The Use of Digital Technology to Frame and Foster Learners' Problem-Solving Experiences

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Abstract The purpose of this chapter is to analyze and discuss the extent to which the use of digital technology offers learners opportunities to understand and appropriate mathematical knowledge. We focus on discussing several examples in which the use of digital technology provides distinct affordances for learners to represent, explore, and solve mathematical tasks. In this context, looking for multiple ways to solve a task becomes a powerful strategy for learners to think of different concepts in problem-solving approaches. Thus, the use of a dynamic geometry system such as GeoGebra becomes important to represent and analyze tasks from visual, dynamic, and graphic approaches.

Keywords Digital tools • Mathematical problem solving • Tool affordances and appropriation process

Introduction

The developments and availability of digital technologies have been transforming the way people communicate, obtain information, socialize, develop, and comprehend disciplinary knowledge. A digital technology such as a GeoGebra can improve and eventually transform cognitive abilities we already possess and help us develop new ones. People usually develop these cognitive abilities when they represent and explore tasks through these technologies. A cognitive technology makes its mark in our mind through steady work and after a while it becomes part of our cognitive resources. A key historical example is writing. As Donald (2001, p. 302) has explained it, literacy skills transform the functional architecture of the brain and have a profound impact on *how people perform their cognitive work*. The complex neural components of a literate vocabulary, Donald explains, have to be hammered by years of schooling to rewire the functional organization of our thinking. Similarly, the decimal system (Kaput & Schorr, 2008, p. 212) first enlarged access to computation

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and eventually paved the way to Modern Age. Today, an architect begins using specific software to design his buildings. Taking profit from the plasticity of the visual images that the software provides, the architect can imagine a new plan, a new design. Gradually he will begin thinking of his design *with* and *through* the software. He will incorporate the tool affordances as part of his thinking and one day, the software will have *disappeared*. Now it is *coextensive* with his thinking while solving his design tasks. The tool has become an instrument and the design activities are *instrumented* activities. Throughout this chapter, we argue that the systematic use of digital technologies plays an important role in teachers and students' ways to comprehend mathematical ideas and to engage in problem-solving activities.

Conceptual Foundations: Learning from, Through, and with the Others

Action does not belong (exclusively) to the user and neither does it to the environment; both the user and environment are actors and reactors. We understand *dragging* as our hands within the environment, where it is possible to *touch* and transform mathematical entities living in the digital environment. The user and environment are, from the point of view of agency, *coextensive*. Thus, we can speak of *coaction* between the user and the environment, not just between the user and the artifact (Moreno-Armella & Hegedus, 2009). Coaction is the broader process within which an artifact is being internalized as a cognitive instrument. Yet, in the social space of the classroom, there can be a collective actor. One participant can observe how another drives the technology at hands and, the former, incorporates into her strategies what she observed. At the end participants can act and react to the environment in ways that are essentially different from their initial ones. We can learn *from*, *through*, and *with* others. So the traditional triangle user-technology-task has to be enlarged: coaction becomes embedded in a social structure. The ways in which people appropriate technological artifacts cannot be separated from the cultural matrix they live in, and vice versa, technology cannot be separated from culture.

Engaging in practical use of tools begins to build in the user a cognitive resource for thinking about the world as a scene for the potential application of this tool. Artifacts operate in a two-sided manner, both providing resources for acting on the world and for regulating thinking about the world.

Human beings do not interact directly with their environment but through mediating artifacts. This is true in particular when considering cognitive activities as mathematical problem solving. Historically, writing and the decimal system are the most basic mediating cognitive technologies. They were instrumental to pave the Renaissance and Modern Age.

We want to explore this general setting in the important case of contemporary mathematics learning. Consequently, we will be referring to digital artifacts that are transforming the educational landscape and the mathematics curricula.

Béguin (2003) pointed out the design of artifacts does not finish until the tool or object fulfills material and technical requirements; it should include how users transform the artifact into an instrument to solve problems. Moreno-Armella and Santos-Trigo (2016) argue that artifacts are not neutral as they deeply modify our ways of thinking once we have internalized them into our cognitive structures. On their side, Koehler and Mishra (2009) have pointed out that the (cognitive) technologies "have their own propensities, potentials, affordances, and constraints that make them more suitable for certain tasks than others" (p. 61). Some of those affordances and constraints are inherent to the technology design, but also users impose others during their implementation as well as they can eliminate those constraints due to the innovative use of technology. The coordinated use of digital technologies allows for diverse ways to identify, formulate, represent, explore, and solve problems situated in different fields or contexts. Consequently, new routes can emerge for learners to construct and comprehend disciplinary knowledge. *How will learning environments be transformed in order to cope and take advantages of digital developments?*

The discussion of this question becomes important in order to properly explore learning scenarios in which learners rely systematically on the coordinated use of digital technologies to develop new versions of disciplinary knowledge and problem-solving skills. To illustrate what the use of technology could bring to learning environments, we will discuss, in the next sections of this chapter, some mathematical tasks that will enable inductive and deductive reasoning through the digital media. In every one of the activities discussed, a goal is clear: to provide opportunities for learners to engage in mathematical thinking and problem-solving experiences.

Teachers play an important role in providing opportunities for students to use technology in problem solving. As suggested by Mishra and Koehler (2006), teachers need to know ways to use technology in learning environments in addition to deep knowledge about the subject (mathematics) and teaching practices. This is a complex demand for the teachers as it requires "an understanding of the representation of concepts using technologies… and how technology can help redress some of the problems that students face…and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones" (p. 1029).

Tasks are the vehicle for learners to focus on fundamental concepts that are developed through one's own actions and social interactions (Santos-Trigo, 2010). With the use of digital technologies, learners become active participants in the learning process since they offer a rich diversity of opportunities to represent and explore the tasks from distinct perspectives.

Recently, the incorporation of mathematical action technologies (GeoGebra, for instance) has provided solid ground to transform static learning materials into enlivened dynamic media. We select, for the next sections, a set of problems we have amply discussed with teachers in our academic programs of teachers' education at Cinvestav-IPN, Mexico.

The Use of Digital Technology in Looking for Solutions of Mathematical Tasks

A key principle to structure and foster problem-solving activities in learning environments is to help learners pay attention to what is essential in identifying and grasping mathematical concepts and how to use those concepts during the process of solving problems. Searching for alternative ways to represent and solve problems is a powerful strategy for students to identify and contrast the role played by concepts and their representations across the whole problem-solving process. In Chinese classrooms this teaching strategy is called "one problem, multiple solutions" (Cai & Nie, 2007), and it is widely used in mathematical instruction. Students will not only recognize and value multiple paths used to represent and explore problems, they will also have an opportunity to reflect on the extent to which concepts are connected or used to achieve the solution. In addition, Gardner (2006) recognizes that for people to develop problem-solving creativity, they need to pose new questions and to look for novel problem solutions.

How can one distinguish that a solution to the "same" problem is different from others? Leikin (2011) suggests that solutions can be judged as different if they involve (a) the use of different representations of concepts to explore and solve the problem; (b) the use of different theorems, mathematical relations, or auxiliary constructions to support conjectures; and (c) the presentation of different arguments and ways of reasoning about concepts to achieve the problem's solution. Thinking of different ways to solve problems could also become important to transform routine problems into a set of nonroutine activities (Santos-Trigo & Camacho-Machín, 2009).

How can students use digital technology to look for different ways to solve mathematical tasks? We discuss a simple mathematical task that involves the construction of an equilateral triangle, in which the use of a dynamic geometry system (GeoGebra) becomes important to think of different concepts and ways of reasoning to represent, explore, and solve the task.

The task: Given a vertex of an equilateral triangle and a line to which the other two vertices belong, find the location of the other two vertices. Can you show different ways to construct such a triangle?

We have used this task in our problem-solving seminar with high school teachers. The goal of the seminar is to work on mathematical tasks through the use of different digital technologies and to analyze ways of reasoning that emerge during the solution process. Here, we illustrate some approaches to the task where the use of GeoGebra was essential to construct a dynamic model of the task. Also, we show some examples of solutions where the tool is used as straightedge and compass to solve the problem.

Dynamic Models

Subjects need to comprehend key information and concepts involved in the task statement. What data are provided? What does it mean that one side of the triangle lies on a given line? What do I know about equilateral triangles? Can I construct a family of triangle holding partial conditions (isosceles) by moving a particular vertex on the given line? Is there any way to relate the family of isosceles triangles with the construction of an equilateral one? The discussion of these types of questions becomes important for learners to construct a dynamic representation of the task.

Focusing on the Construction of a Family of Isosceles Triangles

Polya (1945) pointed out that relaxing initial conditions of a problem is an important heuristic to construct/explore the behavior of a partial condition of the problem. Figure 1 shows a movable point P on line *l* and a circle *c* with center at point *C* (the given vertex) and radius *CP*. Triangle *PCQ* is isosceles, since *CP* and *CQ* are radii of the same circle. It is observed that when point P is moved along line *l*, a family of isosceles triangles is generated. At what position of *P* does triangle *PQC* become equilateral? One way to respond to this question is to rely on the property that *in any equilateral triangle, height, perpendicular bisector, angle bisector, and median coincide*. To this end, the perpendicular bisector of segment *PC* is drawn; then, this must pass through point *Q* when triangle PQC becomes equilateral. Points *R* and *S* are the intersections of the perpendicular bisector and circle c. With the use of the tool, it is found that the locus of each point (R and S) when point P is moved along line l is a line (Fig. 1). Then, the intersection of each locus and line *l* determines vertices *P* and *Q*, to form triangle PQC equilateral (Fig. 2).

An Approach That Relies on Symmetry Properties and a Locus of Objects

Dynamic models involve constructing auxiliary objects and analyzing behaviors of some elements when moving particular points. Figure 3a shows a perpendicular line to the given line *l* that passes through point C, a movable point A on line *l* and point B is the symmetric point of A with respect to the perpendicular to *l*. Point D is the intersection of line CB and the perpendicular bisector of segment or side AC. Figure 3a also shows the locus of point D when point A is moved along line l. This locus intersects line *l* at two points. When points A and B coincide with those intersection points, respectively, then triangle ABC is equilateral (Fig. 3b). This is because there, d(A, B = D) = d (B, C) (definition of perpendicular bisector) and also d(A,C) = d(C, B).

Models That Involve Relations and Geometric Properties

The models explore connections between properties and results and the construction of the triangle. The tools' affordances are important to represent and visualize the results. For instance, the accuracy of involved construction allows learners to visualize a *hot point* to pay attention to or possible relationships.

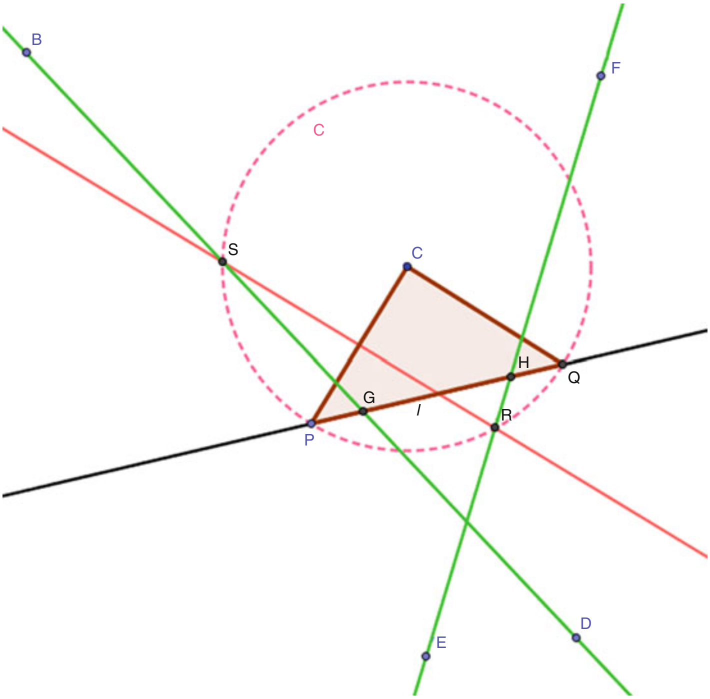

Fig. 1 Finding the loci of points R and S when point P is moved along line *l*

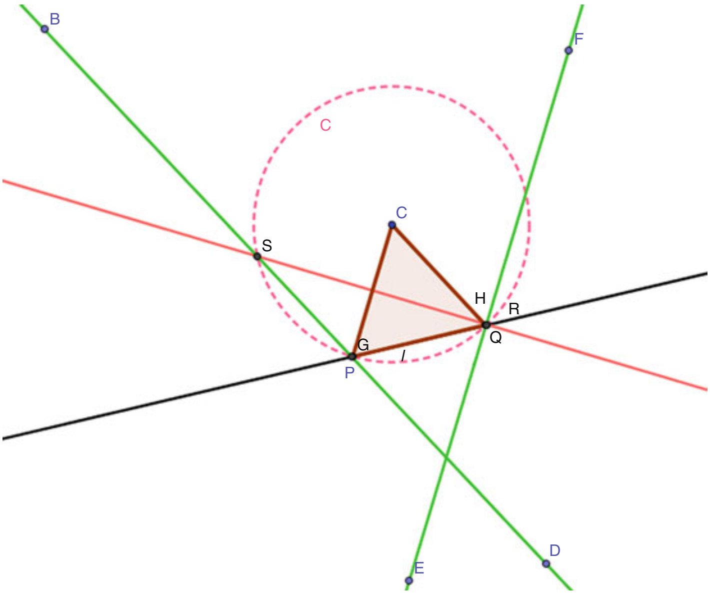

Fig. 2 The construction of an equilateral triangle

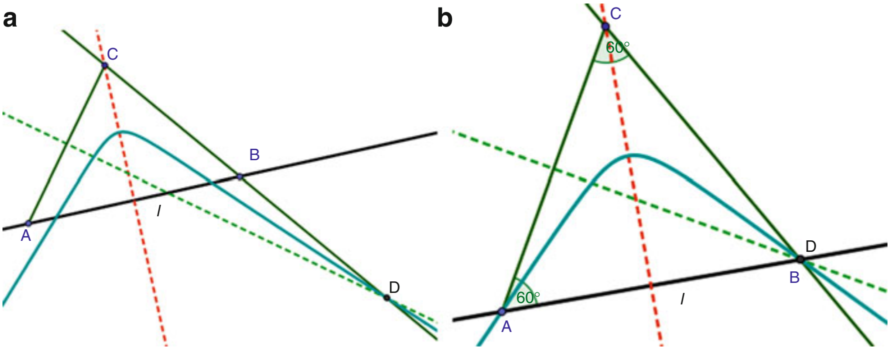

Fig. 3 (**a**) Drawing the locus of point D when point A is moved along line *l*. (**b**) When points A and B coincide with the intersection points of the locus, then triangle ABC is equilateral

A Viviani's Theorem Approach (http://tinyurl.com/VivianiTheorem)

The idea is to use the theorem: *for any interior point P in an equilateral triangle, the sum of the distances from P to the sides of the triangle is the length of the height of the triangle*. A crucial issue here is where to locate the interior point P to connect the theorem and the construction of the triangle. An option can be to locate the interior point on the segment drawn from point C and that is perpendicular to the given line. Thus, segment CM is the height of the required equilateral triangle, point P is any point on segment MC, and point Q is the middle point of segment PC (Fig. 4). Two circles are drawn: circle c with center at point P and radius PQ and circle d with center point Q and radius QP. Points G and H are the intersection points of both circles. Lines CH and CG intersect line *l* at points A and B, respectively. It is observed that triangles PHC and PGC are right triangles because side PC is the diameter of circle d. Thus, triangle ABC is equilateral; it holds that the sum of segments PM + PG + PH corresponds to the height MC (Fig. 4).

An Approach Based on Similarity of Triangles

Figure 5 shows an equilateral triangle PQR whose side PQ is any parallel line to the given line *l*. Then two parallel lines to QR and PR that pass through point C are drawn. These parallel lines intersect line *l* at points A and B, respectively. Triangle ABC is equilateral because corresponding angles of triangles ABC and PQR are congruent (Fig. 5).

Comment: Looking for several ways to represent and solve a task is a key problem-solving strategy where learners are encouraged to think of the task from diverse angles or perspectives. For instance, dynamic models become important to

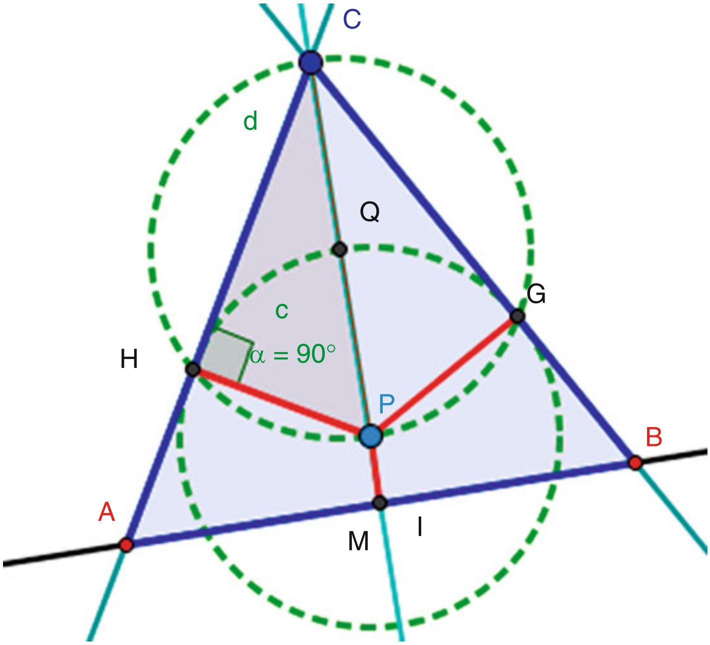

Fig. 4 Using Viviani's theorem to construct the equilateral triangle ABC

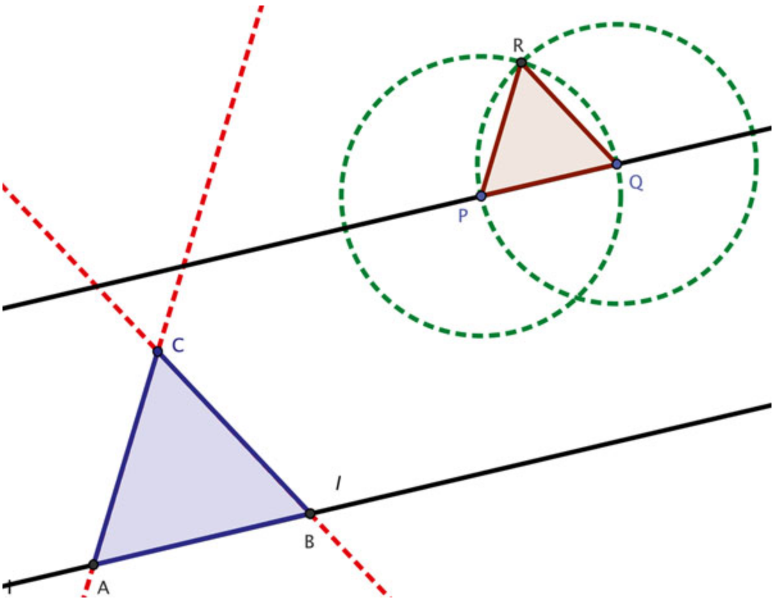

Fig. 5 Drawing equilateral triangle ABC through parallel properties

explore mathematical behaviors of a family of objects through dragging, finding loci, and quantifying attributes or graphic affordances. This object exploration not only provides information regarding invariants or patterns involved but also ways to justify emerging relations or conjectures.

Widening the Scope: How Digital Affordances Offer Opportunities for Learners to Engage in Mathematical Thinking

The use of digital technologies plays a central role in widening students' ways to represent and explore mathematical concepts. In this section, we show examples where the use of GeoGebra not only is central to assemble a dynamic configuration but also becomes important to visualize and support mathematical results.

An Exploration of Basic Geometric Properties

Figure 6a shows that the measure of angle BOC is twice the measure of angle BAC. By focusing on triangle AOB, one observes that the measure of exterior angle BOC is the sum of the measures of angles BAC and ABO; consequently, angle BAC is equal to angle ABO, that is, triangle AOB is isosceles which implies OA = OB. What is the locus of point B when ray AB is moved on the plane? Figure 6b shows that is a circle with center at O and radius OA. The converse is a classical theorem about the angle subtended by an arc in a circle.

Now consider the problem: In a circle centered at O and a chord AB with midpoint C (Figure 7), what is the locus of C as B travels free along the circle?

Chords AE and AD show the position of the midpoint; when the chord is a diameter, the midpoint coincides with the center of the circle (Fig. 7).

Figure 8 makes it visible that the locus is a circle with center O′. In fact, drawing O′C parallel to OB makes the angle CAO′ half the angle CO′O for every position of B. Then the point C describes a circle as shown below.

But problem-solving activities always include extending the results obtained at certain moments: What if C is not the midpoint of AB?

In this case, if we draw the parallel line to OB through C, we obtain Fig. 9:

Again, for each C on AB, angle CAO is half the angle CO′O which means that C will describe a circle with center O′.

It is important to make explicit that students can explore and visualize these results while working in a dynamic environment such as GeoGebra. Then, the problem-solving activities will blend the dynamic exploration with the *geometric reasoning on the figure*.

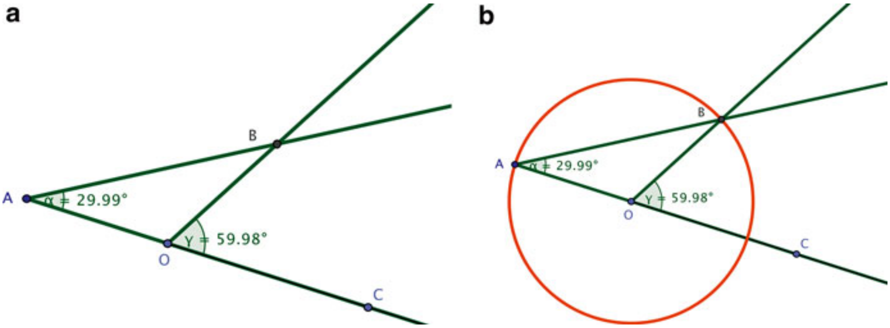

Fig. 6 (**a**) $m\sphericalangle \mathrm{BOC} = 2\sphericalangle \mathrm{BAC}$. (**b**) What is the locus of point B when ray OB is moved on the plane?

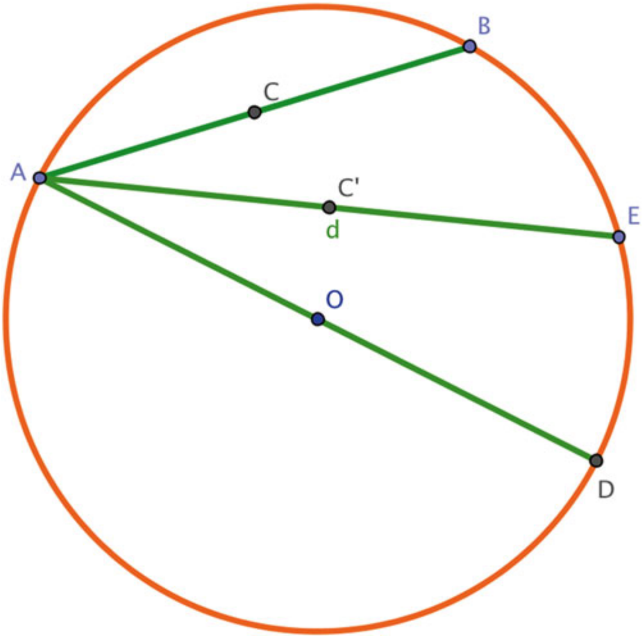

Fig. 7 When the chord is a diameter, its midpoint is the center of the circle

A Triangle and a Variation Task

Sketching a variation phenomenon, without making explicit its algebraic model, is an important problem-solving strategy that learners can apply in a technology environment. We illustrate a dynamic model where the length variation of a triangle side can be explored graphically and through geometric properties.

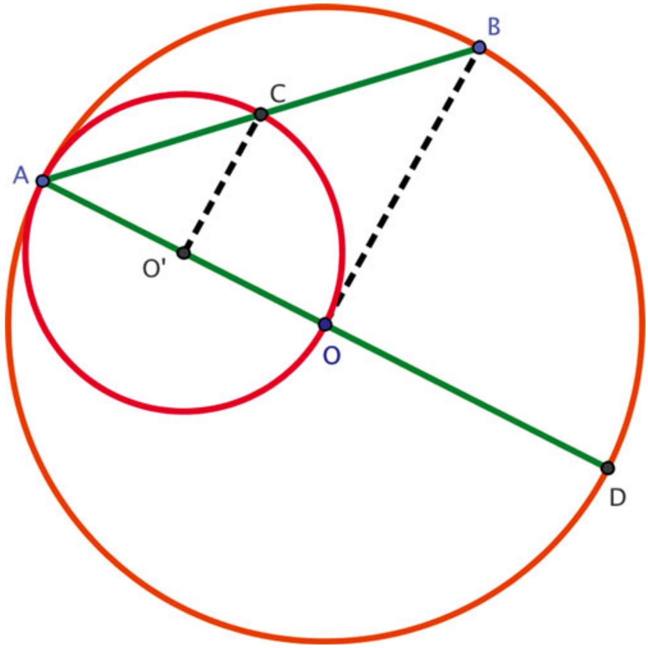

Fig. 8 Locus of point C when point B moves along the circle is a circle centered at O′

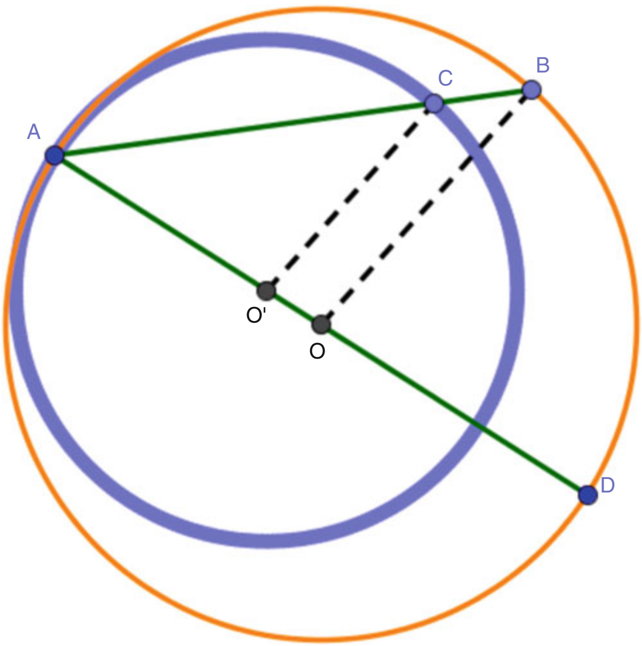

Fig. 9 What is the locus of point C when B is moved along the circle?

Given a triangle ABC and a point P on side BC. From P we draw perpendicular lines to AB and AC and points E, F are the intersection points of those perpendiculars and sides AB and AC.

Find the position of P such that the segment EF is the shortest (Fig. 10).

In a dynamic system such as GeoGebra, we can proceed thus: Draw a perpendicular line from P on side BC as shown in Fig. 11:

The segment PQ has the same length as EF. Then the locus of Q as we move P on BC renders a visual evidence of how the length of EF varies—as P travels on BC. We could name this activity as the heuristic phase that allows the student to become acquainted with the problem. The visual information suggests that the position that renders the minimum length for EF occurs when PQ coincides with the height drawn from A. That is, when segment PQ is a height of the triangle.

Problem solving is surely the kernel of mathematical thinking at school. In our work we encourage our students to develop a drive for blending different approaches to a problem. In the present case, if P is the foot of the height from A, we discover (the students discover after a long discussion in the classroom) that the quadrilateral AEPF is cyclic. Previously, we discussed the conditions under which this fact is realized: the sum of the opposite angles in the quadrilateral equals 180° (Fig. 12).

The quadrilateral is a dynamic object, that is, it changes (as well as the corresponding circle) as P moves on BC. Playing with the circle, one discovers that when AP is the height, then the circle is tangent at P as the following figure illustrates (Fig. 13).

This one is the smallest circle as P moves on BC; consequently, the chord EF has the minimum length. Loci of points I and D represent the area variation of the family of triangles (PEF) and quadrilaterals (PFAE) when point P is moved along BC, and students could explore at what position of P the areas of those triangles and quadrilaterals reach a maximum value.

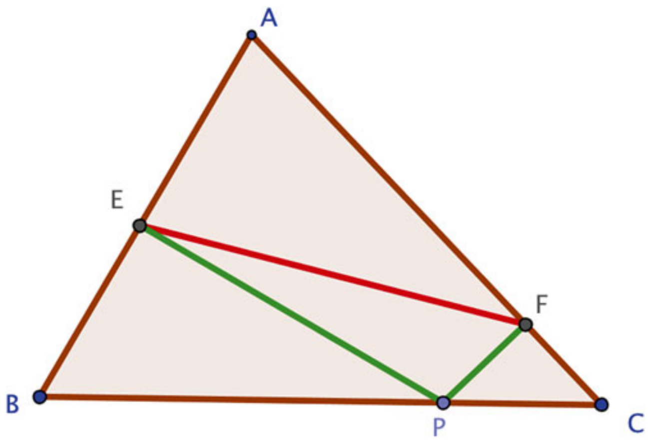

Fig. 10 At what position of P (P moves along segment BC) does segment EF reach the shortest length?

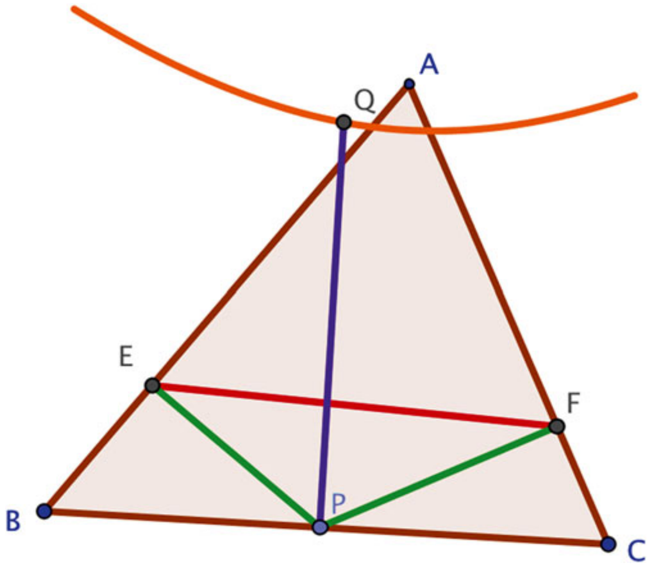

Fig. 11 Graphic representation of length EF

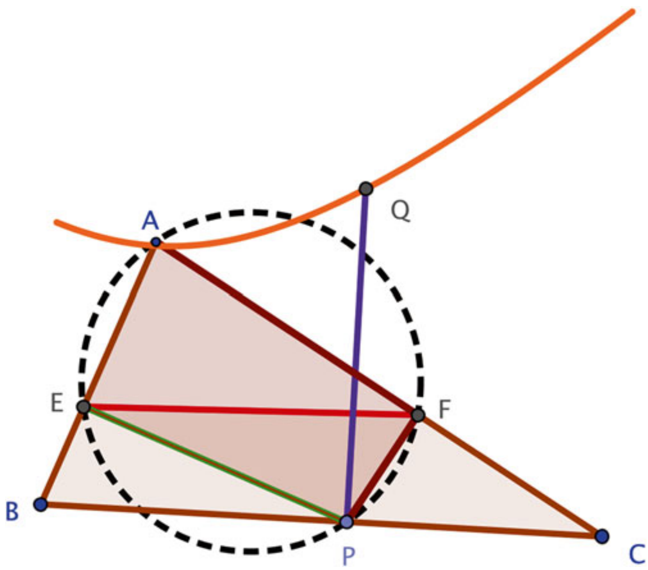

Fig. 12 Multiple representation of length EF

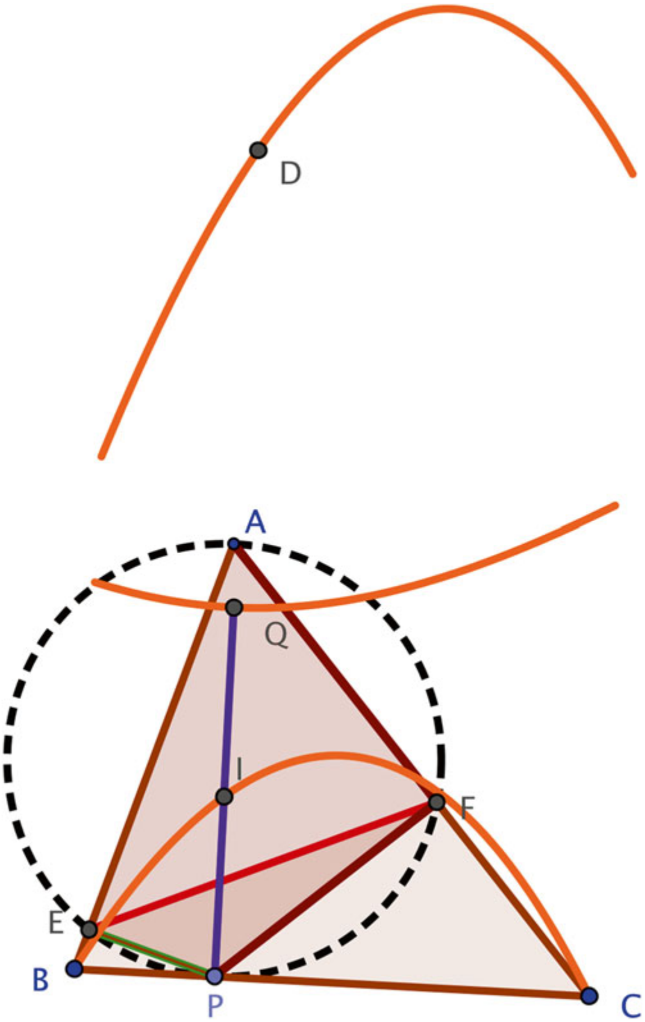

Fig. 13 Finding the solution

The heuristic and deductive phases are complete. It is important to emphasize that the role of the dynamic medium is not ancillary. In fact, the heuristic phase guided by the digital affordances gradually contributes to develop new strategies for problem solving. The blending of paper and pixel is crucial in today's classroom. There is a stable tradition at school, namely, paper and pencil, that will be gradually transformed by the presence of the digital armamentarium.

As we use a new artifact, we feel the *resistance* that it displays. Someone who intends to learn how to use a word processor knows this. Gradually, one overcomes the basic difficulties and begins to *internalize* the artifact—in the present example,

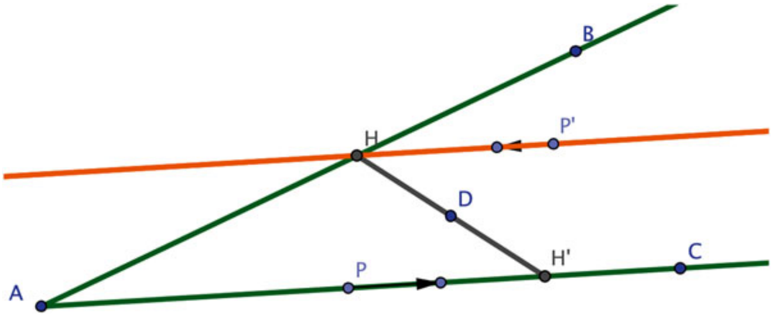

Fig. 14 H solves the problem

the word processor. But the artifact is not passive. With time, its presence will impact the strategies we use to solve problems without that artifact, and the end result is our own transformation as problem solvers. Using an artifact begins to build in the user a cognitive resource for thinking about the problems she/he intends to solve. Artifacts provide resources for acting on problems and, simultaneously, for regulating our thinking as problem solvers.

Locus and the Concept of Central Symmetry

Figure 14 shows another example that illustrates how the availability of digital affordances can redirect or open a new path for exploration. Point D is inside of angle BAC. How to draw a segment from side AC to side AB such that the chosen point D is the midpoint of the segment?

We choose any point P on side AC and reflect it with respect to D to obtain P′. The locus of P′ when point P moves along AC is a parallel line to AC. H is the intersection point of the locus and side AB and H′ is the reflected point of H with respect to D. Segment HH′ is the solution. Let us insist that the availability of a flexible transformation as the central symmetry molds the solution we find for this problem. Our way of thinking is transformed by the presence of the mediating artifact.

The Best View Task: Combining Graphic and Geometric Representation

We will close this section with a problem we first learned from Polya's classic *Mathematics and Plausible Reasoning* (vol. 1, pp. 122–123, 1954). This task was also analyzed in Santos Trigo and Reyes-Rodriguez (2011).

Let us suppose we are walking along a line and we want to determine the position on this line from which we have the best view of a segment AB as in Fig. 15:

The best view of the segment is obtained when the angle at P is largest. If we are to the far right, the angle will be very small, and as we approach walking to the left (as suggested in the figure), the angle will increase. The digital medium allows a first exploration that consists in dragging the point P (walker's position) and see how the angle varies. This experience will suggest the explorer that there is a position where the angle is largest. However, this still does not allow to clearly identify that special position. We can go a step further by representing the measure of the angle by a perpendicular segment to the walking line. Then, we get Fig. 16:

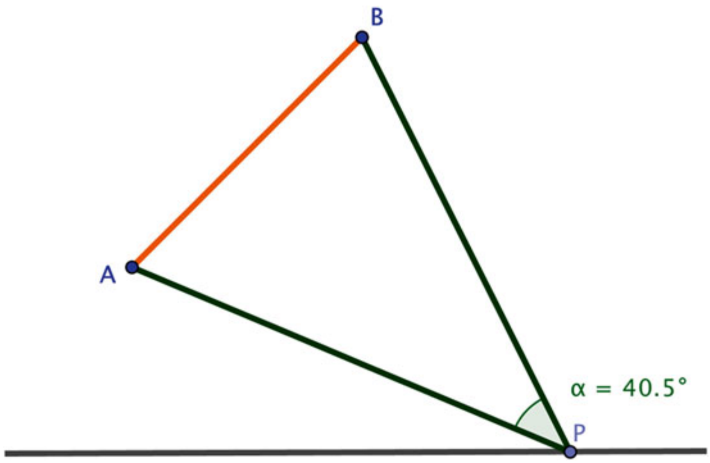

Fig. 15 Explaining the problem

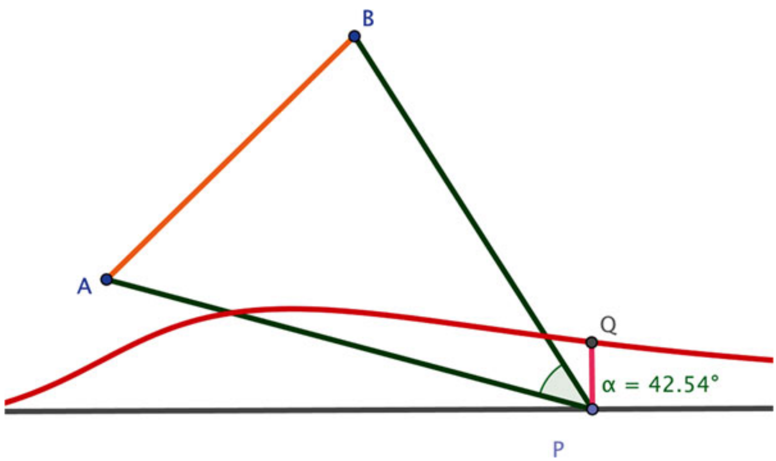

Fig. 16 Cartesian representation of the problem

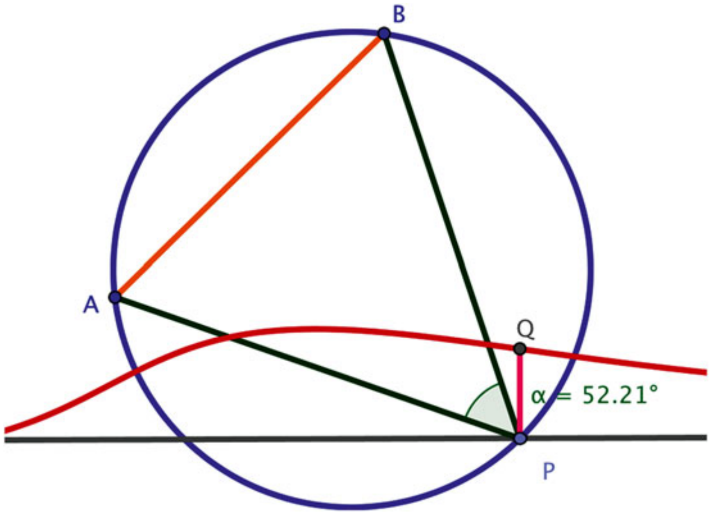

Fig. 17 Euclidian dynamic representation of the problem

Walking to the left eventually one will arrive at a position from where one only sees the point A, as suggested by the graph representing the set of angle measurements. The digital affordances prove their cognitive power: the student is thinking *through* the artifact and *with* the artifact. The optimal position clearly exists. How could we provide a geometric characterization of this position? As we are trying to optimize an angle, previous experience suggests interpreting these angles as subtended by the segment AB as in Fig. 17.

The angle at P is interpreted as an angle subtended by the segment AB. The circle will vary as we move point P on the line. The segment PQ and the circle are two different ways of representing how the angle varies. The segment PQ suggests where the optimal position occurs, and this position coincides with the circle being tangent to the walking line:

This circle has the smallest possible radius and as it contains segment AB as a chord, the corresponding angle at P will be largest. The task exploration, once again, exhibits the virtues of blending *paper and pixel* as a starting point to enrich students' ways of developing their mathematical thinking (Fig. 18).

Concluding Remarks

Throughout this chapter, it is argued that the learners' appropriation process of digital tools could offer them diverse opportunities to develop mathematical thinking and problem-solving competencies. In particular, the tasks we presented illustrate that the use of several problem-solving strategies such as "relaxing task conditions,"

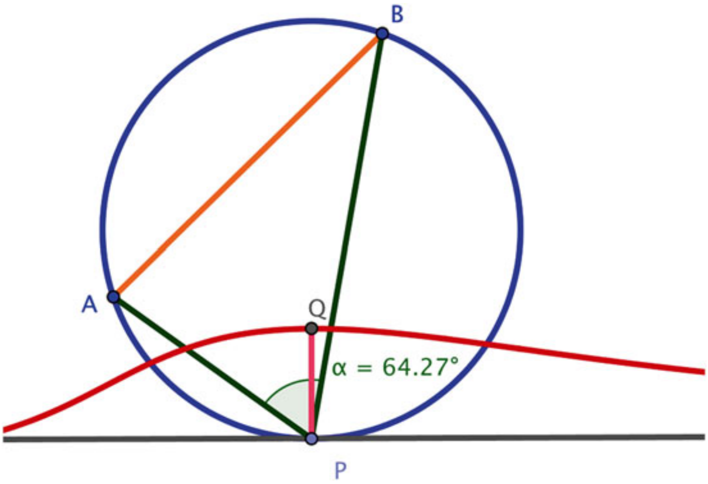

Fig. 18 The solution

"looking for special cases," "assuming the problem is solved," etc., can be extended, compared with the paper and pencil scope, when a dynamic geometric system is used (GeoGebra). To this end, we showed that the construction of dynamic models exhibits novel forms to explore object behaviors that involve "dragging objects," "finding loci," "quantifying objects attributes," "graphic representations," and "visualizing patterns behaviors." Likewise, it is argued that during the process of representing and exploring the tasks, students can use digital technologies to examine embedded concepts and to apply them in problem-solving activities. Indeed, digital technologies open the ways to fertile reinterpretations of existing concepts, and these new forms of the concepts lead to transformations of the meanings of those concepts. The objects we traditionally have drawn on the paper are now for *filming* due to the executable nature of the digital representations. Consequently, digital technologies are leading to new epistemologies, not only affecting students' approaches to problem solving but reshaping the cultural nature of mathematics.

The coordinated use of digital technologies offers diverse opportunities for learners not only to communicate and discuss mathematical tasks and ways to formulate problems but also to represent and explore the tasks from diverse angles and perspectives. Although these digital technologies are not yet fully incorporated in the school culture, their presence is eroding the traditional paper and pencil ways of thinking while confronting a mathematical problem. Nevertheless, this is a rather slow process due to the force of tradition that makes practices resistant to change. This reflects the welcome stability of "good old practices."

The design of digital technologies involves the collaboration of experts' communities working on different fields, and an important element in the design is the users' appropriation process of the tool. Thus, designers should include or rely on information about how users internalize an artifact into their cognitive structures to solve problems and incorporate it into their practices.

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