Infusing Mathematical Problem Solving in the Mathematics Curriculum: Replacement Units

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There are many reports on how problem solving is successfully carried out in specialised settings; relatively few studies report similar successes in regular mathematics teaching in a sustainable way. The problem is, in part, one of boundary crossings for teachers: the boundary that separates occasional (fun-type) problem solving lessons from lessons that cover substantial mathematics content. This chapter is about an attempt to cross this boundary. We do so by designing "replacement units" that infuse significant problem solving opportunities into the teaching of standard mathematics topics.

Introduction

In Singapore, mathematical problem solving has been established as the central theme of the primary and secondary mathematics curriculum since the early 1990s. The Singapore Ministry of Education (MOE) syllabus document states explicitly the importance of problem solving: "Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems" (MOE, 2007, p. 3).

Over the last two decades, mathematics teachers in Singapore have become aware of the importance of problem solving and in bringing the notion of heuristics and Pólya's model into their professional discourses. The success in promulgating mathematics problem solving is, however, limited. While there are many local research undertakings conducted within the field of mathematics problem solving,

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few studied the actual teaching of problem solving *in the classrooms*. In one such recent study, Teong et al. (2009) noted that when teachers were avowedly conducting problem solving lessons, only a narrow set of heuristics was reinforced for usually closed problems. In other words, "problem solving" is restricted to an activity separate from usual teaching of mathematics content and carried out mainly towards the end of a topic where "challenging questions" are encountered. This portrait of problem solving instruction hardly coheres with the vision of the centrality of problem solving as set forth in the intended curriculum. This mismatch is a common worldwide phenomenon: Writers who research in problem solving under different jurisdictions assert that, despite decades of curriculum development, problem solving instruction still requires significant improvement (Kuehner & Mauch, 2006; Lesh & Zawojewski, 2007; Lester & Kehle, 2003). Similarly, Stacey (2005) noted that problem solving remained "elusive."

In line with MOE's curricular goal, this is our project team's vision of problem solving instruction in Singapore classrooms: solving unfamiliar problems is a regular activity in the classroom; teachers provide scaffolds to help students not only to solve problems but also to make extensions beyond the original boundaries of the problems (i.e. carry out Pólya's Stage 4 on "Look Back"); instead of being a separate activity unrelated to the learning of usual mathematics content, problem solving is weaved into the instructional development of mathematics topics so that it is an integrated part of students' learning of mathematics; problem solving processes become unreflectingly the tools of choice when encountering difficulties with mathematics. We term the classroom realisation of this vision as *infusion* of mathematics problem solving. Infusion is one of the primary goals of our design research. In the remaining sections of this chapter, we explicate our design approach towards infusion.

We recognise that an endeavour so onerous—and erstwhile so elusive—as infusion of problem solving is a complex enterprise that needs to take into consideration a confluence of numerous design factors such as the nature of problems, the cognitive and affective orientation of students, and the repertoire of classroom practices that would support problem solving. We thus state at the outset that this chapter focuses only on the overarching theoretical, curricular, and structural elements of this enterprise as they were brought together in our design experiment. Nevertheless, we think that these broad-grained features are critical for infusion. We begin with the theoretical considerations underpinning the design experiment.

Infusion and the Conceptions of Teaching Mathematics Problem Solving

In considering infusion, we make reference to the well-known three conceptions by Schroeder and Lester (1989) which are still widely used in the literature (e.g. Ho & Hedberg, 2005; Stacey, 2005):

- Teaching mathematics *for* problem solving
- Teaching *about* mathematics problem solving
- Teaching mathematics *through* problem solving

We think that these conceptions remain useful as descriptions of still common enactments of mathematics problem solving and as such can serve as an appropriate starting point in clarifying our stance on infusion and in locating the problems associated with infusion.

The inquiry begins at the third conception which represents the final (and most challenging) hurdle of infusion in the classroom. Indeed, infusion certainly must involve teachers (ultimately) utilizing problem solving as a means to help students learn standard mathematics content. When this takes place, then problem solving truly becomes an activity that is tightly integrated into (instead of being separate from) the learning of mathematics. When teaching mathematics *through* problem solving takes place regularly, problem solving becomes an essential part of teachers' and students' conception of "doing mathematics."

However, upon closer examination and taking a curricular design perspective, the "through problem solving" approach may not *always* be the preferred way of teaching mathematics. Take the example of definitions. While it may be argued that even definitions can be "discovered" through suitable problem solving activities, it may not be the most appropriate course of instructional action as the teacher may want to concentrate the problem solving activity on the applications rather than the "discovery" of the definitions. In which case, the sensible approach would be to state the definitions with suitable examples and shift the emphasis on utilizing the knowledge of these definitions in problem solving. Moreover, due to the realistic constraints of curriculum time and the need to fulfil other instructional goals (such as helping students gain sufficient fluency with basic mathematics skills), not all mathematics can be taught *through* problem solving. Thus, while the third conception captures much of our vision of infusion, it does not equate to infusion.

The reality of classroom teaching in Singapore is such that teachers see it as their social responsibility to cover "problems" that appear in high-stakes or national examinations. These problems are the ones that are usually found at the end of textbook chapters (and thus, concomitantly, the end of an instructional unit). These problems are also usually tied to the content covered in the topic; as such, to "get to" these problems, teachers will need to help students learn the requisite mathematics content for solving the problems; in this sense, this practice is teaching mathematics *for* problem solving. For example, problems of this kind include: Given that the median is 6 for the data set: 3, 5, 4, 7, 8, 19, 11, x, state the minimum value of x. To solve this problem, it is clear that students need to first learn about "median" (and thus the need to teach it *for* problem solving). There is a tendency for teachers to immediately prescribe a technique to deal with this type of problems (followed by repeated practice of related "problems") as a matter of efficiency. However, this approach is described by Schroeder and Lester (1989, p. 34) as a "narrow" view of teaching mathematics *for* problem solving:

> [A] solution of a sample … problem is given as a model for solving other, very similar problems. Often, solutions to these problems can be obtained simply by following the pattern established in the sample, and when students encounter problems that do not follow the sample, they often feel at a loss.

This brings us to our conception of teaching *for* problem solving within the scheme of infusion. We do not challenge the practical realities to cover these problems usually found at the end of textbook chapters. When these problems arise, instead of always directly teaching the problem-specific technique, we think they are opportunities for students to attempt them as genuine problems. For example, referring back to the above problem on median, this could be a good juncture to allow students to explore the problem terrain better by, say, substituting values of x as a way to understand the problem before devising a plan and so on.

Thus, teaching *for* problem solving is to us about teaching students the mathematics content—the "resources" in the words of Schoenfeld (1985)—necessary to solve the later *unfamiliar* problems. In other words, the "problems" we have in mind are not exercises that vary slightly from earlier-practised exercises; they are problems in the usual understanding of it in the literature: non-routine and where the solution strategy is not immediately discerned. The mathematics resources learnt earlier are thus necessary but not sufficient to solve the problems. To be successful at solving these problems, the students need to not "feel at a loss"; instead, they are required to use heuristics to help them understand the problem and devise productive plans to move forward in their attacks at the problem. In short, the students need to access problem solving strategies like the ones advocated by Pólya (1957). This leads us to the place of teaching *about* mathematics problem solving.

To us, teaching *about* problem solving involves the explicit instruction of the use of Pólya's (1957) four-stage model in problem solving as well as Schoenfeld's (1985) developments of the problem solving framework. We will describe how these models are used in our design in a later section. At this point, we state our position that it is important to teach *about* mathematics problem solving prior to attempts at teaching mathematics *for* and *through* problem solving. Without problem solving skills, students take a long time to solve problems successfully. Thus, attempts to teach much of mathematics *through* problem solving, though ideal, has not been realistic given the limitations of curriculum time. We think that teaching *about* problem solving first as a separate module is a good investment of time in terms of the "returns" we may obtain later—as in, having learnt problem solving skills, students are more likely to make significant headway in a shorter time when presented with unfamiliar problems meant to help them learn mathematics content. In addition, teaching *about* problem solving introduces to both teachers and students a means or a language to talk about problem solving. The language introduced—for example, the language of "solve a simpler problem" —can then be more easily transferred and reinforced when solving other problems later.

We should perhaps clarify at this point that our conception of teaching *about* problem solving does not divorce the teaching of problem solving strategies from the teaching of mathematics content. In other words, in teaching *about* problem solving, the teacher does not teach problem solving processes devoid of content;

rather, teachers use problems containing mathematical conditions and requiring mathematical solutions. However, the focus is on using the problems and their solutions to *foreground* repeatedly the usefulness of the process such as the Polya's stages, not the other way round. As such, the primary goal in teaching *about* problem solving is thus the learning of the problem solving process and language.

This is how we conceive of the infusion process in relation to the three conceptions of mathematics problem solving: first, teach *about* mathematics problem solving in a separate introductory module to familiarise students with the language and tools of problem solving; second, within the standard mathematics curriculum time, provide regular opportunities to teach mathematics *for* problem solving (i.e. solve unfamiliar problems utilizing the mathematics content learnt in the topic), teach mathematics *through* problem solving (i.e. solve problems that will lead to learning content meant to be covered in the topic), and, in the process of solving these problems, revise and expand the tools to acquire *about* mathematics problem solving. The project entitled Mathematics Problem Solving for Everyone (MProSE) is a design experiment that seeks to study this infusion process.

MProSE Design Experiment

MProSE uses design experiment as the overarching methodological approach. Design experiment starts off with a clear set of product specifications—also known as "parameters"—to guide and evaluate the degree of success of the innovation. Guided by well-established theories, the process of design then undergoes iterative cycles of testing and refinement in localised conditions with a view of improving its fit to the parameters and its "transportation" potential to other relevant contexts.

Design experiment appealed to us in that it allows for the unique demands and constraints of the schools to be met. The methodology's advocacy of an implement-research-refine iterative approach to educational design appeared to us to hold potential in dealing with the complexity of school-based innovations. A design experiment can be described as the "creation of an instructional intervention [in our context, a problem solving emphasis in instruction] on the basis of a local theory regarding the development of particular understandings" (Schoenfeld, 2009). We based our design experiment on the methodology and terminology of Middleton, Gorard, Taylor, and Bannan-Ritland (2006).

MProSE parameters and brief justification (for details, the reader can refer to Quek, Dindyal, Toh, Leong, and Tay, 2011): (1) model of problem solving follows the theoretical basis of Pólya and Schoenfeld. The well-known cornerstones of Pólya's (1957) stages and heuristics as well as Schoenfeld's (1985) framework of problem solving are well accepted by the professional community. As such we seek to build on their contributions, focusing especially on the work of translating these models into workable practices; (2) mathematical problem solving must include the Look Back stage of Pólya's model. This point is really included in the earlier point under Polya's model but is emphasised here as, over time, "Look Back" has become variously interpreted. Mathematicians do indeed solve problems that they encounter; however, they do not stop at the solution of the immediate problem; rather, they use the solution strategy of the problem as a sort of kernel to generate solutions to related problems. Thus, it is this disposition of mathematicians with regard to problems—where they extend, adapt, and generalise problems—that we find essential to build into a school curriculum that seeks to inculcate mathematical thinking; (3) mathematics problem solving is a valued component in school assessment. What we see as the root of the lack of success for previous attempts to implement problem solving in the classroom is that problem solving is not assessed. Because it is not assessed, students and teachers do not place much emphasis on the processes of problem solving; students are more interested to learn the other components of the curriculum which would be assessed; (4) mathematics problem solving must be part of the mainstream curriculum. To "downgrade" mathematics problem solving to a form of enrichment or optional programme for students violates the value of mathematical problem solving; (5) teacher autonomy is important in the carrying out of problem solving lessons. While the beginning stages of innovation may include the involvement of expertise outside the school, ultimately, for the innovation to take root and sustain, teachers' capacity must be built to a point where they own the innovation and possess the ability to carry out problem solving lessons on a regular basis.

At the time of writing, MProSE is entering its sixth year as a design experiment, and in the process, we have undergone several iteration cycles from the original design. It is not realistic, given the constraints of space, to detail the full journey in this chapter. The focus of this chapter is to bring the readers to our current stance with regard to infusion. As such, we will briefly describe the first phase of MProSE infusion (the development of teaching *about* problem solving) and then discuss more substantially our current progress within the second phase of infusion (bringing problem solving into the regular work of teaching standard mathematics).

First Phase of MProSE: Teaching About Problem Solving

Based on the parameters, we designed a module on problem solving in which students are explicitly taught the language and strategies used in problem solving. This refers to the second conception of problem solving, that is, teaching *about* problem solving. The translation of design parameters into actual curricula features will not be discussed. For details, the reader may refer to Leong et al. (2011).

The entire module consists of ten lessons. The duration of a typical problem solving lesson consists of 55 min. Each lesson consists of two main segments. The first segment of the lesson involves the teacher explaining one particular aspect of problem solving (such as one of the four stages in the Pólya's model) and discussing the homework problem of the previous lesson. The second segment emphasises one particular mathematical problem that is illustrative in demonstrating that particular aspect of problem solving. Throughout the entire lesson, only one problem is highlighted in great depth. The students have to work out the problem on "the practical worksheet" guided by the instructions in the worksheet. The worksheet initially consisted of four pages, with each page corresponding to each of Pólya's stages.

The practical worksheet is an important part of the design as it is a tangible embodiment of the problem solving process for teachers and students. Its introduction into the classroom is meant to fulfil at least two roles: guide and reinforce the problem solving process along the lines of Pólya's stages and heuristics and signal a switch from other modes of instruction to a problem solving paradigm. Due to its practical importance in the overall infusion process, the practical worksheet has undergone a number of refinements in the course of the project. Figure 1 shows the compressed version of the three-page practical worksheet in its current form. For the detailed description of the evolvement of the module and the practical worksheet, readers could refer to Dindyal et al. (2013).

In this phase of teaching *about* problem solving, the module is taught separately from the usual teaching of mathematics in regular lessons. As our MProSE project has now moved to a juncture where there is evidence of stability in the implementation of the module, moving to the next phase of infusion—where problem solving is to be a regular feature in the teaching of standard mathematical content—becomes a natural progression. In the sections following, we highlight the progress and challenges in this next phase of infusion. In particular, we focus on the curricula and structural tweaks in response to challenges in teacher development for problem solving.

Practical Worksheet

Problem

I Understand the problem
Use some heuristics such as Draw a Diagram, Restate the Problem, Use Suitable Numbers, etc. to help you.

I have understood the problem. (Circle your agreement below.)
Strongly Disagree Neutral Strongly Agree
1 2 3 4 5
You may proceed to the next page to work out a solution/partial solution.

II&III Devise a Plan and Carry it out
a) *State your plan clearly, for example: (i) Use Suitable Numbers and Look for Patterns; or (ii) Find the areas of all smaller triangles and work out their ratios.*
b) *Number each plan as Plan 1, Plan 2, etc.*
c) *Carry out the plan that you have stated.*

<u>Plan 1</u> Statement of Plan:

<u>Carry out Plan</u> 1

IV Check and Expand
a) *Check your solution.*
b) *Write down a sketch of any alternative solution(s) that you can think of.*
c) *Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.*

Fig. 1 Practical worksheet (compressed by removing spaces for writing)

Second Phase MProSE: The First Implementation

After students had been exposed to instruction *about* mathematical problem solving through the MProSE module, the next infusion step is to use the problem solving skills acquired regularly in the learning of mathematics content in usual mathematics lessons.

One of our key principles was that inclusion of problem solving in regular mathematics should help teachers improve in the teaching of mathematics. If the teachers were satisfied with existing ways of teaching a mathematics topic, then there was no motivation to switch to the problem solving approach. A logical step to begin would be for teachers and researchers to select a difficult topic to teach or a concept where students always made mistakes. In the process of applying the skills and strategies that students had acquired in the MProSE module, they need to solve the mathematics problems through struggling using problem solving—the exploration of which would help them understand the topic better when it is taught later.

We also strongly advocated the continued use of the practical worksheet whenever students are instructed to attempt problems in this phase. We think that the ten lessons in the earlier MProSE module, while substantial, are not sufficient yet in bringing about a habit in the students of applying Pólya's processes when confronted with mathematics problems. Through the followed-up use of the practical worksheet within the teaching of topics, there is continuity in the learning and application of the problem solving skills over an increasingly broad range of mathematics problems. In the process of struggling through problem solving, it was hoped that the problem solving process will become part of the students' learning habit.

In addition, we suggested to the teachers the following guidelines for implementation:

- Infusion problems are to be worked on a practical worksheet.
- Problem is to be worked on for exactly 1 h.
- An infusion problem is to be given as homework in the following situations:
  - On the last lesson prior to teaching a new topic, to prepare the student to constructively develop some feeling for the topic (preparation)
  - Within the span of a topic to allow the student, to explore the difficult nuances of the topic (exploration)
  - At the end of a topic, to consolidate his/her understanding of the topic (consolidation)
- These are the parameters for deciding on the suitability of an infusion problem:
  - Difficult enough to take at least 30 min.
  - Allows student to discover some aspect of the topic: for example, a technique taught is superior to other techniques, or a particularly difficult aspect becomes clearer after enough time is spent exploring it, etc.
  - Very amenable to expansion (Pólya's Stage 4).

We also supplied suitable problems for each of the "difficult" topics that the teachers brought up. We were always in close contact with the teachers through email and school visits.

Second Phase MProSE: Evaluating the First Implementation

In keeping with the features of a design experiment, we examined implementation of the plan in order to refine the design for the next iteration. We held regular meetings with the teachers to discuss pre-implementation details—the need to account for students' affect, the suitability of the problems, and the instructional emphases intended for each problem—as well as post-implementation reflections. From these meetings, it became clear to us that teachers were facing a number of challenges with regard to their attempts at infusing problem solving into their regular lessons:

1. *Instructional goals behind problems*. Much time during the meetings was taken up to discuss the actual "location" of the problems in their teaching schedule. Questions that were addressed included: "Should it be given as an introductory problem at the beginning of the topic, or somewhere in the middle, or towards the end?" "Should the problem be done fully in class, purely as homework, or start as homework and completed in class?" On the surface, these questions appeared to be about the most natural or logical points to insert the problems along the content developmental track of the topic; upon deeper analysis, it revealed the teachers' as-yet unclear instructional goals about what each problem can potentially fulfil. To illustrate this point, we review the meeting discussions over the "cat problem": "5 cats take 5 days to catch 5 mice. How many cats will it take to catch 2 mice in 2 days? How long will it take 1.5 cats to catch 1.5 mice?" First, the teachers shared that they inserted this problem at different junctures in their teaching of the ratio/proportion topic—Karen did it as an introductory problem; Siva used it as a problem at the end of the topic; Mariam also used it at the end but she did only part of it in class and the rest as homework for students. Second, when asked for their reasons for their respective decisions, they brought up mainly considerations related to availability of time pockets for in-class problem solving, but not about the goals that we originally in-built into the problem, such as the opportunity for students to learn conceptual distinctions between direct and indirect proportion through exploring the problem terrain instead of through teacher's direct telling. In particular, the teachers seemed unaware that, by putting the problem at the end of the topic where direct and indirect proportion were explicitly covered through numerous practice questions, the "problem" lost its problem status—unfamiliarity and thus the need to apply problem solving processes—to the students, rendering it more like a routine exercise to them. This went against the original goal of infusing "problem solving" into the topic.

2. *Affect-efficacy of the problems*. A number of the infusion problems were designed to account for students' affect in mathematics through problem solving. An example of the "multiple choice problem" would illustrate this better. The problem statement is: "A student had to take a test consisting of 100 multiple choice questions. Each correct answer is given 5 marks while each wrong answer will have 3 marks deducted. Unanswered questions are given 0 marks. The student attempted all the 100 questions and obtained 444 marks. How many questions did the student get wrong?" We expected students to find this problem accessible: substituting small numbers for correct and wrong answers can help them understand the problem easily; the familiar heuristic of "guess and check" can be utilised to obtain a solution to the problem. Thus, we anticipated that students would find success at solving it and thus have a positive emotional orientation towards the problem. In addition, teachers could use the "check and expand" stage to ask questions—such as, "what if the numbers in the question are changed? Is there an alternative method that can take care of such changes in the question more easily?"—to provide the motivational link to the topic of algebraic equations. However, during the meetings, the teachers shared that their students were generally not motivated to solve the problems. We thought that more concrete strategies in scaffolding students' attempt towards productive approaches would help teachers encourage more success in problem solving—a necessary ingredient for students' long-term buy-in to problem solving.
3. *Time consumption of the problems*. The teachers were very conscious of class time taken up for problem solving. And they were aware that meaningful problem solving takes time. [The Singapore mathematics syllabus is seen by most teachers as heavy content-wise. Coupled with the need to prepare students for high-stakes examinations, it is not uncommon for teachers to feel the constant time pressure to "cover syllabus" (e.g. Leong & Chick, 2007)]. They saw it as a dilemma: if they used up class time to do problem solving, it would reduce the already limited time to "cover syllabus"; if they left problem solving as homework (to free up class time), teachers would then not be at hand to help the students and it would exacerbate the problem of low levels of students' motivation at problem solving. We saw it differently: it was not a case of problem solving versus content coverage; as described in the cases of the "cat problem" and the "multiple choice problem," the problems could be used to explore content, deal with the problems within content, as well as provide motivational links to the more formal treatment of content. However, we understood that, unless teachers could see how the problems can indeed fulfil these roles within the actual content development of a given topic, it would become increasingly harder for teachers to willingly "give up" class time for problem solving.

We thought that the challenges that the teachers faced were significant, and we needed to address them in the next iteration of this phase. In summary, the infusion must include these features: (a) apart from the problems, there should be additional details that will help teachers realise the intended goals behind the problems. This also implies that teachers should be directly involved in the planning that leads to the rationale and finalisation of the problems; (b) the planning should not be restricted to the problems and its immediate temporal surrounds; teachers need to see how the problem(s) fit logically and developmentally within the entire topic progression; in other words, the unit of planning is "zoomed-out" to the whole topic; this will help teachers see how time spent on problem solving IS a form of content coverage; (c) motivational elements should be integrated into the unit planning. In determining a strategy to incorporate these elements in the refinement of the design for the second phase, we also took into account broader structural challenges relating to policy and curriculum.

In addition to the practical challenges the teachers faced, we highlight some structural challenges that we need to confront when considering a refinement of the envisioned infusion. The first is the largely centre-to-periphery model of curriculum dissemination in Singapore. The effectiveness of this dissemination approach depends on, among other factors, "the strength of the central resources," the number of peripheral elements, and their distance from the centre (Kelly, 2004). One critical step in this centre-to-periphery process is the teachers' interpretation of the official curriculum (in the form of a syllabus document) and its translation into classroom practices. It is through these classroom experiences that students learn not just content for national examinations but content imbued with the disciplinarity of mathematics. However, while teachers are consulted by curriculum planners and developers, they nevertheless remain at the far end of the change process (in the eye of the storm, safe from the fury of the blast). The curriculum as an end product is conveyed to teachers in the form of training workshops by the people at the centre. Thereafter, it is left very much to the teachers in a school to implement the curriculum, within the given guidelines, and in view of the vision of the school. In this sense, school-based teacher professional development in interpreting and translating the local mathematics curriculum is key to ensuring the realisation of the overarching curricular goal of mathematical problem solving for all students. This school-based approach to teacher participation in developing a problem solving-centric curriculum—as a form of teacher development—is the model adopted by the MProSE team.

Another challenge to teaching problem solving is the lock-step grid of fixed teaching schedules. Teachers are hard pressed into adhering to these schemes of work to prepare students for term tests or national examinations. In such a context of high time pressure to "cover syllabus," it is not uncommon for teachers to have the mind-set that problem solving is an unessential distraction. In addition (and perhaps related to teachers' perception of limited time), teachers' preferred mode of instruction is the teacher exposition type of teaching that is entrenched in many classrooms. Hattie and Yates (2014), citing Larry Cuban and Nathaniel Gage, pointed out that this teaching methodology, also known as the initiate-response-evaluation approach or conventional-direct-recitation, has survived "considerable criticism and attacks for over two centuries" (p. 44). It is not surprising to find it a common approach in local classrooms. One reason is the easily recognisable and established roles and norms for both teachers and students in the classroom.

As part of our design experiment approach, we need to "accommodate" these practical challenges and structural givens in teachers' preferences and mind-sets. By accommodation, we mean a type of change we make to the design for meeting the (localised or systemic) constraints faced by teachers (Quek et al., 2011). Our approach as a result was to use "replacement units" to bring about instructional change and teacher capacity building within this preferred teaching approach and the lock-step planned curriculum.

Refinement of Second Phase MProSE: The Replacement Unit Strategy

In this last section of the chapter, we bring the readers to the most up-to-date refinement of our MProSE infusion programme: use of the replacement unit strategy. Although this strategy was developed independently during our other projects (see, e.g. Leong et al., 2013), the term "replacement unit" (RU) is attributable to Cohen and Hill (2001). While working on designing an RU, we develop—in consultation with the teachers—a redesign for an entire mathematics topic. This redesign involves restructuring of content and development of all the relevant instructional materials to accommodate the integration of problem solving without changing the original allocated time for the unit. As such, it is an authentic "replacement unit"—in the sense that teachers can replace the original way of teaching the unit by this RU without upsetting the overall teaching schedule.

Cohen and Hill (2001) reasoned that the replacement units were an important innovation in the sense that "[curriculum] developers would be able to ground teachers' professional education in the improved student curriculum that teachers would teach" (p. 47). Linking teachers' professional development with a proposed improved curriculum was a novel way which differed from usual attempts which typically focused on one element at a time. They reported that workshops for teachers on the materials and the pedagogy of the replacement units "had appreciable depth and allowed teachers to investigate more seriously individual mathematical topics, like fractions, in the context of student curriculum" (p. 55). They also reported the positive potential of replacement units for education reform:

> Teachers who took workshops that were extended in time and focused on students' tasks—either the replacement units created for the reforms or new assessment tasks and students' work on them—reported more practices that were similar to those which reformers proposed. In contrast, teachers who took workshops more loosely focused on hands-on activities, gender, cooperative learning, and other tangential topics were less likely to report such practices. (p. 88)

An RU, usually spanning 4–8 h in duration, is a realistic and reasonable period of engagement with teachers for each attempt at curricular redesign. This avoids the onerous task of redesigning the entire curriculum all at once. Moreover, focusing the efforts on one RU at a time allows both the researchers and the teachers to trial (and retrial, if necessary) and to refine the RU as well as to gain familiarity with its underlying design principles over time. This setup of studying and redesigning an RU based on a topic that is covered within realistic time limitations in the teachers' teaching schedule provides the platform to accommodate the structural challenges discussed earlier. Teachers' active involvement from initial discussion to implementation and refinement of the RU also helps close the gap between curriculum planning and practice.

More importantly, an RU is a suitable "unit" to infuse problem solving. It is in the redesigning of a unit that the relevance and place of problem solving can be found. The RU is of appropriate size for problem solving to be weaved seamlessly with the development of mathematics content and thus allowing teachers to see, for example, how motivational elements can be inserted to connect problem solving to content to be learnt, how problem solving can be realistically employed within time constraints, or how problem solving IS the learning of content. In other words, the RU strategy addresses the local challenges discussed earlier.

At the time of writing, we are in the early stage of implementing the RU strategy in the MProSE project. As such, we are unable at this point to provide an analysis of the outcomes of its implementation and follow-up further refinements. Nevertheless, as an infusion strategy that we have come to develop based on our experiences with a number of schools we worked with over more than 5 years, we think it holds promise. A summary of an RU on quadratic equations that we designed together with the teachers is given in the Appendix for the readers' reference.

Conclusion

We think that the current big question in mathematics problem solving research is this: How do we make meaningful problem solving a regular feature in mathematics classrooms? We recast this as the "infusion problem." There are many reasons why the classroom is so "resistant" to change, including change towards problem solving infusion. In this chapter, we focus our discussion of infusion hurdles on existing macro-issues such as the pressure towards content coverage, teachers' readiness towards a problem solving approach, and the lock-step grid of teaching schedules that renders additional curriculum time for problem solving unrealistic. Through our MProSE design experiment, we have come to learn that the way to tackle some of these challenges is not merely through minor tweaks in the way teachers teach; what is needed is a paradigm shift that requires changes to be implemented at the curricular and structural level in the school's mathematics programme. In short, we think the intervention can be carried out in two steps: First, familiarise students with the processes and language of problem solving through a separate module designated to foreground the teaching *about* mathematics problem solving. This intensive learning about problem solving is needed for both teachers and students; thereafter, follow up with integrating problem solving in the teaching of regular mathematics content through RUs. We argue that the RU strategy is a feasible way forward in realising the curricular and structural changes that need to be made;

the strategy also provides a suitable platform for teachers' participation in learning and curriculum redesign.

We readily acknowledge that these broad design features are necessary but not sufficient to deal with the infusion problem. Although not detailed in this chapter, we developed and tweaked classroom implements alongside these structural changes to help teachers and students cope with this "new" problem solving way of learning mathematics. There are ongoing efforts to refine the nature of problems to meet the intellectual and affective needs of the students; at the same time, we modify the practical worksheet so that it easier to use for both teachers and students. We are also currently working on the repertoire of teachers' craft skills that are supportive of the teaching of problem solving. This includes the clarity in teacher's visual representations of the problem solving processes in the whole class setting and the kinds of scaffolds that teachers can use in table-table instruction to help students experience empowerment through problem solving.

In short, we can approach the infusion problem through various loci of study—such as curriculum redesign, teacher development, and classroom task implements. The next stage of our research will involve a careful examination and integration of these factors in a way that fits the local conditions of respective schools so that it results in successful infusion.

Appendix: Description of an RU on Quadratic Equations

Under the topic of "Solving word problems that are reducible to quadratic equations," a common observation among teachers is that some students struggle with translation of the statements in the "word problems" to equivalent equations. The frequently used trajectory can be summarised as such: Teacher demonstrates the steps involved in translating statements to equations over different types of word problems; students can usually follow the steps; but when asked to do it on their own, they are "stuck," especially when confronted with an unfamiliar type of "word problem." The usual response by teachers to such student difficulty is more demonstration and more fine-grained breakdown of steps with the intent of making the skill acquisition process for students more gradual. Here, we propose the problem solving approach within the context of an RU.

We think the problem students encounter is not merely that of lacking familiarity with the different types of word problems; more fundamentally, it is the lack of opportunity for authentic exploration of the word problems—a necessary step for students to make sense of the problems and to appreciate the power of the algebraic approach. In other words, we need to "prepare the ground" so that when the algebraic method is "planted," it will "take root"—students will receive it and learn it better instead of seeing it as a method forced upon them. In particular, we infuse problem solving.

For this RU on quadratic equations, instead of being taught a method of solving word problems right from the start, students are given time to attempt such a word problem on the practical worksheet. In so doing, they are given the opportunity to explore the word problem and hence figure out its underlying structure. At the same time, we are conscious of all the realistic constraints—such as the need to cover standard content and the lock-step schedule which were elaborated in the earlier sections—and we are bound by the redesign of the RU. The lessons in this topic are thus reorganised in this way:

Lesson 1: Solve a word problem reducible to quadratic equation with a practical worksheet.

The "Employee problem": A company wants to employ as many workers as it can afford to complete a project within a short timeframe. If the company pays each worker \$6 per hour, there are only 30 applicants for the job. However, the company needs more workers. It is known that for every \$1 increment in the hourly pay, it will attract two more applicants for the job. The company can only afford a maximum of \$504 per hour in total. How much should the company offer to pay per hour in order to attract the maximum number of workers?

The main goal is to let students re-familiarise with the practical worksheet and feel a sense of empowerment at solving the problem when they use Pólya's stages and heuristics. Note that to solve the problem, students need not use algebra. Students are expected to use other methods such as systematic listing and other heuristics such as "substitute values" to solve the problem.

At the fourth stage of Pólya, we can provide a motivation for algebra by asking, "What happens if we have an owner with greater resources beyond \$504? Can your solution be easily adjusted to cope with this adaptation?" The point is to provide a link to the algebraic representation/solution, which is the scope of the next few lessons.

Lesson 2: Revision of quadratic factorisation and using it to solve quadratic equations.

The main goal is to help students use "zero product rule" and factorisation to solve quadratic equations with integer coefficients. After revision of quadratic factorisation, students are to be taught the steps in solving quadratic equations by factorisation. They then practise the method to gain fluency. In other words, this is a "standard" lesson geared towards mastery of technique—a type of teaching that teachers are familiar with.

Lesson 3: Solve another given word problem using a practical worksheet.

The "Consecutive Numbers problem": "Four consecutive even numbers are such that the product of the smallest and the largest is 186 more than the sum of the other two. What are the four numbers?"

Students are expected to use the resources gathered, both the experience in Lesson 1 on using the algebraic method as well as the method of solving quadratic equations in Lesson 2, to make productive attempts at solving the problem in this lesson. Under Stage 4, students can consider generalizing a standard procedure for solving "word problems" that are reducible to quadratic equations. The intended link from the working for this problem and the more generalised method is illustrated in Table 1.

Table 1 From working to "methodizing" as part of Pólya's Stage 4

Working	Methodizing
Let the first number be x	Step 1: Determine the variable and let it be x
Therefore, the four numbers are x, $x+2$, $x+4$, and $x+6$	Step 2: Express the other variables in terms of x
Product of the smallest and largest number $= x(x+6)$	Step 3: Establish the relationships between x and the terms based on the word phrases
Sum of the other two numbers $= (x+2)+(x+4)$	
$x(x+6)=186+[(x+2)+(x+4)]$	Step 4: Form an equation from the key sentence
$x^2+4x-192=0$	Step 5: Simplify the equation and equate it to zero
$x=-16$ or $x=12$	Step 6: Solve for x

Lesson 4: Apply the general procedure abstracted in Lesson 3 to solve other "word problems."

The main goal is to help students apply the general method in the right column of Table 1 to a variety of other word problems reducible to quadratic equations. The instructional approach is one of practising a learnt method—a style of teaching which is standard for teachers.

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