, in view of (4.16), one has $F(r) \equiv 0$. Hence, F only depends on the odd part $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ of f. Since for small ε the dynamics of (4.1) is determined by G and F, there is no loss of generality in assuming that f is an odd function. In this sense, (4.28) represents the simplest nonlinear feedback function (at least in the ring of polynomials). Together with the results of the previous section, it is seen that simple delayed feedback schemes can be quite powerful in oscillation control.

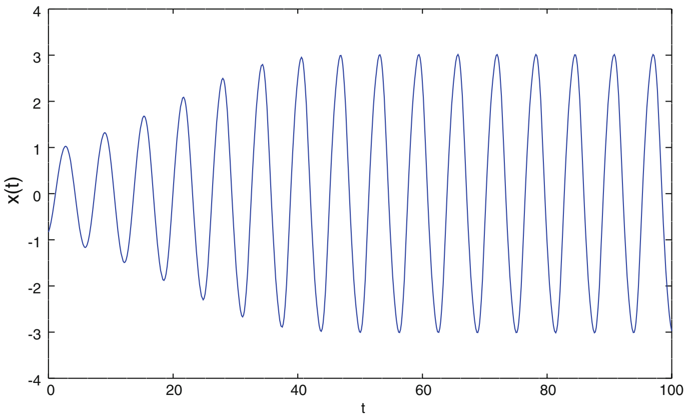

Fig. 4.2 The modified van der Pol oscillator (4.38) under nonlinear feedback exhibiting stable limit cycle oscillations at the prescribed amplitude 3. Parameter values are $\tau = \pi/2$, $\varepsilon = 0.1$, $k_1 = -2$, and $k_3 = 0.48$; random initial conditions

4.5 Controlling the Frequency of Oscillations

The results of the foregoing sections indicate that the control law (4.28) can be effective for controlling the stability of periodic solutions as well as specifying their amplitude. However, so far we have not discussed controlling the *frequency* of the oscillations. For this latter goal, it turns out that the feedback magnitude in (4.1) needs to be modified. Namely, we need to relax the assumption that the forcing term on the right hand side of (4.1) is of order ε. Therefore, we will now consider the slightly modified equation

$$\ddot{x} + x + \varepsilon g(x, \dot{x}, \varepsilon) = f(x(t - \tau)). \tag{4.41}$$

The reason for this change of right hand side can be understood as follows. The previous system (4.1) was viewed as an ε-perturbation of a simple harmonic oscillator. There, by using a suitable feedback function, we were able to create a stable limit cycle at a prescribed amplitude because the harmonic oscillator has periodic solutions of all amplitudes. However, all these solutions have the same frequency 1. Therefore, forcing the system with a feedback magnitude of order ε cannot change the frequency appreciably. In the case of (4.41), however, the unperturbed system

$$\ddot{x} + x = f(x(t - \tau)). \tag{4.42}$$

is no longer the simple harmonic oscillator; in fact, it is not a planar system anymore if $\tau \neq 0$. This may offer more possibilities for choosing a desired periodic solution at a certain amplitude *and* frequency. The price to be paid is that (4.42) is an infinite-dimensional system.

We can proceed in a similar way using averaging theory and the insight gained from the previous sections. From Sect. 4.4 we know that a cubic function of the form $f(x) = \varepsilon(k_1 x + k_3 x^3)$ can be used in (4.42) to control amplitude of oscillations, and we have observed that we would need a feedback term of higher magnitude if we are to have any hope of modifying frequencies significantly. We are therefore naturally led to trying the following feedback form

$$f(x) = kx + \varepsilon(k_1 x + k_3 x^3) \tag{4.43}$$

in (4.41).

With the choice (4.43) and small ε, (4.41) can be viewed as an perturbation of the linear system

$$\ddot{x} + x = kx(t - \tau). \tag{4.44}$$

Just like for the harmonic oscillator, we would like to know what variety of stable periodic solutions (4.44) has. For this purpose we seek purely imaginary solutions of the corresponding characteristic equation

$$\chi(\lambda) := \lambda^2 + 1 - ke^{-\lambda\tau} = 0. \tag{4.45}$$

The following result summarizes the frequency range about such solutions; for a proof see [6].

Lemma 6 ([6]) *Let $\Omega \in (\sqrt{2/5}, \sqrt{2})$ and $k = \Omega^2 - 1$. Let τ be an arbitrary non-negative number if $\Omega = 1$, otherwise let $\tau = \pi/\Omega$. Then the characteristic equation (4.45) has precisely two roots $\lambda = \pm i\Omega$ on the imaginary axis and no roots with positive real parts.*

Thus, unlike the simple harmonic oscillator which has periodic solutions only with a single frequency, (4.44) has periodic solutions with a range of frequencies in the interval $(\sqrt{2/5}, \sqrt{2})$. Coming from a linear equation, these solutions can have arbitrary amplitudes since any multiple of a solution is also a solution. We are now in a familiar setting: after fixing one of these frequencies by choosing k as in the above Lemma, we can add $\mathcal{O}(\varepsilon)$ terms to the feedback to account for $\mathcal{O}(\varepsilon)$ nonlinearities in order to obtain stable limit cycles in (4.41) with a prescribed amplitude. In other words, we activate the coefficients k_1 and k_2 in the feedback law (4.43). We note, however, that the calculation of the averaged equations involves quite a different technique than the previous sections, namely the projection of the dynamics of (4.41) onto a center manifold corresponding to the roots $\lambda = \pm i\Omega$ of the characteristic equation (4.45). As the theory of center manifold reduction for delay differential equations is beyond our scope here, we refer the interested reader to [6] for details. The important thing to note is that the averaged quantities F and G are now given by

$$F(r) = \frac{1}{2\pi} \frac{4\Omega \sin(\Omega\tau) - 2\tau \left(1 - \Omega^2\right) \cos(\Omega\tau)}{\tau^2(1-\Omega^2)^2 + 4\Omega^2} \int_0^{2\pi} \cos t \, f\,(r \cos t)\, dt \quad (4.46)$$

$$G(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{4\Omega \sin t + 2\tau \left(1 - \Omega^2\right) \cos t}{\tau^2(1-\Omega^2)^2 + 4\Omega^2} \, g\,(r \cos t, \Omega r \sin t, 0)\, dt \quad (4.47)$$

instead of (4.11) and (4.12). With this change, the averaged equation still has the form (4.10) and the stability of its fixed points can be calculated as before.

More concretely, a nonlinear feedback function (4.43) can be constructed as follows: Given amplitude $R > 0$ and frequency $\Omega \in (\sqrt{2/5}, \sqrt{2})$, take $k = \Omega^2 - 1$, and choose $\tau = \pi/\Omega$ if $\Omega \neq 1$ (see Lemma 6) or take any τ such that $\sin\tau \neq 0$ if $\Omega = 1$. For $\mathcal{O}(\varepsilon)$ linear and cubic terms (4.28) in the feedback, (4.46) gives

$$F(r) = \frac{1}{2}\gamma k_1 r + \frac{3}{8}\gamma k_3 r^3, \quad (4.48)$$

where

$$\gamma = \frac{4\Omega \sin(\Omega\tau) - 2\tau \left(1 - \Omega^2\right) \cos(\Omega\tau)}{\tau^2(1-\Omega^2)^2 + 4\Omega^2}, \quad (4.49)$$

which has a form similar to (4.29) with γ replacing $\sin\tau$. As in Sect. 4.4, choose the feedback coefficient k_1 satisfying (4.36) and determine k_3 through the formula (4.37), this time using (4.47) and (4.48) to calculate G and F. This determines all the feedback coefficients in (4.43). Then the averaging theorem yields that, for all sufficiently small $\varepsilon > 0$, the system (4.41) under the nonlinear feedback (4.43) has an asymptotically orbitally stable periodic solution of the form $x(t) \approx R\cos(\Omega t)$.

Remark 7 Recall that, by our standing assumption, time is rescaled in (4.41) so that the uncontrolled system ($f \equiv 0$) has frequency 1 in the rescaled time. Thus, the fact that the feedback term can set the frequency of the limit cycle to any $\Omega \in (\sqrt{2/5}, \sqrt{2})$ implies that it can reduce the frequency of the uncontrolled oscillator by as much as about 37 % or increase it by about 41 %.

Example 8 We return to the van der Pol oscillator used in Example 2, this time with the aim of changing *both* the frequency and amplitude of oscillations by delayed linear feedback. The controlled system is given by

$$\ddot{x}(t) + \varepsilon(x^2 - 1)\dot{x} + 1 = (k + \varepsilon k_1)x(t - \tau). \quad (4.50)$$

From $g(x, \dot{x}, \varepsilon) = (x^2 - 1)\dot{x}$ and (4.47) we calculate

$$G(r) = \frac{4\Omega^2}{\tau^2(1-\Omega^2)^2 + 4\Omega^2} \times \frac{r}{2}\left(\frac{r^2}{4} - 1\right)$$

(compare with (4.24)), and from (4.48) and (4.49) we have $F(r) = \frac{1}{2}\gamma k_1 r$. If $\Omega \neq 1$ and τ is to be chosen according to Lemma 6 as $\tau = \pi/\Omega$, then (4.49) simplifies to

$$\gamma = \frac{2\pi\Omega\left(1-\Omega^2\right)}{\pi^2(1-\Omega^2)^2+4\Omega^4},$$

and the averaged equation (4.10) becomes

$$\dot{r} = -\varepsilon\frac{\Omega^4}{\pi^2(1-\Omega^2)^2+4\Omega^4}\times\frac{r}{2}\left(r^2-4+\frac{2\pi\left(1-\Omega^2\right)}{\Omega^3}k_1\right).$$

There exists a positive fixed point

$$R = \sqrt{4-2\pi\left(1-\Omega^2\right)k_1/\Omega^3}, \tag{4.51}$$

provided the radicand is positive. Note that $\bar{G}'(r) > 0$ for all $r > 0$, as in Example 2, so R is a stable fixed point. From (4.51) the value of k_1 can be determined as

$$k_1 = \frac{\Omega^3(4-R^2)}{2\pi\left(1-\Omega^2\right)} \tag{4.52}$$

for given values of R and Ω. For instance, to create a stable limit cycle at about 75 % of the frequency ($\Omega = 3/4$) and twice the amplitude ($R = 4$) of the uncontrolled van der Pol oscillator, we calculate $k = \Omega^2 - 1 = -7/16$ and $\tau = 4\pi/3$ from Lemma 6 and $k_1 = -81/14\pi$ from (4.52). Figure 4.3 shows the resulting limit cycle oscillations obtained for $\varepsilon = 0.01$ and random initial conditions.

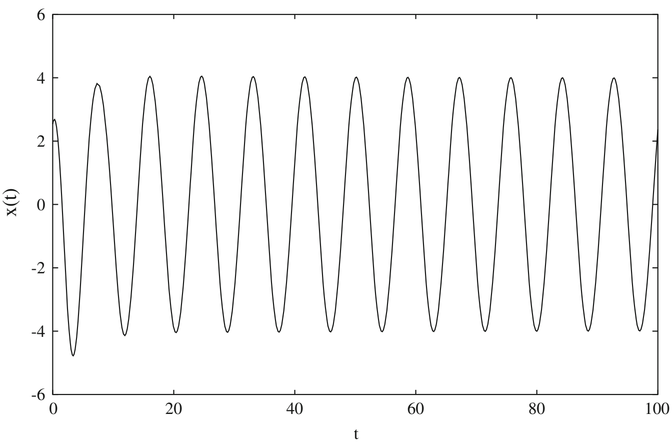

Fig. 4.3 Van der Pol oscillator of Example 8 exhibiting stable limit cycle oscillations at reduced frequency and increased amplitude

4.6 Conclusion

We have shown how delayed output feedback can be effectively used in the control of oscillatory behavior in weakly nonlinear systems. While the local stability of an equilibrium solution can be studied through a linear stability analysis, controlling periodic behavior in general requires nonlinear techniques. Here we have seen that linear feedback is capable of stabilizing the zero solution. Moreover, by adding nonlinear terms to the feedback function, it is possible to create stable limit cycle oscillations with any prescribed amplitude. In addition, delayed feedback can also modify the frequency of oscillations to a certain extent. In many cases these feats cannot be accomplished by undelayed feedback of position, exhibiting a *positive* use of delays in control.

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