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Posing and Solving Mathematical Problems

Advances and New Perspectives

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Introduction

Systematic research on problem solving in mathematics can be seen to have begun over 70 years ago with the work of George Pólya, whose most famous publication was likely the book *How to Solve It* (Pólya, 1945). Today there is a huge literature on mathematical problem solving that includes research studies, descriptions, surveys, and analyses. Among the most influential publications have been (and still are) the book by Mason, Burton, and Stacey (1985); the book by Schoenfeld (1985); and the paper by Kilpatrick (1987). The Mason et al. (1985) book emphasizes the importance of creativity and highlights the many cul-de-sacs in problem solving as well as the importance of a solver's persistence. The book by Schoenfeld (1985) is a well-known sourcebook. Younger researchers call it the "black book" of problem solving. Kilpatrick's (1987) paper underlines the connection between problem solving and problem posing, giving special emphasis to problem formulation. These publications form part of the foundation on which this book rests.

The chapters in the book are based on presentations at the final workshop of a comparative research project from 2010 to 2013 between the University of Chile and the University of Helsinki. The project, whose title was *On the Development of Pupils' and Teachers' Mathematical Understanding and Performance when Dealing with Open-Ended Problems*, was initiated by Prof. Erkki Pehkonen (Helsinki) and Prof. Leonor Varas (Santiago). In 2009, the Chilean CONICYT (Comisión Nacional de Investigación Científica y Tecnológica) and the Finnish Academy opened a cooperative program in educational research. Profs. Pehkonen and Varas worked together on an application for a research grant whose leading idea was pupils' development with open-ended problem solving. The project was funded and operated for 3 years. The final workshop, an integral part of the joint research project, was originally designed as a forum to discuss the main results of the project.

However, with support from the Center for Advanced Research in Education (CIAE) and the Center for Mathematical Modeling (CMM), both at the University of Chile, a grant was obtained that enabled the workshop to be expanded well beyond the project participants. The grant supported the invitation of more than 20 international specialists in the field of mathematical problem solving to join the workshop. In the selection of additional participants, we tried to get a broad group of specialists from different parts of the world. After the workshop, all presenters were offered an opportunity to contribute a chapter to the book, and almost all accepted the invitation. Each paper was blind reviewed by two people—in most cases an author of a different chapter, but in some cases an outside reviewer.

The program of the 4-day problem-solving workshop at the University of Chile (Santiago) in December 2013 was as follows:

	Tuesday 10	Wednesday 11	Thursday 12	Friday 13
9:00–9:45		Yan Ping Xin *United States*		Leonor Varas *Chile*
9:45–10:30		Peter Liljedahl *Canada*	Salomé Martinez *Chile*	Teachers' workshop (CF)
11:00–11:45	Masami Isoda *Japan*	Hähkiöniemi *Finland*	Andras Ambrus *Hungary*	Teachers' workshop (CF)
11:45–13:00	Jeremy Kilpatrick *United States*	Jinfa Cai *United States*	John Mason *England*	Markku Hannula and Liisa Näveri (CF) *Finland*
15:00–15:45	Erkki Pehkonen *Finland*	Torsten Fritzlar *Germany*	Yew Hoong Leong *Singapore*	Valentina Giaconi and María Victoria Martínez (CF) *Chile*
15:45–16:30	José Carrillo *Spain*	Susan Leung *Taiwan*	Wim van Dooren *Belgium*	Alejandro López and Paulina Araya (CF) *Chile*
17:00–17:45	Rosa Leikin *Israel*	Patricio Felmer *Chile*	Markku Hannula *Finland*	
17:45–18:30	Bernd Zimmermann *Germany*			Closing ceremony with music from "*Los Bosquinos Band*"

In the case of several authors, usually the first one gave the presentation.

The book is divided into three parts: (I) Problem Posing and Solving Today; (II) Students, Problem Posing, and Problem Solving; and (III) Teachers, Problem Posing, and Problem Solving.

Part I begins with the summary of the role of mathematical textbooks in problem posing by Jinfa Cai et al. In the next paper José Carrillo and Jorge Cruz discuss the role of problem posing and solving. Affect is also an important factor in problem solving; this is dealt with by Valentina Giaconi et al. in the frame of Chilean elementary students. Nicolas Libedinsky and Jorge Soto Andrade examine the cooperation between affect and problem solving. Jeremy Kilpatrick opens a new aspect in problem solving, discussing problem solving and inquiry. The section is closed by Bernd Zimmermann who looks at the history of mathematics and reveals interesting problems. The section review is given by John Mason.

Part II begins with Jinfa Cai's and Frank Lester's overview on problem-solving research results. Then András Ambrus and Krisztina Barczi-Veres consider the situation of problem solving in Hungary, especially from the viewpoint of average students. Torsten Fritzlar explains the results of an exploratory problem implemented by him. The next paper is from Erkki Pehkonen et al. who describe a new data gathering method used in the Chile–Finland research project. Manuel Santos-Trigo and Luis Moreno-Armella have used technology in order to foster students' experiences in problem solving. In the chapter of Tine Degrande et al., the modeling aspects of problem solving are under focus. Yan Ping Xin deals with model-based problem solving. Here Masami Isoda has written the section review.

Part III begins with John Mason's considerations where he examines the concept of problem from a new viewpoint. The paper of Patricio Felmer and Josefa Perdomo-Díaz discusses Chilean novice teacher in problem solving. Leong Yew Hoong et al. deal with problem solving in the Singaporean curriculum. Problem posing in the elementary school program is examined by Shuk-kwan S. Leung. Edward A. Silver discusses problem solving in teachers' professional learning. Peter Liljedahl explains on the conditions of teaching problem solving. The section review is given by Kaye Stacey.

Finally we would like to thank a lot of peoples for their helping hands. Especially we are grateful for those anonymous reviewers who helped us to improve the chapters in the book. But above all we thank Gladys Cavallone for her huge job in practically organizing the workshop at the university and her efficient handling of the papers of the book.

Santiago, Chile — Patricio Felmer
Helsinki, Finland — Erkki Pehkonen
Athens, USA — Jeremy Kilpatrick

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Contents

Part I
Problem Posing and Solving Today

How Do Textbooks Incorporate Mathematical Problem Posing? An International Comparative Study

Jinfa Cai, Chunlian Jiang, Stephen Hwang, Bikai Nie, and Dianshun Hu

Abstract This study examines how standards-based mathematics textbooks used in China and the United States implement problem-posing tasks. We analyzed the problem-posing tasks in two US standards-based mathematics textbook series, *Everyday Mathematics* and *Investigations in Number*, *Data*, *and Space*, and two Chinese standards-based mathematics textbook series, both titled *Shuxue* (Mathematics), published by People's Education Press and Beijing Normal University. All four textbook series included a very small proportion of problem-posing tasks. Among the four series of textbooks, the majority of the problem-posing tasks were in the content strand of number and operations, with a few in other content strands. Significant differences were found between the Chinese and US textbook series as well as between the two textbook series used in each country. Implications for the inclusion of mathematical problem-posing tasks in elementary mathematics textbooks are discussed.

Keywords Problem-posing tasks • Curriculum • Textbooks • Mathematics education reform • Comparative studies • China • United States

In recent years, interest in incorporating problem posing in school mathematics instruction has grown steadily among mathematics education researchers and practitioners (Australian Education Council, 1991; Cai, Hwang, Jiang, & Silber, 2015; Singer, Ellerton, & Cai, 2013). Although historically, problem solving has been more central than problem posing in school mathematics and mathematics education

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research, over the past several decades, curriculum reforms in many countries around the world have begun to raise the profile of problem posing at different educational levels (e.g., van den Brink, 1987; Chinese Ministry of Education, 1986, 2001a, 2011; English, 1997; Hashimoto, 1987; Healy, 1993; Keil, 1964/1967; Kruteskii, 1976; National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010). In part, this has been reflective of a growing recognition that problem-posing activities can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interest and curiosity (Cai et al., 2015; NCTM, 1991). Because problem posing and problem solving are often interwoven activities (Silver, 1994) and success with one has been shown to be associated with success with the other (Cai & Hwang, 2002; Silver & Cai, 1996), it makes sense to consider how problem posing can be integrated as an effective part of mathematics instruction.

However, for problem posing to play a more central role in mathematics classrooms, teachers must have access to resources for problem-posing activities. In particular, mathematics curriculum materials should feature a good representation of problem-posing activities. Although supplemental materials can partially address the situation (e.g., Lu & Wang, 2006; Wang & Lu, 2000), it is important to have problem-posing activities in the curriculum materials that teachers regularly use, as curriculum can be a powerful agent for instructional change (Cai & Howson, 2013; Howson, Keitel, & Kilpatrick, 1981). Thus, the significance of including productive and robust problem-posing activities in curriculum materials should not be overlooked.

Yet there is at present a lack of research that focuses on problem posing in the textbooks that students and teachers actually use, as opposed to the curriculum frameworks on which those textbooks are based. How has the inclusion of problem posing in curriculum frameworks played out in real textbooks? Given the variety of ways to engage students in one form or another of problem posing, how exactly do textbooks include problem posing? What kinds of choices have textbook writers and curriculum developers made in creating existing materials? In order to begin addressing these questions, this study took an international perspective to examine four mathematics textbook series, two of which are used in China and two of which are used in the United States. All four series are based on reform curriculum standards from their respective countries (Chinese Ministry of Education, 2001a; NCTM, 2000) which include problem posing as an important element.

Both China and the United States have engaged in similar reforms regarding mathematics education, and problem posing has been explicitly included in the reform documents that have guided the reforms in each country. Moreover, the overall role of curriculum is quite similar in these two countries. It serves to determine what students are taught and, with respect to the design of textbooks, it conveys the ideas underlying the educational reforms. Thus, it seemed fruitful to conduct a comparative study between the textbooks of the two countries in order to provide an international perspective on the integration of problem posing into commonly used curriculum materials. Indeed, the field has long been interested in such comparative studies between China and the United States, whether they address curriculum, classroom instruction, teacher education, or a myriad of other aspects of the educational system (Cai, 1995). This research lies squarely in this comparative tradition, taking a curricular perspective to analyze problem posing.

Background

Mathematical Problem Posing and Student Learning

A primary goal of research in mathematics education, including problem posing, is to improve student learning. Researchers have noted the potential for problem posing to benefit student learning, both in mathematics (English, 1998; Lavy & Shriki, 2010; Silver, 1994; Toluk-Uçar, 2009) and in other areas such as reading (Rosenshine, Meister, & Chapman, 1996). Problem-posing activities are often cognitively demanding tasks (Cai & Hwang, 2002) that can require students to stretch their thinking beyond problem-solving procedures to improve their understanding by reflecting on the deeper structure and goal of the task. As tasks with different cognitive demands are likely to induce different kinds of learning (Doyle, 1983), the high cognitive demand of problem-posing activities can provide intellectual contexts for students' rich mathematical development.

In particular, because problem posing involves the generation of new problems and questions aimed at exploring a given situation as well as the reformulation of a problem during the process of solving it (Silver, 1994), encouraging students to generate problems is likely to foster both student understanding of problem situations and the development of more advanced problem-solving strategies. Indeed, using eight open-ended problem-solving tasks, Silver and Cai (1996) found a high correlation between students' mathematical problem-solving performance and their problem-posing performance. More successful problem solvers were those who generated more, and more complex, problems. Similarly, Cai and Hwang (2002, 2003) found links between students' strategy use in problem solving and the types of problems students posed. Clearly, the relationships between problem posing and problem solving provide a rationale for recommendations to incorporate problem posing into school mathematics at different educational levels (Chinese Ministry of Education, 1986, 2001a, 2003, 2011; NCTM, 2000).

Problem Posing, Mathematics Curricula, and Curriculum Reform

Given the potential positive impact of including problem-posing activities in the mathematics classroom, it is useful to consider how curriculum might support such activities. Curriculum has historically been seen as a powerful agent for instructional change in the face of changing societal demands on the education system (Cai & Howson, 2013; Howson et al., 1981). For example, a number of countries including China and the United States have been undertaking similar mathematics education reforms. The overarching goals of the reforms have been to improve students' learning of mathematics and to nurture students' innovation and creativity (Chinese Ministry of Education, 2001b; NCTM, 2000). In the United States, NCTM (2000) has placed a strong emphasis on students' thinking, reasoning, and problem solving. It calls for students to "formulate interesting problems based on a wide variety of situations, both within and outside of mathematics" (NCTM, 2000, p. 258). In China, students' thinking and reasoning have also been emphasized in the mathematics education reform. One of the six objectives of the new curriculum reform is for students to be actively involved in inquiry-based activities in order to develop their abilities to collect and process information, to attain new knowledge, to analyze and solve problems, and to communicate and cooperate (Chinese Ministry of Education, 2001b). At the 9-year compulsory education stage, students are expected to learn how to pose problems from mathematical perspectives, how to understand problems, and how to apply their knowledge and skills to solve problems so as to increase their awareness of mathematical applications (Chinese Ministry of Education, 2001a). The high school mathematics curriculum is intended to enhance students' abilities to pose, analyze, and solve problems from mathematical perspectives, to express and communicate mathematically, and to attain mathematical knowledge independently (Chinese Ministry of Education, 2003). An additional goal is for students to change their learning styles from passive to active through being engaged in problem posing and problem solving (Chinese Ministry of Education, 2001a, 2003).

Yet if, as these curriculum reform documents advocate, problem-posing activities are to become a more central part of mathematics classrooms, there must be resources ready for problem-posing activities. Although teachers can take it upon themselves to transform the problems and tasks in their existing curriculum materials into problem-posing tasks, it is reasonable to posit that having ready-made problem-posing resources available would facilitate teachers' implementation of problem-posing activities in their classrooms. One approach is to provide such activities as supplementary materials. Lu and Wang (2006; Wang & Lu, 2000) launched a project on mathematical situations and problem posing. They developed supplementary teaching materials based on mathematical contexts and used them to enhance students' problem-posing abilities. These teaching materials were not intended to replace textbooks; instead, they were used to supplement regular textbook problems. Although helpful and potentially effective, it remains the case that teachers have easiest and most ready access to materials that are in their existing curriculum materials. Moreover, particularly in countries like China in which teachers carefully study their textbooks to guide and improve their teaching (Cai & Nie, 2007), the inclusion of problem-posing resources in those textbooks should be particularly powerful influences on classroom practice.

How, then, is problem posing represented in the mathematics textbooks that teachers regularly use? Many current textbooks have been designed to implement reform curriculum standards. For example, the NSF-supported projects that developed reform mathematics curricula in the United States based on the 1989 NCTM Standards produced materials that were markedly different from the traditional textbooks that had preceded them. Among other features, the reform textbook series included many more problems set in realistic contexts and more problems that could be solved using multiple strategies (Senk & Thompson, 2003). Similarly, Chinese textbook materials also evolved in response to reform guidelines in China. For example, the 2004 edition of the Chinese elementary mathematics textbook series published by the People's Education Press (PEP) included a larger percentage of problem-posing tasks than the 1994 edition (Hu, Cai, & Nie, 2014). However, more generally it is not so clear where and how textbooks that have been designed to implement reform curriculum standards include problem-posing tasks. Are problem-posing tasks found broadly and systematically across the textbooks with respect to both mathematical content and grade level, or are they distributed unevenly across grade and content? To what extent do the textbooks embody the stances of the reform standards toward problem posing? If reform standards portray problem posing as a theme that should run throughout mathematics education, it is useful to examine the degree to which the actual textbooks exhibit this perspective.

Moreover, it is useful to consider whether the inclusion of problem-posing tasks in reform-guided curriculum materials reflects a systematic approach to the development of problem-posing abilities in students. For example, the inclusion of sample problems within problem-posing tasks may provide a window into the intent of textbook designers. In earlier versions of Chinese mathematics textbooks, problem posing was not included as a topic in its own right. Rather, problem posing was treated as an intermediate step in problem solving. Newer, reform-oriented revisions of the textbooks have included problem posing as a learning goal. To that end, textbook designers have had to incorporate materials that can guide students through the process of posing problems. One way to do this is to include sample problems within problem-posing tasks for students to emulate. Thus, the degree to which problem-posing tasks in textbooks include sample problems can be an indicator of how intentional textbook designers were in building problem posing from the curriculum standards.

Similarly, there are several types of problem-posing tasks that have been identified in research on problem posing. Based on work by Stoyanova (1998) and Silver (1995), Christou, Mousoulides, Pittalis, Pitta-Pantazi, and Sriraman (2005) describe five such types defined by the nature of the problem students are asked to pose: a problem in general (free situations), a problem with a given answer, a problem that contains certain information, questions for a problem situation, and a problem that fits a given calculation. In addition, different problem-posing tasks may present given information to students in several ways, including the use of visual and symbolic modes of representation that may or may not be influenced by and consonant with other design and pedagogical choices for a given textbook. Different types of tasks thus reflect different qualities and priorities in problem-posing task design, such as the degree to which the task is constrained for the student (e.g., Stoyanova, 1998) or the role the task may play in relationship with problem solving (e.g., Silver, 1995). Therefore, the manner in which different types of problem-posing tasks are incorporated into textbooks can provide further information about the degree to which these materials systematically integrate problem posing from the curriculum standards and to which they aim to develop particular aspects of problem posing for students.

On the whole, further work is needed to understand whether and how problem posing is integrated into textbooks and the degree to which different ways of doing so is effective in achieving the goals of curriculum reform. Of course, even when problem posing is intentionally built into curriculum materials, it is still necessary to study how problem-posing tasks are implemented by teachers in actual classrooms. The work that teachers do in transforming written curriculum materials into live instruction depends on many other factors, including teachers' knowledge and beliefs. Nevertheless, as yet there has not been a substantial body of research examining whether and how the curricula themselves incorporate problem posing (Cai et al., 2015). This study is intended to address the gap between the knowledge about the incorporation of problem posing in curricula and textbooks. Specifically, we address the following research question:

How are different problem-posing tasks included in recent US and Chinese reform-oriented mathematics textbooks?

This study will provide researchers, curriculum developers, and textbook writers with rich information about how to incorporate problem posing into school mathematics.

Method

Materials

We examined two series of elementary mathematics textbooks used in China and two series used in the United States. Of the two Chinese textbook series, one was published by PEP, and the other was published by Beijing Normal University (BNU). Both curricula were developed based on the new mathematics curriculum standards (Chinese Ministry of Education, 2001a). We chose two popular series for the textbooks used in the United States: *Everyday Mathematics*, developed by the University of Chicago School Mathematics Project (UCSMP, 2012a, 2012b) and *Investigations in Number*, *Data*, *and Space* (hereafter shortened as *Investigations*), published by TERC, Cambridge, MA (TERC, 2008a, 2008b, 2008c, 2008d, 2008e, 2008f). These two series are generally taken to be examples of *standards*-based curricula (Riordan & Noyce, 2001; Senk & Thompson, 2003). In all four cases, the textbooks represent the most widely adopted elementary mathematics curriculum materials in their respective countries.

Task Analysis

We first checked every task in the four textbook series to identify those that were problem-posing tasks, including those cases where problem posing was included as a component of a larger problem-solving task or activity. We then analyzed each problem-posing task in terms of its (a) grade level, (b) content area, (c) presentation of given information (e.g., with/without graphs, figures, tables, etc.) and whether there were sample questions that students could imitate, and (d) types of problem-posing tasks.

With respect to the types of problem-posing tasks, we classified each problem-posing task according to what it required students to do, relative to the information provided in the task. These types were specified based on a holistic analysis of the requirements in a problem-posing task. Special attention was paid to whether a problem poser needed to provide information as givens and whether there was a sample question that a problem poser could emulate to reproduce similar ones. Five types of problem-posing tasks were identified. We describe these types below, roughly ordered from the problem-posing task types that are the most mathematically constrained to those that are least mathematically constrained:

1. *Posing a problem that matches the given arithmetic operation*(*s*). Students are asked to make up a story or a word problem that can be solved with a given arithmetic operation. Tasks of this type provide the student with an explicit arithmetic operation, and the student is expected to provide a context and pose a problem that matches the operation. For example, *write a story problem for* 65×35. *Then solve the problem and show how you solved it* (TERC, 2008d, Unit 8, p. 29).
2. *Posing variations on a question with the same mathematical relationship or structure*. Given a sample problem or problem situation (it is not necessary for the sample to include a question), students are asked to pose a similar problem complete with given information and question. The student can change the context, the specific numbers, or even which quantity is the unknown quantity, but the fundamental mathematical relationship or structure must mirror the sample. For example, *if six people share three apples*, *each person will get ½ of an apple. Make up a problem about equal shares so that each person gets one fourth of something* (TERC, 2008c, Unit 7, p. 35).
3. *Posing additional questions based on the given information and a sample question*. Students are asked to pose additional problems after solving a given problem with sample question(s). The additional problems are expected to involve the given information but are not required to mirror a particular mathematical relationship. Although students may choose to provide additional information, they may not change the given information. For example, *on weekends*, *a father and his son went climbing. The distance from the ground to the top of the mountain*

is 7.2 km. It took them 3 h to climb up and 2 h to walk down. What are the speeds going up and going down? Can you pose additional mathematical questions (People's Education Press, 2001, 5a, p. 20)?

4. *Posing questions based on given information*. Students are provided with a problem context and information but no sample problem. They are expected to generate questions based on the given information. For example, *four children* (*A*, *B*, *C*, *and D*) *are practicing Chinese typing. The following table shows their practice time every day and their records on a test where each of them could select an article to type. Based on the data source*, *please pose two questions and try to answer them* (Beijing Normal University Press, 2001, 4a, p. 72).

		A	B	C	D
Practice time every day (in minutes)		20	30	35	60
Test records	Time (minutes)	12	19	18	13
	No. of words typed	384	931	846	728

5. *Unconstrained problem-posing tasks*. These tasks ask students to pose problems to show the application of mathematics in real life but otherwise do not provide given information or constraints on the structure of the problem. For example, *what mathematical problems could you find in your life*? *Please write them down. Can you solve them*? (Beijing Normal University Press, 2001, 1b, p. 98).

To establish interrater reliability for the coding of the problem-posing tasks, 30 problem-posing tasks from Chinese textbooks and 26 problem-posing tasks from US textbooks were randomly selected and coded by two coders who are proficient in both Chinese and English. For the Chinese textbooks, the two coders reached the following levels of agreement in each of the categories: (a) content area (100 %), (b) use of various representations for the given information (e.g., with/without graphs, figures, tables, etc.) (92 %) and whether there were sample questions that students could imitate (89 %), and (c) types of problem-posing tasks (82 %). Similarly, for the US textbooks, the two coders reached the following levels of agreement in each of the categories: (a) content area (89 %), (b) use of various representations for the given information (e.g., with/without graphs, figures, tables, etc.) (88 %) and whether there were sample questions that students could imitate (81 %), and (c) types of problem-posing tasks (77 %). The discrepancies were resolved through discussion.

Results

Number of Problem-Posing Tasks at Different Grade Levels

The two Chinese textbook series and the US *Everyday Mathematics* series were written for children in grades 1–6. However, the *Everyday Mathematics* textbooks for children at grades 1 and 2 are combined. The US *Investigations* series was written

Table 1 Total number of problems and percentage of problem-posing (pp) tasks in the four mathematics textbooks series from grades 1–6

	China				United States			
	PEP		BNU		Investigations		Everyday	
Grade	*n*	% PP	*n*	% PP	*n*	% PP	*n*	% PP
1	527	3.61	570	5.96	490	0	–[b]	–
2	565	6.73	549	5.65	741	1.62	1651	1.03
3	589	3.40	541	2.77	832	0.72	1322	1.06
4	621	4.83	561	2.85	760	1.97	1565	1.28
5	659	2.12	619	2.75	726	2.62	1896	1.16
6	627	1.75	545	2.94	–[a]	–	1673	0.42
Total	3588	3.68	3385	3.81	3549	1.47	8107	0.99

Note: [a]Investigations does not have grade 6 textbooks
[b]For Everyday Mathematics of grades 1 and 2, we combined the data because there is only one combined Student Reference Book for the two grades

for children in grades 1–5. For each textbook series, the total number of tasks (*n*) and the percentage of those that were classified as problem-posing tasks are shown in Table 1.

Overall, the percentages of problem-posing tasks were quite small for all four textbook series. However, there were some differences across the series. The percentages of problem-posing tasks in the two Chinese textbook series were more than double those in the two US textbook series. The problem-posing tasks in the two Chinese textbook series made up similar percentages of the total numbers of tasks in those series, whereas the two US textbook series were significantly different from each other in terms of percentage of problem-posing tasks. Specifically, a higher percentage of the tasks in the *Investigations* textbooks was problem-posing tasks compared with that in the *Everyday Mathematics* textbook series ($z=2.25$, $p<0.05$).

The percentages of problem-posing tasks were also very different across different grade levels. No grade had the largest percentage of problem-posing tasks across the four series, and indeed the percentage rose and fell from grade to grade within most of the series (although the grade-to-grade fluctuations within *Everyday Mathematics* were comparatively small). Between the two textbook series in each country, we compared the percentage of problem-posing tasks at each grade level. There were no significant differences except between *Investigations* and *Everyday Mathematics* at grade 5 ($z=2.69$, $p<0.01$).

Number of Problem-Posing Tasks in Different Content Areas

We classified the problem-posing tasks in the four textbook series by the content area in which they were situated: number and operations, algebra, geometry, measurement, and data analysis and probability, following the content areas used by

Table 2 Percentage distribution of problem-posing tasks in different content areas in the four mathematics textbook series

	China		United States	
Content area	PEP ($n = 132$)	BNU ($n = 105$[a])	Investigations ($n = 52$)	Everyday ($n = 80$)
Numbers and operations	73.48	76.19	90.38	91.25
Algebra	0	1.90	5.77	1.25
Geometry	3.79	2.86	0	1.25
Measurement	0.76	2.86	0	0
Data analysis and probability	21.97	16.19	3.85	6.25

Note: [a]In several review sections in the BNU textbook series, there are problems like "What mathematical problems have you found in your life? Write them down and try to solve them." Therefore, the content areas they are related to cannot be determined. Twenty-four such problem-posing tasks were excluded in this analysis

NCTM (2000) (Table 2). However, in several review sections in the BNU textbook series, there were questions like "What mathematical problems have you found in your life? Write them down and try to solve them," for which the content area could not be determined. The 24 free-structured problem-posing tasks of this type in the BNU series were therefore omitted from the content area analysis. The percentage distribution of problem-posing tasks in the five content areas was significantly different across the four textbook series (chi-square $= 31.22$, $df = 12$, $p < 0.01$). However, no significant difference was found between the two textbook series in each country.

For all four textbook series, the majority of the problem-posing tasks were related to number and operations. The percentages of number and operations problem-posing tasks in the US textbook series were higher than those in the Chinese textbook series (*Investigation* vs. *PEP*: $z = 2.50$, $p < 0.05$; *Everyday Mathematics* vs. *PEP*: $z = 3.15$, $p < 0.01$; *Investigation* vs. *BNU*: $z = 2.13$, $p < 0.05$; *Everyday Mathematics* vs. *BNU*: $z = 2.68$, $p < 0.01$). However, the difference in the percentages of problem-posing tasks in number and operations in the two textbook series in each country was not significant.

For the two Chinese textbook series, the second highest percentage of problem-posing tasks was related to data analysis and probability. The difference in the percentages of problem-posing tasks in data analysis and probability in the two textbook series in each country was not significant. However, the percentages of data analysis problem-posing tasks in the two Chinese textbook series were significantly higher than those in the two US textbook series (*PEP* vs. *Investigations*, $z = 2.96$, $p < 0.01$; *PEP* vs. *Everyday Mathematics*, $z = 3.02$, $p < 0.01$; *BNU* vs. *Investigations*, $z = 2.23$, $p < 0.05$; *BNU* vs. *Everyday Mathematics*, $z = 2.07$, $p < 0.05$).

For all four textbook series, very few problem-posing tasks were related to algebra, geometry, or measurement, with the percentages all less than 6 %.

Table 3 Percentages of types of problem-posing tasks in the four mathematics textbook series

Types of problem-posing tasks	China		United States	
	PEP ($n=132$)	BNU ($n=129$)	Investigations ($n=52$)	Everyday ($n=80$)
Posing a problem that matches the given arithmetic operation(s)	3.79	3.88	84.62	68.75
Posing variations on a question with the same mathematical relationship or structure	0	6.20	13.46	23.75
Posing additional questions based on the given information and a sample question	65.91	56.59	1.92	5.00
Posing questions based on given information	30.30	14.73	0	2.50
Unconstrained problem-posing tasks	0	18.60	0	0

Types of Problem-Posing Tasks

The problem-posing tasks in the four textbook series were classified into the following five types based on what they required the student to do: (1) posing a problem that matches the given arithmetic operation(s), (2) posing variations on a question with the same mathematical relationship or structure, (3) posing additional questions based on the given information and a sample question, (4) posing questions based on given information, and (5) unconstrained problem-posing tasks. The percentages of the problem-posing tasks of each type are shown in Table 3.

The data in Table 3 showed large discrepancies between the Chinese and US textbook series and between the two textbook series in each country. Recall that the types of problem-posing tasks were roughly ordered from most constrained to least constrained. The percentages in Table 3 suggest that the Chinese textbooks had larger percentages of problem-posing tasks that were comparatively less constrained, whereas the US textbooks had larger percentages of tasks that were comparatively more constrained.

For the two Chinese textbook series, the majority of the problem-posing tasks required students to pose additional questions for given information after presenting students with sample questions (e.g., *On weekends, a father and his son went climbing. The distance from the ground to the top of the mountain is 7.2 km. It took them 3 h to climb up and 2 h to walk down. What are the speeds going up and going down*? *Can you pose additional mathematical questions*?). Although the percentages of problem-posing tasks of this type were not significantly different between the two textbooks within either country, the percentages in the two Chinese textbook series were significantly higher than those in the two US textbook series (*BNU* vs. *Everyday Mathematics*: $z=7.52$, $p<0.001$). In contrast, for the two US textbook series, the majority of problem-posing tasks required students to pose problems that matched the given arithmetic operations (e.g., *Write a story problem for* 65×35. *Then solve the problem and show how you solved it*). The percentages of problem-posing tasks

Table 4 Percentages of problem-posing tasks with/without sample questions and with/without information presented in pictures, figures, or tables (PFT)

Textbook series	With sample questions		Without sample questions	
	With PFT	Without PFT	With PFT	Without PFT
PEP ($n=132$)	33.33	32.58	12.88	21.21
BNU ($n=131$)	56.59	0.78	20.16	22.48
Investigations ($n=60$)	3.85	5.77	3.85	86.54
Everyday Mathematics ($n=81$)	17.50	10.00	6.25	66.25

of this type were not significantly different between the two Chinese textbook series, but the percentage of problem-posing tasks of this type in *Investigations* was significantly higher than that in *Everyday Mathematics*. The percentages in the two US textbook series were significantly higher than those in the two Chinese textbook series (*BNU* vs. *Everyday Mathematics*: $z=10.08$, $p<0.001$).

For the *PEP* textbook series, the second most common type of problem-posing task was posing questions based on given information. The percentage of such tasks in PEP was significantly higher than that in the *BNU* textbook series ($z=3.01$, $p<0.01$), although this type of problem-posing task was the third most common type in *BNU*. In turn, the percentage of such tasks in *BNU* was significantly higher than that in the *Everyday Mathematics* textbook series ($z=2.86$, $p<0.01$). For the *BNU* textbook series, the second most common type of problem-posing task was unconstrained problem-posing tasks. There were no such tasks in the other three textbook series.

For the *Everyday Mathematics* textbook series, the second most common problem-posing task was posing variations on a question with the same mathematical relationship or structure. Although this percentage was not significantly higher than that in the *Investigations* textbook series, it was significantly higher than those in both Chinese textbook series (*BNU*: $z=3.68$, $p<0.001$). However, the percentages of reformulation problem-posing tasks in *BNU* and *Investigations* were not significantly different.

Presentation of Problem-Posing Tasks and Inclusion of Sample Questions

Table 4 shows the degree to which the four textbooks included sample questions in problem-posing tasks and to which they presented information in these tasks using pictures, figures, or tables. Significant differences existed among the four textbook series in both aspects (chi-square $=167.78$, $df=9$, $p<0.001$). There were also significant differences between the two Chinese textbook series (chi-square $=49.15$, $df=3$, $p<0.001$) but not between the two US textbook series.

Specifically, the two Chinese textbook series (*PEP* 66 %, *BNU* 57.37 %) had higher percentages of problem-posing tasks with sample questions than the US textbook series (*Investigations* 9.62 %, *Everyday Mathematics* 27.50 %). The differences between the two Chinese textbook series regarding inclusion of sample questions were not significant. However, they are significant between the two US textbook series.

Of the problem-posing tasks included in the US mathematics textbooks, less than half were presented with information in pictures, figures, or tables (*Investigations* 7.70 %, *Everyday Mathematics* 23.75 %). This was a lower percentage than in the two Chinese textbook series (*PEP* 46.21 %, *BNU* 76.75 %). The two textbook series within each country were significantly different in their percentages of problem-posing tasks that included information presented in pictures, figures, and tables (*PEP* vs. *BNU*, $z = 5.06$, $p < 0.001$; *Investigations* vs. *Everyday*, $z = 2.38$, $p < 0.05$).

Discussion

Problem Posing and Curriculum Reform

Curriculum reform has often been viewed as a powerful tool for educational improvement because changes in curriculum have the potential to change classroom practice and student learning (Cai & Howson, 2013). Reform-guided mathematics curricula in both China and the United States have put great emphasis on problem posing because of its potential to develop students' creative thinking and ability to innovate in the new century. Consequently, both Chinese and US textbook developers have made some effort to integrate problem-posing tasks into curriculum materials. Although our data show that the Chinese textbooks we examined do contain a greater percentage of problem-posing tasks than the US textbooks, the percentage of such tasks in each of the four textbooks we examined is still quite low.

The comparatively small representation of problem-posing tasks among a large sea of problem-solving tasks may reflect, to some degree, the relative emphases and placement of problem posing in the reform curriculum guidelines of the two countries. Problem posing was explicitly included as part of the problem-solving standard for each grade band in NCTM's (1989) *Curriculum and Evaluation Standards* that guided the development of US reform mathematics curricula in the 1990s. In the subsequent *Principles and Standards for School Mathematics* (NCTM, 2000), problem posing was again part of the problem-solving standard in each grade band. Given the strong focus on increasing the role of problem solving in reform mathematics curricula, it may be the case that problem posing was overshadowed. Indeed, the recent *Common Core State Standards for Mathematics* (NGACBP & CCSSO, 2010) only mentions problem posing once, whereas problem solving permeates the document (Ellerton, 2013). The Chinese reform curriculum standards also include problem posing as part of the overall objectives regarding problem solving (Chinese Ministry of Education, 2011). In addition, they discuss the role of problem posing in assessment and instruction. This broader inclusion of problem posing across the Chinese reform curriculum guidelines may be connected to the somewhat greater inclusion of problem posing in the two Chinese textbook series we examined.

Content Areas, Grade Levels, and Intentionality of Design

It is clear that the distributions of problem-posing tasks across different content areas and different grade levels in the four textbook series are extremely uneven. More specifically, the problem-posing tasks are heavily concentrated in the number and operations content area. Of course, number and operations has traditionally been a primary focus of elementary mathematics, and the designers of even reform-oriented mathematics textbooks may be deliberately focusing attention on this area to accord with the traditional expectations teachers have of elementary mathematics curricula. However, the degree of concentration of problem-posing tasks in number and operations exceeds what would be expected based on the overall distribution of content of the textbooks in this study. In particular, the dearth of problem-posing tasks related to geometry and measurement is out of proportion to the coverage of geometry and measurement topics in the textbooks. This is somewhat puzzling, given the degree to which geometry, in particular, is amenable to conjecturing and forming hypotheses (Yerushalmy, Chazan, & Gordon, 1990). Indeed, geometry is also an area in which technological tools such as dynamic geometry software have been shown to be supportive of problem posing (Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2005). It is notable that none of the textbook series examined here make use of such technological tools to promote students' problem posing, though not entirely surprising given the relatively slow pace of textbook development and the comparatively fast pace of technological development.

In some of the textbook series we examined, a few content areas other than number and operations include a somewhat more substantial percentage of problem-posing tasks. For example, problem posing is somewhat better represented in the data analysis and probability portions of the Chinese textbooks. This may be due, in part, to an expanded emphasis on data analysis and probability in China (Chinese Ministry of Education, 2001b). Looking to the reform curriculum guidelines in China (Chinese Ministry of Education, 2011), problem posing is explicitly mentioned with respect to data analysis and probability: "To develop basic knowledge and skills in statistics and probability as well as to be capable of solving simple problems through experience in problem posing…." Indeed, higher percentages of problem-posing tasks are integrated into this content area in both Chinese textbook series than in the US textbook series. In parallel fashion, in recent years more emphasis has been put on early algebraization in the United States (Cai & Knuth, 2011). In particular, the focus of the *Investigations* series is on algebra (Cai et al., 2005). Thus, it is not overly surprising that *Investigations* includes more algebra-related problem-posing tasks than the other three textbook series.

The uneven distribution of problem-posing tasks across content areas is mirrored in the way tasks are distributed across grade levels. The distribution of tasks across grades shows a great deal of variability in every curriculum other than *Everyday Mathematics*, which has a comparatively low percentage of problem-posing tasks in every grade. As Fig. 1 shows, even though the percentages of problem-posing tasks in the curricula are generally small within *PEP*, *BNU*, and *Investigations*, they rise and fall markedly from grade to grade. There does not appear to be any trend toward

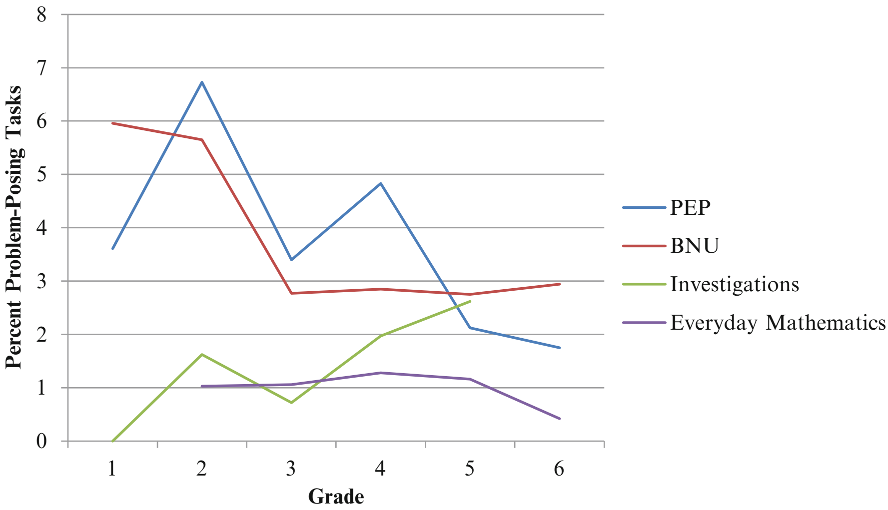

Fig. 1 Percentage of problem-posing tasks in each grade by curriculum

increasing or decreasing problem posing as students progress through the elementary grades, nor do the textbooks maintain a regular level of problem posing from grade to grade.

The general lack of consistency in the inclusion of problem-posing tasks, both across content areas and across grades, suggests a need for greater intentionality in the planning and design of how problem posing should be embedded in textbooks. Although there have clearly been some intentional efforts to incorporate problem posing in these textbook series, the inconsistency of implementation may not be helpful for making problem posing a classroom routine.

Types of Problem-Posing Tasks

The distributions of problem-posing tasks into the five types that we identified are also uneven. However, in this case there is a pattern to the unevenness, specifically regarding the degree to which the tasks are more or less mathematically constrained that appears to be related to whether the textbooks are US or Chinese. The majority of the problem-posing tasks in the two Chinese textbook series are tasks in which the student is given some information and a sample question and is then asked to pose additional questions based on the given information. Although a sample question is provided in these tasks, the student is not necessarily expected to mirror the mathematical structure of the given problem. The PEP textbooks also include a substantial proportion of problem-posing tasks in which the student is expected to pose additional questions based on given information but without a sample question. These tasks give the student a great deal of latitude in choosing the mathematical structure of their problem, although the context is fixed. The BNU textbooks include tasks of this type as well as even more unconstrained problem-posing tasks in which students are prompted to pose questions relating mathematics to real life. In contrast, the majority of problem-posing tasks in both US textbook series have much stronger constraints, requiring students to pose problems with solutions that match the given arithmetic operations. In these tasks, the student may choose a context relatively freely, but the mathematical structure of the problem is already fixed. Indeed, the most common types of problem-posing tasks in the US textbooks are those in which the mathematical structure of the problem is largely fixed and given to the student.

It is not immediately clear why there should be a difference in the level of task constraints between problem-posing tasks in Chinese textbooks and those in US textbooks. One potential explanation for these differences might lie in differences in how teachers and textbook designers view the use of problem-posing tasks for mathematics teaching, such as teaching a new concept versus practicing a new approach. However, this would need to be further investigated with respect to how these problem-posing tasks are actually used in mathematics classrooms.

Use of Representations and Sample Questions

Problem-solving research has shown that US textbooks generally have more problems that include information represented in pictures, figures, and tables than Chinese textbooks (Zhu, 2003) and that US students are more likely to solve mathematical problems using visual representations than Chinese students (Cai, 1995, 2000). In this study, we examined the use of pictures, figures, and tables to represent information in the problem-posing tasks from the four textbook series. The data show clear differences between the Chinese and US series. However, these differences do not mirror the trend identified in the problem-solving literature. Both *Investigations* and *Everyday Mathematics* feature a smaller percentage of problem-posing tasks that include pictures, figures, and tables than the two Chinese textbook series. In particular, the *BNU* series uses such representations in over three-quarters of its problem-posing tasks. The disjunction between these results and the findings from problem-solving research may be related to the prevalence of tasks in the US textbooks that asked students to pose problems whose solutions matched a given operation. The problem-posing tasks in the Chinese textbooks tend to be less mathematically constrained and thus perhaps may afford greater latitude to employ diverse representations.

With respect to the inclusion of sample questions, the problem-posing tasks in the two Chinese textbooks series are again more likely than their US counterparts to exhibit this feature. As we noted above, sample questions may be included in problem-posing tasks as a way to guide students as they learn how to pose their own problems. The Chinese reform curriculum guidelines have made problem posing a learning goal in its own right (Chinese Ministry of Education, 2011). Thus, it makes sense that textbook designers would intentionally include examples for students to study and emulate as they learn how to formulate their own problems.

Implications and Directions for Future Research

Problem posing has been lifted up as an important component of mathematics learning in reform mathematics curriculum documents in both the United States and China. However, our examination of four textbook series from these two countries shows that there is still a very small proportion of problem-posing tasks built into the materials that students use every day. If curriculum is a major agent of change for the teaching and learning of mathematics, there simply may not be enough problem posing in current curriculum materials to realize the goals stated in the reform documents. More specifically, although textbook writers have clearly made some efforts to include problem posing in textbooks, these efforts have resulted in uneven inclusion, both with respect to content area and to grade level. The results of this study suggest that in order to better support teachers as they attempt to fulfill reform recommendations to engage their students in problem-posing activities and to develop their students' mathematical dispositions around problem posing, curriculum developers will need to carefully examine the quantity and types of problem-posing tasks that are included at every grade level. In particular, the dearth of problem-posing tasks related to geometry and measurement is somewhat perplexing and requires attention. Even though we believe that the proportion of problem-posing tasks in the textbooks is very small, it is still an open question what proportion might be appropriate.

Curriculum operates on several levels. This study has focused on the intended curriculum as embodied in textbooks. It provides information from one perspective about what is happening with problem posing in mathematics education in China and the United States. Thus far, there have been no studies that have reported on the actual use of problem-posing tasks from regular textbooks in real classrooms. Looking forward, future studies must also attend to the implemented curriculum—to how teachers and students actually make use of regular curriculum materials to engage in problem posing (or not) in their classrooms (Cai et al., 2015).

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Problem-Posing and Questioning: Two Tools to Help Solve Problems

José Carrillo and Jorge Cruz

Abstract This paper analyses the solutions put forward by two secondary school pupils to two mathematical problems. The task of working out the solutions was framed by two questionnaires aimed at encouraging self-reflection (completed before and after the activity). The pupils were also asked to pose a new problem with a similar structure to each of the original problems. The results from the different data collection instruments are mutually congruent, from which we can conclude that the methodology is suitable for the design, implementation and evaluation of problem-posing and problem-solving. This methodology can be useful in terms of both research and teaching itself.

Keywords Problem-solving • Problem-posing • Metacognitive questioning

Introduction

Problem-solving has for some time occupied a prominent position in mathematics education research and the mathematics curriculum (Törner, Schoenfeld & Reiss, 2007).

In Portugal, where the research presented here was undertaken, problem-solving continues to receive significant attention throughout the education system as can be seen in syllabus content and teacher training, as well as in systems of evaluation, whether via standardised tests or continuous assessment.

By contrast, problem-posing has received less attention in the syllabus than problem-solving, as it is a more recent approach (Brown & Walter, 1983; Kilpatrick, 1987, cited in Cai & Hwang, 2002) and represents an emergent area of research as well as a significant new tool for teaching (Singer, Ellerton & Cai, 2013).

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Cruz (2003) found that in a study of students aged 12 and 14, the latter were not always the best able to mobilise resources or to apply heuristic and checking techniques. On the other hand, those with the best grades in mathematics (irrespective of age) showed better results in problem-solving in terms of the categories considered in the study (Cruz & Carrillo, 2004). The awarding of grades followed the *Principles and Standards for School Mathematics* (NCTM, 2000), which are consistent with the Portuguese assessment guidelines, both focusing on problem-solving, reasoning and proof, making connections, oral and written communication and uses of mathematical representation.

In the light of this, further studies into how students tackle problem-solving activities would be valuable, in particular through observation of the behaviour of those who habitually achieve the best grades. The decision to select students with good school grades draws on studies to be found in Lesh and Zawojewski (2007). These studies show that such students have at their disposal better structuring of ideas, a wider range of strategies and more representations and are more adept in creating an image of the problem. Consideration of the problems formulated by such students could provide insights into the links between problem-solving and problem-posing and suggest directions that could be followed in teaching in terms of characterising what makes a good problem-solver.

Problem-posing, included in the *Standards* since 1989 as an activity of value to mathematics teaching (NCTM, 1998, p. 163), can be regarded as a task which, by virtue of its design, allows student solutions to be evaluated in terms of their quality (Kilpatrick, 1987, cited in Cai & Hwang, 2003; Goldenberg & Walter, 2006). Cai and Nie (2007) consider the teaching of problem-solving in China and describe three types of tasks, one of which requires students to pose new problems modelled on an original (cf. also Cai et al.'s contribution in this book). The researchers found that such tasks helped students to make connections and make sense of mathematics. Kontorovich and Koichu (2009) suggest a framework for characterising problem-posing [PP]. Amongst the four aspects that these researchers consider are resources, in which the stimulus for PP is regarded as essential. One means of furnishing this stimulus is through an original problem which serves as the basis for formulating a new problem, and it is this option which is followed in this study. In this regard, we consider whether (metacognitive) reflection and problem-posing help to bring students to a clearer understanding of the problem to be solved. This question can be broken down into two related questions:

(a) Does reflection enhance pupils' awareness of the structure of the problem to be solved? (The structure of the problem can be understood as the configuration of relationships, concepts, procedures and degree of difficulty. Several structures could be linked to the same problem, corresponding to different solutions or approaches to the problem.)
(b) Do the problems devised by the pupils benefit from the prior solution of similarly structured problems and vice versa?

Method

Participants

The study focussed on two subjects, both 14 years old, chosen from a group of 27 pupils according to two criteria: good academic grades in mathematics and a positive attitude towards mathematics and problem-solving.

Design

A questionnaire for identifying mathematical beliefs and attitudes to problem-solving (Villa, 2001) was completed by the group of 27 students. The aim of the questionnaire was to identify students who were both academically successful in mathematics and had a positive attitude to problem-solving. Three students obtained scores (academic results and mathematical beliefs) significantly above their classmates, suggesting a favourable disposition towards problem-solving. These students were doing exceptionally well in mathematics and so were ideally suited to participating in the study. This is consistent with Schoenfeld's (1985, 1992) model, in which a good problem-solving profile includes appropriate resources, strategies, control and a favourable system of beliefs and affects. Of these four dimensions, the latter was the least likely to be guaranteed by purely good academic results, for which reason the questionnaire for identifying mathematical beliefs was employed.

Before and after the solution of each problem, the students completed a questionnaire (pre-PS and post-PS) specifically designed to gather data on the students' understanding of the problems. The questionnaires played a significant role in encouraging students to question their own reasoning and procedures, a process which according to Flavell (1976) can be described as underpinning the capacity for metacognitive reflection. Nevertheless, it should be borne in mind that the study concerned students with good academic grades, in the expectation of finding indicators within the data of their ability to achieve the structure of the problems (considered beneficial to finding solutions).

After solving each problem, the students were then invited to pose a new problem which could be solved using the same method. The analysis of these posed problems drew on two instruments (Cai & Hwang, 2003 and Leung, 1997), whilst the analysis of the solutions themselves followed Carrillo (1998).

The present article is concerned with the information provided by the students via protocol sheets completed during the process of solving the two problems. The students also answered pre-PS and post-PS questionnaires and attempted to pose new problems with the same structure as the original.

Instruments

A Set of 12 Problems, of Which This Paper Deals with Just Two

The first problem consisted of two questions, the first not too difficult and the second somewhat more demanding. In order to answer the questions, it was not necessary to use an algebraic model of the two contracts (see below), although the recognition that such a structure was implicit in the problem was important for posing a new problem along the same lines.

P2

Based on the advertisement, which type of contract would be preferable:

(a) For 2 years' employment
(b) For 9 years' employment

Propose a problem which could be solved using a similar method. (It is not necessary to provide a solution.)

International company seeks engineer
REQUIREMENTS:
- Degree in Chemical Engineering
- Age up to 35
- Good knowledge of English

CONDITIONS:
Contract A
- Annual salary in 1st year of € 25,000.
- Annual salary increment of € 3,000.

Contract B
- half-yearly salary in 1st 6 month period of € 10,000.
- half-yearly increment of € 1,000.

Send *CV* to situations vacant n° 1251 in this magazine.

The reason for choosing this problem was its algebraic structure, which the students had to demonstrate they understood if they were to propose a similar problem.

P8 (selected from GAVE, 2004)

Table 1 Questionnaire prior to problem solving

QUESTIONNAIRE PRIOR TO PROBLEM SOLVING:

After carefully reading the first problem, answer the questions below:

1. How far do you think you have understood this problem: (Tick the appropriate box)

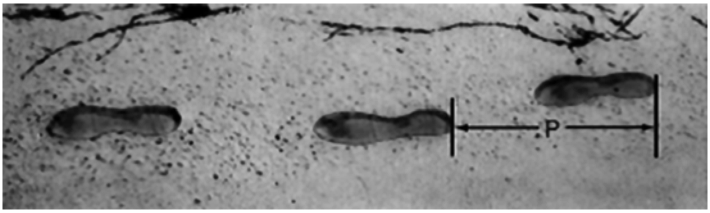
Fully ☐ The main points ☐ A little ☐ Not at all ☐

2. How do you think you can solve this problem?

3. Do you think you are lacking anything at the moment to solve the problem? What?

4. Where do you think the difficulty of this problem lies?

Now try to solve the problem.



The figure on the right shows a man's footprints. The length of the stride (pace-length), P, is the distance between the heel marks of two consecutive footprints.

For men, the formula $\frac{n}{P} = 140$ establishes an approximate relation between n and P, in which

n = the number of steps per minute and

P = length of stride in metres.

– If this formula is applied to Pedro's stride and he takes 70 steps per minute, what is Pedro's stride length?
– Bernardo knows that the length of his stride is 0.80 m. Apply the formula to Bernardo's walking. Calculate the speed he walks in metres per minute and kilometres per hour.
– After you have solved this problem, formulate a problem which can be solved in the same way.

A problem from the PISA test was selected in order to enable comparison between PISA and the Portuguese syllabus in terms of problem-solving. The larger study, from which this paper is drawn, is currently scrutinising results for possible interpretations with regard to PISA performance. This problem was of particular interest for its algebraic structure, which the students had to demonstrate they understood if they were to propose a similar problem (Table 1-3).

Table 2 Questionnaire subsequent to problem solving

QUESTIONNAIRE SUBSEQUENT TO PROBLEM SOLVING:
1.How far do you think you managed to solve the problem: (Tick the appropriate box)
Completely ☐ Not very well ☐
The main elements ☐ Didn't solve or the answer was unsatisfactory ☐
2.Did you solve the problem as you initially expected to? If not, what changed your expectations?
3.What main difficulties did you find while solving the problem?
Thank you for participating.

Pre-PS and Post-PS Questionnaires About the Processes of Problem-Solving and Problem-Posing

Instruments for Processing/Analysing the Information

The data provided by the pre- and post-questionnaires was processed using an adapted version of Efklides' (2006) analysis (adapted) for classifying metacognitive knowledge and experiences. Efklides' concept of metacognition, based on Flavell (1979), sees it as knowledge acting upon an objective world (the task) at a metalevel through monitoring and checking. She proposes, in a summary table, three characterising features (metacognitive knowledge, metacognitive experience and metacognitive competencies/skills), along with their manifestations, grouped under monitoring and checking, although she recognises certain difficulties in distinguishing these (Efklides, 2006, p. 4). This instrument, however, was not applied across the full range of features, as the design of the pre- and post-PS questionnaires did not intend to supply such comprehensive data. Rather, the questionnaires aimed to capture the structure of the problem which each student managed to construct after carefully reading the rubric. It was to meet this requirement that the instrument needed to be adapted (Table 2).

This adaptation consisted in omitting the category of *metacognitive competencies*, in which Efklides included procedural knowledge, that is, actions for controlling cognition. As these take place in action, specifically the context of solving the proposed task, it was possible to analyse them separately via the students' solution protocols using other instruments. Finally, it should be noted that the post-PS questionnaire was not designed to evaluate how the task has been carried out but simply to measure the degree to which the original ideas about the problem had been carried through. The classification instrument—omitting metacognitive competencies—is set out in the Appendix (Table 4).

The solutions themselves were analysed using a slightly adapted version of Carrillo's (1998) scheme, omitting the category concerning the personal characteristics of the solver and giving prominence instead to the tactical features of the process. This scheme, set out below, consists of five analytical dimensions, each with five levels of acquisition (for an example of these levels, see Table 5 in the Appendix).

For dealing with the reformulation of problems, we drew on the instruments for classifying *problem-posing* offered by Cai and Hwang (2003) and Leung (1997). Cai and Hwang classify the formulation of the new problem according to its similarity with the original problem and its structure. Leung, on the other hand, focuses on the plausibility of the new problem in terms of the quality of information included in the reformulation.

According to Cai and Hwang (2003), a newly formulated problem can be classified as 'extensive', 'not extensive' or 'other'. If extensive (E), it follows the structure of the original problem but is more demanding in terms of the mathematical work required to solve it. If it is not extensive (NE), it fully patterns the structure of the original problem and maintains the same level of difficulty, and if other (O), it fails to follow the structure of the original problem. Leung, on the other hand, classifies the problems as follows: not a problem, that is, the suggested situation is descriptive only and fails to ask a question that can be answered; non-mathematical problem, in which the question posed falls outside the scope of mathematics; implausible mathematical problem, according to which the problem falls within the scope of mathematics but the data involved or the solution do not make sense in the context; insufficient plausible mathematical problem, whereby the problem can receive a mathematical treatment but the data involved are insufficient to arrive at a solution; and sufficient plausible mathematical problem, in other words, a well-formulated problem that can be solved.

Employing these codes, we propose the following categorisation:

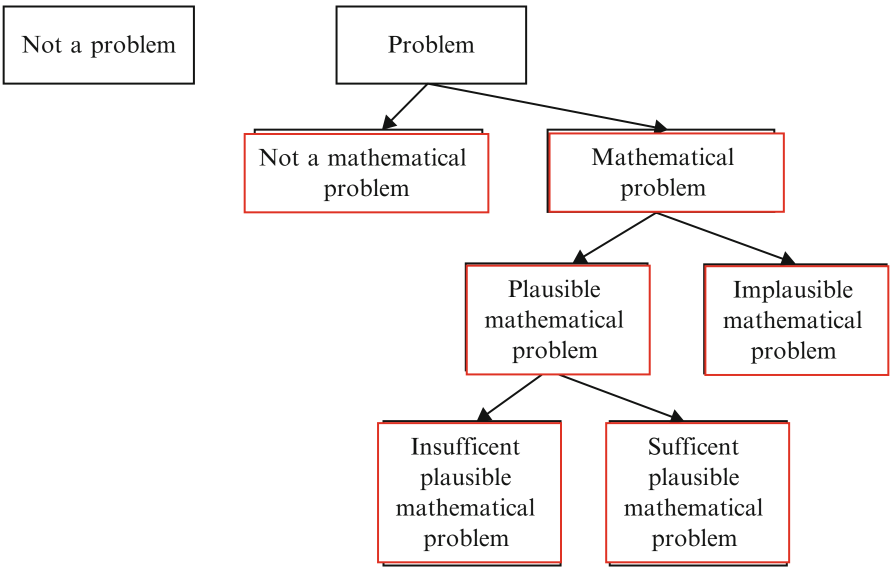


Table 3 Tactical characteristics of the process (efficacy of the action)

2.	Tactical characteristics of the process (efficacy of the action)
2.1.	Obtaining a meaningful representation (including obtaining the structure of the problem)
2.2.	Efficacy and adequacy of the planning (with data possibly provided by the use of intuition)
2.3.	Efficacy and adequacy of the execution (with data possibly provided by the use of generalisation of objects, relations and calculations and by reasoning with mathematical symbols and spatial relations)
2.4.	Efficacy in the use of revision
2.5.	Quality of the final version of the solution (including clarity, simplicity and economy, as well as reasonableness of the solution and rationale for the process)

To summarise, the students' answers to the pre- and post-questionnaires were considered and coded according to Efklides' (2006) system of analysis (Table 4, Appendix). Various relevant information units (student responses) have been included in order to substantiate the commentaries and coding (U for unit and P for problem). The students' progress in solving the problems were analysed using Carrillo's (1998) instrument (Table 3), again supported with illustrative samples of the students' solutions, accompanied by appropriate excerpts from the descriptors of the five analytical dimensions. Finally, the analysis of the new problems posed by the students was effected using the instruments developed by Cai and Hwang (2003) and Leung (1997). Some of the rubric from the problems has been included.

Analysis and Results

This section presents the analysis of the data provided by the students Clara and Rafael.

Clara

Her level of achievement is reflected in the pre-PS questionnaire, where she shows knowledge of the type of problem (task) and the demands of the task (explanation of the abbreviations see above):

> I think I can solve the problem calculating the amount I would receive in the four situations. (U1P2)

She also knows which strategy to use (strategy) and seems to have the appropriate mathematical knowledge (knowledge) to find a solution:

> calculating the values of the right unknown, substituting certain values in the formula (U1P8)

and estimates where the greatest difficulty might lie (difficulty):

but the fact that the calculations don't involve any unknowns, just specific numbers, means it shouldn't cause any difficulties. (U2P2)

Likewise, in the post-PS questionnaire, she expresses recognition of some difficulty in the task (difficulty):

the main difficulty in the solution… was the number of values and their size. (U3P2)

Her solutions to the two problems are almost uniformly situated at the highest level descriptors of the instrument used.

In problem 2, she shows a good ability to achieve a meaningful representation, so that the descriptor corresponding top level 5 ('the solver understands the structure of the problem perfectly and usually retrieves the mathematics underlying the data in the problem statement') best describes the work she produces:

'Contract A: $25000+(25000+3000)=53000$'.

In problem 8, the implementation of her solution is effective and appropriate, corresponding to level 5 of the descriptor ('execution is consistent with planning and is effective in contributing key results towards the overall solution') best describes the work she produces, in particular, when she converts 89.6 m/min to 5.376 km/h:

$$\text{“}112\times 0.80 = 89.6 \text{ m/min}$$
$$(112\times 60)\times\left(\frac{0.80}{1000}\right)=\ldots\text{”.}$$

The new problems she poses represent sufficient plausible mathematical problems, with an identical structure to the original problems, as required.

Her awareness of the difficulties shown in the questionnaires can be seen in the problem she poses based on problem 2, as is her confidence in the solution. On the one hand, she reduces the amounts involved, but, on the other hand, because she perfectly understands the structure of problem 2, she is sufficiently confident to change the amount to be added in the original to an amount to be subtracted in the new problem. This is the problem she poses:

'Manuel decided to buy a car on special offer, making payments of €350/month in the first year, after which, every year the payments are reduced by €25. How many years would it take to pay for a €20,000 car, assuming that the price includes interest?'

In the case of problem 8, she added a question, which represents an extension of what was originally required.

In summary, the data from the pre-PS and post-PS questionnaires coincided in revealing a high level of metacognitive knowledge in terms of the indicators for (mathematical) knowledge and (estimate of) difficulty. The task is tackled coherently throughout, without any difficulties arising, and correct solutions arrived at. The posed problem is completely consistent with the original, having an identical structure and even adding an extension.

Rafael

In the pre-PS questionnaire, he shows he understands the task (task):

> I put in what I earn by the end of the year with contract A (already done) and then I put in what I earn by the end of the year with contract B. (U1P2)
>
> Substituting the letters with numbers. (U1P8)

He shows uncertainty as to how to proceed during the task (strategy):

> I think what can make this problem difficult is finding a way to solve it. (U2P2)

and insecurity as to his chances of success (confidence).

In the post-PS questionnaire, he shows himself uncertain of his solution (validity):

> Check that the money was always added to what he had earned. (U3P2)

The solution to problem 2 reveals a good mental representation, and consistent planning and execution, but also reveals difficulties when it comes to reviewing his work, in that he fails to notice an error in calculation which accumulates over the course of working out the final total. As he adds up the half-yearly salary increments in contract B, he misses out the 16th payment and so miscalculates the total, which should be 333,000€ instead of 306,000€:

"…	13º	14º	15º	17º	18º		"
	22000	23000	24000	25000	26000	= 306000	

His work would benefit with greater attention given to the final two items of the instrument (2.4. Efficacy in the use of review strategies and 2.5. Quality of the final version of the solution), given that, as the questionnaires show, he felt little confidence in the validity of his work.

His solution to problem 8 also shows difficulties in the last two levels of action described by the instrument (2.4 and 2.5). He again fails to spot an error and does not attempt to find a solution through other means, despite having also given signs of a lack of confidence and uncertainty in his answers to the questionnaires.

With respect to problem 2, he is unable to reformulate it and presents an insufficient plausible mathematical problem, the structure of which fails to mirror that of the original. He proposes the following problem:

'Make a formula for each contract'.

This formulation, which goes little way to meet the requirements of the task, clearly shows that the difficulties revealed in the questionnaires and in solving this problem were real. The student is unable to extract the mathematical structure of the problem, which would enable him to pose another with a comparable structure, although he is aware that there is one.

In reformulating problem 8, he presents a sufficient plausible mathematical problem, but fails to preserve the mathematical structure of the original. The original problem concerns a direct proportional relation, whilst the reformulation is based on inverse proportionality.

In summary, the pre-PS and post-PS questionnaires reveal metacognitive knowledge in terms of the indicators for task, strategy, confidence and validity. The student's working through of the task is consistent with the data supplied by the questionnaires, as revealed by his inability to overcome perceived difficulties, and the solutions to the problems themselves. The posed problems provide further confirmation, as they lack any content or mathematical structure (in the first instance) and any mathematical structure (in the second).

Conclusions

The use of a questionnaire prior to attempting the problem strengthens the students' questioning of aspects of the problem and the resources and strategies to be employed. It also provides data on their self-awareness and decision-making capacity. Unfortunately, although in the global study questionnaires were used with some problems and not with others, it was not possible to draw a comparison of the same problem with and without a questionnaire, which could have provided further information of interest.

The data revealed by the pre-PS questionnaire, with regard to awareness of task typology, strategies, type of knowledge involved and identification of difficulties (or not), are consistent with the students' actual solutions to the problems and with the new problems they posed based on the original tasks. We acknowledge that this correlation represents what was to be expected, but it is worth noting as it indicates the methodological consistency of the following scheme for fieldwork we would like to propose: (a) pre- and post-PS questionnaires, (b) task of solving a problem and (c) posing a problem with the same structure as the original. It also indicates that the instruments used in the analysis produced consistent data.

In the case where the problem-solving task was not fully completed, the post-PS questionnaire indicated less knowledge regarding the student's own capabilities, little confidence in the validity of the revealed knowledge and a sense of difficulty (which might vary during the task). In the case where the problem-posing task was not successfully accomplished, the proposed problems fell short of what was desired, either because the data were insufficient (Rafael's reformulation of problem 2) or because its structure did not pattern the structure of the original problem (Rafael's reformulation of problems 2 and 8).

The case of Clara shows an evident relation between the results of the questionnaires (knowledge of task, strategy, mathematical content) and her grasp of the tactical aspects of the process of problem-solving. The problems she posed respected the mathematical structure of the original problems and even introduced small changes and extensions which indicated a total understanding of the structure of the problems which were given to her to solve and reformulate. In this case problem-posing represents a good indicator of the PS process. At the same time, it promotes an amplification in the understanding of this process.

If we consider the problems posed by the students, it becomes evident that, when they respected the structure of the original problems (implying the creation of a sufficient plausible mathematical problem), this meant an understanding of the structure of the original problem, which had a fairly complete solution in the tactical aspects of the process (see the case of Clara). The opposite of this affirmation is also true (see the case of Rafael).

The results of this study lead us to conjecture that:

– The use of pre- and post-PS questionnaires improves problem-solving (presumably because it urges questioning), and problem-solving, in its turn, benefits problem-posing.
– The formulation of problems by the students (with appropriate data and structure) indicates the use of more meaningful representations, gaining access to the structure of the problem.

Further studies into how this method could improve the problem-solving abilities of lower achievers would also be valuable.

Appendix

Table 4

Efklides (2006)	Meaning as understood in this study
1. Metacognitive knowledge	Declared knowledge of cognition, deriving from long-term memory
Ideas, beliefs, 'theories'	May be explicit or implicit
1.1. Person/self	Concerns oneself and one's possibilities (related to 1.2, 1.3 and 1.4)
1.2. Task	Concerns categories (classes) of tasks and the means of solving them
1.3. Strategies	Concerns general ways (modes) for acting (may be heuristic)
1.4. Goals	Concerns objectives or the type of solution
1.5. Cognitive functions	Concerning memory or thinking (what they are and how they act), attention
1.6. Validity of knowledge	Concerns epistemological knowledge, quality of knowledge
2. Metacognitive experience	Involves aspects of the mobilisation of knowledge
2.1. Feelings	Considered as products of monitoring (good functioning)
2.1.1. Familiarity	Denotes previous occurrence of a stimulus (frequency) and fluency in the mode of action
2.1.2. Difficulty	Results from complexity of task, context, personal characteristics (cognitive), self-image, affective factors, extrinsic feedback such as positive or negative sensations. May vary during task, may be illusory
2.1.3. Knowing	Concerns appropriate mathematical knowledge for task
2.1.4. Confidence	Derives from previous experience, hesitation vs. overconfidence
2.1.5. Satisfaction	Monitors the personal criteria and standards by which the quality of response is judged
2.2. Judgements/estimates	
2.2.1. Estimate of effort	Monitors work to be carried out, related to difficulty
2.2.2. Estimate of time	Monitors time taken, related to difficulty

Table 5

2.1. Obtaining a meaningful representation
1. The solver never obtains a representation for the problem and does not understand the given situation, which is totally unfamiliar to him or her. Because the solver does not understand the structure of the problem, he or she is unable to articulate the reasoning by which the problem is introduced (sometimes the solver does not even state it)
2. The structure of the problem is occasionally understood, usually imperfectly; that is to say, he or she is able to express in his or her own words some, but not all, elements of the problem; abstract reasoning is not used
3. Understanding of the problem extends to all or most elements, though not in depth. The basic structure of the problem is usually understood, though sometimes imperfectly; both concrete and abstract reasoning are used
4. Largely understands the problem, although there may be an element which is not understood. The structure of the problem may be understood, but the posing of a new, similar problem causes difficulty
5. The solver obtains a highly meaningful representation of the situation, allowing a successful planning process to begin, after formulating the problem in his or her own terms. Therefore, the solver may understand the structure of the problem perfectly and usually retrieves the mathematics underlying the data in the problem statement. The solver abstracts, starting with concrete relations and moving towards formal structures

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Affective Factors and Beliefs About Mathematics of Young Chilean Children: Understanding Cultural Characteristics

Valentina Giaconi, María Leonor Varas, Laura Tuohilampi, and Markku Hannula

Abstract Understanding the affective factors and beliefs of young students about mathematics is a complex task. This is especially important in the framework of problem solving where these kinds of beliefs are related to various learning processes and influence achievement. In this work, we present the analysis of a questionnaire regarding beliefs about self-competence, self-confidence, mastery goal orientation, effort, difficulty of mathematics, and enjoyment of mathematics applied to Chilean third graders. Exploratory factor analysis leads us to the conclusion that it is possible to measure these kinds of beliefs with a Likert-type questionnaire and that there is an inverse item effect. We tested two confirmatory factor analysis models that allowed us to understand the behavior of inverse items in relation to the mathematics-related affect traits. These models suggest that the inverse item effect is a response style of Chilean children and the affect structure is consistent with the theoretical one considering this effect.

Keywords Students' mathematics-related beliefs • Inverse items • Method effect • Response style

Introduction

Affective components have a significant role in the learning of mathematics and problem solving (Hannula, 2011; Leder, 2006; Op' t Eynde, de Corte, & Verschaffel, 2002). For instance, in the problem solving area, there are studies that show the

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relation between beliefs and problem solving. Some examples are the positive relation between self-efficacy beliefs about problem solving and accuracy, response time, and efficiency in solving addition and multiplication problems (Hoffman, 2010; Hoffman & Schraw, 2009; Hoffman & Spatariu, 2008) or the positive relation between self-efficacy and problem solving, controlled by mental ability (Pajares & Kranzler, 1995). Nevertheless, the measurement of the affective components in different cultures and age groups poses challenging issues (Tuohilampi et al., 2014).

In this article we present an analysis of the structure of mathematics-related beliefs of very young Chilean children. These children were starting third grade and belonged to 14 class groups in nine schools from Santiago. The data came from a questionnaire which originated in Finland and was adapted for use in a bilateral Chilean-Finnish project. This instrument presented risks for two reasons, the cultural differences and the age of the students, both of which could be a pitfall for understanding the items. However we preferred to use this instrument with its known properties that would allow us to make comparisons, in particular with the Finnish children participating in the Chilean-Finnish project. This information about the instrument and its application in other populations is an important referent that gives us tools to understand what is different from the expected results. In a prior work (Tuohilampi et al., 2014), we presented the results of this application to a sample of Chilean and Finnish students who were starting third grade. Through an exploratory factor analysis, the congruence of the obtained results with the theoretical model corresponding to the questionnaire was examined.

In both countries most of the obtained factors were distinguishable and their contents made sense, allowing for their characterization. But there was an exception in the Chilean results, a characteristic not present in Finnish solutions: in the Chilean results, the inversely and directly formulated items tended to load in different factors. This was the most remarkable difference between the structures in the two countries. There could be various interpretations for this behavior. One possible explanation is that the Chilean pupils had greater difficulty understanding inverse statements. This is in line with the argument of Metsämuuronen (2012) and other authors who propose that the functioning of inverse items is related to higher difficulty in comprehension. Another possible interpretation is a cultural characteristic regarding language (Tuohilampi et al., 2014) that could be present in the results as a response style.

The purpose of the research presented in this paper is to find a model that fits to the Chilean data and that therefore represents the mathematics-related affects and belief structure of the participating third graders and which explains the behavior that differs from the expected results, particularly in the case of inverse items.

Theoretical Framework

Mathematics-related affect has been studied using a variety of frameworks. In our study we are interested in the relatively stable affective traits of the individual. The conceptualization by Hannula (2012) identifies three dimensions of mathematics-related affect, which we will refer to in regard to our study. In the first dimension, our study focuses on affective traits rather than dynamically changing states. In the second dimension, we draw from psychological theories rather than embodied or social theories. In the third dimension, our study aims to cover all three aspects of affect: emotional, cognitive, and motivational traits. In the emotional dimension, we look at student enjoyment of mathematics (EOM). In the cognitive dimension, we are interested in students' self-efficacy, self-confidence, and perceived difficulty of mathematics (DOM). Lastly, in the motivational dimension, we are interested in students' effort and mastery goal orientation (MGO). We are aware that these dimensions tend to be correlated with each other (e.g., Roesken, Hannula, & Pehkonen, 2011). However, it is still an open question as to how universal these dimensions are and how strongly they correlate in different populations.

We are concerned with the methodological issue of how to determine the relations between beliefs. The exploration of these relations can be done with exploratory and confirmatory factor analysis (EFA and CFA, respectively). These methods explain the correlation matrix of the items and propose a linear relation between the items and factors or latent variables. The relation is determined by the matrix of loadings. In EFA, this matrix is estimated, and its analysis allows us to know the relation between the items and the factors and later to identify a factor with a theoretical variable or construct. In CFA, the form of the matrix of loadings and the correlations between the factors are proposed by the researcher and make it possible to test theoretical hypotheses. Both methods are complementary tools, because the information given by exploratory factor analysis of how the items are clustered is done without any input or preconceived ideas from the researcher. This information is powerful evidence that allows us to design and propose hypotheses we later test with CFA.

For the purposes of this article, it is necessary to define direct and inverse items in a mathematics-related affect questionnaire. We define a direct item as one that is generally considered to imply a positive affect (e.g., *I am sure that I can learn math*). In comparison, inverse items are those that imply a negative affect (e.g., *Mathematics is difficult*, or *I am not good at mathematics*). An inverse item is not the same as one with a negation, for example, *I am not unhappy* would be a direct item with a negation. There are different opinions about how adequate it is to use inverse items in questionnaires. It is recommended for avoiding acquiescence bias which is the tendency to agree with all the items regardless of content (Podsakoff, MacKenzie, Lee, & Podsakoff, 2003). Sometimes inverse items are used to better capture a concept. For example, research on anxiety focuses on the negative dimension of mathematics-related affect. On the other hand, there is research that shows that inverse items form scales with lower reliabilities (Chamberlain & Cummings, 1984), have lower discrimination parameters (Sliter & Zickar, 2013), have different distributions, and form artifactual factors (Spector, Van Katwyk, Brannick, & Chen, 1997). There are many hypotheses about the mechanism that makes inverse items form their own factors. Examples of this are respondents' lack of ability to understand negatively worded items and their carelessness in reading the items (Spector et al., 1997, Woods, 2006). Metsämuuronen (2012) found that when students with the lowest level of achievement scores were analyzed separately, inverse items formed their own factor. However, other studies have found that inverse items form their own factors with regular populations, for example, Horan, DiStefano, and Motl (2003), Schriesheim and Eisenbach (1995), and Pilotte and Gable (1990). So, it is not clear that the different functioning of negatively worded items can be explained only by respondents' low ability or understanding of the items. Probably, inverse items are responded differently because of their formulation. Spector et al. (1997) argue that when we measure a construct, people agree with items that are close to their level of the construct and disagree with items that are far away from this point in either direction. This would suggest that even if a person agrees with an item, he or she might not necessarily disagree with the same item inversely worded. That means that an inverse item recoded is not exactly equivalent to a positively worded item. This idea is not contradictory to the fact that direct and inverse items are measuring the same construct; it only says that the pattern of response to inverse items after recoding does not have to be the same as the pattern of response to direct items.

We are not able to test if the inverse items are measuring negative dimensions of affect, because we do not have enough inverse and direct items in any of the traits measured. But we can use CFA models to test if the inverse items present a method effect, which refers to "the influence of a particular method that inflates a correlation among different traits measured with the same method" (Marsh & Grayson, 1995). To do this, we are going to use CFA models proposed in Marsh and Grayson (1995), who suggested a way to separate the effect of the method from the effect of the traits measured. We have two methods: measurement with a direct item and with an inverse item. The effect of each trait is modeled by a factor, and the method effect is modeled as correlations between the unique factors of the items measured with the same method and by method factors, where each method factor loads in all the items measured with the respective method.

If a method effect is found in different traits and it can be modeled by one factor, this can be interpreted as a response style (Horan et al., 2003). According to Bentler, Jackson, and Messick (1971), a response style is "a potentially measurable personality variable or trait" and "a style refers to a behavioral consistency operating across measures of several conceptually distinct content traits." A response style is different from a method artifact, because it is a characteristic of the people who respond to the instrument.

Methodology

The questionnaire for this study is an adaptation of a questionnaire developed originally for Finnish fourth grade students (Hannula & Laakso, 2011). The scales were based on previous instruments (Fennema & Sherman, 1976; Midgley et al., 2000) and a qualitative study (Pietilä, 2002). In order to make responding to the questionnaire simpler, a 3-point Likert scale was chosen. Moreover, the language of some items was simplified to better suit the young age of the respondents. Details of the

Table 1 Example and number of items in each trait measured

	Trait measured	Example of an item	Direct items	Inverse items	Total items
Cognitive	Self-competence (*Competence*)	*I have made it well in mathematics*	1	3	4
	Self-confidence (*Confidence*)	*I am sure that I can learn math*	4	0	4
	Difficulty of mathematics (*DOM*)	*Mathematics is difficult*	1	2	3
Motivational	Mastery goal orientation (*MGO*)	*On every lesson, I try to learn as much as possible*	5	0	5
Behavioral	Effort (*Effort*)	*I always prepare myself carefully for exams*	3	1	4
Emotional	Enjoyment of mathematics (*EOM*)	*I have enjoyed pondering mathematical exercises*	2	3	5
Total number of items in the questionnaire			16	9	25

instrument development are reported in Tuohilampi et al. (2014). The previous studies suggested that the questionnaire would identify the following scales: self-efficacy in mathematics (*Confidence*), self-concept in mathematics (*Competence*), *MGO*, effort (*Effort*), *EOM*, and *DOM*. The items were originally formulated in either English or Finnish, and they were translated into English, Finnish, or Spanish.

The scales of inverse items were recoded to have the same direction as the direct items. The classification and the number of inverse and direct items in each dimension are shown in Table 1. In the items of the Spanish version of the questionnaire, none of the direct items have a negation, and two of the nine inverse items have a negation (both in the competence dimension).

The data used in this paper are part of the baseline of a longitudinal research project on open-ended problem solving, implemented in Finland and Chile. The number of participants was 901 third graders from 14 classes that belonged to nine schools. The data were collected from March to April 2011 (Santiago) which is the beginning of the academic year in Chile. In Chile, there are private, semiprivate, and public schools. The data from Chile were collected from all three types of schools. On the whole, we assume that the data can be considered to be representative at least of urban pupils. Moreover, as the purpose of the present article is to test two alternative hypotheses for the already observed effect of inverse items, even nonrepresentative data would be acceptable.

Statistical Analysis

We used descriptive statistics to analyze the distribution of the items. The statistics used were average, standard deviation, and skewness. The skewness gives information about the asymmetry of the distribution of each item.

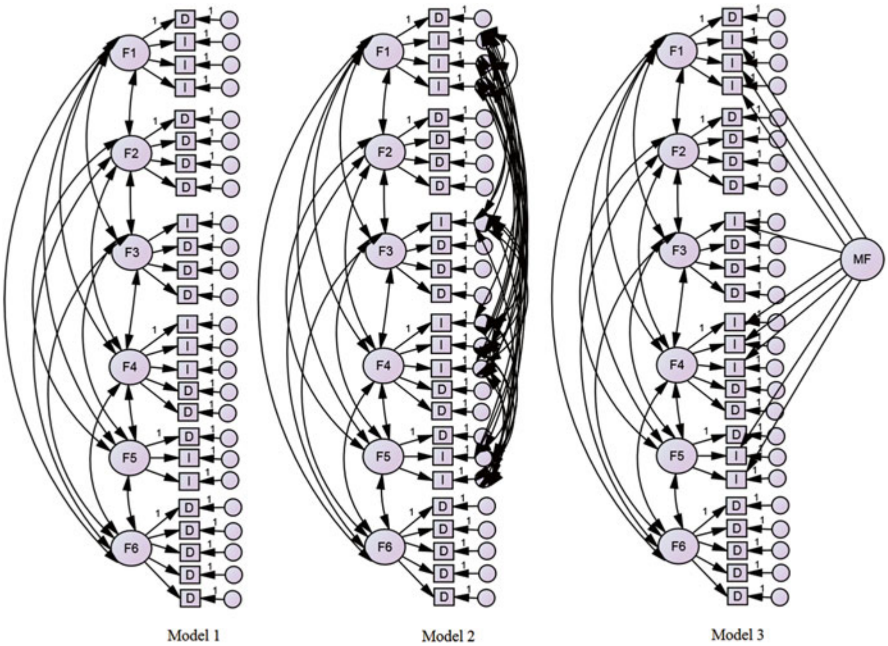

Fig. 1 CFA models estimated. Each rectangle represents an item of the questionnaire, the label I means inverse item, and the label D means direct item. F1, F2, F3, F4, F5, and F6 represent the trait factors for *Competence*, *Confidence*, *Effort*, *EOM*, *DOM*, and *MGO*, respectively. MF represents a method factor. The *empty circles* pointing to items represent unique factors

Tuohilampi et al. (2014) have done an exploratory factor analysis of the data and reported the structure and interpretation of the factors. We show the same analysis here to focus only on the behavior of inverse and direct items. To get a more easily interpretable solution, we used the varimax rotation and the principal component analysis estimation method. The analysis was done over the matrix of correlations because the scale of the items was not important. The criterion for determining the number of factors was the number of eigenvalues greater than 1.

To model the effect of inverse items, we use CFA models. The models estimated are a subset of the models used in Marsh, Scalas, and Nagengast (2010) and Lindwall et al. (2012). To state a baseline model for future comparisons, we employed Model 1 which represents the theoretical model and does not take into account the inverse item issue. In this model each item loads in the factor representing the trait measured (trait factor) and the correlations between trait factors are free parameters estimated by the model (Fig. 1).

To model the method effects, we estimated Model 2 and Model 3. Model 2 explains the method effect as correlations between the unique factors of the inverse items. Model 3 models the method effect as a method factor that loads in all the inverse items. In this research, there are two methods of measurement: direct or inverse. The models used here only test an inverse item effect. It is also possible to model a direct item effect (the models are analogous to those used for inverse items).

There are also models that account simultaneously for inverse and direct item effects. Examples of models that account for an inverse item effect, a direct item effect, and both effects can be found in Lindwall et al. (2012) and Marsh et al. (2010). Models that consider direct item effects were also estimated but showed worse fit or were not identified.

Another possibility to account for the inverse item effect would have been to test the hypothesis of a negative and a positive vision of affect in each trait. However, this method would require more direct and inverse items in each dimension than we had.

Descriptive statistics, t-test, and exploratory factor analysis were estimated with the software SPSS 21. The CFA models were estimated with the software AMOS 21. Since the items are ordinal variables which take three possible values, we could not assume that they are normally distributed. That is why we used bootstrapping with the maximum likelihood (ML) option to estimate the model. ML with non-normal variables gives consistent and unbiased estimates, and the bootstrap allows us to do statistical testing without distribution assumptions in the data (West, Finch, & Curran, 1995). AMOS 21 is not able to do bootstrapping with missing data, so the CFA models were estimated with the data of $N = 578$ students that have complete responses. The method used to make inferences about the parameters of the model was bias-corrected percentile method, and the bootstrap took 500 samples.

In confirmatory factor analysis, there are several fit indexes. We choose the χ^2 statistics and also the GFI, AGFI, and RMSEA which give information about the absolute fit of the model to the data. CFI gives information about how good the model is in comparison to a baseline model; AIC and BIC are fit indexes that include not only the fit of the model but also the number of parameters and permit comparison between non-nested models (Byrne, 2010, Hu & Bentler, 1995).

A value of RMSEA lower than 0.05 is considered a good fit, between 0.05 and 0.08 is considered an acceptable fit. In the case of CFI, a value higher than 0.97 is interpreted as good fit and between 0.95 and 0.97 an acceptable fit. For GFI and AGFI, a value higher than 0.95 means a good fit and between 0.90 and 0.95 an acceptable fit. Finally for the χ^2/d.f. index, a value lower than 2 shows a good fit and between 2 and 3 shows an acceptable fit (Schermelleh-Engel, Moosbrugger, & Müller, 2003). AIC and BIC are useful for making comparisons between models. In both indexes a lower value implies a better fit.

We used different kinds of fit indexes because each one emphasizes a different aspect of good fitting. Also, to use all of them gives us a good perspective about the fit of the model with the data.

Results

For each item descriptive statistics were estimated (average, standard deviation and skewness). We separated the items in two groups: direct items (N=16) and inverse items (N=9) and then compared the items statistics in each group. Table 2 shows that the inverse items and the direct items were statistically significantly different in

Table 2 Average, standard deviation, and t-test of different item statistics for direct and inverse items

	Average of item statistics		Standard deviation of items statistics		t-test for comparing item statistics between direct items and inverse items		
Statistic	Direct items	Inverse items	Direct items	Inverse items	t	d.f.	*p*-value
Item average	2.60	2.24	0.17	0.25	4.27	23	.00
Item standard deviation	0.56	0.74	0.07	0.03	−7.39	23	.00
Item skewness	−1.33	−.49	0.81	0.53	−2.76	23	.01

all the descriptive item statistics (average, standard deviation, skewness). This does not mean that the items measure different constructs or traits, but shows that something worked differently with these items.

The EFA analyzed here is the same as in Tuohilampi et al. (2014). Tuohilampi and colleagues analyzed two solutions of the Chilean data, a 3-factor and a 6-factor solution. In the 3-factor solution, the inverse items formed their own factor. We present in Table 3 the matrix of loadings of the 6-factor solution. The labels of the factors correspond to the analysis of the factor structure made in Tuohilampi et al. (2014). In the present paper, we are going to focus only on the different behavior of inverse and direct items.

The dimensions *Confidence* and *MGO* do not have inverse items, so we are not going to analyze them here. We are going to focus on *Competence*, *EOM*, *Effort*, and *DOM*.

The dimension *Effort* only has one inverse item, and this item loads in the *Effort* factor, but also has a high loading in the *EOM* factor. The dimension *DOM* has two inverse items and one direct item, and the behavior of their items does not look related with an inverse item effect. Finally the dimensions *Competence* and *EOM* have three inverse items each. The inverse items of both these dimensions loaded together and separately from direct items.

The behavior of *Competence* and *EOM* items supports the hypothesis that the inverse items were understood by the students because without students' understanding them, we probably would not have the inverse items of a dimension loading in the same factor.

Also the tendency to have inverse items loading in different factors from direct items shows that there is an inverse item effect, and that this effect is over different dimensions.

Having the evidence from the exploratory factor analysis of an inverse item effect, we proceeded to model this with confirmatory factor analysis (CFA) to understand this effect. We cannot expect a good fit from Model 1, because we know that there is an inverse item effect. We can see the fit in Table 4, where the fit indexes of Model 1 show a fit acceptable or bad. The two models that account for the method effect are significantly better than Model 1 because the decrease of the χ^2 statistics is significant. It is important to remark that they are better taking into

Table 3 Factor loadings for a varimax rotation of an exploratory factor analysis

Item	EOM	MGO	Confidence	Easiness and fun	Competence (inverse)	Effort
1.d Effort: hardworking	−.004	−.040	.285	.076	−.059	**.698**
5.d Effort: preparing carefully for exams	.008	.287	.024	.218	.031	**.607**
14.d Effort: much working	.059	.410	.022	.142	.004	**.500**
19.i Effort: working too little	**.446**	.011	−.030	−.172	.312	**.458**
6.i Competence: not that good	.204	.092	.076	−.009	**.706**	−.010
8.d Competence: have made it well	−.041	.171	.136	**.586**	.249	.152
9.i Competence: not the type who can	.057	.031	.062	.116	**.748**	−.022
16.i Competence: weakest subject	.374	.002	.005	.199	**.495**	.094
3.d EOM: enjoy pondering	.180	.313	.094	**.431**	−.093	.209
15.d EOM: pleasant to calculate	.193	.195	.038	**.591**	−.023	−.073
18.i EOM: has been something of a core	**.705**	.075	.020	.104	.200	−.115
22.i EOM: boring to study	**.772**	.192	.048	.106	.059	.138
23.i EOM: mechanical and boring subject	**.743**	.117	.102	.161	.083	.034
17.d DOM: easy	.106	−.051	.196	**.652**	.087	.183
20.i DOM: laborious	.281	−.201	.087	**.371**	.086	−.011
24.i DOM: difficult	**.434**	−.066	.170	.314	.275	.108
4.d Confidence: can get good grade	−.097	.160	**.585**	.209	.204	.158
11.d Confidence: can succeed	.081	.062	**.701**	.134	.091	.164
13.d Confidence: would handle more difficult	.051	.318	**.631**	.039	.010	.089
21.d Confidence: confident that can learn	.256	.222	**.538**	.136	−.076	−.088
2.d MGO: want to learn lots of new things this year	−.115	**.681**	−.016	.223	.109	.069
7.d MGO: want to understand perfectly all the tasks	.053	**.549**	.196	.050	.104	.153
10.d MGO: try to understand as much as possible	.086	**.518**	.164	−.017	.094	.258
12.d MGO: intend to develop mathematics skills	.166	**.615**	.252	−.039	−.010	−.061
25.d MGO: intend to learn lots of new mathematics skills	.311	**.533**	.261	.053	−.190	−.020

Inverse items are labeled with the form XX.i and direct items are labeled XX.d
The total variance explained by the factors is 10.5, 9.6, 7.7, 7.5, 7.0, and 6.7 % for the *EOM*, *MGO*, *Confidence*, *easiness and fun*, *Competence* (inverse), and *Effort* factors, respectively

Table 4 Model fit indexes

	Number of parameters	χ^2	d.f.	χ^2/d.f.	GFI	AGFI	CFI	RMSEA	AIC	BIC
Model 1	65	661.7**	260	2.55[a]	.90[a]	.88[a]	.85	.052[a]	791.66	1075.02
Model 2	101	330.6**	224	1.48[g]	.96[g]	.94[g]	.96[g]	.029[g]	532.63	972.95
Model 3	74	399.2**	251	1.59[g]	.95[g]	.93[g]	.94[g]	.032[g]	547.23	869.84

The indexes with [g] indicate a good fit and indexes with [a] indicate an acceptable fit
**Indicates p-value < 0.01

account their greater number of parameters because Model 1 has higher values of AIC and BIC indexes than Models 2 and 3, which means that Model 1 has a worse fit. This supports the hypothesis of an inverse item effect.

The models that account for the method effect present a good fit for the data (Table 4). So we were able to describe the effect of inverse items with two models. Regarding the comparative fit indexes, AIC favors Model 2 and BIC favors Model 3, so they do not help to make a choice between these models, only to inform us that they are better than Model 1. Therefore, considering that both models have a good fit, we chose the simpler one, Model 3. This model is much more parsimonious, and even though mathematically it is not nested in Model 2, theoretically, if Model 3 explains the data well, this implies that Model 2 is also going to do it. In Model 3, we are explaining with one factor the effect of inverse items in four different traits. This model proposes that the effect of the method is one-dimensional. In Model 2 the effect of the inverse items could be specific and different in each trait.

Regarding the loadings of the items in the factors, we can see in Table 5 that the loadings that relate items to trait factors are significant in all models. In Model 3, all the loadings from the method factor were significant. Regarding correlations, all the correlations between trait factors were positive and significant in the three models (Table 6). In Model 3 the highest correlation between trait factors was found between *Competence* and *DOM*. This is coherent with the exploratory factor analysis made with Finnish data where *DOM* was merged with the items of *Competence*. The explanation is that the impression of mathematics difficulty in general merges into the impression of own skills in mathematics (Tuohilampi et al., 2014). In Table 6, we can see that the estimated correlation between *Effort*, *Competence*, and *EOM* with the *MGO* and *Confidence* factors was lower in Model 1 than in Models 2 and 3. So, if we ignore the effect of inverse items, the correlations between factors having inverse items with factors having only direct items are underestimated. For space reasons we do not show the correlations between the unique factors of inverse items in Model 2, but as a summary from the 36 estimated correlations, 35 were statistically significant considering p-values lower than 0.05. It is important to remark that mostly all the parameters in the models were significant, this means that parameters are not redundant and that there is an influence of the traits and the methods.

Table 5 Standardized structural coefficients and *p*-values for the three models estimated

	Model 1	Model 2	Model 3
Parameter	Estimate	Estimate	Estimate
8.d ← Competence	.450**	.685**	.655**
9.i ← Competence	.494**	.315**	.390**
16.i ← Competence	.589**	.314**	.380**
6.i ← Competence	.548**	.286**	.372**
21.d ← Confidence	.489**	.488**	.489**
13.d ← Confidence	.574**	.575**	.574**
11.d ← Confidence	.586**	.581**	.582**
4.d ← Confidence	.568**	.573**	.572**
19.i ← Effort	.359**	.297**	.333**
14.d ← Effort	.517**	.546**	.545**
5.d ← Effort	.533**	.547**	.548**
1.d ← Effort	.472**	.489**	.487**
18.i ← EOM	.647**	.281**	.357**
22.i ← EOM	.755**	.430**	.511**
23.i ← EOM	.727**	.399**	.468**
3.d ← EOM	.360**	.572**	.574**
15.d ← EOM	.347**	.436**	.439**
17.d ← DOM	.528**	.682**	.668**
24.i ← DOM	.670**	.513**	.548**
20.i ← DOM	.369**	.263**	.294**
25.d ← MGO	.554**	.561**	.562**
12.d ← MGO	.517**	.511**	.511**
10.d ← MGO	.491**	.485**	.485**
7.d ← MGO	.528**	.528**	.527**
2.d ← MGO	.483**	.487**	.488**
24.i ← MF			.339**
20.i ← MF			.232**
19.i ← MF			.437**
6.i ← MF			.389**
9.i ← MF			.277**
16.i ← MF			.470**
18.i ← MF			.578**
22.i ← MF			.563**
23.i ← MF			.558**

The *p*-values were calculated with bootstrapping
**Indicates p-value < 0.01

Conclusion and Discussion

In mathematics-related affects, inverse statements are very informative and cannot always be replaced by direct items. It is important to use inverse items because of their content, not only as a methodological tool to avoid acquiescence bias.

Table 6 Estimated correlations between latent variables and *p*-values for the three models

	Model 1	Model 2	Model 3
Parameter	Estimate	Estimate	Estimate
EOM ↔ DOM	.670**	.721**	.681**
Competence ↔ MGO	.313**	.434**	.428**
Competence ↔ Confidence	.445**	.627**	.614**
Confidence ↔ EOM	.395**	.669**	.635**
Confidence ↔ Effort	.591**	.583**	.586**
Effort ↔ MGO	.694**	.683**	.682**
Confidence ↔ MGO	.696**	.696**	.696**
EOM ↔ MGO	.420**	.687**	.649**
DOM ↔ MGO	.351**	.338**	.353**
Competence ↔ Effort	.438**	.559**	.541**
EOM ↔ Effort	.440**	.631**	.599**
Effort ↔ DOM	.516**	.544**	.545**
Competence ↔ DOM	.777**	.754**	.764**
Competence ↔ EOM	.660**	.750**	.605**
Confidence ↔ DOM	.524**	.577**	.582**

The *p*-values were calculated with bootstrapping
**Indicates *p*-value <0.01

However, it is clear that using inverse items can produce problems as was the case in our study. In this work we found a different functioning of inverse items in comparison to direct items, but this difference was a systematic behavior in most of the traits measured with inverse items. Horan et al. (2003) found similar results in a model of self-esteem, school attitudes, and locus of control (a systematic effect of inverse items in all the traits) and interpreted it as a response style effect and not as a methodological artifact. Response style is something more substantial than artifacts because it is a characteristic of the people and can be associated with a potentially measurable personality variable or trait (Horan et al., 2003). Because of the systematic behavior of inverse items and our ability to model the traits measured when we took into account this effect, we conclude that Chilean third graders understood the questionnaire and that the different results for inverse items were not due to lack of understanding. If this had been the case, the behavior would have been more chaotic. Also in the Chilean data, lower reliabilities were found compared with the Finnish data (Tuohilampi et al., 2014). The method effect may be one cause for the lowered reliabilities in the scales that contain inverse and direct items.

The important fact is that we found a good model (Model 3) for the structure of mathematics-related affects and beliefs of Chilean third graders considering fit, parsimony, and the theoretical assumptions. In this model the data are well explained considering trait factors that represent the measured theoretical dimensions and a method factor that accounts for the inverse item effect.

We were not able to test the hypothesis of a negative and a positive vision of affect. To do this, it would have been necessary to have several direct and inverse items in each dimension, and this was not the case in our questionnaire. It is therefore important to validate these results with other types of research techniques that allow us to understand the process involved in answering inverse items that change the pattern of responses. Also it would be very interesting to explore why this pattern was not present in the Finnish data. With the models tested here, we know that the inverse item effect could be model as a one-dimensional factor uncorrelated with trait factors, but we cannot reject the hypothesis of a positive and negative view of affect. The key result is that the effect of inverse items in different traits could be modeled with one factor. This shows us that inverse items systematically elicit a different behavior from Chilean students that we interpreted as a response style. Why Chilean third graders have this response style is something to be researched. One hypothesis is that in Chile, the subject mathematics is special; it is not the same to fail in mathematics than in other subjects; usually, being good in mathematics is seen as being "intelligent." This focus on mathematics is reinforced by national assessments that evaluate mathematics and Spanish each year at many grade levels (history and sciences are also evaluated, but less frequently). The PISA 2012 study shows that Chile is a country where students have high levels of mathematics anxiety (OECD 2013, p. 102). This could make students react differently with the inverse items of mathematics-related affect regardless of the specific trait measured.

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On the Role of Corporeality, Affect, and Metaphoring in Problem-Solving

Nicolás Libedinsky and Jorge Soto-Andrade

Abstract We explore the role of corporeality, affect, and metaphoring in problem-solving. Our experimental research background includes average and gifted Chilean high school students, juvenile offenders, prospective teachers, and mathematicians, tackling problems in a workshop setting. We report on observed dramatic changes in attitude toward mathematics triggered by group working for long enough periods on problem-solving, and we describe ways in which (possibly unconscious) metaphoring determines how efficiently and creatively you tackle a problem. We argue that systematic and conscious use of metaphoring may significantly improve performance in problem-solving. The effect of the facilitator ignoring the solution of the problem being tackled is also discussed.

Introduction

The relevance of problem-solving for the teaching and learning of mathematics has become commonplace nowadays. In the Western world, this has been triggered to a great extent by the pioneering taxonomy of Pólya (1945), as reported in first person by Schoenfeld in Arcavi, Kessel, Meira, and Smith (1998), Appendix A. Different approaches to problem-solving in mathematics and mathematics education have emerged in the course of time (Schoenfeld, 1985, 1992, 2010, 2012; Silver, 1985), some of them having their roots before the twentieth century, like the Japanese problem-solving approach, described in Isoda and Nakamura (2010) and Isoda and Katagiri (2012).

Our main working hypotheses regarding problem-solving concern the role of metaphoring, cognitive mode switching, and embodiment. More precisely, we claim that metaphoring may arise naturally as a response to a problematic situation the learners are involved in, implying quite often a change in the cognitive mode or

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style of the learner. Moreover, we claim that corporeality plays also a fundamental role in problem-solving since we do not just tackle or solve problems "in our heads" but through the body, mind, and affect (Hannula, 2013).

Our purpose in this paper is to bring grist to the mill of our hypotheses by presenting various down-to-earth cases where the implementation of the sort of approach we intend to foster makes a dramatic difference to the learner's understanding, feeling, and performance.

To this end, we report on some case studies of problem-solving with a wide spectrum of learners, ranging from average or gifted regular students majoring in science as well as in social science and humanities to primary school teachers from rural areas in Chile and juvenile offenders engaged in a social reinsertion program.

Let us recall first some basic facts and references regarding metaphoring, cognitive modes, and corporeality.

The "metaphorical approach" we adhere to in mathematics education has been progressively laid down during the last decades (English, 1997; Lakoff & Nuñez, 2000; Presmeg, 1997; Sfard, 1997, 2009; Soto-Andrade, 2006, 2007, 2013, vom Hofe, 1995, and many others), as (conceptual) metaphors are not being regarded as simply rhetorical devices as they classically were but as powerful cognitive tools helping us to build or grasp new concepts, as well as to solve problems in an efficient way.

Well-known examples of conceptual metaphors in mathematics education are "subtraction is going backwards;" "an equation with one unknown is a balanced pair of scales with one incognito weight;" "probabilities are weights, or masses;" "a random walk is a fission process, or an iterated splitting or sharing;" and "a polygon is a closed space between crossing sticks."

The concept of cognitive modes, or "cognitive styles" in French, emerged from work by Luria (1973) and was further developed by Flessas (1997) and Flessas and Lussier (2005), who pointed out to their impact on the teaching–learning process. A cognitive mode is defined nowadays as one's preferred way to think, perceive, and recall, in short, to cognize. It reveals itself particularly in problem-solving. To generate what they call the four basic cognitive modes, Flessas and Lussier (2005) combine two dichotomies: verbal–nonverbal and sequential–nonsequential (or simultaneous), closely related to the left–right hemisphere and frontal–parietal dichotomies in the brain (Luria, 1973). This affords four basic cognitive modes: *verbal–sequential*, *verbal–simultaneous*, *nonverbal–sequential*, and *nonverbal–simultaneous*. This may be supplemented with Schwank's dichotomy *predicative–functional*, also described as *structural–dynamic* (Schwank, 1999) to provide eight cognitive modes in all.

As said before, one of our hypotheses is that the most meaningful and significant metaphors arising in a problematic situation will involve a cognitive mode switch for the learner. Moreover, we hypothesize that the ability to switch from one way of cognizing to another is trainable.

Regarding corporeality, one of our basic tenets, i.e., the importance of bodily attitude (e.g., standing and working on nonpermanent vertical surfaces) in cooperative problem-solving, has already been highlighted by Liljedahl (2014). For an authoritative survey on the role of affect in problem-solving, we refer to Hannula (2013).

We now proceed to present our case studies.

Problem-Solving by Juvenile Offenders: A Multiple Case Study

A big challenge in Chilean society is the reeducation and reinsertion of juvenile offenders, guilty of various felonies as well as misdemeanors. This challenge is being addressed, among other actions, by a joint program run by the National Office for Minors (Servicio Nacional de Menores (SENAME)) and the Faculty of Sciences of the University of Chile that involves mathematical training workshops held at the University for small groups of minors from SENAME. Usually, these minors are dropouts from high school whose education is scanty and fragmentary, to say the least.

To work with these young persons (aged 18–22), we have implemented a highly metaphorical, enactive, and visual approach (Presmeg, 1997, 2006; Soto-Andrade, 2006, 2007, 2013, Soto-Andrade & Reyes-Santander, 2012), eliciting a much higher motivation than traditional teaching.

We report here on the outcomes and performance of the juvenile offenders in a workshop carried out in 2011–2012 and 2014, where an open-ended Finnish problem (Pehkonen, 1995) was proposed. The workshops lasted one semester and they had 7–10 students.

Our main specific working hypothesis was that student-centered, open-ended, and creative problem-solving activities that can relate to the personal and social needs of the pupils and their past experiences, in the sense of Pehkonen (1995) and Järvinen and Twyford (2000), were especially suited to the case of young subjects, like our juvenile offenders. Since they have developed remarkable skills to survive in hostile or repressive environments, we hypothesized that creativity and metaphorical activity may be more spontaneous in them than in "regular" students and that emerging idiosyncratic metaphors might be of significant help for them to solve the challenges proposed.

The methodology consisted in observing and interviewing the students as they carried out the activity described below. Records of this observation comprised videos, written and drawn production of the students, and some transcriptions.

In 2011, 2012, and 2014, we carried out a 60-min work session on the following Finnish open-ended problem (Pehkonen, 1995):

Partition a square in four equal (*i.e.*, *congruent*) *pieces in four different ways*.

A sample of solutions figured out by the juvenile offenders in 2011–2012 is given in Figs. 1 and 2. A whiteboard of solutions obtained in 2014 is shown in Fig. 3.

First, they found quickly the most obvious three ways to partition the square and realized that horizontal and vertical stripes were "the same," but they had a hard time finding a fourth (essentially) different way. During a lapse of approximately 20 min, they generated however an interesting array of wrong answers (see figures 1,2 and 3 above), each on his own. Since no correct solution emerged for a while, the facilitator of the workshop (JSA) had the idea to share their wrong solutions on the whiteboard, in particular the "absurd" concentric squares solution shown below.

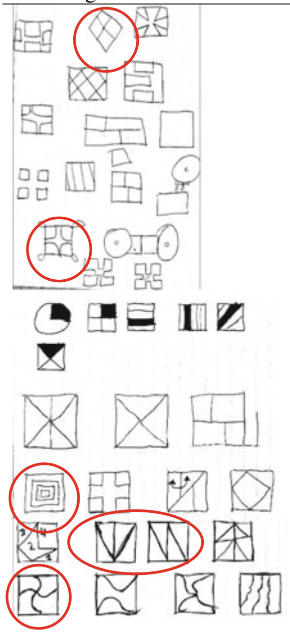
Drawing	Comments
	Juvenile offenders:
	Claudio 1
	Notice the incomplete partition.
	Claudio 2
	Notice the absurd concentric solution, with central symmetry
	Two classical solutions.
	A remarkable solution with central symmetry

Fig. 1 Partitioning the square, juvenile offenders 2011–2012

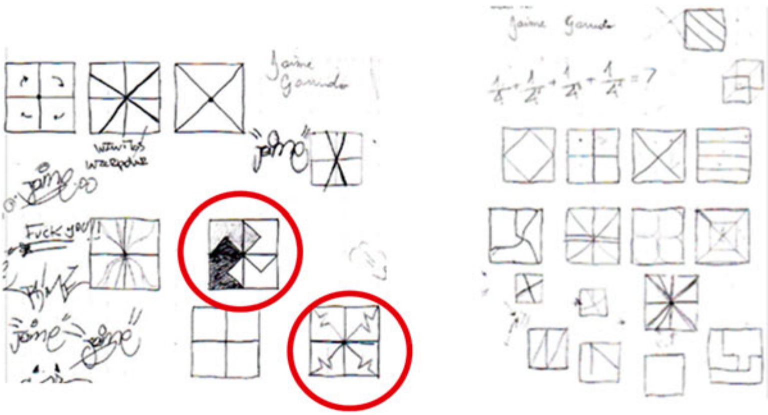


Fig. 2 Jaime’s partitions

Shortly afterward, one student got the idea of drawing a line upward from the center of the square to the border, “deviating from the straight line upwards, turning a bit to the left” in his own words. Notice here the heavy metaphorical content of this description that applies to his own condition: in Spanish, indeed, “desviarse del camino recto” (“to deviate from the straight or righteous path”) is a very common

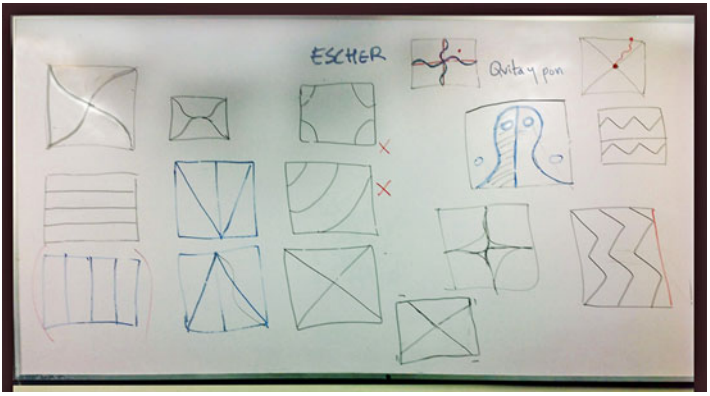


Fig. 3 Square partitions, juvenile offenders 2014

expression. Then all sorts of "deviations" popped up (see Fig. 1 and Fig. 2), providing a handful of different solutions. One offender, who had worked this out independently, when asked how did he get the idea of going out from the center of the square, answered: "I got it from the wrong concentric squares solution, but for me the centre is not an origin, but a 'punto de fuga'" (a usual expression in perspective drawing, in Spanish, meaning literally "escape point"). He explained then that the wrong solution of his mate appeared as a framed aisle in perspective to him and that he very much liked to draw in perspective. Notice again the metaphor for his life condition.

Others like Jaime (Fig. 2) also realized that infinitely many solutions may be obtained by "adding and taking away" (first red circle in Fig. 2), instead of deforming the straight paths from the center to the border. Notice however his very original partition in the second red circle, obtained by "perturbing the straight line with a shiver, or a frisson." In this case, we also see some evidence of his affective mood and tensions in his writings next to the squares. This suggests a close relationship between creativity and affect and emotion (Hannula, 2013).

One interesting phenomenon is that among these juvenile offenders, dropouts from school, who perform very poorly in standard (TIMSS-like) assessment tests, the same clever idea emerged (take any path from the center of the square to its border, and rotate it thrice in a quarter of a turn!) as the one Ragnar (a case study in Soto-Andrade, 2006, who majored later in anthropology) had during a work session on this problem with Prof. Pehkonen himself, 2 years ago, in Santiago. The point is that Ragnar's cognitive and educational background (starting at a Waldorf school) is wide apart from our juvenile offenders'.

Notice the incorrect curved partition in the lower right corner. It has the merit however of being the first partition suggested in this session that used curved lines! Its author was reluctant to share it, because another offender pointed out immediately that it was wrong, but he realized afterward that this partition opened up the way to many correct curved partitions, in particular the one provided by the hooded figure—"encapuchado" in Spanish—in the second row. It was christened that way

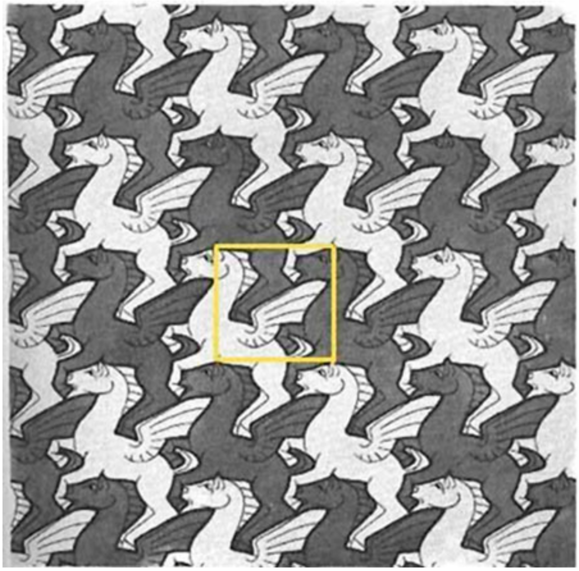


Fig. 4 Escher's pegasi

by his author, as a humorous allusion to hooded youngsters that after a pacific civil demonstration often initiate riots by throwing stones and burning devices to police forces (something familiar for these young offenders).

Moreover they also remarked, working in an interactive way, that new partitions may be obtained from very simple ones, like the one of four squares, by "stealing away" a bit from one square on one side and "giving it back" on the other side. So they rediscovered by themselves Escher's method of tessellation by compensation and were able to figure out very quickly how to construct the tessellation shown in Fig. 4, where the superimposed yellow square tile was figured out by them and not given in advance!

Problem-Solving by University Students: Multiple Case Studies

When working with these second and third year university students, we tried hard to choose exciting and hard problems and to reinforce sense of humor in the classroom. This motivated the students significantly.

We report here on some important aspects of three workshops carried out in 2013 and 2014. Our main specific working hypothesis was that group work is extremely important when trying to solve difficult problems (much more than in other contexts) mostly because what is crucial in solving hard problems is a metamathematical attitude related partly to self-esteem and partly to "know what to do when you don't know what to do."

The background for our experimental research was the following. Each one of the three experiences was a one-semester workshop (3 h a week) for second or third year university students majoring in mathematics or mathematics education. The first workshop had 30 students, the second one 7, and the third one 10.

The methodology consisted in observing and interviewing the students as they carried out the activity described below. Records of this observation comprise written and drawn production of the students and some written observations of them.

The work methodology that we are going to explain was developed over the years in different workshops of problem-solving and problem invention. In particular, one important workshop that will not be described here took place in 2001 in a high school. Just 1 year after the workshop, the three participating students obtained the three gold medals in the Chilean mathematics Olympiad. That was an important moment, where the facilitator realized that this methodology had some interest.

Our working methodology was work sessions were 180 min long. Students worked in self-defined groups, standing up, in front of a blackboard, the facilitator behind them. They worked on hard problems that needed a whole 180-min work session to be solved. More or less half of the time, problems remained unsolved at the end of the work session. Answers or solutions were never *given* by the session facilitator (NL). Hints toward a solution were only given when the students looked demotivated. Problems were selected because of their amusing and interesting character (to foster motivation among the students) according to the facilitator's appreciation, and then, in the following workshops, the opinions of the students of the first workshops were taken into account. Half of the time the facilitator did not know the solutions of the proposed problems. When a group quickly solved the problem, they were asked to generalize the problem or to find a variation.

Examples of problems given in the first sessions are:

1. Resolve and generalize the "towers of Hanoi" problem.
2. The SEND + MORE = MONEY problem.
3. The prisoners and hat puzzle in the ten-hat variant and generalizations.

Motivation is one of the main driving forces for development in mathematics and can be developed in different ways, one of which is to find beauty in mathematics. But some students have to enact situations in order to "see" the beauty. We give here one example.

The facilitator gave the following problem: prove that the sum over "faces" of all dimensions of a hypercube is 3 to the power of the dimension of the hypercube. For example, in a usual cube, the sum of the vertices (8) plus the number of edges (12) plus the number of faces (6) plus the number of cubes (1) is 27, that is, 3 to the third power.

The idea is that this is a good lighthouse to discover what a hypercube of more than three dimensions is. Students had many different approaches to this problem, but one group (we call it group A) was confused about the idea of what a "face" in a 4-hypercube looked like. They imagined that it should look like a usual cube, but they were not able to "see" this. At some point, one of the students in group A said "in a cube, a face is *all the ways to walk* from one vertex to another vertex that is

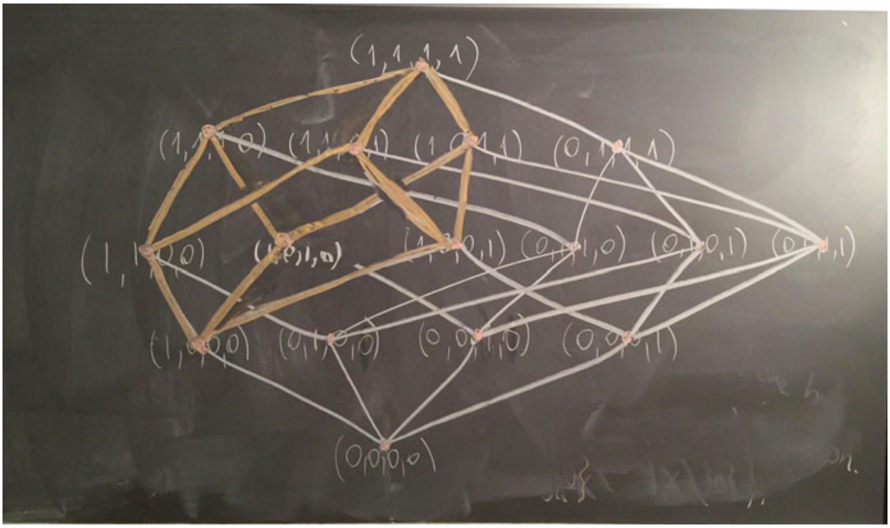


Fig. 5 Student's view of a 3-cube inside a 4-cube

two steps apart." So they used the metaphor that the cube is a "place" where you can walk, and "objects" (e.g., faces) inside this place are defined by your possible movements. This follows exactly the two patterns that Thurston teaches us (Thurston, 1998) when he explains how to imagine 3-manifolds: you have to imagine yourself inside of the manifold and imagine that you are more or less the same size of the geometric object you try to imagine.

Then they drew a 4-hypercube in the blackboard by using a natural definition of the 4-hypercube, sequences of four 0s and 1s, where an edge joins two vertices if the corresponding sequences differ in exactly one position. Finally, they drew in orange all the "ways to walk from one vertex to another that is three steps apart," as one student said, and suddenly they "saw" a 3-cube (a face) inside the 4-hypercube as shown in Fig. 5.

Something remarkable about that moment was that one student remained completely silent with eyes wide open. She then told the facilitator that it was the first time she had ever experienced something beautiful in mathematics. After that moment, her attitude toward mathematics changed dramatically, and she became, until the present day, much more interested in mathematics in general.

A second remark about these workshops was internal to the facilitator (NL) and has to do with the implicit set of metaphors underlying positioning in the classroom. The facilitator had set up a "classical classroom" where students were sitting on their chairs and the professor wrote on the blackboard. The implicit metaphor in that situation is that the professor has "something to give to the students," while in the setting explained before, a reasonable metaphor of what is happening is "the professor is behind the students to support them if they fall."

The important point about this is that the facilitator felt, in the first case (classical classroom), that students were not smart in general, because they had a hard time understanding the theorems. While in the second case, the facilitator was mostly impressed by how smart the students were, because their creativity came into play and they invented lots of solutions of a different nature to those that the facilitator would have thought of. This makes a huge difference in the motivation the students will develop, because the opinion the teacher has about their students is something intangible but somehow understood by them.

Another interesting thing to be remarked was the change in attitude toward the problems. More or less in the fourth session of every workshop, the students stopped being demotivated if they didn't solve the problem after a few minutes; they began to realize that the objective of the work sessions was to think all together and not to immediately solve the problem. Many times they stayed more than the 3 h just because they were curious. One girl once told the facilitator that even though she liked the workshop, she was thinking to stop coming because Monday night (the workshop was held on Mondays) she could not sleep if she had not solved the current problem.

Reflecting on the role of corporeality in these workshops: the fact that students were standing up for 3 h (with the possibility of sitting down from time to time) was quite important for them. Many of them bear witness that this was in part the reason why they were so active in the sessions. Some groups that sat down reported to have much less and slower communication between them. Also we remarked that they tend to have a contemplative attitude toward the problem if the problem is hard and to lose concentration (having other students thinking with you seems to increase concentration levels).

Another instance of corporeality involved problems related to surfaces. In one problem, the students were asked to visualize the geometry of the "Whitney umbrella" (the zeros of the equation $x^2=y^2z$). For a large number of students, the task of imagining the surface is impossible without gesturing. For example, one group explained the surface to one another by moving their hands outward, with joined hands corresponding to the locus $z=$constant. It is also interesting whether the students "see" this surface as an umbrella (again a metaphor).

One activity in one of the workshops was to play a game where metaphors, corporeality, and affect were deeply intertwined. It is the following game (invented by Sebastian Libedinsky) that we call SL game. There are two players. They both have 30 "objects." Each round they have to say a natural number (including zero) bounded by 30, both at the same time. The biggest number wins the round. Now for the next round, each player "loses" the number of objects he said in the first round. For example, if player one says 7 and player two says 4 in the first round, then player one wins the round, but he has only 23 objects left for the next rounds, while player two lost the round but he/she has 26 objects for the next rounds. In each round, you cannot say more than the number of objects you still have. The player that first wins three rounds wins the game.

This game was very fun to play for the students and they laughed a lot playing it, so that their sense of humor was highly activated. The atmosphere allowed interesting phenomena to arise. For example, each player's strategy depended on what metaphor he used for the objects (matches, water, fingers, etc.). They had to choose one in order to play, but they usually chose one because they thought it was funny. Most of them did not realize that their choice would be so determinant. For example, players who imagined objects as being water were more likely to invent probabilistic strategies. Players imagining the object as matches were much more likely to try to win the first round. One student said laughing: "I want to start a fire." The players using their fingers to represent objects were more likely to say little numbers (for evident reasons). The game was usually won by players applying some probability rule, so "water" usually won over "fire." Another important feature of this game is that less gifted students usually beat the more gifted students.

Let us make a final remark regarding these workshops. Many times the facilitator did not know the answer of a proposed question. Not knowing the answers entailed a lot of fun for the facilitator, which translates in a good state of mind of the whole class. He didn't try to solve the problems with the students, but he helped them in developing their ideas without knowing himself if a particular way they engaged in would lead to a solution. We believe that the fact of ignoring the solution of the problems was extremely important for the dramatic change in attitude the students had, because they saw a lot of meta-mathematical reasoning from the facilitator. He really pondered with them whether their approaches had chances to succeed. If he had known the answer beforehand, they could have never seen an honest reasoning of this type from him.

Finally, in these workshops we asked for problem posing. We can remark that when students try to invent problems they quickly appreciate how difficult it is to find a problem which is both feasible and pleasant for them and their fellow students. This leads them to appreciate much more the problems posed to them by the teacher. They start developing sensitivity toward the art of creating or modifying problems. And this is a key to deep mathematical thinking, as can be seen by the fact that in mathematical research half of the problem is to find a fascinating problem that usually comes in the form of conjecturing a fascinating phenomenon. But to obtain this, you have to solve other problems beforehand.

As an example of this point, there was a group of students that realized that in the power set of a finite set, the operations "intersection" and "symmetric difference" are like adding and subtracting in "certain ring." This was a starting point for many natural questions about ring theory that they tried to solve in the course of the workshop.

Problem-Solving by Primary School Teachers: What About the Sum of All Exterior Angles of a Polygon?

We worked on this problem in 2010 in a session of a professional development program in Puerto Montt, Chile, with three groups of 30 in-service primary school teachers coming from rural areas in the south of Chile.

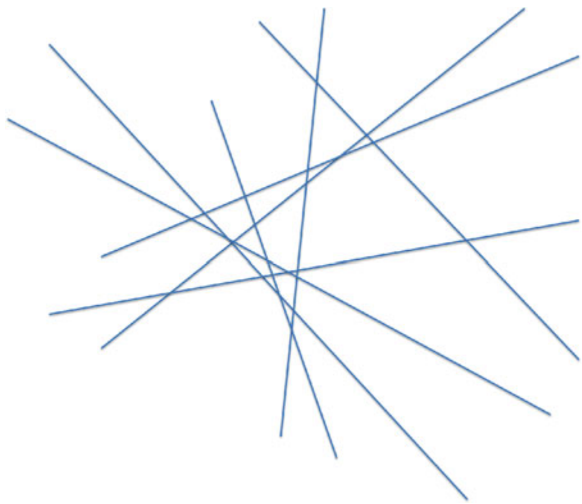


Fig. 6 Crossing sticks metaphor for a polygon

The usual approach to this problem, observed among secondary school teachers in our country and elsewhere, is to calculate first the sum of interior angles of the polygon that depends on its number of sides, then express exterior angles in terms of interior angles, calculate diligently, and finally arrive to the conclusion that the requested sum measures 360° and say, together with their students: We are done! We however share Schoenfeld's claim that we are not done! (Schoenfeld, 2012). To explore other ways to approach the problem, we suggested to the teachers to metaphorize the ingredients of the problem. That is, to try to figure out different metaphors for a polygon, to begin with. Then, do the same for the exterior angles, which they found almost unanimously less friendly than the interior angles. After 10–15 min, some interesting metaphors emerged, like the following:

- A polygon is an enclosure bounded by crossing sticks! (Fig. 6)
- A polygon is a "German closed path," i.e., a chain of rectilinear segments, no curves.

These metaphors were often expressed in gestural language.

The teachers enacted then these metaphors. When enacting the first one with 20 cm long sticks, they got the idea of moving the sticks; sliding them first, to better see the exterior angles (Fig. 7); and then moving them parallel to themselves, so as to reduce the size of the selected enclosure, preserving its shape, an idea more likely to emerge when you enact your metaphor with concrete material than when you work with pencil and paper or just recite the scholastic definition of a polygon. In this way, they "saw" immediately that the sum of exterior angles is a whole angle.

They also enacted the second metaphor: one of them gave instructions to another one to follow a given polygonal closed path: begin here, go straight ahead 5 steps, stop, then turn to your left in 45° (an exterior angle, much more relevant in this context than the interior angle!), then go on for 8 steps, etc. until the walker came

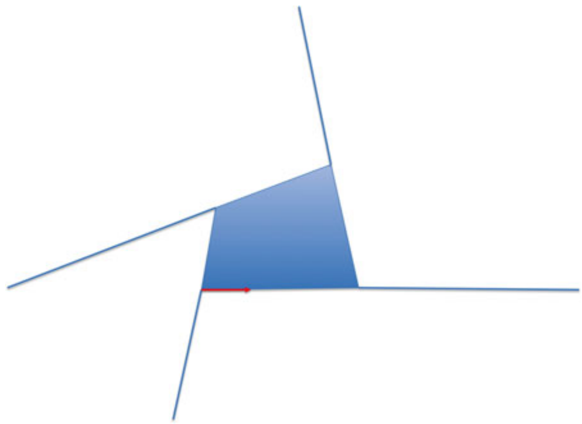


Fig. 7 Sliding the sticks

back to its initial position with her nose pointing in the same direction as when she started. This teacher “felt” then that she had made in all a whole turn! We see then a friendly, metaphoric, and enactive way to figure out the sum of all exterior angles of (any) polygon.

Discussion

We have seen that these approaches elicit positive cognitive and affective reaction from our students (juvenile offenders, university students, and primary school teachers). After the workshop sessions, they bear witness to a completely new experience of mathematics, when comparing with their previous mathematical instruction.

In the case of juvenile offenders, they appear as a group far more creative and autonomous than regular students and also teachers, with the exception of students like Ragnar, who have had a first-rate educational and cognitive experience since his childhood. This convergence of bright ideas, emerging independently in subjects so wide apart in personal life histories, socioeconomic status, and educational studies, when confronted with open-ended problems, with a strong visual, motoric, and metaphorical component, is a phenomenon that deserves further study, in our opinion. It is also remarkable how metaphors emerging from their life condition (breaking the law, punishment, full-time and part-time imprisonment, etc.), like “deviating from the straight path” or taking advantage of an “escape point,” play the role of tools helping them to solve the proposed cognitive challenge. Our findings also suggest that further research should be carried out on the cognitive and therapeutic effects of the metaphorical approach to mathematical challenges in juvenile offenders engaged in a reinsertion process as well as on exploring various means to free the expression of creativity in regular students.

We argue that the examples shown in this paper (hypercube, exterior angles of a polygon, Whitney umbrella, objects in SL game) show how crucial to problem-solving the role of enactive bodily metaphoring might be.

Notice that also in the problem of partitioning the square, when you try to enact the procedure of partition, you may realize that you are unconsciously metaphoring it as slicing a pizza with a knife, i.e., doing straight cuts, not curved ones. An alternative metaphor is however dividing a paperboard square with scissors that opens up the possibility of curved portions and so on.

Other relevant aspect of corporeality is how important it seems to be in long problems. We have remarked that standing up makes the whole body work and this bodily attitude helps concentrating on the problem and also helps the students to be more courageous regarding the ideas they have. It is also easier to share their excitement about a new idea they had, since they can move more easily and excitement is usually shared through body language.

We think that to work in groups while solving a problem has usually various advantages. A fundamental part of problem-solving is self-esteem, and it is a long and hard individual process to develop it. If you work on hard problems in groups, your fellow students help in this sense, because they are excited every time you have a good idea, probably more than the teacher would be, even if he is very sensitive, because as sensitive as he can be, he will never be able to understand in detail what things are difficult for her students.

The fact that the teacher tries to go with the flow of student's thoughts (not imposing his own way of solving a problem) usually makes him discover, as we said before, how creative students can be. The resulting teacher's excitement, honestly communicated to the students, turns out to be very inspiring for them, because excitement in this kind of matters is difficult to fake. It is very important that the teacher feels and expresses admiration of his students (a point strongly emphasized in Japanese problem-solving approach). It is very important (and usually underestimated) in a lesson that both the teacher and the students are motivated. The fact of not knowing the answer to the problems proposed makes the job of the teacher especially stimulating and fun, but we believe that for the teacher to feel comfortable doing this, he has to have a good level in solving problem, ideally much higher than the students.

One thing we have observed is that the difficulty level of a problem is a fundamental issue in problem-solving. Someone can be very good in solving "easy problems" (problems that take him 20 min or less to solve) but very bad in solving harder problems or even extremely easy problems (3 min problems). We have seen examples of people being extremely good at solving hard Olympiad problems (that can take them a couple of hours) and quite poor to solve research-level problems or very easy problems. We believe that in a class, it is paramount to have all levels of problems, because they use different skills for different levels and they can be motivated in different ways. We hypothesize that the fundamental differences in responses to these different classes of problems are due to self-esteem and the frustration threshold.

We claim then that by working on problem-solving with enactive metaphors like these, learners of all walks of life may generate their own creative approaches and think mathematically, something that otherwise would be accessible just to a happy few.

We would like to finish by saying that the expression “problem-solving” seems to be a bad metaphor for what students might do. It implies that the problems are “outside” in the world and that their relationship to them has to be to solve them, something like a gatherer, eating the food he finds in the wilderness. We prefer the expression “problem development” or “problem looping” that suggests the idea that problems are constantly solved and invented in a circular never-ending process involving subjects and a world that co-determine each other (Varela, 1987, 1999).

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Reformulating: Approaching Mathematical Problem Solving as Inquiry

Jeremy Kilpatrick

Abstract The Gestalt psychologist Karl Duncker (1903–1940) characterized problem solving as getting from where you are to where you want to be through successive reformulations of the problem until it becomes something you can manage. That view can be seen in a recent European project to promote inquiry as a means of learning mathematics and attracting students to its study. It can also be seen in increased research efforts to study problem formulation in mathematics. Considerations of how to educate teachers of mathematics to approach problem solving as inquiry should include attention to questions of metacognition and to the cognitive demand of a problem—in much the same manner as George Pólya (1887–1985) promoted such attention. For Duncker and Pólya, we solve problems by replacing our unsuccessful efforts by successful ones through a heuristic inquiry process.

One of the first times I rented a car with a GPS (global positioning system), I was following my younger son as we drove from the Portland, Oregon, airport to his house. The GPS gave me what it thought was the shortest route, but my son took several shortcuts that the GPS apparently did not know about. Each time he would take a shortcut, the GPS would inform me—in what I thought was an increasingly exasperated voice—that it was "recalculating" the route. The refrain "recalculating, recalculating" echoed in my ears as we made our way to my son's house.

I realized that the GPS was trying to solve the problem of getting me to my son's house while minimizing the distance traveled, and it struck me that what the GPS voice was calling *recalculating* was much the same as what Karl Duncker (1945) had labeled *reformulating* in the problem-solving process. In fact, Duncker characterized *problem solving* as a sequence of reformulations:

> The finding of a general property of a solution means each time a *reformulation of the original problem*. (p. 8)
>
> The final solution is mediated by successive reformulations of the problem, and . . . these reformulations or solution-phases are in their turn mediated by general heuristic methods. (p. 47)

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Every time the GPS was "recalculating" the route to my son's house, it was recasting the problem from one of getting from point A to point B via the shortest route to one of getting from point A′ to point B via the shortest route. The goal was staying the same in this case, but the initial condition, and therefore the solution path, was changing every time I departed from the proposed route. Looking back on the episode, I could see that the original problem was being broken into a sequence of new problems, each arising from the previous one that collectively provided a solution to the original.

I have long thought that Duncker's depiction of problem solving captures the essence of what happens when we work at solving a mathematical problem. We need to take an active stance toward the problem, recognizing that it may be easier to solve if we can break it into parts or recast it in another form. In this paper, I consider some of the virtues of looking at problem solving in mathematics as an inquiry process.

PRIMAS

In 2007, Michel Rocard and his colleagues on the European Commission's High Level Group on Science Education published a report calling for the incorporation of inquiry methods into science and mathematics teaching in Europe as a means of dealing with what they saw as seriously declining interest by young people in the study of science and mathematics. In response, an international project—*PR*omoting *I*nquiry in *M*athematics *A*nd *S*cience education across Europe or PRIMAS—was established in the Seventh Framework Program (FP7) of the European Commission. PRIMAS involved the participation of researchers from 14 universities in 12 European countries: Cyprus, Malta, Slovakia, Denmark, the Netherlands, Spain, Germany, Norway, Switzerland, Hungary, Romania, and the United Kingdom. From 2007 to 2013, these researchers worked together to promote the implementation and use of inquiry-based learning (IBL) in mathematics and science.

A special issue of ZDM (Maaß, Artigue, Doorman, Krainer, & Ruthven, 2013) contains discussions of issues raised by the implementation of IBL as well as reports of large-scale implementation efforts in various European countries. The term *IBL* "refers to a teaching culture and to classroom practices in which students inquire and pose questions, explore and evaluate" (Maaß & Doorman, 2013, p. 887). The IBL concept emerges from a long tradition of education research and practice, much of it from the teaching of science, but varying across country contexts. The PRIMAS project did much to implement and explore the IBL concept in mathematics teaching.

The Context of PRIMAS

One of the important features of the PRIMAS project was its recognition of the need to tailor professional development for teachers to the circumstances in each country. Maaß and Doorman (2013) note that "widespread implementation of IBL [requires] a process of scaled-up professional development initiatives" (p. 888). They argue for "a model for dissemination and implementation that both addresses core principles of IBL and has the flexibility for implementing and scaling up professional development in various national contexts" (p. 888). Consequently, they designed a model for PRIMAS that was based on the principles of design research.

As Schoenfeld and Kilpatrick (2013) note, design research needs to draw upon "local" rather than "global" theory, so that any new materials are accompanied by a theoretical rationale as to why those materials should work in a specific situation: "Attempts at intervention should… provide two things: revised materials and a revised theoretical understanding of how students come to grips with that particular topic" (p. 908). Global theory, which deals with general cases of professional preparation and development, needs to be replaced in this case by a local theory suited to the features and needs of the country in which the professional preparation and development is being given.

As an example of the challenge faced by the implementation of IBL, Schoenfeld and Kilpatrick (2013) cite two recent documents from the United Kingdom containing mathematics frameworks: (a) the National Curriculum in England (Department for Education & United Kingdom, 2013b) and (b) the General Certificate of Secondary Education (GCSE) in mathematics (Department for Education & United Kingdom, 2013a). In both documents, the focus is much more on skills than inquiry, and they pose considerable obstacles for anyone attempting to implement IBL on a large scale in the United Kingdom.

A major argument for implementing IBL stems from declining enrollments in STEM (science, technology, engineering, and mathematics) classes in European secondary schools and the resulting need to change students' career objectives. As Rocard et al. (2007) observe:

> In recent years, many studies have highlighted an alarming decline in young people's interest for key science studies and mathematics. Despite the numerous projects and actions that are being implemented to reverse this trend, the signs of improvement are still modest. Unless more effective action is taken, Europe's longer term capacity to innovate, and the quality of its research will also decline. Furthermore, among the population in general, the acquisition of skills that are becoming essential in all walks of life, in a society increasingly dependent on the use of knowledge, is also under increasing threat. (p. 2)

Wake and Burkhardt (2013) argue that efforts to achieve the goal of increasing interest and enrollments by implementing IBL have been ineffective. They identify a number of features that have been acting as barriers to that goal. None of the studies reported in the ZDM special issue (Maaß et al., 2013) provides any evidence that changes in instruction by implementing IBL can change students' career objectives. Increased enrollments in school subjects and thereby the pursuit of certain careers have ordinarily resulted more from changes in market forces and degree requirements than from changes in teaching, and that phenomenon is likely to persist. Nonetheless, the goal of implementing IBL in mathematics ought to remain central to reform efforts.

Inquiry and Problem Solving in Mathematics

Artigue and Blomhøj (2013) point out that John Dewey was responsible for introducing *reflective inquiry* as a basis for pedagogical practice and also that there is significant overlap between the concepts of *inquiry* and *problem solving*. These concepts may overlap, but they have somewhat different connotations. For example, problem solving in mathematics assumes the presence of a focus for one's efforts, whereas inquiry in science encompasses all aspects of an investigation.

As Schoenfeld and Kilpatrick (2013) observe, in the United States, inquiry is very much in the province of science education, whereas problem solving resides in mathematics education. In the 1989 standards of the National Council of Teachers of Mathematics, for example, the first standard listed was "mathematics as problem solving," whereas the term *inquiry* does not appear in the document. In contrast, in the document containing the US National Science Education Standards (National Research Council, 1996), the term *problem solving* does not appear in the index, whereas the term *scientific inquiry* has a half page of index entries (Schoenfeld & Kilpatrick, 2013, p. 905).

Outside the United States, the two concepts differ as well. For example, consider the assessment frameworks for mathematics and science used in the 2015 Trends in International Mathematics and Science Study (TIMSS; Mullis & Martin, 2013), in the 2012 Program for International Student Assessment (PISA; Organisation for Economic Co-operation and Development [OECD], 2013a), and in the 2015 PISA (OECD, 2013b, 2013c). Table 1 shows the frequency of appearance of the terms *problem solving* and *inquiry* or *enquiry* in the documents containing those frameworks. The term *problem solving* appears in all three mathematics framework documents and much less—or never—in the science framework documents. In contrast, neither *inquiry* nor *enquiry* appears in any of the mathematics framework documents, whereas both terms appear in the science framework documents, with *enquiry*

Table 1 Frequency of "problem solving" or "inquiry/enquiry" in each of three framework document pairs

Framework	"Problem solving"	"Inquiry" or "enquiry"
TIMSS 2015		
Mathematics	6	0
Science	0	6
PISA 2012		
Mathematics	4	0
Science	2	23
PISA 2015		
Mathematics	6	0
Science	1	62

Note. The TIMSS 2015 data are from Mullis and Martin (2013), the PISA 2012 data are from the Organisation for Economic Co-operation and Development (OECD, 2013a), and the PISA 2015 data are from the OECD (2013b, 2013c)
The TIMSS frameworks use *inquiry*; the PISA frameworks use *enquiry*.

becoming increasingly common in the PISA science frameworks. The PISA science frameworks repeatedly refer to "scientific enquiry" as a major process involved in becoming scientifically literate. In the draft 2015 science framework (OECD, 2013c), one of the three competencies required for scientific literacy is "Evaluate and Design Scientific Enquiry" (p. 8).

Although the field of mathematics education has traditionally focused much more on problem solving than on inquiry, projects such as PRIMAS suggest that a focus on inquiry might provide some important opportunities for getting students involved in the study of mathematics whether or not it can change their career objectives. Scientific inquiry is not the same thing as mathematical problem solving, but their common features might stimulate new approaches to the teaching of mathematics.

Problem Formulating

The topic of problem formulating in mathematics has drawn increased attention in recent years, especially because of the work of Ed Silver and Jinfa Cai (see Singer, Ellerton, & Cai, 2015, for examples of that work and its consequences). Almost three decades ago, I posed the question, "Where do good problems come from?" (Kilpatrick, 1987). My answer was that they come from teachers and textbooks but too rarely from students. (Today, one might add that they come from the Web—which is where many poor problems come from as well.) One of the points I tried to make in the article in which I posed the question was that problem formulating ought to be both a goal and a means of mathematics teaching.

Consequently, as one moves to implement IBL in mathematics, one needs to consider the role played by problem formulating in inquiry. A teacher can introduce a situation to learners as providing the source of a problem. Once the learners have constructed a mathematical model of the situation, they can use that model to formulate a problem. The reformulating process then can begin immediately as the learners check both the model and its adequacy for the situation. A simple example of a situation might be one in which the ages of three people—Ann, Brian, and Carl—are given (7, 10, and 13 years, respectively), and the following relations are given: Brian is 3 years older than Ann, and Carl is 3 years older than Brian. For a mathematical model of the ages, one might choose a, b, and c as the three ages, respectively, and write $b=a+3$ and $c=b+3$ as the model for the given relations. Then one example of a problem posed given this model (and hiding the ages) might be that if Carl is 13 years old, how old is Ann? A more difficult problem would be that if the sum of the three ages is 30 years, how old is Carl? As a follow-up, one might explore questions of what relations between the ages are determined and what relations are undetermined. How might the situation be modified to yield another model and other problems? Such questions can help learners see how problems are constructed as well as how they are solved.

Some of the work on problem formulation has addressed the use of so-called open-ended problems in mathematics instruction (Cifarelli & Cai, 2005; Cooney, Sanchez, Leatham, & Mewborn, 2002; Pehkonen, 1997b; Silver, 1995). Pehkonen (1997a) considered mathematics problems as either open or closed (i.e., exactly explained) with respect to either the starting situation or the goal situation. Problems can be open in any of three ways, with either or both situations open, but as Pehkonen observed, school mathematics problems are ordinarily closed with respect to both situations. He defined an "open-ended problem" as having a closed starting situation and open goal situation (p. 9). He noted that the so-called open-approach method—using open-ended problems in the classroom to promote mathematical discussion—was developed in Japan in the 1970s (p. 7). At roughly the same time, the use of open-ended investigations was being promoted in the United Kingdom. Since that time, efforts to promote the use of open-ended problems—as well as open problems of other types—have spread around the world (p. 7). It would appear that any kind of open or closed mathematics problem can be used to stimulate inquiry, but the experience in various countries, including the countries in the PRIMAS project, suggests that problems that are open in at least one respect seem likely to provide the most fertile sources of IBL.

Teacher Preparation

Because IBL is not a common practice in most school mathematics classrooms, teachers need to acquire experience using it in their instruction. Although the PRIMAS project focused on professional development, an especially opportune time for teachers to gain experience with IBL is while they are still being prepared to teach. That way, they can see, analyze, and explore various facets of IBL in an environment that allows, and should promote, innovative instruction. They need to understand what IBL is meant to be as well as how they might put it into their practice. They need a teacher preparation program that focuses their attention on the learning of the students they are teaching.

John Dewey (1904/1964) pointed out that teaching demands preparation in both theoretical and practical work: A teacher needs to be prepared to address questions of the relationship between subject-matter knowledge and educational theory and simultaneously to manage the daily routines of classroom practice. Consequently, teacher preparation needs two foci. The first focus concerns the more theoretical aspects of the job: what might be termed *the laboratory approach*, a forward-looking approach that is "local, particular, situated" (Shulman, 1998, p. 512). Such a focus is essential if the prospective teacher is to understand what IBL entails. The second concerns preparation for the practical aspects of the job: *the apprenticeship approach*, a traditional approach in which past performance serves as a model for future performance. One weakness of teacher preparation programs in Dewey's time as well as today is that those programs are dominated by the apprenticeship approach, whereas a balance is needed between the two approaches.

In a study by Philipp et al. (2007), the beliefs of prospective teachers who studied children's mathematical thinking (some by watching videos only, some by also working with individual children) underwent more change than the beliefs of prospective teachers who only visited classrooms. The laboratory approach was only modestly better than an apprenticeship approach in improving the prospective teachers' mathematical knowledge, but it strikingly improved their beliefs about children's learning and thinking. By controlling the mathematical tasks used with children, Philipp et al. increased the likelihood that the prospective teachers encountered situations that had the potential to affect their beliefs.

In response to this and other work on teacher preparation, we have revised our courses at the University of Georgia for prospective teachers of secondary mathematics to include more laboratory-approach activities that are designed to provide IBL. Two important features of those courses are assisting the prospective teachers to pay attention to metacognition and cognitive demand in their teaching. Both metacognition and cognitive demand are constructs associated with IBL.

Metacognition

Schoenfeld (1987) posed the question, "What's all the fuss about metacognition?" He then attempted to explain, to people puzzled about the use of the term by researchers in mathematics education, what it meant. Schoenfeld defined *metacognition* in terms of a set of questions:

1. Your knowledge about your own thought processes. How accurate are you in describing your own thinking?
2. Control or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from those observations to guide your problem-solving actions?
3. Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way that you do mathematics (p. 190)?

Nonetheless, *metacognition* has remained for some time a term used almost exclusively by educational researchers. The authors of the National Research Council report *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) avoided using it, arguing that most teachers and lay readers would not understand it. It has, however, begun to creep into documents in education. For example, Singapore's Mathematics Framework used metacognition as one of five so-called process priorities (see Ginsburg, Leinwand, Anstrom, & Pollock, 2005, pp. 15–16). It has come to be used extensively in discussions of mathematical problem solving.

Garofalo and Lester (1985), like some other researchers, see metacognition as a binary phenomenon: "Metacognition has two separate but related aspects: (a) knowledge and beliefs about cognitive phenomena, and (b) the regulation and control of cognitive actions" (p. 163). Surveying the literature on metacognition in mathematics problem solving, they conclude:

> Lester (1983) and Schoenfeld (1983) believe that the failure of most efforts to improve students' problem-solving performance is due in large part to the fact that instruction has overemphasized the development of heuristic skills and has virtually ignored the managerial skills necessary to regulate one's activity. (p. 173)

Garofalo and Lester (1985) view the work of George Pólya on problem solving as failing to address metacognition: "Unfortunately, Pólya's conceptualization considers metacognitive processes only implicitly" (p. 169). That judgment strikes me as simplistic. Even though Pólya (1945, 1966, 1981) did not use the term *metacognition* in his writings on problem solving, he certainly considered questions of regulation and control of one's cognitive efforts. The heuristic questions and suggestions he gives in *How to Solve It* (Pólya, 1945) are intended to help the problem solver guide his or her work. Here is a problem from one of the Stanford University Competitive Examinations in Mathematics together with hints drawn from Pólya's writings that illustrate how he wanted the problem solver to think about the problem:

> PROBLEM: Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the two other angles.
>
> HINTS: What is the hypothesis? What is the conclusion? Let a, b, and c denote the sides, and A, B, and C the opposite angles, respectively. Then the hypothesis is that $a<(b+c)/2$ and the conclusion is that $A<(B+C)/2$. Look at the conclusion. Could you restate it? (Pólya & Kilpatrick, 1974/2009, pp. 11, 27)

The questions in the hints—What is the hypothesis? What is the conclusion? Could you restate it?—are designed to help guide the solver's work. They are metacognitive questions that get the solver outside the confines of the problem so as to analyze its components.

Pólya developed his interest in pedagogy at an early age. While a postgraduate student at the University of Vienna (1910–1911), he was tutoring a boy in solid geometry and had an unforgettable experience that radically affected his approach to problem solving. At the beginning of the second volume of *Mathematical Discovery*, he recounts that experience:

> It happened about fifty years ago when I was a student; I had to explain an elementary problem of solid geometry to a boy whom I was preparing for an examination, but I lost the thread and got stuck. I could have kicked myself that I failed in such a simple task, and sat down the next evening to work through the solution so thoroughly that I shall never again forget it. Trying to see intuitively the natural progress of the solution and the concatenation of the essential skills involved, I arrived eventually at a geometric representation of the problem-solving process. This was my first discovery, and the beginning of my lifelong interest, in problem solving. (Pólya, 1981, vol. 2, p. 1)

Pólya then shows graphically, using a problem on the volume of the frustum of a right pyramid, how the solution can be visualized as a sequence of connections, building a bridge between what is given and what is unknown (Pólya, 1981, Vol. 2, p. 9). Pólya's (1919) first publication on problem solving and heuristics made use of this means of expressing how a solution might progress. Two years earlier, when he was only 30, he had delivered a speech on teaching at the city hall in Zürich

(Alexanderson, 1987, p. 18), and his publications elaborated the argument he had given in that speech (Pólya, 1938, p. 119).

The heuristic suggestions that Pólya (1945) lists in *How to Solve It*—inside the front and back covers of the original edition and in the front matter of later editions—can also be seen as metacognitive advice to the problem solver, including the advice he gives for understanding the problem and devising a solution plan:

- What is the unknown? What are the data? What is the condition?
- Draw a figure. Introduce suitable notation.
- Do you know a related problem?
- Do you know a theorem that could be useful?
- Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.
- Here is a problem related to yours and solved before. Could you use it?
- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem?

Pólya's advice for carrying out the plan and looking back at the solution is also of a metacognitive nature:

- Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?
- Can you check the result? Can you check the argument?
- Can you derive the result differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

Even though Pólya does not differentiate between advice that is heuristic and advice that is managerial, it is clear from these two lists that his advice has both qualities. For example, to find a more accessible related problem, the solver needs to understand how problems might be related and how the proposed problem might be modified to make it more accessible. To see the result of a problem-solving plan at a glance, the solver needs to be aware of the thinking that led to the result. At the University of Georgia, we have found that acquainting prospective secondary mathematics teachers with Pólya's metacognitive suggestions for problem solving helps them introduce their students to problem solving as an inquiry process.

Cognitive Demand

Stein, Smith, Henningsen, and Silver (2000) identified four levels of cognitive demand that a mathematical task can make on the person confronted with the task. It can require any of the following:

- Memorization (of previously seen material)
- Procedures without connections (to understanding, meaning, or concepts)
- Procedures with connections (to understanding, meaning, or concepts)
- Doing mathematics (with complex, nonalgorithmic thinking)

In our courses for prospective teachers, we give them opportunities to observe tasks being implemented at different levels in the mathematics classes they are observing at local schools. By familiarizing them with the four levels—even though it is not always easy for them to distinguish the levels—we are providing them with a framework they can use in observing and planning lessons. They quickly see how easy it is for a teacher to lower the demand of a task being set for the students, and they also see how difficult it can be to raise that demand when they are planning a lesson or are in the middle of teaching one. By focusing on cognitive demand, we are encouraging an inquiry approach to solving mathematical problems.

For example, we asked our prospective teachers to show us how they might raise the cognitive demand of one or more tasks. One prospective teacher chose the following three tasks from the Balanced Assessment in Mathematics Program (2001):

> A circle of radius 8 units is drawn on graph paper with A and B the endpoints of its horizontal diameter. A figure is given that shows the circle and the two points.
>
> 1. The first task is to draw a triangle ABC with C located anywhere on the circle so that the area of the triangle is a maximum. The task involves drawing and shading the triangle as well as calculating its area and explaining why it is a maximum.
> 2. The second task is to draw a triangle with half the maximum area and explain why it was drawn that way.
> 3. The third task is to find a triangle ABC with minimum area.

As given on the Web site, the first task includes drawings of two triangles ABC that do not have maximum area so as to show students locations for C that do not solve the problem. In her paper, the prospective teacher modified the first task by removing the drawings, arguing that the task would be more challenging if students were asked to create their own examples. The creators of the Balanced Assessment tasks were simply trying to help students along toward a solution, but the modification does raise the cognitive demand in a useful way.

As in so many respects when it comes to problem solving, Pólya (1945) anticipated the question of cognitive demand. He distinguished between routine and nonroutine problems, with the latter making a greater cognitive demand even though he did not use that terminology: "Routine problems, even many routine problems, may be necessary in teaching mathematics but to make the students do no other kind is inexcusable" (p. 158):

> There are problems and problems, and all sorts of differences between problems. Yet the difference which is the most important for the teacher is that between 'routine' and 'nonroutine' problems. The nonroutine problem demands some degree of creativity and originality from the student, the routine problem does not. … I shall not explain what is a nonroutine mathematical problem: If you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed such tension and triumph in one of your students, look for another job and stop teaching mathematics. (Pólya, 1966, pp. 126–127)

Problem Solving as Reformulation

For some final remarks on reformulation, both Karl Duncker and George Pólya made useful observations. Duncker (1945) saw problem solving as productive reformulation:

> We can … describe a process of solution either as development of the solution or as development of the process. Every solution-principle found in the process … functions from then on as reformulation, as sharpening of the original setting of the problem. *It is therefore meaningful to say that what is really done in any solution of problems consists in formulating the problem more productively*. (pp. 8–9)

Similarly, Pólya (1945) noted that we should modify our reformulations to yield a more accessible problem:

> We often have to try various modifications of the problem. We have to vary, to restate, to transform it again and again till we succeed eventually in finding something useful. We may learn by failure; there may be some good idea in an unsuccessful trial, and we may arrive at a more successful trial by *modifying* an unsuccessful one. What we attain after various trials is very often … a more accessible auxiliary problem. (pp. 185–186)

Both Duncker and Pólya recognized that problem solvers need to take an active stance toward a problem, using the tool of reformulation to yield a solution.

Coda

On a Saturday morning in July several years ago, my family was heading to a reunion in upstate New York. We were in two cars, with my younger son and his family leading the way in a rental car equipped with a GPS, and the rest of us following in our car. As we started out of my older son's neighborhood in Brooklyn, he said, "I hope the GPS isn't going to take us across the Brooklyn Bridge into Manhattan," which was of course exactly what it did. When we reached Manhattan, he said, "Oh, no. We shouldn't go down Canal Street"—a street clogged with Saturday shoppers and traffic. Again, that was where the GPS sent us. Once we had crept all the way across Lower Manhattan, we managed to signal my younger son to pull over and convinced him to let us lead the way out of Manhattan.

The lesson: When reformulating a problem of minimizing a variable, in this case, time, one needs to consider factors that might prevent one's minimizing another variable, in this case, distance, from providing an optimal solution. The GPS we had may have known how to minimize distance as a way of minimizing time, but it did not know how to take another variable such as traffic congestion into account (although there are now GPSs with a live traffic update capability). Reformulating requires an awareness of all dimensions of a problem.

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Improving of Mathematical Problem-Solving: Some New Ideas from Old Resources

Bernd Zimmermann

Abstract The main focus of this chapter is on improvement of teaching of problem-solving.

In the introduction important and to some extent neglected issues of teaching problem-solving will be presented and discussed.

On this background the following questions and possible answers will be addressed:

How to combine the traditional mathematical curriculum with problem-solving, understanding, and creativity?

Pupils can construct rules for calculating with fractions themselves, conjectures as well as proofs of the Pythagorean and related theorems.

How history might help to better understand and foster mathematical problem-solving processes?

Observed problem-solving processes of pupils involved in well-known problems from calculating the area of a circle, fractions and calculus can be interpreted as reinvention of old and important heuristic strategies.

Finally, some recommendations are made for teacher education and teachers.

Introduction

In spite of the tremendous affords concerning improving mathematical problems-solving, there are still a lot of obstacles and "problems" with problem-solving.

Cycles "*problem-solving-back to basics*": During the last years some members of the problem-solving community (Schoenfeld, 1992, p. 336, Lesh, 2006, p. 18, Lesh/Zawojewski, 2007, p. 763) deplored that there was no sustainable progress in research of mathematical problem-solving. Some people claim that at least in the USA there are repetitions of 10-years cycles during the last 50 years, going from focusing on teaching problem-solving to teaching the basics and back again (Schoenfeld, 1992, Lesh, 2006, Lesh/Zawojewski 2007).

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Lesh/Zawojewski (2007) conclude that there might be more than these two options. Furthermore, Lester (2013) deplores that very often there are only invented new names for old concepts.

The situation seems to be different in Europe (and, surely, much more different in several other countries), due to more complex political developments especially during the last 25 years and different systems of values, especially concerning education and the image of teachers (cf. Pehkonen, Ahtee & Lavonen, 2007).

There are some further issues, which seem to prevail in most countries:

Constraints of teaching: A pertaining discrepancy between reality of teaching, subjected to test-driven curriculum prescriptions, time limitation, social challenging multicultural classes, and high expectations of school-board and parents—e.g., to teach according to new "Standards"—puts a lot of pressure and stress on many teachers, which might lead again and again to the well-known tradition of teaching the basics, not only in the USA. Additionally, there are some problems with poor teacher education (cf., e.g., Schoenfeld, 2004, p. 91, cf. also Hattie, 2003, p. 7).

The role of the teacher: Problems as the aforementioned ones might help to clarify the importance of the role of the teacher and the quality of teaching—not only with respect to problem-solving. Rather often, this issue is not handled in an appropriate way (cf. Lester, 2013, p. 252):

> "As has been noted in the USA in recent years, it is by (such a) focus on the attributes of excellent teachers that more faith is being restored in the public school system—which has taken a major bashing. The typical redress has been to devise so-called "idiot-proof" solutions where the proofing has been to restrain the idiots to tight scripts—tighter curricula specification, prescribed textbooks, bounded structures of classrooms, scripts of the teaching act, and all this underpinned by a structure of accountability. The national testing movements have been introduced to ensure teachers teach the right stuff, concentrate on the right set of processes (those to pass pencil and paper tests), and then use the best set of teaching activities to maximise this narrow form of achievement (i.e., lots of worksheets of mock multiple choice exams)." (Hattie, 2003, p. 1)

But

> "Interventions at the structural, home, policy, or school level is like searching for your wallet which you lost in the bushes, under the lamppost because that is where there is light. The answer lies elsewhere—it lies in the person who gently closes the classroom door and performs the teaching act—the person who puts into place the end effects of so many policies, who interprets these policies, and who is alone with students during their 15,000 hours of schooling." (Hattie, 2003, p. 3).

Therefore, Hattie (2003, p. 4) concludes:

> "I therefore suggest that we should focus on the greatest source of variance that can make the difference—the teacher. We need to ensure that this greatest influence is optimised to have powerful and sensationally positive effects on the learner. Teacher can and usually do have positive effects, but they must have exceptional effects."

According to Hattie, 2003, 6 pp, expert teachers have (and should have) the following properties:

Table 1 Characteristics of good teaching

- They have deeper and more integrated content knowledge and are more able to connect it with other domains according to the needs of their students and to their goals
- They adopt a problem-solving stance to their work
- They are very flexible
- They like to improvise
- They make a difference between important and not important decisions
- They care for optimal classroom climate
- They have a multidimensional complex perception of classroom situations
- They are more context dependent and have high situation cognition
- They are more adept at monitoring student problems and assessing their level
- They give useful feedback to their students
- They are proficient in developing and testing hypotheses about learning difficulties or instructional strategies
- They are more automatic
- They have high respect for students
- They are passionate about teaching and learning
- They engage students in learning and develop in their students self-regulation, involvement in mastery learning, enhanced self-efficacy and self-esteem as learners
- They provide appropriate challenging tasks and goals for students
- They have positive influences on students' achievement
- They enhance not only surface, but also deep learning (with understanding)

Lester presents some similar duties, which a skillful mathematics teacher should fulfill (Lester, 2013, p. 262):

Table 2 Characteristics of good teaching of mathematical problem-solving

1. Designing and selecting appropriate tasks for instruction
2. Making sense of and taking appropriate actions after listening to and observing students as they work on a task
3. Keeping tasks appropriately problematic for students
4. Paying attention to and being familiar with the methods students use to solve problems
5. Being able to take the appropriate action (or say the right thing) at the right time
6. Creating a classroom atmosphere that is conducive to exploring and sharing

Of course, no teacher is able to fulfill all of these criteria in his teaching in every lesson, every situation, and every moment. But it seems useful to try to concentrate at least on some of these properties and reflect about them from time to time.

Understanding teaching of problem-solving as complex problem-solving (cf. also Mason, 2016). It is obvious that problem-solving is a real complex and challenging mathematical activity (Lester, 2013, p. 255). This statement holds even more for the teaching of problem-solving: It is a very complex and challenging problem to be solved by teachers, especially if you think about the long lists of challenges of the teacher mentioned in Table 2!

> "The relative ineffectiveness of instruction to improve students' ability to solve problems can be attributed to the fact that problem solving has often been conceptualized in a simplistic way." (Lester, 2013, p. 254, cf. also Lesh/Zawojewski 2007, p. 764).

Therefore it seems reasonable to augment ones understanding of problem-solving into the direction of "complex problem-solving," a paradigm, which has been constituted by the cognitive psychologist Dietrich Dörner. He deals with problems included by the following questions: Why does it happen so often, that so many mistakes are done when trying to improve conditions for environment, running a company, governing a city, or running a nuclear power plant (cf. Dörner, 1997)? All these problems have in common to contain many variables, which permanently change, interact, and very often are in-transparent or hidden. That means such problems are very complex. It is quite obvious to assume that the same holds for teaching, even more for teaching mathematical problem-solving.

Furthermore, Dörner and his team developed several computer programs simulating possibilities of interaction with objects just mentioned. They analyzed the problem-solving behavior of subjects using these programs (cf. Frensch & Funke, 1995).

Fritzlar tried to transfer these ideas in his dissertation to the teaching of mathematical problem-solving (Fritzlar, 2003). He developed a very complex computer program which student teachers could use to simulate some important parts of teaching. He placed main emphasis on cognitive activities of the pupils and instruction activities of the teacher. The subjects, using this program, could select a class and some learning goals. When starting the virtual lesson, they had many options to choose an intervention for their not quite virtual class, because the reactions were chosen from a database, constituted by recorded real reactions of pupils of some 50 classes (grade 4 und 5), which tried to solve a given problem (fold-and-cut problem, cf. Fritzlar in Zimmermann, Fritzlar, Haapasalo & Rehlich, 2011). The main goal of this program was to support student teachers to become more sensitive for the complexity of problem-solving and its teaching. This instrument might also help to gain a deeper understanding especially of beginning teachers' difficulties in teaching mathematical problem-solving. Furthermore, it might be used as a training instrument to improve student teachers' ability to cope with complex teaching situation. Of course, such program can be only an additional tool. It cannot substitute comprehensive classroom praxis, which remains, of course, to be the most important factor in teacher training.

One of the advantages of this program is given by the fact that the user could revise a decision, if its effect proved to be not productive. He or she can run this program several times with different goals, different teaching methods, or different classes, too. Most of such possibilities cannot be carried out in reality.

A summary of this project can be found in Zimmermann, Fritzlar, Haapasalo & Rehlich, 2011.

Implementation of problem-solving into the classroom. There is a lack of appropriate literature which helps teachers effectively to **implement** results of the problem-solving research into their everyday classroom-praxis (Lesh/Zawojewski, 2007, pp. 763, 766; Lester, 2013, p. 250, Kilpatrick, 1987, p. 300). This might be also one reason that normal classroom-teachers stick or come back to teaching the basics. Furthermore, it may not be sufficient if textbooks are strongly problem oriented to help to make problem-solving alive in the classroom. On the other hand, several textbook authors are still reluctant to follow this line too strictly because of the constraints of reality (see above and cf. also the paper of Cai, Jiang & Hu, 2016).

A lot of checklists (with teaching resp. problem-solving strategies) and corresponding training for teachers (for teaching) as well as for pupils (for problem-solving) seem to be without sustainable effect (cf., e.g., Bauersfeld, 1993, p. 79, Lesh, 2006, p. 16, Begle, 1979, cf. also Freudenthal 1991, p. 46).

Therefore—because of the aforementioned lack of appropriate literature and methods which could help to implement mathematical problem-solving into the classroom—we try to focus in this chapter especially on the teacher.

We want to revitalize a very old and well-known method: Presenting authentic (original) examples of problem-solving lessons, guided by classroom discussions, reconstructed by the author from his notes and by his memory, augmented by some remarks or reflections concerning courage, patience, self-esteem, flexibility, understanding, selecting appropriate problems, heuristics, discovery, invention, creativity, constructivist activities, questioning or listening and timing (cf. Tables 1 and 2).

There is a long tradition in such writing style in the scientific literature: think, e.g., of the "Menon-Dialogue" from Platon (Anderson & Osborne, 2009), Galilei's "Discorsi" (Galilei, 1985), Rényi's "Dialogues about Mathematics" (Rényi, 1967), and Lakatos's "Proofs and Refutation" (Lakatos, 1976). But our "dialogues" were not invented, but occurred in real classrooms. The aforementioned examples were all invented to trigger in the reader a "holistic" picture of the respective "philosophy"—e.g., Platon: learning is remembering, Lakatos: Mathematics is created by conjectures, proofs, and refutations. We share this holistic view, too, with respect to mathematics teaching and learning.

Therefore, we try to avoid the well-known method of operationalizing all kind of teaching and problem-solving activities by separating them into "tiny, watertight compartments" (Hilton, 1981).

This holistic view should be supported by referring to ideas from history of mathematics. "History teaches us how mathematics was invented" (Freudenthal 1991, p. 48). This view does not only help to better understand mathematics of today, but it might help the teacher to understand better the re-inventing processes and problem-solving processes of pupils, too.

Our examples should invite the readers to think about their teaching or to probe similar lessons rather than to learn a new vocabulary to talk in a "scientific manner" about teaching problem-solving.

Therefore we prefer more a textbook style, writing about real experience of other teachers or my own.

So, when trying to do research, I think we must never forget the simple question: "What is the use of it"? (Freudenthal 1991, p. 149).

Thus, one might refer to the following well-known statement in relation to mathematics education, too:

"The philosophers have only interpreted the world in various ways. The point, however, is to change it" (Karl Marx).

Part 1: Teaching Mathematical Problem-Solving and the Traditional School Curriculum

As to the experience of the author, many teachers—e.g., even more in Eastern than in Western Germany—take challenging problems—like those from the well-known books of Martin Gardner, Averbach/Chein, 1980 or Mottershead, 1985 or even of Pólya (1954, 1973, 1980) and Mason, Burton & Stacey, 2010—as not belonging to the curriculum they are used to. Thus, for many of them such problems constitute more an "unnormal" curriculum, distracting them from their duty to cover all the content-stuff of mathematical concepts and rules in a given time (cf. Zimmermann, 1991a, 1997).

As a consequence of poor test-results of PISA and TIMSS new curriculum-standards had been developed in Germany since a couple of years, partly in accordance to the standards of the NCTM, 2000. But it takes a lot of time to change teaching traditions.

Thus, one can assume that much more teachers would appreciate the problem-solving approach, if they could experience that teaching the traditional school curriculum—which still tends to be oriented towards classical content mainly—could be combined in a natural way with teaching problem-solving.

We want to demonstrate that this is possible by presenting some real-classroom examples from the classical contents "fractions" and "Pythagorean Theorem."

Example 1: Comparing Fractions We are in a class of sixth-graders in a Hamburg comprehensive school of a workers-district with some 80 % of migrants. Many of them are not capable to speak German fluently—so, obviously there was no elitist situation! In the previous lessons fractions were treated, the pupils had learned already how to expand and reduce fractions.

The following lesson was the first one about ordering of fractions.

At the beginning of the lesson the teacher wrote at the blackboard and asked the pupils:

Problem 1
How would you arrange the following fractions according to their size: $\frac{5}{6}, \frac{3}{7}, \frac{2}{3}$?

We invite the reader to stop reading here for a short moment and solve this little problem her- or himself.

Now please think for a while about your strategies you applied!

Of course you know the following often practiced schoolbook-strategy:
"Compare two fractions in the following way:

1. Determine their common denominator.
2. Multiply the respective numerator of each fraction by the same factor by which you have to multiply its denominator to get the common denominator.
3. Order now the new numerators according to their size.
4. The order of these fractions yields the order of the corresponding equivalent original fractions."

Your own experience and that one of several teachers may be often *quite different* from the following procedure, which was suggested by pupils of our class:

The fraction $\frac{3}{7}$ is less than $\frac{1}{2}$, the fraction $\frac{2}{3}$ is larger than $\frac{1}{2}$, therefore $\frac{3}{7} < \frac{2}{3}$.

$$\text{Furthermore, } \frac{5}{6} \text{ is closer to 1 than } \frac{2}{3} \text{ is close to 1, therefore } \frac{2}{3} < \frac{5}{6}.$$

Therefore, the final order is $\frac{3}{7} < \frac{2}{3} < \frac{5}{6}$.

Looking back:

Lester, 2013, p. 260, refers to a similar situation as "teaching via problem-solving."

The pupils created their own strategies "referring to the cornerstones $\frac{1}{2}$ and 1"—without "heuristic-checklists" or any other advice by the teacher—and came to the result which they presented at the blackboard. They did not know the rule quoted above in advance.

It is very important to select simple examples for fractions to increase the probability that pupils can invent such strategies.

Of course, after some additional similar examples, the teacher can stimulate his pupils to reach out for a general, always working rule by letting them search for further examples with increasing numerators and denominators.

Example 2: Dividing Fractions We are still in the same class some time later. The pupils just learned in the previous lesson the rule how to multiply fractions:

> Two fractions are to be multiplied by multiplying the corresponding numerators and denominators.

At the beginning of the next lesson the teacher asked the pupils to repeat the rule they just learned about the multiplication of two fractions. Then he asked:

Problem 2

Could you conjecture a rule how to divide one fraction by another fraction?

After some time of discussion between the pupils one pupil gave the following answer:

> "To divide two fractions, I have to divide the numerator of the first fraction by the numerator of the second fraction; then, divide the result of 'the denominator of the first fraction divided by the denominator of the second fraction'."

Once again I would like to invite the reader too think about this answer and possible reaction if she or he would be the teacher.

After you thought about some possible reaction (e.g., one could be to say nothing and wait for the reaction of the other pupils! At the same time you can think about the mathematical content) perhaps you might think about the question, why the teacher presented problem 2 immediately after he let the pupils repeat the rule they learned during the previous lesson.

The teacher never heard about a rule like this one! Therefore it was a real problem-solving situation for our teacher in two senses: 1. Mathematics: What about the mathematical value of this answer? 2. Education: How to react to this statement?

Of course he intended to come to the well-known rule:

"You have to divide one fraction by another one by multiplying the first one by the reciprocal of the second one."

I (the author of this contribution) presented this scene to my student teachers and asked them: How would you react if you would have been the teacher of this class?

Answers came as "I would try to help the pupils to understand, why this conjecture is wrong."

The teacher of this class wasn't sure about this conjecture either, but he was a very experienced and sensitive teacher with sufficient courage and self-confidence. So he said: "Let's check this conjecture by simple examples."

Let us start with $\frac{4}{9}:\frac{2}{3}$.

The pupils calculated $\frac{4:2}{9:3}=\frac{2}{3}$. Of course one has to check the result by reversing the process: as $\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$, therefore the equation $\frac{4}{9}:\frac{2}{3}=\frac{2}{3}$ is true. So the conjecture is true, if the numerator and denominator of the second fraction divide the numerator and the denominator of the first fraction, respectively.

What can we do, if the situation is not that easy? Let us continue carefully! E.g., what happens in case of $\frac{5}{9}:\frac{2}{3}$? When the door squeaks, you have to lubricate the hinges, to make them work smoothly again!

As 5:2 "squeaks," perhaps we can make the division by 2/3 work by "lubricating" the fraction $\frac{5}{9}$ by appropriate expansion (by 2): $\frac{5\cdot 2}{9\cdot 2}:\frac{2}{3}\overset{?}{=}\frac{(5\cdot 2):2}{(9\cdot 2):3}=\frac{5}{6}$. This equation holds, as the test yields $\frac{5}{6}\cdot\frac{2}{3}=\frac{5}{9}$. Therefore the conjecture is true also in this case.

What to do now with $\frac{5}{11}:\frac{2}{3}$?? We apply again the expanding strategy, as it proved already to be successful in the previous case:

$$\frac{5}{11}:\frac{2}{3}=\frac{\big((5\cdot 2)\cdot 3\big)}{\big((11\cdot 2)\cdot 3\big)}:\frac{2}{3}=\frac{\big((5\cdot 2)\cdot 3\big):2}{\big((11\cdot 2)\cdot 3\big):3}\overset{?}{=}\frac{5\cdot 3}{11\cdot 2}=\frac{5}{11}\cdot\frac{3}{2}. \qquad (*)$$

As the first term equals the last term in (*), the class gained also the well-known rule: To divide one fraction by another one, the first fraction is to be multiplied with the reciprocal of the second. The rule is correct as it is demonstrated by the last check: If we multiply the last term in (*) by $\frac{2}{3}$, we get $\frac{5}{11}\cdot\frac{3}{2}\cdot\frac{2}{3}=\frac{5}{3}$. And indeed: The conjecture is always true, as the whole calculation in (*) is completely independent from the type of natural numbers we use for numerators and for denominators. Therefore, it would not make a structural difference if we would use letters (names for variables) instead of concrete numbers. This would not change the mathematical outcome, but probably the learning outcome (esp. the understanding!) of the pupils!

Looking back

At the beginning of the lesson the teacher let the pupils repeat the rule they have learned in the previous lesson. By asking now for a conjecture for division of two fractions, he provoked his pupils to make an interesting conjecture.

It is important that the teacher creates an appropriate climate in the class, which helps to support self-confidence and courage to make conjectures. Especially it should be clear to the pupils that it is no problem to make mistakes.

These attitudes and properties are important for the teacher to foster problem-solving processes of his pupils (cf. Tables 1 and 2).

After the conjecture has been formulated, it is very important that the teacher helps his pupils (if necessary) to check the conjecture. E.g., the teacher could ask for appropriate examples. If the pupils cannot find such examples themselves, it is important to suggest such examples, stepwise increasing their complexity.

In this way we have the scenario of guided re-invention of a well-known rule.

Example 3: The Pythagorean Theorem This is another standard issue of the traditional curriculum in nearly every country. The didactical problem is not how to prove the theorem—there are several hundred proofs (cf. Loomis, 1972), but how to create such classroom setting, that pupils have a chance to re-invent the theorem (the corresponding conjecture) by themselves, including its proof.

Sometimes I learned about introductory scenarios for the Pythagorean Theorem as follows:

Pupils had to explore several rectangular triangles of different size and shape by measuring the lengths of the corresponding sides a, b, and c. Then they have to make a table with six columns and record their results for a into the first column, the results for b into the third, and for c into the fifth column. Subsequently the teacher asks his pupils to calculate a^2, b^2, and c^2 and record it into the neighboring empty columns to the previous noted values of a, b, and c. Then the teacher asks the pupils: What can you observe?

"Discovery learning: i.e., uncovering what was covered by somebody else—hidden Easter eggs." (Freudenthal 1991, p. 46)!

Thus, the main reason for the following approach is guided by the constructivist philosophy, namely to look for opportunities which might help teachers to let pupils reinvent some mathematics, especially from the traditional curriculum.

We try to present here one possible approach via similarity of triangles.

Let us assume that we are in grade 8 or 9 and the class has sufficient knowledge about similarity of triangles. They know especially:

> If in two triangles corresponding angles are congruent, then the triangles are similar. The ratios of corresponding lengths are equal.

Thus, let the pupils draw a rectangular triangle ABC and let them separate it into two triangles by drawing the altitude h. These triangles and the large one are similar because corresponding angles are congruent (Fig. 1):

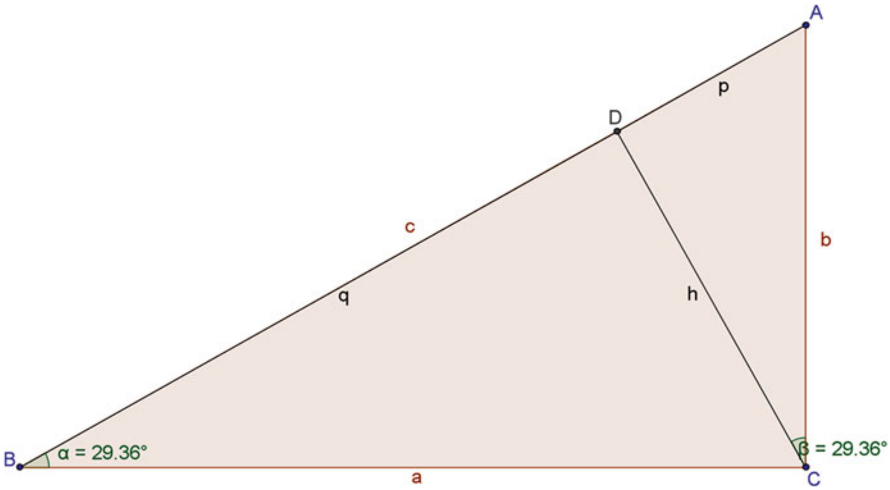


Fig. 1 Nearly all pupils can figure out this result, if they are accustomed with similarity

Problem 3
Now let the pupils investigate ratios of lengths of sides, which have to be equal because of similarity of corresponding triangles.

E.g., pupils may figure out that $q{:}h = h{:}p$. As to the teaching experience of the author, it is now important that the pupils have learned during their algebra courses to look for visual representations (interpretations) of algebraic terms, e.g., $a \cdot b$ could be interpreted as area of a rectangle with sides a and b. The equation between ratios can be interpreted as an equation between fractions, too. The pupils know at that age how to multiply in such equations "crosswise," which let them find the equation $h^2 = p \cdot q$. This is the well-known "**altitude-theorem**." Both sides of the equation can be interpreted as areas of rectangles: The square formed by the altitude h is equal to the rectangle, formed by the projections of the two legs p and q of the rectangular triangle on its hypotenuse.

In higher grades this equation could be also interpreted as an equation of a square-root-function (for clarification let $p = x$ variable, $q = 1$ constant, $h = y$).

Furthermore this situation could be found in history of mathematics, when more than 2000 years ago Menaechmus tried to solve the famous problem of doubling the cube graphically. In this context the "altitude-theorem" might be interpreted as the beginning of the rectangular coordinate system (cf., e.g., Bretschneider, 1870, Zimmermann, 1991a).

More examples for equations between ratios and (well known for the reader) graphic interpretations might be found as follows:

$b : p = c : b \Rightarrow b\dagger = c \cdot p$ (**cathetus theorem 1**),

$a : q = c : a \Rightarrow a\dagger = c \cdot q$ (**cathetus theorem 2**).

If we add both equations at right we get $a^2 + b^2 = c \cdot (p + q) = c^2$, the **Pythagorean Theorem**.

After some experience with several classes I came to the conclusion: It is true that pupils can explore and result in those theorems, getting conjectures and proofs nearly at the same time.

Now let us look ahead of these traditional results. Can the pupils find more equal ratios, which might be interpreted in a similar way, giving more insights or rules?

E.g., one can find $b{:}c = h{:}a$ which lead at once to $a \cdot b = h \cdot c$, dividing by 2 yields two possibilities to calculate the area of the large triangle, which might be a good experience of success for mediocre pupils.

If we consider the equation $a{:}b = h{:}p$, one can derive $a \cdot p = h \cdot b$ which does not make much sense. Of course one can try also $a \cdot (c - q) = h \cdot b$ which might be transformed into $a \cdot c = h \cdot b + a \cdot q$, which seems to be not very helpful either. Can you find some sense?

Looking back

Given that pupils are well acquainted with similarity of triangles—they have a very good chance, to reinvent and prove at the same time well-known theorems of the standard curriculum by this approach. The author practiced this approach for several times in his classes during his time as a schoolteacher.

This approach is not very time-consuming—so it should be attractive for teachers.

It is very important that the pupils have got sufficient previous experience in representing simple formulae in a geometric way and vice versa.

Given such experience, there is a good opportunity to learn more about the power of the heuristic "change of representation" (from geometry to algebra and back) by finding and proving theorems. The teacher should carefully think about the possibility to let his pupils reflect about this heuristic explicitly! Could such discussion help to improve pupil's problem-solving competence? Could there be an appropriate moment to do so?

Once again we have an example, where the teaching of standard content can be combined with problem-solving and investigative situations for pupils.

Example 4: The Pythagorean Theorem: Al Sijzī and Problem Fields We refer to a generalization of the Pythagorean Theorem by the Persian mathematician Abu Sa'id Ahmad ibn Muhammad ibn 'Abd al-Jalil al-Sijzī, who lived in the tenth century. The corresponding paper (al-Sijzī/Brentjes 1996) had not been published until now. This is an excerpt of a paper of Zimmermann, B., Fritzlar, T., Haapasalo, L. & Rehlich, H., 2011.

This problem has also been tested in real classrooms for several times, too, but it may be somewhat harder than the foregoing ones. You can let your pupils try to cope with it at the end of a corresponding teaching unit.

Problem 4

Given a rectangular triangle with its Thales-circle. "Move" A and B in such a way out of the Thales-circle that the new points A′ and B′ are also located symmetrical to its center M. So you get the new triangle A′B′C with sides a′, b′, c′.

What can you figure out about the sum of the squares $(a')^2+(b')^2$?—Hint: Imagine moving C on the old Thales-circle to C′. What does happen to $(a')^2+(b')^2$?

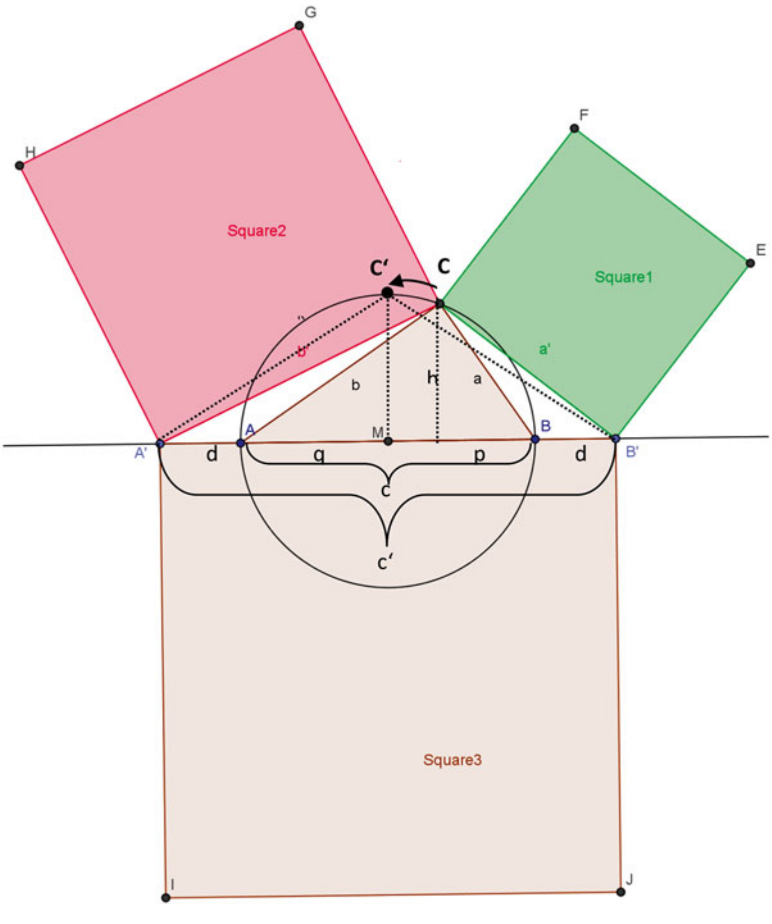


Fig. 2 Comparing the size of the squares when moving C along the circle and therefore the whole triangle to other positions; cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziPythagorasGeneralized0.ggb

If you delete the squares in Fig. 2, you will come to the geometric situation in Fig. 3, which represents a draft from the original version of this problem:

Fig. 3 Part of the manuscript of al Sijzī (copy from a microfilm)

Solutions Pupils come rather often to the conjecture $a'^2+b'^2=\text{const}$ spontaneously. In case of no ideas -but do have enough patience to wait for them!- dynamic geometry software (DGS) can help. But I recommend to use it only in this situation!

A proof for the conjecture $a'^2+b'^2=\text{const}$ can be carried out in several ways and so has been done by al-Sijzī, too (cf. al-Sijzī/Brentjes 1996). One possibility is given here. By examining the special degenerated case by dragging the point C to the point B (cf. Fig. 2, applying the strategy "examine special cases"!), the conjecture can be posed more precisely in the following way:

Theorem $(a')^2+(b')^2=(c+d)^2+d^2(=const)$.

Proof Focusing on the altitude h (cf. Fig. 2) and applying four times the Pythagorean Theorem, we get the following two equations:

$$(a')^2=h^2+(p+d)^2=h^2+p^2+2pd+d^2 = a^2+2pd+d^2$$

$$(b')^2=h^2+(q+d)^2=h^2+q^2+2qd+d^2=b^2+2qd+d^2,$$

we add these equations and receive

$$(a')^2+(b')^2=a^2+b^2+2d(p+q)+2d^2$$

and finally, again applying the Pythagorean Theorem and using $p+q=c$, we receive

$$(a')^2+(b')^2=c^2+2dc+2d^2=const. \qquad \text{(q.e.d.)}$$

Let us take another "view" on this problem:

This theorem can be represented graphically by the following Fig. 4 ($(c+d)^2+d^2=c^2+2cd+2d^2$). The application of the strategy "change of representation" (from algebra to geometry) may not only be a nice exercise for your students, but can also help them to better understand the theorem. Now they can "see," why c^2 is less than c'^2 and what makes the difference.

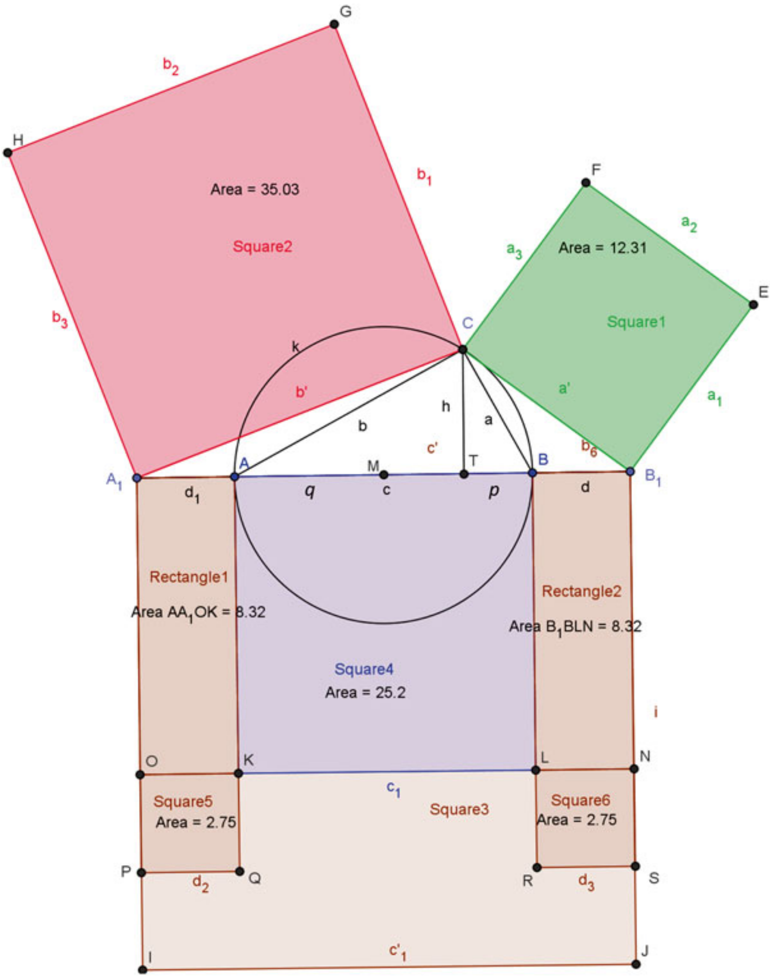


Fig. 4 Representing the area of square1 and square2 by parts of square3; cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziPythagorasGeneralized1.ggb

This solved problem can be the starting point to ask your pupils for more generalizations and variations, so expanding the initial situation to a whole "problem-field":

Al-Sijzī presents a first simple generalization himself: He "moves" the points A, B *into the inner* of the Thales-circle. He proves that $(a')^2 + (b')^2 = \text{const.}$ holds also in this case.

This could be another nice exercise for your pupils.

Furthermore, one can converse problem 4:

Problem 5

Given two fixed points A and B in the plane. What is the locus of all points C in the plane with the property $AC^2 + BC^2 = \text{const.}$?

After finding (and proving) the answer (a circle), one can continue to generalize this problem:

Given two fixed points A and B in the plane. What is the locus of all points C in the plane with the property

$$AC^n + BC^n = \text{const.}, n \in N\,?$$

In this way one arrives at Fermat-curves and higher mathematics, which might be especially interesting in a course for gifted pupils (more details are outlined in Zimmermann, B., Fritzlar, T., Haapasalo, L. & Rehlich, H., 2011).

Looking back

Pupils—and not only the gifted ones!—are able to *create and apply* very often very reasonable strategies to solve elementary problems like ordering fractions (*without* previous training in a specific method!).

They are also able to create standard rules—guided carefully by a talented teacher.

The learning of routine techniques and their reasonable application might also be improved by this approach.

Creating situations where pupils have a good chance to create their methods might help the pupils to *learn that learning* (*mathematics*) *means*, too, *to construct something actively within their heads* and not only absorption and transmission of *outside* knowledge *into* their heads (the main idea of *constructivism*).

This can be achieved by referring to suitable problems from history of mathematics and modern technological tools, too.

In these ways, the *motivation* of the pupils might be reinforced, because they learn that *their* thinking is the main focus and generator of mathematics instruction.

Creativity, problem-solving, and teaching the traditional curriculum are no excluding alternatives!

In any case lessons like those presented here require teachers, who are educated appropriately which means sensitive and open towards the possibilities of pupils as well as to the subject and who are themselves sufficient curious, courageous, flexible, and creative.

Pupils need such teachers to become (or stay!) courageous and creative themselves.

This should be one of the main concerns of teacher education!

Part 2: Identification and Fostering of Problem-Solving Processes via "Lenses" from History of Mathematics

There are many good possibilities and reasons to learn from history of mathematics to teach mathematics (cf., e.g., Zimmermann, 2009).

One neglected aspect could be to analyze history of mathematics as a cognitive long-term study to figure out which methods of thinking, esp. problem-solving strategies, proved to be especially successful during the last 5000 years with respect to solving and creating new and important problems. One could expect that such strategies could be useful also in future and as an additional background knowledge of teaching (cf. Zimmermann, 1990, Zimmermann, 1991a).

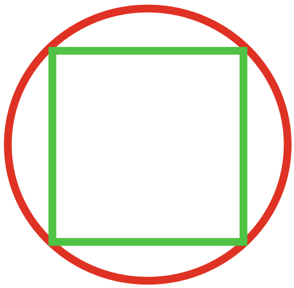
Therefore we focus now on the history of mathematical problem-solving processes in order to get an additional tool to better identify and foster problem-solving processes of pupils of today.

The following examples come all from real teaching lessons and they were first published in Zimmermann, 2009.

Example 1: Area of a Circle à la Antiphon and Bryson Some years ago I taught my sixth-graders to calculate the areas of some simple geometrical figures as squares, rectangles, and rectangular triangles. The following dialogue with a quite clever and creative student (let us call him Jens) took place in the classroom at the end of this teaching unit, which I paraphrase according to my memory:

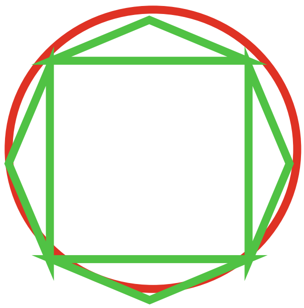
Jens (*asking the teacher*): Why not calculate also the area of circle?
Teacher: That's too difficult now, we'll get it later!
Jens: Why not do it in the following way? (He went to the blackboard and started drawing with the comments as follows.)

Here we have a circle and we inscribe a square as follows:



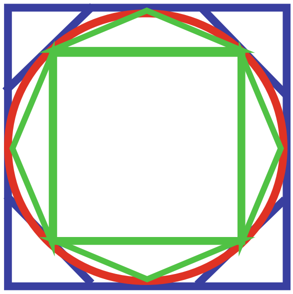
The area of the square is a rough approximation of the area of the circle.

We can improve this approximation by putting the following "hats" on the sides of the square:



We can continue this process of refinement on the sides of the new "hats."

But we can carry out a similar process also from outside. Therefore, let us additionally circumscribe a square around the circle:



The mean between the area of the inscribed square and that of the circumscribed square should yield a better approximation of the area of the circle than each of both squares alone. If we cut off the corners from the outside square in a symmetric way such that the cuts are tangent to the circle, one gets a better approximation of the square by a circumscribed regular octagon. The mean of the areas of the circumscribed octagon and the inscribed octagon should be an even better approximation of the area of the circle than the mean of the areas of the inscribed and circumscribed squares. This process can be continued as far as we want.

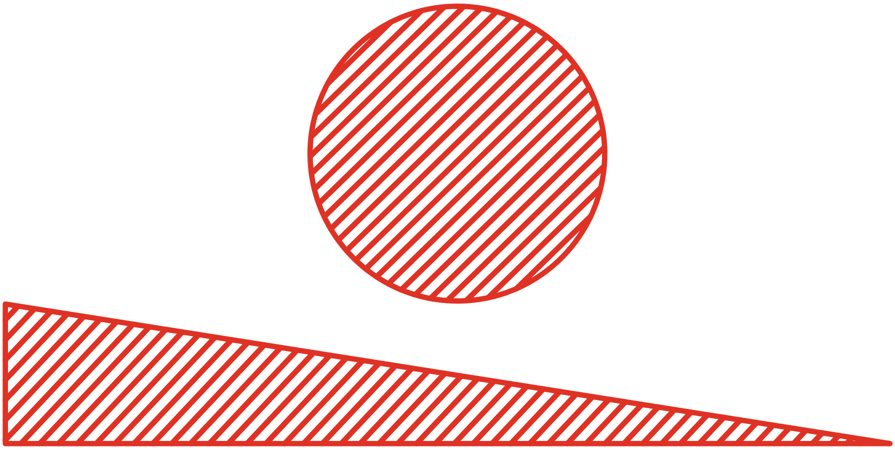
Looking back

Of course, I was baffled by this splendid idea of Jens. I became even more impressed when I learned later on from history of mathematics that this idea was a reinvention of some of the ideas of Antiphon and Bryson which they presented more than 2000 years ago, when they tried to solve the famous problem of squaring the circle (cf. Bretschneider, 1870, pp. 100; 126). One can interpret the method Jens is approaching here the calculation of the area of a circle as a method of successive approximation (cf. Kilpatrick, 1967). Antiphon and Bryson used it for another goal.

Archimedes used these ideas in his famous "Measurement of a circle" to find a proof for the assumption about the area of the circle, which constituted a method for proving, which was called later "method of exhaustion" (cf. Heath, 2002).

Example 2: Area of a Circle à la Archimedes Resp. Kepler Some years ago, when introducing integral calculus, I presented the following theorem of Archimedes to my pupils (grade 12):

"*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius* (R)*, and the other to the circumference* (C) *of the circle*." (cf. Heath, 2002, p. 91)

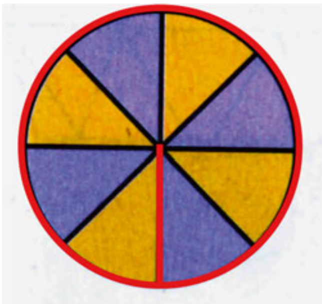


Then I asked them (I quote from my memory again):

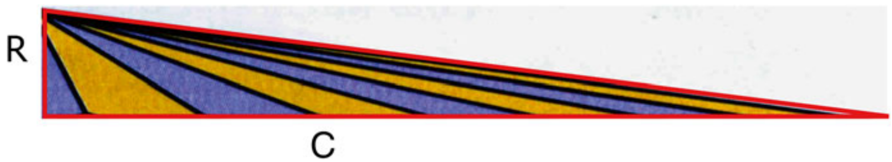
"What do you think, how Archimedes might have come to the idea to transform the given circle into this rectangular triangle?"

A girl (Birte) gave me the following answer:

"Let us assume the circle is divided in as many small triangles as we want. All triangles have the center of the circle as one point in common; one side—nearly straight, if the number of triangles is very high—is part of the circumference, having the same length for all triangles:



Now I assume that the endpoints of all basic sides of the triangles are connected by elastic bands with the center of the circle. I make a cut along one side of two adjacent triangles to this center. Then I "take" the left end of the circumference of the circle—assuming it to be a wire—and bow and stretch it until it is a straight line:



So we get a rectangular triangle, which has the circumference C of the circle as one side and its radius R as another side. All triangles are transformed into triangles of the same basis and altitude (the radius) as the triangles of the circle. So the area of the rectangular triangle is the same as the area of the circle."

Looking back:

Very often professional teachers react to ideas like these in such a way that they say that this a nice (heuristic) idea, but they stress that this is by no means a proof, because the basis of the little triangles are never straight and the altitude of these triangles is always somewhat smaller than the radius of the circle. So this idea has to be improved by or augmented to a rigor argument.

Of course, this statement is correct. But—as to my opinion—it does not take into account the importance of the quality of ideas sufficiently. That this idea is a real great one can be underlined by the fact that it had been expressed by Kepler nearly in the same way some hundred years ago (cf. Struik, 1986, p. 194; Kepler, 2000, p. 15). The mathematical thinking of Kepler was completely bound to the geometric tradition constituted by Euclid and Archimedes. By thoughts as the quoted ones he could determine the volume of many geometric solids.

It should be also mentioned that the method of Birte (and Kepler) can be interpreted as the use of atomistic methods, which constituted not only a heuristic, but also a quite different approach to analysis, which run parallel to the classical Cauchy-Riemann approach (which began with Archimedes) from Democrit, Cavalieri, and Leibniz and culminated in modern nonstandard analysis (cf. Zimmermann, 1991a).

Furthermore, one can assume that the reason why Archimedes came to the idea to transform the given circle into a rectangular triangle was initiated by the attempts of Hippocrates to square at least parts of the circle (the lunes) by finding rectangular triangles of the same area (cf. Heath, 1981, pp. 195, 196).

As in case of our pupils, so it is in history of mathematics: you can really understand neither modern pupils nor modern mathematics on the basis of actual time-events, but only on the basis of previous time-events!

Therefore, knowledge about possible historic background as in this example might help to even better understand and appreciate the thoughts of pupils.

Example 3: Introducing Addition of Fractions Our student teachers in Jena had to make a course on teaching practice in which they went each week to school during their term to held prepared lessons to get first teaching experience. Within this course one of our student teachers presented in the first teaching-lesson of her life the following standard schoolbook-problem (which she selected herself) to the class (grade 6), which should help to introduce addition of fractions. The pupils were already familiar with the addition of fractions with equal denominator.

Peter gets a specific amount of pocket money for 1 week. Now he takes away the following portions from this starting budget: He spends 3/10 for a ticket for a movie. He buys a cake for ¼. He likes ice-cream very much, therefore he buys one for 1/5. Finally he buys chewing-gum for 1/10. At the end of the week there is 1.50€ left.

How much pocket money Peter got in this week?

Remark: This problem has a long history. First testimonies can be found in a collection of problems from the twelfth century very often in form of poems called "Leelavati." Here it deals about a pearl-necklace (cf. Srinivasiengar, 1988).

One has to take into account that the pupils had the following *prerequisite knowledge* about fractions:

- They learned some basics about fractions and different representations.
- They knew how to expand and how to reduce fractions.
- They made first experience with the addition of fractions with the same denominator.

After some 10 min of cooperative work two girls presented the following solution at the blackboard (P = pupil):

P: Let's assume that he receives 20€ in this week. Now we use and check the given data.

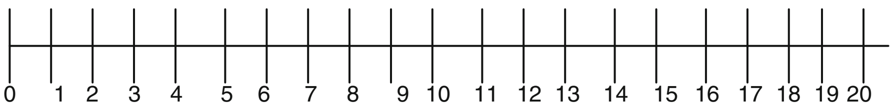
The student teacher (T)—who knew and prepared for THE right solution, hesitated strongly and said to the pupils:

T: But you don't know the answer until now and you cannot be sure that 20€ is the right solution.

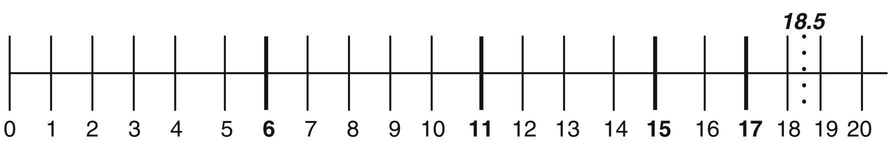
One of the pupils answered:

P: We don't mind. We assume this and let's see what will happen if we continue now!

One pupil drew a long (part of the number-) line at the blackboard and marked 0 at the beginning and 20 at the end of this line. Subsequently, she divided it into 20 parts of equal length:



Then they started—beginning at 0—by marking the appropriate amount of money after changing all given fractions into fractions with denominator 20:



P: First Peter spent 6€ for the movie (first fat line and so on), then he bought a cake for 5€. Furthermore the ice-cream was 4€. Finally he had to pay for the chewing-gum 2€. So we come to 17€. Then there should be a rest of 1.50€. Therefore, there is now a total of 18.50€ (dotted line). But there would be still 1.50€ left!?

Stop reading now and think about this situation!
How would you react as a teacher in this moment?

T: You see now, that the solution of 20€ you assumed must be wrong. Thus we have to look for another solution.

In this way the student teacher brought the process of the pupils to an end. She presented now a complete different attempt to the pupils without using hardly any ideas of them.

Another possible continuation of this discussion, including the ideas of the pupils, could have been as follows (T = teacher):

T: So you have still some money left. Or: So you don't have left 1.50€ but 3€—twice as much you should have. What to do now?

P: Possible reaction: If the amount left is twice as much we should have, perhaps we assumed also twice as much pocket money than Peter should have at the beginning of the week. So we have to divide 20€ by 2 and get 10€.

It might follow now a check (verification) of this new assumption.

The process which had been offered by the pupils can be interpreted to a considerable extent as the beginning of the procedure of the method of "false position," which is well known from history of mathematics.

The method of *false position* had been applied already by ancient Egyptian mathematicians (cf., e.g., Chace, 1986) and can be seen, e.g., in relation to the "regula falsi," the method of successive approximation (up to modern forms of iterations) and working backwards (cf., e.g., Zimmermann, 1995).

To get an idea of this method, we quote here an English translation of a typical example from Rhind Mathematical Papyrus, written about 1560 B.C. with content from about 1800 B.C. (cf. Clagett, 1999, p. 141, remarks in brackets [...] added by the editor. Cf. also Chace, 1986, p. 69).

"[Problem 26]

A quantity with ¼ of it added to it becomes 15.

[Assume 4]

[That is] multiply 4, making ¼, namely 1, [so that the] total is 5 [proceeding in the usual manner]:

\1	4
\¼	1
Total	5

[As many times as 5 must be multiplied to make 15, so many times 4 must be multiplied to give the required number.]

Operate on 5 to find 15

\1	5
\2	10
Total: 3	

Multiply 3 times 4.

1	3
2	6
\4	12

This becomes 12. [And find its ¼:]

1	12
\¼	3
Total	15

[Hence] **the quantity is 12** and its ¼ is 3 and **the total is 15**.

[This checks out since the sum agrees with what was originally specified.]"

Looking back:

It is possible to interpret the pupils' approach as a re-invention of the method of false position (cf. also Chabert et al., 1999). Furthermore, by drawing a number line, one can also say that the pupils made three *changes of representation*: first, they changed the problem from a problem with fractions into a problem with integers; second, they made a visual representation of the numbers; and, third, they represented the addition of numbers by the composition of lines of appropriate lengths.

Altogether: The suggestions of the pupils have been full of productive heuristics.

One might conclude that using history of mathematics with focus on problem-solving processes can yield to more insight in the importance of heuristic strategies. But it is not necessary to make them explicit, because the examples demonstrate that pupils can (re-)invent such methods themselves (without any checklists!). The teacher has carefully to think about an additional use for the pupils which may or may not have an explicit discussion about such strategies.

This example demonstrates in a particular way that it is possible on the background of history of mathematics (besides well-known learning theories) to interpret, to understand, and to esteem problem-solving processes of pupils appropriately.

Summary

One might get the following possible insights and conclusions by the previous sections:

The teaching of contents like fractions and Pythagorean Theorem from the traditional curriculum can be very often combined with teaching of problem-solving and nearly every pupil can be reached. Appropriate questions and careful guidance can help to stimulate pupils to (re)invent classical rules and strategies in their own way. To do so, teachers should be strongly encouraged and motivated to orient themselves even more towards listening and being sensitive and open towards the possibilities of their pupils as well as to the subject.

Analysis of 5000 years of history of mathematical problem-solving unravelled several heuristics having been extraordinarily successful. When observing problem-solving processes of pupils of today it is sometimes possible to interpret them, e.g., as (reinvention of) the method of successive approximation, atomistic method, and false position. All these methods proved to be very successful heuristics across time (Zimmermann, 1990, Zimmermann, 1991a). Thus, such knowledge about history could help teachers as an additional tool to identify, appreciate, motivate, and foster pupils' problem-solving abilities.

Teachers are invited to think even more about questions like: If I expect my pupils to be more curious, courageous, flexible, and creative when solving mathematical problems, what about me when teaching problem-solving?

Some Conclusions

Trying to improve problem-solving has to focus on some recommendations mainly for teachers and their education.

There seems to some extent a demand for synchronization between teacher education and pupil education.

Teachers should have the opportunity in their studies to solve (and pose) appropriate problems of similar—sometimes also more demanding—type they use as a teacher later on at school.

They should solve problems also in working-forms, which they have to arrange in their later work as a teacher (single, partner, group, outdoor-work).

Additional to systematic course in single subjects of education and mathematics, they should become acquainted with examples of lessons, e.g., like from the TIMSS-video-study.

The beginning teachers should have in their seminars the opportunity to simulate little parts of lessons, teaching their fellow students solving a problem, who should try to act for a short while like pupils at school.

A special emphasis should be placed on the traditional curriculum which should be analyzed with respect to its potential to be combined with problem-solving.

There should be additional courses of history of (elementary) mathematics. Of course—in case that there is no time—parts of history should be integrated into the normal content courses.

Teachers should mutual visit one another in their classes, especially when teaching problem-solving. They should prepare lessons together and discuss approaches to problem-solving (e.g., also video-taped lessons).

Of course, teacher should have enough time for such activities, so we need a reduction of bureaucracy, but this is surely an everlasting wish from all sides!

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Part 1 Reaction: Problem Posing and Solving Today

John Mason

Abstract The chapters in this section, drawn together as a whole, illustrate the complexity of mathematical problem solving, both as an activity itself and as a way of teaching mathematics. Attention is paid to affect, to problem posing, to the largely absent support from textbooks for a problem solving, much less problem-posing stance, and to the effectiveness of adopting an inquiring stance to mathematics teaching, allowing students to bring to the fore their own metaphors for construing and creatively resolving mathematical problems.

Centrality of Problem Solving in Mathematics

That mathematics is about solving problems (with origins both within mathematics itself and in the material world) has been attested to by many mathematicians. Paul Halmos is perhaps one of the most direct:

> The mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions. (Halmos, 1980 p. 519)
>
> One of the hardest parts of problem solving is to ask the right question, and the only way to learn to do so is practice. (Halmos, 1980 p. 524)

Note that this means practising asking questions, not simply practising on routine exercises.

> The major part of every meaningful life is the solution of problems … it is the duty of all teachers, and all teachers of mathematics in particular, to expose their students to problems much more than to facts. (Halmos, 1980 p. 523)
>
> A teacher who is not always thinking about solving problems—ones he does not know the answer to—is psychologically simply not prepared to teach problem solving to his students. (Halmos, 1985 p. 322)

Poincaré, Whitehead, Freudenthal and a host of others have testified similarly.

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However, problem solving is not a single ‘thing’ which explains why, in common with other slogans, policy makers, educationalists, researchers, textbook authors and teachers are all able to claim that they are promoting problem solving despite a widely diverse range of practices under that heading.

Problem Solving as a Complex Activity

As a slogan, *problem solving* is much more honoured in the breach than in the observance as has been noted about the four-phase description given by George Pólya (1962). ‘Problem solving’ comes and goes as a focus of attention, as Bernd Zimmerman (this section) indicates. Policy makers periodically alight on it, but examination systems followed by textbooks and school senior management place teachers under impossible pressures of time and learner performance. These pressures appear to make it difficult to engage students mathematically in a problem solving vein. The force of tests and examinations is to impel teachers to get students to practise routine procedures in an attempt to score highly, even though such ‘learning’ may not be robust or stable over time. Of course there are and have been notable exceptions, for example, in the 100 % coursework project in the 1980s (Ollerton & Watson, 2007) in the UK, the ‘real problem solving movement’ epitomised by Mellin-Olsen (1987), and recently in a project in both Chile and Finland, reported on in this section by Valentina Giaconi and colleagues.

Teaching Problem Solving

As Zimmerman (this section) suggests, teaching problem solving is in itself a complex problem which needs a complex approach and will not succeed through attempts to distil it to a single simple essence.

The complexity of problem solving as a description of an activity arises from the complexity of the human psyche. Teaching is about people, not about machines; teaching problem solving is about evoking mathematical thinking together with positive affect to enable learners to make use of their own natural powers. In order for problem solving to become an integral part of learners’ experience in school and university, all aspects of the human psyche, cognition, affect, behaviour, attention, will and metacognition or witnessing must be involved. Focusing on only one or two aspects is simply inadequate and very unlikely to lead to full-scale integration into learners’ ways of being in the world.

In their chapter in this section, Nicolás Libedinsky and Jorge Soto-Andrade provide evidence that both juvenile offenders and university undergraduates can, if given sufficient time in a supportive environment, make use of metaphors to reach unexpected insights, insights which in turn promote concentration and engagement not hitherto displayed. In this sense they instantiate Jeremy Kilpatrick’s claim in his chapter that problem solving is a way of being which promotes inquiry into what lies behind mathematisable phenomena using learners' natural powers and enabling these to be developed over time. The use and development of learners' own powers is in itself rewarding. Promoting metaphoric thinking may, as they suggest, contribute to enriching learners' experience of mathematics and particularly problem solving.

Problem Posing

Problem solving involves reformulating problems and, indeed, posing them for yourself, as Jeremy Kilpatrick points out in his chapter. Ask a mathematician a technical question arising from some problem you are working on and almost always you will be asked the origin and context of your question, so that the mathematician can have an overview and an opportunity to reformulate the problem for themselves.

José Carrillo and Jorge Cruz demonstrate in their chapter in this section that asking learners to pose questions 'like the ones in an exercise set' is a powerful method of revealing the range and scope of what the learners' sense of the domain being sampled by that set of exercises. They confirm a proposal of Watson and Mason (2002) and experience of Rahman (2006) that student construction of examples is both enriching for the learners and revealing for the researcher or teacher.

Teaching Problem Solving

One might hope that textbooks, often seen as principal agents in teaching mathematics, might support and promote problem solving and problem posing. However Jinfa Cai and colleagues reveal in their chapter that the textbooks they considered are particularly weak in stimulating problem posing and often pretend that practising routine exercises amounts to problem solving when it is at best peripheral. To turn a set of exercises into a problem solving opportunity requires a skilled teacher who can direct attention to the generality instantiated by the exercises (Watson & Mason, 2006).

There have been many suggestions as to how to teach problem solving effectively. For example, Zimmerman here promotes access to historical accounts, and several authors recommend an inquiry stance. There is no doubt that learners' affect is a major influence, as noted by Carol Dweck (2000). But there are difficulties in probing deeply into affect and sense of self, as illustrated by Valentina and colleagues in their chapter where they encounter possible cultural factors which may indicate interinfluences between the culture at large (as presented in the media), the historical legacy in both the global and the local culture in the classroom and the ethos established locally.

Future Prospects

Researching problem solving has a very long history, but simplistic attempts to isolate core features have not brought universal success. There are difficulties enough in drawing on the whole of the human psyche when promoting mathematical problem solving, trying to demonstrate effectiveness or to shed light on the complexity encounters methodological difficulties.

For example, Valentina Giaconi and colleagues bring to the surface assumptions that have to be made when analysing self-report responses. Is the respondent interpreting the items according to their current state or a remembered state, and is that state typical? Is the respondent considering the item deeply or reacting habitually or automatically according to cultural values espoused by the peer group, by others such as parents and teachers, or according to the respondents' assumptions about what is expected of them?

José Carrillo and Jorge Cruz demonstrate that it is not only possible but fruitful to combine enquiry into learners' affect both before and after working on problems and to relate these with their sense of possible actions beforehand and reflections on actions that proved to be effective. As with problem posing, not only does this shed light on learners' thinking, on the focus of their attention, but it serves to alert learners to responding to problems by considering possible actions, rather than reacting spontaneously out of habit with unconsidered actions.

It has long been known that withdrawing from action and considering the nature of that action can make a major contribution to learning. It may turn out that the best way to maintain complexity, both in teaching and in researching problem solving, is through engagement and reflection, both of which may need to be supported by teachers calling upon the full scope of the human psyche in its social setting. Promoting personal narratives as adjuncts to concept images could be of real benefit to learners, for as David Hilbert (1900) noted:

> A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

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Part II
Students, Problem Posing, and Problem Solving

Can Mathematical Problem Solving Be Taught? Preliminary Answers from 30 Years of Research

Frank K. Lester Jr. and Jinfa Cai

Abstract In this chapter, the authors note that during the past 30 years there have been significant advances in our understanding of the affective, cognitive, and metacognitive aspects of problem solving in mathematics and there also has been considerable research on teaching mathematical problem solving in classrooms. However, the authors point out that there remain far more questions than answers about this complex form of activity. The chapter is organized around six questions: (1) Should problem solving be taught as a separate topic in the mathematics curriculum or should it be integrated throughout the curriculum? (2) Doesn't teaching mathematics through problem require more time than more traditional approaches? (3) What kinds of instructional activities should be used in teaching through problems? (4) How can teachers orchestrate pedagogically sound, problem solving in the classroom? (5) How can productive beliefs toward mathematical problem solving be nurtured? (6) Will students sacrifice basic skills if they are taught mathematics through problem solving?

Keywords Problem solving • Problem posing • Successful problem solver • Teaching through problem solving • Assessment • Instructional tasks • Classroom discourse • Beliefs • LieCal project • Problem-based curriculum

A considerable amount of research on teaching and learning mathematical problem solving has been conducted during the past several decades, and taken collectively, this body of work provides useful suggestions for both teachers and curriculum writers. The past 30 years were an especially productive period in the history of

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problem solving in school mathematics. Indeed, they mark a period when the first author was most actively engaged in research, and the second author got his start. In this chapter, we take a reflective look at what we learned in the past 30 years. In particular, we will note that, as is the nature of research, much has been learned but much remains to be learned (for a more recent update on research in this area, see Lester (2013) and Schoenfeld (2013)).

Key Questions About Teaching Students to Be Successful Problem Solver

During the past 30 years, there have been significant advances in our understanding of the affective, cognitive, and metacognitive aspects of problem solving in mathematics and other disciplines (e.g., Frensch & Funke, 1995; Lesh & Zawojewski, 2007; Lester, 1994, 2013; Lester & Kehle, 2003; McLeod & Adams, 1989; Schoenfeld, 1985, 1992, 2013; Silver, 1985). There also has been considerable research on teaching mathematical problem solving in classrooms (Kroll & Miller, 1993; Lesh & Zawojewski, 2007; Wilson, Fernandez, & Hadaway, 1993), as well as teaching mathematics through problem solving (Lester & Charles, 2003; Schoen & Charles, 2003). On the other hand, reviews of problem-solving research clearly point out that there remain far more questions than answers about this complex form of activity (Cai, 2003; Lesh & Zawojewski, 2007; Lester, 1994, 2013; Lester & Kehle, 2003; Schoenfeld, 1992, 2013; Silver, 1985; Stein, Boaler, & Silver, 2003). In fact, although there is a great deal of consensus within the mathematics education community that the development of students' problem-solving abilities should be a primary goal of classroom instruction (National Council of Teachers of Mathematics, 1989, 2000), there is no consensus about what we teachers should do in classrooms to reach this goal. However, even though "[w]e clearly have a long way to go before we will know all we need to know about helping students become successful problem solvers" (Lester, 1994, p. 666), many of the issues associated with problem-solving instruction have been studied extensively, and research-based insights for improving students' problem solving through classroom instruction are now available (Cai, 2010). In this analysis, we discuss these results in light of our own and others' research and suggest what they might mean for practice.[1] The discussion is organized around six practice-based questions.

[1] We wish to emphasize that due to the complex nature of problem solving, there are no hard and fast rules concerning what students can learn about problem solving or how it should be taught. Indeed, the main theme of this analysis is that the suggestions we provide are meant as guidelines for teachers' to consider seriously not directives that should be rigidly followed.

Question 1: Should Problem Solving Be Taught as a Separate Topic in the Mathematics Curriculum or Should It Be Integrated Throughout the Curriculum?

Lesh and Zawojewski (2007) point out that there is little or no evidence that students' problem-solving abilities are improved by isolating problem solving from learning mathematics concepts and procedures. They challenge the oft-held assumption that a teacher should proceed by:

> First teaching the concepts and procedures, then assigning one-step "story" problems that are designed to provide practice on the content learned, then teaching problem solving as a collection of strategies such as "draw a picture" or "guess and check," and finally, if time, providing students with applied problems that will require the mathematics learned in the first step. (p. 765)

In fact, the evidence has mounted over the past several decades that such an approach does not improve students' problem solving to the point that today no research is being conducted with this approach as an instructional intervention (e.g., Begle, 1973; Charles & Silver, 1988; Lester, 1994; Schoenfeld, 1979, 1985). But there is mounting evidence to support thinking of mathematics teaching as a system of interrelated dimensions: (1) the nature of classroom tasks, (2) the teacher's role, (3) the classroom culture, (4) the mathematical tools to aid learning,[2] and (5) the concern for equity and accessibility (Hiebert et al., 1997; Lester & Charles, 2003; Schoen & Charles, 2003). When classroom instruction is thought of as a system, it no longer makes senses to compartmentalize problem solving—or any other aspect of mathematical activity—as a separate part of the curriculum. The implication of this change in perspective is that if we are to help students become successful problem solvers, we first need to change our views of problem solving as a topic that is added onto instruction after concepts and skills have been taught. One alternative is to make problem solving an integral part of mathematics learning. This alternative, often called *teaching through problem solving*, adopts the view that there is a symbiotic connection between problem solving and concept learning (Lambdin, 2003).

A bit of elaboration on the notion of *teaching through problem solving* is in order. In teaching through problem solving, learning takes place during the process of attempting to solve problems in which relevant mathematics concepts and skills are embedded (Lester & Charles, 2003; Schoen & Charles, 2003). As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned or that they can construct on the spot, and justify their ideas in ways they feel are convincing. The learning environment of teaching through problem solving provides a natural setting for students to present various solutions

[2] Hiebert et al. (1997) describe "mathematical tools" as the collection of language, materials, and symbols that students have available when they engage in mathematical activity.

to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives on the concept or idea they are learning. Empirically, teaching mathematics through problem solving helps students go beyond acquiring isolated ideas toward developing increasingly connected and complex system of knowledge (e.g., Cai, 2003; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cobb et al., 1991; Hiebert et al., 1996; Hiebert & Wearne, 1993; Lambdin, 2003). The power of problem solving is that obtaining a successful solution requires students to refine, combine, and modify knowledge they have already learned.

To reiterate, the teaching-through-problem-solving approach has been shown to result in students improving their problem-solving performance not because they learned general problem-solving strategies and heuristics but because they had deep, conceptual understanding of mathematics. And, at the same time, research has indicated that teaching students to use general problem-solving strategies and heuristics has little effect on students' being better problem solvers (Lesh & Zawojewski, 2007; Lester, 1994; Schoenfeld, 1979, 1985, 1992). Research also has clearly shown that teaching with a clear focus on understanding can foster students' development of problem-solving abilities (Hiebert, 2003; Lambdin, 2003).

In summary, developing students' ability to solve problems is not an isolated instructional act or a topic that is covered separately from the rest of the math curriculum. Instead, it is an integral part of mathematics learning across content areas. Learning of substantive mathematical content and developing problem-solving skills cannot be separated from one and another; problem solving should be infused into all aspects of mathematics learning.

Question 2: Doesn't Teaching Mathematics Through Problem Require More Time Than More Traditional Approaches? Specifically, What Sort of Time Commitment Is Required to Teach Through Problem Solving?

To help students become successful problem solvers, teachers must accept that students' problem-solving abilities often develop slowly, thereby requiring long-term, sustained attention to making problem solving an integral part of the mathematics program. Moreover, students must also buy into the importance of regularly engaging in challenging activities (Lester, 1994; Lester & Charles, 2003; Schoen & Charles, 2003).

Developing students' abilities to solve problems is not only a fundamental part of mathematics learning across content areas, but it also is an integral part of mathematics learning across grade levels. Beginning in preschool or kindergarten, students should be taught mathematics in a way that fosters understanding of mathematics concepts and procedures and solving problems. In fact, there is strong evidence that even very young students are quite capable of exploring problem situations and inventing strategies to solve the problems (e.g., Cai, 2000; Carpenter et al., 1998; Kamii, 1989; Maher & Martino, 1996; Resnick, 1989). Helping students become successful problem solvers should be a long-term instructional goal, so that effort is made to reach this goal in every grade level, every mathematical topic, and every lesson.

The most effective way for students to learn to solve problems is for them to solve a variety of problems: a lot of them but not simple repetitions, both in and out of school (Cai & Nie, 2007; Gu, Huang, & Marton, 2004; Lester, 1994). The long-term commitment students need to make is a willingness to engage in problem-solving activities and to form habits of mind such as thinking about word meanings, justifying claims and conjectures, analyzing answers and solution strategies, using alternative representations, and acquiring a toolkit of problem-solving strategies (Goldenberg, Shteingold, & Feurzeig, 2003; Levasseur & Cuoco, 2003). Indeed, according to the winner of Fields Medal (often described as the Nobel Prize in mathematics) in 2006, Terence Tao, his remarkable accomplishment in mathematics is related to his hard work with many difficult problems at an early age. Tao is purported to have said:

> When I was a kid, I had a romanticized notion of mathematics, that hard problems were solved in "Eureka" moments of inspiration… With me, it's always, "Let's try this. That gets me part of the way, or that doesn't work. Now let's try this. Oh, there's a little shortcut here." You work on it long enough and you happen to make progress towards a hard problem by a back door at some point. At the end, it's usually, "Oh, I've solved the problem." (Press release of the UCLA newsroom, August 22, 2006; http://newsroom.ucla.edu/releases/Terence-Tao-Mozart-of-Math-7252)

In addition to making problem solving a regular part of everyday instruction, homework can extend learning opportunities and engage students in independent problem solving. Research shows that doing homework has a positive impact on students' achievement, although the impact varies across grade levels. According to the meta-analysis by Cooper (1989a, 1989b), the positive effect of homework on students' achievement increases steadily from the elementary grades through high school. Also, after reviewing all of the available research literature, Marzano, Pickering, and Pollock (2001) concluded "homework does produce beneficial results for students as low as 2nd grade" (p. 62). Although there is no clear answer to the question of how much homework is the right amount, international studies have shown that students in the United States spend significantly less time on homework than students from other countries, especially from Asia (Stevenson & Lee, 1990).

In summary, students cannot become successful problem solvers overnight. Helping students become successful problems solvers should be a long-term instructional goal for teachers to reach in every grade level, every mathematical topic, and every lesson. Teaching today's students to become the thinking and caring leaders who will be able to solve the world's increasingly complex and quantitative problems requires a total commitment (Committee on Prospering in the Global Economy of the 21st Century, 2007).

Question 3: What Kinds of Instructional Activities Should Students Be Used in Teaching Through Problems?

Before providing our answer to this question, let us state what we mean by “mathematics problem.” In simplest terms for us, a mathematics problem is a task presented to students in an instructional setting that poses a question to be answered but for which the students do not have a readily available procedure or strategy for answering it.

Researchers refer to the mathematical activities in which students engage as tasks that can be defined broadly as projects, questions, problems, constructions, applications, and exercises. Mathematical tasks provide intellectual environments for students’ learning and the development of their mathematical thinking. Doyle (1988) argued that tasks with different cognitive demands are likely to induce different kinds of learning because they govern not only students’ attention to particular aspects of content but also their ways of processing information. Mathematical tasks that are truly problematic have the potential to provide intellectual contexts for students’ mathematical development. Such tasks can promote students’ conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity (Cai, 2014; Hiebert, 2003; Hiebert & Wearne, 2003; Marcus & Fey, 2003; NCTM, 1991; Van de Walle, 2003). Researchers recommend that students should be exposed to truly problematic tasks so that mathematical sense making is practiced. Unfortunately, there is considerable evidence that many US mathematics teachers think that they have the responsibility to remove the challenge (and the struggle) for their students when they are engaged in problem solving. In their study of eighth-grade students who were part of the Third International Mathematics and Science Study (TIMSS), Hiebert and colleagues (2005) found that US teachers almost always intervened to show students how to solve the problems they had been asked to solve, leaving the mathematics they were left to do rather straightforward. This stands in direct contrast to teachers in Germany and Japan, who allowed students much greater opportunities to struggle with the more challenging parts of problems.

A number of studies have provided clear evidence to support the connection between the nature of tasks and student learning (Cai, 2014; Hiebert & Wearne, 1993; Stein & Lane, 1996; Stein, Remillard, & Smith, 2007). In these studies, students who showed the biggest gains were those in classrooms using cognitively demanding tasks. Also, Cai and Merlino (2011) found that engaging students in solving challenging problems led them to enjoy doing mathematics.

Mathematical problems that are truly problematic and involve significant mathematics have the potential to provide the intellectual contexts for students’ mathematical development. However, only “worthwhile problems” give students the chance to solidify and extend what they know and stimulate mathematics learning (Marcus & Fey, 2003; Van de Walle, 2003). But what is a worthwhile problem? Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Most importantly, worthwhile mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals. Lappan and Phillips (1998) developed a set of criteria for a good (worthwhile) problem that they used to develop their middle school mathematical curriculum (*Connected Mathematics*), and there has been some research supporting the effectiveness of this curriculum for fostering students' conceptual understanding and problem solving (Cai, Wang, Moyer, Wang, & Nie, 2011). The fact that the curriculum as a whole has been shown to be effective suggests that teachers might want to attend to this set in designing and choosing problems.

Criteria for a Good Mathematics Problem (Lappan & Phillips, 1998)

- The problem has important, useful mathematics embedded in it.
- Students can approach the problem in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.
- The problem requires higher-level thinking and problem solving.
- The problem contributes to the conceptual development of students.
- The problem connects to other important mathematical ideas.
- The problem promotes the skillful use of mathematics.
- The problem provides opportunity to practice important skills.
- The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.

The role of teachers is to select and develop tasks that are likely to foster the development of understandings and mastery of procedures in a way that also promotes the development of abilities to solve problems and reason and to communicate mathematically (NCTM, 1991).

It is very common for students to be asked to interpret a story or solve a story problem, but students are less often asked to make up stories in classrooms or pose mathematical problems based on given situations. Writing stories to go with number sentences may help students focus on the meaning of the procedures involved. We can examine students' thinking from a different perspective if we ask them to generate their own mathematical problems. Research shows that students are capable of generating interesting mathematical problems, and there is a direct link between students' problem-solving and problem-posing skills (Cai & Hwang, 2002; Silver & Cai, 1996). Therefore, writing story problems to match number sentences or posing mathematical problems based on situations can engage students in learning important mathematics and develop their problem-solving abilities.

It goes without saying that the most important criterion of a worthwhile mathematical problem is that the problem should serve as a means for students to learn important mathematics. Such a problem does not have to be complicated or have a fancy format. As long as a problem fosters students' learning of important mathematics, it is a worthwhile problem. As Hiebert et al. (1997) have noted, a problem as simple as finding the difference in heights between two children, one 62 and the other 37 inches tall, can be a worthwhile problem if teachers use it appropriately for students' learning of multi-digit addition. Teachers must decide what mathematical tasks to select or develop according to the specific learning goals of a lesson. Textbooks can be a useful resource for selecting worthwhile mathematical tasks. In fact, teachers can develop worthwhile mathematical tasks by simply modifying problems from the textbooks.

Question 4: How Can Teachers Orchestrate Pedagogically Sound, Active Problem Solving in the Classroom?

Accepting the premise that mathematics instruction should be thought of as a system of interrelated dimensions, it is clear that selecting worthwhile mathematical tasks alone does not guarantee students' learning. Indeed, there is considerable evidence that even when teachers have good problems they may not be implemented as intended. Stein, Grover, and Henningsen (1996) found that only about 50 % of the tasks that were set up to require students to apply procedures with meaningful connections were implemented effectively by the teacher. A task applying procedures with meaningful connections has various features, such as it focuses students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. Therefore, in the classroom, students' actual opportunities to learn depend not only on the type of mathematical tasks that teachers pose but also on the kinds of classroom discourse that takes place during problem solving, both between the teacher and students and among students; too often it seems that teachers do not allow students to struggle with challenging tasks (Cazden, 1986). Discourse refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in instructional tasks. Considerable theoretical and empirical evidence exists supporting the connection between classroom discourse and student learning. The theoretical support comes from both constructivist and sociocultural perspectives of learning (e.g., Cobb, 1994; Hatano, 1993; Hiebert et al., 1997). As students explain and justify their thinking and challenge the explanations of their peers and teachers, they are also engaging in clarification of their own thinking and becoming owners of "knowing" (Lampert, 1990). The empirical evidence supporting the positive relationships between teachers' asking high-order questions and students' learning can be found in the work of Hiebert and Wearne (1993) and of Redfield and Rousseau (1981).

What then is desirable discourse in mathematics teaching? To answer this question, we refer to a study developed by Thompson and his associates involving two classes of grade seven students (Thompson, Philipp, Thompson, & Boyd, 1994). A primary focus of their study was to contrast two teaching episodes in which the students in the two classes discussed their work on the same task. There were a number of similarities between the two teaching episodes Thompson et al. analyzed. For example, both teachers opened their lessons with the same problem and with similar instructions to the students. Both teachers pressed their students to give rationales for what they did. However, the two teaching episodes differed significantly in terms of how the teachers led the classroom discussion. For example, students in one class began to give explanations that were grounded in their conceptions of the situation (i.e., in making sense of the situation presented in the problem). By contrast, the explanations given by students in the other class remained strictly procedural. In addition, one teacher was less persistent than the other in probing the students' thinking. He accepted solutions consisting of calculation sequences. However, the other teacher persistently probed students' thinking whenever their responses were cast only in terms of calculations. The actions of the two teachers were driven by different images of the tasks and of the pedagogical goals it served. For one teacher, the goal was to solve a problem, whereas for the other teacher, the goal was to encourage students to reason and reflect on their reasoning and to demonstrate their understanding of the mathematics concepts embedded in the task.

Thompson et al.'s analysis clearly showed that mathematical tasks can be implemented differently, depending on the nature of the classroom discourse. Indeed, there are a number of factors that can influence the implementation of worthwhile problems in classroom (e.g., Henningsen & Stein, 1997). One of the predominant factors is the amount of time allocated to solving and discussing the problem. For example, more than 40 years ago, Rowe (1974) found that the mean time that teachers waited between asking a question and, if no answer was forthcoming, intervening again was only 0.9 s. A wait time of less than 1 s prevented most students from taking part in the classroom discussion. No wonder many students believe that every problem should be solvable with little or no thinking (Lesh & Zawojewski, 2007).

Another influential factor is that sometimes teachers remove the challenges of a mathematical task by taking over the thinking and reasoning by telling or showing students how to solve the problem. In addition to selecting and developing worthwhile mathematical tasks, teachers are also responsible for listening carefully to students' ideas and asking them to clarify and justify their ideas orally and in writing, as well as monitoring their participation in discussions and deciding when and how to encourage each student to participate. The questions that teachers ask are also critical for orchestrating sound classroom discourse (Rasmussen, Yackel, & King, 2003; Stephan & Whitenack, 2003).

Question 5: How Can Productive Beliefs Toward Mathematical Problem Solving Be Nurtured?

One of the most striking results presented in the various national reports of students' mathematics achievement is represented by a large number of high school students who avoid taking advanced mathematics courses. These students often drop advanced math classes, not necessarily due to a lack of ability but largely based on their negative attitudes toward mathematics and on their perception of their future career opportunities. According to National Assessment of Education Progress

(NAEP) results, in grade levels 4, 8, and 12, students who agreed that they like mathematics and who think mathematics is useful for solving problems scored higher than did the students who disagreed (Kloosterman & Lester, 2004; Silver & Kenney, 2000). Yet even among those students who expect to become scientists, less than 75 % of those students believe that advanced mathematics or science courses are necessary for their future careers (e.g., Ma, 2006).

In order to help students become successful problem solvers, we need to nurture productive beliefs toward mathematics in general and problem solving in particular. All too often students hold the belief that there is only one "right" way to approach and solve a problem (Lester, Garofalo, & Kroll, 1989; McLeod & Adams, 1989). The results from both national (Lindquist, 1989; Lubienski, McGraw, & Struchens, 2004) and international (Lapointe, Mead, & Askew, 1992) assessments show that many students do not view mathematics as a creative and intellectually engaging activity, but rather as a set of rules and procedures that they must memorize in order to quickly follow the single correct way to obtain the single correct answer. For example, on the 2003 NAEP assessment, nearly one-third of US eighth-grade students reported that learning mathematics is mostly memorizing, and about one-fifth of eighth-grade students agreed with the statement that there is only one correct way to solve a mathematics problem (McGraw & Lubienski, 2007).

Students' beliefs about the nature of problem solving are not restricted to how problems are supposed to be solved. Many students also have firmly held beliefs about what is expected of them when their teachers give them problems to solve. For example, in solving an absurd problem like "There are 26 sheep and 10 goats on a ship. How old is the captain?" 10 % of Belgian kindergartners and first graders "solved" this problem by adding the numbers to get the captain's age (Verschaffel & De Corte, 1997). The percentages of students who "solved" the problem in this way increased to 60 % for Belgian third and fourth graders and 45 % for fifth graders. The more formal education the third-, fourth-, and fifth- grade students had, the less attention they paid to making sense of the problem and their solutions, in contrast to the first graders. A similar problem was administered to a group of Chinese fourth graders, seventh graders, eighth graders, and twelfth graders. About 90 % of the Chinese fourth graders, 82 % of the seventh and eighth graders, and 34 % of the twelfth graders "solved" this problem by combining numbers in it without realizing the absurd nature of the problem (Lee, Zhang, & Zheng, 1997). When these Chinese students were asked why they did not recognize that the problem was meaningless, many students responded that any problem assigned by a teacher always has a solution (Cai, 2003). This sort of result has been documented consistently by researchers (e.g., Lester et al., 1989; McLeod & Adams, 1989).

Students' beliefs about problem solving can also be revealed when they are asked to solve a problem using alternative strategies. Research suggests that some students seem unconcerned about getting different answers for a problem with a unique answer when they are asked to solve the problem using different strategies. For example, in a study by Silver, Leung, and Cai (1995), Japanese and US fourth-grade students were asked to find multiple ways to determine the total number of marbles that had been arranged in a certain way. Some students obtained different numerical answers when they used alternative solution strategies; surprisingly, they seemed unconcerned about getting different answers.

On the other hand, studies by a number of researchers (e.g., Carpenter et al., 1998; Cobb et al., 1991; Verschaffel & De Corte, 1997) suggest that it is possible to change students' beliefs about mathematics and problem solving using alternative instructional practices, such as teaching through problem solving. For example, in contrast to students in a control group, Cobb et al. (1991) found that students in their problem-centered project held more positive beliefs about the importance of understanding, in addition to being the better problem solvers, than a comparison group of students. Therefore, it is quite possible for teaching with a focus on understanding and problem solving to provide a healthy learning environment for students to form positive beliefs about mathematics and problem solving and develop problem-solving skills.

That teachers' beliefs about mathematics impact their teaching is well documented (Philipp, 2007; Thompson, 1992). Teachers who hold different beliefs about mathematics teach differently. Research shows that engaging students in problem-posing and problem-solving activities in the classroom has a positive influence on students' problem-solving performance and their attitudes toward mathematics (Cai, 2003; Cai & Hwang, 2002; Cai & Merlino, 2011; Rosenshine, Meister, & Chapman, 1996; Silver & Cai, 1996). In such classrooms, students become active participants in the creation of knowledge rather than passive receivers of rules and procedures.

Question 6: Will Students Sacrifice Basic Skills if They Are Taught Mathematics Through Problem Solving?

In teaching through problem solving, the focus is on conceptual understanding, rather than on procedural knowledge; it is expected that students will learn algorithms and master basic skills as they engage in explorations of worthwhile problems (Cai, 2003). However, many people, parents and teachers alike, worry that the development of students' higher-order thinking skills in teaching through problem solving comes at the expense of the development of basic mathematical skills. Obviously, both basic skills and high-order thinking skills in mathematics are important, but having basic mathematical skills does not imply having higher-order thinking skills or vice versa (Cai, 2000; Hatano, 1988; Steen, 1999; Sternberg, 1999). Therefore, it is reasonable to wonder if students will sacrifice basic skills in a learning environment involving teaching through problem solving (Battista, 1999; Schoenfeld, 2002). It is also reasonable to wonder if students who use *Standards*-based curricula can truly develop conceptual understanding and higher-order thinking skills.

There are two lines of relevant research to address this question. The first line includes those studies conducted on reform curricular programs funded by the US National Science Foundation (NSF) that teach mathematics through problem solving and that have been implemented by teachers. The *NSF-funded* programs are problem-based curricula, and the intent is to teach mathematics and to build students' understanding of important mathematical ideas through explorations of real-world situations and problems (National Research Council, 2004; Senk & Thompson, 2003). The second line of research includes studies based on innovative materials developed by researchers in specific content areas (e.g., Carpenter et al., 1998; Cobb et al., 1991; Hiebert et al., 1997; Hiebert & Wearne, 1993; Stein & Lane, 1996; Wood & Sellers, 1997). Unlike the first line of research, in this second line, researchers usually focus on teaching grade-specific mathematical topics using a problem-solving approach.

The results from these two lines of research generally point in the same direction: on standardized tests measuring computational skills and procedural knowledge, students using problem-solving approaches performed at least well as students using traditional curricula. In addition, students using problem-solving approaches performed better than students using traditional curricula on tests specifically designed to measure conceptual understanding and problem solving (Cai et al., 2011; Carpenter et al., 1998; Cobb et al., 1991; Fuson, Carroll, & Drueck, 2000; Hiebert & Wearne, 1993). For example, Cobb et al. (1991) examined the performance on a standardized mathematics achievement test of ten classes, whose students had participated in a year-long, problem-centered mathematics project and compared them with eight non-project classes. They also studied the performance of these same classes of students on instruments designed to assess students' computational proficiency and conceptual development in arithmetic. They found that levels of computational performance between project and non-project students were comparable, but the project students had higher levels of conceptual understanding in mathematics than did non-project students. Other studies—involving elementary school students (e.g., Carpenter et al., 1998; Hiebert & Wearne, 1993)—have obtained similar results: students learning mathematics through problem solving do at least as well as those students receiving traditional instruction on both basic computation and conceptual understanding. Similarly, the few existing studies involving middle and high school students (Cai et al., 2011; Reys, Reys, Lapan, Holliday, & Wasman, 2003; Riordan & Noyce, 2001; Schoen & Charles, 2003; Tarr et al., 2008) have shown that students who receive problem-based instruction have higher levels of mathematical understanding than students with more traditional instruction, and there are comparable basic number skills between the two groups. In the LieCal Project,[3] the results showed that the students using the Connected Mathematics Program (CMP) had growth over the three middle school years that was similar to those students using other commercial, more traditional, curricula on items assessing simple word problem solving, computations, and equation solving. However, on items assessing conceptual understanding and complex problem solving, CMP

[3] LieCal Project (Longitudinal Investigation of the Effect of Curricula on Algebra Learning) was funded by the National Science Foundation. It investigated whether the Connected Mathematics Program (CMP) can effectively enhance student learning of algebra. The LieCal Project investigated not only the ways and circumstances under which the CMP curriculum can or cannot enhance student learning, but it also looked at the characteristics of the curriculum and implementation that lead to student achievement gains.

students showed significantly greater growth over the three middle school years than did non-CMP students (Cai et al., 2011). The LieCal Project has also used various student learning outcome measures to examine the impact of middle school curriculum on students' learning in high school. On all of the measures, CMP students performed equally well or better than the non-CMP students when they were in high school (Cai, 2014; Cai et al., 2013).

There is an exception, however. Ni and his associates conducted a longitudinal study to investigate whether or not the curriculum reform in Mainland China brought about desirable student learning outcomes in elementary mathematics (Ni, Li, Cai, & Hau, 2015). As in the United States, the most important feature of the reformed mathematics curriculum in China is that it is problem based. Improved performance was observed in the students of both groups over time on the measures of computation and problem solving for which the tasks involved both process-constrained and process-open questions.[4] However, the reform group performed better than did the non-reform group on complex problem-solving tasks. On the other hand, the non-reform group did better than the reform group on computation and simple problem solving. The few studies that have examined the reformed curricula in the United States have obtained similar findings (National Research Council, 2004).

The findings appear to suggest that in some instances students' conceptual understanding does come at the expense of the development of basic mathematical skills when using reformed curriculum, but is the trade-off worth it? For the reformed group, the success rate for solving multiple-choice tasks increased from 80 to 87 % over 2 years and 75–92 % for the non-reformed group in the study by Ni et al. (2015). The reformed group showed reasonably good growth on the measures of basic mathematical skills. The world has been changing dramatically, and these changes are happening much faster than we anticipated. Today, possessing a large amount of knowledge and information is not sufficient. Instead, in this continually changing world, the most important qualities we can help our students develop are the abilities to think independently and critically, to learn, and to be creative. In this sense, it seems worthwhile to sacrifice a few percentage points on basic mathematical skills to gain considerable measures of higher-order thinking skills.

On balance then, the research evidence suggests that teaching mathematics with a problem-solving perspective considerably enhances students' problem solving and conceptual understanding while at the same time producing moderate to excellent improvement in their basic computational and other procedural skills.

[4] A process-constrained problem requires a student to carry out a procedure or a set of routine procedures to solve the problem. In other words, the problem is set in such a way that it constrains a student's solution to a rather limited process. Usually, a process-constrained problem can be solved by applying a "standard algorithm." On the other hand, a task that is process open may not require an execution of a procedure or a set of procedures; instead it requires an exploration of the problem situation and then finding the solution to the problem. Therefore, the task is set in such a way that it allows students to use alternative, acceptable solution strategies. Usually, a process-open task cannot be solved by following a "standard algorithm." See Cai (2000) for details.

Summary

How can we teachers help students become successful problem solvers? This analysis of research conducted primarily since 1985 provides some partial answers to six practice-based questions. Research clearly suggests that problem solving should not be taught as a separate topic in the mathematics curriculum. In fact, research tells us that teaching students to use general problem-solving strategies has little effect on their success as problem solvers. Thus, problem solving should be taught as an integral part of mathematics learning, and a significant commitment should be made to include problem solving at every grade level and with every mathematical topic. In addition to making a commitment to problem solving in the mathematics curriculum, teachers need to be strategic in selecting appropriate tasks and orchestrating classroom discourse to maximize learning opportunities. In particular, teachers should engage students in a variety of problem-solving activities: (1) finding multiple solution strategies for a given problem, (2) engaging in problem posing and mathematical exploration, (3) giving reasons for their solutions, and (4) making generalizations (Cai, 2010). Focusing on problem solving in the classroom not only impacts the development of students' higher-order thinking skills but also reinforces positive attitudes. Finally, there is little evidence that we should worry that students sacrifice their basic skills if we teachers focus on developing their mathematical problem-solving skills.

At the beginning of this analysis, we noted that we need to conduct much more research before we will have answers to all the questions teachers have about teaching mathematics with a problem-solving perspective. For now, it suffices to say that, although we do not know the one BEST way to teach students to be better problem solvers, research has begun to provide compelling evidence to support some methods over others.

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Teaching Mathematical Problem Solving in Hungary for Students Who Have Average Ability in Mathematics

András Ambrus and Krisztina Barczi-Veres

Abstract Fostering talented students from Grade 1 to Grade 12 has always dominated Hungarian mathematics education. Another main characteristic of our education is whole class teaching, but the results of Hungarian students on international mathematics tests show that teaching mathematics to everybody should hugely differ from teaching mathematics to the talented. How can we change this one-sided dominance in a traditional mathematics teaching culture? In our study, we analyse the role of memory systems, which are decisive factors of problem solving. Here, we find a lot of differences between the talented and the average. Following this, we present a case study concerning teaching mathematical problem solving for average students: using guiding questions and hints in problem solving and applying cooperative techniques in teaching problem solving. The research was implemented by a secondary school mathematics teacher.

Introduction

If somebody looks at Hungarian mathematics education from the outside, it seems very promising. Thanks to G. Pólya, mathematical problem solving and fostering talented students always stand in the centre of school mathematics. Some world famous names, like P. Halmos, J. Neumann, P. Erdős, L. Lovász, etc. with Hungarian origin, serve as an evidence for the effectiveness of our mathematics education. Moreover, we still train very bright young mathematicians. But these facts show only one side of the coin. How about the other side? John Mason mentioned in a personal discussion that "The much-vaunted Hungarian Mathematics Teaching has not spread significantly into the mainstream". Furthermore, Laurinda Brown from the University of Bristol, who has visited Hungarian mathematics lessons many times, formulated her opinion as "You in Hungary are teaching mathematics, we in England children". So, what are the main problems in current Hungarian

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mathematics teaching? To get some ideas in the following, we analyse the results of Hungarian students on PISA 2012 mathematics tests, on the mathematics maturity exams and on some university tests.

On *PISA 2012* Mathematics test, out of the 65 participants, Hungary reached the 39th place with an average of 477 points (OECD average is 494 points). The top rating level 6 was reached only by 9.3 % of the Hungarian students; the lower levels 2 or 1 were achieved by 28.1 %. On the creative problem solving PISA 2012 test, out of the 44 participants, Hungary placed 33rd. Here, 35 % of the Hungarian students reached level 2 or level 1 (PISA, 2012). Hungary has always participated in PISA tests right from the beginning, usually achieving OECD average results, but these results show a decreasing tendency. As a reaction to these results, the Hungarian government has introduced the so-called competence tests for grades 4, 6, 8 and 10. These tests contain so-called PISA-like problems, and the schools are evaluated based on their students' results on these tests. A direct consequence of this is that some schools prepare their students directly for the test, practising problems taken from earlier competence tests, to make the style of the problems known to their students. So, the results of these tests do not reflect the whole reality.

Another important test in Hungarian mathematics education is the central maturity exam at the end of year 12. The exam has two levels, middle vs. higher level. The higher level one has a written and an oral part; differential and integral calculus are only part of the higher level schemes of work. Most of the higher level problems are complex; many of them are modelling problems, while the middle level problems usually include basic mathematical tasks, algorithms or procedures. In a year, about 95,000 students take the mathematics maturity exam, out of whom about 3500 students take the higher level exam (3.6%!). The problem is that most of these students are not prepared to solve complex problems. From the above-mentioned data, we may ask why so few students choose the higher level exam. Universities training engineers, information technologists, mathematicians, economists, mathematics teachers and architects have places to offer for about 20–25 % of a year group, and only about 3.6 % of a year group takes the higher level maturity exam in mathematics, although a high level of mathematical knowledge is a requirement for these majors.

First-year students have to take a test at the beginning of their university or college studies in which the problems are based on the middle level maturity exam requirements. Based on their results, most of the students must participate in an "adjustment" course where they go through the basic secondary mathematical concepts, algorithms, procedures, etc. Without going into details, we mention some findings of an elite Faculty of Technical University Budapest (Csakany, 2011): most of the students do not understand relationships; do not know key ideas; have very weak analysing abilities; their work is hard to follow; their knowledge is unstable; they have very weak modelling abilities; and their imaginative abilities are weak too. We do not need to emphasise how important these factors are in effective problem solving. To summarise, it seems that the whole system—the Hungarian elementary and secondary mathematics education—does not work very effectively.

What can we do to achieve some changes? We concentrate only on one but an important issue: how can we reach not only the top 10 % of students in mathematics lessons but also "the rest", the less able ones too? Our hypothesis is that the teaching style of mathematics that works for the top 10 % is not effective for the next 10–20 % and for the remaining part. In this article, we concentrate on the mentioned 10–20 %, because we need to prepare them much better for higher studies.

Most mathematics problems in Hungarian mathematics teaching are closed problems, and a lot of students cannot start solving them without help because they mostly need to apply top-down deductive methods. For these students, opening a problem gives a chance to take individual steps towards the solution, for example, investigating some concrete cases, which is more of a bottom-up, inductive method. After some theoretical considerations about the cognitive architecture of human mind, cognitive load theory and cooperative teaching methods, we will report about a teaching experiment in which cooperative teaching techniques, opening problems and using guiding questions were applied. We would like to emphasise that we made our experiment in a very rigid Hungarian teaching environment—class teaching, closed problems—to demonstrate that there is a more effective way than the traditional one.

Theoretical Background

One of the problems in Hungarian mathematics education is that the science of learning is often neglected (We are teaching mathematics, not children!). If we analyse the human mind architecture, we may come closer to understanding the reasons of the underlying problems.

The Cognitive Architecture of the Human Mind: Memory Structures

Most neuroscientists accept Baddeley's model of memory structures (Baddeley, Eysenk, & Anderson, 2009): perceptual (sensory) memory, working memory and long-term memory. From our point of view, the working and long-term memory are important as basic places for human cognition.

In *working memory*, the conscious human processing occurs (comprehension, understanding, critical thinking, problem solving, etc.). It is called the workbench of our brain; it is the active problem space. It has four components: phonological loop to hold and rehearse verbal information, visual-spatial sketchpad to hold visual and spatial information, episodic buffer which connects the verbal and visual-spatial information directed by the central executive with the help of the information taken from the long-term memory. The central executive is the so-called supervisory attention system, because it monitors, controls and directs the information processing in our brain. Our working memory constructs plans, uses transformation strategies, analogies, metaphors, brings together things in thought, abstracts and externalises mental representations. In problem solving, students need a clear mental representation of the task (understanding the problem). While seeking a strategy (solution method), the students need to hold the conditions and the goal in their memory, and taking this into consideration, they should monitor their progress in the solution, inhibit wrong, unsuccessful ideas and control their results. It is very hard to make these components appear in class teaching.

The *WM has* very limited *capacity* holding 7 ± 2 info units, maybe four plus/minus one in case of people who are below the average. When processing information, the number of items which can be processed is maybe two or three depending on the information being processed. The *time limit* for holding information without rehearsal is about 18–20 s. In problem solving, the goal maintenance and inhibiting irrelevant information are very important factors (Baddeley et al., 2009; Clark, Kirschner, & Sweller, 2006).

The long-term memory is the storehouse of our knowledge. It holds information in schemas. Schemas are mental structures that we use to organise and structure knowledge. We bring schemas from long-term memory to working memory to understand the situations and problems. We build schemas in working memory and integrate them into existing schemas in long-term memory. Long-term memory does not a have capacity or time limit. The connection between long-term and working memory is very important in learning as schemas build one information unit for working memory, so they can free working memory sources. Another important aspect is the schema automation, because when using automated schemas, there are no working memory capacity demands.

Cognitive Load Theory

Cognitive load can be defined as the load imposed on the *WM* by presenting information. It is based on following assumptions: First, the capacity of the *WM* is limited. Moreover, *long-term memory* stores information as schemas which represent units of information, and automaticity of these schemas in *LTM* can be achieved. Finally, learning requires active conscious processing in *WM*.

According to Chipperfield (2006), CLT is based on the following principles of cognitive learning:

- Capacity of STM (working memory) is limited—7 informational units.
- LTM is unlimited in capacity—where all information and knowledge is stored.
- Knowledge stored in LTM—schemas or schemata.
- Schemas, no matter how large or how complex, are treated as a single entity in working memory.
- Schemas can become automated.

Types of Cognitive Loads

Intrinsic cognitive load depends on the elements that must be processed simultaneously. For example, when solving word problems, these elements are reading the problem, concluding what the problem asks and solving the problem. These are elements that interact. Intrinsic cognitive load is embedded in the problem; teachers cannot influence them.

Extraneous Cognitive Load

It depends on the manner of presenting information. This may include superfluous information that is not necessary for learning the presented material, such as background music or holding mental representations of facts or figures. For example, the fact that a geometric figure and the corresponding written statements are separated may be hard to comprehend for some students.

Germane Cognitive Load

It means the cognitive load placed on *WM* during schema formation, integration and automation. This explains the differences between students in experiences, ability level and content knowledge.

To sum it up, the cognitive load is the sum of the intrinsic load, the extraneous load and the germane load (cognitive load = intrinsic load + extraneous load + germane load). When planning teaching, teachers must take the possible cognitive loads into consideration.

Some Consequences for Teaching Problem Solving

We mentioned above that class teaching is dominant in Hungarian mathematics education. Furthermore, there is a tradition, maybe thanks to G. Pólya, that the so-called problem-oriented style is used frequently. However, as mentioned in the introduction, the efficacy of this style is not obvious (Chandler & Sweller, 1991).

> In real classrooms, several problems occur when different kinds of minimally guided instruction are used. First, often only the brightest and most well-prepared students make the discovery. Second, many students, as noted above, simply become frustrated. Some may disengage, others may copy whatever the brightest students are doing—either way, they are not actually discovering anything. Third, some students believe they have discovered the correct information or solution, but they are mistaken and so they learn a misconception that can interfere with later learning and problem solving. Even after being shown the right answer, a student is likely to recall his or her discovery—not the correction. Fourth, even in the unlikely event that a problem or project is devised that all students succeed in completing minimally guided instruction is much less efficient than explicit

> guidance. What can be taught directly in a 25-minute demonstration and discussion, followed by 15 minutes of independent practice with corrective feedback by a teacher, may take several class periods to learn via minimally guided projects and/or problem solving (Clark, Kirschner, & Sweller, 2012).

Based on 50 years of practice in Hungarian mathematics education, we may agree. For novices, when learning new information, guided instruction is more effective. Of course, the experts (10 % at the top) can solve a lot of new problems individually (expertise reversal effect).

Reduction of Cognitive Load

Based on the Cognitive Load Theory, there are some instructional techniques developed with the aim to reduce cognitive load. Sweller, van Merrienboer and Pass analyse in their study (Sweller, 1994) seven instructional effects: the goal-free effect, worked example effect, completion problem effect, split-attention effect, modality effects, redundancy effect and variability effect. In our study, we concentrated on the goal-free effect, worked example effect and the cooperative method. The cooperative method is mentioned in many places as a tool to reduce CL:

1. *Use of worked examples*: "Research has provided overwhelming evidence that, for everyone but experts, partial guidance during instruction is significantly less effective than full guidance" (Clark et al., 2012). Furthermore, to help developing problem-solving skills, both Pólya (1973) and Schoenfeld (1985) suggest using so-called helping or guiding questions. In our experiment, using guiding questions and prompts is not fully guided but strongly guided instruction.
2. *Goal-free (open) problems*: In some problems, the distance between the starting phase and the goal is very big. It is desirable to ask students to find all the data they can find. For example: *In a triangle, two sides are given 7 and 11 cm. The angle between them is 57°. Find all the data in this triangle what you can.* In our experiments, opening problems go in this direction.
3. *Applying cooperative teaching methods*: Research experiments show that in group work, the *WM* capacities of the members are added together, so the cognitive load is not very high for the individuals. Our classroom experiment is based on group work.

In our experiment, we focused on students who are not in the top 10–15 % of a class, and our aim was to reduce their cognitive load to help them to become more successful problem solvers. To try to achieve this desired aim, we decided to combine the above-mentioned three factors. The reasons for opting for cooperative techniques were that it helps facilitate active student participation, encourages communication between students and gives them opportunity to be creative. As mentioned before, in Hungarian mathematics education, the most widespread teaching method is frontal teaching which gives the opportunity for the less talented or less active students just to sit around in class and copy stuff from the board without grasping the material fully or joining in the process of problem solving. Moreover, students are becoming less and less independent when it comes to solving problems individually. They find it difficult to come up with a method that leads to the solution and are not very creative when they have to solve non-typical mathematical problems. Another reason for combining the three factors is that when teaching problem solving, teachers try to use guiding questions instead of just telling the students what to do, but in frontal class work, this method does not seem to be effective enough. Combining cooperative teaching techniques with helping questions and open problems helps making asking these questions a more conscious process for the students.

Cooperative Teaching

What is Cooperative Teaching?

Cooperative teaching and learning is an arrangement where students work together to solve a problem or to achieve another common goal. The success of their work depends on whether the group members are able to cooperate, to respect each other and to trust each other. They depend on each other that is why mutual support is inevitable for progress (Kagan, 2004). In Hungary, József Benda thought that applying cooperative learning might be the answer for issues in education such as raising achievement, integration and developing school work (Józsa & Székely, 2004).

Many teachers think that if they simply arrange students in small groups and provide them with a task, they are using cooperative teaching, but cooperative teaching is not simply group work. The main difference between the two teaching formats is that if students are put together in groups, it is their task to find out who is responsible for which part of the task. This might result in some students doing the bulk of the work while others just observing or completely staying out. But in cooperative learning, certain structures were created so that the following four principles are always present: *p*ositive interdependence, *i*ndividual accountability, *e*qual participation and *s*imultaneous interactions (*PIES*). These four principles make sure that every member of the group participates in the work to the best of his or her ability (Johnson & Johnson, 1994).

Cooperative Structures

The structures designed to ensure the presence of the four principles were given catchy names that makes them easy to remember (Kagan, 2003). Here, we present some of these structures that were used during the experiment.

Pair Check: "(1) Partner A works the first problem as Partner B coaches and praises. (2) Partner B works the next problem as Partner A coaches and praises. (3) Pairs compare and discuss answers. (4) Teams celebrate correct answers or resolve differences … (5) Pairs repeat steps as they complete the worksheet".

Expert Jigsaw: The main steps of this structure are: (1) Each team has a task to work on, so the members become the experts of the given task or topic; (2) New teams are formed, so that each new team has an expert of every task; (3) In the new teams, the members share their knowledge with each other; (4) Everybody goes back to their original team and share the new pieces of information (Slavin, 2010).

Think-Pair-Share: This structure was designed to ensure equal involvement of the team members. In mathematics, it is very useful in problem solving. The problem is presented to a group. The members have some time to think about the solution and they note down their ideas. They pair up within the group and share their ideas—instead of this, *Round Robin* can be used to share ideas (Kagan, 2004).

Round Robin: This structure was designed to give every group member the chance to contribute to the work and to share their ideas. To ensure this starting with one member, each person gets some time (1–3 min) going clockwise (or anticlockwise) to present their views (Kagan, 2004).

What Does the Teacher do in a Cooperative Lesson?

In cooperative teaching, the role of the teacher changes significantly. Instead of being the person who dominates and leads the lesson, the teacher becomes more of a coach or a tutor who mainly observes the students' activity and provides help if needed (Burns, 1990). As a result of this, planning a cooperative lesson requires more time and creativity. Furthermore, it is the teacher's responsibility to maintain an atmosphere where effective work is possible and the students do not misuse the opportunity to chat with each other (Dees, 1990).

Groups in a Cooperative Lesson

According to Crabill (1990), the ideal group size is a group of four. The grouping can be done in many different ways. Each group can contain a weaker student, a more able student, a quiet student and a more talkative student, or we can aim for more homogeneous groups.

The Experiment

Background Information: The School, the Students, the Class

The experiment carried out was an action research, which indicates that the researcher was the teacher of the class as well. This type of experiment is popular amongst classroom teachers who are interested in professional development-related research. As Koshy (2005) defines action research, it is a kind of an enquiry which constantly aims at refining practice thus contributing to the teacher's professional development. According to this author, action research is researching one's own practice; therefore, it is participatory and situation-based, it is emergent and it is mainly about improvement. Furthermore, through action research, the goal of mathematics teachers and researchers of maths education can be brought closer (Zimmermann, 2009).

In the experiment being presented, the question of using control groups might arise. However, according to Slavin (1996), the problem with comparing the outcome of cooperative learning to the outcome of other methods is that the studies being compared might differ in many factors—such as the subjects (different students in different groups), the duration (cooperative work is not used in each class and students had been learning with other methods for a longer time), the measures, etc.—that can account for the different outcomes. So, that is the reason why control groups were not used.

The *school* where the experiment took place is a mixed comprehensive secondary school whose students' achievement is outstanding in the region. There were 16 students participating in the experiment who were 16–17 years old. The *class* they attended specialised in science subjects and foreign languages. These students have 5 years to complete their secondary school studies—which is normally 4 years in Hungary, one preparatory year followed by four "normal" years. The leader of the experiment was the teacher of this group as well. The action research took place in the academic year 2012/2013, which was the students' third year in the school. In the preparatory year, their timetable contained three Maths lessons per week, and this number increased to four—four lessons per week in the following 2 years. These numbers are higher than the number of weekly maths lessons in an average class, that is why we often had the opportunity to discuss certain topics in greater detail or to solve competition-like problems in class. The *students* were not necessarily gifted in mathematics, but most of them definitely had a great interest in the subject and were usually keen to solve challenging problems, so they were motivated and easy to activate in class. Their grades ranged from satisfactory (3) to excellent (5) in the five-scale Hungarian grading system. Some of them were regular members of the weekly group study sessions and they took part in mathematics competitions as well (Barczi, 2013).

Methods of Data Collection

Before commencing the action research, the students were asked to fill in different questionnaires related to communication skills, attitude towards learning, attitude towards working in groups, attitude to maths and a mathematical pretest (Ambrus, 2004; Tóth, 2007). The first part of the experiment (*see*: *The problems*) was followed by a mathematical post-test, and at the end of the school year, the students had to repeat the above-mentioned tests and had to complete a delayed mathematical test too.

Additionally, a video record was made of about half of the lessons, and the group discussions were voice recorded as well. During the lessons, the teacher made notes about the behaviour and work of the students. Each student had an exercise book, a so-called reflection booklet in which they could record not only the solutions of the problems but their feelings, best ideas, strategies they used, etc. These books were used to reflect on the problem solving as well (Pólya, 1973). First, the students tried to describe the problem-solving process with their own words. For example, they said that they started solving the problem from the "last" information then working backwards. In the case of another problem, they said that they systematically tried some concrete values trying to generalise their ideas later.

The Problems

During the first part of the experiment, five curriculum-based mathematical problems were discussed, and to avoid making students feel that they are merely having fun in these lessons, the problems were carefully chosen so that they were in line with what had been covered in maths and they developed certain mathematical competencies. The problems were chosen from different fields of mathematics—combinatorics, algebra, geometry, number theory—moreover, they were either investigations or open problems. In Hungarian mathematics education, these types of problems are hardly used; therefore, the way they were presented was surprising for the students. Some problems could be extended by varying the problems (Kilpatrick, 1987); the options for this were suggested either by the students or by the teacher. In this way, for the detailed discussion of a problem, two or three lessons were needed. Each lesson was planned using cooperative teaching techniques.

In the second part of the experiment, the students continued their work based on the Year 10 scheme of work. In the rest of the school year, once every 2 or 3 weeks, there was one lesson planned with cooperative techniques, and the content of the lesson was based on the curriculum.

Two Problems and Helping Questions

One of the aims of the experiment was to figure out whether cooperative techniques help to activate "the rest" of the students and not only the top 10 or 15 % and whether the helping questions contributed to the more successful problem solving of the less able students. In this section we will present some problems that were covered using cooperative techniques while the focus was on using helping questions. The lesson plans and the names of the cooperative structures used in the lessons are included. In the class that was examined, there were 16 students, so four groups of four were formed. The teacher chose to form heterogeneous groups and the "difficult" students were group mates of the patient ones.[1]

Primes and Factors

This problem was one of the five problems discussed during the first part of the experiment. It is a curriculum-based topic since its solution involves factorising algebraic expressions and working with divisibility.

Starter problem: Think of a two-digit number. Swop the digits then subtract the smaller number from the bigger one, e.g. 42–24 = 18. Try with more numbers. Do you have any prime numbers amongst your answers? Is it possible to have a prime as the result? Can you explain your answer?

Problem: Find the biggest number that divides each term of the following sequence:

$$1^5 - 1, 2^5 - 2, 3^5 - 3, \ldots, n^5 - n.$$

Helping questions:

- Examine the second term. Can you factorise it?
- Using the factorised form, can you tell which numbers divide the product?
- Is it divisible by 2, 3 or 5?
- Try to do the previous steps on the 3rd and the 4th terms.
- Can you do the same for the general term? $n^5 - n$.

Lesson Plan

The starter problem was discussed using the *Pair Check* method which was followed by a short class discussion—in these discussions, not only the solution but solution strategies were mentioned as well. When solving the main problem, the structure used was *Pair Check* (see above). Students were originally organised in groups of four which were divided further into two pairs. One student had to try to solve the problem while telling his ideas to his pair and explaining what he did and why. The helping questions were printed on separate pieces of paper and were placed face down in front of the pair so that the students could not see the questions. The questions were ordered. The instructions given to the student who had to check his pair's solution were as follows: If your pair gets stuck and you can't help him,

[1] To give students the opportunity to work with as many other students as possible, the groups were changed once in the twelve-lesson period. On the other hand, for effective cooperative work, the students need time to 'learn' how to work together, that is why the group settings were changed only once.

then turn the topmost question over. Read it and see whether it helps to carry on the solution. The pairs were given a set time to try solving the problem. After their time was up, the pairs came together and discussed their solutions—not only the answer but the method as well—using *group discussion*. Finally, a short class discussion was held to make sure that everybody had understood the problem and to summarise the most important steps in the solution process.[2]

Word Problems

These problems were covered during the second part of the experiment. Solving word problems with the help of quadratic equations is part of the curriculum, and understanding word problems has always been difficult for students.

The four groups were given two word problems, each one of which was easier, while the other was a bit more challenging. For the first problem, the students were given helping questions which were based on Pólya's (1962) and Schoenfeld's (1992) guidelines but were more concrete and topic specific. The different groups received different problems:

1. We have two copper alloys. The first one contains 6 kg copper; the second one contains 12 kg copper. The copper content of the first alloy is 40 % less than in the second alloy.
2. When mixing the two alloys, we obtain an alloy with 36 % copper content. What percentage of the first alloy is copper? What percentage of the second alloy is copper?

   Helping questions:

   - What are we looking for? What is the unknown?
   - Use a figure! Can you represent the given information on a drawing?
   - Could you express the weight of the alloys? (in terms of the unknown)
   - What percentage of the mixture is copper?
   - How much does this weigh?
   - Could you write an equation?

3. Three pipes fill up a pool in 3 h if they all "work" together. How long does it take to fill up the pool for each pipe on its own if the second pipe needs twice the time of the first pipe and the third pipe needs 5 h more than the first pipe?

   Helping questions:

   - What do you choose for unknown?
   - How can you summarise the given data in a table?

[2] When discussing the problems, we focused not only on the final solution but using Pólya's suggestion, we also reflected on the solution method highlighting the most difficult parts and discussing how different students managed to overcome the obstacles.

- Can you express other unknown quantities in terms of x?
- Do you remember the "good idea"? (units)
- What is the connection between these quantities?
- Can you write an equation?

4. Two boats leave the harbour at the same time. One of them travels towards the North while the other towards the East. Two hours later, their distance is 60 km. Find the speed of the boats if one of them covers 6 km more than the other in an hour.

   Helping questions:

   - Draw a figure!
   - What do you choose for unknown?
   - Can you express the distance covered by the boats with the help of your unknown?
   - What is the connection between the distance covered by the boats and their distance?
   - Can you write an equation?

5. A tourist walked 80 km altogether. If he had walked 4 km less every day, his trip would have taken 1 day longer. How many km did he walk originally in a day?

   Helping questions:

   – Can you summarise the given information in a table?
   – What do you choose for unknown?
   – How can you express the speed in each case?
   – Can you express the time in terms of what you have found?
   – Can you write an equation?

Lesson Plan

The groups were given the above-listed problems with the helping questions. They had to solve them using the *Pair Check* method.[3] This time, the helping questions were handed out as a list but only the "checking" students were allowed to look at them. The solution was followed by a group discussion and a whole class discussion. Following this, new problems of similar type were distributed but this time without the questions. The students first had to solve them, but also their task included writing helping questions for the new problems. The method was *Pair Check* again. Finally, the new problems with the helping questions were collected and redistributed amongst the groups who had to solve them using the helping questions. Due to lack of time, this last task was given out as homework.

[3] This lesson was a practice lesson. In the previous lesson, we revised solving these types of problems.

Students' Questions

- What kind of table could you use? What would you include in it?
- What could you use instead of a table? (mixing problems)
- How can you use a table? What should go in the rows and columns?
- What is x? What can you express in terms of x?
- Which piece of information haven't you used yet?
- How can you make an equation from this?
- From the two results, which one is really a solution?
- Check your work!

Results

Students' Comments from their Reflection Booklets

"It is easier to work in groups because the different ways of thinking met; however, after 12 lessons it became a bit boring". **K-M.M.**

"Working in groups is useful and easier than working alone". **O.R.**

"I really liked working in groups; however, towards the end, my group mates became annoying. … This method could be used again just for shorter time". **M.T.**

"Working in groups is good. Many of us found the solution faster". **H.P.**

"The group work would have been better if I had been together with the right people". **K.D.**

"I am very glad that I could try working in groups because I was able to get to know my classmates better". **B.T.**

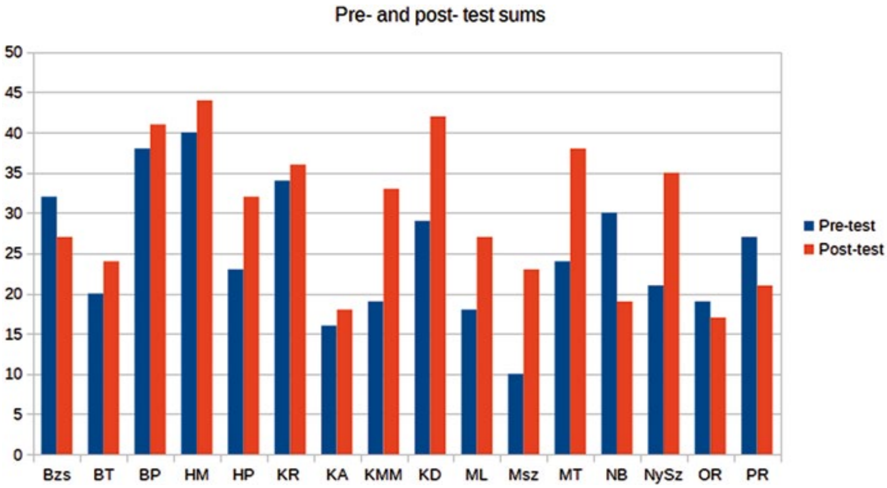
"Group work confirmed my feeling that I prefer working alone. I'd rather work following my teachers explanations". **Ny.Sz.**

The comments were chosen so that they reflect the ratio of opinions from the groups, namely, here the majority of the comments show that students liked working in groups and this opinion could be found in the majority of the reflection booklets. There were only a few students who agreed with the last comment, but those who did agree were from the better achievers. Many students mentioned that one of the main benefits of working together was that students with different ways of thinking were able to find the solution easier—which refers to the reduction of the cognitive load of individual students. On the other hand, a lot of students wrote that the exclusive use of cooperative techniques might cause boredom on the students' side. All in all, the students' comments suggest that teachers should use cooperative teaching but should be careful not to "overuse" it.

Pre- and Post-test Results

The students had to complete a mathematical pretest (Appendix 1) before the experiment that measured the mathematical knowledge and the knowledge of heuristic strategies needed for solving the problems that were to be presented in the first part

Table 1 Total points on the pre- and post-tests



of the experiment. After finishing the first part of the experiment, the students completed a mathematical post-test (Appendix 2). Some tasks of the two tests were identical; however, the post-test included questions in which the problem-solving skills learnt during the first part had to be applied. The diagram below shows the mathematical pre- and post-test results of the students (Table 1).

It can be clearly seen that 12 students out of the 16 improved their results and only four of them achieved a lower sum on the post-test than on the pretest. Students B. Zs., B. P., H. M. and K. R. already had a sum above 30 points on the pretest. Two of these students (B. P. and K. R.), based on their previous achievement in mathematics, their previous contribution to class work and their achievements on previous mathematics competitions, can be considered as talented students in mathematics. On the other hand, K. A., O. R. and M. Sz. are weaker students, and their pretest results are lower compared to the whole group. However, after solving mathematical problems with cooperative techniques, two of these students (K. A. and M. Sz.) managed to improve their test results, especially M. Sz., who doubled his total points on the post-test. Out of the four students who received lower points on the post-test, one (O. R.) can be considered as a weak student, while the others (B. Zs., N. B. and P. R.) are average ability students.

For each task, the students were awarded a maximum of five points. Table 2 shows the mean average point of each student on the two tests and the standard deviation (SD in the table) of the received points on each test.

On the pretest, the mean average score is the highest for H. M. and B. P., and their standard deviation is amongst the low ones. This means that these students received high marks for each task on the whole test and they attempted to answer each question. As mentioned before, B. P. is a talented student, so it is not surprising that he made an improvement on his mean average as well. K. M. M. and M. Sz. also have a low standard deviation, but their mean averages are rather low. These data

Table 2 Pre- and post-test statistics

	Pretest mean	Pretest SD	Post-test mean	Post-test SD
Bzs	3.56	1.94	3.00	2.09
BT	2.22	1.99	2.67	2.24
BP	4.22	1.72	4.56	0.73
HM	4.44	1.13	4.89	0.71
HP	2.56	2.40	3.56	2.40
KR	3.78	1.99	4.00	2.35
KA	1.78	2.44	2.00	2.03
KMM	2.11	1.45	3.67	2.12
KD	3.22	2.05	4.67	1.69
ML	2.00	2.40	3.00	2.24
Msz	1.11	1.69	2.56	1.87
MT	2.67	2.24	4.22	2.13
NB	3.33	2.18	2.11	2.40
NySz	2.33	1.87	3.89	1.32
OR	2.11	2.03	1.89	1.58
PR	3.00	2.18	2.33	1.72

suggest that these two students attempted most tasks but received low marks for them. M. Sz., who is one of the least talented students in the group, managed to raise his mean average mark to 2.56 on the post-test. The stronger students (B. P., H. M., K. R.) achieved better average marks on the post-test; one of the weaker students (O. R.) achieved a lower mean average mark, while the others (K. A. and M. Sz) achieved a higher mean average mark on the post-test.

The Teacher's Comments

The use of cooperative techniques definitely had an impact on the students' behaviour in class and it brought some change into the routine of Maths lessons. First of all, in the "cooperative lessons" everybody participated. There was no rush for the slower students but at the same time, the quick ones could work on extra problems while waiting for everybody to finish. The different problems could be discussed in more detail with the possibility of problem variation or opening the problem; furthermore, as a result of cooperative work, more solutions were presented for a problem. The first problem presented in the article was an open problem in the solution of which students had the opportunity to experiment with given values and building a general statement based on their findings. In small groups, the weaker students were more confident in sharing their ideas or asking questions. Moreover, it was easier for the teacher to help those who were stuck in the problem solving because the rest of the students were busy working on their own problems. However, there were some drawbacks of using cooperative teaching. It is more time consuming than "normal" teaching; because of this, less problems could be discussed. Group work is inevitably noisy and if more students need help at the same time, it might be difficult to handle.

Taking everything into consideration, the advantages of using cooperative teaching techniques outweigh the disadvantages. Students need to be put into situations in which they have to communicate with each other and in which they are "forced" to be active participants of the teaching and learning situation. Cooperative techniques seem to be a rather effective tool for achieving this.

Conclusion

Using open problems: as the teacher's comments say, open problems contribute to the development of problem-solving skills as they give the opportunity for students to come up with different solutions; they give a chance to weaker students to try concrete values and develop a generalised idea later, thus helping them to break the problem into smaller parts and reduce their cognitive load.

Helping questions: helping questions gave the chance to average ability or weaker students to get started and helped them to work through the solution of the given problem, again breaking the original problem into smaller bits, therefore reducing the cognitive load of the students.

Cooperative techniques: both the students' and the teacher's comments show that using cooperative techniques along with other teaching methods is beneficial for teaching problem solving not only for the talented but also for the average students.

In sum, it can be seen from the pre- and post-test results that combining the above-mentioned features in teaching mathematical problem solving has a positive impact on the students' achievement. Furthermore, they contribute well to reducing the cognitive load of individual students.

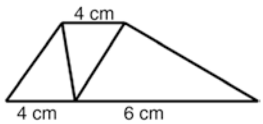
Following the above-described experiment, further action researches can be designed with special attention on using helping questions and open problems.

Appendix 1: Mathematical Pretest

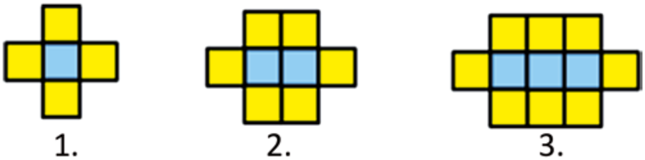
1. The bigger cogwheel of a bike has 35 teeth and the smaller one has 15. How many times do we need to turn the pedal so that both cogwheels get back to their original position? (The pedal is on the bigger wheel.)
2. The sides of a cuboid are whole numbers in centimetres. The areas of two of the faces are 24 and 36 cm^2. Find the volume of the cuboid.
3. In a jewellery shop on Monday, half of the stock and four pieces of jewellery were sold. On Tuesday, half of what was left and further two pieces were sold.

On Wednesday, the shop assistant sold five pieces of jewellery. On Thursday, two less than half of what was left was sold. At the end, there were eight pieces of jewellery in the shop. How many pieces of jewellery did the shop have on Monday?

4. How many different four-digit numbers can we form so that all digits are elements of the set {1; 2; 3; 4; 5; 6; 7}?
5. Snow White and the seven dwarfs have dinner around a round table. In how many different ways can they sit down next to each other?
6. The sum of two numbers is 2250. Twelve percent of the first number equals to 18 % of the second. Find the two numbers.
7. Laci got a pay rise of 15 % so his current salary is 241,500 HUF. How much was his original salary?
8. Work out the area of the triangles on the figure if you know that the area of the trapezium is 21 cm^2 (The shape is not drawn to scale).



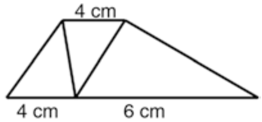
9. Look at the following shapes. Without drawing, find the number of yellow squares in the 4., 5., 6. shape. How many yellow squares are there in the 100. shape? How many in the nth shape?



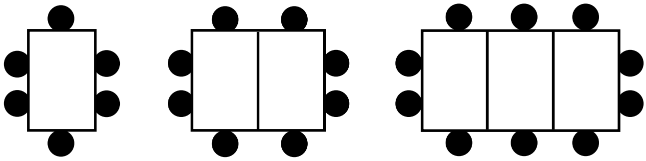
Appendix 2: Mathematical Post-test

1. The bigger cogwheel of a bike has 35 teeth and the smaller one has 15. How many times do we need to turn the pedal so that both cogwheels get back to their original position? (The pedal is on the bigger wheel)
2. Which numbers are always factors of the product of three consecutive numbers? Why?
3. In a jewellery shop on Monday, half of the stock and four pieces of jewellery were sold. On Tuesday, half of what was left and further two pieces were sold. On Wednesday, the shop assistant sold five pieces of jewellery. On Thursday, two less than half of what was left was sold. At the end, there were eight pieces of jewellery in the shop. How many pieces of jewellery did the shop have on Monday?

4. In how many ways can we arrange three red and three blue beads in a circle?
5. How many different four-digit numbers can we form so that all digits are elements of the set {1; 2; 3; 4; 5; 6; 7}?
6. Think of a three-digit number, write it down, then reverse the digits (e.g. 756 and 657). Subtract the smaller number from the bigger one. What do you notice? Prove your assumption.
7. Work out the area of the triangles on the figure if you know that the area of the trapezium is 21 cm². (The shape is not drawn to scale $A = \frac{a+c}{2} \cdot m$.)



8. Work out the area of the inscribed circle of an equilateral triangle if its sides are 1 m long.
9. In a restaurant, the guests sit around tables as you can see on the figure. Continuing the pattern, how many guests can sit around four tables? Around five tables? Around 100 tables? Around *n* tables?



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“Memorable Diagonals”: Exploratory Problems as Propositions for Doing Mathematics

Torsten Fritzlar

Inductive Working in Mathematics

Learning by experience is of essential importance in everyday life and also in the academic field. Academically engaging with experience can be called induction or inductive reasoning, which Klauer explains as “recognising regularity or order in the only apparently non-ordered while giving awareness to disruptions or non-order in the only apparently ordered” (Klauer, 1991, p. 137; translated by T.F.). Even if it might seem surprising, induction according to Pólya plays an important role especially in mathematics:

> *Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science.* (Pólya, 1971, p. vii)

He was able to demonstrate with numerous examples that inductive procedures are of particular significance especially for the genesis of (subjectively) new mathematics (e. g., Pólya, 1954a, 1954b). Pioneering in this context is certainly also Lakatos’ mathematics historical case study (1976) on developments in polyhedron geometry from the eighteenth to the twentieth century. But also in current sociological studies concerning mathematics, the high status of the quasi-empirical approach is emphasized (e. g., Heintz, 2000a, 2000b). Finally, from the point of view of mathematics as a scientific discipline, it can be claimed that in the light of Gödel’s incomplete theorems, experimental and inductive reasoning is a necessary complement to the deductive approach (Putnam, 1979). Therefore, Swiss mathematician Armand Borel speaks of mathematics as a “humanities-based science” (geistige Naturwissenschaft; Borel, 1984, p. 29).

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Pólya also claims credit for detailed analyses of inductive mathematical working for which he distinguishes two phases.[1] At the center of a first *phase of exploration* is the engagement with examples, observing, organizing, and searching for patterns and interrelations. It (ideally) culminates in a conjecture with regard to a general regularity. In a second *test phase*, this conjecture is tested on further examples; if necessary, special cases or in some way extreme examples can be included herein. With every test that results positively, the subjective trust in the validity of the hypothesis can increase until it is (preliminarily) *verified* (but of course not considered *proven* deductively). What also matters for Pólya in this context is an "inductive attitude," that is, a readiness to adapt one's own assumptions to one's own experiences as efficiently as possible and to continuously step from observations up to generalizations and down again to (critical) observations (Pólya, 1954a, 1954b, p. 7).

What implications do these reflections and analyses have for teaching mathematics? Working deductively is a particular strength of mathematics as a scientific discipline. Its dominance, mainly in the depiction of "finished" mathematics, at the same time induces a one-sided perception that seeks to be completed (Leuders, Naccarella, & Philipp, 2011). For a stronger process orientation in class, an appropriate wealth of experience with regard to doing mathematics, and a balanced view of the subject, pupils should be enabled to work experimentally and inductively. They should be allowed to learn that everyday experiences and common sense can also be used in mathematics classes. They also should learn that even mathematics knowledge is not absolute and that mathematics is in no sense as "otherworldly" as it often seems from the outside (Chazan, 1990). But how can this be encouraged?

Mathematical Exploratory Problems

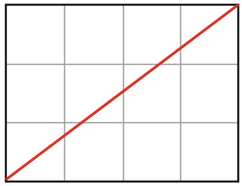
In mathematics educational literature, various kinds of problems are described. For example, Pólya distinguishes between two rather general types in Euclid's tradition—"problems to find" and "problems to prove" (Pólya, 1962). For inductive working, *exploratory problems* seem particularly suitable to me. These are mathematically rich situations whose processing can be characterized by the following: exploring examples ideally with regard to self-derived questions, gathering and analyzing data, constructing relations or patterns, and conjecturing and verifying hypotheses (Fritzlar, 2010). Thereby, exploratory problems combine and extend aspects of both classical problem types described by Pólya.

From the exemplary work, two students have undertaken on the "memorable diagonals" (Fritzlar, Rodeck, & Käpnick, 2006). In Fig. 1, it can be seen that these diagonals are a problem of this kind. In the following paragraph, the pupils' work is described in parts.

[1] American philosopher and mathematician Charles Sanders Peirce distinguishes in this context between induction and abduction.

Memorable diagonals

In the picture you can see a rectangle consisting of small 3×4-squares. They form a quadratic grid. If a diagonal is inserted, the grid lines are intersected by this diagonal.



How many points of intersection arise in this 3×4-rectangle?
How many points of intersection are there in other rectangles, for example in a 3×5-, a 4×5- or a 4×6-rectangle?
Come up with more rectangles yourselves and determine the number of points of intersection!
Can you also give the number of points of intersection for very large rectangles?

Fig. 1 Memorable diagonals

Memorable Diagonals

In the context of a research project, we examine how pupils in the last year of primary school (grade 4) and the first 2 years of secondary school deal with exploratory problems of this type. Their problem solving was carried out in tandem and was accompanied by a tutor (T) and videotaped. By now, we were able to investigate in detail how five tandems worked on this and other problems of this type. To give a first impression, the work on "memorable diagonals" done by Liam (L) and John (J) is described in excerpts and condensed. Both pupils were in the fourth grade of primary school and visited a fostering project for mathematically interested and talented primary school children at the University of Halle-Wittenberg. For better readability, oral utterances have been softened slightly, and an outline in paragraphs has been arranged.

(A) The first question is worked on with reserved guidance by the tutor. From initially different answer suggestions, a lively discussion develops between the two pupils. Finally, after about 2 min, they come to an agreement about what a point of intersection between the diagonal and the grid line is. After this, they conclude that there are five points of intersection for the given 3×4 rectangle.

(B) J: **I am doing 3×5 right now.**
L: **Then I'll do 4×6 and 4×5.**

The pupils embark on the next stage separately at first. Starting from a small number of examples, initial conjectures are made and put to the test immediately:

J: **So, this one works, my theory. Because here with 4×5 there were six and with 4×6 there are seven.**[2]

[2] The pattern supposed by John cannot be reconstructed. For a 4×5 rectangle, there are seven points of intersection.

L: **And with 3 × 5?**
J: **There are […] six.**
L: **Well, then your theory mightn't be right. Because twice six and once seven.**

(C) The pupils work on some further examples. Without observable intention, Liam among others also chooses the 2 × 2 square.

L: **I think 2 × 2 only has two [points of intersection]. This only has two.**
J: **No, it meets directly in the middle. You didn't draw precisely. If you draw precisely, it should meet in the middle.**
L: **Then it meets exactly once. One point of intersection. Is 4 × 4 three? See here, I have a theory: Because 2 × 2 is one, right? If 2 × 2—and always subtract minus one equals one. If 4 × 4 is three now, this could be right.**

Based on only one example, Liam apparently develops a hypothesis about the number of points of intersection for squares. This is initially specified and put to the test for an easily accessible example, the 4 × 4 square: **Three! My theory is right. Three.**In order to raise the certainty of this hypothesis, another example is investigated:

L: **8 × 8—if you subtract one from eight, it should be seven. I'll test this now. If it's right, my theory is exactly right.**

Through the correctly predicted number of points of intersection, the hypothesis becomes more convincing,[3] but at the same time, the confined scope comes into mind:

J: **And what are you going to do with a 3 × 9 or something like that?**
L: **This is tough. But the theory we just had is right for now, […]**
J: **I'll do a 20 × 20 now. Let's see if it's correct.**
L: **Well, your 20-thingy should be exactly 19. 19 points of intersection.* […]*
J: **It is.* […]*
J: **It's just that there is no theory with 2 × 3 or something like that.**

(D) The pupils investigate further examples and develop ideas about potential relations, which however cannot be confirmed.

J: **My theory isn't working!**
L: **Seems like everything was just a waste of time.**
J: (To the tutor) **Do you have a solution?**

(E) Seeing that the boys are stuck at this point and that their motivation is in danger of dropping after several minor failures, the tutor addresses John's question and recommends organizing the results obtained.

T: **Why don't you write down which rectangles you have already investigated and what the results of those were.* […]*

[3] Indeed, Liam is not yet sure regarding the universality of the pattern assumed (cf. the following remarks).

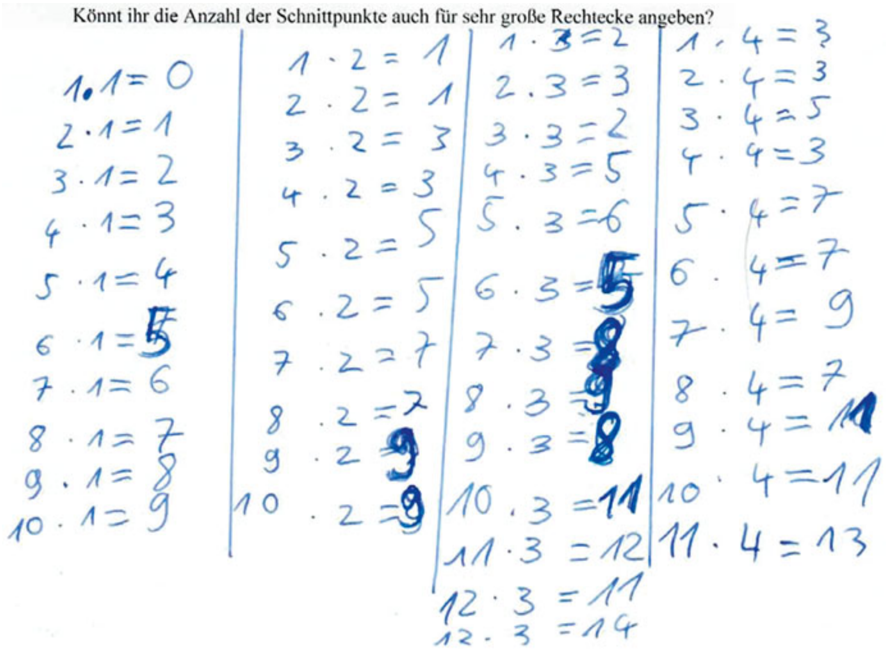


Fig. 2 Results of problem solving by Liam and John

L: **We've already had 2×2, 4×4, 3×3 … that's simple really.**
T: *Exactly. *You have all the squares.**
L: **Well, we don't have all of them, but according to my theory they're known already. That's why they don't have to be investigated now.**

Naturally, Liam knows that not all squares have been tested. Yet, he is also convinced that no additional tests are necessary due to his hypothesis having been confirmed by numerous examples.

(F) At the same time John starts to write a schema of the results obtained and begins to complete the emerging columns (cf. Fig. 2).

J: **What was the result for a 2×1? I think for a 2×1, we'll get one.* […]*
J: **And what is the result for a 3×1? Hmm, well this is a good start! For the 4×1 […]* […]*
J: **Slowly we could get to a theory. For the ones [□×1 rectangles] […] the minus ones are the numbers of points of intersection.**
L: **For 10×1, we should get a good few.**
J: **We should get exactly nine for that.**
L: **According to your theory. We'll put it to the test!**
L: **[…] 7, 8, 9. It is exactly nine, like you predicted. Fine, so we have another theory!* […]*

J: * Now we'll have to do the same with the 2, 3, 4, 5, 6, 7, 8, 9 and 10. Have fun doing that!*

(G) John and Liam now turn their attention to the two column-wide rectangles.

L: *You take notes, and I'll test the theories.*

J: *Try 1×2! […] No, we have 1×2 already, it equals one. You try 2×2! No, 2×2 is also one. Can you do 3×2 now, please! And please use the ruler […]*

L: *Fine. But you can already guess what that equals. Three.*

J: *Hopefully not, that way the latest theory would be gone. Unless, no, it wouldn't be gone.*

L: *Yes, that's three. So, 4×2. If this also equals three, I'd have a new theory. Three! So, 5×2.*

J: *5×2 would be five. And 6×2 also five.*

J: *And 7×2 seven.*

L: *And 8×2 also seven. And 9×2 also nine. And 10×2. Right, we'll have to test that now.*

Based on the first examples, the two students develop the same conjecture simultaneously but independently from one another. They both agree that it has to be verified for further examples.

J: *6×2. […] According to the theory, it is correct! 7×2?*

L: *And yet again a new theory, I bet!*

J: *Hopefully! Otherwise we'd have to try everything out. For 7×2, it's also seven. So, every second time it changes by two. Here in the two times line. […] In the three times line it should be every third time//* […]

L: *Your theory is right, definitely!*

J: * We should just check the nine again.*

Once more their conjecture is confirmed, and Liam and John go on to check rectangles of larger width in a similar way. Step by step, the following overview emerges:

(H) After about 60 min working time, both pupils struggle to concentrate on the problem at hand. The tutor tries to motivate them once again for the search of an all-embracing pattern.

T: *I'd have one more question if you can bear with us that long. Is there also a theory about the theories? What I mean is this: Do the theories somehow belong together? Could you make one big theory from them?*

L: *One huge theory for all theories? Well, we'd have to check them next time.*

It can be concluded that Liam and John strive persistently to find a solution in spite of some setbacks. They commit themselves intensely to the examples and are mostly organized; the examples are then arranged and emerging gaps are filled. The pupils quickly come up with several hypotheses, respectively, which are tested numerous times. A wide variance of testing examples is strived for. Along with the duration and intensity of the pupils' work, the variety of the constructed patterns also seems remarkable to me.

Working on Exploratory Problems

For a detailed analysis, problem solving of this kind is videotaped, transcribed, and analyzed through theoretical coding (Strauss & Corbin, 1990). In a firstly open coding process, codes had been assigned to individual parts of the transcript or the problem-solving process. Thereby, we considered questions such as what, who, how, with what, what for, and why. Based on this coding, comprehensive process elements were constructed with which pupils’ problem-solving processes can be described. A description (not only) of Liam’s and John’s work on the “memorable diagonals” seems possible by the following cycle (Fig. 3). However, not all phases have to be passed through necessarily.

This cycle could be reconstructed also for other tandems and other exploratory problems although it was normally run through in a lower frequency. Altogether, our study demonstrated that experimental and inductive work can be already encouraged in primary school students by using appropriate exploratory problems. Figure 3 also illustrates essential similarities with the approaches of research mathematicians, as described by Pólya. Parallels can also be seen with studies by Philipp (2013). Additionally we were able to identify occasional[4] planning processes (in episodes

[4] And therefore visualized lighter in Fig. 3

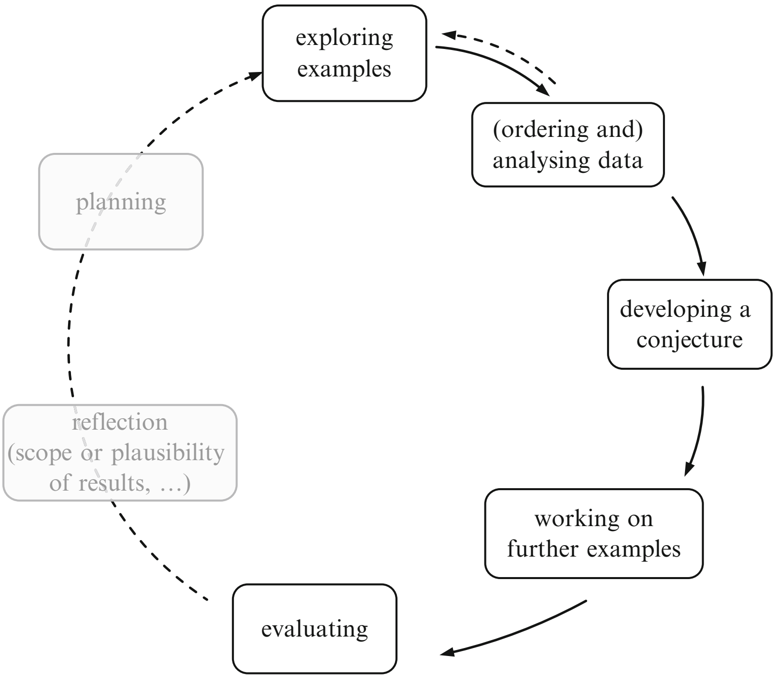


Fig. 3 Cycle of exploratory work

Table 1 Könnt ihr die Anzahl der Schnittpunkte auch für sehr groβe Rechtecke angeben?

1 • 1=0	1 • 2=1	1 • 3=2	1 • 4=3
2 • 1=1	2 • 2=1	2 • 3=3	2 • 4=3
3 • 1=2	3 • 2=3	3 • 3=2	3 • 4=5
4 • 1=3	4 • 2=3	4 • 3=5	4 • 4=3
5 • 1=4	5 • 2=5	5 • 3=6	5 • 4=7
6 • 1=5	6 • 2=5	6 • 3=5	6 • 4=7
7 • 1=6	7 • 2=7	7 • 3=8	7 • 4=9
8 • 1=7	8 • 2=7	8 • 3=9	8 • 4=7
9 • 1=8	9 • 2=9	9 • 3=8	9 • 4=11
10 • 1=9	10 • 2=9	10 • 3=11	10 • 4=11
		11 • 3=12	11 • 4=13
		12 • 3=11	
		13 • 3=14	

B, F, G) and reflections, for example, about the plausibility and scope of results or possible consequences for further working (in episode C). It seems important that already primary school students are able to carry out such higher-level processes which facilitate the realization of potentials of the inductive approach.

These experiences are a further confirmation of the French mathematician Jacques Hadamard's view: *Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature* (Hadamard, 1949, p. 104). Yet, he thereby assumes that pupils are granted access to appropriate exploration opportunities and that they are given the freedom of experience.

During their entire problem-solving processes, Liam and John did not ask once for explanations or final proofs of the patterns they constructed. This might be another example of younger children having a rather empirical view on mathematics. For them, results won deductively do not have a greater or different value than results won empirically per se. On one hand, this can be seen as an argument to strengthen experimental and inductive working in school; on the other hand, it emphasizes the importance of making accessible the complementarity of induction and deduction.

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Pupils' Drawings as a Research Tool in Mathematical Problem-Solving Lessons

Erkki Pehkonen, Maija Ahtee, and Anu Laine

Abstract Firstly, we describe a research project on problem-solving implemented in 2010–2013 in the Department of Teacher Education at the University of Helsinki. But we are especially concentrating on the results of one background study in the project—pupils' drawings in a mathematics lesson. Pupils' drawings seem to be a powerful method to gather information from small children. With the aid of drawings, one may investigate different topics in children's thinking. Here, we focus on pupils' and teachers' communication, the emotional atmosphere of the class and the types of work used in class. The drawing studies offer three different channels to pupils' conceptions in problem-solving.

Keywords Problem-solving • Mathematics lesson • Pupils' drawings • Research tool

In Finland, there is a 9-year comprehensive school where all children study in heterogeneous groups, including in mathematics. Teaching in schools is regulated by the national curriculum (National Board of Education, 2004). The national curriculum emphasises the importance of creating a learning environment having an open, encouraging, easy-going and positive atmosphere and that the responsibility to maintain this environment belongs to both the teacher and the pupils. Teaching mathematics in elementary grades is usually concentrated on the use of textbooks. Details on mathematics teaching in Finland can be found, for example, in the book by Pehkonen, Ahtee, and Lavonen (2007). Here, we aim to clarify third graders' (about 9 years old) conceptions of mathematics and mathematics teaching for problem-solving through their drawings.

Pupils' conceptions are considered from the viewpoint of classroom communication, emotional atmosphere and types of work. It is important to grasp what is happening in Finnish schools, because pupils' attitudes in mathematics get increasingly worse after Grade 3 (Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014, p. 285).

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The background study on pupils' drawings came up surprisingly as an interesting new method to gather and interpret data on pupils' mathematics-related conceptions in a research project on problem-solving. When we have presented our results at international conferences, many researchers from different countries have been interested in implementing the same task in their own country.

Therefore, we focus here on pupils' drawings. The purpose of this study was to explore what can be said, based on the pupils' drawings, about how third graders experience their mathematics lessons. In particular, we were curious about what communication on mathematics is like between the teacher and her pupils as well as the pupils' emotional position.

In order that the teacher can be successful in guiding pupils' problem-solving lesson, he/she should be aware of pupils' mathematical conceptions. These form basis for teaching problem-solving. As background information for the drawing study, we first present the research project on problem-solving from which our results come.

The Research Project

The 3-year research project was run in the Department of Teacher Education at the University of Helsinki in the years 2010–2013.[1] The objectives of the project were to clarify the development of pupils' and teachers' mathematical understanding and problem-solving skills during 3 years—from Grade 3 to Grade 5—when open problems were used regularly once a month. It was a joint comparative research project with Chile, and we aimed to parallel the Finnish and Chilean teaching practices in mathematics.

Data Gathering

For the research project, we selected two groups of Grade 3 classes: one experimental group and one control group, for a total of ten teachers. The experimental classes were from cities surrounding Helsinki (Vantaa, Espoo and Kirkkonummi), and the control classes were from Helsinki. In both groups, the same background studies were implemented, but in the experimental group, once a month, there was an additional lesson on open problem-solving that two researchers videotaped. In the control group, only initial and final measurements (background studies) were implemented.

[1] The research project was financed by the Academy of Finland (Project 135556).

Background Studies

For the background studies of the research project (the initial measurements in 2010), we used different methods to uncover pupils' mathematics-related beliefs and mathematical knowledge: a questionnaire of pupils' conceptions of mathematics, pupils' drawings in a mathematics lesson, a test on pupils' mathematics knowledge and problem-solving skills and a postal survey of teachers' conceptions of problem-solving.

Implementation of the Project

In the experimental group, the teachers taught in average one mathematics lesson per month dealing with open-ended problems, and for all other lessons, they used their own teaching method. The teachers in the control group applied only their conventional methods for mathematics teaching. Data were gathered from pupil and teacher questionnaires, pupils' drawings, teacher interviews, classroom observations and field notes during the implementation of the open-problem-solving lessons, videotaped work, thinking-aloud protocols and videotaped discussions.

We never expected that such a short interference in teaching (only once a month an open-problem lesson and everything else conventional teaching) would result in a big change in mathematical knowledge or teaching habits. But we anticipated that when the teachers and pupils experienced open mathematics teaching (open problem-solving), that would offer them an idea on an alternative way of teaching that might, with time, help them to change their understanding of mathematics and its teaching.

The experimental tasks used in the project were open problems where either the starting situation or the ending situation or both contained some additional options. Therefore, the problems did not have one definite answer, but they might have many different answers depending on the auxiliary conditions the solver put forward. Thus, solving these problems required that the solver must combine in a new way the information already familiar to him or her.

These tasks were introduced beforehand in the experimental group teachers' and the researchers' monthly joint meetings. In the meetings, the teachers helped us provide a proper wording and presentation mode for the tasks. But finally, every teacher pondered for herself the implementation of the task in her own teaching group and gave us her lesson plan before the experimental lesson. Altogether, 20 different open problems were dealt with in 3 years; they were on various topics of elementary mathematics: arithmetic, combinatorics and geometry.

There are some published papers that describe the research project in more detail (e.g. Pehkonen, Näveri, & Laine, 2013).

Here, we will restrict our discussion to the Finnish part of the project, and more specifically, to one of the background studies: the Pupils' Drawing Task.

Pupils' Drawings

The research study we are describing in this chapter is on pupils' drawings and what we can gain from the drawings. We have selected the dealing with three different aspects in the drawings: (1) communication, (2) emotional atmosphere and (3) types of work. The order of these aspects is chronological. We began about 5 years ago our research on drawings with the case of classroom communication and teaching methods (cf. Pehkonen, Ahtee, Tikkanen, & Laine, 2011). Secondly, our interest focused on affective factors seen in the drawings (cf. Laine, Näveri, Ahtee, Hannula, & Pehkonen, 2013). And recently, we have tried to improve research methods in the case of teacher-centred vs. pupil-centred teaching (cf. Ahtee, Pehkonen, Näveri, Hannula, Laine, Portaankorva-Koivisto, & Tikkanen, submitted).

As an aspect of the analysis, we developed an a priori coding scheme to be applied to all the drawings that focused on aspects of communication in the drawings. There is another classification for pupils' emotional position. Thus, our research question was as follows:

What can we reveal via pupils' drawings on mathematics teaching in their class?

And we developed from this question three more specific subquestions:

1. *How do the teacher and the pupils communicate with each other as seen in third graders' drawings?*
2. *What kind of emotional atmosphere in a mathematics lesson can be seen in third graders' drawings?*
3. *How can we identify the type of work done during a mathematics lesson as seen in third graders' drawings?*

Theoretical Framework

Here, we consider the main concepts of our drawings study at a theoretical level with the help of the existing literature. The most important concept is the use of drawings as a source for information. Another important concept is the emotional atmosphere in class.

Pupils' Drawings as Research Tools

Drawing is an alternative form of expression for children. Barlow, Jolley, and Hallam (2011) have noted that freehand drawings help children recall and express more details about events they illustrate. Drawings tend to facilitate the recalling of events that are unique, interesting or emotional but not routine events or isolated bits of information that are not part of a narrative. Pupils' drawings open a holistic way to evaluate and monitor pupils' understandings of classroom climate, where there are several facets of communication and types of work.

Ruffell, Mason, and Allen (1998), in addition to Bragg (2007), challenge the written questionnaire as a method for studying children, as children do not necessarily understand the words and statements used in a questionnaire in the way that the researcher has intended. In an ideal situation, the child under study has an opportunity to verbalise his or her own concepts. According to Hannula (2007), it is not easy to get linguistically rich responses from young children. Thus, it is a challenge to develop for teaching mathematics research methods that would be suitable for young children and that take the children's views into account.

Many researchers (e.g. Dahlgren & Sumpter, 2010; Ludlow, 1999; Remesal, 2009; Tikkanen, 2008) emphasise that one way to evaluate the teaching of mathematics is to ask pupils to draw a picture about the lesson: Pupils who have received teacher-centred teaching often draw a blackboard and a teacher in front of the class. The pictures less often include references to communication between pupils. The drawings also tell us about beliefs, attitudes and feelings that have to do with mathematics. When pupils have taken part in the kind of teaching that activates them, they produce pictures that emphasise activities and communication between pupils.

Drawings help pupils to overcome the difficulties in disclosing their thoughts, feelings and opinions to an adult researcher (Zambo & Zambo, 2006). According to Weber and Mitchell (1996), pupils' classroom drawings form rich data to study children's conceptions on teaching. Pupils' drawings have made an alternative and complementary contribution to conventional research methods by conveying their images about mathematics, mathematics teaching, their teacher and their peers and classrooms in mathematics lessons.

Both meaning making and interpretation have a central role in analysing drawings. According to Blumer (1986), the meanings given by the pupils to various situations and things guide their actions, how they interpret different situations and what they include in their drawings. Giving meaning is a continuous process, which in this study takes place particularly in the social context of the mathematics lesson. Different pupils will find different meanings in the same situations and things. The meanings may have to do with physical objects, such as the classroom blackboard or a desk; social interaction, such as working alone or in a group; or with abstract matters, such as the concepts of mathematics or the feelings that are elicited by teaching. The methods used in teaching organise both the actions between the teacher and the pupils as well as the actions between pupils. Based on the influences of the teaching, the pupil may evaluate himself or herself as poor and his or her classmates as good in mathematics.

Tikkanen (2008) compared Finnish and Hungarian fourth graders' experiences with mathematics teaching. The data consisted of pupils' drawings and narratives. Three types of classrooms were identified according to their mathematical contents and the style of narration. Regardless of teaching methods, most of the pupils had a positive attitude towards mathematics and a positive self-concept.

In the framework of motivation theory, Dahlgren and Sumpter (2010) compared second and fifth graders' conceptions of mathematics and mathematics teaching *via* drawings with a written questionnaire in Sweden. All pupils presented mathematics teaching as an individual activity with a focus on the textbook. Most of the second graders had a positive attitude towards mathematics, whereas a larger proportion of the fifth graders had a negative one.

Rolka and Halverscheid (2011) analysed fifth and sixth graders' drawings, texts and interviews for studying their mathematical worldviews. They tried to classify the drawings into the three categories proposed by Ernest (1991): instrumentalist view, Platonist view or problem-solving view. Their conclusion was the following: 'Considering the picture alone as the data source for extracting the underlying mathematical world view is related to a large amount of subjectivity in interpretation and will certainly not allow for an unambiguous classification' (Rolka & Halverscheid, 2011, p. 522). Therefore, they considered also text and interviews in order to study pupils' mathematical world views.

In our research project, we used as background tests besides pupils' drawings a questionnaire of pupils' conceptions of mathematics and a test on pupils' mathematics knowledge and problem-solving skills.

On the Emotional Atmosphere in a Classroom

Teachers have a central role in advancing the affective atmosphere and social interaction in their class. Harrison, Clarke, and Ungerer (2007) summarise that a positive teacher-pupil relation advances both pupils' social accommodation and their orientation to school, and it is thus an important foundation for the pupils' academic career in the future.

Evans, Harvey, Buckley, and Yan (2009) define three complementary components of classroom atmosphere: (1) academic, referring to pedagogical and curricular elements of the learning environment; (2) management, referring to discipline styles for maintaining order; and (3) emotional, the affective interactions within the classroom. In this study, we concentrate on the last component; that is, the emotional atmosphere, which can be noticed, for example, as an emotional relation between the pupils and the teacher. The state of the pupils' emotional atmosphere is an important background factor in problem-solving.

The emotional atmosphere within the classroom can be regarded either from the viewpoint of individuals in the class (psychological dimension) or from the viewpoint of a community (social dimension). Whilst the individual perspective looks at the individual experiences in the class, the social perspective looks at the class more holistically with a focus on social interaction, communication and norms. Furthermore, a distinction can be made between two temporal aspects of affect: state and trait. *State* refers to the emotional atmosphere in a specific moment in the class, whilst *trait* refers to a more stable condition or property (cf. Hannula, 2011). In this study, our perspective is holistic, connecting pupils' individual dimensions.

Different affective dimensions can be studied also using social level concepts at the level of community; that is, of a classroom. Rapidly changing affective states include, for instance, a social interaction connected to a certain situation, communication related to this and the emotional atmosphere present in the classroom. When similar situations happen repeatedly in a classroom, pupils may form more stable affective traits typical to a certain classroom. Social norms (Cobb & Yackel, 1996), social structures and the atmosphere in a classroom are such traits. Pupils will 'learn' that during mathematics lessons, homework is always checked in the same way, and a certain norm is developed. When also other parts of the mathematics lesson happen repeatedly in the same kind of atmosphere, the atmosphere may become general and include all mathematics lessons, possibly also lessons of other subjects.

Methods

The results of the study are based on pupils' drawings that were gathered in autumn 2010 in Greater Helsinki. The teacher of the class gave the following task instruction to her pupils who worked independently, and then she collected the drawings for the researchers.

The third graders in question (about 9 years old) came only from the nine teachers in the Greater Helsinki area. The drawings from the class of one teacher had to be put aside, since the pupils had produced them in pairs.

The drawing task:

Draw your teaching group, the teacher and the pupils in a mathematics lesson. Use balloons for speech and thought to describe conversation and thinking. Mark the pupil that represents you by writing on it ME.

The drawings by 133 pupils were analysed, of which there were 72 from boys and 61 from girls. About two-thirds of the pupils had added into their drawings some balloons for speech and thought. Thus, they enabled us to investigate communication between the teacher and the pupils as well as among the pupils.

Pupils have marked in many drawings the pupils' and the teacher's faces. With the help of those facial expressions, we were able to conclude how the pupil who did the drawing has experienced the emotional atmosphere in class. Thus, deciding in each drawing the pupil's attitude, the person who did the drawing and with the help of facial expressions, we can add up the emotional atmosphere in the whole class.

Since this chapter is a compound of three substudies, they are dealt with separately. And therefore, the methods used in each substudy are presented apart later on in detail in a proper place.

Result of Pupils' Drawings

Communication

We wanted to find out how the pupils experienced the kind of communication used in problem-solving during mathematics lessons. Thus, we sought answers to the following question:

> *How do the teacher and the pupils communicate with each other as seen in third graders' drawings?*

The starting point of the classification of pupil drawings was the analysis method developed by Tikkanen (2008) in her doctoral dissertation. According to this method, a drawing as data source for observation can be divided into content categories. A content category means the phenomenon on which data is gathered. We have chosen here the following as content categories: (1) teacher's communication and (2) pupils' communication.

For analysis, the content categories were operationalised into the following subcategories:

1. Teacher's communication: gives instructions, keeps order, teaches, gives feedback, and observes quietly, whilst the pupils work
2. Pupils' communication: a pupil makes/asks/or thinks a remark in connection to teaching; a pupil asks for help; pupils discuss with each other; or a pupil makes/or thinks an improper remark

Two researchers classified the pupils' drawings, and in the case of a difference of opinion, both researchers re-examined and discussed the drawing in question together. All the drawings ($N = 133$) were carefully classified. The evaluation of agreement was elicited by calculating the classifiers' differences.

The method of analysing the drawing was a mixed method, and it can be classified as inductive content analysis (Patton, 2002), as we were trying to describe the situation in the drawing without letting our own interpretations influence it. Each drawing was carefully examined in order to find all subcategories of the main content category. In every content category, the last subcategory was 'not recognisable'. The agreement between two classifiers in all subcategories was very good; that is, over 90 % (range 91–95 %).

In many drawings, one can see only stick figures; in some cases, hands are beginning from the head, and in some drawings, there are only pupil desks representing pupils. However, some of the third graders were very talented in drawing, and then in the pupils' drawing, one can see several details. The example in Fig. 1 of the pupils' drawings is very informative. In speaking bubbles, the pupils present their memory pictures about mathematics lessons and their atmosphere. But the pupils' method of presenting a saying (loud or whispering) and thinking is not always consequent.

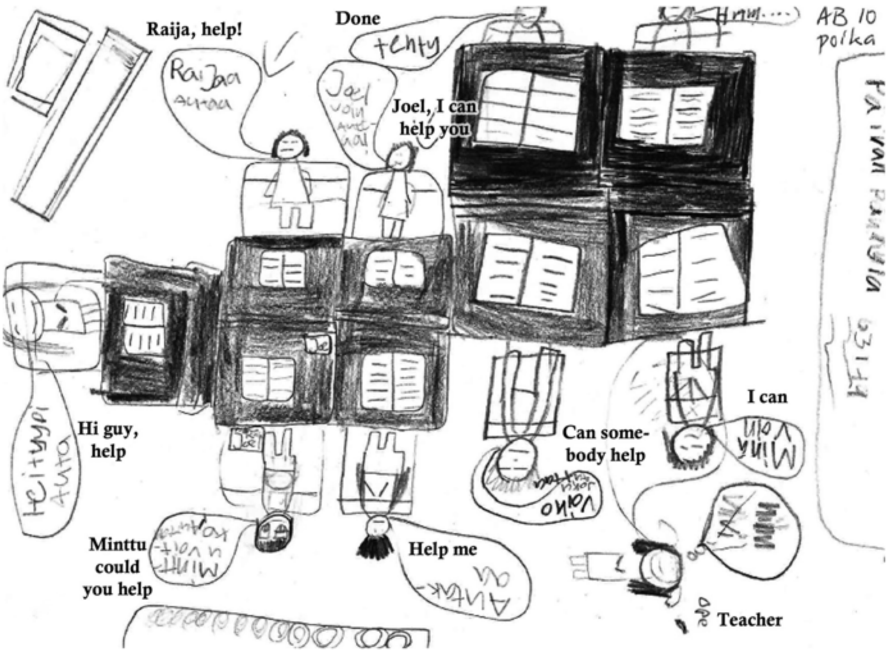

Fig. 1 An example of a pupil's drawing; its analysis is below

An interpretation of the drawing in Fig. 1 is presented using the content categories previously described. In many content classes, there is not only one feature but many (cf. the second content category): (1) teacher's communication (the teacher observes quietly) and (2) pupils' communication (pupils make remarks in connection to teaching; pupils ask for help).

Results

In this study, we tried to answer the research question with the help of the drawing analysis. It is helpful to notice that in the categories of the classification, the frequency is larger than the number of the pupils, since in many drawings, one can find several features.

Firstly, we will deal with the content category 'teacher's communication' (cf. Table 1). Since in the drawings of many pupils there were several indicators, the total frequency was 145. This totality is divided rather uniformly between several factors. In the parentheses, we give first the absolute frequency and then the relative frequency in a percentage.

In 'teacher's communication', the mode value (36; 25 %) is 'teaches' that contains both a teacher's own questions and expository teaching. But the frequencies are almost as large in the subcategories 'follows quietly pupils' working' (33; 23 %)

Table 1 The relative frequencies in the content categories: A teacher's communication ($N = 145$), Pupils' communication ($N = 191$)

A teacher's communication (%)		Pupils' communication (%)	
Teaches	25	Connected to teaching	34
Follows quietly	23	Is improper	25
Maintains order	13	Asks for help	15
Gives orders	10	Pupils discuss	12
Gives feedback	10		
Not recognisable	19	Not recognisable	10

and 'not recognisable' (28; 19 %). Thus, most of the pupils convey an impression that a teacher asks questions and delivers knowledge in mathematics lessons. There are many drawings where the teacher is not drawn at all.

Secondly, we take the content category 'pupils' communication' (cf. Table 1). Since in the drawings there were several indicators, the totality here is 191. The largest frequency is in the subcategory 'a pupil makes/asks/or thinks a remark in connection to teaching' (65; 34 %). The next largest frequency (48; 25 %) is in the subcategory 'a pupil makes or thinks a improper remark'. The frequencies of the rest of the three subcategories are under half of the maximum frequency. Therefore, we could say that in the drawings, pupils' communication is a compound of pupils' remarks where the largest share form the remarks connected to teaching or learning of mathematics, but there are also a great many improper remarks.

In Table 1, there are relative frequencies of communication in class. If we select from the content categories the most popular one (the mode category), we receive from the third graders' communication in mathematics lessons the following prototypic picture: According to the pupils' drawings, a teacher's communication consists mainly of teaching (25 %). It is interesting to notice that also another quarter of the pupils' experiences (23 %) carry another idea of teaching: The teacher quietly follows her pupils' working. Pupils' communication is clearly connected with teaching (34 %).

Conclusion

No negative attitude to the teacher could be found in these drawings. This finding is different from the results of the study by Picker and Berry (2000). They found that pupils often ask teachers for help. Of course, a teacher commands somewhat when maintaining order. The interaction between a teacher and pupils seems to be positive in the drawings, and that is important, since pupils are in cooperation with their teacher for about 4–5 lessons during a school day.

Altogether, two-thirds (67 %) of the third graders produced drawings where the pupils' thinking, speaking and action can be seen. According to Tikkanen (2008), mathematics lessons seem to contain many actions that pupils include in their drawings from working environment to mathematics drawn in the blackboard and in the speech bubbles.

In the published paper (Pehkonen et al., 2011), the communication results are presented more in detail.

Emotional Atmosphere

In the second substudy, we wanted to find out what kind of emotional atmosphere the pupils convey in their drawings of a mathematics lesson. Thus, we sought answers to the following question:

> *What kind of emotional atmosphere in a mathematics lesson can be seen in third graders' drawings?*

In this study, we were concentrating only on *the holistic evaluation of the emotional atmosphere in a classroom,* which is based on all the pupils' and the teacher's moods seen in a drawing as well as on the pupils' speech and thought bubbles in the picture. The pupils' mood and the teacher's mood are determined from the form of the mouth (smiling, neutral, sad/angry, not visible) and on their utterances or thoughts. The emotional atmosphere was classified as one of the following:

1. Positive (all persons smile or think positively, some part can be neutral)
2. Ambivalent (positive and negative), if at least one contradicting (positive or negative) facial or other expression is found in the drawing
3. Negative (all persons are sad or angry or think negatively; some can be neutral)
4. Neutral (all facial or other expressions are neutral)
5. Unidentifiable (when it is impossible to see any facial or other expressions)

In order to get an overview of the emotional atmosphere of the whole class, we made a summary of the holistic evaluation of the individual pupils' drawings. It is important to notice that we were interested in the general atmosphere during mathematics lessons and not in any specific feelings towards mathematics activities.

Results

The emotional atmosphere in a mathematics lesson is taken as an entirety that consists of the pupils' and the teacher's facial expressions and their utterances or thoughts in the drawings. The observations are classified using the scale: positive, ambivalent, negative, neutral and unidentifiable. The result of the analysis is presented in Table 2.

The mode value of the emotional atmosphere in mathematics lessons was classified as positive, since it can be seen in 50 (38 %) of the drawings. For example, the

Table 2 Emotional atmosphere in a mathematics lesson in third grade (frequency; percent)

	Positive	Ambivalent	Negative	Neutral	Unidentifiable
Total (133)	50 (38 %)	44 (33 %)	13 (10 %)	20 (15 %)	6 (5 %)

drawing in Fig. 2 (Appendix 1) was classified as positive because all the pupils as well as the teacher are smiling. Furthermore, both the teacher's and the pupils' speech or thought bubbles are either positive or neutral.

The number of the pupils (44; 33 %) who portrayed the emotional atmosphere in their mathematics lesson as ambivalent is almost the same as the number of pupils who described it as positive. An example of an ambivalent case is presented in Fig. 3 (Appendix 1). The pupils are sitting in rows, and there are both positive and negative facial expressions in the drawing.

A tenth of the pupils pictured the emotional atmosphere as negative; that is, they drew sad or angry faces or the speech bubbles contained negative (or neutral) thoughts. In Fig. 4 (Appendix 1) there is an example of a drawing showing a negative emotional atmosphere.

In 15 % of the drawings, the emotional atmosphere was classified as neutral because the persons' facial or other expressions were neither positive nor negative. Pupils and teachers were normally talking only about mathematical tasks as in Fig. 5 (Appendix 1).

As a summary, we can conclude that the mode value of the emotional atmosphere in the pupils' drawings of mathematics lessons is positive in 50 cases (38 %), where both the teacher and all the pupils are smiling (or some of them are neutral) or thinking positively or neutrally (cf. Table *2*). A third of the pupils have drawn the emotional atmosphere in the classroom as ambivalent, which means that in their drawings, there is at least one person whose facial expression is sad or angry or who says (or thinks) something that is interpreted as negative. The difference between the positive and ambivalent subcategories is not large, as the latter category contains also the drawings in which among many smiling pupils, there is at least and perhaps only one pupil showing a sad face. It can thus be said that in these third graders' drawings, the principal mood in mathematics lessons is positive.

Next, we looked at a classroom-specific emotional atmosphere in the mathematics lessons found in the third graders' drawings from the classes of nine different teachers. We made a summary of the holistic evaluation of the individual pupils' drawings in order to get an overview of the emotional atmosphere of the whole class. The summary of emotional atmosphere in the different classrooms is presented in Table *3*.

Even though the emotional atmosphere in pupils' drawings on mathematics lessons is mostly positive in the total data (cf. Table *2*), there are large differences among the different classrooms. It is possible to look at the mode of the emotional atmosphere in every classroom (cf. Table *3*), but it is important to notice that this mode does not reveal the whole truth. The profiles of the emotional atmospheres also vary widely within the classrooms.

Table 3 The distribution of emotional atmosphere in mathematics lesson in the nine classes (frequency; percent)

	Positive	Ambivalent	Negative	Neutral	Unidentifiable
A (15 pupils)	8; 53 %	4; 27 %	3; 20 %	0; 0 %	0; 0 %
B (14 pupils)	7; 50 %	1; 7 %	1; 7 %	3; 22 %	2; 14 %
C (19 pupils)	9; 47 %	7; 37 %	2; 11 %	0; 0 %	1; 5 %
D (18 pupils)	8; 44 %	6; 33 %	0; 0 %	2; 11 %	2; 11 %
E (16 pupils)	4; 25 %	2; 13 %	1; 6 %	9; 56 %	0; 0 %
F (17 pupils)	5; 29 %	4; 24 %	0; 0 %	8; 47 %	0; 0 %
G (17 pupils)	2; 12 %	5; 29 %	5; 29 %	4; 24 %	1; 6 %
H (11 pupils)	4; 36 %	5; 46 %	1; 9 %	1; 9 %	0; 0 %
I (6 pupils)	2; 33 %	4; 67 %	0; 0 %	0; 0 %	0; 0 %
Average (133 pupils)	49; 37 %	38; 29 %	13; 10 %	27; 1 %	6; 4 %

In this study, our summary is that the emotional atmosphere in these mathematics classes at Grade 3 seems to be mainly positive, although there are big differences between classes. More details on the study of emotional atmosphere can be read in the published paper (Laine et al., 2013).

Types of Working

As said earlier, a drawing gives a 'snapshot' of how the pupil who did the drawing has experienced his or her teacher's and his or her classmates' activities during mathematics lessons. Here, our aim was to find out how the pupils saw what type of work is done in mathematics lessons. Therefore, we needed to create a method to analyse young pupils' drawings in order to find answers to the following question:

> *How can we identify the type of work done during a mathematics lesson as seen in third graders' drawings?*

The concepts of teacher-centredness vs. pupil-centredness are actually rather complicated ideas, and they contain a wide range of meanings (cf. Neumann, 2013).

Data Analysis

Since our aim was to develop a research method, we made several experiments to elaborate the range of teacher-centredness vs. pupil-centredness used in the classroom as seen in the pupils' drawings. After several trials, we listed together from the drawings in one classroom all possible teachers' and pupils' activities during mathematics lessons as well as the types of work used in the classrooms as seen in the pupils' drawings.

Two researchers then completed these lists by going through all the third graders' drawings at hand ($N = 133$). In this way, three different lists were formed, but we deal here only with the third one: Types of work during a mathematics lesson as seen in the pupils' drawings. The final list is given in Appendix 2.

This list contains three different types of work: namely, independent work, group work and work with the teacher in charge. When pupils are working independently, they seem to be solving by themselves problems from the textbook or those that the teacher has written on the blackboard or given as a spreadsheet.

When pupils are working with the teacher in charge, the teacher is teaching—for example, asking questions to the whole class—or all pupils seem to be concentrating on the same task. In the case of group work, the pupils are discussing their tasks with their classmates, and the teacher is more a supporter than a foreman. In the case of 'impossible to say', there is no indication of pupils' work. Furthermore, we have also listed whether the pupils are sitting alone beside their tables or in pairs or bigger groups.

Examples

Here we point out Figs. 2–5 (Appendix 1) as examples that will illustrate the coding in the category types of work. Additionally, we use Fig. 1 as a model example. In Appendix 2 are the categories in Types of work. For example, the abbreviation TW12 refers to the second subcategory in the content category TW1 (pupils are working independently).

In Fig. 2 (Appendix 1), there are 18 pupils sitting in groups. The smiling teacher is sitting behind her desk and praises them, saying, '*I am very satisfied with my pupils*'. The tasks on which the pupils are working are from the textbook. Almost all the pupils are working on these tasks at their desks. Thus, we can say that the pupils are working independently (TW12).

In Fig. 3 (Appendix 1), the teacher is standing beside the blackboard and asking questions. The pupils are sitting by themselves and working with the teacher in charge (TW31).

In Fig. 4 (Appendix 1), seven pupils are sitting in pairs and working independently (TW12), but the teacher is maintaining order.

In Fig. 5 (Appendix 1), the teacher stands beside the blackboard and questions the pupils. Therefore, the situation in the drawing is classified as working with the teacher in charge (TW31).

Furthermore, the type of work in Fig. 1 seems to be group work (TW2). The teacher is sitting quietly, and the nine pupils are working in two groups.

Figures 2–5 show that the organisation of pupils' desks does not indicate the type of work.

Results

In Table 4, one can see the distribution of the three different types of work—independent work, group work and work with the teacher in charge—found in the third graders' drawings.

According to the pupils' drawings, the most usual type of work during the mathematics lesson is independent work (57 %) even though the pupils are more frequently sitting in groups (68 %) than in rows (32 %). This coincides with the traditional image according to which pupils are solving the given tasks mostly from the textbook by themselves. Next comes the type of work in which the teacher is standing in front of the classroom and pointing at the task on the blackboard (26 %). In this case, there is no difference in how the pupils are sitting, whether in rows or groups.

Only in four drawings (3 %) could we find the pupils doing group work; that is, the different groups had different tasks, the pupils were discussing together and the teacher was going around giving advice and guiding the work. However, there were altogether 19 drawings (13 %) in which it was impossible to say what the type of work in the classroom was like. For example, in one drawing, there were only three girls talking together.

Discussion and Conclusions

The main aim of this study is to introduce a method to analyse teachers' and pupils' activities from young pupils' drawings. With the help of the list we collected from third graders' drawings, it is possible to identify types of work during a mathematics lesson. We started this research in order to find out which method—teacher or pupil-centred (see, e.g. Thomas, Pedersen, & Finson, 2001)—the teachers use more in their mathematics lessons.

However, we came to the conclusion that it is impossible to decide from the drawings whether they show teacher or pupil-centredness because these drawings are snapshots from a certain event, though perhaps quite usual situations during mathematics lessons. For example, work with the teacher in charge certainly belongs to every teacher's repertoire when she or he is introducing new topics.

A paper that describes the study more in detail has been submitted to an international journal (cf. Ahtee et al., submitted).

Table 4 The types of work during a mathematics lesson as seen by the third graders in their drawings ($N=133$)

TW1	*Independent work*	57 %
TW11	Sitting alone	32 %
TW12	Sitting in a group	68 %
TW2	*Group work*	3 %
TW3	*Work with the teacher in charge*	26 %
TW32	Sitting alone	51 %
TW32	Sitting in a group	49 %
TW4	Impossible to say	13 %

Conclusions

The drawings collected contain rich information from which we have selected only a small part for our purpose. The instruction given to the pupils was quite open, thus there is large variability in the drawings.

Here, we first provide a summary of the results in order to answer the research questions. Secondly, we discuss the reliability of the drawing study.

Pupils' drawings reveal important information on pupils' behaviour that is difficult to obtain from young children using more conventional methods (cf. Pehkonen et al., 2011; Weber & Mitchell, 1996). Especially by connecting words and images, the pupils who did the drawings reflect their feelings and attitudes towards their teacher, other pupils and situations. They also express the group values that are prevalent within their specific environment. Thus, the method developed in this study gives us a tool to find out how young pupils see teachers' activities as well as pupils' activities in mathematics lessons. Therefore, it gives researchers and school authorities the possibility to see what is happening in classrooms. It also gives the opportunity to compare, for example, different grades, different systems and even different countries.

Summary of Results

The first research question was '*How do the teacher and the pupils communicate with each other as seen in third graders' drawings?*' In about a half of the drawings, the pupils convey that a teacher teaches (25 %) and quietly follows the class (23 %). This finding is understandable, since that is the reason for teachers to be with pupils in the classroom. Usually for some part of a lesson, the teacher teaches a new topic or questions the old knowledge. And a part of the lesson is dedicated to the pupils' independent work (practising new tasks), and therefore, the teacher follows the class quietly.

The second research question was '*What kind of emotional atmosphere in a mathematics lesson can be seen in third graders' drawings?*' In these third graders' drawings, the mode value of the emotional atmosphere in the mathematics lesson was positive. This finding matches the result for learning outcomes in mathematics in the beginning of the third grade (cf. Huisman, 2006); namely, that third graders' collective attitude towards studying mathematics is fairly positive. However, it seems possible to obtain more information on this many-sided question with the aid of pupils' drawings (e.g. Kearney & Hyle, 2004).

The third research question was '*How can we identify the type of work used during a mathematics lesson as seen in third graders' drawings?*' According to the drawings by the third graders, pupils are working independently twice as often as working with the teacher in charge. So, it seems fair to conclude that third grade teaching seems to be fairly teacher-centred.

On an Enlargement of the Drawing Study

As we realised the possibilities of drawings to reveal pupils' conceptions on mathematics teaching and learning, we began actively to enlarge our database. Our aim is to develop an international comparison project on pupils' drawings of mathematics lessons. Thus, we will be able to compare mathematics teaching in different countries in order to single out similarities and differences.

Today we have, beside Finnish and Chilean drawings, pupils' drawings from the United States (Georgia), Germany (Sachsen-Anhalt) and England. There is even a journal paper based on the US material; that is, a bilateral comparison between Finland and Georgia, USA (cf. Hart, Pehkonen, & Ahtee, 2014). Furthermore, we are expecting to receive comparative drawing material from Albania and Italy. The size of each sample is about 100–200 drawings, and all are from third graders.

On Reliability

In thinking about the reliability and validity of drawings in mathematics, Stiles, Adkisson, Sebben, and Tamashiro (2008) concluded that drawings enable stronger and more personal expressions than an opinion about statements in a questionnaire, such as 'I like mathematics'. In this way, pupils may draw hearts if they love mathematics or an assault rifle to destroy mathematics to convey not liking it. In addition, Dahlgren and Sumpter (2010) consider pupils' drawings to be a reliable method for assessing pupils' concepts about mathematics teaching.

It is evident that in pupils' drawings, there are many kinds of influences. These drawings were made in the beginning of the third grade (September 2010). When evaluating a teacher's effect in this study, one has to take into account that the third graders made their drawings at the beginning of a new school year when they had gone to school for only 1 month after the summer holiday. On the one hand, the pupils' conceptions of mathematics lessons had been affected mainly by the two previous school years. Thus, they might have been thinking about their teacher in Grades 1 or 2. On the other hand, pupils' affective conditions and properties affect how they interpret different situations during mathematics lessons (Hannula, 2011).

Additionally, many third graders seem to have difficulties in drawing, and therefore, they might concentrate on drawing only situations that are easy to draw for them. To overcome these difficulties, the teacher might ask a pupil to explain his or her drawing. Another solution could be a whole class discussion on drawings. Ruffell et al. (1998) support these solutions, since they emphasise that the child under study should have an opportunity to verbalise his or her own concepts.

Endnotes

As a summary, drawings seem to be a versatile way to collect information about emotional atmosphere in mathematics lessons (see also Harrison et al., 2007). The method offers a single teacher the possibility to obtain and evaluate information about how his or her pupils experience mathematics and mathematics lessons. And the method can also hint to which features the teacher should pay more attention and how the teaching should be developed. Furthermore, it is fairly easy to open such a 'window' on pupils' thinking in a lesson without much additional work by the teacher.

Pupils' drawings reveal important information as to what kind of view the pupils have extracted from their lessons. Especially by connecting words and images, the pupils who did the drawings reflect their feelings and attitudes towards their teacher, other pupils and situations. When all the pupils' drawings in a classroom or a random sample in a country are collected, it is possible to obtain a view that is prevalent in this specific environment. Thus, by analysing the pupils' drawings, we can find out how young pupils see their mathematics lessons.

Appendix 1: Four Examples of Drawings

Fig. 2 A positive emotional atmosphere; the type of work is independent

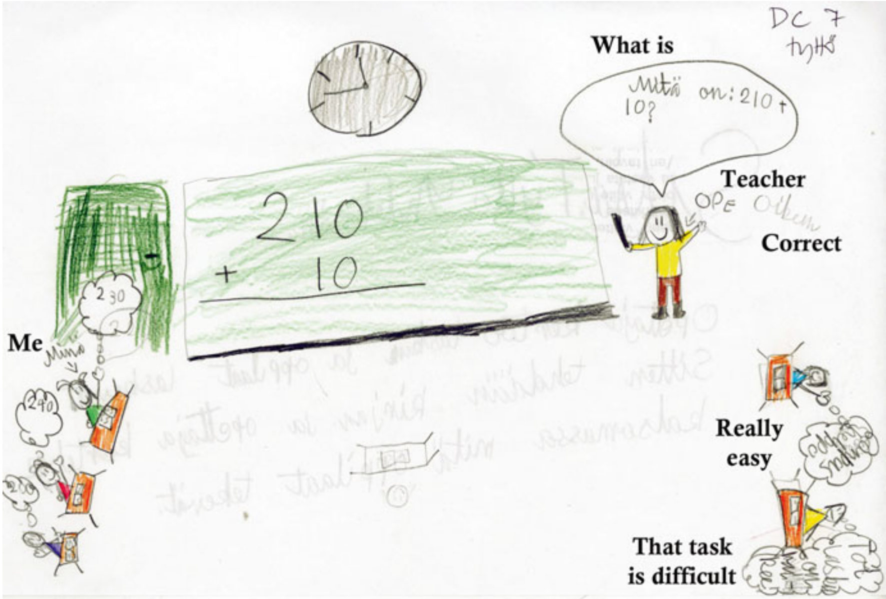

Fig. 3 An example of an ambivalent emotional atmosphere; the type of work is the teacher in charge

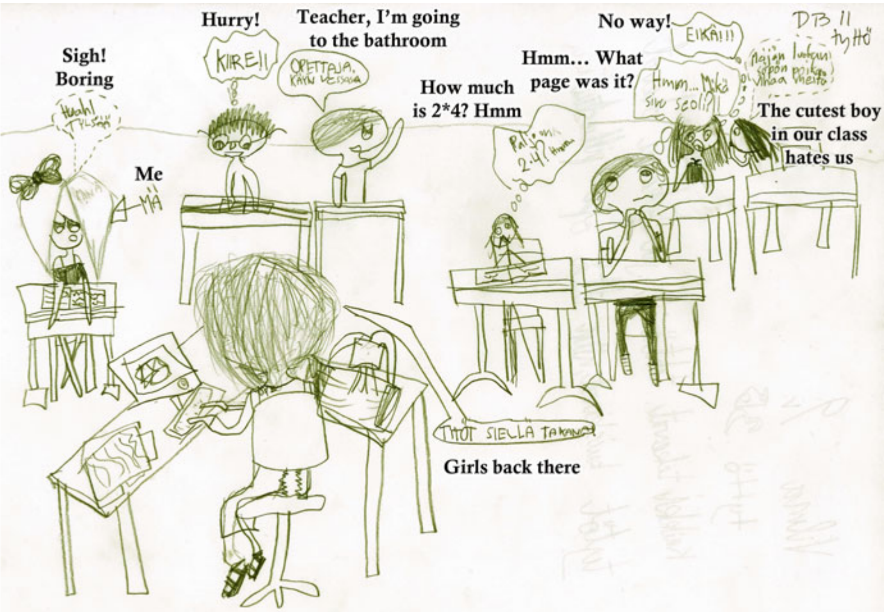

Fig. 4 An example of a negative emotional atmosphere; the type of work is independent

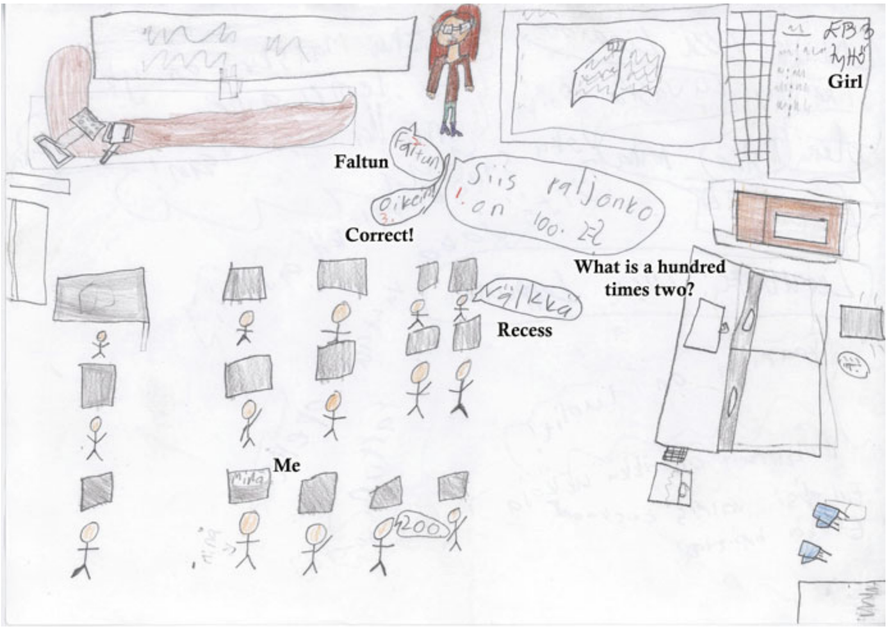

Fig. 5 An example of a neutral emotional atmosphere; the type of work is the teacher in charge

Appendix 2: The Type of Work During a Mathematics Lesson as seen in the Pupils' Drawings

Code	Title	Comment
TW1	*Working independently*	Pupils are solving the same problem or working with different tasks in their own pace. 'May I go and check?'
TW11	Pupils are sitting by themselves	
TW12	Pupils are sitting in pairs or in bigger groups	
TW2	*Working in groups*	Pupils are working in pairs or bigger groups. The groups may have different tasks. The teacher does not have a central role
TW3	*Work with the teacher in charge*	All the pupils are thinking about the same part of the task
TW31	Pupils are sitting by themselves	
TW32	Pupils are sitting in pairs or in bigger groups	
TW4	*Impossible to say*	It is impossible to conclude the type of work
TW41	Pupils are sitting by themselves	
TW42	Pupils are sitting in pairs or in bigger groups	

Extra remark: In some cases, there were clearly two different types of work in the same classroom, and then they were both accepted

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The Use of Digital Technology to Frame and Foster Learners' Problem-Solving Experiences

Manuel Santos-Trigo and Luis Moreno-Armella

Abstract The purpose of this chapter is to analyze and discuss the extent to which the use of digital technology offers learners opportunities to understand and appropriate mathematical knowledge. We focus on discussing several examples in which the use of digital technology provides distinct affordances for learners to represent, explore, and solve mathematical tasks. In this context, looking for multiple ways to solve a task becomes a powerful strategy for learners to think of different concepts in problem-solving approaches. Thus, the use of a dynamic geometry system such as GeoGebra becomes important to represent and analyze tasks from visual, dynamic, and graphic approaches.

Keywords Digital tools • Mathematical problem solving • Tool affordances and appropriation process

Introduction

The developments and availability of digital technologies have been transforming the way people communicate, obtain information, socialize, develop, and comprehend disciplinary knowledge. A digital technology such as a GeoGebra can improve and eventually transform cognitive abilities we already possess and help us develop new ones. People usually develop these cognitive abilities when they represent and explore tasks through these technologies. A cognitive technology makes its mark in our mind through steady work and after a while it becomes part of our cognitive resources. A key historical example is writing. As Donald (2001, p. 302) has explained it, literacy skills transform the functional architecture of the brain and have a profound impact on *how people perform their cognitive work*. The complex neural components of a literate vocabulary, Donald explains, have to be hammered by years of schooling to rewire the functional organization of our thinking. Similarly, the decimal system (Kaput & Schorr, 2008, p. 212) first enlarged access to computation

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and eventually paved the way to Modern Age. Today, an architect begins using specific software to design his buildings. Taking profit from the plasticity of the visual images that the software provides, the architect can imagine a new plan, a new design. Gradually he will begin thinking of his design *with* and *through* the software. He will incorporate the tool affordances as part of his thinking and one day, the software will have *disappeared*. Now it is *coextensive* with his thinking while solving his design tasks. The tool has become an instrument and the design activities are *instrumented* activities. Throughout this chapter, we argue that the systematic use of digital technologies plays an important role in teachers and students' ways to comprehend mathematical ideas and to engage in problem-solving activities.

Conceptual Foundations: Learning from, Through, and with the Others

Action does not belong (exclusively) to the user and neither does it to the environment; both the user and environment are actors and reactors. We understand *dragging* as our hands within the environment, where it is possible to *touch* and transform mathematical entities living in the digital environment. The user and environment are, from the point of view of agency, *coextensive*. Thus, we can speak of *coaction* between the user and the environment, not just between the user and the artifact (Moreno-Armella & Hegedus, 2009). Coaction is the broader process within which an artifact is being internalized as a cognitive instrument. Yet, in the social space of the classroom, there can be a collective actor. One participant can observe how another drives the technology at hands and, the former, incorporates into her strategies what she observed. At the end participants can act and react to the environment in ways that are essentially different from their initial ones. We can learn *from*, *through*, and *with* others. So the traditional triangle user-technology-task has to be enlarged: coaction becomes embedded in a social structure. The ways in which people appropriate technological artifacts cannot be separated from the cultural matrix they live in, and vice versa, technology cannot be separated from culture.

Engaging in practical use of tools begins to build in the user a cognitive resource for thinking about the world as a scene for the potential application of this tool. Artifacts operate in a two-sided manner, both providing resources for acting on the world and for regulating thinking about the world.

Human beings do not interact directly with their environment but through mediating artifacts. This is true in particular when considering cognitive activities as mathematical problem solving. Historically, writing and the decimal system are the most basic mediating cognitive technologies. They were instrumental to pave the Renaissance and Modern Age.

We want to explore this general setting in the important case of contemporary mathematics learning. Consequently, we will be referring to digital artifacts that are transforming the educational landscape and the mathematics curricula.

Béguin (2003) pointed out the design of artifacts does not finish until the tool or object fulfills material and technical requirements; it should include how users transform the artifact into an instrument to solve problems. Moreno-Armella and Santos-Trigo (2016) argue that artifacts are not neutral as they deeply modify our ways of thinking once we have internalized them into our cognitive structures. On their side, Koehler and Mishra (2009) have pointed out that the (cognitive) technologies "have their own propensities, potentials, affordances, and constraints that make them more suitable for certain tasks than others" (p. 61). Some of those affordances and constraints are inherent to the technology design, but also users impose others during their implementation as well as they can eliminate those constraints due to the innovative use of technology. The coordinated use of digital technologies allows for diverse ways to identify, formulate, represent, explore, and solve problems situated in different fields or contexts. Consequently, new routes can emerge for learners to construct and comprehend disciplinary knowledge. *How will learning environments be transformed in order to cope and take advantages of digital developments?*

The discussion of this question becomes important in order to properly explore learning scenarios in which learners rely systematically on the coordinated use of digital technologies to develop new versions of disciplinary knowledge and problem-solving skills. To illustrate what the use of technology could bring to learning environments, we will discuss, in the next sections of this chapter, some mathematical tasks that will enable inductive and deductive reasoning through the digital media. In every one of the activities discussed, a goal is clear: to provide opportunities for learners to engage in mathematical thinking and problem-solving experiences.

Teachers play an important role in providing opportunities for students to use technology in problem solving. As suggested by Mishra and Koehler (2006), teachers need to know ways to use technology in learning environments in addition to deep knowledge about the subject (mathematics) and teaching practices. This is a complex demand for the teachers as it requires "an understanding of the representation of concepts using technologies… and how technology can help redress some of the problems that students face…and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones" (p. 1029).

Tasks are the vehicle for learners to focus on fundamental concepts that are developed through one's own actions and social interactions (Santos-Trigo, 2010). With the use of digital technologies, learners become active participants in the learning process since they offer a rich diversity of opportunities to represent and explore the tasks from distinct perspectives.

Recently, the incorporation of mathematical action technologies (GeoGebra, for instance) has provided solid ground to transform static learning materials into enlivened dynamic media. We select, for the next sections, a set of problems we have amply discussed with teachers in our academic programs of teachers' education at Cinvestav-IPN, Mexico.

The Use of Digital Technology in Looking for Solutions of Mathematical Tasks

A key principle to structure and foster problem-solving activities in learning environments is to help learners pay attention to what is essential in identifying and grasping mathematical concepts and how to use those concepts during the process of solving problems. Searching for alternative ways to represent and solve problems is a powerful strategy for students to identify and contrast the role played by concepts and their representations across the whole problem-solving process. In Chinese classrooms this teaching strategy is called “one problem, multiple solutions” (Cai & Nie, 2007), and it is widely used in mathematical instruction. Students will not only recognize and value multiple paths used to represent and explore problems, they will also have an opportunity to reflect on the extent to which concepts are connected or used to achieve the solution. In addition, Gardner (2006) recognizes that for people to develop problem-solving creativity, they need to pose new questions and to look for novel problem solutions.

How can one distinguish that a solution to the “same” problem is different from others? Leikin (2011) suggests that solutions can be judged as different if they involve (a) the use of different representations of concepts to explore and solve the problem; (b) the use of different theorems, mathematical relations, or auxiliary constructions to support conjectures; and (c) the presentation of different arguments and ways of reasoning about concepts to achieve the problem’s solution. Thinking of different ways to solve problems could also become important to transform routine problems into a set of nonroutine activities (Santos-Trigo & Camacho-Machín, 2009).

How can students use digital technology to look for different ways to solve mathematical tasks? We discuss a simple mathematical task that involves the construction of an equilateral triangle, in which the use of a dynamic geometry system (GeoGebra) becomes important to think of different concepts and ways of reasoning to represent, explore, and solve the task.

The task: Given a vertex of an equilateral triangle and a line to which the other two vertices belong, find the location of the other two vertices. Can you show different ways to construct such a triangle?

We have used this task in our problem-solving seminar with high school teachers. The goal of the seminar is to work on mathematical tasks through the use of different digital technologies and to analyze ways of reasoning that emerge during the solution process. Here, we illustrate some approaches to the task where the use of GeoGebra was essential to construct a dynamic model of the task. Also, we show some examples of solutions where the tool is used as straightedge and compass to solve the problem.

Dynamic Models

Subjects need to comprehend key information and concepts involved in the task statement. What data are provided? What does it mean that one side of the triangle lies on a given line? What do I know about equilateral triangles? Can I construct a family of triangle holding partial conditions (isosceles) by moving a particular vertex on the given line? Is there any way to relate the family of isosceles triangles with the construction of an equilateral one? The discussion of these types of questions becomes important for learners to construct a dynamic representation of the task.

Focusing on the Construction of a Family of Isosceles Triangles

Polya (1945) pointed out that relaxing initial conditions of a problem is an important heuristic to construct/explore the behavior of a partial condition of the problem. Figure 1 shows a movable point P on line *l* and a circle *c* with center at point *C* (the given vertex) and radius *CP*. Triangle *PCQ* is isosceles, since *CP* and *CQ* are radii of the same circle. It is observed that when point P is moved along line *l*, a family of isosceles triangles is generated. At what position of *P* does triangle *PQC* become equilateral? One way to respond to this question is to rely on the property that *in any equilateral triangle, height, perpendicular bisector, angle bisector, and median coincide*. To this end, the perpendicular bisector of segment *PC* is drawn; then, this must pass through point *Q* when triangle PQC becomes equilateral. Points *R* and *S* are the intersections of the perpendicular bisector and circle c. With the use of the tool, it is found that the locus of each point (R and S) when point P is moved along line l is a line (Fig. 1). Then, the intersection of each locus and line *l* determines vertices *P* and *Q*, to form triangle PQC equilateral (Fig. 2).

An Approach That Relies on Symmetry Properties and a Locus of Objects

Dynamic models involve constructing auxiliary objects and analyzing behaviors of some elements when moving particular points. Figure 3a shows a perpendicular line to the given line *l* that passes through point C, a movable point A on line *l* and point B is the symmetric point of A with respect to the perpendicular to *l*. Point D is the intersection of line CB and the perpendicular bisector of segment or side AC. Figure 3a also shows the locus of point D when point A is moved along line l. This locus intersects line *l* at two points. When points A and B coincide with those intersection points, respectively, then triangle ABC is equilateral (Fig. 3b). This is because there, d(A, B = D) = d (B, C) (definition of perpendicular bisector) and also d(A,C) = d(C, B).

Models That Involve Relations and Geometric Properties

The models explore connections between properties and results and the construction of the triangle. The tools' affordances are important to represent and visualize the results. For instance, the accuracy of involved construction allows learners to visualize a *hot point* to pay attention to or possible relationships.

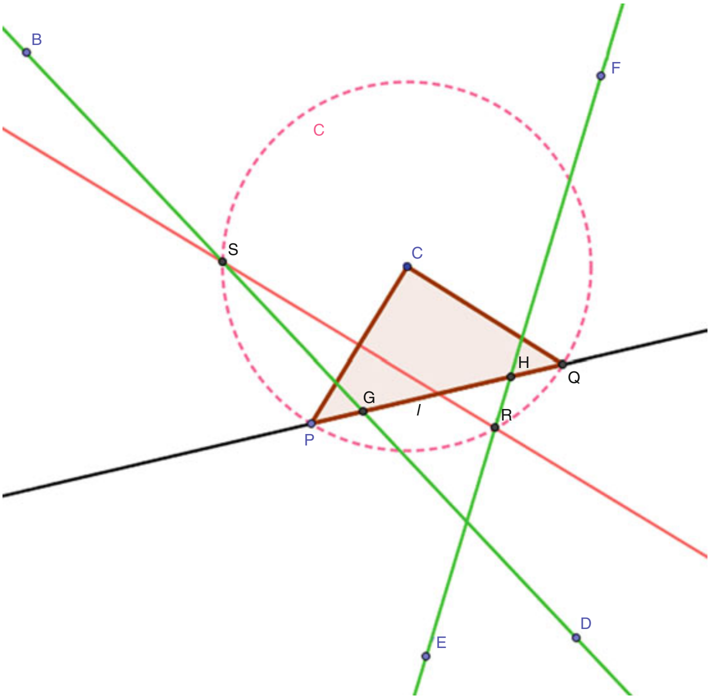

Fig. 1 Finding the loci of points R and S when point P is moved along line *l*

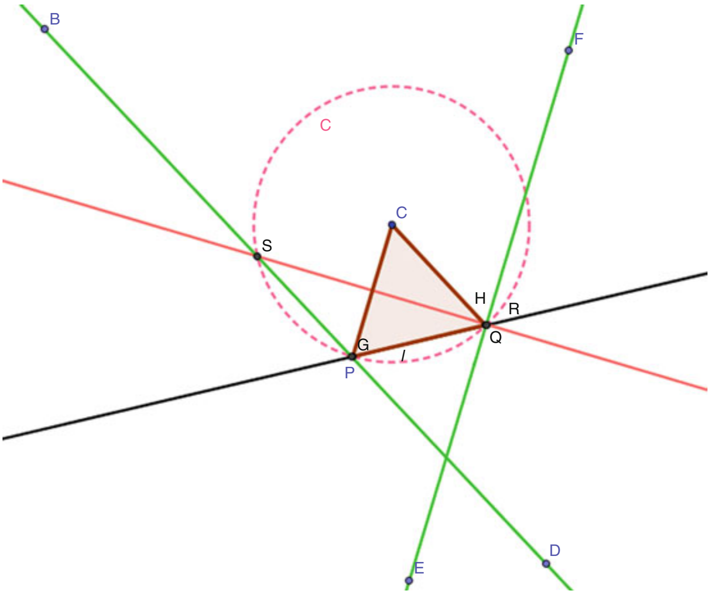

Fig. 2 The construction of an equilateral triangle

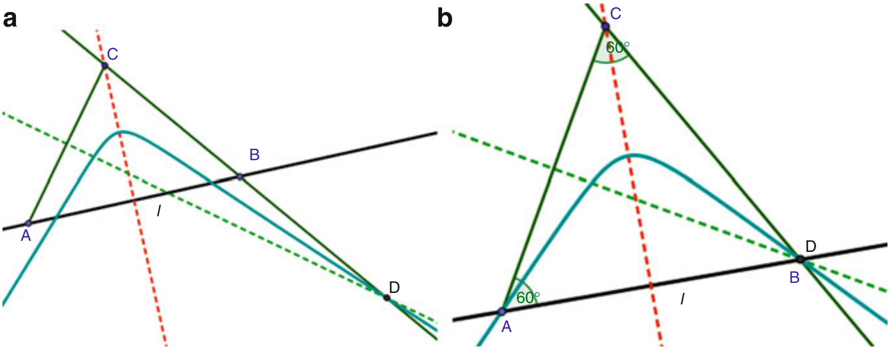

Fig. 3 (**a**) Drawing the locus of point D when point A is moved along line *l*. (**b**) When points A and B coincide with the intersection points of the locus, then triangle ABC is equilateral

A Viviani's Theorem Approach (http://tinyurl.com/VivianiTheorem)

The idea is to use the theorem: *for any interior point P in an equilateral triangle, the sum of the distances from P to the sides of the triangle is the length of the height of the triangle*. A crucial issue here is where to locate the interior point P to connect the theorem and the construction of the triangle. An option can be to locate the interior point on the segment drawn from point C and that is perpendicular to the given line. Thus, segment CM is the height of the required equilateral triangle, point P is any point on segment MC, and point Q is the middle point of segment PC (Fig. 4). Two circles are drawn: circle c with center at point P and radius PQ and circle d with center point Q and radius QP. Points G and H are the intersection points of both circles. Lines CH and CG intersect line *l* at points A and B, respectively. It is observed that triangles PHC and PGC are right triangles because side PC is the diameter of circle d. Thus, triangle ABC is equilateral; it holds that the sum of segments PM + PG + PH corresponds to the height MC (Fig. 4).

An Approach Based on Similarity of Triangles

Figure 5 shows an equilateral triangle PQR whose side PQ is any parallel line to the given line *l*. Then two parallel lines to QR and PR that pass through point C are drawn. These parallel lines intersect line *l* at points A and B, respectively. Triangle ABC is equilateral because corresponding angles of triangles ABC and PQR are congruent (Fig. 5).

Comment: Looking for several ways to represent and solve a task is a key problem-solving strategy where learners are encouraged to think of the task from diverse angles or perspectives. For instance, dynamic models become important to

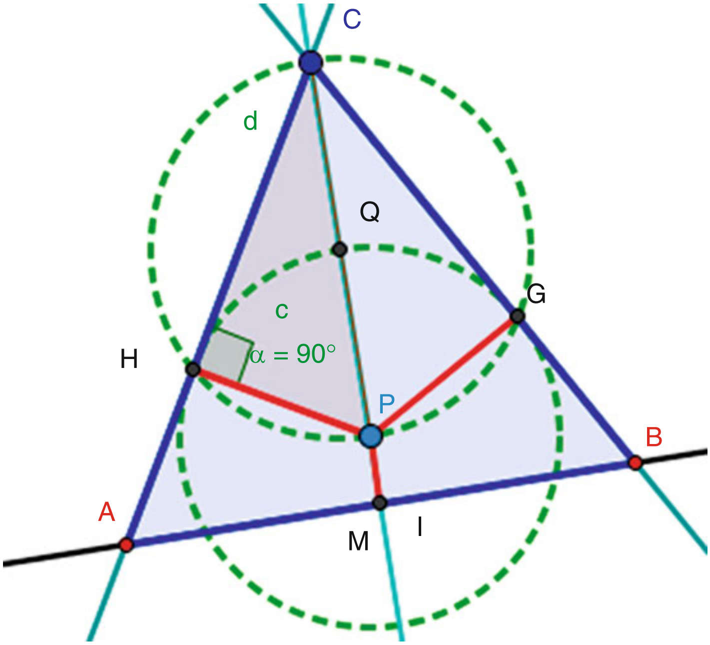

Fig. 4 Using Viviani's theorem to construct the equilateral triangle ABC

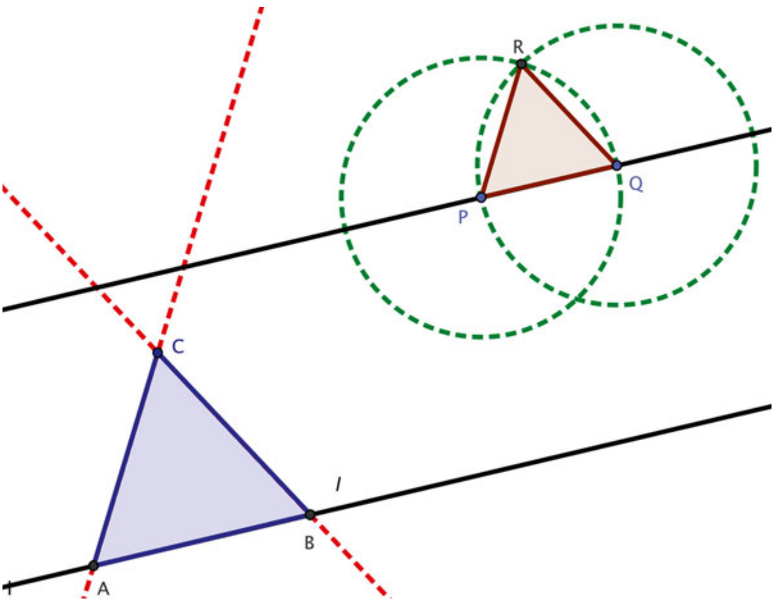

Fig. 5 Drawing equilateral triangle ABC through parallel properties

explore mathematical behaviors of a family of objects through dragging, finding loci, and quantifying attributes or graphic affordances. This object exploration not only provides information regarding invariants or patterns involved but also ways to justify emerging relations or conjectures.

Widening the Scope: How Digital Affordances Offer Opportunities for Learners to Engage in Mathematical Thinking

The use of digital technologies plays a central role in widening students' ways to represent and explore mathematical concepts. In this section, we show examples where the use of GeoGebra not only is central to assemble a dynamic configuration but also becomes important to visualize and support mathematical results.

An Exploration of Basic Geometric Properties

Figure 6a shows that the measure of angle BOC is twice the measure of angle BAC. By focusing on triangle AOB, one observes that the measure of exterior angle BOC is the sum of the measures of angles BAC and ABO; consequently, angle BAC is equal to angle ABO, that is, triangle AOB is isosceles which implies OA = OB. What is the locus of point B when ray AB is moved on the plane? Figure 6b shows that is a circle with center at O and radius OA. The converse is a classical theorem about the angle subtended by an arc in a circle.

Now consider the problem: In a circle centered at O and a chord AB with midpoint C (Figure 7), what is the locus of C as B travels free along the circle?

Chords AE and AD show the position of the midpoint; when the chord is a diameter, the midpoint coincides with the center of the circle (Fig. 7).

Figure 8 makes it visible that the locus is a circle with center O′. In fact, drawing O′C parallel to OB makes the angle CAO′ half the angle CO′O for every position of B. Then the point C describes a circle as shown below.

But problem-solving activities always include extending the results obtained at certain moments: What if C is not the midpoint of AB?

In this case, if we draw the parallel line to OB through C, we obtain Fig. 9:

Again, for each C on AB, angle CAO is half the angle CO′O which means that C will describe a circle with center O′.

It is important to make explicit that students can explore and visualize these results while working in a dynamic environment such as GeoGebra. Then, the problem-solving activities will blend the dynamic exploration with the *geometric reasoning on the figure*.

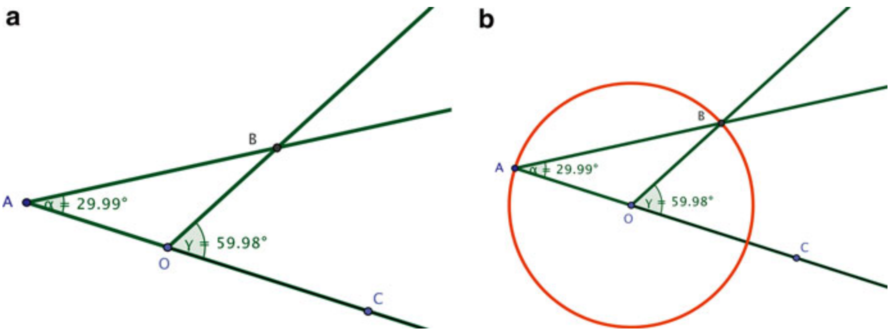

Fig. 6 (**a**) $m\sphericalangle BOC = 2\sphericalangle BAC$. (**b**) What is the locus of point B when ray OB is moved on the plane?

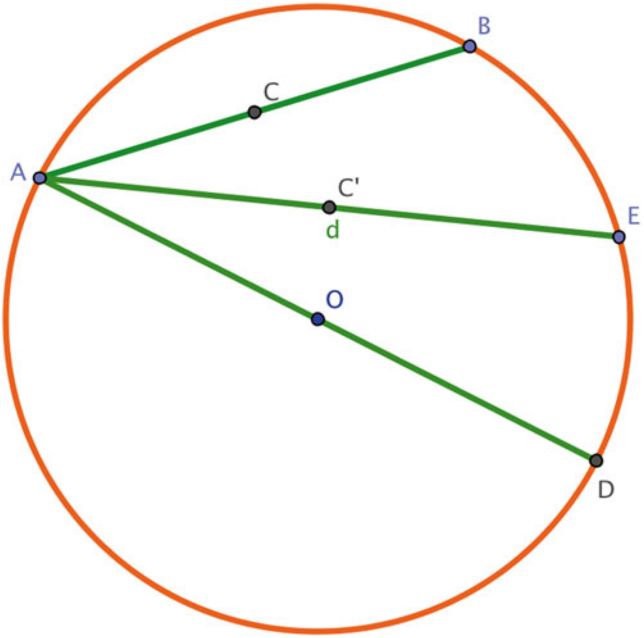

Fig. 7 When the chord is a diameter, its midpoint is the center of the circle

A Triangle and a Variation Task

Sketching a variation phenomenon, without making explicit its algebraic model, is an important problem-solving strategy that learners can apply in a technology environment. We illustrate a dynamic model where the length variation of a triangle side can be explored graphically and through geometric properties.

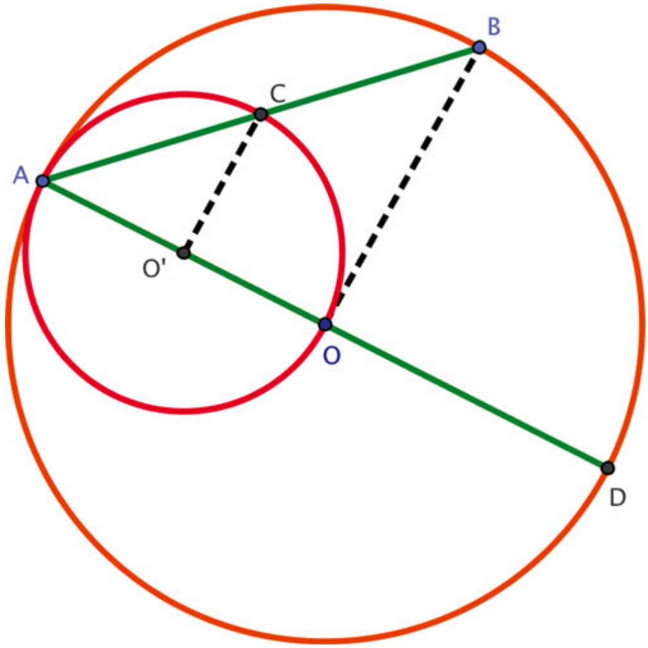

Fig. 8 Locus of point C when point B moves along the circle is a circle centered at O′

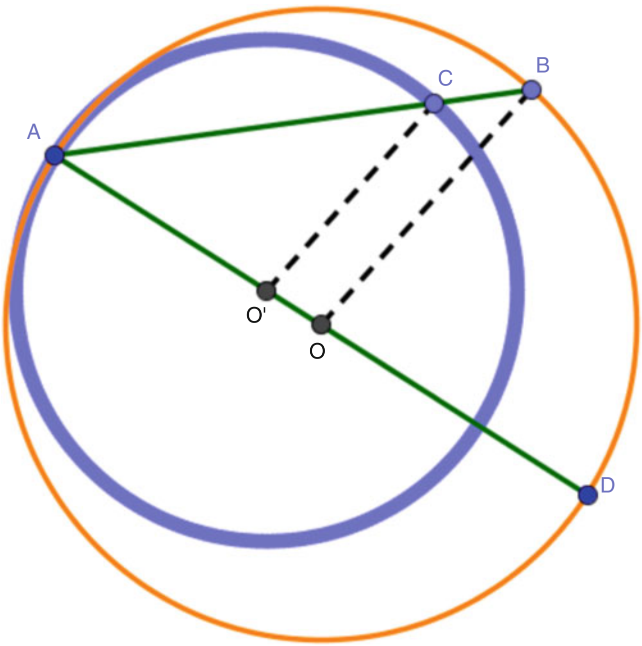

Fig. 9 What is the locus of point C when B is moved along the circle?

Given a triangle ABC and a point P on side BC. From P we draw perpendicular lines to AB and AC and points E, F are the intersection points of those perpendiculars and sides AB and AC.

Find the position of P such that the segment EF is the shortest (Fig. 10).

In a dynamic system such as GeoGebra, we can proceed thus: Draw a perpendicular line from P on side BC as shown in Fig. 11:

The segment PQ has the same length as EF. Then the locus of Q as we move P on BC renders a visual evidence of how the length of EF varies—as P travels on BC. We could name this activity as the heuristic phase that allows the student to become acquainted with the problem. The visual information suggests that the position that renders the minimum length for EF occurs when PQ coincides with the height drawn from A. That is, when segment PQ is a height of the triangle.

Problem solving is surely the kernel of mathematical thinking at school. In our work we encourage our students to develop a drive for blending different approaches to a problem. In the present case, if P is the foot of the height from A, we discover (the students discover after a long discussion in the classroom) that the quadrilateral AEPF is cyclic. Previously, we discussed the conditions under which this fact is realized: the sum of the opposite angles in the quadrilateral equals 180° (Fig. 12).

The quadrilateral is a dynamic object, that is, it changes (as well as the corresponding circle) as P moves on BC. Playing with the circle, one discovers that when AP is the height, then the circle is tangent at P as the following figure illustrates (Fig. 13).

This one is the smallest circle as P moves on BC; consequently, the chord EF has the minimum length. Loci of points I and D represent the area variation of the family of triangles (PEF) and quadrilaterals (PFAE) when point P is moved along BC, and students could explore at what position of P the areas of those triangles and quadrilaterals reach a maximum value.

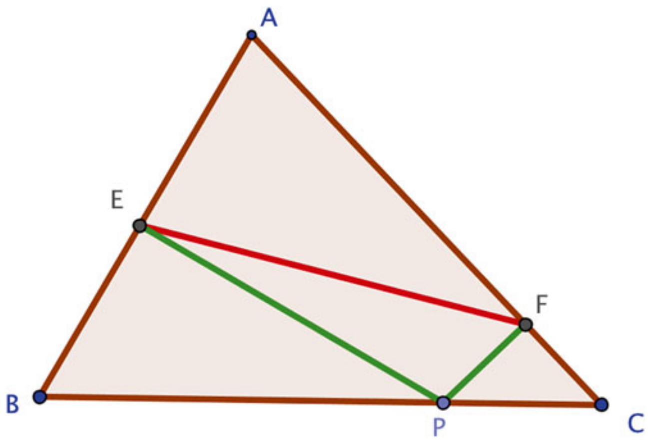

Fig. 10 At what position of P (P moves along segment BC) does segment EF reach the shortest length?

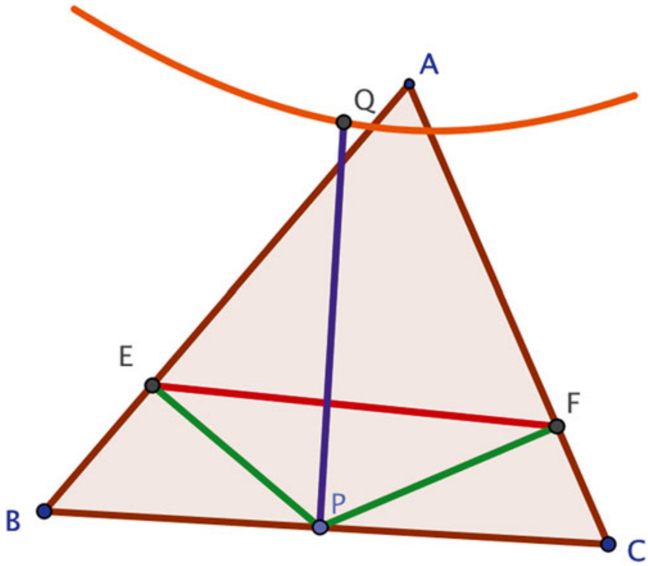

Fig. 11 Graphic representation of length EF

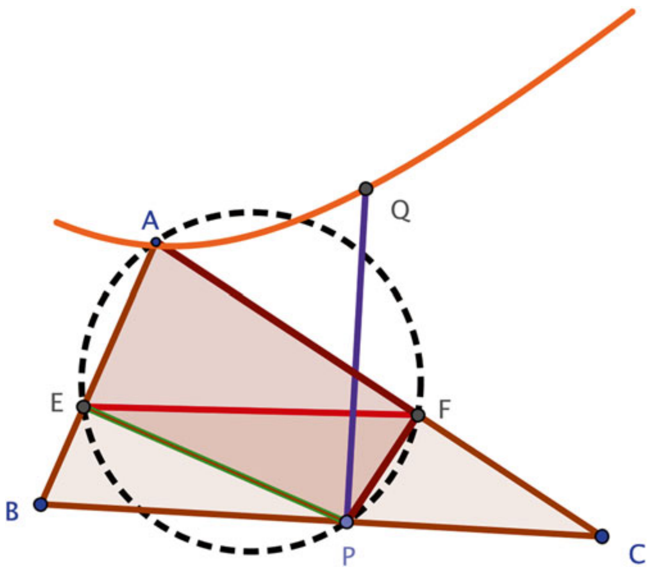

Fig. 12 Multiple representation of length EF

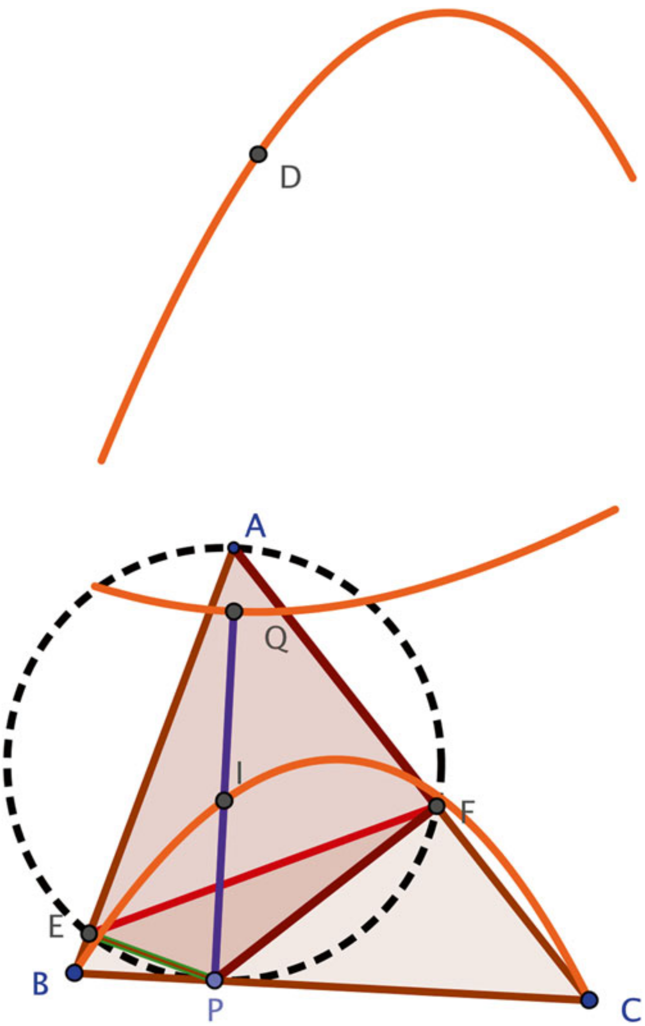

Fig. 13 Finding the solution

The heuristic and deductive phases are complete. It is important to emphasize that the role of the dynamic medium is not ancillary. In fact, the heuristic phase guided by the digital affordances gradually contributes to develop new strategies for problem solving. The blending of paper and pixel is crucial in today's classroom. There is a stable tradition at school, namely, paper and pencil, that will be gradually transformed by the presence of the digital armamentarium.

As we use a new artifact, we feel the *resistance* that it displays. Someone who intends to learn how to use a word processor knows this. Gradually, one overcomes the basic difficulties and begins to *internalize* the artifact—in the present example,

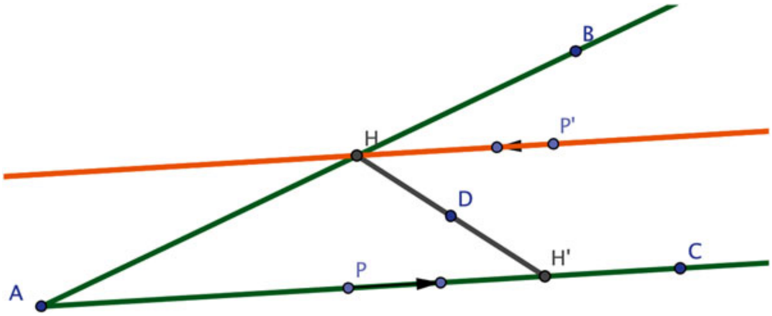

Fig. 14 H solves the problem

the word processor. But the artifact is not passive. With time, its presence will impact the strategies we use to solve problems without that artifact, and the end result is our own transformation as problem solvers. Using an artifact begins to build in the user a cognitive resource for thinking about the problems she/he intends to solve. Artifacts provide resources for acting on problems and, simultaneously, for regulating our thinking as problem solvers.

Locus and the Concept of Central Symmetry

Figure 14 shows another example that illustrates how the availability of digital affordances can redirect or open a new path for exploration. Point D is inside of angle BAC. How to draw a segment from side AC to side AB such that the chosen point D is the midpoint of the segment?

We choose any point P on side AC and reflect it with respect to D to obtain P′. The locus of P′ when point P moves along AC is a parallel line to AC. H is the intersection point of the locus and side AB and H′ is the reflected point of H with respect to D. Segment HH′ is the solution. Let us insist that the availability of a flexible transformation as the central symmetry molds the solution we find for this problem. Our way of thinking is transformed by the presence of the mediating artifact.

The Best View Task: Combining Graphic and Geometric Representation

We will close this section with a problem we first learned from Polya's classic *Mathematics and Plausible Reasoning* (vol. 1, pp. 122–123, 1954). This task was also analyzed in Santos Trigo and Reyes-Rodriguez (2011).

Let us suppose we are walking along a line and we want to determine the position on this line from which we have the best view of a segment AB as in Fig. 15:

The best view of the segment is obtained when the angle at P is largest. If we are to the far right, the angle will be very small, and as we approach walking to the left (as suggested in the figure), the angle will increase. The digital medium allows a first exploration that consists in dragging the point P (walker's position) and see how the angle varies. This experience will suggest the explorer that there is a position where the angle is largest. However, this still does not allow to clearly identify that special position. We can go a step further by representing the measure of the angle by a perpendicular segment to the walking line. Then, we get Fig. 16:

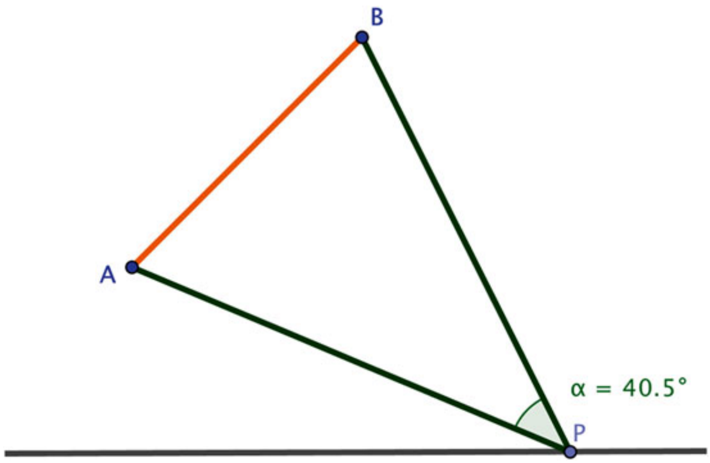

Fig. 15 Explaining the problem

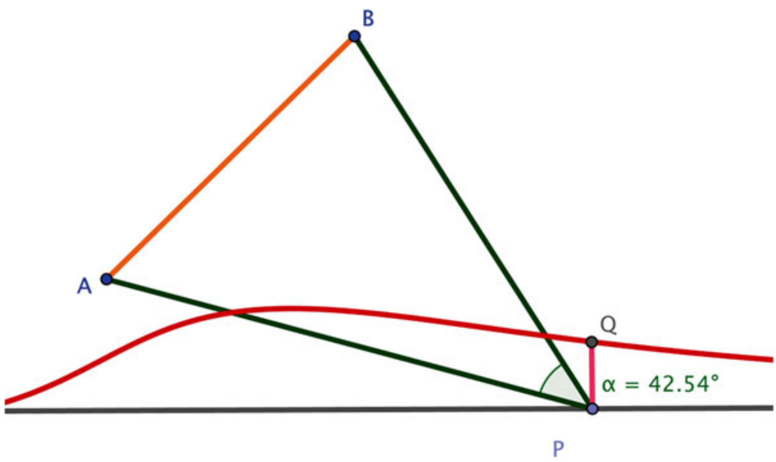

Fig. 16 Cartesian representation of the problem

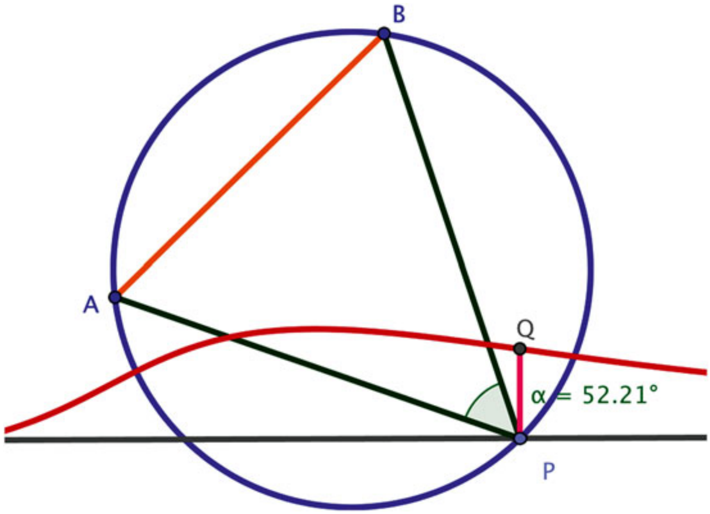

Fig. 17 Euclidian dynamic representation of the problem

Walking to the left eventually one will arrive at a position from where one only sees the point A, as suggested by the graph representing the set of angle measurements. The digital affordances prove their cognitive power: the student is thinking *through* the artifact and *with* the artifact. The optimal position clearly exists. How could we provide a geometric characterization of this position? As we are trying to optimize an angle, previous experience suggests interpreting these angles as subtended by the segment AB as in Fig. 17.

The angle at P is interpreted as an angle subtended by the segment AB. The circle will vary as we move point P on the line. The segment PQ and the circle are two different ways of representing how the angle varies. The segment PQ suggests where the optimal position occurs, and this position coincides with the circle being tangent to the walking line:

This circle has the smallest possible radius and as it contains segment AB as a chord, the corresponding angle at P will be largest. The task exploration, once again, exhibits the virtues of blending *paper and pixel* as a starting point to enrich students' ways of developing their mathematical thinking (Fig. 18).

Concluding Remarks

Throughout this chapter, it is argued that the learners' appropriation process of digital tools could offer them diverse opportunities to develop mathematical thinking and problem-solving competencies. In particular, the tasks we presented illustrate that the use of several problem-solving strategies such as "relaxing task conditions,"

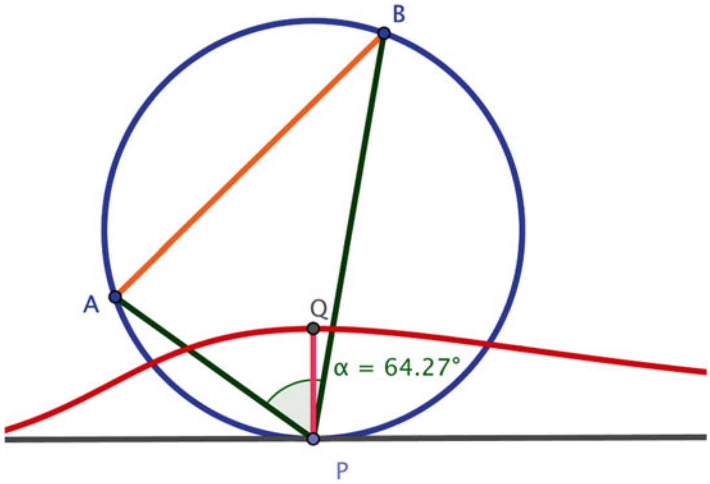

Fig. 18 The solution

"looking for special cases," "assuming the problem is solved," etc., can be extended, compared with the paper and pencil scope, when a dynamic geometric system is used (GeoGebra). To this end, we showed that the construction of dynamic models exhibits novel forms to explore object behaviors that involve "dragging objects," "finding loci," "quantifying objects attributes," "graphic representations," and "visualizing patterns behaviors." Likewise, it is argued that during the process of representing and exploring the tasks, students can use digital technologies to examine embedded concepts and to apply them in problem-solving activities. Indeed, digital technologies open the ways to fertile reinterpretations of existing concepts, and these new forms of the concepts lead to transformations of the meanings of those concepts. The objects we traditionally have drawn on the paper are now for *filming* due to the executable nature of the digital representations. Consequently, digital technologies are leading to new epistemologies, not only affecting students' approaches to problem solving but reshaping the cultural nature of mathematics.

The coordinated use of digital technologies offers diverse opportunities for learners not only to communicate and discuss mathematical tasks and ways to formulate problems but also to represent and explore the tasks from diverse angles and perspectives. Although these digital technologies are not yet fully incorporated in the school culture, their presence is eroding the traditional paper and pencil ways of thinking while confronting a mathematical problem. Nevertheless, this is a rather slow process due to the force of tradition that makes practices resistant to change. This reflects the welcome stability of "good old practices."

The design of digital technologies involves the collaboration of experts' communities working on different fields, and an important element in the design is the users' appropriation process of the tool. Thus, designers should include or rely on information about how users internalize an artifact into their cognitive structures to solve problems and incorporate it into their practices.

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Proportional Word Problem Solving Through a Modeling Lens: A Half-Empty or Half-Full Glass?

Tine Degrande, Lieven Verschaffel, and Wim Van Dooren

Abstract We discuss two studies related to upper elementary school pupils' use of additive and proportional strategies to solve word problems, in order to shed a light on pupils' modelling disposition (i.e. both their abilities and their inclination) in the context of proportional reasoning. In Study 1, we used word problems that were clearly additive or proportional, while in Study 2 the problems were formulated with Greek symbols so that pupils had no access to the actual contents of the problems. Both studies yielded very similar results. 3rd graders initially are strongly inclined to reason additively to missing-value word problems (whether they are additive, proportional, or incomprehensible) and 6th graders are strongly inclined to reason proportionally. In the intermediate stage pupils heavily rely on the numbers appearing in the word problems in order to decide to apply a proportional or additive method. Even though the results were very similar, different nature of the tasks in both studies reveals a different aspect of pupils' modelling disposition. The first study showed how pupils largely neglect the actual model underlying a word problem, and consistently apply the same model across situations. The second study indicates that already at a young age, a substantial number of leaners is inclined to give answers based on quantitative analogical relations.

Introduction

Contemporary math education curricula consider it as an important goal that pupils can use mathematics to solve real-world problem situations. This process of applying the appropriate mathematical structures and operations, in order to make sense of realistic real-life problems and to solve them, is otherwise termed mathematical modeling (Van Dooren, Verschaffel, Greer, & De Bock, 2006). Traditionally, mathematical modeling and applied problem solving are taught through word problems (Verschaffel, Greer, & De Corte, 2000). Word problems can be described as "verbal

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descriptions of problem situations (…) wherein a question is raised, the answer to which can be found by performing mathematical operation(s) with the numbers in the problem" (Verschaffel et al., 2000, p. ix). Importantly, the concept of "word problem" does not necessarily imply that every word problem represents a true problem for a pupil, in the sense that there is no routine solution method available and that the activation of (meta)cognitive strategies is therefore required (Verschaffel, Depaepe, & Van Dooren, 2014). Neither does it necessarily imply that the pupil considers the question being posed as a (personally) interesting or relevant one and has a desire to find the solution. Whether or not a word problem represents a true and/or attractive problem for a pupil depends on several factors, such as the familiarity of a pupil with the problem, his prior knowledge and skills, etc.

However, during the last two decades, it has been shown that pupils perceive word problem solving as a puzzle-like activity with little grounding in the real world. One of the problems is that in classroom practice, pupils can often successfully decide which operation to perform to solve a word problem in a textbook or a test without first having to gain a deep understanding of the problem situation: superficial characteristics of the word problems already lead them to the correct answer (Van Dooren et al., 2006; Verschaffel & De Corte, 1997). Arguably, word problems in textbooks and tests may thus largely overestimate the actual modeling abilities of pupils. Good performance on these word problems does not necessarily indicate that pupils acquired a good modeling disposition, but may merely reflect a tendency in pupils to cope with these problems in a stereotyped and superficial way.

In this chapter, we will focus on pupils' modeling disposition in the context of proportional and nonproportional word problems. We will use the notion of disposition, because we want to refer not only to pupils' abilities but also to their inclination for a specific way of reasoning (De Corte, Greer, & Verschaffel, 1996). We will answer the following questions: Are pupils indeed often undeservedly successful in answering such word problems, while they do not show a true disposition toward modeling? Or are pupils who correctly solve these word problems despite their aforementioned superficial coping strategy (and even pupils who incorrectly solve those word problems due to superficial modeling behavior) still showing some initial but important modeling dispositions? To put it in other words, when looking at proportional word problem solving through a modeling lens, we ask ourselves the typical question whether the glass is half full or half empty. Before we answer these questions by means of results of two empirical studies, we will give an overview of previous studies and theorizing about proportional reasoning in proportional and nonproportional word problems.

Theoretical and Empirical Background

Because of its wide applicability in mathematics and science, proportional reasoning is a major topic in primary and secondary math education. Typically, from 3rd or 4th grade on, pupils are increasingly confronted with missing-value proportionality problems, in which three numbers are given and a fourth has to be determined (Kaput & West, 1994).

Studies indicate that pupils associate such missing-value word problems with the proportionality scheme, even when it does not appropriately model the problem situation in the nonproportional word problem (De Bock, Verschaffel, & Janssens, 2002). One of the most extensively documented cases relates to lower secondary pupils' tendency to (improperly) give proportional answers to geometry problems like "Farmer Gus needs 8 h to fertilise a square pasture with sides of 200 m. How much time will he approximately need to fertilise a square pasture with sides of 600 m?" (answering "24 h" in this case) (De Bock et al., 2002; De Bock, Van Dooren, Janssens, & Verschaffel, 2007; Modestou, Gagatsis, & Pitta-Pantazi, 2004). But also upper secondary and even university students overuse proportionality in various other domains like probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). Consider, for example, the coin problem that Fischbein (1999) gave to 5th to 11th graders: "The likelihood of getting heads at least twice when tossing three coins is smaller than/equal to/greater than the likelihood of getting heads at least 200 times out of 300 times" (p. 45). Fischbein found that the number of erroneous proportional answers of the type "equal to" increased with age: 30 % in grade 5, 45 % in grade 7, 60 % in grade 9, and 80 % in grade 11. Other cases of improper proportional reasoning were found in calculus (Esteley, Villarreal, & Alagia, 2004), physics (De Bock, Van Dooren, & Verschaffel, 2011), and economics (De Bock, Van Reeth, Minne, & Van Dooren, 2014).

In all the above cases, knowledge about the specific—and sometimes rather advanced—concepts and principles in mathematics and science (e.g., concept of area, concept of chance, etc.) is required in order to unmask the inappropriateness of the proportional answer. Still, several studies point out that even in solving arithmetic word problems where such advanced mathematical and scientific knowledge is not required, pupils give proportional answers (e.g., Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Vleugels, & Verschaffel;, 2010). Moreover, the tendency to improperly use proportional methods increases with age (at least until fifth grade), as shown by Van Dooren et al. (2005). In the latter study, a large group of pupils from grades 2–8 were offered proportional problems (e.g., "In the shop, four packs of pencils cost 8 euro. The teacher wants to buy a pack of pencils for every pupil. He needs 24 packs. How much must he pay?") and three kinds of nonproportional word problems:

- Additive problems, e.g., "Ellen and Kim are running around a track. They run equally fast, but Ellen started later. When Ellen has run 5 laps, Kim has run 15 laps. When Ellen has run 30 laps, how many has Kim run?" (proportional answer: $30 \times 3 = 90$ laps, correct answer: $30 + 10 = 40$ laps) (this problem is basically the same as that used by Cramer, Post, and Currier, 1993).
- Constant problems, e.g., "A group of five musicians plays a piece of music in 10 min. Another group of 35 musicians will play the same piece of music. How long will it take this group to play it?" (proportional answer: $10 \times 7 = 70$ min, correct answer: 10 min).

- Affine problems, e.g. “The locomotive of a train is 12 m long. If there are four carriages connected to the locomotive, the train is 52 m long. If there were eight carriages connected to the locomotive, how long would the train be?” (proportional answer: 2×52 m = 104 m, correct answer: 12 m + (8×10 m) = 92 m).

The study showed that the tendency to improperly give proportional answers to the nonproportional word problems was already present before formal instruction in proportional reasoning. Already in second grade, 26 % of all nonproportional problems were answered proportionally. Moreover, the tendency to give incorrect proportional answers to nonproportional word problems increased considerably to 51 % in 5th grade, with a decrease thereafter, but still 22 % in 8th grade. Remarkably, for the additive problem, there was even a *decrease* in the number of correct answers throughout elementary school. In other words, the overuse of proportional methods is already present in lower primary school pupils, becomes more prominent when missing-value proportionality problems are central in pupils’ classroom practice, and decreases again around the end of primary school however without disappearing completely.

Moreover, not only the tendency to improperly give proportional answers was already present at a young age, also the mathematical knowledge and skills to correctly answer the proportional problems were already clearly present in the first years of primary school. Many 3rd graders (53 %) gave correct answers to the proportional problems, and even 2nd graders (who only got variants involving small numbers and allowing simple scalar solutions) mostly gave correct answers. Performance on proportional word problems considerably improved until 8th grade (with 93 % correct answers), with most learning gains being made between 3rd and 5th grade, the period wherein proportionality is being systematically taught and practiced.

Rationale

In the remainder of this chapter, we will present two empirical studies that followed up on the results of Van Dooren et al. (2005). The rationale for selecting these two studies is their contrasting (but at the same time complementary) perspective. We are especially interested in what these studies tell us about pupils’ modeling disposition in the context of proportional reasoning.

The first study (see Van Dooren, De Bock, Evers, & Verschaffel, 2009) built on previous studies in the domain of proportional reasoning, as it especially focused on task characteristics that are associated with pupils’ choice for a proportional or another type of reasoning. This way, it illustrated pupils’ superficial approach to mathematical modeling, in the domain of proportional reasoning. Specifically, we showed how pupils often rely on a superficial, irrelevant problem characteristic in order to decide which model needs to be applied to a specific word problem, namely, the numbers given in a problem, while they ignore the mathematical model that is actually underlying it. The findings of this study provided an indication that we often overestimate pupils’ true mathematical modeling capacities—as pupils can be undeservedly successful in a variety of word problems—and that proportional reasoning capacities cannot be measured merely by means of a set of missing-value word problems.

In the second study (see Degrande, Verschaffel, & Van Dooren, 2014), we took a different perspective that sheds another light on pupils' modeling disposition. In that study, we administered a specific kind of word problem that we consider "mathematically neutral," so neutral with respect to the underlying mathematical model. More specifically, in the tasks used in the second study, the problem context was unreadable and the mathematical model underlying the word problem was made inaccessible to pupils. Therefore, solutions to these problems did not so much give us insight into pupils' proportional reasoning abilities, but rather allowed us to get a view on pupils' spontaneous inclination toward what we will call "quantitative analogical reasoning," i.e., looking for a mathematical relation between two magnitudes that are given in a word problem and applying this relation to a third given magnitude. This mathematical relation could be a proportional relation or any other relation (e.g., exponential, quadratic, additive, etc.). In the discussed paper, additive relations were of specific interest, next to proportional relations. By focusing on the common denominator "quantitative analogical reasoning," we explicitly acknowledged that additive reasoning and proportional reasoning have several conceptual similarities and that additive reasoning—even improper additive reasoning—that may occur in younger pupils may be a valuable precursor for the development of true proportional reasoning skills, while it is not always considered as such.

Study 1: How Numbers Affect Pupils' Solutions to Proportional and Nonproportional Word Problems

Introduction

This first study focuses on one issue that has been largely overlooked in research on the overuse of proportionality for a long time: the *nature of the numbers* in the proportional and nonproportional problems. The nature of the numbers in the word problem may have an impact on pupils' tendency to use proportional or other methods. The finding that pupils' choice for a solution method is based on this irrelevant problem characteristic (i.e., nature of the number ratios), instead of on the underlying mathematical model of the word problem, may thus be an indication for pupils' superficial approach to mathematical modeling in this domain.

The importance of this issue can be clarified by considering the literature on proportional reasoning. A frequently reported error on missing-value proportionality tasks (e.g., Hart, 1984; Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b) is the so-called "constant difference" or "additive" strategy. In this strategy, the relationship within the ratios is computed by subtracting one term from a second, and then the difference is applied to the other ratio (instead of considering the multiplicative relationship). For example, "One mixture has 2 oranges to 7 parts of water. Another mixture tastes the same and has 5 oranges. How many parts of water does it have?" The most prominent explanation for this error is that it is a kind of "fall-back" strategy (especially for younger and less skilled proportional reasoners) to deal with proportionality problems with non-integer ratios (see, e.g., Karplus et al., 1983, who call this the "fraction avoidance syndrome").

In sum, correct reasoning on proportional (missing-value) tasks sometimes is affected by the nature of the numbers. Particularly younger, less skilled proportional reasoners perform worse if ratios in proportional problems are non-integer. The claim underlying the present study is that this finding also applies to the use of proportional methods to solve *non*proportional problems. The nonproportional problems in many of the abovementioned studies (e.g., De Bock et al., 2002; Van Dooren et al., 2003, 2005; Verschaffel et al., 2000) contained "easy" numbers: both the ratio of quantities of the same nature (i.e., the internal ratio a/c) and the ratio of quantities of different nature (i.e., the external ratio a/b) were integer. Although the problems had no proportional structure, the given numbers thus somehow invited pupils to conduct proportional calculations. Linchevski, Olivier, Sasman, and Liebenberg (1998) found some indications that such integer ratios could "trigger" unwarranted proportional reasoning (an error they call the "proportional multiplication error"), but they did not systematically test this hypothesis. The goal of the present study is to test this hypothesis and, this way, to gain further insight in the determinants of pupils' tendency to overuse proportional methods.

Method

508 4th, 5th, and 6th graders from five randomly chosen Flemish primary schools participated in this study. They received a test containing eight missing-value word problems presented in random order. The problems were identical to those used by Van Dooren et al. (2005). The design of the test is shown in Table 1 and examples of word problems are given in the left column of Table 2. The test contained one type of *proportional* problems (for which proportional strategies provide the correct answer) and three types of *nonproportional* problems (for which another strategy must be applied to find the correct answer). The three types of nonproportional problems had different mathematical models underlying them: additive, constant, and affine (i.e., a model of the form $f(x)=ax+b$). For each category, two items were included.

Central to this study was that the numbers in (all four types of) word problems were experimentally manipulated, as clarified in Table 2. The manipulation was in such a way that the internal and external ratios between the numbers were either integer (I) or non-integer (N).

Table 1 Design of the eight test items

	Item I	Item II
Proportional (PR)	1	2
Nonproportional		
Additive (AD)	3	4
Constant (CO)	5	6
Affine (AF)	7	8

Table 2 Examples of word problems and manipulation of numbers in the II-, NI-, IN-, and NN-versions

Example of word problem		Numbers and solutions for each version[a]			
		II	NI	IN	NN
PR	In the shop, *a* packs of pencils cost *b* euro	9 27	9 24	9 27	9 24
	The teacher wants to buy *c* packs. *How much* does she have to pay?	18 $C:54$	18 $C:48$	12 $C:36$	12 $C:32$
AD	Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run *a* rounds, Kim has run *b* rounds	16 32 48 $C:64$	16 24 48 $C:56$	16 32 36 $C:52$	16 24 36 $C:44$
	When Ellen has run *c* rounds, *how many* has Kim run?	$P:$ 96	$P:$ 72	$P:$ 72	$P:$ 54
CO	A group of *a* musicians plays a piece of music in *b* minutes	25 75	25 40	25 75	25 40
	Another group of *c* musicians will play the same piece of music. *How long* will it take this group to play it?	50 $C:75$ $P:$ 150	50 $C:40$ $P:$ 80	35 $C:75$ $P:$ 105	35 $C:40$ $P:$ 56
AF	The locomotive of a train is 12[b] m long	4 44	4 42	4 44	4 42
	If there are *a* carriages connected to the locomotive, the train is *b* m long in total	8 $C:76$	8 $C:74$	10 $C:92$	10 $C:90$
	If there would be *c* carriages connected to the locomotive, *how long* would the train be?	$P:$ 88	$P:$ 84	$P:$ 110	$P:$ 105

[a]Numbers are schematically represented as $\begin{matrix} a & b \\ c & x \end{matrix}$ (*C*: correct solution, *P*: proportional solution)

[b]For the NI and NN-version, this value was 10

This manipulation led to four different versions of each item:

- II-version: external ratio (*a*/*b*) integer and internal ratio (*a/c*) integer
- NI-version: external ratio (*a*/*b*) non-integer but internal ratio (*a/c*) integer
- IN-version: external ratio (*a*/*b*) integer but internal ratio (*a/c*) non-integer
- NN-version: external ratio (*a*/*b*) non-integer and internal ratio (*a/c*) non-integer

For example, the II-version of the additive (AD) word problem in Table 2 was "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 16 rounds, Kim has run 32 rounds. When Ellen has run 48 rounds, how many rounds has Kim run?" A correct reasoning for this II-version is to focus on the (constant) difference between the numbers: *Kim is initially running 16 rounds ahead of Ellen. This remains the same, so when Ellen has 48 rounds, Kim has* $48 + 16 = 64$ *rounds*. When reasoning proportionally here (which is improper, of course), one needs to focus on the ratios between the numbers: either on the external ratio *a/b* (*initially, Kim has twice as many rounds as Ellen (32/16), so when Ellen has 48 rounds, Kim has* $48 \times 2 = 96$ *rounds*) or on the internal ratio *a/c* (*at the end, Ellen has three times as many rounds as initially (48/16), so by that time, Kim has 32 × 3 = 96 rounds*).

When comparing this II-version to the NN-version of the same word problem (see Table 2), the correct approach is comparably easy (only the exact constant difference to work with differs). Proportional reasoning, however, is considerably more complex here, because both the internal and external ratio are non-integer. The multiplicative "jump" from 16 to 24 is far less evident than that from 16 to 32, but for a skilled proportional reasoner, it is still feasible. Reasoning proportionally for the NN-version might be, for example, *initially, Kim has 3/2 times as many rounds as Ellen, so when Ellen has run 36 rounds, Kim has run 36 × 3/2 = 54 rounds*.

The tests were manipulated so that—on a random basis—two of the eight word problems (in Table 1) were in the II-version, two in the NI-version, two in the IN-version, and two in the NN-version. In other words, a test was created in which 8 out of the 16 types of problems (see Table 2) were offered to pupils. Pupils' answers to the problems were classified as either *correct* (C, correct answer was given), *proportional error* (P, proportional strategy applied to a nonproportional item), or *other error* (O, another solution procedure was followed).

Hypotheses

Due to space restrictions, we limit ourselves here to comparing the "extreme" versions of the proportional and nonproportional items, i.e., the II- and NN-versions with, respectively, *both* (internal and external) ratios integer and *no* ratios integer (for the full paper including the IN- and NI-versions, see Van Dooren et al., 2009).

A first set of hypotheses relates to pupils' performances on the *proportional problems*. Based on the literature on proportional reasoning mentioned above, we expect that proportional problems with non-integer ratios (NN-version) will cause more errors (i.e., less correct (C) answers) than proportional problems with integer ratios (II-version) (HYP 1A). Additionally, we anticipate that this effect will be stronger in younger, less experienced proportional reasoners, so we predict that the different performance on the II- and NN-versions will be most pronounced in 4th grade, and that it will gradually diminish through 5th and 6th grade (HYP 1B).

The second set of hypotheses deals with the *nonproportional word problems*. As argued above, we expect that problems with non-integer ratios (NN-version) will elicit less unwarranted proportional (P) answers than problems with integer ratios (II-version) (HYP 2A). We expect that particularly for the additive items (AD), the decrease in P-answers will result in more correct (C) answers—because the "additive" strategy that pupils often *erroneously* apply to non-integer proportional problems is exactly the *correct* strategy for AD-items, whereas for the constant (CO) and affine (AF) items, the decrease in P-answers might as well result in more other errors (O-answers) (HYP 2B). Finally, as for the proportional items, we expect that differences in the number of P-answers on the NN- and II-versions of the nonproportional items will be the strongest in the 4th graders and will gradually diminish through 5th and 6th grade (HYP 2C).

Results

Table 3 shows the percentage of correct answers to the *proportional problems*. As expected (HYP 1A), the NN-versions of the proportional problems elicited less correct answers (56.8 %) than the II-versions (82.1 %). A repeated measure logistic regression analysis showed that this difference was significant, as there was a main effect of "number type," $\chi^2(1, N=508)=52.51$, $p<.0001$.

The analysis also revealed a "number type" × "grade" interaction effect, $\chi^2(2, N=508)=166.59$, $p<.0001$. In line with HYP 1B, the difference between the II- and NN-version was very strong in 4th grade (65.2 % correct answers to the II-version and only 23.6 % on the NN-version), less strong but still significant in 5th grade (with 86.3 % and 63.8 % correct answers, respectively), and not significantly different in 6th grade (96.4 % and 85.5 % correct answers, respectively).

In Table 4 we have split up the results for the three different types of *nonproportional problems*. It shows that the NN-versions elicited considerably less P-answers than the II-versions, and this was true for each type of nonproportional problem. For the additive (AD) problems, the II-versions elicited 29.3 % P-answers, and the NN-versions only 12.3 %, $\chi^2(1, N=508)=23.41$, $p<.0001$. For the constant (CO) items, the II-version elicited 61.7 % P-answers vs. 36.0 % in the NN-version, $\chi^2(1, N=508)=34.03$, $p<.0001$. Finally, for the affine (AF) items, percentages were 56.6 % and 34.4 %, respectively, $\chi^2(1, N=508)=31.54$, $p<.0001$. So HYP 2A was confirmed.

Table 4 suggests that also HYP 2B was confirmed. For the AD-items, as expected, the decrease of P-answers resulted in an increased number of C-answers: the II-versions got only 51.6 % C-answers, whereas the NN-versions got 73.0 %, $\chi^2(1, N=508)=24.71$, $p<.0001$, while there was no significant difference in the number of O-answers (19.0 % and 14.7 %, respectively).

Table 3 % correct answers on the proportional problems in the II- and NN-version

	4th grade	5th grade	6th grade	Total
II	65.2	86.3	96.4	82.1
NN	23.6	63.8	85.5	56.8

Table 4 % correct, proportional, and other answers on the nonproportional problems in the II- and NN-version

		4th grade			5th grade			6th grade			Total		
		C	P	O	C	P	O	C	P	O	C	P	O
AD	II	57.3	23.6	19.1	48.8	35.0	16.2	48.1	30.1	21.7	51.6	29.3	19.0
	NN	80.9	0.0	19.1	68.8	12.5	18.8	68.7	25.3	6.0	73.0	12.3	14.7
CO	II	6.9	57.4	35.6	13.6	63.0	23.4	5.7	64.8	30.0	8.6	61.7	29.7
	NN	17.2	8.1	74.7	12.4	38.3	49.4	6.8	61.3	31.8	12.1	36.0	52.0
AF	II	13.8	54.0	32.2	23.5	54.3	22.2	28.4	61.4	10.2	21.9	56.6	21.5
	NN	12.6	12.6	74.7	18.5	38.3	43.2	31.8	52.3	15.9	21.1	34.4	44.5

For the CO- and AF-items, the decrease in the number of P-answers led to a significantly higher number of O-answers: For the CO-items, there was an increase from 29.7 to 52.0 %, $\chi^2(1, N=508)=25.99$, $p<.0001$, and for the AF-items, the increase was from 21.5 to 44.5 %, $\chi^2(1, N=508)=33.82$, $p<.0001$. No significant differences were found in the number of C-answers, neither for the CO-items (8.6 and 12.1 %) nor for the AF-items (21.9 and 21.1 %).

Finally, HYP 2C was confirmed too, because the differences in the number of P-answers to the NN- and II-versions were the largest in the 4th graders. In 5th and especially 6th grade, differences were considerably smaller or even completely gone:

- AD-items: The "number type" × "grade" interaction effect for P-answers, $\chi^2(2, N=508)=25.19$, $p=.0003$, indicates that 4th graders gave significantly more P-answers to the II-variant (23.6 %) than to the NN-variant (0.0 %). The difference was still present in 5th grade (35.0 % vs. 12.5 %), but 6th graders gave almost equal numbers of P-answers to the II- and NN-variant (30.1 % vs. 25.3 %).
- CO-items: A similar "number type" × "grade" interaction effect was found, $\chi^2(2, N=508)=40.60$, $p<.0001$. In 4th grade, the II-variant elicited much more P-answers (57.4 %) than the NN-variant (8.1 %). In 5th grade the difference was smaller but still significant (63.0 % vs. 38.3 %), but in 6th grade, the difference had disappeared (with 64.8 % and 61.3 % P-answers, respectively).
- AF-items: Again, a "number type" × "grade" interaction effect, $\chi^2(2, N=508)=32.83$, $p<.0001$, showing a large difference in P-answers in 4th grade (54.0 % on the II-variant vs. 12.6 % on the NN-variant), a smaller difference in 5th grade (54.3 % vs. 38.3 %), and a nonsignificant difference in 6th grade (61.4 % vs. 52.3 %).

Conclusion

Study 1 explicitly addressed the claim that the nature of the numbers in the word problem might trigger superficial modeling processes by experimentally manipulating the integer or non-integer character of the ratios in the word problems. The results on the proportional problems replicated those reported in the proportional reasoning literature: problems with non-integer ratios elicited less correct proportional answers than variants with integer ratios. Moreover, as expected, this effect was particularly strong in 4th grade, while it became less influential in 5th and especially 6th grade.

With respect to the nonproportional problems, our findings confirmed the hypothesis that pupils are less inclined to overuse proportional methods when the given numbers do not form integer ratios. Also in line with our expectations, the decrease of unwarranted proportional answers resulted in better performances on problems with an additive structure, as the "additive strategy"—which is often erroneously applied on non-integer proportional problems—is correct for solving this kind of word problems. For constant and affine word problems, the decrease in proportional answers did not result in better performances. Instead, pupils started to commit more other errors. Finally, we also found the expected interaction effect: 4th graders were particularly sensitive to the presence of non-integer ratios in nonproportional problems, whereas 5th and especially 6th graders were hardly or not affected by this task characteristic. In other words, with age pupils became less prone to superficial mathematical modeling approaches based on the nature of the numbers in the non-proportional problem.

Study 2: Toward an Appreciation of Pupils' Quantitative Analogical Reasoning in Mathematically Neutral Word Problems

Introduction

Several previous studies, including the first study presented in this chapter, provide a rather negative view on pupils' modeling capacities, by showing that the (over)use of both additive and proportional methods is strongly determined by task (number characteristics) and subject (grade) characteristics (Van Dooren et al., 2005; Van Dooren, De Bock, & Verschaffel, 2010). First, and as also shown by Study 1, the application of proportional methods occurs more frequently when the numbers in the word problem form integer ratios. Pupils thus ignore the mathematical model underlying a word problem and instead rely on a superficial, irrelevant problem characteristic in order to decide which model needs to be applied to a specific word problem, namely, the numbers given in a problem. This finding indicates that we often overestimate pupils' true mathematical modeling capacities. Second, the overuse of proportional methods to additive problems tends to increase with age during elementary school and the first years of secondary school, and the overuse of additive methods to proportional problems occurs more frequently by younger pupils (see introduction of Study 1). Moreover, between the stage where younger pupils overuse additive methods on proportional problems and the stage where they overuse proportional methods on additive problems, there is a stage of simultaneous overuse of additive and proportional methods. Pupils in this intermediate stage give additive answers to word problems with non-integer ratios and proportional answers to problems with integer ratios, independent of the actual mathematical model of the problem they solve.

However, the glass is not only half empty, as young pupils may have more modeling dispositions than is usually suggested by older research. Most research on the development of proportional reasoning considered pupils' (improper) additive reasoning as an indicator of not having reached the stage of proportional reasoning yet (or at least not yet completely). While we agree with this finding, we also want to argue that pupils who reason additively in proportional word problems—even if they herewith relied on the nature of the numbers in the word problem—have already taken a valuable step in their development toward proportional reasoning, as compared, for instance, to pupils who just add all the given numbers. Kaput and West (1994) already emphasized that pupils who improperly use the additive approach for proportional reasoning problems of the missing-value type still "distinguish the quantities, construct units, and correctly identify the unknown quantity" (p. 251). In other words, improper additive reasoners still demonstrate—consciously or unconsciously—insight into the different known and unknown magnitudes and the fact that they are analogously related. They focus on the quantitative relation between two magnitudes that are given in the word problem and apply this relation to a third given magnitude in order to calculate the missing one. So, regardless of the correctness for a given problem, or whether or not they relied on the number characteristics of the word problem for their choice, additive and proportional missing-value reasoning have in common that pupils focus on the analogical relations between the four magnitudes in the word problem. Thus, both additive reasoning and proportional reasoning are types of quantitative analogical reasoning (hereafter abbreviated as QA reasoning). In sum, whereas previous studies focused on the differences between both types of reasoning and pupils' inability to correctly distinguish where to apply each of them, we depart from the similarities between additive QA reasoning and proportional QA reasoning. After all, the only difference between proportional and additive reasoning is that the latter focuses on a different kind of mathematical relation between a and b (i.e., a difference instead of a ratio, as argued by Nunes & Bryant, 2010).

In this study, we wanted to investigate the development of pupils' spontaneous inclination toward QA reasoning. We applied a novel approach to investigate the development of spontaneous QA reasoning, namely, by giving pupils word problems that are unreadable to them. Although this might seem at first sight a strange methodological choice, we will explain the rationale for it. In all aforementioned previous studies into pupils' choice for an additive or proportional solution method, word problems with an underlying mathematical model that could be determined clearly and unquestionably by carefully reading and processing the word problem were used. However, one of the basic ideas of the present study is that, in order to get a complete picture of the development of pupils' (additive or proportional) spontaneous QA reasoning, one needs tasks that are totally open to both additive and proportional reasoning. Only in those tasks, pupils' inclination is fully spontaneous and not directed by the mathematical model underlying the problem. In the tasks used in the present study, the problem context is unreadable and the mathematical model underlying the word problem is thus inaccessible. Therefore, we call these tasks mathematically neutral, i.e., neutral with respect to the underlying mathematical model. These mathematically neutral tasks allow us to get a view on pupils' general and spontaneous inclination toward QA reasoning and, in case such reasoning occurred, which type of QA reasoning then would be used (additive or proportional). We designed such neutral problems by posing them in Greek literal symbols which were completely inaccessible to the (Flemish) pupils involved in our study. The numbers were of course accessible as they were presented in their usual Arabic form. Still, pupils were asked to try to solve these "incomprehensible" word problems. In doing so, we were not interested whether pupils were *able to* make sense of the problem and to translate it in the appropriate mathematical structures and operations. We were rather interested in their *spontaneous inclination* toward additive or proportional reasoning. More specifically, our intention was to find out to what extent they would look for a quantitative analogical relation between the given numbers and, if so, if they would opt for an additive or a proportional one.

Research Questions and Hypotheses

Our first research question was: To what extent do pupils apply quantitative analogical reasoning in neutral word problems, and how is this affected by age? Because of elementary school pupils' increasing classroom experiences with solving missing-value word problems, we expected that even those neutral word problems would elicit a substantial amount of QA reasoning (*HYP 1*), and that this amount would increase with age (*HYP 2*).

Our second research question was: What is the nature of pupils' QA reasoning, and how is it affected by age and by number characteristics? Given that both additive and proportional types of answers to missing-value problems were observed in previous research, we hypothesized that we would observe both types of QA answers to our neutral word problems (*HYP 3*). Furthermore, based on the aforementioned previous research results about clearly additive and proportional word problems, we anticipated that among the QA answers, there would be a development with age, from a dominance of additive answers toward a dominance of proportional answers for neutral word problems too (*HYP 4*). We also expected a reliance on the characteristics of the numbers in the word problem. More specifically, we predicted that problems containing non-integer ratios would lead to a higher number of additive answers than problems with integer ratios, and that the latter problems would lead to a higher number of proportional answers than problems with non-integer ratios (*HYP 5*). Finally, we anticipated that the sensitivity to the numbers in the problem would be the strongest in the intermediate stage of pupils' development, between the initial stage, with mainly additive answers, and the final stage, wherein mainly proportional answers were expected (*HYP 6*).

Method

Participants were 325 pupils from 3rd to 6th grade from two primary schools in Flanders (88 3rd graders, 78 4th graders, 81 5th graders, and 78 6th graders). The number of boys and girls was approximately equal in the sample. The pupils solved two neutral word problems. These neutral word problems were part of two larger paper-and-pencil tests. Each of these tests contained one neutral word problem, along with some buffer items (related to various parts of the pupils' curriculum). Both neutral word problems were stated in Greek literal symbols, but the numbers were given in the usual Arabic form as shown in Fig. 1. Flemish pupils could absolutely not read nor understand the text of these problems, so neither the proportional nor the additive solution method—nor any other solution method—could be considered as correct or incorrect. The two word problems only differed with respect to the numbers used in the problem: the given numbers formed integer (internal and external) ratios (e.g., 4, 16, and 8 as given magnitudes) for one problem and non-integer (e.g., 4, 14, and 6 as given magnitudes) for the other one. To minimize the influence

This word problem is a Greek one. Try to fill in a number on the dotted line.

Αδα καλκα πορελαντορα λικτουν κοττορ.
Νοϖεργανιχα τινεσταρι 4 ποσστορ ιο χηιον ανπερα τον ϖορχον 16 στατον εστανο τυϖ μαγχανετο.
Προβαλεντι μογρονατεσ 8 ογροντ ο γνοστον καλκονο τοτ λινδεναν, ναγ κιφ νισϖορ κ σχκρινον λοπεναδο μαορν εωεινστ?

Answer:

Γελομαλ λοπανδορα ριτ .... νιφφ τοτο.

Fig. 1 "Greek" word problem

of the specific numbers in both problems, several sets of numbers forming integer and non-integer ratios were used.

The two tests were administered on two separate moments, with 1 week in between. The researcher told pupils that the test was aimed at assessing their general mathematics achievement. For the neutral word problems, the test merely mentioned that the problems were in Greek but that pupils were nevertheless invited to try to fill them in.

Results

Quantitative Analogical Reasoning

In a first step of the analysis, the responses to the two neutral word problems were classified as "QA answers" when either proportional or additive operations were executed on given numbers (i.e., calculating x in $b/a = x/c$ or in $b-a = x-c$). All answers other than the ones we were interested in were classified as "other answers" (i.e., when the given numbers were combined in another way than specified above or when the problem was left unanswered). While coding the responses, a third category, namely, "sum-of-three" answers, was added for coding cases wherein the three given numbers were added (i.e., calculating x as $x = a + b + c$). This solution method is not of specific interest for the present study (as it is not a QA answer in the sense explained above), so we will not further discuss it in the results section. However, it was still included as a separate coding category, because a large number of pupils had used it. This is not that surprising, as it is a systematic solution method based on a part-whole structure, which is well documented in the word problem solving literature (e.g., Vergnaud, 1982; Wolters, 1983). Traditionally, three categories of elementary addition and subtraction word problems that can be analyzed in terms of this part-whole structure have been further distinguished, based on the semantic relations underlying these word problems: change problems (i.e., problems in which an event changes the value of a quantity), combine problems (i.e., problems in which two amounts are combined), and compare problems (i.e., problems in which two amounts are compared) (Riley, Greeno, & Heller, 1983; Verschaffel & De Corte, 1993, 1997).

Table 5 gives an overview of the percentage of all QA, other, and sum-of-three answers in different grades. This table reveals that 20.5 % of all answers were QA answers. Another 42.6 % was of the sum-of-three type, and the remaining 36.9 % were other answers. So, in line with HYP 1, we found a substantial number of QA answers, especially given that the two neutral word problems were completely incomprehensible to these pupils. However, even more interesting is the effect of age on the percentage of QA answers.

A generalized estimating equations analysis revealed that pupils' age affected their answers. The percentage of QA answers significantly increased from 9.1 % in 3rd grade to 41.1 % in 6th grade ($\chi^2(3)=43.858$, $p<.001$), which was in line with HYP 2. As shown in Table 1, the initially low percentage of QA answers was due to the remarkably large percentage of sum-of-three answers. Almost half of the answers (48.9 %) were characterized as such in 3rd grade and still almost a quarter in 6th grade ($\chi^2(3)=24.579$, $p<.001$). The percentage of other answers also decreased with age, from 42.0 % in 3rd grade to 35.9 % in 6th grade, but this decrease was much smaller and nonsignificant.

Proportional or Additive Quantitative Analogical Reasoning

In a second step, we focused on the subset of answers being coded as QA answers (20.5 % of all answers, i.e., 133 out of 650), to answer our second research question about the precise nature of QA reasoning. All QA answers were further categorized as "proportional answers" (when multiplicative operations were executed on given numbers, i.e., calculating x in the expression $b/a=x/c$) or "additive answers" (when additive operations were executed on given numbers, i.e., finding x in $b-a=x-c$).

Table 6 gives an overview of the percentage of additive and proportional answers. As expected (HYP 3), the neutral word problems elicited both proportional and additive answers. Of all QA answers, half were additive (49.6 %), whereas the other half were proportional (50.4 %). Moreover, the percentage of additive and proportional answers differed depending on pupils' grade and on the nature of the numbers. The results of a generalized estimating equations analysis indicated that the percentage of proportional answers significantly increased with age, from 25.0 % in

Table 5 Percentages of quantitative analogical (QA), other, and sum-of-three answers given by pupils in different grades

	QA	Other	Sum of three
3rd grade	9.1	42.0	48.9
4th grade	7.7	35.9	56.4
5th grade	25.3	33.4	41.4
6th grade	41.1	35.9	23.1
Total	20.5	36.9	42.6

Table 6 Percentages of additive and proportional answers given by pupils in different grades

	N	Additive	Proportional
Integer			
3rd grade	12	60.0	40.0
4th grade	11	80.0	20.0
5th grade	20	26.1	73.9
6th grade	23	17.6	82.4
Total	66	30.6	69.4
Non-integer			
3rd grade	4	100.0	0.0
4th grade	1	100.0	0.0
5th grade	21	77.8	22.2
6th grade	41	56.7	43.3
Total	67	72.1	27.9
All problems			
3rd grade	16	75.0	25.0
4th grade	12	91.7	8.3
5th grade	41	48.8	51.2
6th grade	64	35.9	64.1
Total	133	49.6	50.4

3rd grade to 64.1 % in 6th grade ($\chi^2(3) = 884.927$, $p < .001$, see Table 2). Accordingly, the percentage of additive answers significantly decreased from 75.0 % in 3rd grade to 35.9 % in 6th grade. These findings were consistent with HYP 4. Second, the nature of the numbers affected the kind of QA answers, as expected in HYP 5. The integer problem evoked significantly more proportional answers than the non-integer problem (69.4 % vs. 27.9 %, $\chi^2(1) = 1349.979$, $p < .001$). Third, the number effect interacted significantly with the effect of grade ($\chi^2(2) = 452.825$, $p < .001$), which was in line with HYP 6. The number effect was the largest in 5th grade (leading to a difference of 51.7 % between the percentages of proportional answers to the integer and non-integer variant) and decreased toward 6th grade (39.1 %). However, the difference in 3rd grade (40.0 %) and 4th grade (20.0 %) was not reliable, due to the very low absolute number of QA answers.

Conclusion

This study focused on pupils' spontaneous inclination toward quantitative analogical (QA) reasoning in word problems that could be considered neutral in terms of their underlying mathematical model, given the completely unknown alphabet and language in which they were posed. In a first step, we analyzed pupils' tendency to give answers based on QA reasoning. This kind of analysis is rather unique, because previous research into this topic has mainly focused on either additive reasoning or proportional reasoning, without explicitly recognizing the common nature of these two types of reasoning. Our study indicated that the neutral word problems did elicit answers based on QA reasoning, in approximately one out of five cases. This percentage considerably increased with age, which provides a more positive picture of pupils' modeling disposition. Consciously or not, older pupils more frequently looked for a relation between two given numbers in the word problem and applied this to the third number, in order to calculate a fourth one.

In a second step, we investigated on which kind of quantitative relation the quantitative analogical reasoners relied. The same overall percentage of answers was additive or proportional, but the percentage of additive answers decreased with age, while that of proportional answers increased. Furthermore, problems with integer ratios evoked more proportional than additive answers, whereas there reverse was true for problems with non-integer ratios. This number effect was most prominent in 5th grade.

Discussion

We presented two studies related to upper elementary school pupils' use of nonproportional (i.e., additive) and proportional strategies to solve word problems, in order to shed a light on pupils' modeling disposition (i.e., both abilities and inclination) in the context of proportional reasoning. In Study 1, we used word problems that were clearly additive or proportional, while in Study 2 the problems were formulated with Greek symbols so that pupils had no access to the actual contents of the problems. Notwithstanding the important difference in the nature of the problems, both studies yielded very similar results. While 3rd graders initially are strongly inclined to reason additively to missing-value word problems (whether they are additive, proportional, or incomprehensible) and 6th graders are strongly inclined to reason proportionally, in the intermediate stage, pupils heavily rely on the numbers appearing in the word problems in order to decide to apply a proportional or additive method.

Even though the results were very similar, the perspective of both studies—and what each perspective shows about pupils' modeling disposition in the context of proportional reasoning—is quite different. This is exactly what the title of the chapter refers to: the discussion between those who state that the glass of wine is half empty and those who state it is half full. This classical contrast between the optimistic and pessimistic way of perceiving a given situation also seems to apply to the phenomenon we investigated in this chapter. The first study showed how pupils largely neglect the actual model underlying a word problem. Their solution method either is consistently the same across all problems, whatever the actual underlying model is, or it is based on superficial and irrelevant problem characteristics. Thus, one may argue that these pupils are unable to deal with the modeling aspect of solving these missing-value word problems.

Some important broader theoretical, methodological, and practical implications of this pessimistic perspective are listed below. *Theoretically*, it further documents the variety of superficial cues pupils rely on while solving word problems (Sowder, 1988). Not only problem formulations or keywords but also particular number combinations can be associated with certain solution methods (here, proportional methods). This association moreover interacts with pupils' mathematical knowledge: for older, more experienced proportional reasoners, a missing-value format seems a "sufficient condition" to apply proportionality, whereas for less experienced pupils, the "necessary condition" is that the numbers must have an integer multiplicative structure. *Methodologically*, our study warns against the assessment of the overuse of proportionality merely using problems whose numbers have integer ratios (e.g., in Van Dooren et al., 2005). Nevertheless, this warning especially seems to hold for the assessment of younger, less experienced proportional reasoners (or proportional reasoners in the so-called intermediate stage). *Practically*, our results suggest that the classroom teaching of proportionality might benefit from explicitly discussing the criteria that pupils use (or do not use) when deciding on the appropriateness of proportional solution methods.

In contrast to Study 1, the second study, which focused on pupils' inclination toward additive or proportional QA reasoning (rather than their abilities to do so), favors a more optimistic view on pupils' modeling disposition. Though these results are very similar to those in Study 1, they are interpreted in a different way. Given that additive and proportional reasoning in typical missing-value problems also have strong similarities, we considered it worth focusing on the evolution of common denominator which we called QA reasoning. The results from our study indicated that the neutral word problems—where both additive and proportional answers could be considered meaningful reactions—did elicit answers based on QA reasoning in approximately one out of five cases. This was a substantial number of QA answers, especially given that the word problems were posed in an inaccessible language and thus completely nonsensical to the pupils. The percentage of QA answers moreover considerably increased with age. Consciously or not, older pupils more frequently looked for a relation between two given numbers in the word problem and applied this to the third number, in order to calculate a fourth one. This finding indicates an early inclination toward quantitative relational reasoning, closely related to the notion of SFOR introduced by McMullen, Hannula-Sormunen, and Lehtinen (2013). They studied what they call pupils' "Spontaneous Focus On quantitative Relations" (SFOR) by using specific non-explicitly mathematical tasks, whereas we have conceptualized QA reasoning in the context of missing-value word problems, which are explicitly mathematical tasks. Future research should study the relation between their SFOR construct and our notion of QA reasoning. Moreover, and as also remarked by McMullen et al. (2013), future research on pupils' spontaneous focusing on quantitative (analogical) relations should include a measure of pupils' actual quantitative skills, to confirm that the increase in their (analogical) relational answers is mainly due to an increase in whether and how they perceive a situation in quantitative (analogical) relational terms, rather than an increase in their skill to operate with quantities and quantitative relations. In this second study, we assumed that younger pupils' insufficient quantitative skills may not have been a prominent explanation for the observed lack of QA answers, because several previous studies (reviewed by Nunes & Bryant, 2010) have already shown that pupils at the ages involved in our study typically succeed in solving both additive and proportional reasoning problems.

So, the findings of the two studies presented in this chapter are not irreconcilable, just like optimists' and pessimists' reactions to the glass of wine being half full or half empty. Both would agree that there is room for more wine in the glass, and even that more wine is desirable. Regardless of the fact that additive analogical reasoning often inappropriately occurs in proportional missing-value problems and might be based on the number characteristics of number in the word problem, it is still a valuable step in pupils' development toward proportional reasoning. Additive reasoning is after all already a way of QA reasoning. Therefore, we suggest that both additive and proportional missing-value problems should be offered in the elementary school curriculum, and that pupils repeatedly should be stimulated and helped to distinguish between additive and proportional problems. We are convinced that offering both types of missing-value problems would help pupils to gain an understanding of what quantitative analogical reasoning means, as well as what it thus implies to determine the precise nature of that relation in a word problem.

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Conceptual Model-Based Problem Solving

Yan Ping Xin

Abstract While mathematics problem-solving skills are well recognized as critical for virtually all areas of daily life and successful functioning on the job, many students with learning disabilities or difficulties in mathematics (LDM) fail to acquire these skills during their early school studies, thereby subjecting themselves to lifelong challenges with mathematical problem solving. This chapter will introduce a conceptual model-based problem-solving (COMPS) approach that aims to promote elementary students' generalized word problem-solving skills. With the emphasis on algebraic representation of mathematical relations in cohesive mathematical models, the COMPS program makes connections among mathematical ideas; it offers elementary school teachers a way to bridge the gap between algebraic and arithmetic teaching and learning. The COMPS program may be especially helpful for students with LDM who are likely to experience disadvantages in working memory and information organization. Findings from a series of empirical research studies will be presented, and implications for elementary mathematics education will be discussed pertinent to all students meeting the new Common Core State Standards for Mathematics (CCSSM, 2012).

About 5–10 % of school-age children have been identified as having mathematics disabilities (Fuchs, Fuchs, & Hollenbeck, 2007), and students whose math performance was ranked at or below the 20–35 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM) (Bryant et al., 2011; Fuchs et al., 2007). Students with LDM lag behind their peers very early on in their educational trajectory and continue to fall further behind as they transition from elementary to secondary school. According to the 2011 National Assessment of Educational Progress (NAEP) mathematics assessment, 64 % of eighth graders with disabilities who participated in the assessment scored below the basic level compared to 22 % of students without disabilities. The most recent NAEP results show that, from 2011 to 2013, score gains were seen in mathematics at grades four and eight for higher-performing students at the 75th and 90th

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percentiles, but there were no significant changes over the same period for lower-performing students at the 10th and 25th percentiles (NAEP, 2013).

In conjunction with this lack of growth in mathematics learning among students with disabilities, expectations for all students, including those with LDM, have been elevated in today's educational climate. In particular, the Common Core State Standards for Mathematics (CCSSM, Common Core State Standards Initiative [CCSSI], 2012) emphasize conceptual understanding of ideas and the connections between mathematical ideas. The CCSSM also emphasizes that students "model with mathematics." For instance, recommended instruction could start from conceptual modeling that is situated in a real-world problem context (e.g., draw pictures or diagrams to semantically represent the problem with the story context), and finally decontextualize the mathematical relationship to "represent it symbolically and manipulate the representing symbols" (CCSSI, 2012) to find the solution. In particular, the Common Core emphasizes higher-order thinking and reasoning as well as algebra readiness throughout elementary mathematics.

It should be noted that the Common Core Standards do not intend to provide a comprehensive guideline for a full range of supports appropriate for learners with diverse needs nor specific intervention approaches/materials for students with learning disabilities or difficulties in order for them to meet the standards (CCSSI, 2012). There is a need to explore potential intervention support that addresses this new emphasis to facilitate *all* students' access to higher-order thinking and meeting the Common Core Standards.

In this chapter, I will introduce a conceptual model-based problem-solving (COMPS) approach that has a focus on algebraic conceptualization of mathematical relations in model equations. With the COMPS (see section "The COMPS Intervention Program" in this chapter for more detail), the emphasis is on representing story problem in mathematical model-based diagram equations (e.g., Part + Part = Whole; Unit Rate x # of Units = Product, Xin, 2012) on the basis of students' understanding of the mathematical relationship in the problem. Findings from preliminary empirical studies including both single-subject design and group comparison design (Xin, Wiles, & Lin, 2008; Xin et al., 2012; Xin & Zhang, 2009; Xin et al., 2011) indicate that COMPS has shown promise in improving students' problem-solving skills as well as pre-algebra concept and skills.

Traditional Instructional Practice

One of the distinctive features of the traditional instruction (TDI) was its focus on the *choice of operation* when dealing with problem solving. Historically, "inability to select and apply the appropriate arithmetic operation" was considered the primary difficulty in children's word problem solving (Jonassen, 2003, p. 268). Excerpt 1 below, taken from a study conducted by Xin et al. (2013), reflected this common focus in teaching practice:

Table 1 Sample keyword strategies (adapted from an online resource for enVisionMath)

Addition	Subtraction	Multiplication	Division
Plus	Dropped	Product of	Per
Increased by	Decreased by	Times	Out of
More than	Lost/fell	Twice	Sharing
Combined	Change	Multiplied by	Each/every
Altogether/in all	Difference		Divided by
Total	Less/less than/fewer than		

Excerpt 1 (May 29, 2012)
Each problem that I went through with the children I began by having a student read aloud the problem. From that point on, we always had a conversation about whether or not we should be using multiplication or division.

To determine the choice of operation, it is not uncommon to see that students rely on a "keyword" strategy (e.g., the word *times* in the problem would cue an operation of multiplication) for making a decision on the choice of operation. Table 1 illustrates some of the keywords recommended/taught by the school curriculum.

The keyword strategy, which has been the practice in the United States for generations (Cathcart, Pothier, Vance, & Bezuk, 2006; Sowder, 1988), directs students' attention toward isolated "cue" words in the problem. The keyword strategy might be a "quick and dirty" way to "fix" word problem solving; however, it is at odds with contemporary approaches to word problem solving that stress conceptual understanding of mathematical relations in a problem *before* attempting to solve it with an operation (Jonassen, 2003). In particular, the keyword strategy does not orient students' attention to a problem's underlying mathematical structure and relations or encourage mathematical modeling that is emphasized by the new Common Core Standards (CCSSI, 2012). Further, applying the keyword strategy might contribute to students being prone to "reversal operation" errors when encountering the so-called "inconsistent language" problems (e.g., "Tara solved 21 problems. She solved three times as many problems as Pat. How many problems did Pat solve?"), where students might mistakenly multiply, when they need to *divide*, for solution due to the keyword "times" (Xin, 2007).

Other strategies commonly used include drawing a picture, using repeated addition to solve multiplication (as well as division) problems, or repeated subtraction to solve division problems. See examples in Excerpt 2 and Excerpt 3 (Xin et al., 2013).

Excerpt 2 (May 29, 2012)
For this type of problem [see the problem presented as part of Fig. 1], I would sometimes draw a picture similar to this one [see the picture] so that students could visually see and understand whether or not we were doing multiplication or division. Since many students know that multiplication is somewhat like repeated addition, a picture likes this one helped them to see this. Once I began to show students a problem using a picture like the one below, they saw they did in fact need to multiply.

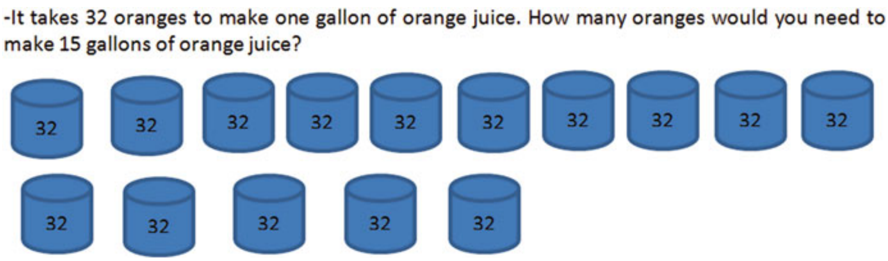


Fig. 1 An example of *drawing the picture* for solving problems

Excerpt 3 (June 4, 2012)

Edwin received a total of $374 to buy basketballs for the basketball team. Each basketball costs $34. How many basketballs can he buy?

We talked about what they were asking and how we could find out how many basketballs there would be. Then we figured out we could do repeated addition until we got to 374. We knew we couldn't go over it. We also decided we could do repeated subtraction until we didn't have any left. We then would count up the number of sets of 34 that were either added or subtracted depending on the strategy they used.

In addition, "guess and check" is another strategy encouraged in teaching and used by students in solving mathematics word problems. Excerpt 4 (Xin et al., 2013) supported the use of this strategy:

Excerpt 4 (May 29, 2012)

Since this problem [i.e., the same problem as in *Excerpt* 3] needed to use division that involved two-digits, it posed quite a challenge for my fourth graders. We did a lot of *guessing and checking* as we worked through a division problem of this sort.

Another thing I try to stress to them is whether or not our numbers for our answer are going to be increasing (multiplication) or decreasing (division).

When solving partitive division problems (such as "There are 126 spring rolls to be placed on 42 platters. How many spring rolls will be on each platter?"), students might also be taught, in conjunction with the keyword strategies (e.g., "each," "per," "every"), that "division is all about sharing" or "dividing out evenly" to groups. While repeated addition or subtraction may help students in understanding the concept of multiplication and division in a concrete way (often accompanied by semi-concrete pictures), Schwartz (1988) expressed concern with an overly simple perception of multiplication as "repeated addition" and division as "solving problem of equitably sharing" some set of objects (p. 46) and highlighted the importance of modeling in teaching and learning mathematics.

In summary, the keyword strategy may not be a robust strategy, and the strategies of repeated addition/subtraction, drawing pictures, or "guess and check" might be useful strategies during the beginning stage of the learning. However, without

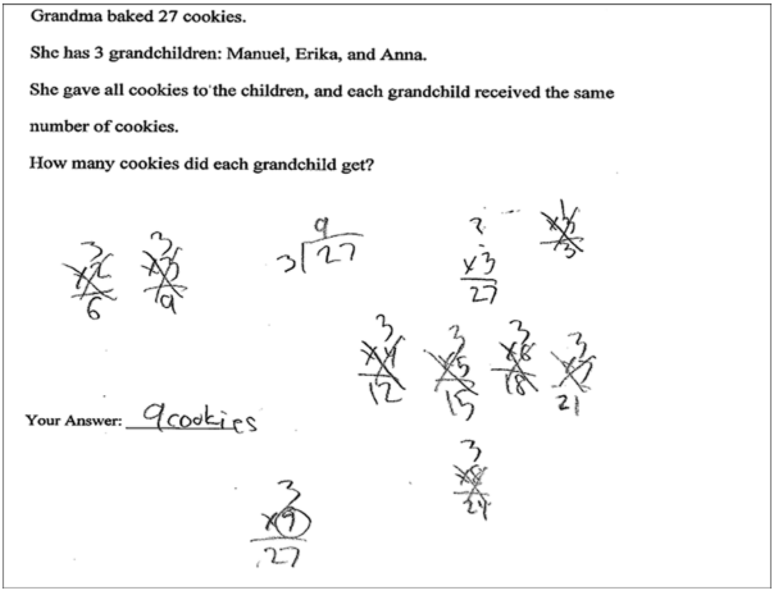


Fig. 2 A fourth-grade student's problem-solving process

advancing students to mathematical model-based problem solving, these strategies may easily become cumbersome and error-prone, particularly when the numbers in the problem become large. Figures 2 and 3 present sample works of two elementary students, who were enrolled in an after-school program where the above strategies were taught (Xin et al., 2013). Specifically, the after-school program was designed to boost students' math problem-solving performance. Figure 2 shows a fourth-grade student, who used a "guess and check" strategy in solving a partitive division problem. She started with number facts from 1×3 to 2×3, 3×3 …, all the way to 3×9 and finally got the total (i.e., 27) given in the problem. She then checked her answer (9) using division. It should be noted that when the numbers in the problem are small, as in the case of this problem, it might be manageable to correctly solve the problem using such "guess and check" strategy. However, when the numbers become large, such problem-solving process may become cumbersome or not efficient.

Figure 3 shows a third-grade student's reasoning about why "$28 + 4 = 32$" should not be the correct answer to a quotitive division problem (see the problem as part of Fig. 3) and how she used *repeated subtraction* for finding the answer to this problem. It seems that she either started with an incorrect number or left out the first "4" she subtracted mentally perhaps and, therefore, reached an incorrect answer.

The TDI strategies illustrated above were considered best practices by experienced school teachers who taught the after-school intervention program (Xin et al., 2013). Historically, special education teachers were expected to collect "a bag of tricks" to hopefully "fix" the "problems" of students with special needs. The strategies illustrated

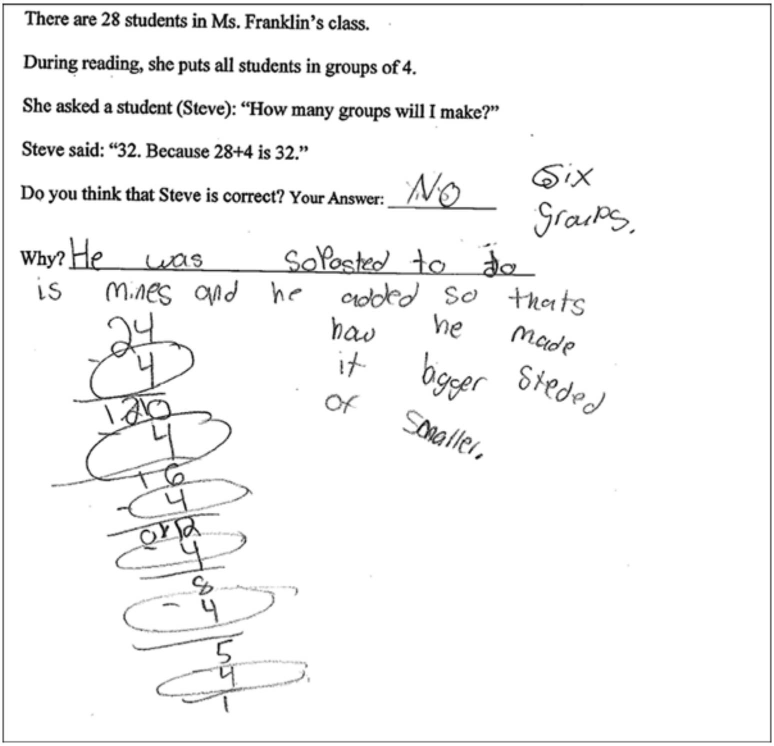


Fig. 3 A third-grade student's reasoning and problem solving

above were used by the teachers as a collection of strategies to hopefully boost the performance of students with LDM; they were not used as a coherent form of instruction and they were not connected to each other through mathematical modeling.

Mathematical Modeling

Recently, Blum and Leiss (2005) provided a framework for modeling (see Fig. 4). In this modeling cycle, one must (a) read and understand the task, (b) structure the task and develop a real situational model, (c) connect it to and/or represent it with a relevant mathematical model, (d) solve and obtain the mathematical results, (e) interpret the math results in real problem context, and (f) validate the results (either end the task or re-modify the math model if it does not fit the situation). In light of research in mathematics education, many students have difficulties in making the transition from a real situational model to mathematical model; and it is a weak area in students' mathematical understanding (Blomhøj, 2004).

In short, modeling involves translation or representation of a real problem situation into a mathematical expression or model. Mathematical models are an essential part of all areas of mathematics including arithmetic and should be introduced to all age

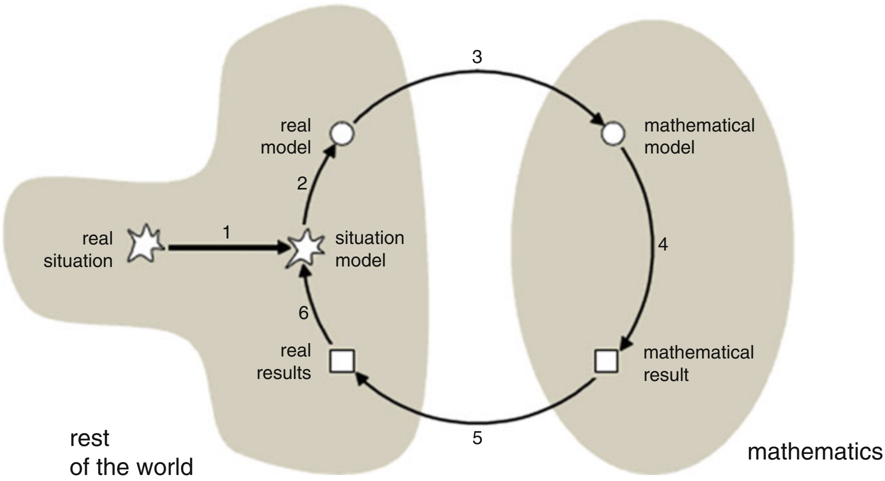


Fig. 4 Modeling cycle (Blum & Leiss, 2005)

groups including elementary students (Mevarech & Kramarski, 2008). It should be noted that engaging students in the modeling process does not necessarily mean to engage students in the discovery or invention of mathematical models or complex notational systems. According to Lesh, Doerr, Carmona, and Hjalmarson (2003), engaging students in the modeling process can be interpreted as when such models or systems are given to the students, "the central activities that students need to engage in is the unpacking of the meaning of the system" (p. 216), representation of the real problem situation in a mathematical expression or model, and the flexible use of the model to solve real problems.

Conceptual Model-Based Problem Solving

Contemporary approaches to story problem solving have emphasized the conceptual understanding of a story problem before attempting any solution that involves selecting and applying an arithmetic operation for solution (Jonassen, 2003). Because problems with the same problem schema share a common underlying structure and hence require similar solutions (Chen, 1999; Gick & Holyoak, 1983), students need to learn to understand the structure of the mathematical relationships in word problems, and students should develop this understanding through creating and working with a meaningful representation of the problem (Brenner et al., 1997) and mathematical modeling (Hamson, 2003).

The representation that models the underlying mathematical relations in the problem, that is, the conceptual model, facilitates solution planning and accurate problem solving. The conceptual model *drives* the development of a solution plan that involves selecting and applying appropriate arithmetic operations. Building on the theoretical framework of conceptual models (e.g., Lesh et al., 1983) as well as cross-cultural curriculum analyses (e.g., Xin, Liu, & Zheng, 2011), I have developed the *conceptual model-based problem-solving* (COMPS) program (Xin, 2012). One distinguishable feature of the COMPS program is that it focuses on representing the word problem in a defined mathematical model (the stage of "mathematical model" as it is presented in Blum and Leiss's modeling cycle, see Fig. 4), which is expressed in an algebraic equation that drives the solution plan. To facilitate a better understanding of the COMPS intervention program, the next section presents additive and multiplicative problem structure in elementary mathematics.

Additive and Multiplicative Word Problem Structure and It's Model Expression

Within the context of elementary mathematics, additive word problems entail a range of *part-part-whole* and *additive compare* problem structures. A *part-part-whole* (PPW) problem describes an additive relation between multiple parts and the whole (i.e., parts make up the whole). It includes problems such as combine (e.g., *Christine has five apples. John has four apples. How many apples do they have together*?), change-join (e.g., *Christine had five apples. John gave her four more apples. How many apples does Christine have now*?), and change-separate (e.g., *Christine had nine apples. Then she gave away four apples. How many apples does she have now?*) (Van de Walle, 2004). Placement of the unknown can be on the *part* or on the *whole* (see nine variations of PPW problems in Table 2). An *additive compare* (AC) problem compares two quantities, and it involves a comparison sentence that describes one quantity as "more" (AC-more) or "less" (AC-less) than the other quantity (e.g., "Christine has nine apples. She has five more apples than John. How many apples does John have?" or "Christine has nine apples. John has four less apples than Christine. How many apples does John have?"). Placement of the unknown can be on the *big*, *small*, or *difference* quantity (see six variations of AC problems in Table 2).

The most basic multiplicative word problems entail various *equal groups* (EG) as well as *multiplicative compare* (MC) problem structure (other problem structures include *Cartesian product and rectangular area*, Greer, 1992; Schmidt & Weiser, 1995). An *equal groups* (*EG*) problem describes a number of equal sets or units. The placement of the unknown can be on the *unit rate* (# of items in each unit or group or *unit price* as in money-related story contexts), *number of units* or sets, or the *product* (see three variations of *EG* problems in Table 3). A *multiplicative compare* (*MC*) problem compares two quantities, and it involves a comparison sentence that describes one quantity as a multiple or part of the other quantity. Placement of the unknown can be on the *compared* set, the *referent* set, or the *multiplier* (i.e., multiple or part) (see three variations of *MC* problems in Table 3). It should be noted that the MC problems in Table 3 only include those with *multiple* NOT *part* relations such as "2/3."

Table 2 Variations in *additive* word problems (adapted from Xin et al. (2012))

Problem type	Sample problem situations
Part-part-whole (PPW)	
	Combine
Part (or smaller group) unknown	1. Jamie and Daniella have found out that together they have 92 books. Jamie says that he has 57 books. How many books does Daniella have?
Whole (or larger group) unknown	2. Victor has 51 rocks in his rock collection. His friend, Maria, has 63 rocks in her collection. How many rocks do the two have altogether?
	Change-join
Part (or smaller group) unknown	1. Luis had 73 candy bars. Then, another student, Lucas, gave him some more candy bars. Now he has 122 candy bars. How many candy bars did Lucas give Luis?
	2. A girl named Selina had several comic books. Then, her brother Andy gave her 40 more comic books. Now Selina has 67 comic books. How many comic books did Selina have in the beginning?
Whole (or larger group) unknown	3. A basketball player ran 17 laps around the court before practice. The coach told her to run 24 more at the end of practice. How many laps did the basketball player run in total that day?
	Change-separate
Part (or smaller group) unknown	1. Davis had 62 toy army men. Then, one day he lost 29 of them. How many toy army men does Davis have now?
	2. Ariel had 141 worms in a bucket for her big fishing trip. She used many of them on the first day of her trip. The second day she had only 68 worms left. How many worms did Ariel use on the first day?
Whole (or larger group) unknown	3. Alexandra had many dolls. Then, she gave away 66 of her dolls to her little sister. Now, Alexandra has 63 dolls. How many dolls did Alexandra have in the beginning?
Additive compare (AC)	
	Compare-more
Larger quantity unknown	1. Denzel went out one day and bought 54 toy cars. Later, he found out that his friend Gabrielle has 56 more cars than he bought. How many cars does Gabrielle have?
Smaller quantity unknown	2. Tiffany collects bouncy balls. As of today she has 93 of them. Tiffany has 53 more balls than her friend, Elise. How many balls does Elise have?
Difference unknown	3. Logan has 117 rocks in his rock collection. Another student, Emanuel, has 74 rocks in his collection. How many more rocks does Logan have than Emanuel?
	Compare-less
Larger quantity unknown	1. Ellen ran 62 miles in one month. Ellen ran 29 fewer miles than her friend Cooper. How many miles did Cooper run?
Smaller quantity unknown	2. Kelsie said she had 82 apples. If Lee had 32 fewer apples than Kelsie, how many apples did Lee have?
Difference unknown	3. Deanna has 66 tiny fish in her aquarium. Her dad Gerald has 104 tiny fish in his aquarium. How many fewer fish does Deanna have than Gerald?

Table 3 Variations in *multiplicative* word problems (adapted from Xin (2012))

Problem type	Sample problem situations
Equal groups (EG)	
Unit rate unknown	A school arranged a visit to the museum in Lafayette Town. It spent a total of $667 buying 23 tickets. How much does each ticket cost?
Number of units (sets) unknown	There are a total of 575 students in Centennial Elementary School. If one classroom can hold 25 students, how many classrooms does the school need?
Product unknown	Emily has a stamp collection book with a total of 27 pages, and each page can hold 13 stamps. If Emily filled up this collection book, how many stamps would she have?
Multiplicative compare (MC)	
Compared set unknown	Isaac has 11 marbles. Cameron has 22 times as many marbles as Isaac. How many marbles does Cameron have?
Referent set unknown	Gina has sent out 462 packages in the last week for the post office. Gina has sent out 21 times as many packages as her friend Dane. How many packages has Dane sent out?
Multiplier unknown	It rained 147 in. in New York one year. In Washington, DC, it only rained 21 in. during the same year. The amount of rain in New York is how many times the amount of rain in Washington, DC?

Generally speaking, part-part-whole (or "part + part = whole") is a generalizable conceptual model in addition and subtraction word problems where *part*, *part*, and *whole* are the three basic elements. In contrast, factor-factor-product (or "factor × factor = product") is a generalizable conceptual model in multiplication and division arithmetic word problems where *factor*, *factor*, and *product* are the three basic elements.

It should be noted that the COMPS emphasizes mathematical model-based analysis and problem solving. As such, it is different from Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999) where word (or story) problems are divided into subtypes on the basis of semantic analysis of the story situations. That is, in CGI, a part-part-whole problem type is distinct from compare, join, and separate problems (see Table 2 for sample story situations) due to semantic differences in these word problem story situations. In contrast, the COMPS model considers join, separate, and compare problems a subset of part-part-whole because the mathematical model underneath the various cover stories is the same.

On the other hand, because of the semantic differences across a range of additive or multiplicative word problems, the three basic elements (in either the part-part-whole or factor-factor-product model) will have unique denotations when a specific problem subtype applies. For example, in a *combine* problem type (e.g., *Emily has four pencils and Pat has eight pencils. How many pencils do they have all together?*), the number of pencils Emily has and the number of pencils Pat has are the two *parts*; these two parts make up the combined amount (i.e., "all together") or the *whole*. In contrast, in an *additive compare* problem type (e.g., *Emily has nine stickers, Pat has four fewer stickers than Emily. How many stickers does Pat have?*), the number of stickers Emily has is the bigger quantity (or the *whole* amount), whereas the number of stickers Pat has is the smaller quantity (or one of the parts) and the difference between Emily and Pat is the other smaller quantity (the other part); combining these two parts is the bigger quantity (or the whole).

The COMPS Intervention Program

Instructions for carrying out the COMPS will be delivered in two phases: *problem representation* and *problem solving*. During the phase of *problem representation*, word stories with no unknowns will be used to help students understand the problem structure and the mathematical relations involved (e.g., two parts make up the whole or "part + part = whole"). Specifically, students will learn to decontextualize the mathematical relations involved in the problem and represent it in the corresponding model equation. Figure 5 presents conceptual models for the *part-part-whole* (Fig. 5a) or *additive compare* (Fig. 5b) problem structure and Fig. 6 for the *equal groups* (Fig. 6a) or *multiplicative compare* (Fig. 6b) problem structure. During this phase of instruction, as all quantities are given in the story (no unknowns), students will be able to map all the given information in the model equation and then check for the "balance" of the equation (i.e., whether the left side of the equation is equal to the right side of the equation) to validate the truthfulness of the model. In the meantime, the concept of "equality" and the meaning of an equal sign are reinforced.

Problem representation instruction will be followed by *problem-solving* instruction. During *problem-solving* instruction, word problems with an unknown quantity will be presented. When representing a problem with an unknown quantity in the COMPS diagrams, students can choose to use a letter (can be any letter they prefer) to represent the unknown quantity. Students are encouraged to use the *DOTS* checklist (see Fig. 7) to guide the problem-solving process.

Overall, the instruction requires explicit strategy explanation and modeling, dynamic teacher-student interaction, guided practice, performance monitoring with corrective feedback, and independent practice. As supported by Pressley (1986), "the less mature the learner, the more explicit teaching must be" (p. 145). Making explicit the problem-solving process, when teaching mathematics to students with LDM, reduces ambiguity and promotes success (Baxter, Woodward, & Olson 2001). For more detailed description of the COMPS program, please refer to a recently published book (Xin, 2012); this book also includes modeling PowerPoint presentation slides for instructors' use and guided practice and independent worksheets for students' use. It is suggested that the COMPS model equations be provided on all modeling and guided practice worksheets or even on the independent practice worksheets in the beginning stage of the instructional program. However, they should be gradually faded out on the independent worksheets once students have internalized the models; and it should not be provided during pre- or post-intervention assessment. Below, I briefly summarize three empirical studies that examined the effect of the COMPS program.

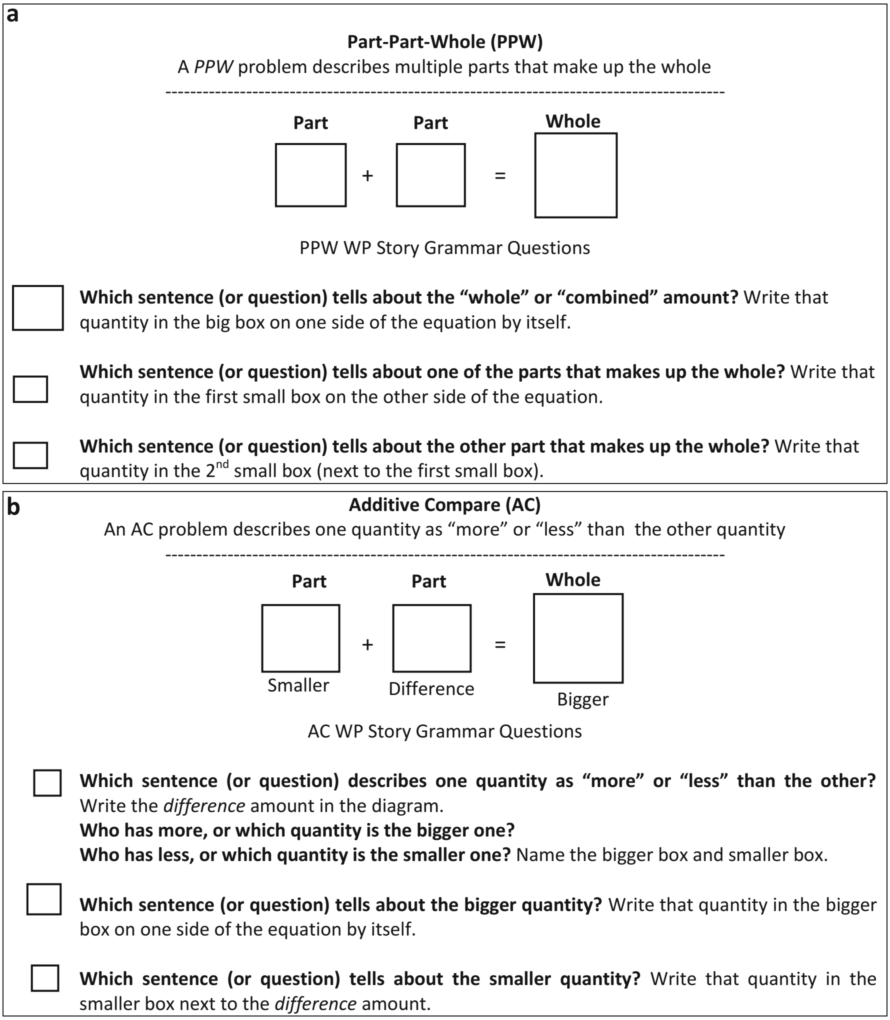


Fig. 5 (**a**) Conceptual model of part-part-whole word problems (Xin, 2012, p. 47). (**b**) Conceptual model of additive compare word problems (Xin, 2012, p. 67)

Empirical Studies that Support the COMPS Intervention Program

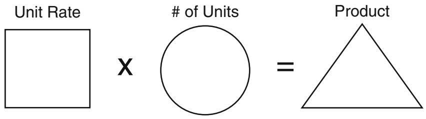
Study 1

Xin, Wiles, and Lin (2008) is the study where I conceptualized and implemented the COMPS program with elementary students with LDM to enhance their additive and multiplicative word problem-solving skills. In this study, I developed a set of

a

Equal Group (EG)

An *EG* problem describes number of equal sets or units



EG WP Story Grammar Questions

□ **Which sentence or question tells about a Unit Rate (# of items in each unit)?** Find the unit rate and write it in the Unit Rate box.

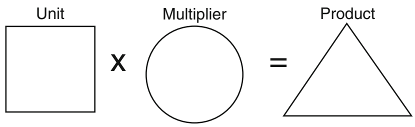
○ **Which sentence or question tells about the # of Units or sets (i.e., quantity)**? Write that quantity in the circle next to the unit rate.

△ **Which sentence or question tells about the Total (# of items) or ending product**? Write that number in the triangle on the other side of the equation.

b

Multiplicative Compare (MC)

A MC problem describes one quantity as a multiple or part of the other quantity



MC WP Story Grammar Questions

○ **Which sentence (or question) describes one quantity as a multiple or part of the other?** Detect the two things (people) being compared and who (the compared) is compared to whom (the referent UNIT). Name "whom" and "who" in the diagram. Fill in the relation (e.g., "2 times" or "½") in the circle.

□ **What is the referent UNIT?** Write that quantity in the referent unit box.

△ **What is the compared quantity or product?** Write that quantity in the triangle on one side of the equation by itself.

Fig. 6 (**a**) Conceptual model of equal groups word problems (Xin, 2012, p. 105). (**b**) Conceptual model of multiplicative compare word problems (Xin, 2012, p. 123)

DOTS

(Word Problem Solving Checklist)

- Detect the problem structure.
- Organize the information using conceptual model diagrams.
- Transform the diagram into a meaningful math equation.
- Solve for the unknown quantity in the equation and check your answer.

Fig. 7 DOTS checklist (Xin, 2012, p. 107) *word problem (WP) story grammar* heuristic questions (please refer to Figs. 5 and 6) to guide students' mapping of either additive or multiplicative word problem to its corresponding mathematical model equation. Although *story grammar* has been substantially researched in reading comprehension (Boulineau, Fore, Hagan-Burke, & Burke, 2004), *WP story grammar* has never been explored in math word problem understanding and solving. By definition, *story grammar* in reading comprehension literature refers to a typical structure shared by most narrative stories. Similarly, a word problem story structure that is common across a group of word problem situations can be defined as *WP story grammar*. Borrowing the concept of story grammar from reading comprehension literature, I designed a set of *WP story grammar* self-questioning prompts to facilitate conceptual understanding of mathematical relations in word problems and represent such relations in mathematical model equations.

The specific purpose of this study was to assess the effect of conceptual model-based problem solving (COMPS, the intervention), facilitated by *WP story grammar* heuristic questions, on improving additive and multiplicative word problem-solving performance as well as pre-algebra concept and skills of elementary students with LDM.

Design and Participants

An adapted multiple probe design (Horner & Baer, 1978) across participants was employed to evaluate the functional relationship between the intervention and students' word problem-solving performance. Single-subject research design was chosen because the design provides a methodological approach well suited to the investigation of single cases or groups (Kazdin, 1982). In particular, with the multiple probe design, intervention effects can be demonstrated by introducing the intervention to different participants at different points in time. "If each baseline changed when the intervention is introduced, the effects can be attributed to the intervention rather than to extraneous events" such as history, maturation, testing, etc. (Kazdin, 1982, p. 126).

Participants were five fourth- and fifth-grade students with LDM. On the basis of students pretests' performance, three students were identified as needing intervention in additive word problem solving, and they were engaged in solving *part-part-whole* (PPW) and *additive compare* (AC) problems; two students were identified as needing intervention in multiplicative word problem solving, and they were instructed to solve *equal groups* (EG) and *multiplicative compare* (MC) problems.

Intervention Procedure

Participating students received intervention in COMPS three times a week, with each session lasting for approximately 20–35 min. Each student received three to six sessions of instruction on PPW or EG, two to three sessions on AC or MC problem instruction, and one to two sessions on solving mixed word problems including both PPW and AC or EG and MC types.

Students were assessed on either an additive problem-solving criterion test, which involved 14 variously constructed addition and subtraction word problems (similar to those presented in Table 2), or a multiplicative problem-solving criterion test, which involved 12 variously constructed multiplication and division problems (similar to those presented in Table 3). Calculators were allowed throughout the study to accommodate participants' skill deficit in calculation.

Results and Discussion

Effect on additive word problem solving. During baseline condition (prior to the intervention), average performance across three participants on the criterion test was 21 %, 28 %, and 28 % correct, respectively. Following the intervention, the two students who completed COMPS instruction on additive word problem solving (note: one student did not complete the program) performed 79 % correct during post-intervention assessment (a 58 % increase from the baseline performance of 21 % correct) and 86 % correct (a 58 % increase from the baseline of 28 % correct), respectively.

Effect on multiplicative problem solving. During the baseline, average performance across the two participants on the criterion test was 3 % correct and 0 % correct, respectively. After the intervention, both participants' performance were 100 % correct indicating a 97 % and 100 % increase, respectively, from the baseline.

Effect on pre-algebra concept and skills. Two pre-algebra probes were used to assess potential improvement of students' pre-algebra concept and skills. The *solve equations* probe required students to find the value of an unknown quantity (i.e., letter *a*) that makes the equation true (e.g., $93 = 79 + a$; $196 = a \times 28$). Positions of the unknown were systematically varied across three terms in the equation (i.e., the augend, addend, and sum; or the multiplicand, multiplier, and product). Six items were included in either the addition/subtraction probe or multiplication/division probe. In addition, the *algebraic model expression* probe was designed to test students' algebraic expression of mathematical relations or ideas. Twelve items (e.g., "Write an expression or equation. Choose a variable for the unknown. Shanti had some stamps. She gave 23 to Penny. Shanti has 71 stamps left") were included in the addition/subtraction probe; five items (e.g., "Antoni has collected 84 autographs. He filled 14 pages in his news autograph album. Each page holds an equal number of autographs. Write an equation with a variable to model this problem") were included in the multiplication/division probe. These items were directly taken from a commercially published mathematics textbooks being adopted by the participating schools (Maletsky et al., 2004).

Findings from this study indicated that (a) on the *solve equations* probe, from pre- to post-intervention, the two participants who completed the *additive* problem-solving intervention improved from 33 % to 67 % correct and 0 to 100 % correct, respectively. The two participants who completed the multiplicative problem-solving intervention improved their performance from 0 to 67 % correct and from 0 to 100 % correct, respectively; (b) on the *algebraic model expression* probe, the participants of this study had no knowledge of what they were asked to do and made no attempts during the pretest. After the intervention, the two participants who completed the additive problem-solving intervention scored 71 % and 83 % correct, respectively, on the corresponding *algebraic model expression* probe. The two participants who completed the multiplicative problem-solving intervention both scored 100 % correct on the corresponding *algebraic model expression* probe.

Study 2

Xin, Zhang et al. (2011) conducted a group comparison study to further evaluate the effectiveness of the COMPS program when compared to a general heuristic instructional approach (GHI) for teaching multiplication/division word problem solving to elementary students with LDM.

Design and Participants

A pretest-posttest comparison group design with random assignment of participants to groups was used to examine the effects of the two word problem-solving instructional approaches: COMPS and GHI. Participants included a group of 29 fourth graders with LDM from two elementary schools in the Midwestern United States.

Intervention Procedure: The Two Comparison Conditions

The intervention for the COMPS condition is consistent with the description provided in the section titled "the COMPS Intervention Program." However, the delivery of the COMPS program was assisted by the PowerPoint (PPT) presentation of the COMPS featuring animations. The comparison condition, GHI, was guided by a general heuristic five-step problem-solving checklist, SOLVE. SOLVE was taken from the participating schools' enacted curriculum and teaching practice. SOLVE required students to: (a) *s*earch for the question, (b) *o*rganize the information, (c) *l*ook for a strategy, (d) *v*isualize and then work the problem (draw a picture, make a table, write an equation, etc.), and (e) *e*valuate your answer. For the second step, *o*rganize the information, the instructor guided students to highlight the keywords (e.g., "times" as in the *multiplicative compare* [MC] problems; "each" or "per" as in the *equal groups* [EG] problems). For the third step, *l*ook for a strategy, the instructor asked students to think about "What is the best way to solve the problem?" and specifically "Which operation to use?" For the fourth step, *v*isualize and then work the problem, students were engaged in visualizing the problem situation. Students were also instructed that they could draw a picture (to describe the given information in the problem), make a table (to organize the given information), write an equation or math sentence, etc.

Students in both conditions engaged in the assigned strategy learning three times a week, with each session lasting for approximately 30–45 min (one school with 30 min and the other school with 45 min). The COMPS group received three sessions on introduction, six sessions each on EG and then MC problem representation and solving, and three sessions on mixed review. While students in the GHI group also received 18 sessions of instruction, they engaged in solving both types of problems in each session. Students in both conditions solved the same number and type of problems. Calculators were allowed in both groups throughout the study to accommodate participants' skills deficits in calculation.

Results and Discussion

Findings from this study showed that the COMPS group improved significantly more than the GHI group from pretest to posttest on the criterion word problem-solving tests (similar in structure to the problems presented in Table 3). There was a significant interaction effect between *group* and *time of testing*, $F(2, 50) = 4.499$, $p = 0.016$. These findings support and extend previous research regarding the effectiveness of the COMPS instruction in solving arithmetic word problems (e.g., Xin et al., 2008; Xin, 2008). More importantly, the results indicate that only the COMPS group significantly improved, from pretests to posttests, their performance on the criterion test (the effect size, when COMPS was compared to the TDI group, was 0.66). Similarly, only the COMPS group significantly improved their performance on the algebraic model expression test (similar to the one used in Xin, Wiles, and Lin, 2008, as described in *Study 1*). The effect size, when COMPS was compared to the TDI group, was 0.86. The results of this study suggest that the COMPS approach produced better outcomes than the traditional general heuristic instructional strategy "SOLVE" when it was used as an intervention program to help students with LDM.

Study 3

There is a need to explore computer-assisted mathematics intervention programs that employ research-based best practice to facilitate individualized Response-to-Intervention (RtI) programs for students with LDM, as well as to address a shortage of qualified teachers to teach mathematics (Hutchison, 2012). Xin et al. (2012) conducted a single-subject design study to examine the effect of a computer-assisted COMPS program on promoting multiplicative problem-solving skills and algebra readiness of eight elementary students with LDM.

Design and Dependent Measures

The design and the measures employed in this study were similar to *Study 1* as described above.

Intervention Procedure

Intervention components. The COMPS computer tutorial program was developed on the basis of the program described in *Study 1* (Xin et al., 2008). The COMPS computer program involved three modules (A, B, and C). Module A introduced the basic concept of equal groups in multiplicative reasoning. Then, students engaged in representing and solving EG problems (see Table 3 for sample problems). Module B extended the problem context to MC problems (see Table 3 for sample problems). Finally, module C provided students with opportunities to solve mixed EG and MC problems.

Before students entered module A, they learned about basic function keys in the program. Participants were all familiar with the basic operations of the computer; they soon learned the operation of the program. In module A, students engaged in activities for them to understand the concept of *equal groups* (see Fig. 8a), which is fundamental for multiplicative reasoning. Both examples (i.e., equal groups) and non-examples (i.e., non-equal groups) were presented to facilitate students' concept formation. In the case of non-example (non-equal groups), students were asked to fix the non-equal groups and make them equal groups (see Fig. 8b). Then, students engaged in representing EG word problem stories (without unknowns) in the conceptual model equation. With the assistance of computer-simulated concrete modeling, the program aimed to establish the connection between concrete visual representations of "real" (simulated) objects and the abstract models and lead students to understand that (a) the unit rate is the number of "things" or items in *each* group, (b) the # of units refers to the number of groups, and (c) the product is the total number of items in all groups.

Next, the program engaged students in representing and solving problems with an unknown. Students used a letter "*a*" to represent the unknown quantity. When the product was the unknown, students would solve the problem by multiplying the two factors as defined in the model equation. When one of the factors was the unknown, dividing the product by the known factor would give the answer to the unknown factor. Students learned necessary conceptual and procedural knowledge regarding solving an unknown in an equation. The program allowed students to use a calculator (built into the program, see Fig. 8c) to find out the unknown in the equation. However, students could also use mental math if able, to solve for the unknown.

In module B, students learned to represent and solve MC problems (e.g., Gary and Victoria are having a party. Gary invited 28 people to his party. Victoria invited six times as many people as Gary. How many people did Victoria invite?) using the

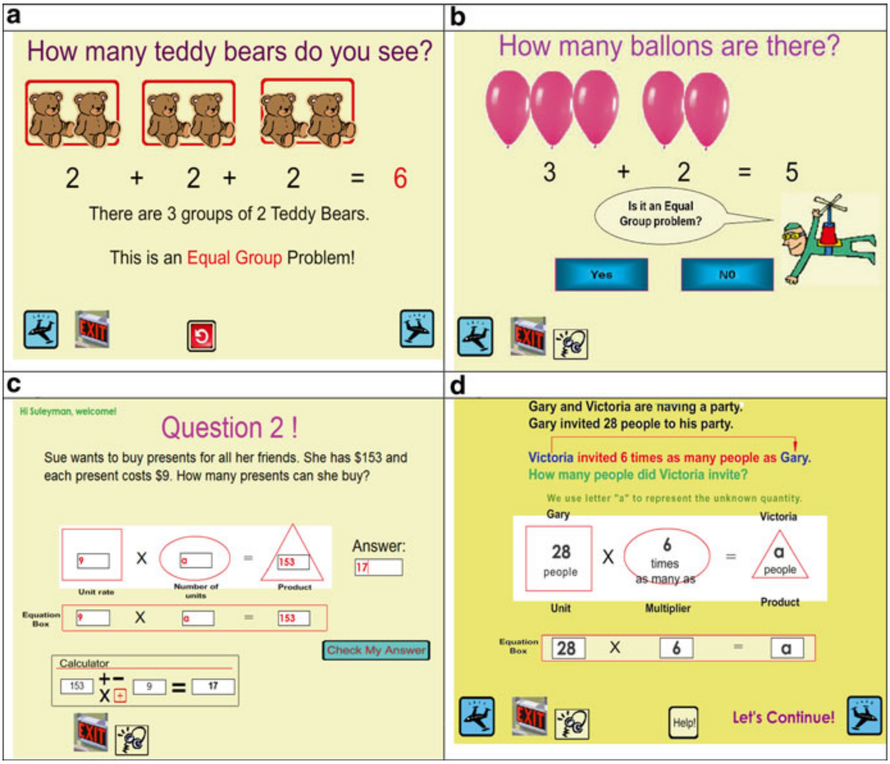


Fig. 8 Computer-assisted COMPS program sample screenshots (from Xin et al. (2012))

similar mathematical model (as presented in Fig. 6b). Nevertheless, students now needed to unpack the meaning of the model within the context of MC problem situation. Particularly, in a MC problem situation, the quantity that is being compared against is the *referent unit*. Given the specific comparison statement in an MC problem—the one presented in the beginning of this paragraph, for instance, as the number of people invited by Victoria is compared to the number of people invited by Gary—Gary is the referent unit (see Fig. 8d). In module C, participating students engaged in representing and solving mixed EG and MC problems. During this phase of instruction, the program faded out the diagrams and only presented the equation boxes (i.e., ___ × ___ = ___) for students to model the mathematical relations and solve the problem.

The COMPS tutoring program adopted a mastery-learning paradigm (Bloom, 1976). That is, if a student performed on a worksheet at a level below 70 % correct, she/he was sent back to corresponding content of instruction and required to repeat the worksheet that she/he failed. As such, each individual worked on various numbers of sessions for the completion of the program. On average, participants worked on module A (EG conception, EG representation and problem solving) for about 11 sessions (range: 6–21) with about 3–4 sessions on EG concept and 7–8 sessions on

EG representation and problem solving. Students worked on module B (MC representation and problem solving) for an average of seven sessions (with a range from 3 to 13). Students also worked on module C (solving mixed EG and MC problem types) for an average of two sessions (range: 1–3).

Results and Discussion

During the baseline, participating students' performance on the criterion test ranged from 8 to 58 % correct with a median of 25 % across eight participants. Following the intervention, students' performance on the criterion test ranged from 88 to 100 %, with a median of 94 % correct. The results indicate that all participants gained percentage points in their percent correct. On the other hand, students' performances on the algebraic model expression test improved from a median score of 0 % correct during the baseline across the eight participants to a median of 67 % correct (range: 67 to 100 %) following the intervention.

General Discussion and Implications

Traditionally, teaching for understanding has seemed to involve concrete object manipulations or representations that are "away from symbolic formalisms" (Sherin, p. 524) due to the concern that symbolic expression would be out of the reach of elementary students. Symbolic expression of mathematical relations in model equations, as presented in this chapter, is not just a mechanical process of translating text to equations. Students' mapping of information in the model equation is based on conceptual understanding of the three key elements or quantities involved and relations among them. That is, formations of symbolic expression or equation are experienced as arising from an understanding of underlying structure of the problem (Thompson, 1989). As Sherin (2001) argued in physics equation learning, "we can strive for conceptual understanding while basing instruction on the use of equations" (p. 529). In fact, "we do students a disservice by treating conceptual understanding as separate from the use of mathematical notations" (Sherin, 2001, p. 482). In short, the use of symbolic expressions can involve significant understanding.

One of the benefits of emphasizing model-based problem solving is to prevent students from over-relying on the "keyword" strategy or other misleading strategies (e.g., "look at the numbers; they will tell you which operations to use. Try all the operations and choose the most reasonable answer…," Greer, 1992, p. 28) for decision-making on the choice of operation for solution. One of the "well-known" keyword strategies includes: If you see the word *times* in the problem, you should apply a multiplication to get the answer. Keyword strategy such as this has been a "robust" yet ineffective and *detrimental* practice for generations (Cathcart, Pothier, Vance, & Bezuk, 2006, Sowder, 1988). The detrimental effect of blindly applying the "keyword" strategy has been reflected in common "reversal errors" US students have been making for over decades (Cawley, Parmar, Foley, Salmon, and Roy, 2001; Lewis, 1989; Xin, 2007).

The COMPS approach presented in this chapter directs students' attention to underlying mathematical relation in the problem and representing such relation in the algebraic model equation. The algebraic equation then directly drives the solution process, that is, applying an appropriate operation (add or subtract or multiply or divide) to solve for the unknown quantity in the equation. During this process, the choice of operation for solving various arithmetic word problems is determined by the model equation (part + part = whole or factor × factor = product). In the case of multiplicative problem solving, EG problems for instance, when the *product* is the unknown, the model equation "factor × factor = product" or specifically "unit rate × # of units = product" (see Fig. 6a) *tells* that multiplying the two factors will give the solution for the unknown product. When the *unit rate* or *# of units* is the unknown, the model equation *tells* that dividing the product by the known factor will solve for the unknown factor. For MC problem solving, when *the compared set* or the product is the unknown, the model equation "unit × multiplier = product" (see Fig. 6b) *tells* that multiplying two factors (*referent* unit and the multiplier) will solve for the *compared set* or the *product*. When the *referent* unit is the unknown, the model equation *tells* that dividing the *product* by the *multiplier* or *scalar* will solve for the unknown quantity.

Findings from empirical studies in COMPS (e.g., Xin, 2008; Xin et al., 2008; Xin, Zhang et al., 2011; Xin and Zhang, 2009; Xin et al., 2012), as illustrated above, indicate that elementary students with LDM can be expected to move beyond concrete operations and toward thinking symbolically or algebraically. Algebraic conceptualization of mathematical relations and model-based problem solving can be taught through explicit and systematic strategy instruction. This was evidenced by participating students' creation and articulation of their own story problems with a particular problem structure (Xin, 2008). To a larger extent, introducing symbolic representation and algebraic thinking in earlier grades may facilitate an overall smoother transition from elementary- to higher-level mathematics learning and improve middle and high school mathematics performance.

Practical Implications and Future Directions

The problems included in the above empirical studies (refer to Tables 2 and 3) represent "the most common form of problem solving" (Jonassen, 2003, p. 267) in elementary school mathematics curricula. Learning to solve variations of these word problems is the basis for solving more complex real-world problems (Van de Walle, 2004). Given the generalized mathematical models for the additive and multiplicative problem structure (see Figs. 5 and 6), a range of arithmetic word problems involving four basic operations can be represented and modeled. In addition, the COMPS (with the assistance of *WP story grammar* in representation) emphasizes symbolic or algebra expressions of mathematical relations in model equations that directly link problem representation to solution; it has the potential to innovatively bridge the gap between learning arithmetic and algebra. In fact, this "bridging" has been emphasized in Chinese elementary mathematics textbooks for decades (Ding & Li, 2014).

To promote model-based problem solving that is built upon students' existing knowledge base, further endeavors to enhance the COMPS program intend to address students' transition from an intuitive understanding of concrete models to a more abstract level of understanding. In fact, the COMPS program has become an important part of two cross-disciplinary research projects (Xin, Tzur, & Si, 2008; Xin, Kastberg, & Chen, 2015) supported by National Science Foundation, which aims to create intelligent tutors to nurture students' mathematical reasoning and to facilitate model-based problem solving.

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Reaction: Students, Problem Posing, and Problem Solving

Jeremy Kilpatrick

This chapter summarizes and responds to the content of the chapters in this section, all of which deal with students and their perspectives on problem posing and problem solving. Issues related to learning mathematical problem solving that are dealt with in the chapters concern addressing the cognitive demands that problems make, helping groups of students work on problems, promoting inductive and analogical reasoning, dealing with learning difficulties, and using information technology in solving problems. Issues related to learning how to pose mathematical problems concern helping students represent problems and formulate related problems. Each chapter in the section provides rich ideas for future research.

Although they almost inevitably touch on questions of teaching, the first seven chapters of Part 2 focus primarily on students' learning of problem posing and problem solving. The students whose thinking and learning the chapters discuss range across the school grades, working on problems that vary from one-step arithmetic word problems to challenging investigations of mathematical patterns. The opening chapter, by Cai and Lester, offers six suggestions from research that are intended to help those students become successful in problem solving. As Cai and Lester note, the research literature on mathematical problem solving is vast but also incomplete and poorly linked to practice, with little agreement in the field as to how problem posing and problem solving ought to be handled in the mathematics class. Their survey attempts to delineate some generalizations from the research literature that have practical value for instruction. The chapter makes a good introduction to the section by raising themes with echoes in the remaining chapters.

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Learning to Solve Problems

Every chapter in the section addresses some issues that arise when students are learning to solve problems in their mathematics class. Some chapters focus on strategies they might use, others on efforts teachers might make to help them with that learning. The following remarks address some of the many issues raised in the section.

Addressing Cognitive Demands

Cai and Lester point out that the mathematical problems that students are given to solve should make a variety of cognitive demands on them so that they learn different ways of making sense out of problematic situations. Many teachers, at least in the United States, however, commonly find various ways of reducing the cognitive demands of the problems they pose so that their students will be more successful. Cai and Lester offer some criteria for evaluating "worthwhile" mathematics problems and some suggestions for raising the cognitive demand of a task, or at least keeping it reasonably high.

In their chapter in this section, Ambrus and Barczi-Veres report a study of "average" Hungarian students, in contrast to their more mathematically talented peers for whom Hungary is well known. The authors explored some ways to address what they term the *cognitive load* of the mathematical problems the students were solving. After a brief introduction to cognitive load theory, they report a teaching experiment in which cognitive load was reduced by having students work cooperatively in groups; setting open-ended problems; and providing worked examples and the sorts of "guiding questions," or heuristic questions, proposed by Pólya (1945) and Schoenfeld (1992). In this study, the goal was to reduce the cognitive demand in a constructive fashion that would allow the students to address a problem without becoming overwhelmed by its demands. Even though Cai and Lester reported that the research literature claims that "teaching students to use general problem-solving strategies and heuristics has little effect on students' being better problem solvers," it appears that the selected, more specific questions in the Ambrus and Barczi-Veres study helped lead to improved problem-solving performance on the part of the participating students.

Working Together

The group work that Ambrus and Barczi-Veres promoted was seen by the students as contributing to their performance:

> Many students mentioned that one of the main benefits of working together was that students with different ways of thinking were able to find the solution easier—which refers to the reduction of the cognitive load of individual students.

Not all students in the study, however, found the group work beneficial, and in particular, some "better achievers" found it boring. The authors' conclusion seems justified: "All in all, the students' comments suggest that teachers should use cooperative teaching but should be careful not to 'overuse' it."

The chapter by Pehkonen, Ahtee, and Laine focuses on the evaluation of the emotional atmosphere in classrooms where so-called open problems are being used. Students' drawings provided the stimuli. As part of that evaluation, Pehkonen et al. looked at what they termed the *type of work* being done: teacher centered or student centered. They were able to classify most of the drawings according to whether they showed students working independently, in groups, or with the teacher in charge. The students appeared almost never to be working in groups. Although they appeared to be working independently twice as often as working with the teacher in charge, Pehkonen et al. concluded that the teacher centeredness of a lesson was not something that could be definitively assessed from the snapshot provided by students' drawings. The drawings give information about the classroom's emotional tone and students' feelings and attitudes, but the drawings clearly need to be supplemented by other data.

In his chapter, Fritzlar reports work done by pairs of students working together to solve what he calls an *exploratory problem*. The problem allowed students at the end of primary school or beginning secondary school to engage in the kind of inductive reasoning promoted by mathematicians such as Pólya (1954). Exploratory problems provide, as Frizlar says, "mathematically rich situations whose processing can be characterized by the following: exploring examples ideally with regard to self-derived questions, gathering and analyzing data, constructing relations or patterns and conjecturing and verifying hypotheses."

Reasoning Inductively and Analogically

Fritzlar's attention to inductive reasoning is echoed in the chapters by Cai and Lester and by Ambrus and Barczi-Veres. Cai and Lester emphasize the value, for even very young students, of "exploring problem situations and inventing strategies to solve the problems." Students need to encounter problem situations in which there is no standard algorithm, so instead they must try out various approaches to see what works. Ambrus and Barczi-Veres, by using open problems and guiding questions, promoted a "bottom-up" inductive method of problem solving. By using worked examples, the authors were encouraging the students in their study to reason inductively, looking across problems to get a better sense of what makes for similar solutions.

In their chapter, Degrande, Verschaffel, and Van Dooren emphasize what they call *quantitative analogical reasoning*, that is, "looking for a mathematical relation between two magnitudes that are given in a word problem, and applying this relation to a third given magnitude." Their focus is on proportional reasoning, but by expanding that focus to include reasoning that involves other relations—in particular, additive reasoning—they are able to provide a more nuanced picture of the thinking of students when they encounter word problems. The chapter is concerned with what the authors term a *modeling disposition* in solving proportional reasoning problems: Are students sometimes undeservedly successful in solving such problems because the numbers in the problem suggest the needed arithmetic operation, or can their attention to those numbers actually demonstrate an "initial but important" modeling disposition? The answer, shown in two carefully designed empirical research studies, is both. The moral appears to be that as students progress from the additive reasoning, they tend to show as third graders in solving missing-value problems, whether additive or not, to the correct use of multiplicative reasoning they tend to show as sixth graders, and in the intermediate stage, they rely on the numbers in the problem as indicators of which operation to perform. Only gradually do students learn to make use of the mathematical model underlying a word problem.

Dealing with Learning Difficulties

As already noted, Ambrus and Barczi-Veres studied ways of improving the problem-solving performance of average students, some of whom undoubtedly had learning difficulties of various types. The chapter by Xin addresses the issue directly, exploring the use of computer-assisted mathematics intervention programs for students identified as having learning disabilities or difficulties in mathematics. Xin has developed what she terms a "Conceptual Model-based Problem Solving (COMPS) approach that aims to promote elementary students' generalized word problem-solving skills." She reports studies of the effectiveness of the approach in helping students avoid such misleading strategies as focusing on so-called keywords and instead use modeling to solve various types of single-operation word problems in arithmetic.

Using Information Technology

The COMPS computer tutorial program developed by Xin is unlike any of the other programs discussed in the section, especially in its use of a mastery-learning paradigm. The program uses algebraic expressions to promote students' conceptual understanding of the problems posed as well as creation of their own word problems. In their chapter, Santos-Trigo and Moreno-Armella explore the use of *mathematical action technologies*—in particular, dynamic geometry software—to promote the solution of one problem in multiple ways. They illustrate their approach with multiple solutions to the problem of constructing an equilateral triangle given one vertex and the line on which the other two vertices are located. Both chapters illustrate the value of information technology in advancing the study of strategies and processes for solving mathematics problems.

Learning to Pose Problems

In her chapter, Xin reports her use of what she terms *problem representation* "to help students understand the problem structure and the mathematical relations involved (e.g., two parts make up the whole, or 'part + part = whole')." She would give students so-called word stories with no unknowns such as the following: "Jane had 34 crayons. Her sister, Sally, gave her 16 more crayons. Now Jane has 50 crayons." Then she would ask them questions such as "What is this story about?" and "How many crayons did Jane have at the beginning?" The questions would lead to the making of a model with bars to represent the quantities in the problem, and then the quantities would be inserted into a figure showing an equation with boxes representing the addends and the sum. After considerable practice making bar models and diagram equations for a variety of situations, students are then given part-part-whole problems to solve.

Although Xin's program does not make room for problem posing, it certainly could be modified to include that activity. Once students have learned to work with situations represented in problem representations, they could be asked to form their own word problems for a situation by introducing an unknown for one of the quantities. An alternative would be to give the students an equation with a missing term and ask them to make up a situation, and thus a problem, that the equation models. Cai and Lester point out that "writing story problems to match number sentences or posing mathematical problems based on situations are . . . the sorts of tasks that can engage students in learning important mathematics and develop their problem-solving abilities."

Santos-Trigo and Moreno-Armella illustrate how information technologies can be used not only to solve problems but also to formulate related problems by using such heuristic techniques as considering special cases or relaxing the conditions of a given problem (Pólya, 1945). As Cai and Lester note, problem posing is much less common in mathematics classes than problem solving is, and it receives much less attention in the chapters in this section than problem solving does. It deserves more consideration by researchers and teachers alike, and information technologies may make that possible.

Reflections

Although in this reaction paper, I have highlighted instances in which two or three chapters in the section addressed the same issue, many more issues were touched on by only one chapter. The theme of students, problem posing, and problem solving obviously has a multitude of aspects still to be addressed by theory, research, and practice. I noted at the outset that we now have a vast but incomplete literature on mathematical problem solving and problem posing, and there is obviously room for much more. Each of the chapters in this section can lead to important new work in countless directions. I hope that readers will see at least a few promising directions for their own work.

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Part III
Teachers, Problem Posing, and Problem Solving

When Is a Problem…? "When" Is Actually the Problem!

John Mason

Abstract Bill Brookes (1976) suggested that something is a problem only when a person experiences it *as* a problem. Ten years later, Christiansen and Walther (1986) suggested, following Vygotsky, that a *task* is what students are offered or inveigled to undertake, and *activity* is what happens as they attempt to carry out their interpretation of the task. Combining these, something or some situation is a problem only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity.

The principal issue then is what makes a situation problematic for some students and not others and what activates activity through having possible actions come to mind. Although *real problem-solving* and *authentic mathematics* are popular slogans, and although it is popular to try to make mathematics "real" for students by drawing on situations from the material world, the proposal here is that the issue is not about whether the situation arises in the material world, but rather, in alignment with the Realistic Mathematics Project at the Freudenthal Institute (*Developing Realistic Mathematics Education*. Utrecht: Freudenthal Institute, 1994), whether the situation can become "real" for students, enabling them to experience a "problem". A great deal can be said about the complex interaction between teacher's demeanour and vision, student initiative, and classroom practices, but it is not the focus of this paper.

The real issue in adopting a problem-solving approach to teaching mathematics is *when* to introduce exploratory tasks, *when* to intervene, and in what way. Thus it is the "when" that is the real problem for teachers, not the "what". There is of course no general theory which tells one how to act, but there are ways to prepare for action.

Title is taken from Brookes (1976).

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Since my interest is in lived experience and practical action, I take a phenomenological stance. Readers are invited to engage in a series of tasks, some mathematical and some reflective, through which they might consider conjectures about the origins and state of problematicity, about how and when students can be invited to engage with problems, about how and when a teacher might intervene, and why.

Introduction

I first became aware of *problem-solving* as a label for an activity when I arrived in the UK in 1970, after completing my Ph.D. in combinatorial geometry in the USA. But it took only seconds to realise that for me that was what mathematics is about: solving problems. As a teenager I posed (and sometimes resolved) my own problems, and I was a reasonably "Moore-oriented" student in that I preferred to work on problems myself rather than seek help, though when I was thoroughly stuck, I did also read other people's solutions in order to learn techniques (Jones, 1977; MAA Videotape, 1966). When I encountered George Pólya (1965) on film, then later in his books, I felt I had met a kindred spirit, and I was gratified to find I was also in alignment with Paul Halmos (1980), and later, with Schoenfeld (1985) amongst many others.

During my 55 years of teaching and 43 years of employment as an academic in mathematics and in mathematics education, I have seen the theme of *problem-solving* comes into focus as the centre of attention and then fade away again, perhaps four or five times. It seems to be back again, rising to the top of the curriculum agenda in many countries. Have we as a community actually learned anything over the years? It seems that each generation has to rediscover and restate in their own vernacular the insights that inform effective teaching, and this includes problem-solving. The book *Thinking Mathematically* (Mason, Burton, & Stacey, 1982/2010) was indeed a recasting of Pólya's sage advice into the vernacular of "processes". The new edition (2010) adds a chapter suggesting that the language of "students' natural powers" is now even more appropriate than the language of processes. In Mason (2012), I look back over his advice and recast it yet again.

My title is supposed to indicate an ambiguity, raising the question of when something actually becomes a problem for people (so that they initiate action to resolve it or avoid it) and proposing that the real "problem" in teaching mathematics is "when to initiate, when to intervene, and when to draw work to a conclusion". Of course, in parallel with "when", there is the question of "how". I eschew the notion of "how best", finding from my experience that there are always choices, and though some may be more effective than others in particular situations, there is no "best" way of doing anything in mathematics education. Often it is the very multiplicity which is powerful.

Quick Summary

There are three possibly fresh ideas that might contribute to the development of mathematical problem-solving in classrooms, which form the core of this paper and which are developed in it. First, there is a difference between reacting and responding, second there is the notion of appropriate challenge for particular students in a particular situation, and third, there is the notion that work can be suitably drawn to a conclusion with one or more conjectures together with evidence, as long as the inner task, the key ideas have been encountered and registered.

Reacting is an automatic somatic function, while responding involves cognition (Leron & Hazzan, 2006; Mason, 2009). We often speak of having an idea that *comes to mind* and of "being stuck" as a state of tunnel vision in which nothing seems to *come to mind*. Here "mind" is usually interpreted as cognition or "thoughts". But the human psyche consists of more than cognition, and it is well known from ancient psychology and from modern neuroscience that in response to a stimulus, the body (enaction) fires first, followed by emotion (affect) and then rather later by intellect (cognition) (Mcleod & Adams, 1989; Norretranders, 1998; Ravindra, 2009). Stimuli cause energy to flow into and through whichever selves (Bennett, 1964) or micro-identities (Varela, 1999) are dominant at the moment, and these activate acts and produce emotions characteristic of that particular self, which in turn enable or block thoughts including access to less routine actions and alternative emotions. Thus, it might be helpful to think in terms of *acts* that *come to action* (enactively), emotions that *come to the heart* (traditionally the heart is the seat of affect/emotions), and ideas, thoughts, or images that *come to thought* (cognitively). All of these contribute to the experience of *coming to mind*. Some suggestions will be made on how this perspective can inform pedagogic choices in the future.

When students are reacting spontaneously and out of habit, the teacher's role is to prompt them to withdraw from the action, emotion, or mind set and become aware of other possibilities. The intervention may be more sociopsychological than mathematical. The temptation to intervene mathematically (directing attention to some mathematical action or structural relationship) may lead to progress locally but may not be of much help globally when the student encounters another situation and reacts in the same way. The whole point of working on tasks is to learn from the experience, but one thing we do not seem to learn from experience is that we do not often learn from experience alone (Mason, 1992). Put another way, "succession of experiences does not add up to an experience of succession" which turns out to be a version of an assertion by James (1890, p. 628) that a succession of feelings does not add up to a feeling of succession. Something else is required, namely, withdrawing from the action and becoming aware of the habitual (re)actions and of other possibilities, what Schön (1983) called *reflecting in action*.

An enriched sense of "coming to mind" contributes a small part to the issue of "when and how to intervene", which is an immense topic well beyond the scope of this chapter. In order to intervene effectively, it is necessary to have some way to speak to the experience of learners. Questions and prompts need to bring to learners' minds, in the full sense, possible actions, positive emotions, and insightful thoughts. As an example, a lesson learned at the Open University over several years, and manifested in the *Thinking Mathematically* (Mason et al., 1982/2010) but often overlooked, is that instead of dropping students into extensive *investigations*, *explorations*, or *open problem-solving*, it is vital first to initiate them into mathematical thinking so that they have access to mathematical acts that can actually help them when they get stuck, whether when working on new ideas, when exploring, or when attempting exercises and assessment tasks.

One effective approach is to give students tasks in which their natural response (if not reaction) is to act in some desirable manner. Their attention can then be drawn to this self-initiated act, a label can be given or negotiated, and then that label can be used in a scaffolding-and-fading form (Love & Mason, 1992; Seeley Brown, Collins, & Duguid, 1989) to inform interactions with the teacher and with mathematics over a period of time, until students have internalised the action as something they can initiate for themselves when stuck. According to van der Veer and Valsiner (1991), Vygotsky's notion of the *Zone of Proximal Development* describes a state in which a learner is on the edge of being able to initiate acts for themself which previously had to be initiated or cued by a relative expert. By fading the directness of prompts, students can be encouraged to internalise and integrate acts into their functioning. In this way learners can develop a repertoire of appropriate mathematical habits, whether using their natural powers (Mason et al., 1982/2010) or as delineated by Cuoco, Goldenberg, and Mark (1996), for example.

Method of Enquiry

In this paper, I want to summarise what I have learned about supporting other people in their problem-solving, or as I like to describe it, "fostering and sustaining mathematical thinking in others" (Mason et al., 1982/2010). My preference is for a somewhat extreme form of phenomenology: offering a series of task exercises. What you get from them is what you notice about yourself (or selves), about how those selves characteristically channel energy into the mathematical use of your natural powers, and about how the different parts of your psyche work together or against each other. Thus the data being offered here will be your own experience. The products of enquiry are refreshed or fresh sensitivity to notice pertinent distinctions and to be guided by those to make informed choices in the future, together with your own modifications, refinements, and additions to the task exercises as you use them in turn with others (Mason, 2002).

It is impossible, however, to resist some commentary on the task exercises, making observations which may resonate (or dissonate) with your experience in such a way as to sharpen, extend, and enrich your awareness and making links with frameworks of distinctions which have proved fruitful for many in the past. Thus I permit myself to indulge in some theorising as background to your noticing.

Some Phenomena

Here are some descriptions of incidents that it might be useful to link to effective pedagogic strategies. In each case, try to find an example in your own experience:

Given a task, some students immediately do the first thing that comes to action.

Given a task, some students immediately decide that they cannot do it (the first thing that comes to the heart); progress is blocked by their affective reaction (see Dweck, 2000 for ways to overcome this).

Given a task, some students wait until the teacher comes round so they can get more specific instructions as to what they are supposed to do (a form of funnelling).

Given a task, students ask "Why are we doing this?" or "When will I ever need to know/do this outside of school?"; this may be a desire for application, but more often it is a plea for help, a statement that "I cannot cope".

Given an extended or vaguely specified task, students unused to anything other than rehearsing recently encountered techniques do not know what to do and may resist taking any initiative, waiting until they are told "what to do".

These phenomena will be addressed in the theorising that follows the core of the paper, which is the data collection based on some immediate experience.

Data Collection

The key point about the following task exercises is to become aware of how you use yourself and of how your psyche reacts and responds, bearing in mind the adage that "A solved problem is as useful to the mind as a broken sword on the battlefield" (Shah, 1970, p. 119).

Task Exercise 1: Five Settings

TE1(a) What would happen if the minimum wage was set at a level judged to be a minimum living wage, or if the government rewarded employers who adopted this?

TE1(b) At the airport you find various stated rates for buying and selling currency, together with the statement that there is no commission for exchanges over some stated amount. For example:

$$\text{Euros Sell}: 1.224\ \text{Buy}: 1.32$$

$$\text{Rand}: \text{Sell}: 13.237\ \text{Buy}: 17.549$$

What is the actual commission at these rates?

TE1(c) When and where might you see a vertical half-moon?

TE1(d) What numbers can be expressed as one more than the product of four consecutive numbers?

TE1(e) Imagine a triangle. Imagine a circle exterior to the triangle and touching it at one point (tangent to it at a point). Allow the circle to roll or slide around the triangle, always keeping one point of the circle in touch with the triangle. What is the length of the locus of the centre of the circle as it moves once around the triangle?

Comment

TE1a requires extensive modelling and knowledge of economics and so is unlikely to attract interest, except for people with specific expertise or interest already. TE1b seems a little out of reach for most people to whom I have posed it and so attracts little interest, despite being of considerable importance when travelling between countries and obtaining relevant currencies. TE1c often attracts interest, but most people are unaware of ever seeing a vertical half-moon!

TE1d moves into pure mathematics. It usually provokes people to try some examples in order to see what might be the case or to write down some algebra and get totally stuck and then resort to specific cases. As such it generates immediate experience of the power of specialising, not simply to get some answers, but as fodder for detecting some underlying structural relationships.

TE1e calls upon imagining before drawing any diagrams and sorting out precisely the implications of the circle remaining in contact with the triangle at all times; the result is then relatively straightforward if slightly unexpected.

The important observation is that interest is (or is not) attracted: it is not the problem that is interesting but rather the person who is or is not, becomes or does not become, interested under particular conditions (at a particular time in a particular place and situation with a particular recent experience and with particular personal propensities and dispositions). Interest is a state of attention: active (taking an interest in …) or passive (being interested by…). The word “interest” is a way of describing investment of energy of a self, which may involve a letting go or backgrounding of other concerns and desires, perhaps even a different self to dominate, and allowing one’s self to be “touched”.

As most teachers and comedians know, interest needs to be aroused and attention attracted within the first few minutes of an encounter, though as students develop trust in their teacher and their teacher’s practices, that period can be extended. Furthermore, what appeals to someone depends on their current state (within a social group, perhaps) which in turn depends on their recent past experience.

One of the factors that affects interest, engagement, and motivation is having a sense of possible actions. I learned through experience at the Open University that starting with tasks that invoke natural powers, that is, natural mathematical actions, students can become aware of what to do when they feel stuck. Having developed a repertoire of actions makes the novel more interesting and reduces the blocking effect of feeling powerless. The next task exercise presents some of the tasks that we found useful at the Open University Summer Schools, but first, think back to your activity in response or reaction to the five tasks:

Did you specialise in order to make sense of any of them?
Did you take the opportunity to vary, extend, or generalise the tasks for yourself?

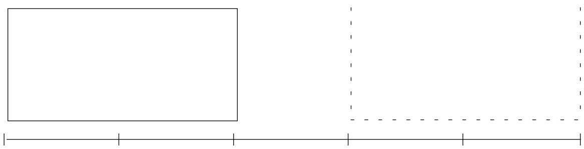
If not, then you overlooked the real contribution that the tasks can make to your future practice.

Task Exercise 2

The following tasks are taken from the Open University Summer Schools that ran from 1971 until 1996 (Mason, 1996). These took place in the middle of the first mathematics course. Again it is essential that the reader attempts each task and pays attention to the ebb and flow of energy in the form of immediate (re)action or response such as interest, surprise, tedium, and subsequent affect when some particular possible actions have been tried.

TE2a: Shifting Heavy Objects

Heavy objects like cupboards, armchairs, and settees can be moved by rotating them about one corner. Suppose we permit *only* 90° rotations, and we wish to move a settee (twice as long as wide) as shown:



What positions can you reach? Don't be satisfied with a yes/no answer: an explanation is wanted.
What happens if the settee is of different proportions?

TE2b: Products

Show that the product of any two numbers, each of which is the sum of two squares of integers, is itself the sum of two squares of integers.

TE2c: Differences

In a certain collection of objects, there are some which differ in colour and some which differ in shape. Must there be objects differing in both colour and shape?

TE2d: Number Patterns

1. I was using a calculator to subtract numbers from their square, when Pat, looking over my shoulder, turned to me and said:
   "I can get your answers by adding numbers to their squares".
   Is she right? Always?
2. What sorts of numbers arise from the sequence:

$$3 \times 5 + 1,\ 4 \times 6 + 1,\ 5 \times 7 + 1, \ldots$$

Convince me!

3. What sorts of numbers arise from the sequence:

$$3\left(5^2 - 4\right) + 1;\ 4\left(6^2 - 5\right) + 1;\ 5\left(7^2 - 6\right) + 1; \ldots$$

4. "Look at this!", said S.P.:

$$\begin{aligned} 10 \times 1 \times (1-2)^2 + 5 \times (1+1)^2 &= 1^4 + 29 \\ 10 \times 2 \times (2-2)^2 + 5 \times (2+1)^2 &= 2^4 + 29 \\ 10 \times 3 \times (3-2)^2 + 5 \times (3+1)^2 &= 3^4 + 29 \end{aligned}$$

"Aha!", said G.E.N., "The general pattern is … ", (he mumbled). Comment! Convince!

5. What sorts of numbers arise when you add one to the product of four consecutive numbers?
6. What sorts of numbers do you get from calculations like:

$$\left(4^2 - 1\right)\left(5^2 - 1\right) + 1;\ \left(6^2 - 1\right)\left(7^2 - 1\right) + 1;\ \left(8^2 - 1\right)\ \left(9^2 - 1\right) + 1; \ldots ?$$

Generalise? Convince!

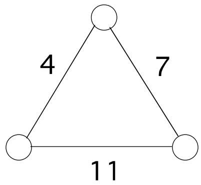
TE2e: One Sum

I have written down two numbers that sum to one. I square the larger and add the smaller. I also square the smaller and add the larger. Which of my results will be bigger: sometimes? always? (Conjecture, then convince.) Try using a diagram.

TE2f: Similitude

What shapes of paper have the property that they can be cut in half by a straight line to yield two pieces each similar to the original?

TE2g: Arithmagons



Hidden at each vertex of the triangle is a number.
The edge numbers are the sums of the numbers on adjacent vertices.
Can you reconstruct the vertex numbers? Can you find a quick rule of thumb to reveal vertex numbers that always works? Convince! Generalise to other polygons.

See Mason and Houssart (2000) for a history of variations to Arithmagons.

TE2h: Fare is Fair

I wish to divide 18 identical chocolate bars equally amongst 30 children. How many cuts *must* be made, and how many pieces *must* there be? Generalise!

These tasks display a range of mathematical domain, yet require very little in the way of technical knowledge. They are intended to bring students into contact with important mathematical themes and to direct attention to the use of their natural powers used mathematically (Mason, 2008).

Themes	Powers
Invariance in the midst of change	Imagining and expressing
Doing and undoing	Specialising and generalising
Freedom and constraint	Conjecturing and convincing
Extending and restricting	

Reflection

Since withdrawing from action and reflecting is of particular importance, it would be wise for readers to pause and ask themselves what powers of their own they used. What made you choose to stop work on each task? Was it when you could "see" your way through to a resolution, or was it as the result of increasingly negative emotions, perhaps connected with a sense of either excessive or inadequate challenge?

Comment

The range of tasks was designed to appeal to students in different states with different dispositions. Each task is intended to generate surprise or curiosity, releasing energy so that students would initiate action to try to find out more, and each was effective for at least some students. Most are intended to invoke the power to specialise to or use particular cases so as to locate, express, and then justify a generalisation. All of the TE2 tasks illustrate the fact that problem-solving is really the same as modelling (Lesh & Fennewald, 2010) in the sense that resolution comes about through finding some way to represent the information to oneself in a manipulable form. Searching for a suitable action to get oneself going on a problem is largely about finding something relevant to manipulate through which underlying structure can be encountered, expressed, and then exploited.

One thing we found, informally, at the Open University, was that how the tutor introduced the tasks made a huge difference. Of importance, but less so, was how they sustained activity, as was how they drew work to a close. This is the "how" and the "when" of intervention. We never tried to be prescriptive, though we did try to influence tutors by getting them to work together on similar tasks at their own level in order to sensitise them to what students might experience and for them to experience types of intervention that they might not have encountered previously.

There are many different ways to introduce a task, too many to summarise or survey here (see Mason & Johnston-Wilder, 2006 for a few). To illustrate some possible variation, consider the following, bearing in mind that simply reading the task may not provide sufficient experience to contrast the different ways in which a task setting can incite or block the flow of energy.

Task Exercise 3: Ride and Tie

Setting A

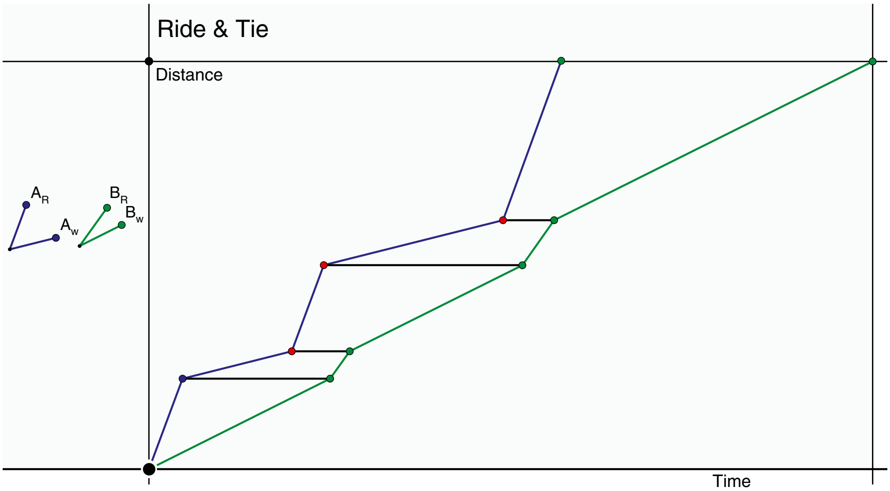
In a Ride and Tie race, two competitors and one horse cover a course as quickly as possible under various constraints (a minimum number of exchanges between walking and riding, minimum rest times for the horse each hour, at most one person on the horse at any time). All three have to cross the finish line to complete their run.

Setting B

In the eighteenth and nineteenth centuries, both in the USA and in the UK, people developed a means of sharing transport on a journey. For example, person *A* sets out walking while person *B* rides a horse. At some point *B* ties the horse and proceeds on foot. When *A* reaches the horse he/she mounts and rides on to a suitable place to tie the horse and again proceeds on foot. They alternate in this way until they reach their destination. Given the speeds of each when riding and walking, how do they minimise their collective travel time?

Setting C

Same as setting B together with: here is a proposed graph of such a journey, with the rates shown as slopes of arrows on the left. However, they do not arrive at the same time. How should they adjust their journey so as to complete the journey in the shortest possible time?



Reflection

What is the same and what is different about these tasks? What sort of recent past experience might be necessary, or be being appealed to in the choice of setting? What made you start thinking about the mathematics rather than about the difference in stimulation of the settings? What led you to stop work and carry on reading?

Comment

The third version of the task is designed to provoke students to reason with and about graphs rather than to treat graphs as the end product of some sequence of tasks. Trying to express the geometric relations algebraically is a bonus. But whether the task attracts, their interest depends on some extent on how the task is presented, hence the three presentations, by way of contrast. Setting *A* requires the reader to specialise by adding in imagined details and then to formulate a task, which may or may not be the task envisaged by the author. Students are expected to ask themselves questions such as how to minimise the time taken and still have *A* and *B* arrive together. Setting *B* offers details which may make it easier for a reader to enter into the situation. Specific details such as variables and relationships are still left open. If mental imagery is invoked, then, as in the *Realistic Mathematics Education* project (Gravemeijer, 1994), students are likely to find themselves involved, whereas if it is presented drily and at length, student interest is less likely to be attracted. Setting *C* focuses attention on a graphical presentation and its interpretation and as such forces specific values for the walking and riding speeds, but with an indication that these could be varied. Since getting students to interpret graphs is an overlooked aspect of graphical presentation, there is more likelihood that students will engage in this than in the other less specific, less focused settings.

Detailed Comment

The first thing to notice is that the graph does not work because the two people do not arrive at the same time, and if one arrives first, an adjustment to the timings could have reduced the time taken. However, it is actually worse than that, because according to the graph, the horse has to be in two places at once!

The second thing is to realise that for purposes of calculation, *A* and *B* need only ride once and walk once, since they can then divide up these periods to exchange more often and give the horse short rests rather than one long rest. Algebraic expressions can then be developed to find relationships to ensure that they arrive together and to calculate how long the horse has been allowed to rest, or in a dynamic environment, the journeys can be adjusted so as to bring them to the destination at the same time. Why will that minimise the time for all three to arrive? The walking and running speeds, together with the total distance, uniquely determine the arrival time when they arrive together and the total horse-resting time.

An additional feature of the core task is the opportunity to generalise firstly by replacing specific ride and walk rates by parameters, secondly by being aware that walking and horse riding could be replaced by any two modes of transport, and thirdly by extending the task to three people and two modes of transport, or beyond. As with any of the tasks, getting an answer to the task as posed is simply a way station in the exploration of a whole domain of related tasks. Thinking in this way helps prepare students for facing unknown and nonroutine tasks when being assessed.

Again it is best to attempt the following tasks, especially in regard to expressing a generality, for only then is it possible to appreciate what the task is offering students.

Task Exercises 4

TE 3(a) What is the same and what is different about the following facts?

$$37+28=38+27 \quad 95+42=92+45 \quad 65+32=62+35$$

Make up some of your own like these and generalise!
TE 3(b) Generalise the following facts:

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=1 \quad \left(1+\frac{2}{5}\right)\left(1-\frac{2}{7}\right)=1 \quad \left(1+\frac{3}{8}\right)\left(1-\frac{3}{11}\right)=1$$

What relationship is being displayed? What do these have to do with percentages, discounts, and taxes?
TE 3(c) Evaluate efficiently:

$$\frac{10\,000\times 10\,004-10\,002\times 9998}{10\,000\times 10\,001-10\,001\times 9999}=?$$

Reflection

Again, did you vary, extend, or generalise the tasks? If not, you overlooked an opportunity to express generality, lack of experience of which lies at the heart of student struggles in mathematics. What was appealing and what put you off? What led to you moving on to the next?

Comment

The first two are intended to invoke students' power to discern details, recognise relationships amongst what is changing and what is not, then express these as instances of a general property and thus to generalise, calling upon or re-encountering fundamental properties of numbers (what I would call mathematics; doing calculations with numbers is at best arithmetic). The second one might be recognised as the relationship associating fractional increase with subsequent decrease in order to return to the original state, and when expressed as decimals, interest and discount or percentage increase and decrease. The third one might provoke "parking" of the first act that comes to action (to calculate, or resistance to calculation) while looking for some way to ease the strain, namely, seeking relationships by treating 10,000 as a place holder and expressing everything in terms of it.

Tahta (1980) suggested that tasks have both an outer aspect (what you are explicitly asked to do) and an inner aspect (what you are likely to encounter, such as the use of your powers, pervasive mathematical themes, and problem-solving strategies).

There is also a meta-aspect (personal propensities and dispositions which can block possibilities of progress). Here the invitations to generalise provide a prompt to try to articulate perceived conjectured relationships and to want to verify that they do indeed always hold true. An inner aspect of the third is that sometimes complex calculations can be avoided by looking for underlying structure *before* calculating.

These tasks are only likely to be useful when students have already begun to develop their awareness (conscious and unconscious) of their own powers and of pervasive mathematical themes.

When, Then, Is a Problem?

When Brookes (1976) raised the question "when is a problem?", I was at first intrigued and then inspired. He pointed out that it is the person who experiences problematicity, who experiences some "thing" as problematic. School "problems" are simply ink on paper and, now, activated pixels on screens. They do not have attributes such as "real", "authentic", "difficult", "routine", "open", "closed", "interesting", etc. These are all attributes of the state of one or more people in response to some stimulus at some time in some situation and under some conditions. Unfortunately the transfer of attributes of a person's state to attributes of some stimulus or situation is a widespread phenomenon which leads to unfortunate conclusions about how learning comes about and how teaching can support and stimulate learning. Competent teachers are well aware that it is unwise to label the child and that it is much better to label the behaviour. So too with tasks: it is unwise to classify or label the task; it is much better to label the behaviour of particular people in a particular situation at a particular time.

As Ride and Tie may have indicated, how a task is proposed can make a difference in whether people find themselves experiencing "a problem", or not, and whether they find their interest attracted. But it is not a simple matter of classifying presentations, because what matters is an alignment between the setting, the situation, the conditions, and the participants. Pavel Shmakov and Hannula (2010) have found that the use of fantasy characters (crocodiles, comic characters) enhances the attractiveness of tasks in primary and early secondary. This is an example of appealing to the subculture of particular students at a particular time. Care is needed; however, adolescents, for example, may not enjoy attempts by adults to enter their world. Indeed, the whole role of schooling, as articulated by Vygotsky (1986, pp. 172–173), is to provide students with experiences that they would not normally have outside of school. This is the philosophy of the Realistic Mathematics Project (Gravemeijer, 1994), where "real problem-solving" is based on what students can "make real" through the use of their imagination.

How Then Do "Problems" Arise? How Is Interest Generated? How Is Attention Attracted?

For me the most important contribution to engaging students is to get them to make use of their own powers, because this releases small amounts of endorphins. To start with, getting students to imagine some situation already invokes their imagination. Getting them to express relationships that underpin the situation that "make it what it is" continues to make use of their powers by drawing attention to a phenomenon. The question then arises as to whether the phenomenon can be explained, and the desire to do this will depend on a combination of the extent of the surprise or uncertainty and the feeling that the challenge is within the scope of possibility for the student. The student has to trust in the teacher or feel for themselves that they have a chance of making sense mathematically; otherwise negative affect will trip in. I hope that readers will have a taste of this from the tasks offered earlier and of the range of ways in which energies flow from and through different selves.

Acknowledged as a state of an individual, problematicity arises when someone's stasis is disturbed. Heidegger (1927/1949) articulated what has been known for probably thousands of years: change and growth are a response to disturbance. Piaget (1971) drew on the biological terms *assimilation* and *accommodation* as ways of describing cognitive as well as enactive response to disturbance; Festinger (1957, p. 3) coined the expression *cognitive dissonance* to refer to cognitive disturbance that can initiate rethinking, and this was developed by Bell (1986, 1991) as *diagnostic teaching*; Bruner (1996, pp. 94–95) noticed that at the heart of a compelling narrative lies some disturbance; Shah (1970, p. 119) noted enigmatically that "sleep is to the hunter as excitement is to students", a *protasis* worthy of contemplation, seeking out multiple and not necessarily compatible interpretations in relation to response to disturbance. For example, preparation is vital (sleep), but falling asleep means you might miss your prey, and excitement in students means they might miss the intended inner tasks or fail to notice their transitions between various states. A similar protasis, also worthy of contemplation through seeking multiple interpretations, is "wounds are to a patient as assessment is to students" (Mason, 1992, 1998). Again it is only through disturbance that an awareness to act is initiated and, in this case, verified.

Movshovits-Hadar (1988) pointed to the surprise that underpins any mathematical result of significance but which becomes ordinary as expertise grows; Vergnaud (1982) proposed that problems are the source of the meaning of mathematical knowledge, but also that intellectual productions turn into knowledge only if they prove to be efficient and reliable in solving problems that have been identified as being important practically (they need to be used frequently and thus economically) or theoretically (their solution allows a new understanding of the related conceptual domain). Calling something a "problem" is therefore a shorthand for "a stimulus that provokes a disturbance" for particular people in a particular situation with particular recent experience.

In order to be disturbed, it is necessary to be in an appropriate state, that is, to have an appropriate self in the dominant role, to be prepared. A self-doubting self makes it harder to initiate action and may be used as an excuse for not acting; an overly confident self with tried and trusted actions may similarly block access to the novel, the creative, and the responsive. An Aristotelian *golden mean* seems desirable. Self-doubt and self-confidence arise from past experience (which contribute to the formation of one or more selves), from affirmation from "significant others" (Mead, 1934), and from having a repertoire of acts, emotions, and thoughts coming to mind (action, emotion, and thought).

When (and How) to Intervene?

The real question in working with others on mathematical problems is deciding when to intervene and in what way. Responses to the question of how and when to propose exploration, to intervene in exploration, or to draw exploration to a conclusion will depend on what exploration is considered to contribute to students' appreciation, comprehension, and understanding of mathematics. Put another way, it will depend on the intended inner task, and the intensity of desire that the inner task actually be realised.

For example, exploration can be seen as a motivational route into a topic, preparing the students to appreciate concepts and techniques which help resolve problems through their not being able to resolve problems initially for themselves (see, e.g., Burn, 2009). One view of tasks set to students is to prepare them to be able to appreciate and comprehend what is then expounded.

Exploration can also be seen as the principal means whereby students come into direct contact with concepts and techniques, and it can be seen as a powerful way to stimulate student sense-making of concepts and techniques to which they have already been introduced, placing the ideas in a more general context and appreciating the scope of the kinds of problems they can be used to resolve. Thus, when to introduce an exploration depends on a student's experience and the teacher's epistemological stance and reading of the particular situation (students, course, timing, importance of the concept or technique, etc.).

Most of the tasks presented earlier can be used in all three ways, as introduction and scene setting, as preparation for encountering fresh ideas or actions, and as review and consolidation of ideas already encountered. To be used effectively, the teacher needs to be aware of the affordances of a task, of the likely themes, powers, and concepts that are likely to arise.

There is, in my view, no "right answer" to the "when and how" question. Rather it is a matter of developing sensitivity to student experience so as to have possible actions come to mind (action, emotion, thought, and will) rather than reacting automatically out of habit. The greater the variety of possible actions that become available to the teacher, and, likewise, to the student, the richer and more extensive the repertoire, the more likely some effective action will become available to be chosen. The best way to develop sensitivity is to engage regularly in personal and collective mathematics to oneself and to spend time actually listening to and observing students.

Sometimes it is helpful to learners to let them become immersed in being stuck (Mason, 2014; Mason et al., 1982/2010); other times it is useful to offer them prompts which can be the subject of scaffolding and fading so that over a period of time, those prompts are internalised by learners; other times it is not relevant to attend to details on which someone might become stuck, but rather to work on developing an overview. To recognise different possibilities requires, in addition to a rich repertoire of mathematically based questions and prompts (e.g., see Watson & Mason, 1998), some awareness of broad mathematical themes and connections with other topics: what might be called *mathematical vision*, together with a suitable classroom ethos or atmosphere, that is, a mathematically based way of working on mathematics with others.

In the case of the Open University Investigation prompts in Task Exercise 2, tutors were free to intervene whenever and in whatever manner seemed appropriate to them. There was a wide variation in what tutors focused on and consequently in their interventions. For example, some took the view that students should be nudged or pushed so that they had some result before break (this was in a 3 h session with a break in the middle); others held back and only moved to plenary discussion near the end, when they would draw attention to various themes and problem-solving processes that had been described in course materials. Mary Boole (Tahta, 1972) captures the tutor dilemma beautifully in her phrase *teacher lust*: that welling up of desire to tell someone what you know. Sometimes it is appropriate, but often it cuts off student access to rich experience of coming to something for themselves. That is why it is vital to allow time for reflection, indeed to insist on occasionally "coming up for air" by withdrawing from the action to consider the effectiveness of that action, for students to reconstruct what they have done, what others have done, and what they would wish to do in the future, so that they genuinely learn from the experience. Tahta (1991) explores the influence of teacher desire that students learn and how that can sometimes support but other times block student development through influencing their disposition and inclination.

One approach to the "when and how" question is to try to live out the adage "try to do for students only what they cannot yet do for themselves". It is an admirable sentiment but difficult to live up to. When the end of time available approaches, it is tempting to succumb to teacher lust and tell students what they might have found. Although it can be frustrating to leave students at the end of a session, or even the end of a topic, without resolving all the uncertainties and misconceptions, it is not clear that "giving results correctly stated" at the end makes any difference to students: if it did, exposition would work much more efficiently and frequently than it does.

Mathematical Vision

Ball (1993) introduced the notion of *mathematical horizon* as part of what teachers need to be aware of in order to teach effectively, as do Guttiérez, Sengupta-Irving, and Dieckmann (2010) and Noss and Hoyles (1996). Of critical importance is awareness of how the components of a specific topic relate to or connect with other topics (see, e.g., the Structure of A Topic framework in Mason and Johnston-Wilder (2004a, 2004b)). One useful way to become aware of connections is through becoming alert to pervasive mathematical themes such as doing and undoing, invariance in the midst of change, freedom and constraint, and local and global characterisation of properties. Another is to be aware of specifically mathematical actions such as the vertical mathematisation (Treffers, 1987) move of isolating characteristic properties, abstracting these, and using them to define or characterise through generalising. Many people have tried to articulate what this might encompass, for example, Cuoco et al. (1996) talk about *habits of mind*, while Dyrszlag (1984) referred to *controlling* (in the sense of both guiding and assessing) students' understanding of mathematical concepts.

If mathematics is seen as a necessary drudge, being forced to work on it is likely to amplify feelings of negativity such as resentment, resistance, and dislike. If mathematics is seen solely as a domain of correct tools for getting correct answers, it is likely to appeal to those aspects of the human psyche with a preference for clear and certain rules. Emotional satisfaction is likely to be associated with correctness rather than with the engagement and creativity. If mathematics is seen as a human endeavour involving the use and development of human powers, it could come to be associated with creativity and release, with insight, and connection making. Using powers such as to imagine and to express what is imagined in words, gestures, drawings, and symbols; to specialise and to generalise; to conjecture and to convince; and to classify and to characterise releases endorphins which can provide a frisson of pleasure. The use of such powers is a way for selves to channel energies in positive directions. Encountering major mathematical themes such as doing and undoing; freedom and constraint; and invariance in the midst of change can provide the basis for making connections between otherwise apparently disparate ideas and again provide pleasure.

Conjecturing Atmosphere

Choosing when to initiate and when to intervene is situation dependent, and the dominant aspect is the ethos or atmosphere of the classroom, which can make a significant difference. Instead of dwelling on being right, on not saying anything in case it might be wrong, and on trying to "get there first", a conjecturing atmosphere in which everything that is said or done is treated as a conjecture can contribute to access to the unexpected, the creative, and the novel and to not being rushed on before one is ready. A conjecturing atmosphere is another way of referring to a classroom rubric (Floyd, 1981), the socio-mathematical norms and values (Yakel & Cobb, 1996) which inform and justify mathematical practices (Jeffcoat et al., 2004; Watson & Mason, 1998).

In a conjecturing atmosphere, you never tell someone they are wrong, merely "invite them to modify their conjecture"; the euphemism can make a significant difference. Those who are sure wait, check, and ask questions that might help others, rather than blurting out an answer; those who are uncertain take opportunities to articulate what they think they do know and to try to articulate what they are uncertain about. Learning is marked by the growing ease and articulateness of self-constructed narratives (Chi & Bassok, 1989; Chi, Bassok, Lewis, Reiman, & Glasser, 1989). Pedagogically, you learn little when always getting answers correct; you learn something when you make a mistake or an incomplete conjecture and then modify it, and you learn a lot when you make and then distrust your conjectures and work at reflecting on the process by which a conjecture was modified and justified.

Important aspects which make problem posing and solving more or less attractive for students include the milieu (conjecturing atmosphere), the way tasks are posed, the way the teacher intervenes in support, the relationship between teacher and mathematics, and the relationship between teacher and students, because these all influence the relationship between student and mathematics.

Paying Attention to Student Attention

The purpose of listening to and observing students is to try to enter their world, to see things from their perspective. Armstrong (1980) shows just how powerful this can be, and it is a common practice in many teacher education courses near the beginning. Unfortunately there is rarely time in a busy schedule to reinforce it by further close observation near the end of a course, when sensitivities and issues have been enriched, and the same is true in the first few years of teaching: there are apparently more pressing activities which displace this most fundamental of all ways to sensitise oneself to students. Davis (1996) goes further, coining the expression *teaching by listening*, and shows how powerful it can be to provoke students to try to resolve issues themselves, or at least to delineate the boundaries, before being presented with someone else's deeply considered approach.

Of course, what is observed is mostly what you are already sensitised to discern. Montaigne (1580, p. 960) put it beautifully: "Human eyes can only perceive things in accordance with such forms as they [already] know". Hanson (1958, p. 19) rephrased it as "There is a sense in which seeing is a 'theory-laden' undertaking", which Goodman (1978, pp. 96–97) extended to "[facts] are as theory-laden as we hope our theories are fact-laden".

Observing the ebbs and flows of one's own attention can prompt observation of how student attention changes, not simply in its focus, but in the nature or form of that attention. In parallel with van Hiele (1986) and in alignment with the structure of observed learning outcomes (SOLO) taxonomy (Biggs & Collis, 1982), some microchanges in the structure of attention can be discerned (Mason, 2003), including holding wholes (gazing), discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties. By sensitising oneself to these different forms of attention in themselves, that is, in their own experience, teachers can sensitise themselves to the movements of their students' attention and thereby gauge the pace and focus of their interactions with students. It also provides a framework for analysing student productions (Molina & Mason, 2009; Scataglini-Belghitar & Mason, 2012).

Another way of working on developing sensitivity to notice opportunities to act freshly rather than out of habit is to make brief-but-vivid accounts of incidents in lessons and then to compare responses with colleagues (Mason, 2002; Tripp, 1993). The website www.lessonsketch.org provides cartoons with classic teaching issues which can be used to stimulate discussion amongst colleagues from which can emerge fresh insights as to possibilities for action.

Choosing when to initiate and when to intervene is an art not a science; it depends on sensitivity to student experience achieved through sensitivity to one's own experience. By developing sensitivity to student experience, by working on the ethos of the classroom, and by seeking to enrich the range of possible actions as a teacher, students can begin to notice themselves developing a disposition to pose their own problems and to enjoy the pleasure that is available from working on challenging problems.

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Novice Chilean Secondary Mathematics Teachers as Problem Solvers

Patricio Felmer and Josefa Perdomo-Díaz

Abstract In this chapter we present a research on a group of 30 novice Chilean mathematics teachers as problem solvers. We study their performance while working on two problems, how they felt when they worked on them and how do they see as problem solver in a self-evaluation.

The analysis we present is part of a larger research project whose general objective is to explore relationships among (a) the opportunities that initial teacher training programs offer them to grow as problem solvers and as teachers able to promote problem solving in their class, (b) the mathematical knowledge of novice mathematics teachers as problem solvers, and (c) their pedagogical practices regarding the way they promote their students as problem solvers.

Keywords Problem solving • Problem solver • Mathematics teacher • Teacher training

Introduction

In modern educational practice, there is an increasing tendency to let students experience in classroom what experts do regularly in their work, in an important portion of their learning activities. These ideas that get roots in Dewey's learning approach (1933, 1938) were devised from the general perspective of philosophy, psychology, and education and propose something that is quiet obvious in music classes, where students are supposed to sing, play instruments, and sometimes even create songs, all regular activities of a professional musician.

This tendency has had an important manifestation in science through the perspective *inquiry-based learning* (Barrow, 2006) where students are invited to face research problems in the spirit of science, proposing hypothesis, getting evidence and analyzing it to confront the hypothesis, and reshaping the preestablished

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knowledge, obtaining new one. A long way has taken place since Dewey presented his ideas and there are still enormous challenges to make students practice science in classrooms. Barrow describes the evolution of the interpretation of *inquiry-based learning* and mentions teacher personal experiences with science as one of the difficulties this endeavor faces. Barrow presents the heart of the matter: before teaching, teachers should experience science by themselves. Preservice teachers should have their science courses where science should be experienced in a realistic way, where no recipe or script is previously known, and where guided demonstration is not considered an experiment. In the same way, in-service teachers that want to change their approach to learning should experience science by themselves; they should be involved in professional development programs that provide opportunities for making authentic experiments, before expecting to implement science in their classrooms. From the side of children, a vast study conducted under the auspices of the National Academies put together research on cognitive and developmental psychology, education, and history and philosophy of science to synthesize the current knowledge about how children learn the ideas of science (Duschl, Schweingruber, & Shouse, 2007). This study gave rise to a book with practical ideas about the implementation of science in school (Michaels, Shouse, & Schweingruber, 2008). It is interesting to notice that this report and subsequent book came out from a scientist organization, showing that these ideas make a lot of sense among scientists as well as among educators.

In mathematics there also exist the tendency of bringing mathematician's work inside school classrooms putting in place practices like representing, conjecturing, defining, arguing, proving, communicating, etc. In this sense, inquiry-based learning is one of the options of teaching in which students are invited to work in a similar way as mathematicians and scientists do (Artigue & Blomhøj, 2013). The dramatic call by Lockhart (2009) is an expression of mathematicians' agreement with these ideas that come together with the educational approach of learning through experience. Research about this tendency is getting more common and Hähkiöniemi (2013) on the use of technology to implement mathematics in the classroom and the work by Rasmussen and Kwon (2007) about the involvement of undergraduate students in differential equations through an inquiry-oriented approach exemplify the diversity of existing research. Another perspective that points to introduce mathematicians' work in the classroom is *modeling*. Mathematical modeling is a regular activity of an applied mathematician and its interest in education is ample and very active (e.g., Galbraith, Henn, & Niss, 2007; Stillman, Kaiser, Blum, & Brown, 2013). But probably the most developed and vastly spread way of introducing mathematician's work into classroom is *problem solving*. This regular activity of professional mathematicians involves also mathematizing (modeling), representation, reasoning and arguing, and communication of mathematics, all competences evaluated every 3 years in PISA international mathematics tests. *Inquiry-based learning*, *modeling*, and *problem solving* are not separate perspectives at all; their main difference is on the lens used. In this chapter we remain on the problem-solving perspective.

It was Polya (1957, 1966) who opened the way to numerous investigations and actions to rethink mathematics education, by setting his famous four steps to solve a problem. However, in too many cases, these steps were popularized and reduced to the extent that is considered a “model” or the recipe to solve problems and teach problem solving (e.g., Kilpatrick, 1987; Santos-Trigo, 2007). This idea of the great model to solve repeated problems may make us forget that Polya understood that in order to teach problem solving, teachers should have experienced problem solving by themselves. The importance of the experience on problem solving that teachers need for teaching has also been emphasized by many other authors. Mason (1992) pointed out how “teacher-proof” materials or guides have not been successful for teaching mathematics. It is necessary that teachers had struggled with a problem to appreciate the struggle that a student may have when solving a problem. In this line, we also quote Kilpatrick (1978), with a review on research and the central ideas of problem solving at that time, Schoenfeld and Kilpatrick (2008) that summarize the key elements for mathematics teaching, and Isoda and Katagiri (2012) describing the tradition of solving problems in Japan.

This idea of bringing mathematician’s world to classroom, to make students experience authentic problem solving as a learning activity and as a motivational activity, makes them experience the emotions a mathematician experiments. In front of a problem, the student, like the mathematician, will face the anxiety of not knowing the way the problem can be solved, the frustration of a failed strategy, and certainly the glory of triumph, the sensation of achievement, and the power of the victorious. But if we want this to really happen in classroom, we cannot forget the teacher that, in the line between the mathematician and the student, has to create the situation, has to propose the problem, and has to ask the questions or create the opportunity for students do it. Thus, the teacher should be just one of them; the teacher should have lived the same emotions while solving problems as mathematicians and students. Regrettably, a teacher passing through years of formation, first as a school student and then as a university student, preparing for being a teacher, had not experienced mathematics in the way mathematicians do.

The purpose of this study is to portray secondary mathematics teachers as problem solvers; we want to know how they solve problems, what they feel while doing it, and how they see themselves as problem solvers. Our aim is to center the attention on the teacher as problem solver, as mathematicians experiencing mathematics. An enormous amount of research has been directed to understand how students experience problem solving and also to pedagogical techniques and approaches for teachers to make students work in problem solving. However, not much attention has been given to teachers as problem solvers, that is, to look at them experiencing the emotion, the frustration and anxiety while working a problem, and the joy and triumph when the problem is solved. The few works we know on teachers are in professional development settings. In the first one, Chapman (1999) reports on six elementary teachers participating in a 30 h program, experiencing problem solving. As a result of the experience, teachers were significantly influenced on their personal meaning of problem solving and obtained a more positive view of themselves as problem solvers. In the second work we know, Yimer (2009) reports on 42 middle school teachers participating in a 2-week intensive refresher course where problem solving was the focus; problems with emphasis on fractions, measurements, and geometry were given to teachers to work during the course and a follow-up pedagogically focused came after for 2 months. As a result of this professional development program, teachers moved from imitating strategies to inventing them, and they developed the feeling that mathematical ideas are developed as a result of mathematical discourse within a community of problem solvers. In both cases, the teachers were considered as problem solvers and encouraged gradually to be like a mathematician, working in a problem not knowing the way to solve it. Our research differs from Chapman and Yimer ones in that teachers were not engaged in a professional development setting, but we just inquire on the way secondary mathematics teachers solve problems, looking for the abilities, feelings, and their view of themselves as problem solvers.

The work described in this chapter was conducted in the context of important developments in all aspects of education in Chile. Regarding school mathematics, Chile has experienced the introduction of problem solving in all areas of mathematical school curriculum in 2009, but with the recently approved new curriculum its status was raised as a clearly distinguished mathematical ability, together with representation, modeling and communication, and reasoning (Mineduc, 2012). This new curriculum puts enormous challenges to the system as a whole and it is very consistent with the ideas of bringing mathematicians' world into classroom. Teacher then is a focus of attention and many questions are posed, like, for example, what do we do to make universities provide opportunities to preservice teachers to experience authentic mathematics and how do we do so that in-service teachers experience the same?

Framework

This research is based on the idea that a *problem* is a mathematical task that a person tries to solve and for which that person does not have a straight and known way to solve it. To be a problem is not an inherent attribute of a mathematical task; it depends on the relationship between the task and the person who is interested on the problem and tries to solve it, the problem solver. In this sense, for being a problem, a mathematical task has to be difficult enough but not too difficult for the problem solver and with a different difficulty from just an operational one (Schoenfeld, 1985). Mathematical tasks posed during the Problem-Solving Workshop (PSW), the source for this research data, were selected having in mind that they could be problems for novice secondary mathematics teachers. Answers of teachers that participated in the PSW show that the proposed tasks were actually problems for them because they did not know a direct way to solve them, the solution could not be obtained straightaway, and difficulties were not just operational.

There exist different models to describe mathematical problem-solving process (e.g., Carlson & Bloom, 2005; Polya, 1957; Yimer & Ellerton, 2010). Most of them agree that problem solving is not a linear and unidirectional process, so different problem solvers can use different ways to solve the same problem correctly. Some of the questions a problem solver asks herself or himself are: How to answer the given question? How to find a strategy? How to know if a strategy will give the solution? How to know if the found solution is correct? How to know if it is the only solution? If there are more solutions, how many are there? How can they be obtained? Schoenfeld (1985) points out that the way persons solve a problem depends on their previous knowledge, heuristics, metacognition, their belief system, and the educative practices in which they have been involved.

In this chapter we are interested on knowing secondary mathematics teachers as problem solvers in a different perspective from that of knowledge and beliefs proposed by Schoenfeld. We are interested on behavior, feelings, and self-perception of novice secondary mathematics teachers relating with problem solving and how similar they are from mathematicians' ones. On the base of our view is the point that mathematics classroom must provide students the opportunity to experience mathematics as mathematicians, and teachers are in charge to make it possible. So, it will be desirable that mathematics teachers consider problem solving in a similar way than mathematicians do. To do this we have to consider how mathematicians behave when solving a problem and how do they feel during the process of solving a problem. Information about mathematicians' work solving problem is limited, and if we restrict to our focus on behavior, feelings, and self-perception, the situation is not better. One source is a book of Burton (2004) who talks about mathematicians as learners and the challenge that mathematics teachers have in translating to students the mathematicians' way of learning. She points out many interesting ideas for our research: the existence of different thinking styles among mathematicians and that every mathematician uses just one of them and the importance of heterogeneity of approaches and that intuition and feelings play a prominent role in the thinking and working of mathematicians.

In our view, a mathematician tries to solve a problem when he/she considers having enough knowledge to do it and he/she is interested or motivated in solving it for some reason. A mathematician tries different strategies and persists in looking for the way to the solution. When a mathematician finds a solution, he/she asks him-/herself if there are more and if there exists a better way to solve the problem. When a mathematician solves the problem, he/she understands the problem in a comprehensive way, and he/she may explain it to others. These ideas were the basis for defining the variables for the analysis of teachers' solutions to the problems proposed in the PSW (see section "Teachers as Problem Solvers: Performance").

Feelings when solving problems and self-evaluation of the process of problem solving belong to the affective domain, which includes beliefs, attitudes, emotions, and motivation (Hannula, 2012; McLeod, 1992). From these four aspects, emotions have been less studied than the others due in part to their nature (Pekrun, 2005). Study of emotions is important for several reasons: (a) emotions reflect success or failure of an individual with a task, so emotions inform about and to the cognitive and motivational domains (Hannula, 2012); (b) emotions are a principal component for decisions making (Schoenfeld, 1998); and (c) emotions are the basis on which attitudes and beliefs toward mathematics are built (Mandler, cited in McLeod, 1992, p. 578). Moreover, to know how a person feels, solving a problem informs about the relationship that such person has had with this type of task (Efklides & Volet, 2005). McLeod (1992) proposes five dimensions to characterize emotions: direction (positive or negative), magnitude, length, level of consciousness, and level of control of the person. In this research we only consider the direction of emotions that secondary teachers report after solving the two problems of the PSW (see section "The Teachers' Feeling While Solving Problems").

The process of self-evaluation includes metacognitive aspects and it is related with the belief system too. In the case of problem solving, a problem solver's self-evaluation is related with their beliefs about mathematics and about self (McLeod, 1992). In this research we are interested on secondary teachers' self-evaluation as a way to know how they see themselves as problem solvers and what they consider important in problem-solving processes (see sections "Participants and the Problem-Solving Workshop" and "Teachers Self-perception as Problem Solvers"). We consider that this point is especially important in the case of teachers because their conception of problem and problem solving would be transmitted to their students.

Participants and the Problem-Solving Workshop

This study is part of a research project whose main goal is the analysis of the opportunities that universities provide to future secondary teachers to experience problem solving and the use of problem solving at schools. The participants were 30 secondary mathematics teachers that finished their university studies, becoming teachers, between 2010 and 2011. They were selected from three leading universities in Chile, 10 from each one, chosen randomly within the universities. Participant teachers were invited to a Problem-Solving Workshop (PSW) that consisted on solving three problems (two individually and one in group), answering four questionnaires with questions about the situation experienced at the PSW and its relation with their university training and a final full group discussion about proposed problems, their feelings while solving them, and the opportunities they have had of experiencing something similar at the university. Those three were the only activities of the workshop.

Attending to the aim of this paper, we just take into account the two individually solved problems and two of the questionnaires. The two problems proposed to the teachers are the following:

The slot machine (Problem 1). Six friends played with 25 tokens in a slot machine. If everyone played a different number of tokens, how many tokens did every player play?

Cristina's cans (Problem 2). Cristina lined up her cans in two rows and one is left alone. Then she tried with three rows and with four rows and in both cases she left one alone. Finally, she treated with five rows and then no can was left alone! How many cans did Cristina have?

These problems were chosen among others because they could be solved with mathematical contents studied in primary and secondary school, they allow for different approaches and strategies to be solved, and they have more than one solution. In Problem 1, the solutions can be listed, while in Problem 2, a formula should be given to describe its infinitely many solutions. These characteristics make these problems appropriate for teachers to exhibit their familiarity with problem solving and their ability and training to solve them. Teachers had 30 min to work on the two problems. The idea of the PSW was to give teachers enough time to work on the problems freely and to display their knowledge so we can observe their mathematical behavior, even though there may be not enough time to solve both of them.

After working on these problems, teachers answered four questionnaires. For this study, we took four questions out of two of the questionnaires:

Q1: *How did you feel solving the problems*?

Q2: *If you had to evaluate your performance in the solutions of these problems*, *what grade would you assign to your work*? (*Give at least three criteria that you take into account for assigning yourself that grade*).

Q3: *For each of the following four topics*, *mark the reasons that may have lead you to assign yourself this grade*.

Topic	Yes	No	Justification
Personal reasons. *For example, I do not like this kind of problems/I got nervous*			
Contextual and/or environmental reasons. For example, short time/working place			
Reasons related to your previous experience. For example, I lack of practice with these kinds of problems/I am well trained with these problems			
Reasons related to your formation. For example, high school subjects/courses			

Q4: *Thinking about your process to solve the problems, assign to yourself a grade in each of the following items and justify.*

	Grade	Justification
Interest on the problems		
Familiarity with the knowledge related with the problems		
Search of different strategies in solving the problems		
Carrying out the selected strategies		
Certainty that the found solution is correct		

Teachers solved the problems on papers that we collected and they answered questionnaires in a Google Doc format. Thus the data to be analyzed consists on documents written by the teachers (on paper and online). In the following sections, we will discuss the way we analyzed the information obtained during the PSW through the instruments we just described and we present the results obtained. We dedicate one section to each of the research questions.

Teachers as Problem Solvers: Performance

In this section we analyze teachers' performance based on the written notes handed in by them after 30 min working on the problems. We consider some of the characteristics described in the framework about mathematicians' behavior when solving problems: to persist looking for the way to solve the problem, try different strategies, and ask itself if there are more than one solution. In relation with the reasons for trying to solve proposed problems, novice teachers try to solve the problem because it was part of the agreement for participating in the research. The point that when a mathematician solves the problem, he/she understands the problem in a comprehensive way and can explain it to others was not possible to be considered in the analysis of data because of time restrictions of the PSW. Based on this, written teachers' answers to each problem were analyzed according to five descriptors of teachers' performance: (*a) find one solution*, *(b) notice the existence of more solutions*, *(c) find more than one solution*, *(d) find all solutions*, and *(e) use more than one strategy*. In case of "*use more than one strategy*," we just consider the cases where to use more than one strategy were necessary, for example, if a teacher notices that a problem has more than one solution and could not obtain them with the strategy used.

In terms of the framework, only 10 % of the teachers reached the complete solution to Problem 1 and the same percentage was observed in Problem 2 (Table 1). All the teachers found one solution for the first problem but just 2/3 of them showed evidence of thinking about the existence of more than one solution and only the half of the teachers found more than one solution. In the case of the second problem, almost 25 % of the teachers did not find any solution; just a bit more than 10 % of the teachers' written work has evidence of looking for another solution and less than 10 % find more than one. Taking into account the nature of the mathematical conceptual knowledge needed to solve the proposed problems, we consider that this analysis suggests that an important number of teachers are not familiar with multiple solution problems. A little percentage of this group of novice secondary mathematics teachers behaves in a similar way than mathematicians do while solving a problem.

Table 1 Teachers achievement solving the two proposed problems

Teacher	Finding at least one solution	Noticing more solutions	Finding more solutions	Finding all solutions
Problem 1	30	20	16	3
Problem 2	23	12	8	3

We classified teachers in three groups: low, intermediate, and high, according to their performance in the two problems they worked out in PSW.

- A teacher was classified as having *low* performance if he/she at most noticed the existence of more than one solution in the two problems.
- A teacher was classified as having *intermediate* performance if he/she found more than one solution in one problem or he/she noticed that at least in one problem there are more than one solution and try to find them with more than one strategy, exhibiting certain mathematical flexibility.
- Finally, a teacher was classified as having *high* performance if he/she solved completely one problem, finding all the solutions, or if he/she found more than one solution in one problem and tried more than one strategy.

This classification takes into account that some teachers may have concentrated in only one problem, solving it and thus exhibiting good performance, and that a low performance is considered when the teacher did not do well in both problems. We use this classification to make some analysis with other variables, but we would like to recall that this is a descriptive analysis, without intending to make statistical statements. The terms low, intermediate, and high used in defining our categories have a meaning relative to our sample, and they do not reflect an absolute appraisal for teachers' performance. In Table 2 we display the teachers belonging to each category.

Although there are an important number of teachers in the high category (40 %), we consider that the number of teachers in low category is quite large (30 %), having into account that the selected problems can be solved with school mathematics knowledge and that teachers in this study have obtained their formation in leading universities in Chile.

Teachers at the Low Category. Teachers classified having low performance are those who did not obtain more than one solution for each problem or realize the existence of more than one solution in just one problem. There are six out of nine teachers in this category that did not even show evidence of thinking about the existence of more than one solution in one of the problems. The other three (P01, P12, P19) thought about the existence of another solution although none of them could find another one. Just one of them (P01) tried to use different strategies in one problem and realized the other has more than one solution. Figure 1 shows two examples of responses of teachers at this category.

Teachers in the Intermediate Category. There are nine teachers classified in this category, the same number as in the low category. Most of them (seven) were classified as intermediate just because in one of the problems they found one more

Table 2 Teachers' performance classification

Category	Description	Teachers	*N*
Low	Both grades below or equal to 3.4	01, 11, 12, 19, 25, 26, 27, 29, 30	9
Intermediate	Both grades below 5.8, one above 3.4	02, 03, 06, 07, 16, 17, 20, 21, 23	9
High	One grade above or equal to 5.8	04, 05, 08, 09, 10, 13, 14, 15, 18, 22, 24, 28	12

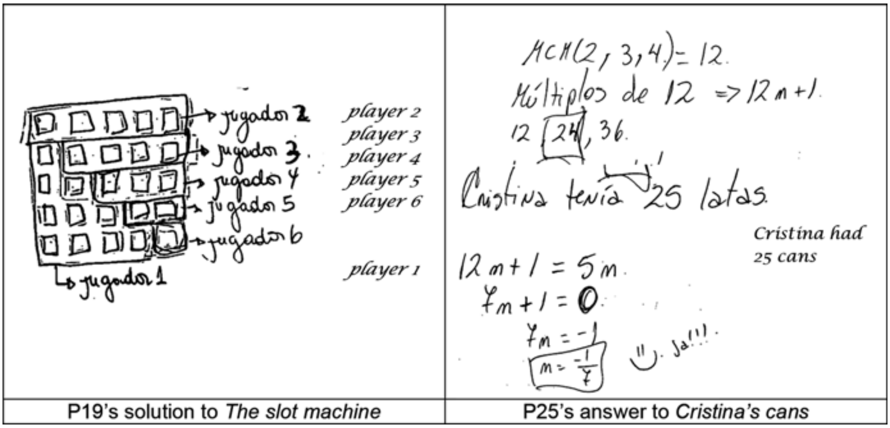


Fig. 1 Answers of low category teachers

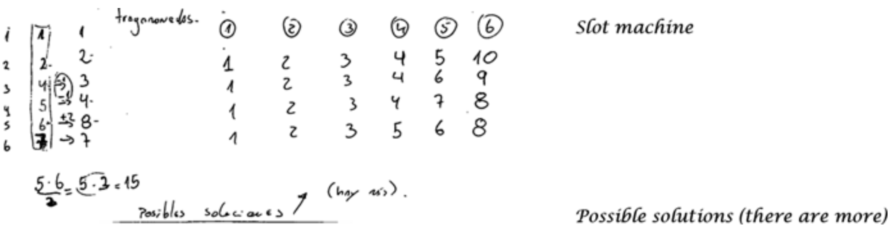


Fig. 2 P20's solution to *The slot machine*

solution, without showing interest on finding more than one in the other problem. This means that the proficiency to solve problems of the teachers in this category is not so different than that of teachers in the low category.

What really makes the difference between teachers in low and intermediate categories is that last ones showed some interest about the possibility that one of the problems could have more than one solution and try to obtain them with different strategies or obtained more than one solution with one strategy. Eight of the nine teachers got more than one solution at least in one of the two problems. Teacher P17 could not to obtain more than one solution, but she tried different strategies. Figure 2 shows an example of the answer of one teacher in this group.

Teachers at the High Category. In this category are teachers that completely solve one of the problems, finding all solutions, found more than one solution in one problem, and tried more than one strategy to obtain the other solutions. This is the largest of the three categories, with 12 teachers in it, but there are big differences in performance in the two problems, between teachers in this category. More than a half of teachers in this category showed a very low level or response in one of the problems, characterized by trying to obtain a second solution but do not reaching it. Two teachers had the maximum grade in the first problem and the minimum grade in the second one (P08 and P09), and according to their response to Q3, in the questionnaire, the main reason for the second grade was lack of time. As an example of this, we have selected teacher P05 answer to Cristina's cans problem. Teacher P05 started with a graphical representation of the situation and follow with a high-level mathematical concept: congruencies (Fig. 3). Although this teacher's answer included the correct general expression for all the solutions of the problem, we did not mark it with the maximum because the teacher hesitated on her solution and did not try neither verify nor look for another strategy.

We conclude with some remarks regarding teachers' performance and our view on what a problem is and how a problem solver should behave in front of a problem, as discussed in the framework. First, in view of teachers' achievement, we may say the two proposed tasks were problems. On the other hand, we see that most teachers did not behave as a mathematician, in the sense that many of them did not try various strategies while intending to solve the problems; they did not question their answer until they understand the problem completely, in particular, try to find all solutions of the problems. As part of an ongoing research, we are relating the findings on performance of teachers while solving problems with their teaching practices and their initial formation.

The Teachers' Feeling While Solving Problems

In this section we describe and analyze the information gathered during the PSW regarding teachers' feelings while solving the two proposed problems. As it was indicated in the framework of this research, feelings while solving a problem give information about the familiarity of teachers with problem solving in general and with the solution of the particular type of problems proposed.

As described in section "Participants and the Problem-Solving Workshop," we asked teachers to answer question Q1: *How did you feel while solving the problems?* They wrote an answer expressing freely about their feelings after having worked for 30 min on the problems. We isolated the main feelings expressed in their writing and we recorded the number of teachers expressing similar feelings, as shown in Table 3.

We observe the variety of feelings expressed and that positive feelings were more frequent. Next we addressed the questionnaire again and we classified teachers in

Table 3 Feelings expressed by teachers after solving problems

Positive feelings	No. of teachers	Negative feelings	No. of teachers
Challenged	11	Confused/doubtful/unsure	6
Motivated/enthusiastic	7	Uncomfortable/nervous	3
Entertained	1	Distressed	2
Quiet/safe	2	Disappointed/frustrated	3
Inspired	1	Disputed	1
Comfortable/at ease/well	6		

Table 4 Classification of teachers according to their feelings

Category	Value	Teachers
Positive	1	01, 02, 04, 07, 09, 18, 22, 19, 20, 21, 23, 25, 26
Neutral or ambiguous	1/2	05, 06, 10, 11, 12, 17, 24, 27, 28, 30
Negative	0	03, 08, 13, 14, 15, 16, 29

three categories depending on the type of feeling they expressed: we assigned (1) if the teacher clearly expressed positive feelings, (1/2) if the teacher expressed neutral feelings or if he/she expressed both positive and negative feelings, and (0) if the teacher clearly expressed negative feelings. With this information, we obtained the following summary (Table 4).

The following examples of answers given by teachers to question Q1 illustrate their expressed feelings and the way we assigned the value.

Examples of Positive Feelings

P19: It was very pleasant; the truth is that since a long time, I have not solved this kind of problems; I think not even in the university; it was a challenge to have found the solutions, through various strategies.

P20: Enthusiastic. I was totally concentrated and immersed in them. I wish to have given a better answer. I didn't realize as time passed.

P25: Each of the problems allowed me to develop a much more playful than structured resolution; so, I felt motivated to respond without the need for elaborated solutions, but a practical answer to the problem.

Examples of Neutral or Positive and Negative Feelings

P05: I felt challenged. I would have liked to have more time to reach to a more satisfactory answer; I did not manage to generalize my mathematical ramblings into mathematical statements.

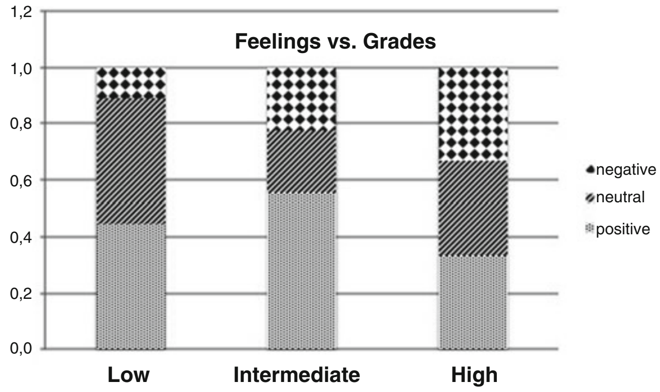
P30: In the slot machine, I felt more convinced about my answers. In the one on cans, I know the answer works, but I was not happy, since I did not find an answer, or better, an algorithm to use in the solution, but I just solved it by trials.

Examples of Negative Feelings

P08: I was frustrated because, despite of all my mathematical knowledge, I could not give a definitive answer to the problems. In any case, I guess this is the feeling students have many times, although in their case, much of the time; it is different because sometimes the needed information is there, but they lack the ability of seeing it, not of using it.

P15: Strange, one normally expects that the problem can be solved with more complex or higher-level mathematical objects; that's why I felt that perhaps other responses were expected from me. Anyway I know that this way of thinking is that often keeps us from mathematics itself when teaching.

It is interesting to see the feelings expressed by the teachers in each of the three categories defined from teachers' performance in solving problems (low, intermediate, and high). Graph 1 shows the proportion of each type of feelings for



Graph 1 Proportion of each type of teachers' feelings for each performance group

each category. It can be observed that teachers who expressed negative feelings are concentrated among those that obtained better grades (recall that the number of teachers in each category is 9–9–12). This may possibly mean that teachers with better grades are more critical than the others. In fact, three of the teachers (P08, P14 and P15) in the high category expressed negative feelings while they got the maximum grade in one of the problems. We would like recall the descriptive nature of this analysis.

Teachers Self-Perception as Problem Solvers

An important portion of our data is concerned with novice teachers' self-perception as problem solvers. This information mainly help to understand the familiarity of teachers with problem solving, complementing the rest of the information. Data analyzed in this section come from questions Q2 to Q4 of the questionnaires that teachers answered during the PSW. The information we gathered is the following:

1. A grade of self-evaluation of teachers' work solving the problems
2. What reasons (among personal, contextual, or environmental experiences and formation) they had to assign themselves this grade
3. A grade of self-evaluation of teachers' work solving problems regarding three criteria: search for different strategies, carrying out the strategies, and certainty of the solution

(1) *Self-evaluation as Problem Solver*. Teachers assigned themselves a grade (S-grade) between 1 and 7, as seen in Table 5. One of the teachers (P15) did not answer this question, so data are about 29 teachers. S-grade average is 5.6, with standard deviation 0.9. We observe that all S-grade are greater than or equal 4, where 4 is the approving grade in Chile. This means that all the teachers in our investigation consider that they did well enough as problem solvers. Moreover, more than a half of teachers' S-grade is 6 or 7 (17 teachers), that means that they consider that they do really well. Other 12 teachers' S-grades are between 4 and 5.5.

Table 5 Teachers' self-grade on solving the two proposed problems

Teacher	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15
Self-grade	6.0	6.0	6.0	6.0	7.0	6.0	6.0	6.0	6.0	4.0	5.0	4.0	4.0	6.0	*
Teacher	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Self-grade	5.5	4.0	5.5	6.0	5.0	6.0	4.0	7.0	5.0	7.0	6.0	6.0	5.0	7.0	5.0

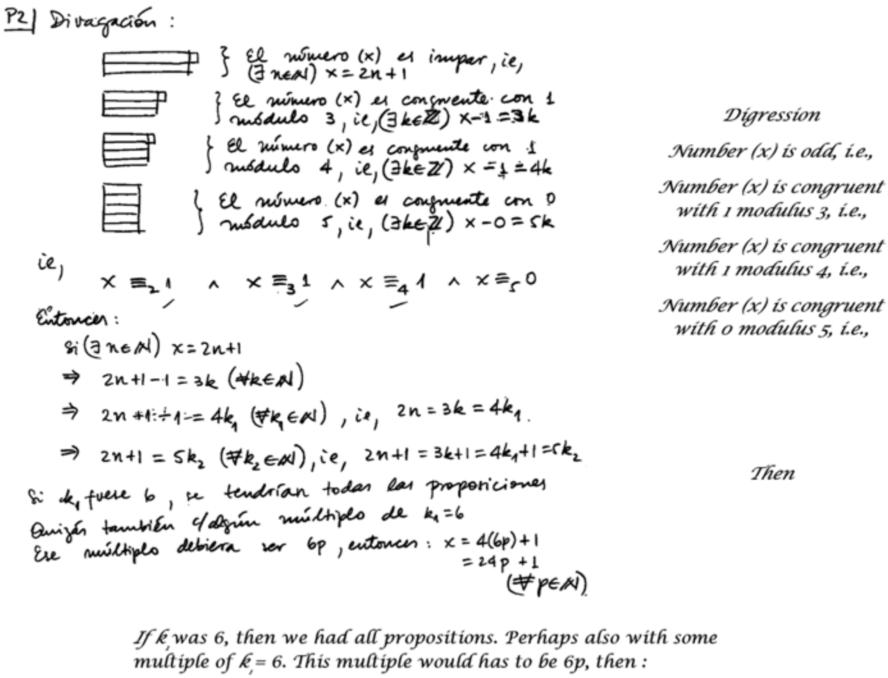
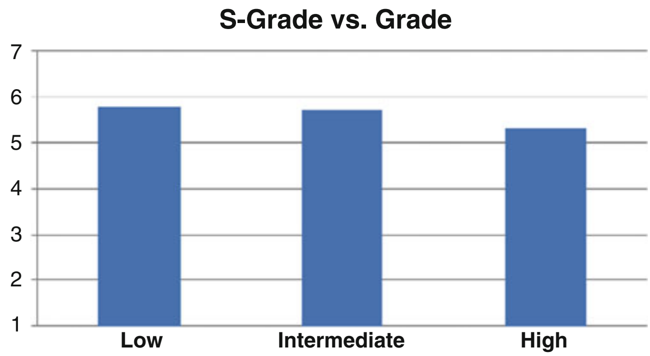


Fig. 3 Teacher P05's work on *Cristina's cans*

We would like to attract attention on those teachers whose S-grade is 7 (P05, P23, P25, and P29). Performance of two of them (P25, P29) was classified in the low category because they obtained only one solution of each problem and they did not show any interest about the existence of other possible solutions (e.g., Fig. 1). P23 is in the intermediate level, according to his performance, and P05 is in the high level, but both of them had a very low performance in the second problem, 2.2 and 3.4, respectively (e.g., Fig. 3).

If we consider the three categories of performance defined in section "Teachers as Problem Solvers: Performance" (low, intermediate, high) and compute the average S-grade obtained by the teacher in each class, we observe that *low* grade teachers evaluated themselves better than the teachers that obtained *high* grade. The teachers that obtained *intermediate* grade self-evaluated slightly lower than the *low* class, but definitely higher than the *high* class (see Graph 2). However, we would like to attract attention about the descriptive nature of this analysis.



Graph 2 Average teachers' self-grade in each of the three teachers' performance groups

Table 6 Reasons and directions for teachers' self-evaluation

Teacher	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Personal	1	1	0	1			−1		1	−1			−1	x	x
Contextual					−1	−1	−1	−1	−1	−1		−1		x	x
Experience	1	0	−1	1	1		−1		1					x	x
Formation		1	1	1	0		−1		1		1		−1	x	x
Sum	2	2	0	3	0	−1	−4	−1	2	−2	1	−1	−2	x	x

Teacher	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Personal		−1	x	1		−1	1	1		1	1	−1	−1	1	
Contextual			x		−1					1				1	
Experience	−1	−1	x				−1	1	−1	1		−1	−1	1	
Formation		−1	x			−1	−1	1		1		−1	−1	1	−1
Sum	−1	−3	x	1	-1	−2	−3	3	−1	4	1	−3	−3	4	−1

(2) *Reasons for Self-Evaluation*. We asked teachers about possible reasons that could have influenced their self-evaluation, related with personal, contextual, experiencing, or formation aspects (question Q3, section "Participants and the Problem-Solving Workshop"). Teachers had to say *yes* or *no* and explain their answer. For analyzing these data, we considered only those teachers that said *yes* in a given aspect and we assigned to each of them 1, 0, or −1 according to:

1: If the teacher reported that the reason affected positively
0: If the teacher reported ambiguous reasons, of being affected positively or negatively
−1: If the teacher reported that the reason affected negatively

A summary of the assignments given to each teacher appears in the following two tables. Empty cells correspond to *no* answers, which is interpreted as the teacher's self-evaluation was not affected by the given reason. Teachers P14, P15, and P18 did not answer the questionnaire (Tables 6 and 7).

Table 7 Resume of the reasons and directions for teachers' self-evaluation

Reasons	YES	NO	Positive	Neutral	Negative	Difference
Personal	18	9	10	1	7	3
Contextual	10	17	2	0	8	−6
Experience	16	11	7	1	8	−1
Formation	17	10	8	1	8	0

Table 8 Contextual reasons influencing negatively

P08: *Time was not enough*; *maybe if I had about 5 more minutes, then I could have explained in a better way the algorithm to determine the quantity of cans*
P09: Time was short, so I could not work on the two problems
P12: I had some ideas for the second problem; with some more time, I could have studied the situation better
P20: Personally, I would have liked to have 10 more minutes

Most positive personal reasons put forward by teachers are because they found the problems challenging, while negative reasons are related to not being motivated or to feel nervous with this type of activity. In the case of contextual reasons, eight teachers reported negative influence, but seven of them justified that on short time to answer the two problems (see Table 8).

About experience reasons, teachers that pointed out negative reasons said that they do not have enough practice with this type of problem and those that exposed positive experience reasons mentioned that they use to solve this type of problem and in some cases they propose them to their students. Formation reasons mentioned by teachers are mainly related with university courses; some teachers recalled secondary level experiences or extracurricular courses. Some examples of the reasons exposed by teachers can be read below. We select extreme examples from teachers that reported three or four reasons influencing negatively their S-grade (P7, P17) and those that reported three or four reasons influencing them positively (P4, P25).

Teacher P7 that assigned S-grade 6.0

Personal	*I worked in slowly manner*
Contextual	*I needed more time*
Experience	*For some time I have not worked in this type of problems*
Formation	*The realization and development of problems need to be addressed in a better form in didactic courses*

Teacher P17 that assigned S-grade 4.0

Personal	*I like these problems, they challenge me, but I cannot deny that I get nervous because I do not know what is expected from my answer*
	I assigned myself 4.0 because I feel that it has been long since I did not work with these challenges and I do not know what I am doing and if I am doing well or not

Contextual	–
Experience	*I do feel that I lack of practice in this type of problems*
Formation	*At this time the courses I have taken are not related to mathematical problems, but to curriculum and assessment*

Teacher P4 that assigned S-grade 6.0

Personal	*I like challenging problems because they test my skills and knowledge. In case I fail to achieve the results, it helps me as an alert to strengthen some aspects of analysis, for example*
Contextual	–
Experience	*I felt I was not short on resources to tackle the problems. The conviction on my knowledge allowed me to work with ease and confidence*
Formation	*I have had a rigorous mathematical formation, from mathematical reasoning to the mechanization of some processes. I think that helps me a lot when facing problems*

Teacher P25 that assigned S-grade 7.0

Personal	*I like problems that generate mathematical-logical reasoning, without the need to solve them in a structured and mechanical way, as when you solve an equation*
Contextual	*The space where the workshop was held was comfortable, no pressure or problems to develop problem solving in a quite manner*
Experience	*I feel that I have been trained in a good way in problem solving, working with case studies and problem-solving workshops in my university days*
Formation	*The possibility of attending mathematics meetings, colloquia and solving-days, allowed me to visualize the problems using methods similar to those used in earlier occasions*

(3) *Self-evaluation with Given Criteria*. At last we consider teachers' self-perception as problem solver with another form of self-evaluation, in this case with three given criteria that we think are important according to our framework (section "Framework"). Data analyzed here come from teachers' answers to Q4 (section "Participants and the Problem-Solving Workshop"), where they assigned themselves a grade considering their work in relation with (1) search for different strategies in solving the problems (S-strategies), (2) carrying out the selected strategies (S-working), and (3) certainty of the correctness of the given solution (S-certainty). Thus, we obtained three self-grades that illuminate the views that teachers have about themselves while solving the problems during the PSW. Here we have the summary of all grades (Table 9).

We first compute the average and standard deviation for each of the self-grades and compare them with the average grades obtained in each of the problems (Table 10). As we observed in the case of the S-grade, all other self-grades are much higher than average grades in problems, and with smaller standard deviation.

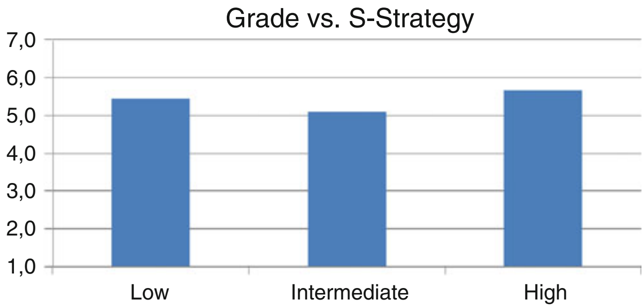
Table 9 Teachers' self-evaluation considering three specific criteria

Teacher	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15
S-strategies	6.0	6.0	5.0	7.0	7.0	5.0	6.0	5.0	5.0	5.0	6.0	5.0	7.0	5.0	6.0
S-working	6.0	7.0	7.0	7.0	6.0	6.0	6.0	7.0	7.0	5.0	6.0	4.0	5.0	7.0	3.0
S-certainty	7.0	5.0	5.0	7.0	7.0	4.0	6.0	6.0	7.0	3.0	7.0	6.0	5.0	7.0	4.0

Teacher	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
S-strategies	4.0	5.0	5.0	5.0	4.0	4.0	5.0	7.0	6.0	5.0	7.0	5.0	5.0	5.0	5.0
S-working	5.0	5.0	6.0	6.0	5.0	5.0	4.0	7.0	5.0	7.0	7.0	6.0	5.0	6.0	4.0
S-certainty	5.0	5.0	4.0	5.0	6.0	7.0	3.0	7.0	7.0	7.0	6.0	7.0	6.0	6.0	7.0

Table 10 Mean and standard deviation of teachers' self-evaluation

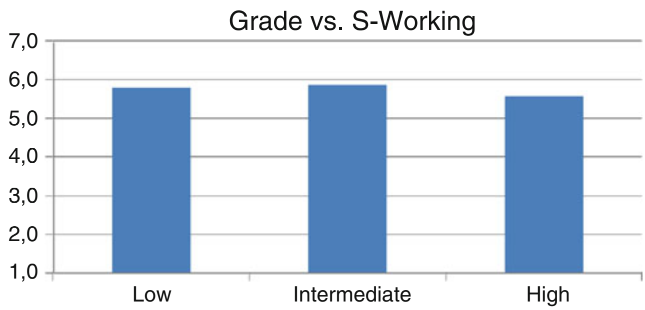
	S-grade	S-strategies	S-working	S-certainty	Problem 1	Problem 2
$\overline{x}$	5.59	5.43	5.73	5.80	4.20	3.24
σ	0.94	0.90	1.11	1.27	1.59	1.88



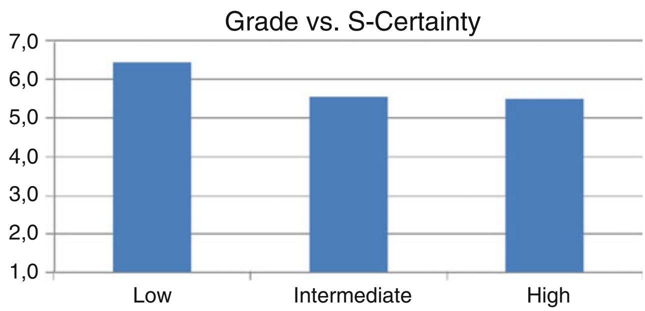
Graph 3 Teachers' self-evaluation in relation with the strategies used by teachers' performance group

A second analysis can be made comparing the grades of each teacher, that is, a measure of its performance in solving the two proposed problems, with the grade they assigned themselves with the three given criteria. In this case we used the classification of the teachers presented in section "Teachers as Problem Solvers: Performance": low, intermediate, and high according to the performance in solving problems. In Graph 3 we observe that the S-grade on the search for different strategies is lower in the *low* class than in the *high* class, which is consistent with the justification they gave for the S-grade. Graph 4 is somehow the opposite of Graph 3, but with less different between the three groups of teachers.

When we compare the grade with the S-certainty, we find that the teachers in the *low* class are more sure that their solution is correct than those teachers that actually belong to the *high* class, with intermediate S-certainty for the *intermediate* class (Graph 5). This situation can be further illustrated with the justification that teachers gave for their assignment of S-certainty. Here some examples, where we also provide the grade they obtained in problems 1 and 2 for completeness.



Graph 4 Teachers' self-evaluation of the way they carry out the selected strategies by performance group



Graph 5 Teachers' certainty self-evaluation by performance group

Teacher	Grades	S-grade	Justification
P08	7.0–1.0	6	*Seeing different situations I could "narrow" the answer and see that it was in a margin which I could develop widely*
P13	5.8–1.0	5	*In the first case I did not reach an elaborated conclusion; in the second one I think I have generalized, but I had the feeling that still there were conclusions to be drawn*
P14	5.8–7.0	7	*I knew it was correct, but I also knew that there were many others*
P16	2.2–4.6	5	*I am convinced about the solutions I found, which are consistent with the given information, but I think I missed others*
P21	4.6–2.2	7	*I am sure that, even if the road is not the "cleanest," the answers are correct*
P27	3.4–2.2	6	*I feel confidence of having reached the result*
P30	2.2–2.2	7	*I'm sure that the answers I gave are correct, since I checked and I got the expected result*

In relation with teachers' self-perception as problem solvers, the main focus of this section, we would like to highlight the high perception than they have about themselves, especially in the case of teachers in the low level. It is important to recall that teachers classified with low performance just obtained one solution for each problem, most of them did not think about the possibility of another solution and just one of the nine teachers of this category tried to use more than one strategy to solve the problems. Their average self-grade in the three criteria given by us is above 5.5 which is quite high in the Chilean system.

Final Discussion

In the context of a broader study, 30 novice teachers participated in a Problem-Solving Workshop, solving two problems, self-evaluating their accomplishment, and answering questions about their feelings while solving the problems. The chosen problems (*The slot machine* and *Cristina's cans*) could be well used in a problem-solving activity with school students; they could be considered nonroutine problems and their solutions are nonunique. One may expect that a teacher trained in problem solving or using problem solving in regular teaching activities would be familiar with this kind of problems, and their performance in solving them, self-perception, and feelings while solving the problems should be consistent with this familiarity. With this in mind we are going to make a final discussion about what we have observed.

The average performance obtained by teachers in the two problems is 3.72 in a scale from 1 to 7, while they self-evaluated their performance with 5.59, showing a quite high difference of 1.87 point. The reason for this difference could be a simple shift in perception of teachers; however, we see that those students obtaining lower performance self-evaluated their work with a higher grade than those obtaining a higher performance (see Graph 2). We think that this is an indication that there is a number of teachers that are not familiar with this type of problems; this is further supported with explicit statements of nine teachers when explaining the influence of their formation in the self-evaluation; they said they did not see these type of problems before or very rarely (see, e.g., P7 and P17 in section "Teachers Self-perception as Problem Solvers").

When looking with more details to the rubric used in grading the problems, we get more evidence of low or no exposure of some teachers to this type of problems. We notice that answers from 33 % of the teachers did not reflect any interest on having more than one solution for *The slot machine* problem. The same happens with answers from 60 % of teachers in Cristina's cans problem. Especially eloquent is the fact that the three teachers that found only one solution for each problem self-evaluated themselves with 7.0, 7.0, and 6.0, somehow not realizing the nature of the problems.

When we asked teachers to explain the influence of their experience in their self-evaluation, we found that 12 of them mentioned explicitly that they do not have enough practice with this type of problems or that they do not use them for a while (see, e.g., P7 and P17 in section "Teachers Self-perception as Problem Solvers"). This raises questions about what is the incidence of problem-solving activities in the classroom of these novice teachers and about the quality of the activities in case they use them. This last point is important since, even if a problem may have good characteristic to let student experience mathematics, the way it is managed in classroom may change completely its nature. The question is how teacher will react to questions like: How to solve the problem? How to find a strategy? How do you know if the strategy will lead to the solution? How to know if the answer is correct? If the solution is unique, how to find more solutions? How to know if all solutions were found?

Even though further research on the actual performance of teachers regarding problem solving in classroom is suggested by our findings. The evidence obtained in this study already call for teacher professional development actions if problem solving is going to be part of the teaching activities. This is especially important in view of the prominence that problem solving has in the current Chilean national curriculum.

On the other hand, if we take into account that teachers participating in our study are novice teachers, it is suggested that some actions should also be taken in the initial formation institutions. It is regrettable that students passing through years of formation at the university, preparing for being teachers, do not experience mathematics in the real way as mathematicians do. It is not about knowing mathematics, advanced mathematics; it is not about knowing pedagogical techniques for mathematics, or special training about implementing problem solving in classroom. It is about the experience of being a mathematician.

Here we come back to arguments by Barrow (2006) for science that will apply to mathematics, saying that it is necessary that teachers wanting to change their approach to learning should experience mathematics first hand; they should be involved in professional development programs that provide opportunity for carrying authentic problem solving, being authentic mathematicians, before expecting to implement mathematics in their classrooms. We conclude with a quite eloquent statement by Polya (1966) " … I shall not explain what is a non-routine mathematical problem: If you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed such tension and triumph in one of your students, look for another job and stop teaching mathematics."

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Infusing Mathematical Problem Solving in the Mathematics Curriculum: Replacement Units

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There are many reports on how problem solving is successfully carried out in specialised settings; relatively few studies report similar successes in regular mathematics teaching in a sustainable way. The problem is, in part, one of boundary crossings for teachers: the boundary that separates occasional (fun-type) problem solving lessons from lessons that cover substantial mathematics content. This chapter is about an attempt to cross this boundary. We do so by designing "replacement units" that infuse significant problem solving opportunities into the teaching of standard mathematics topics.

Introduction

In Singapore, mathematical problem solving has been established as the central theme of the primary and secondary mathematics curriculum since the early 1990s. The Singapore Ministry of Education (MOE) syllabus document states explicitly the importance of problem solving: "Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems" (MOE, 2007, p. 3).

Over the last two decades, mathematics teachers in Singapore have become aware of the importance of problem solving and in bringing the notion of heuristics and Pólya's model into their professional discourses. The success in promulgating mathematics problem solving is, however, limited. While there are many local research undertakings conducted within the field of mathematics problem solving,

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few studied the actual teaching of problem solving *in the classrooms*. In one such recent study, Teong et al. (2009) noted that when teachers were avowedly conducting problem solving lessons, only a narrow set of heuristics was reinforced for usually closed problems. In other words, "problem solving" is restricted to an activity separate from usual teaching of mathematics content and carried out mainly towards the end of a topic where "challenging questions" are encountered. This portrait of problem solving instruction hardly coheres with the vision of the centrality of problem solving as set forth in the intended curriculum. This mismatch is a common worldwide phenomenon: Writers who research in problem solving under different jurisdictions assert that, despite decades of curriculum development, problem solving instruction still requires significant improvement (Kuehner & Mauch, 2006; Lesh & Zawojewski, 2007; Lester & Kehle, 2003). Similarly, Stacey (2005) noted that problem solving remained "elusive."

In line with MOE's curricular goal, this is our project team's vision of problem solving instruction in Singapore classrooms: solving unfamiliar problems is a regular activity in the classroom; teachers provide scaffolds to help students not only to solve problems but also to make extensions beyond the original boundaries of the problems (i.e. carry out Pólya's Stage 4 on "Look Back"); instead of being a separate activity unrelated to the learning of usual mathematics content, problem solving is weaved into the instructional development of mathematics topics so that it is an integrated part of students' learning of mathematics; problem solving processes become unreflectingly the tools of choice when encountering difficulties with mathematics. We term the classroom realisation of this vision as *infusion* of mathematics problem solving. Infusion is one of the primary goals of our design research. In the remaining sections of this chapter, we explicate our design approach towards infusion.

We recognise that an endeavour so onerous—and erstwhile so elusive—as infusion of problem solving is a complex enterprise that needs to take into consideration a confluence of numerous design factors such as the nature of problems, the cognitive and affective orientation of students, and the repertoire of classroom practices that would support problem solving. We thus state at the outset that this chapter focuses only on the overarching theoretical, curricular, and structural elements of this enterprise as they were brought together in our design experiment. Nevertheless, we think that these broad-grained features are critical for infusion. We begin with the theoretical considerations underpinning the design experiment.

Infusion and the Conceptions of Teaching Mathematics Problem Solving

In considering infusion, we make reference to the well-known three conceptions by Schroeder and Lester (1989) which are still widely used in the literature (e.g. Ho & Hedberg, 2005; Stacey, 2005):

- Teaching mathematics *for* problem solving
- Teaching *about* mathematics problem solving
- Teaching mathematics *through* problem solving

We think that these conceptions remain useful as descriptions of still common enactments of mathematics problem solving and as such can serve as an appropriate starting point in clarifying our stance on infusion and in locating the problems associated with infusion.

The inquiry begins at the third conception which represents the final (and most challenging) hurdle of infusion in the classroom. Indeed, infusion certainly must involve teachers (ultimately) utilizing problem solving as a means to help students learn standard mathematics content. When this takes place, then problem solving truly becomes an activity that is tightly integrated into (instead of being separate from) the learning of mathematics. When teaching mathematics *through* problem solving takes place regularly, problem solving becomes an essential part of teachers' and students' conception of "doing mathematics."

However, upon closer examination and taking a curricular design perspective, the "through problem solving" approach may not *always* be the preferred way of teaching mathematics. Take the example of definitions. While it may be argued that even definitions can be "discovered" through suitable problem solving activities, it may not be the most appropriate course of instructional action as the teacher may want to concentrate the problem solving activity on the applications rather than the "discovery" of the definitions. In which case, the sensible approach would be to state the definitions with suitable examples and shift the emphasis on utilizing the knowledge of these definitions in problem solving. Moreover, due to the realistic constraints of curriculum time and the need to fulfil other instructional goals (such as helping students gain sufficient fluency with basic mathematics skills), not all mathematics can be taught *through* problem solving. Thus, while the third conception captures much of our vision of infusion, it does not equate to infusion.

The reality of classroom teaching in Singapore is such that teachers see it as their social responsibility to cover "problems" that appear in high-stakes or national examinations. These problems are the ones that are usually found at the end of textbook chapters (and thus, concomitantly, the end of an instructional unit). These problems are also usually tied to the content covered in the topic; as such, to "get to" these problems, teachers will need to help students learn the requisite mathematics content for solving the problems; in this sense, this practice is teaching mathematics *for* problem solving. For example, problems of this kind include: Given that the median is 6 for the data set: 3, 5, 4, 7, 8, 19, 11, x, state the minimum value of x. To solve this problem, it is clear that students need to first learn about "median" (and thus the need to teach it *for* problem solving). There is a tendency for teachers to immediately prescribe a technique to deal with this type of problems (followed by repeated practice of related "problems") as a matter of efficiency. However, this approach is described by Schroeder and Lester (1989, p. 34) as a "narrow" view of teaching mathematics *for* problem solving:

> [A] solution of a sample … problem is given as a model for solving other, very similar problems. Often, solutions to these problems can be obtained simply by following the pattern established in the sample, and when students encounter problems that do not follow the sample, they often feel at a loss.

This brings us to our conception of teaching *for* problem solving within the scheme of infusion. We do not challenge the practical realities to cover these problems usually found at the end of textbook chapters. When these problems arise, instead of always directly teaching the problem-specific technique, we think they are opportunities for students to attempt them as genuine problems. For example, referring back to the above problem on median, this could be a good juncture to allow students to explore the problem terrain better by, say, substituting values of x as a way to understand the problem before devising a plan and so on.

Thus, teaching *for* problem solving is to us about teaching students the mathematics content—the "resources" in the words of Schoenfeld (1985)—necessary to solve the later *unfamiliar* problems. In other words, the "problems" we have in mind are not exercises that vary slightly from earlier-practised exercises; they are problems in the usual understanding of it in the literature: non-routine and where the solution strategy is not immediately discerned. The mathematics resources learnt earlier are thus necessary but not sufficient to solve the problems. To be successful at solving these problems, the students need to not "feel at a loss"; instead, they are required to use heuristics to help them understand the problem and devise productive plans to move forward in their attacks at the problem. In short, the students need to access problem solving strategies like the ones advocated by Pólya (1957). This leads us to the place of teaching *about* mathematics problem solving.

To us, teaching *about* problem solving involves the explicit instruction of the use of Pólya's (1957) four-stage model in problem solving as well as Schoenfeld's (1985) developments of the problem solving framework. We will describe how these models are used in our design in a later section. At this point, we state our position that it is important to teach *about* mathematics problem solving prior to attempts at teaching mathematics *for* and *through* problem solving. Without problem solving skills, students take a long time to solve problems successfully. Thus, attempts to teach much of mathematics *through* problem solving, though ideal, has not been realistic given the limitations of curriculum time. We think that teaching *about* problem solving first as a separate module is a good investment of time in terms of the "returns" we may obtain later—as in, having learnt problem solving skills, students are more likely to make significant headway in a shorter time when presented with unfamiliar problems meant to help them learn mathematics content. In addition, teaching *about* problem solving introduces to both teachers and students a means or a language to talk about problem solving. The language introduced—for example, the language of "solve a simpler problem" —can then be more easily transferred and reinforced when solving other problems later.

We should perhaps clarify at this point that our conception of teaching *about* problem solving does not divorce the teaching of problem solving strategies from the teaching of mathematics content. In other words, in teaching *about* problem solving, the teacher does not teach problem solving processes devoid of content;

rather, teachers use problems containing mathematical conditions and requiring mathematical solutions. However, the focus is on using the problems and their solutions to *foreground* repeatedly the usefulness of the process such as the Polya's stages, not the other way round. As such, the primary goal in teaching *about* problem solving is thus the learning of the problem solving process and language.

This is how we conceive of the infusion process in relation to the three conceptions of mathematics problem solving: first, teach *about* mathematics problem solving in a separate introductory module to familiarise students with the language and tools of problem solving; second, within the standard mathematics curriculum time, provide regular opportunities to teach mathematics *for* problem solving (i.e. solve unfamiliar problems utilizing the mathematics content learnt in the topic), teach mathematics *through* problem solving (i.e. solve problems that will lead to learning content meant to be covered in the topic), and, in the process of solving these problems, revise and expand the tools to acquire *about* mathematics problem solving. The project entitled Mathematics Problem Solving for Everyone (MProSE) is a design experiment that seeks to study this infusion process.

MProSE Design Experiment

MProSE uses design experiment as the overarching methodological approach. Design experiment starts off with a clear set of product specifications—also known as "parameters"—to guide and evaluate the degree of success of the innovation. Guided by well-established theories, the process of design then undergoes iterative cycles of testing and refinement in localised conditions with a view of improving its fit to the parameters and its "transportation" potential to other relevant contexts.

Design experiment appealed to us in that it allows for the unique demands and constraints of the schools to be met. The methodology's advocacy of an implement-research-refine iterative approach to educational design appeared to us to hold potential in dealing with the complexity of school-based innovations. A design experiment can be described as the "creation of an instructional intervention [in our context, a problem solving emphasis in instruction] on the basis of a local theory regarding the development of particular understandings" (Schoenfeld, 2009). We based our design experiment on the methodology and terminology of Middleton, Gorard, Taylor, and Bannan-Ritland (2006).

MProSE parameters and brief justification (for details, the reader can refer to Quek, Dindyal, Toh, Leong, and Tay, 2011): (1) model of problem solving follows the theoretical basis of Pólya and Schoenfeld. The well-known cornerstones of Pólya's (1957) stages and heuristics as well as Schoenfeld's (1985) framework of problem solving are well accepted by the professional community. As such we seek to build on their contributions, focusing especially on the work of translating these models into workable practices; (2) mathematical problem solving must include the Look Back stage of Pólya's model. This point is really included in the earlier point under Polya's model but is emphasised here as, over time, "Look Back" has become variously interpreted. Mathematicians do indeed solve problems that they encounter; however, they do not stop at the solution of the immediate problem; rather, they use the solution strategy of the problem as a sort of kernel to generate solutions to related problems. Thus, it is this disposition of mathematicians with regard to problems—where they extend, adapt, and generalise problems—that we find essential to build into a school curriculum that seeks to inculcate mathematical thinking; (3) mathematics problem solving is a valued component in school assessment. What we see as the root of the lack of success for previous attempts to implement problem solving in the classroom is that problem solving is not assessed. Because it is not assessed, students and teachers do not place much emphasis on the processes of problem solving; students are more interested to learn the other components of the curriculum which would be assessed; (4) mathematics problem solving must be part of the mainstream curriculum. To "downgrade" mathematics problem solving to a form of enrichment or optional programme for students violates the value of mathematical problem solving; (5) teacher autonomy is important in the carrying out of problem solving lessons. While the beginning stages of innovation may include the involvement of expertise outside the school, ultimately, for the innovation to take root and sustain, teachers' capacity must be built to a point where they own the innovation and possess the ability to carry out problem solving lessons on a regular basis.

At the time of writing, MProSE is entering its sixth year as a design experiment, and in the process, we have undergone several iteration cycles from the original design. It is not realistic, given the constraints of space, to detail the full journey in this chapter. The focus of this chapter is to bring the readers to our current stance with regard to infusion. As such, we will briefly describe the first phase of MProSE infusion (the development of teaching *about* problem solving) and then discuss more substantially our current progress within the second phase of infusion (bringing problem solving into the regular work of teaching standard mathematics).

First Phase of MProSE: Teaching About Problem Solving

Based on the parameters, we designed a module on problem solving in which students are explicitly taught the language and strategies used in problem solving. This refers to the second conception of problem solving, that is, teaching *about* problem solving. The translation of design parameters into actual curricula features will not be discussed. For details, the reader may refer to Leong et al. (2011).

The entire module consists of ten lessons. The duration of a typical problem solving lesson consists of 55 min. Each lesson consists of two main segments. The first segment of the lesson involves the teacher explaining one particular aspect of problem solving (such as one of the four stages in the Pólya's model) and discussing the homework problem of the previous lesson. The second segment emphasises one particular mathematical problem that is illustrative in demonstrating that particular aspect of problem solving. Throughout the entire lesson, only one problem is highlighted in great depth. The students have to work out the problem on "the practical worksheet" guided by the instructions in the worksheet. The worksheet initially consisted of four pages, with each page corresponding to each of Pólya's stages.

The practical worksheet is an important part of the design as it is a tangible embodiment of the problem solving process for teachers and students. Its introduction into the classroom is meant to fulfil at least two roles: guide and reinforce the problem solving process along the lines of Pólya's stages and heuristics and signal a switch from other modes of instruction to a problem solving paradigm. Due to its practical importance in the overall infusion process, the practical worksheet has undergone a number of refinements in the course of the project. Figure 1 shows the compressed version of the three-page practical worksheet in its current form. For the detailed description of the evolvement of the module and the practical worksheet, readers could refer to Dindyal et al. (2013).

In this phase of teaching *about* problem solving, the module is taught separately from the usual teaching of mathematics in regular lessons. As our MProSE project has now moved to a juncture where there is evidence of stability in the implementation of the module, moving to the next phase of infusion—where problem solving is to be a regular feature in the teaching of standard mathematical content—becomes a natural progression. In the sections following, we highlight the progress and challenges in this next phase of infusion. In particular, we focus on the curricula and structural tweaks in response to challenges in teacher development for problem solving.

Practical Worksheet

Problem

I Understand the problem

Use some heuristics such as Draw a Diagram, Restate the Problem, Use Suitable Numbers, etc. to help you.

I have understood the problem. (Circle your agreement below.)

Strongly Disagree		Neutral		Strongly Agree
1	2	3	4	5

You may proceed to the next page to work out a solution/partial solution.

II&III Devise a Plan and Carry it out

a) State your plan clearly, for example: (i) Use Suitable Numbers and Look for Patterns; or (ii) Find the areas of all smaller triangles and work out their ratios.
b) Number each plan as Plan 1, Plan 2, etc.
c) Carry out the plan that you have stated.

Plan 1 Statement of Plan:

Carry out Plan 1

IV Check and Expand

a) Check your solution.
b) Write down a sketch of any alternative solution(s) that you can think of.
c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Fig. 1 Practical worksheet (compressed by removing spaces for writing)

Second Phase MProSE: The First Implementation

After students had been exposed to instruction *about* mathematical problem solving through the MProSE module, the next infusion step is to use the problem solving skills acquired regularly in the learning of mathematics content in usual mathematics lessons.

One of our key principles was that inclusion of problem solving in regular mathematics should help teachers improve in the teaching of mathematics. If the teachers were satisfied with existing ways of teaching a mathematics topic, then there was no motivation to switch to the problem solving approach. A logical step to begin would be for teachers and researchers to select a difficult topic to teach or a concept where students always made mistakes. In the process of applying the skills and strategies that students had acquired in the MProSE module, they need to solve the mathematics problems through struggling using problem solving—the exploration of which would help them understand the topic better when it is taught later.

We also strongly advocated the continued use of the practical worksheet whenever students are instructed to attempt problems in this phase. We think that the ten lessons in the earlier MProSE module, while substantial, are not sufficient yet in bringing about a habit in the students of applying Pólya's processes when confronted with mathematics problems. Through the followed-up use of the practical worksheet within the teaching of topics, there is continuity in the learning and application of the problem solving skills over an increasingly broad range of mathematics problems. In the process of struggling through problem solving, it was hoped that the problem solving process will become part of the students' learning habit.

In addition, we suggested to the teachers the following guidelines for implementation:

- Infusion problems are to be worked on a practical worksheet.
- Problem is to be worked on for exactly 1 h.
- An infusion problem is to be given as homework in the following situations:
  - On the last lesson prior to teaching a new topic, to prepare the student to constructively develop some feeling for the topic (preparation)
  - Within the span of a topic to allow the student, to explore the difficult nuances of the topic (exploration)
  - At the end of a topic, to consolidate his/her understanding of the topic (consolidation)
- These are the parameters for deciding on the suitability of an infusion problem:
  - Difficult enough to take at least 30 min.
  - Allows student to discover some aspect of the topic: for example, a technique taught is superior to other techniques, or a particularly difficult aspect becomes clearer after enough time is spent exploring it, etc.
  - Very amenable to expansion (Pólya's Stage 4).

We also supplied suitable problems for each of the “difficult” topics that the teachers brought up. We were always in close contact with the teachers through email and school visits.

Second Phase MProSE: Evaluating the First Implementation

In keeping with the features of a design experiment, we examined implementation of the plan in order to refine the design for the next iteration. We held regular meetings with the teachers to discuss pre-implementation details—the need to account for students’ affect, the suitability of the problems, and the instructional emphases intended for each problem—as well as post-implementation reflections. From these meetings, it became clear to us that teachers were facing a number of challenges with regard to their attempts at infusing problem solving into their regular lessons:

1. *Instructional goals behind problems*. Much time during the meetings was taken up to discuss the actual “location” of the problems in their teaching schedule. Questions that were addressed included: “Should it be given as an introductory problem at the beginning of the topic, or somewhere in the middle, or towards the end?” “Should the problem be done fully in class, purely as homework, or start as homework and completed in class?” On the surface, these questions appeared to be about the most natural or logical points to insert the problems along the content developmental track of the topic; upon deeper analysis, it revealed the teachers’ as-yet unclear instructional goals about what each problem can potentially fulfil. To illustrate this point, we review the meeting discussions over the “cat problem”: “5 cats take 5 days to catch 5 mice. How many cats will it take to catch 2 mice in 2 days? How long will it take 1.5 cats to catch 1.5 mice?” First, the teachers shared that they inserted this problem at different junctures in their teaching of the ratio/proportion topic—Karen did it as an introductory problem; Siva used it as a problem at the end of the topic; Mariam also used it at the end but she did only part of it in class and the rest as homework for students. Second, when asked for their reasons for their respective decisions, they brought up mainly considerations related to availability of time pockets for in-class problem solving, but not about the goals that we originally in-built into the problem, such as the opportunity for students to learn conceptual distinctions between direct and indirect proportion through exploring the problem terrain instead of through teacher’s direct telling. In particular, the teachers seemed unaware that, by putting the problem at the end of the topic where direct and indirect proportion were explicitly covered through numerous practice questions, the “problem” lost its problem status—unfamiliarity and thus the need to apply problem solving processes—to the students, rendering it more like a routine exercise to them. This went against the original goal of infusing “problem solving” into the topic.

2. *Affect-efficacy of the problems*. A number of the infusion problems were designed to account for students' affect in mathematics through problem solving. An example of the "multiple choice problem" would illustrate this better. The problem statement is: "A student had to take a test consisting of 100 multiple choice questions. Each correct answer is given 5 marks while each wrong answer will have 3 marks deducted. Unanswered questions are given 0 marks. The student attempted all the 100 questions and obtained 444 marks. How many questions did the student get wrong?" We expected students to find this problem accessible: substituting small numbers for correct and wrong answers can help them understand the problem easily; the familiar heuristic of "guess and check" can be utilised to obtain a solution to the problem. Thus, we anticipated that students would find success at solving it and thus have a positive emotional orientation towards the problem. In addition, teachers could use the "check and expand" stage to ask questions—such as, "what if the numbers in the question are changed? Is there an alternative method that can take care of such changes in the question more easily?"—to provide the motivational link to the topic of algebraic equations. However, during the meetings, the teachers shared that their students were generally not motivated to solve the problems. We thought that more concrete strategies in scaffolding students' attempt towards productive approaches would help teachers encourage more success in problem solving—a necessary ingredient for students' long-term buy-in to problem solving.
3. *Time consumption of the problems*. The teachers were very conscious of class time taken up for problem solving. And they were aware that meaningful problem solving takes time. [The Singapore mathematics syllabus is seen by most teachers as heavy content-wise. Coupled with the need to prepare students for high-stakes examinations, it is not uncommon for teachers to feel the constant time pressure to "cover syllabus" (e.g. Leong & Chick, 2007)]. They saw it as a dilemma: if they used up class time to do problem solving, it would reduce the already limited time to "cover syllabus"; if they left problem solving as homework (to free up class time), teachers would then not be at hand to help the students and it would exacerbate the problem of low levels of students' motivation at problem solving. We saw it differently: it was not a case of problem solving versus content coverage; as described in the cases of the "cat problem" and the "multiple choice problem," the problems could be used to explore content, deal with the problems within content, as well as provide motivational links to the more formal treatment of content. However, we understood that, unless teachers could see how the problems can indeed fulfil these roles within the actual content development of a given topic, it would become increasingly harder for teachers to willingly "give up" class time for problem solving.

We thought that the challenges that the teachers faced were significant, and we needed to address them in the next iteration of this phase. In summary, the infusion must include these features: (a) apart from the problems, there should be additional details that will help teachers realise the intended goals behind the problems. This also implies that teachers should be directly involved in the planning that leads to the rationale and finalisation of the problems; (b) the planning should not be restricted to the problems and its immediate temporal surrounds; teachers need to see how the problem(s) fit logically and developmentally within the entire topic progression; in other words, the unit of planning is “zoomed-out” to the whole topic; this will help teachers see how time spent on problem solving IS a form of content coverage; (c) motivational elements should be integrated into the unit planning. In determining a strategy to incorporate these elements in the refinement of the design for the second phase, we also took into account broader structural challenges relating to policy and curriculum.

In addition to the practical challenges the teachers faced, we highlight some structural challenges that we need to confront when considering a refinement of the envisioned infusion. The first is the largely centre-to-periphery model of curriculum dissemination in Singapore. The effectiveness of this dissemination approach depends on, among other factors, “the strength of the central resources,” the number of peripheral elements, and their distance from the centre (Kelly, 2004). One critical step in this centre-to-periphery process is the teachers’ interpretation of the official curriculum (in the form of a syllabus document) and its translation into classroom practices. It is through these classroom experiences that students learn not just content for national examinations but content imbued with the disciplinarity of mathematics. However, while teachers are consulted by curriculum planners and developers, they nevertheless remain at the far end of the change process (in the eye of the storm, safe from the fury of the blast). The curriculum as an end product is conveyed to teachers in the form of training workshops by the people at the centre. Thereafter, it is left very much to the teachers in a school to implement the curriculum, within the given guidelines, and in view of the vision of the school. In this sense, school-based teacher professional development in interpreting and translating the local mathematics curriculum is key to ensuring the realisation of the overarching curricular goal of mathematical problem solving for all students. This school-based approach to teacher participation in developing a problem solving-centric curriculum—as a form of teacher development—is the model adopted by the MProSE team.

Another challenge to teaching problem solving is the lock-step grid of fixed teaching schedules. Teachers are hard pressed into adhering to these schemes of work to prepare students for term tests or national examinations. In such a context of high time pressure to “cover syllabus,” it is not uncommon for teachers to have the mind-set that problem solving is an unessential distraction. In addition (and perhaps related to teachers’ perception of limited time), teachers’ preferred mode of instruction is the teacher exposition type of teaching that is entrenched in many classrooms. Hattie and Yates (2014), citing Larry Cuban and Nathaniel Gage, pointed out that this teaching methodology, also known as the initiate-response-evaluation approach or conventional-direct-recitation, has survived “considerable criticism and attacks for over two centuries” (p. 44). It is not surprising to find it a common approach in local classrooms. One reason is the easily recognisable and established roles and norms for both teachers and students in the classroom.

As part of our design experiment approach, we need to "accommodate" these practical challenges and structural givens in teachers' preferences and mind-sets. By accommodation, we mean a type of change we make to the design for meeting the (localised or systemic) constraints faced by teachers (Quek et al., 2011). Our approach as a result was to use "replacement units" to bring about instructional change and teacher capacity building within this preferred teaching approach and the lock-step planned curriculum.

Refinement of Second Phase MProSE: The Replacement Unit Strategy

In this last section of the chapter, we bring the readers to the most up-to-date refinement of our MProSE infusion programme: use of the replacement unit strategy. Although this strategy was developed independently during our other projects (see, e.g. Leong et al., 2013), the term "replacement unit" (RU) is attributable to Cohen and Hill (2001). While working on designing an RU, we develop—in consultation with the teachers—a redesign for an entire mathematics topic. This redesign involves restructuring of content and development of all the relevant instructional materials to accommodate the integration of problem solving without changing the original allocated time for the unit. As such, it is an authentic "replacement unit"—in the sense that teachers can replace the original way of teaching the unit by this RU without upsetting the overall teaching schedule.

Cohen and Hill (2001) reasoned that the replacement units were an important innovation in the sense that "[curriculum] developers would be able to ground teachers' professional education in the improved student curriculum that teachers would teach" (p. 47). Linking teachers' professional development with a proposed improved curriculum was a novel way which differed from usual attempts which typically focused on one element at a time. They reported that workshops for teachers on the materials and the pedagogy of the replacement units "had appreciable depth and allowed teachers to investigate more seriously individual mathematical topics, like fractions, in the context of student curriculum" (p. 55). They also reported the positive potential of replacement units for education reform:

> Teachers who took workshops that were extended in time and focused on students' tasks—either the replacement units created for the reforms or new assessment tasks and students' work on them—reported more practices that were similar to those which reformers proposed. In contrast, teachers who took workshops more loosely focused on hands-on activities, gender, cooperative learning, and other tangential topics were less likely to report such practices. (p. 88)

An RU, usually spanning 4–8 h in duration, is a realistic and reasonable period of engagement with teachers for each attempt at curricular redesign. This avoids the onerous task of redesigning the entire curriculum all at once. Moreover, focusing the efforts on one RU at a time allows both the researchers and the teachers to trial (and retrial, if necessary) and to refine the RU as well as to gain familiarity with its underlying design principles over time. This setup of studying and redesigning an RU based on a topic that is covered within realistic time limitations in the teachers' teaching schedule provides the platform to accommodate the structural challenges discussed earlier. Teachers' active involvement from initial discussion to implementation and refinement of the RU also helps close the gap between curriculum planning and practice.

More importantly, an RU is a suitable "unit" to infuse problem solving. It is in the redesigning of a unit that the relevance and place of problem solving can be found. The RU is of appropriate size for problem solving to be weaved seamlessly with the development of mathematics content and thus allowing teachers to see, for example, how motivational elements can be inserted to connect problem solving to content to be learnt, how problem solving can be realistically employed within time constraints, or how problem solving IS the learning of content. In other words, the RU strategy addresses the local challenges discussed earlier.

At the time of writing, we are in the early stage of implementing the RU strategy in the MProSE project. As such, we are unable at this point to provide an analysis of the outcomes of its implementation and follow-up further refinements. Nevertheless, as an infusion strategy that we have come to develop based on our experiences with a number of schools we worked with over more than 5 years, we think it holds promise. A summary of an RU on quadratic equations that we designed together with the teachers is given in the Appendix for the readers' reference.

Conclusion

We think that the current big question in mathematics problem solving research is this: How do we make meaningful problem solving a regular feature in mathematics classrooms? We recast this as the "infusion problem." There are many reasons why the classroom is so "resistant" to change, including change towards problem solving infusion. In this chapter, we focus our discussion of infusion hurdles on existing macro-issues such as the pressure towards content coverage, teachers' readiness towards a problem solving approach, and the lock-step grid of teaching schedules that renders additional curriculum time for problem solving unrealistic. Through our MProSE design experiment, we have come to learn that the way to tackle some of these challenges is not merely through minor tweaks in the way teachers teach; what is needed is a paradigm shift that requires changes to be implemented at the curricular and structural level in the school's mathematics programme. In short, we think the intervention can be carried out in two steps: First, familiarise students with the processes and language of problem solving through a separate module designated to foreground the teaching *about* mathematics problem solving. This intensive learning about problem solving is needed for both teachers and students; thereafter, follow up with integrating problem solving in the teaching of regular mathematics content through RUs. We argue that the RU strategy is a feasible way forward in realising the curricular and structural changes that need to be made;

the strategy also provides a suitable platform for teachers' participation in learning and curriculum redesign.

We readily acknowledge that these broad design features are necessary but not sufficient to deal with the infusion problem. Although not detailed in this chapter, we developed and tweaked classroom implements alongside these structural changes to help teachers and students cope with this "new" problem solving way of learning mathematics. There are ongoing efforts to refine the nature of problems to meet the intellectual and affective needs of the students; at the same time, we modify the practical worksheet so that it easier to use for both teachers and students. We are also currently working on the repertoire of teachers' craft skills that are supportive of the teaching of problem solving. This includes the clarity in teacher's visual representations of the problem solving processes in the whole class setting and the kinds of scaffolds that teachers can use in table-table instruction to help students experience empowerment through problem solving.

In short, we can approach the infusion problem through various loci of study—such as curriculum redesign, teacher development, and classroom task implements. The next stage of our research will involve a careful examination and integration of these factors in a way that fits the local conditions of respective schools so that it results in successful infusion.

Appendix: Description of an RU on Quadratic Equations

Under the topic of "Solving word problems that are reducible to quadratic equations," a common observation among teachers is that some students struggle with translation of the statements in the "word problems" to equivalent equations. The frequently used trajectory can be summarised as such: Teacher demonstrates the steps involved in translating statements to equations over different types of word problems; students can usually follow the steps; but when asked to do it on their own, they are "stuck," especially when confronted with an unfamiliar type of "word problem." The usual response by teachers to such student difficulty is more demonstration and more fine-grained breakdown of steps with the intent of making the skill acquisition process for students more gradual. Here, we propose the problem solving approach within the context of an RU.

We think the problem students encounter is not merely that of lacking familiarity with the different types of word problems; more fundamentally, it is the lack of opportunity for authentic exploration of the word problems—a necessary step for students to make sense of the problems and to appreciate the power of the algebraic approach. In other words, we need to "prepare the ground" so that when the algebraic method is "planted," it will "take root"—students will receive it and learn it better instead of seeing it as a method forced upon them. In particular, we infuse problem solving.

For this RU on quadratic equations, instead of being taught a method of solving word problems right from the start, students are given time to attempt such a word problem on the practical worksheet. In so doing, they are given the opportunity to explore the word problem and hence figure out its underlying structure. At the same time, we are conscious of all the realistic constraints—such as the need to cover standard content and the lock-step schedule which were elaborated in the earlier sections—and we are bound by the redesign of the RU. The lessons in this topic are thus reorganised in this way:

Lesson 1: Solve a word problem reducible to quadratic equation with a practical worksheet.

The "Employee problem": A company wants to employ as many workers as it can afford to complete a project within a short timeframe. If the company pays each worker $6 per hour, there are only 30 applicants for the job. However, the company needs more workers. It is known that for every $1 increment in the hourly pay, it will attract two more applicants for the job. The company can only afford a maximum of $504 per hour in total. How much should the company offer to pay per hour in order to attract the maximum number of workers?

The main goal is to let students re-familiarise with the practical worksheet and feel a sense of empowerment at solving the problem when they use Pólya's stages and heuristics. Note that to solve the problem, students need not use algebra. Students are expected to use other methods such as systematic listing and other heuristics such as "substitute values" to solve the problem.

At the fourth stage of Pólya, we can provide a motivation for algebra by asking, "What happens if we have an owner with greater resources beyond $504? Can your solution be easily adjusted to cope with this adaptation?" The point is to provide a link to the algebraic representation/solution, which is the scope of the next few lessons.

Lesson 2: Revision of quadratic factorisation and using it to solve quadratic equations.

The main goal is to help students use "zero product rule" and factorisation to solve quadratic equations with integer coefficients. After revision of quadratic factorisation, students are to be taught the steps in solving quadratic equations by factorisation. They then practise the method to gain fluency. In other words, this is a "standard" lesson geared towards mastery of technique—a type of teaching that teachers are familiar with.

Lesson 3: Solve another given word problem using a practical worksheet.

The "Consecutive Numbers problem": "Four consecutive even numbers are such that the product of the smallest and the largest is 186 more than the sum of the other two. What are the four numbers?"

Students are expected to use the resources gathered, both the experience in Lesson 1 on using the algebraic method as well as the method of solving quadratic equations in Lesson 2, to make productive attempts at solving the problem in this lesson. Under Stage 4, students can consider generalizing a standard procedure for solving "word problems" that are reducible to quadratic equations. The intended link from the working for this problem and the more generalised method is illustrated in Table 1.

Table 1 From working to "methodizing" as part of Pólya's Stage 4

Working	Methodizing
Let the first number be x	Step 1: Determine the variable and let it be x
Therefore, the four numbers are x, $x+2$, $x+4$, and $x+6$	Step 2: Express the other variables in terms of x
Product of the smallest and largest number $=x(x+6)$	Step 3: Establish the relationships between x and the terms based on the word phrases
Sum of the other two numbers $=(x+2)+(x+4)$	
$x(x+6)=186+[(x+2)+(x+4)]$	Step 4: Form an equation from the key sentence
$x^2+4x-192=0$	Step 5: Simplify the equation and equate it to zero
$x=-16$ or $x=12$	Step 6: Solve for x

Lesson 4: Apply the general procedure abstracted in Lesson 3 to solve other "word problems."

The main goal is to help students apply the general method in the right column of Table 1 to a variety of other word problems reducible to quadratic equations. The instructional approach is one of practising a learnt method—a style of teaching which is standard for teachers.

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Mathematical Problem Posing: A Case of Elementary School Teachers Developing Tasks and Designing Instructions in Taiwan

Shuk-Kwan S. Leung

Abstract Getting teachers to enact mathematical problem posing (MPP) and having children do mathematics in the making (*How to solve it?* Princeton University Press, Princeton, NJ, 1945) is not easy. In a prior study (*Educational Studies in Mathematics* 83:103–116, 2013), a teacher educator reported on the development of research-based tasks aligned to the math curriculum and worked with 60 teachers to explore for feasible methods to encourage children to pose mathematical problems. In the present study, three selected teachers continued their journeys and not only developed their own tasks but also designed their own problem-posing lessons. The teacher educator worked closely with these teachers for one whole year. Data collection included teachers' journals, children's written work, teachers' interviews, and focus group interviews. This report includes the results of teachers' actions as well as reflections on the tasks used in problem-posing instruction, suggestions to other teachers, identifying arising problems, and attempts to solve such problems. The investigator will discuss how and why teachers develop and enact tasks to get children to pose mathematical problems and will suggest implications for research and practice in the future.

Keywords Mathematical problem posing • Developing tasks • Teachers as researchers

Background

Educating the next generation is a responsibility for all. Among many school subjects, mathematics is often considered a tough subject. When striving to learn or teach mathematics well, engaging in problem solving often leads to good results. In solving a problem, one must understand, plan, carry out, and look back (Polya, 1945), and through solving a problem, one can learn to make mathematics. This point is agreed upon by both mathematicians and mathematics educators. However, posing problems to solve is also an important activity for making mathematics

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(Polya, 1945); mathematicians not only solve problems but also find problems to solve (Leung, 2013). Hence, individuals can learn mathematics by solving given problems or by posing then solving their own problems. In this chapter, the author does not claim one activity is more important than the other. Both problem-solving and problem-posing activities are important and related, yet not the same, in terms of requiring one to propose a problem structure that can reasonably connect the “given” to the “goal.”

This is a report on a teacher educator’s continual effort since 1993 to implement mathematical problem-posing (MPP) instruction in the elementary schools in Taiwan which also includes the efforts of the school teachers involved. In initial studies on MPP, she studied products and processes of prospective teachers in the United States and Taiwan (Leung, 1993, 1994a, 1994b, 1997) and found that prospective teachers were inexperienced in MPP but that they expressed interest in this activity and showed improvement after they did problem posing themselves. Another effort she made was to develop a systematic assessment (or evaluation) method for MPP (Leung & Silver, 1997). Though there are no standard criteria for evaluating a problem, an analysis of posed problems can hint at what posers see in a given situation and what they consider as mathematical and allow inferences about what they “do know” or “don’t know.” At the teachers’ college where she taught, the teacher educator offered a two-credit course on problem posing for prospective elementary school teachers, using various research-based MPP tasks to familiarize students with MPP over 18 weeks. The course is popular which can be seen from the subsequent student teaching of the students who took the course. Some of these students asked children to do MPP during their student teaching experiences. One student received award funding from the National Science Council of the country and conducted a systematic study during student teaching (Leung, 2001).

As with prospective teachers, in-service teachers have been found to be unfamiliar with MPP. In order to introduce MPP to in-service teachers, the teacher educator incorporated MPP into invited talks for teacher-training workshops at national, state, or district levels. When there was an imbalance in the number of invited talks from states in the North, Central, South, and East, she actively made contact with those institutions that did not send her invitations and suggested giving a talk on MPP. In addition, when she was attending national committee meetings on the development of textbooks, she made suggestions for textbooks to include activities on problem posing. Thus, by extending the introduction of MPP to prospective teachers from her institution to practitioners in different states, she learned more about practitioners’ views on MPP from all over the country. The frequently asked questions she received from in-service teachers pointed her to the need to direct research to teachers.

A survey in Taiwan indicated that teachers barely ever asked children to do MPP; children only solved problems given by teachers or from textbooks (Leung, 1994a, 1994b). In the same survey, in-service teachers commented that they themselves needed to be able to do MPP. In addition, they said that it was difficult to implement unless they (1) had access to tasks or were able to develop their own materials and (2) had guidelines on how to respond to children's posed problems.

A 3-year funded research study was conducted to address teachers' needs (Leung, 2013). In Leung's study, the chief investigator (teacher educator) worked with teachers over time to integrate MPP into the elementary mathematics curriculum. The first year was for the development of an MPP task inventory by grade and content. About 100 elementary school teachers in the district were invited to assist in piloting the tasks. After a year of piloting tasks, a resource book was produced which contained 52 tasks. In the second year of implementation, 60 teachers from year one participated. The 60 teachers attended a series of workshops for instruction on how to implement these tasks so that they could model MPP. The results of year two included findings about how teachers solved problems that arose when enacting such tasks. To make MPP a part of instruction, the teacher educator succeeded in supplying an inventory of tasks and instructional guidelines as conditions for integration of this activity into regular math instruction.

In this report, the focus is on the third year of the project. This involved the action research of the practitioners. Only three teachers from year two were selected to participate. They represented those who were highly motivated and willing to commit. The teacher educator (the chief investigator) decided to include the teachers as coinvestigators; they were to make up their own plans for teaching problem posing. The aim was to gain better information about the actions and reflections of teachers who attempt to develop their own tasks to integrate MPP into ordinary grade 1–6 classroom settings. There were three research questions for this phase of the research: (1) What types of MPP tasks are appropriate to use for children, and how should those tasks be set up? (2) What professional development resulted when teachers participated in this inquiry through action, reflection, and seeking solutions? What are some suggestions for other teachers? (3) What children's learning can be witnessed through MPP instruction?

Literature Review

Mathematical Problem Posing

MPP is often considered an inseparable part of problem solving. The importance of problem solving and problem posing in mathematics education is highlighted in well-known books (Brown & Walter, 1983; Freudenthal, 1973; Polya, 1954) and in handbooks on mathematics education research (Grouws, 1992). Problem-posing reports appeared in a focus issue of *Teaching Children Mathematics* in 2005 and also in a special issue of *Educational Studies in Mathematics* in 2013. The two are also key components in curriculum standards in the West and East (Cai & Nie, 2007; Leung, 2013; NCTM, 2000; Törner, Schoenfeld, & Reiss, 2007).

A number of researchers have proposed categorization schemes for problem-posing situations and structure. In a commentary, Leung (2009) discussed problem structure, solving/posing and justifying solutions. If there is a structure with defined goals and specified givens, then a problem exists. If not, one can perform problem posing and solving, and then additional problems can be posed. Reitman (1965) considered four cases of problem situations; in each case the author assumes MPP can be performed. They are defined by two specifications (defined or undefined) times two states (given and the goal), resulting in the four cases (see Table 1: R1, R2, R3, R4). Besides these four cases, MPP has two meanings (Silver, 1994): the reformulation of known problems (S1) or the formulation of new problems (S2). Silver's reformulation of known problems included presentation known problems again: posing problems properly (Butt, 1980) or making up items in a test (Leung, 1996). In addition, MPP can be classified according to structure (see Stoyanova & Ellerton, 1996): structured (SE1), semi-structured (SE2), or free (SE3). In the inventory of 52 MPP tasks (Leung, 2013), there are six types according to given state: known problem (solve then pose), an algorithm, a text, a figure or table, a math topic, or an answer (pose a problem with the given answer). These different classifications of the nature of problem posing, and to what extent the given or goal is known, are summarized in Table 1.

The above classification is based on whether the given or the goal is defined, or to the extent to which it is defined. However, a word problem can be further classified into two types: (a) posing a mathematics problem using a situation and (b) posing a story problem matching a fixed piece of mathematics. For example, an activity asking one to pose one easy and one difficult percent problem (Van den Heuvel-Panhuizen, Middleton, & Streefland, 1995) belongs to the former type. On the other hand, posing problems matching a given division (12 ÷ 8; answer = 13; 12 ÷ 8; answer = 12; 12 ÷ 8; answer = 12.5; Chen, Dooren, Chen, & Verschaffel, 2011) belongs to the latter type.

Table 1 A summary of the nature of problem posing

Reitman (1965)	Silver (1994)	Stoyanova and Ellerton (1996)	Leung (2013)
Given defined	Reformulate known problems (S1)	Structured (SE1)	1. Known problem
Goal defined (R1)			
Given defined	Formulate new problems (S2)	Semi-structured (SE2)	2. Algorithm 3. Text 4. Figure/table 5. Math topic
Goal undefined (R2)			
Given undefined	Formulate new problems (S2)	Semi-structured (SE2)	6. Answer
Goal defined (R3)			
Given undefined	Formulate new problems (S2)	Free (SE3)	
Goal undefined (R4)			

There have also been classifications of problem-posing according to processes. In a study conducted in Cyprus, four MPP processes were considered: editing, selecting, comprehending, and translating. This classification was used to study quantitatively how the four processes are related to children's performance (Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005). However, it is unclear if classifying tasks by process produces mutually exclusive categories. For example, in doing MPP, can a person exhibit only one process and not the other three?

Besides processes, there has also been a classification based on problem-posing products. In the problems posed and not yet solved, there are four characteristics: (1) idiosyncratic, (2) exhibiting plausible reasoning, (3) formed before/during/after one does problem solving, and (4) insufficient or implausible (Leung, 1994a, 1994b). The categories are not mutually exclusive and a posed problem (product) may contain one or more characteristics.

In short, one cannot tell what problem posing is until one studies closely and fully understands the nature of the task, what is given, what is required to take action, and what product is required (a story situation or a math structure).

Students' MPP has been compared in cross-national studies. For example, a comparison of Chinese and American students found that grade six Chinese children outperformed US students in routine problem-solving tasks, but that this result did not hold for open-ended tasks including problem posing (Cai & Hwang, 2002). The result is just the opposite in another study by Yuan and Sriraman (2011) for students in grades 11 and 12. The Chinese students posed more than double the number of problems posed by American students. However, it is hard to compare cultural differences without mentioning instruction on MPP such as how teachers in each nation taught problem posing and what MPP tasks were used.

In Taiwan, in the two most recent versions of documents on mathematics curriculum standards (Ministry of Education, Taiwan ROC, 2008), problem posing is explicitly mentioned in the part concerning getting students to do inquiry. Even in textbooks, there are activities where children are asked to make up problems. For example, the following activity is found in a grade 2 unit about multiples: Children use given information in a picture of an amusement park and make up mathematics problems. In the amusement park, the number of persons (children/adults) in each game is the same: two in a race car, four in a coffee cup, six in a spinning insect, eight in an octopus, and ten in any merry-go-round. In addition to posing problems related to the information given, children also solve the problems they or peers produce.

Indeed, problem posing is an inseparable part of problem solving. The cycle pose-plan-carry out-look back explains the never-ending process of a person doing posing and solving (Leung, 2009). It is customary for one to pose a problem and then solve it (pose then solve). But it is also a natural problem-posing task to require a student to solve a problem and then pose a problem (solve then pose). Instructional history is needed to explain children's performance in these two activities. A commentary on four studies of problem solving and problem posing in Belgium, China, Sweden, and the United States says that reporting failings in student performance in posing and solving is insufficient, as students might be unprepared. After reporting that children are not competent in posing and/or solving, the next step is to do classroom interventions to promote problem posing and/or solving. Exploring how children improve from not competent to competent in MPP when teachers give explicit instruction and teach children to review one's attempt in solving and posing adds more information to teacher training.

Teachers Participation in Taiwan Curriculum Reform

In the history of curriculum reform in Taiwan, and under different waves of reform, all teachers (teacher educators, teacher leaders, school teachers, and student teachers) have been involved in curriculum reform and have assumed different roles. Some served as national committee members for writing curriculum standards documents; others served on the board for national experimental textbook development, and a few belonged to the national committee for supervision for the country. Whenever there has been a curriculum change, there have been training sessions for state/district school representatives. All representatives, upon completion of the training, would conduct workshops for schools in the state/district. The Ministry of Education would commission a committee to prepare books and video tapes as materials for the workshops. In all curriculum reform/revision, both teacher educators and school teachers were involved.

In fact, a revision in curriculum standards often calls for partnership between teacher educators and teachers. The partnership is established to work on a better understanding of the essence of the curriculum, evaluate textbooks, enact curricular materials, and evaluate implementation. In Bieda's (2010) study on proof-related tasks and enactment, the teacher educators worked with veteran teachers. They met regularly to analyze curriculum materials, classified and selected tasks, and finally implemented tasks. This partnership is a good start in initiating a change. Otherwise, teachers find ideas suggested by teacher educators too theoretical and they cannot imagine how those ideas can be applied to their own classroom settings (Sowder, 2008). When such a partnership is established, the common goal for teacher educators and teachers is to better their knowledge in teaching and learning.

In this partnership, it is favorable for teacher educators to invite teachers to be involved in projects that require teachers to assume different roles with different participation at different levels. Hensen (1996) described teachers' involvement in three levels. At Level One, teachers acted as helpers. The help could be data collection or arranging to let teacher educators "borrow" their students to teach. At Level Two, teachers acted as junior partners and decided on the research agenda together with teacher educators. At Level Three, teachers acted as lone researchers or collaborators/equal partners with teacher educators. That is to say, teachers' participation with respect to conducting a study and/or in using data and findings increased at higher levels. In the present study, when teachers acted as lone researchers, the aim was to conduct research on MPP instruction: to experience MPP, to design their own activities or tasks for instruction, and to analyze children's posed problems.

Method

This action research was a response to curriculum revision. A teacher educator worked with three school teachers and followed action research principles (Kemmis, 1991). After identifying an instructional problem, the teacher planned, took action, and then reflected upon practice. This cycle was repeated until the problem was solved.

Teachers as Action Researchers

The Participation Project. In year 1, 105 teachers administered MPP tasks, asked children to pose problems, then collected children's sheets, and returned them. In year 2, 60 teachers from the original 105 participated. They attended a series of workshops and coinvestigated how MPP can be implemented using the set of 52 tasks developed from year 1. Toward the end of the year, they exchanged ways to implement these tasks in the elementary school curriculum, and 46 returned questionnaires indicating a willingness to participate in the 3rd year.

Three Action Researchers. After interviews three teachers were chosen from the 46 teachers to be involved in the 3rd year. They were teachers who taught in a school nearby, less than 1 h ride from the investigator's institution. The three teachers were willing to participate, to be observed, and to attend biweekly meetings. These three teachers' attendance record was good over the first 2 years. They were not paid; the incentive was to study how MPP ideas could be practiced over the 3rd year. All three are female and have taught for at least 10 years in the district near the teacher educator's institution. In Taiwan, the six grades in elementary school are divided into lower grades (grades 1 and 2), middle grades (grades 3 and 4), and higher grades (grades 5 and 6) where the same teacher teaches the same class for 2 consecutive years. In this study, the first teacher taught grade two (Lin, lower grades), the second taught grade 4 (Ma, middle grades), and the third teacher taught grade 5 (Han, higher grades).

Data Sources

There were five data sources: teachers' journals (TJ: "TJ1" for Ms. Lin, "TJ2" for Ms. Ma, and "TJ3" for Ms. Han), children's scripts (CS), teacher interviews (TI), children's interviews (CI), and classroom observations (CO).

Teachers' Journals. Throughout the inquiry process, teachers kept journals and shared with investigators by mail, phone, or in person. The format of the journal was free. In addition, the investigator gave five guided questions for reflecting upon practice on MPP:

1. What tasks did I use?
2. How did my children perform?
3. What were things that my children did not understand?
4. If a colleague uses my tasks/instruction, what suggestions would I give?
5. What is worth writing down in my journal?

Children's Written Work. When children did problem posing in class, they were asked to write down the problems they posed. Teachers kept a record of the work in written form. Teachers used the coding scheme introduced in workshops they attended the year before and analyzed the children's posed problems using five categories: (1) Not a problem, (2) Non-math problem, (3) Implausible math problem, (4) Insufficient math problem, (5) Sufficient math problem (Leung, 1997). Below is an example of a task (the Temperature Task) and categories of posed problems. The following example explained the five categories. When a problem is implausible, it cannot be solved using the information given. For a problem that is insufficient, it cannot be solved unless one adds the missing information.

The Temperature Task and Five Categories of Posed Problems

1. *The temperature of the cup of tea is 50* °C. (1) Not a problem
2. *Is there soup or tea in this cup*? (2) Non-math problem
3. *The temperature of the cup is 45* °C; *the volume is 450 cm³; how many altogether?* (3) Implausible problem
4. *The temperature of the cup of tea is 50* °C*, it is cooled down in 2 h*, *what is the final temperature*? (4) Insufficient problem
5. *If the cup of tea is 49* °C *and heated to 5 degrees warmer*, *what is the final temperature*? (5) Sufficient problem

Teachers sent the results of children's work and the investigator commented, gave suggestions, and collated and sent the information to all three teachers on a monthly basis.

Teacher interviews (*conducted by teacher educator*). During school visits, the investigator interviewed the teachers after observing a lesson. The interview was semi-structured and was conducted in a free format, with a basis on children's scripts, the teachers' journals, and observation notes. The interviews were audio-taped.

The teacher educator read teachers' journals before she conducted individual interviews.

Children's interviews (*conducted by teachers*). A meeting was held for the investigator and the three teachers to decide on how to do these interviews. The teachers randomly chose eight children and paired them into four group interviews. Each teacher decided on the date and place to do the interviews. Ten guided questions were used to ask about children's views on MPP, on mathematics problems, and what they thought they learned from MPP. The teacher asked if the questions were difficult to answer, asked the question again (if needed), and made sure the child answered after they had understood the teacher's questions. The interviews were audio-taped.

Classroom observations (*conducted by teacher educator*). During the second term of the year, the investigator arranged to visit the three teachers. She visited each teacher three times: The first visit was to get to know the case; the second was to observe and audio-tape a lesson and then interview the teacher; the third was to plan a lesson collaboratively and then observe and tape the lesson. The teacher educator made observation notes each time she observed a lesson and the lesson was audio-taped.

Data Analysis

The five sources of data, teachers' journals (TJ), children's scripts (CS), teacher interviews (TI), children's interviews (CI), and classroom observations (CO), were analyzed qualitatively, with triangulation, to address to three research questions concerning the actions and reflections of these teachers in developing tasks and designing instruction:

1. What types of MPP tasks are appropriate to use? How should those tasks be set up?
2. What professional development resulted when teachers participated in this inquiry through action, reflection, and seeking solutions? What are some suggestions for other teachers?
3. What children's learning can be witnessed through MPP instruction?

Results and Discussion

MPP Tasks Used in Instruction

What MPP tasks did teachers use or develop when they designed MPP instruction? The tasks were taken from multiple sources: adapted from textbooks (TJ3), suggestions from colleagues (TI1-3), and from children's or teachers' own ideas (TJ1, TJ2, TJ3). With respect to the information given in the task, they included algorithms (e.g., horizontal: $38+16=54$; vertical addition of two one-digit decimals, TJ1), a math topic (e.g., “make a problem regarding approximation” TJ2), a figure (e.g., a calendar of a month with colors representing public holidays, TJ2; or a row of lamps along a road with each lamp being the same distance from another lamp, TJ3), and a text which is a typical word problem without a specific goal or with a goal that children solve then pose another problem using the word problem description (TJ2, TJ3). The inventory of 52 tasks from the preceding year (TJ1–3, TI1–3, CS1–3), given in a website, was also a source for these three teachers.

What criteria did teachers use to decide on a task? The teachers expressed that the choice of tasks depended on the mathematical content she was teaching that week (TI1), and the source was usually from textbooks (TI1; TI2) or another teacher. However, she converted the tasks from requiring solving to posing according to children’s interests and level of proficiency in reading and in mathematics (TI3). For lower grades, the teacher tended to use real-life situations with figures or even photos (TI1). For upper grades, the teacher would opt to link to mathematical expressions or even symbols (TI3; CO3: distance between lamp posts).

How did teachers set up and use MPP tasks? During observations, the tasks used involved shapes and statistics (CO1-Ms. Lin); quadrilaterals, cubes, and cuboids (CO2-Ms. Ma.); and approximation and distance/time/speed (CO3-Ms. Han.). The transcripts from the visits showed the effectiveness of Ms. Lin and Ms. Ma in using a 40 min session (grade 2; grade 4). In setting up MPP tasks, the teachers also asked children to pose problems in groups and then present their problems, discussing the merits or faults of the problems and finishing by solving them. Ms. Lin allowed drawing the problem out or orally presenting the posed problem. Ms. Lin pasted four pairs of cutout shapes (isosceles triangle, rectangle, squares, and circles) to the chalkboard and asked children to do MPP in groups. Ms. Ma used teaching aids and passed boxes of unit cubes (about 100) for each group to make up cubes and cuboids of varying size and shapes and then write a problem on a sheet of paper for another group to come to see the figures and solve the problems (CO2). The higher grade teacher, Ms. Han, conducted an MPP activity on two consecutive days (TJ3, CO3). She presented a textbook problem on distance/time/speed and asked each student to make up an item for her to consider for a test paper. She collected all questions on day one and went home to prepare a bit as there were contributions from over 30 children. She divided the problems posed by the children into types and then used them accordingly the next day. For plausible problems (categories 4 and 5), she asked them to solve the problems on day 2 or actually included them on a test. For other problems (category 1, not a problem; category 2, non-math problem; category 3, implausible problem), she showed children her selected examples carefully one by one. For each chosen example, she invited children to spot what was wrong and gave hints (e.g., “What happens when we add 50 cm^3 to 30 °C?”) for children to make the problems into plausible ones. In all, teachers used their own creativity in using tasks, setting up tasks, and evaluating the suitability of tasks after MPP teaching and after children’s work. In terms of frequency of MPP, it depends on the topic that was taught. Over the school year, teachers did problem posing approximately once a week. After each session during which they used MPP, they reflected upon their practice, analyzed children's problems, and considered using them. Finally they revised the way they attended to MPP tasks the next time they taught.

Professional Growth Through Teachers' Actions and Reflections

Toward the end of the year, the three teachers remarked: "This activity is neat and I'll be great. I want to be a great teacher" (TI2; CO2). "I once read a posed problem I could not solve…I did not know what to do next" (TI3; TJ3). "On raining days when they did not have Physical Education lessons, they visited peer's problems on the notice boards and solved them. The author graded solutions" (*TJ1*; *CS2*).

During the focus group interview toward the end of the first term, all three teachers expressed that they did not use extra instructional time on MPP. "If I use extra lessons, then I worry that I cannot finish teaching the topics using the allocated time each week." Therefore, they integrated it into the ordinary lessons and followed the order of contents that appeared in the mathematics curriculum. Their specific actions and reflections were given in journals (TJ1–3) and interviews (T11–3) and could also be inferred from the investigator's observation of lessons during school visits. In their journals, the teachers wrote specific tips for other teachers. They also related their experiences, giving suggestions to peers. At the end of the year, the three teachers and the teacher educator shared what they learned from action and reflections on MPP instruction. The teachers' growth can be seen in the list of five suggestions they gave to other teachers who are considering MPP instruction. These suggestions are discussed below.

One suggestion is the use of teaching aids (*TJ1*, *TJ2*, *TJ3*). The investigator asked teachers to explain this suggestion further during interviews. Ms. Lin gave examples of teaching aids, such as using photographs of a stationery shop with marked prices.

> Children were asked to be in a group of two; they then played the role of either a shop keeper or the customer. The photo on the blackboard was for all groups to read. Later, they marked their own prices in the small group. This was so much fun for them especially when they decided on the stationery items in the shop and the price of each item. (TI, CS1, TI1)

Ms. Ma used centimeter cubes in a grade 4 lesson on volumes.

> I used boxes of centimeter cubes and asked each group to make cubes, cuboids of different size, then finally other shapes. Children were able to make up fancy problems on finding volumes. Compared to textbook problems, these problems were real and each figure was three-dimensional. The 6 groups of children walked around and each visited the other five groups, rotated and solved altogether 6 real problems on volumes of solids in a 40-minute lesson.… Very effective use of instructional time using teaching aids. (TI2; TJ2, CS2)

The higher grade teacher, Ms. Han, used meter rules and toothpicks as teaching aids.

> After solving a problem with the number of lampposts and the whole distance I asked them to do MPP. But I know drawing figures of varying distance apart took up a lot of the time.

> So, I gave them a meter rule and toothpicks: they could pose problems on any desirable given length and their preferred number of trees then asked their friends to find the distance between two trees; or vice versa. If they want to pose a new problem, they just removed the toothpicks and placed them on the meter ruler the second time (TI3, TI3, CO3). I like posing problems on distance apart using toothpicks and meter rule. (C13, CS3)

A second suggestion is to design worksheets to go with a task. This was especially a need for Ms. Ma and for Ms. Han, as the tasks' instructions for middle and higher grades were complicated.

> Sometimes, I forget what I was supposed to ask them to do and write, so, I made up a worksheet and asked them to follow the steps. When facing 30 excited faces you got lost…. (TJ3; TI2)
>
> A good thing about worksheets is, we can collect them and work on that after the bell rang. Sometimes, we could not think of any good ideas on how to respond to children's work and so we collected the written work and left the extended discussion to future lessons. (TJ2; TJ1, TI3)

Another point for the use of worksheets was to study the work further with children.

> Children forgot what they produced quickly, when we talked about their work, we referred to the worksheet and they remembered….

Finally, all three replied that sharing work in class was easy when there were worksheets. When there were common things to refer to the whole class would have a rich discussion session.

The third is <u>not to</u> give examples. All teachers voiced that giving examples spoiled the fun and restricted children to posing problems that were similar.

> For once, they asked me to give an example. I did. Then most of them posed a problem identical or similar to mine! (TJ1; TJ2, TJ3) For $36+31=?$ My example is an orange costs $36 and an apple costs $31, then most of the problems posed by children are about fruit or money. There was no exception in a class of 30, can you imagine! (TJ1)

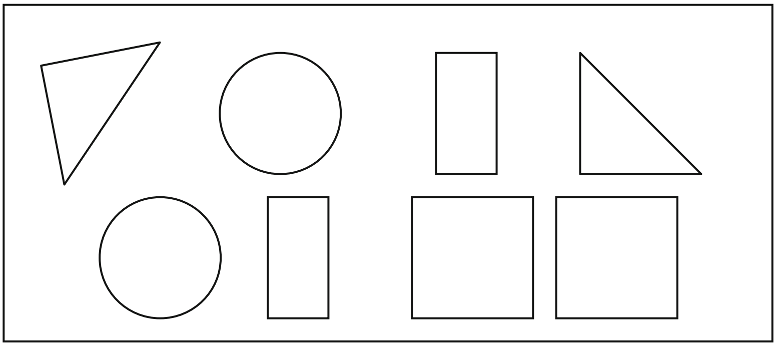
They remembered not getting any examples when they attended the teacher workshop in the previous year offered by the teacher educator.

> Now I remember when we attended workshops in year two, we were advised not to give examples nor hinted a direction. Now we forgot that completely! (TJ2, TJ3)

These teachers forgot and gave examples, then reported the consequences of giving examples. The consequence is children followed examples and did not give any new ideas.

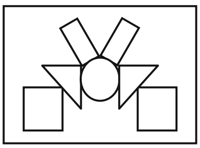
The fourth is to show interest in children's work and publish the work. Teachers shared ideas by displaying children's work to other children in class on the notice board (TJ3), hanging up colorful problem cards along corridors outside the classrooms (TJ1), writing on transparencies and pasting them to windows, or taking photos of the figures (TI2). At the end of a unit which included MPP, Ms. Han and the children edited their posed problems into a book. "Do not overlook the value of given problems that could not be solved" (TJ2, TI3, CS2, CS3). This is especially true for grade 4 and grade 5 children. The teacher and children could do follow-up work. "I select problems that cannot be solved and talk about ways to convert those problems into mathematics problems that can be solved" (TI3). "When they came across problems that could not be solved, I invited revisions to sufficient problems and to open for more opportunities in learning" (TI3). However, teachers must direct children's attention to the problematic problem when children failed after one or two attempts. For example, in binding two pieces of ribbon, the length will be shortened and so adding two lengths is improper. Also, when the children's critique was superficial ("the characters are too small" or "the handwriting is ugly"), then the teacher would ignore the comment or simply say "thank you, what else," thus, not encouraging comments not related to problem posing/solving (TI1, TJ1, CO1).

The fifth is to attend to opportunities to learn: "If you do not hit you miss!" (*TI1*). From classroom observations, the investigator captured how a teacher's decision would create or inhibit opportunities to learn. For example, the teacher set up an MPP task for grade 2 (30 children were divided into seven groups). She posted eight cutout geometric shapes on the blackboard.



During the time when the teacher educator visited the class, the children worked in groups as usual. They read the blackboard and made up a problem together in a group. There were altogether seven problems; one from each group.

1. Which shape can be used to make a square?
2. Which figures are the same? Please circle them.
3. Name the figures posted on the chalkboard.
4. What figure can we make using these shapes?
5. Which cards are identical?
6. Which ones are squares? Which ones are rectangles?
7. How many shapes are there in the following figure?



Ms. Lin, in addition to collecting the seven problems, purposely invited each group to present its problem. Each group used a marker to write the problem on a piece of

A3 size paper (horizontal). One representative came to the blackboard and pinned it to the blackboard using magnets. The representative presented the problem orally, asked if peers understood, edited, and then asked friends to solve the problem. These actions represented an intention to expand from focusing only on what problems were posed to communicating peers' ideas and to add new knowledge by editing and solving problems.

In the discussion of the problem, "Which shape can be used to make a square?", a friend came up and used two triangles to form a square (⧅). Ms. Lin thought that was the only solution, but later another child made an attempt and made another square using two rectangles (◫). The teacher then directed the child who posed the problem to change the question stem "Which shape…" to "Which shapes…" as there were two possible shapes that could make squares. When similar problems were given, such as problems 2 and 5, the teacher took this opportunity to teach congruence although it was not intended for grade two, but she did not use the term "congruence." Ms. Lin went to the chalkboard and used overlapping shapes to show why the shapes were the "same" or "identical," using the children's words. She praised the students, noting that all the problems were creative, and then added, "In addition to asking for shapes that were the same/identical, there were problems involving naming shapes and combining shapes too." In all, she manipulated the responses and used them to develop into discussions that led to deeper understanding of the topic. She constantly collected children's work using the copying machine or asking children to hand copy the work. "Displaying work reminded them of the successes in MPP" (TJ1). "I found that asking kids to hand copy is more efficient (faster than copying machine) and more reliable (you may forget to copy the next day) (TJ2; TI1).

MPP Instruction and Children's Learning

Wait time is needed for children to be familiar with MPP activity. The investigator got this message from the journals of teachers. At the beginning stage (first 3 weeks) of introducing MPP, children tended to ask many questions. The frequently asked questions were: "What problem do I need to pose?" "Do I need to solve the problem?" "What actually is a problem?" "Who is going to solve this problem?" "Can I draw a figure?" However, after they got used to MPP, children were exceedingly involved and devoted time to discussion. In their given problems, they included persons they liked (talent contest winner in the school, superman in movies) and interesting things they do (building sand castles). To them, a problem consisted of information given in text. Later, they learned that they can "see" and pose mathematics problems from almost any given text or drawing.

Children attended more to the structure and reasonableness of a mathematics problem after doing MPP. The teachers witnessed the children's recognition of data in problems after they incorporated MPP into instruction. Before MPP was introduced, children solved textbook problems or problems given by teachers. They were unaware of problems with insufficient data and did not notice what missing data was needed. After children attempted MPP, they challenged if the textbook problems were relevant to the unit (TJ3), paid more attention to presentation of problems given by friends (TJ2), and learned to reply to a query (TJ1, TJ2) and to spot mistakes in test items (TI2, TJ3, CS3).

Children learned to attend to both the realistic and mathematics parts of a problem. Children learned to link mathematics to real-life situations. "Did you ever hear a song that lasted for 4 h? Children did not connect problem posing to real situations?" (TI3). When the problem is "The height of Ming is 35 cm and the height of his father is 2 cm taller," Ms. Lin explained to grade 2 children that when a baby is born, its length is already more than 35 cm. "I must teach children to relate lengths to real life." In a problem on air temperature, one posed problem was "The air temperature at noon is 30° and the air temperature at midnight is 20, find the total temperature." Ma explained to children that merely adding the two numbers and getting 50 would not add any knowledge on air temperatures. After the teacher's explanation, children changed the goal to find the difference. The children knew that the answer to this new problem represented a drop of temperature from noon to midnight.

There was one interesting result on posing division problems. Attention to real-life situations can enable children to find answers without doing any division. "I counted 40 legs and 10 frogs in a cage, how many legs are there in a frog?" While children were busy with doing division, one child exclaimed, "Wait a minute, there is no need to divide (40 by 10). A frog has 4 legs!" The whole class laughed their hearts out. This attention to the realistic part of a problem was also a theme in a prior study on division (see Chen et al., 2011).

Building up socially shared meanings and ending up with consensus. English (1997) has acknowledged the social as well as psychological value of in-class problem-posing activities. In this study, children learned to express ideas when working in groups. When sharing problems, they learned to express themselves clearly in writing so that friends did not argue or misunderstand. One teacher commented that when children befriended other children then the lessons went very smoothly (TJ2). One child remarked, "My presentation was not the best but I already tried my best and the whole class applauded!"

In one instance, the teacher invited the students to submit test items. After a test made with their peers' contributed test items, the children complained that one problem was really cool but very difficult and asked who the author was. The teacher kept the information secret. After a few minutes, the child who posed that problem boldly admitted that the item was prepared by him. "It's ME, I made up this item. I like problems that are challenging…" (CS3; CI3). Teachers remarked that in MPP children shared joy as well frustrations in solving but they expressed how they liked this sharing and enjoyed the friendship.

Children's scripts of problem posing indicated the mathematical knowledge they possessed or missed. Based on children scripts, teachers could trace what children knew. For the Number Strand, in a task with a horizontal addition algorithm ($38+16=54$), children were able to ask change problems (e.g., There are 38 insects in the bush, 16 flew in, how many in all?) and group problems (Two rows of toy cars are combined to form one; if there are 38 in the first row and 16 in the second, how many are there in the combined row?). For grade two, children could pose two-step problems matching a horizontal addition algorithm ($25+8-7$) using a bus stop situation (e.g., There are 25 children in a school bus. Eight get on the bus at the first stop and seven get off at the second stop. "How many children are in the bus?" For quantity and measurement, a calendar (April) was used for children to make up problems (grade 4). Children initially asked about the meaning of the shadings in the calendar (which meant public holidays) but they were not related to mathematics. The others were mathematics problems related to reckoning the total number of days in April. In addition, the problems allowed teachers to become aware of the children's attention to festivals, wedding parties, and paying a fine for overdue books. There were imaginative problems as well (e.g., Peter Pan flew to Paris on Monday and returned home in 3 days. What day of the week was it?). For grade 5, in a speed relationship situation, children manipulated the given state (known) and the goal state (unknown) to make more problems. After MPP they were excited to find a rule: given two of the three (distance, time, speed), they could find the third. The problem "Given the time and speed then find the distance" could be converted to two other problems. The first would be "Given the speed and distance then find the time" and the second would be "Given the distance and time then find the speed." When the students posed a problem for peers to solve, the teacher would suggest the children to solve the problem at their ability levels. For example, if the posed problem was "40 divided by 7" children could either reply "5 and 5/7" or "5, remainder = 5." From the children's scripts responding to each teaching unit and over time, the teachers reported a reduction in ill-posed problems and an increase in the creation of feasible mathematics problems.

Conclusions and Implications

As stated in Leung (2013), problem-posing research is needed to explore the conditions that allow students to perform well in MPP and for teachers to implement it in instruction. The findings in this study suggest that given ample time for intervention and to involve active teachers, problem posing can be integrated into the elementary mathematics curriculum. In this study, teachers used tasks that were structured or semi-structured and did not use an entirely free format. They also extended their knowledge of research-based tasks (Leung, 2013; Reitman, 1965; Silver, 1994; Stoyanova & Ellerton, 1996) to incorporate MPP into specific content in the unit they were teaching (e.g., speed, addition, volume, or division).

When teachers are given MPP tasks and are willing to introduce MPP instruction, what else is needed to make children competent in problem posing, and in turn, contribute to mathematics learning? A promising MPP task and a proficient teacher is a start. But a task cannot guarantee that learning will happen. The way teachers set up tasks, how teachers interact with students, and the way children work with peers in groups all contribute to enhancing learning (e.g., modeling and problem posing; English, Fox, & Watters, 2005). Finally, the teachers in this study made problem posing and solving a combined activity. The key to children being able to actively construct sensible problems relies on the actions of teachers in enacting tasks, reflections upon practice, and inventing innovative steps in MPP instruction.

The recruitment of the three teachers in action research improved the teacher educator's knowledge on practitioners' know-how regarding MPP implementation. The increasing involvement of teachers over time (Hensen, 1996) made a difference. The teacher educator also confirmed the importance of having a common goal in a partnership for collaborative action research; the teacher educator and teachers work together toward a specific goal (e.g., a graduate thesis). Working with the results from all 3 years of the project, the teacher educator and teachers shared what they learned with colleagues and also parents (Leung, 2012, 2013). Materials from the project were also shared on the website of the teacher educator and in books (Leung, 2008, 2015). After having conducted research on children engaging in MPP, and after investigating how to develop tasks for in-class MPP, future studies on MPP include the possibility of working with parents to extend MPP from school to home.

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Mathematical Problem Solving and Teacher Professional Learning: The Case of a Modified PISA Mathematics Task

Edward A. Silver

Problem solving is a core activity in mathematics classrooms at all levels of schooling across the world. Problems are central to mathematics teaching and learning and constitute the basis for intellectual activity in the classroom (Lampert, 2001; Stein, Smith, Henningsen, & Silver, 2000). Thus, mathematics problems form the foundation of students' opportunities to learn mathematics. In turn, the anticipation, examination, and evaluation of students' work on problems constitute a substantial portion of the work of mathematics teachers. Thus, consistent with the so-called practice-based approach to teacher professional learning, the anticipation and examination of students' solutions to mathematics problems should be a strategic site for teachers to learn in and from their instructional practice (Kazemi & Franke, 2004; Krebs, 2005). Yet, teacher learning does not occur as an automatic consequence of their using mathematics problems with students or witnessing the attempts of students to solve problems. Opportunities for teacher learning in and through close examination of aspects of instructional practice appear to be dependent on if and how professional development cultivates teacher inquiry and reflection (Little, Gearhart, Curry, & Kafka, 2003).

The notion that teacher learning can emerge from focused inquiry into and reflection on aspects of their normal is often called practice-based professional development. Ball and Cohen (1999) suggested that the everyday work of teachers could be a rich source for the development of a curriculum for professional learning grounded in the tasks, questions, and problems of practice. To accomplish this goal, they argued that records of authentic practice (e.g., tasks used in instruction or assessment, samples of student work) should become the core of professional education, providing a focus for sustained teacher inquiry and investigation. Other scholars have also pointed to the potential benefits of having teachers learn in and through

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professional practice (e.g., Ball & Bass, 2002; Crockett, 2002; Lampert, 2001; Little, 1999, 2002, 2004; Smith, 2001; Stein et al., 2000; Wilson & Berne, 1999).

This paper focuses on a group of secondary mathematics teachers in the USA whose work with a mathematics problem in the context of a teacher professional development program provided them with an opportunity for inquiry into and reflection on their practice. In particular, I describe the teachers' interactions with the mathematics task over time, including their own solution of the task, their anticipation of how students would solve (or attempt to solve) the problem at various grade levels, their unstructured examination and evaluation of student work on the problem, and their structured consideration of student work with a particular focus on students' strategies and representations. Through this detailed examination of teachers' work with this mathematics problem, I illuminate how mathematical problem solving, as a core activity in mathematics classrooms, can be a strategic site for those interested to practice-based approaches to teacher professional learning.

The Teachers

The teachers whose engagement and learning are described in the paper were participants in DELTA (Developing Excellence in Learning and Teaching Algebra)—a 3-year, multifaceted professional development initiative intended to support teachers in mathematics in the middle and secondary grades (grades 7–11) in Oakland County, Michigan, with a particular focus on improving instruction in algebra. In this paper, I draw on a slice of work undertaken during the first 2 years by teachers and professional development specialists involved in one component of DELTA—*Who's On First? Building Coherence and Connections Across Grade Levels*, hereafter referred to as the curricular coherence component of DELTA.

The goal of the curricular coherence component of DELTA was to assist teams of teachers from participating school districts to develop a *coherent* vision of algebra concepts, skills, and reasoning as it might be taught and learned across grades 7–11. Almost 100 teachers of middle grades and high school mathematics participated in at least a portion of the curricular coherence component of DELTA during the 2-year period of interest in this paper. There were 56 participants in year 1, 26 of whom continued in year 2, when 66 new participants joined them. The teachers were drawn from 13 different school districts that were demographically varied with respect to the socioeconomic status of the school communities and the ethnic composition of the student population.

The Apples Task

The Apples task and related student work used with the DELTA project teachers was an adapted version of an item [M136, Apples] that originally appeared on the mathematics assessment portion of the Programme for International Student

Assessment (PISA). It was one of the 50 PISA mathematics tasks that were publicly released in 2006 (OECD, 2006).

PISA is a collaborative effort of member countries of the Organization for Economic Cooperation and Development (OECD). The main objective of PISA is to provide policy-relevant data on the *yield* of education systems. The assessed population is 15-year-olds, an age that marks the end of compulsory schooling in most OECD member countries. PISA assesses how well 15-year-old youth are able to use the knowledge and skills they have acquired in school to meet the literacy-related challenges they are likely to face outside of school as adult citizens. PISA focuses on literacy—the ability to use and apply knowledge and skills to real-world situations encountered in adult life—in the key subject areas of reading, mathematics, and science. The frameworks guiding the PISA assessments reflect a consensus across the OECD countries regarding the skills and abilities that demonstrate literacy in these areas. For the 2003 assessment, PISA defined mathematical literacy as follows:

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individuals' life as a constructive, concerned and reflective citizen.* (OECD, 2003, p. 24)

Compared to the original Apples task (OECD, 2006, M136, pp. 11–14), the DELTA version (see Fig. 1) incorporated two variations. One was minor: replacing the word conifer with the word pine, thereby using a word thought to be more familiar to students in Michigan than the original wording. The other was a major revision of the wording of question 3.2.

The rewording of question 3.2 was intended to make the task more accessible to middle school students. Question 3.2 appeared as follows in the PISA version of the Apples task:

> There are two formulae you can use to calculate the number of apple trees and the number of conifer trees for the problem described above:
>
> Number of apple trees = n^2
>
> Number of conifer trees = $8n$
>
> Where n is the number of rows of apple trees.
>
> There is a value of n for which the number of apple trees equals the number of conifer trees. Find the value of n and show your method of calculating this.

The task modifications were intended to increase comprehension and accessibility for middle school students without affecting other key features of the task. In particular, the variation preserved the treatment of standard content in novel ways (e.g., juxtaposing a linear and quadratic pattern in the same problem context, including basic pattern finding with sophisticated reasoning about rates of change in the same item) and the cognitive complexity of the task (e.g., the use of multiple representations; calling for a range of processes, including analyzing, generalizing, and comparing). In fact, it could be argued that the modification may have increased the cognitive complexity of the task by making it more open ended than the original version.

Adapted from Mathematics Sample Tasks
OECD's 2009 PISA Assessment

Mathematics Unit 3: Apples

A farmer plants apple tree in a square pattern. In order to protect the apple trees against the wind he plants pine trees all around the orchard.

Here you see a diagram of this situation where you can see the pattern of the apple trees and the pine trees for any number (n) of the rows of apple trees:

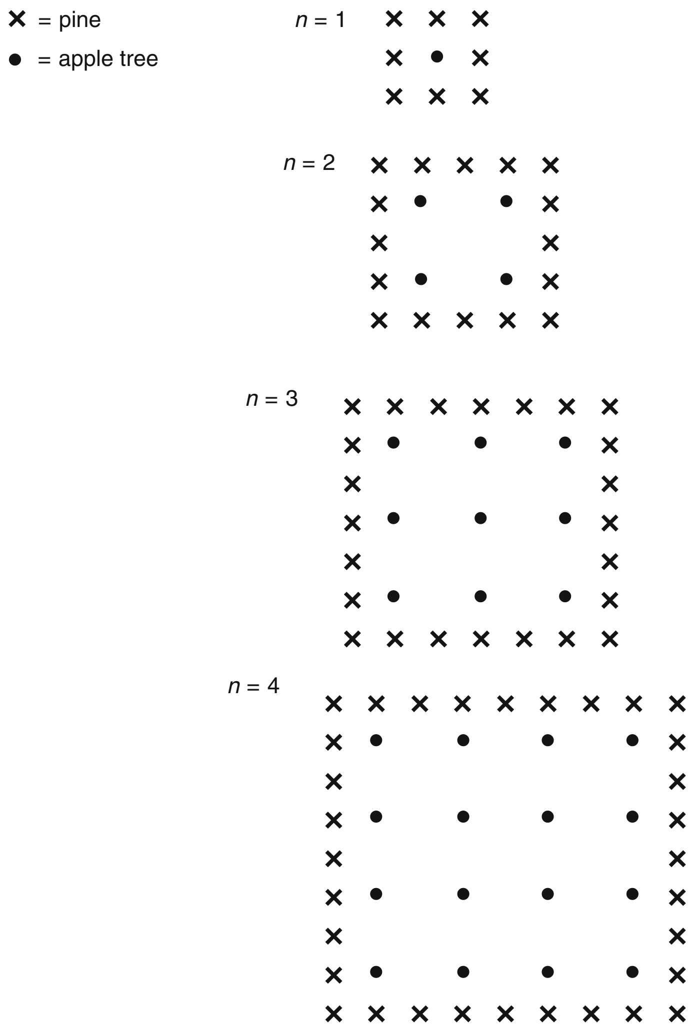


Fig. 1 The Apples task used with the DELTA project teachers

Adapted from Mathematics Sample Tasks
OECD's 2009 PISA Assessment

Question 3.1

Complete the table:

n	Number of apple trees	Number of pine trees
1	1	8
2	4	
3		
4		
5		

Question 3.2 [Note: different wording than in original PISA task]
Describe the pattern (words or symbols) so that you could find the number of apple trees for any stage in the pattern illustrated on the previous page:

Describe the pattern (words or symbols) so that you could find the number of pine trees for any stage in the pattern illustrated on the previous page:

For what value(s) of *n* will the number of apple trees equal the number of pine trees. Show your method of calculating this.

Question 3.3
Suppose the farmer wants to make a much larger orchard with many rows of trees. As the farmer makes the orchard bigger, which will increase more quickly: the number of apple trees or the number of pinet rees? Explain how you found your answer.

Fig. 1 (continued)

Using PISA Tasks in Teacher Professional Development

A central premise of a project that I direct—Using PISA to Develop Activities for Teacher Education (UPDATE)—is that PISA tasks can be a valuable resource for practice-based professional development aimed at assisting mathematics teachers to inquire into and reflect upon their instructional practice. In the UPDATE project, we have been exploring some potential uses of PISA tasks and data.

The project rests on a belief that PISA tasks and data can be useful in much the same way that the (US) National Assessment of Educational Progress tasks and data have long served as key resources for the US mathematics education community (e.g., Blume, Zawojewski, Silver, & Kenney, 1998; Brown & Clark, 2006; Kouba et al., 1988; Silver & Kenney, 2000). In UPDATE we have developed some prototype, PISA-based materials and partnered with other professionals to use the materials in initial teacher preparation settings and teacher professional development contexts with teachers of mathematics in grades 7–11.

Using PISA Tasks in the DELTA Professional Development Project

The teacher professional development episodes described in this paper were the result of a partnership between the UPDATE project and the DELTA project. As noted above, DELTA was a mathematics teacher professional development project conducted under the auspices of Oakland (Michigan) Schools. DELTA consisted of multiple components involving several hundred teachers from schools in Michigan; one component involving more than 100 teachers for 2 years focused on enhancing curricular coherence in the treatment of algebra topics across grades 7–11.

Among the set of tasks used with teachers in the curricular coherence component of the DELTA project, there were two PISA tasks employed. In this paper I summarize what transpired with one of the tasks, the Apples task. Silver and Suh (2014) provide a more complete account of the DELTA project and the use of the Apples task in the project. Interested readers can find there more information than I provide here.

PISA items were deemed to be good candidates for use in the curricular coherence component of the DELTA project for several reasons. First, many PISA tasks require the use of algebra skills, concepts, or processes. The Apples task, for example, involves legitimate algebraic content, including both linear and nonlinear (quadratic, in this instance) relationships and encompasses a range of algebraic skills, concepts, and processes, as a solver analyzes, generalizes, and compares two distinct patterns.

Second, PISA tasks often involve multiple forms of representation. The Apples task, in particular, involves a verbal representation of a situation, associated with a corresponding visual representation. Tabular and symbolic representations are used in subtasks, and the last subtask asks students to explain their reasoning verbally.

Third, because PISA tests the residual, usable knowledge gained by 15-year-old students, the tasks tend to involve applications of knowledge to problems that are embedded in real-world contexts and are not tied to specific formats and exercise types associated with particular curriculum topics in mathematics courses. In the case of the Apples task, there is a contextual embedding, though it is not as interesting or authentic as in many PISA tasks, but the task exemplifies well the way that PISA tasks often step outside curriculum boundaries. In particular, the task involves both linear and quadratic relationships, and the third part of task moves beyond simple equations to consider rates of change in a manner that approaches topics taught in calculus.

The modified Apples task was used in several different ways with the DELTA project teachers on multiple occasions. We summarize here the varied uses of this single task in this professional development initiative because we think they illustrate a range of possible uses of many PISA tasks in teacher professional education settings.

DELTA Teachers Work with the Apples Task

The Apples task was introduced in the third professional development session during the first year of the curriculum coherence component. Prior to this session, participants had examined the state curriculum objectives for grades 7–11, with particular attention to proportionality, linear and quadratic relations, and functions. They had also begun to formulate teaching/learning trajectories for these topics. Teachers began their work with the task at this session, and they continued engaging with the task over several months in several different ways that I summarize here. For more details see Silver and Suh (2014).

Teachers Solve the Problem

When the Apples task was first presented to the DELTA teachers, they were asked to solve the problem individually. Then they met in small groups to discuss and compare solution approaches. This provided teachers an opportunity to familiarize themselves with the mathematical concepts and skills associated with the problem. In this way, they were able to establish the relevance of the task to the mathematics they teach, even though the task presentation likely differed from what they would find in the textbooks used in their classrooms.

Teachers Predict How Students Will Solve the Problem

After solving the problem, DELTA teachers were asked to anticipate what students at the grade level they taught would be likely to do if asked to solve the problem. After working individually, they met in small grade-alike groups to develop a list of shared anticipations for students at each grade level. The expectations were recorded on posters and displayed for general discussion.

The record of initial expectations came to play an important role in the learning of the DELTA project teachers. Subsequent examination of student work on the problem confirmed some of the teachers' expectations and challenged others. As we saw in a subsequent session, the surprises afforded especially important opportunities for teacher learning.

Teachers Examine the PISA Scoring Rubric

Teachers were provided with the PISA scoring guide, which is available for each of the publicly released tasks. They could see in the guide how PISA assigned points for various kinds of responses. Because the PISA task was modified when used in the DELTA project, only the portion of the rubric pertaining to the first and third questions was considered.

Teachers Collect Student Work on the Problem

As a homework following the session in which they solved the Apples task, DELTA participants were asked to administer the task to at least one class of students, if feasible to do so. Collecting student work allows teachers to watch their own students solve the problem. It also provides a set of student responses that can be pooled across teachers to get more substantial sample of responses within and across grades. In DELTA, the teachers collected more than 900 responses from students in classrooms ranging from grades 5 to 12 and enrolled in a variety of mathematics courses (e.g., grade 7, Algebra I, Algebra II, Pre-calculus). The diversity of student responses provided a rich resource for subsequent examination and analysis.

Teachers Examine the Student Work on the Problem

DELTA teachers were asked to examine the solutions produced by their students and then to meet with a grade-level colleague to examine all the student responses at their grade level. In their initial examination, they were asked to identify what the responses reveal about what students appear to understand and appear not to understand and what implications their observations might have for instruction. During the session, the grade-level group observations were recorded on poster paper and displayed in the room to facilitate a large group discussion that occurred later in the day.

Just as students can sometimes make discoveries while exploring problem situations that influence their sense of identity and agency, this type of activity on the part of teachers—a minimally guided exploration of student work—may yield profound insights. But some teachers may benefit from a more structured approach.

For a variety of reasons discussed by Silver and Suh (2014), including an emphasis on content coverage rather than on developing individual student understanding, many secondary school mathematics and science teachers tend to focus almost exclusively on correctness when examining student work. Davis (1997) characterizes this as an evaluative rather than interpretive orientation toward teaching. According to Crespo (2000), secondary school mathematics teachers tend to have an evaluative orientation in which they listen to students' ideas in order to judge them correct or incorrect and to detect and correct misunderstandings, similar to what Otero (2006) called a "get it or don't" conception of assessment.

An evaluative orientation was quite apparent among many DELTA teachers when they initially examined the student work on the Apples task. The poster displays and the grade-level and whole group discussions focused almost exclusively on right/wrong categories and an elaboration of students' errors and apparent misunderstandings, such as difficulties in setting up an equation to solve question 3.2c, missing 0 as a solution, rendering the repeated addition of 8 as $n+8$ rather than as $8n$, and confusing quadratic and exponential growth patterns. The professional development leaders had hoped for more attention to students' understandings, so they decided that it would be beneficial to return to the student work one more time in a future session, with an eye toward shifting teachers' attention to aspects of student performance other than correctness.

Teachers Analytically Examine the Student Work on the Problem

The research team from the UPDATE project undertook an independent analysis of the student work on the modified Apples task, paying particular attention to students' use of representations and strategies on questions 3.2 and 3.3. Two general observations emerged from our examination of the student work that we judged to have potential to engage the DELTA participants:

- When making claims and representing generalizations, students in upper grades and advanced classes were more likely to use mathematical symbolism and equations than middle school and lower-level mathematics class students who relied more often on verbal descriptions. Yet, even in upper-level classes, a sizeable number of students used verbal descriptions to express generalizations in this problem.
- Some students at all grade levels used recursive strategies to solve subtasks 3.2a and 3.2b, with more using recursion for subtask 3.2b; students using recursion used only verbal descriptions rather than symbolic expressions to express their generalizations.

This analysis suggested a scheme that might be useful in drawing teachers' attention to more than right/wrong aspects of student work on the problem. Following our analysis of the student work, we created packets of student responses that contained specific examples to reflect the major strategies and representations evident in the full sample of student work: recursive description, recursive equation, explicit description, and explicit equation. Figure 2 provides sample responses of each type.

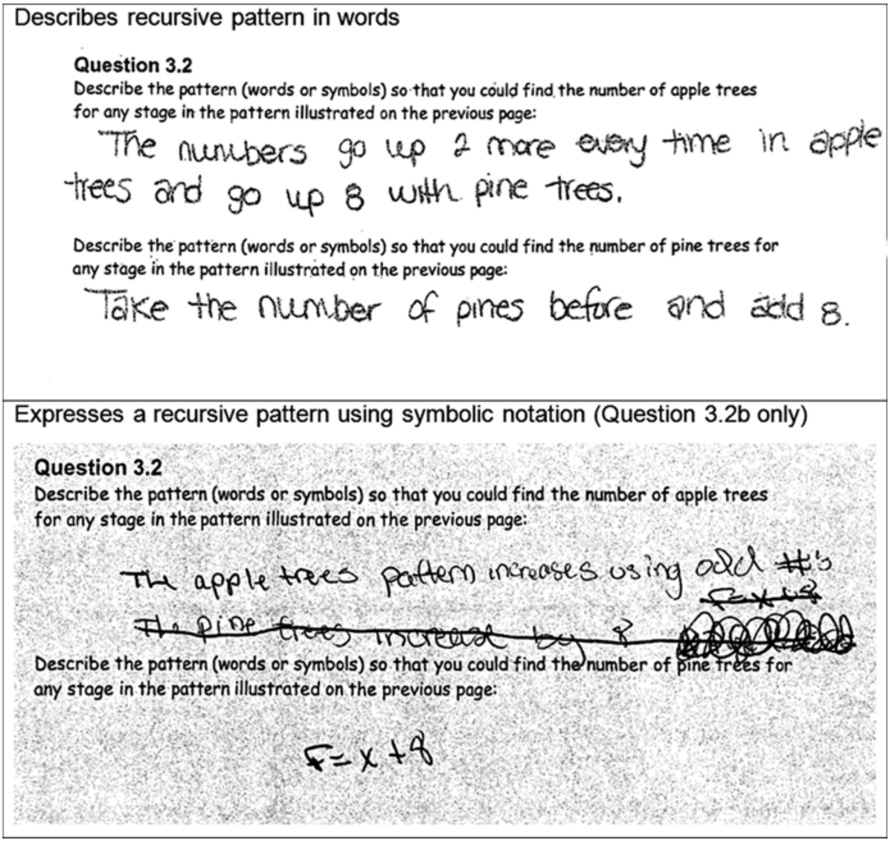
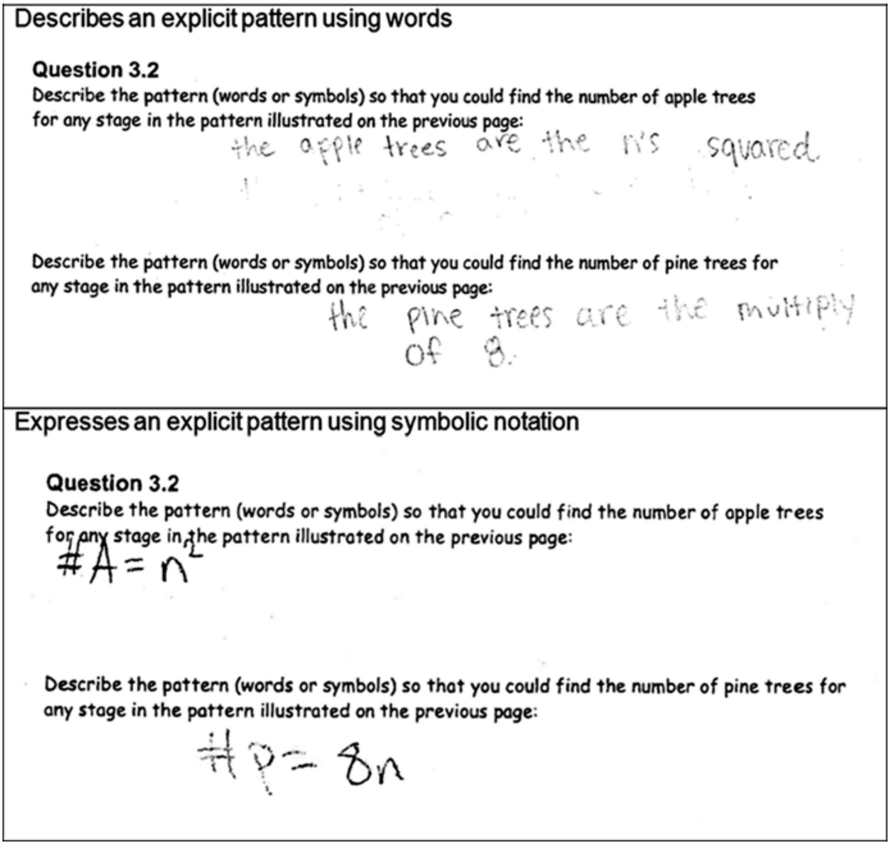


Fig. 2 Sample student responses

The response packets were used with the DELTA teachers in a subsequent professional development session when the Apples task and student work once again became a focus of attention. Teachers were given the packets of student responses, and they were asked to sort the responses to questions 3.2a and 3.2b into the following categories: describe recursive pattern in words, (try to) express a recursive pattern using symbolic notation, describe an explicit pattern using words, and express an explicit pattern using symbolic notation.

Teachers Predict Frequency of Response Types

DELTA teachers were also asked to predict the percent of students who would be likely to produce each type of response at the grade level they teach (i.e., 15 % of grade 8 students will use words to describe a recursive pattern in question 3.2a). Teachers worked individually at first, then in pairs, and finally in grade-level groups to compare and refine their predictions.


Describes an explicit pattern using words

Question 3.2

Describe the pattern (words or symbols) so that you could find the number of apple trees for any stage in the pattern illustrated on the previous page:

the apple trees are the n's squared.

Describe the pattern (words or symbols) so that you could find the number of pine trees for any stage in the pattern illustrated on the previous page:

the pine trees are the multiply of 8.

Expresses an explicit pattern using symbolic notation

Question 3.2

Describe the pattern (words or symbols) so that you could find the number of apple trees for any stage in the pattern illustrated on the previous page:

#A = n²

Describe the pattern (words or symbols) so that you could find the number of pine trees for any stage in the pattern illustrated on the previous page:

#P = 8n

Fig. 2 (continued)

Grade-level predictions were shared and discussed briefly in a whole group session. In general, the predictions were that, as students progressed across the grades and through mathematics courses, they would become far more likely to express generalizations explicitly rather than recursively, and they would also be far more likely to use symbolic expressions and equations rather than verbal descriptions. Once again, by having the teachers make such predictions, the professional developers hoped that the presentation of actual findings might include some surprises that could stimulate teacher learning.

Teachers Consider a Comprehensive Analysis of Student Responses

The UPDATE team presented its coding and analysis of the entire set of more than 900 student responses. For questions 3.2a and 3.2b, graphs were created to depict the frequency of student responses that expressed the generalization explicitly or

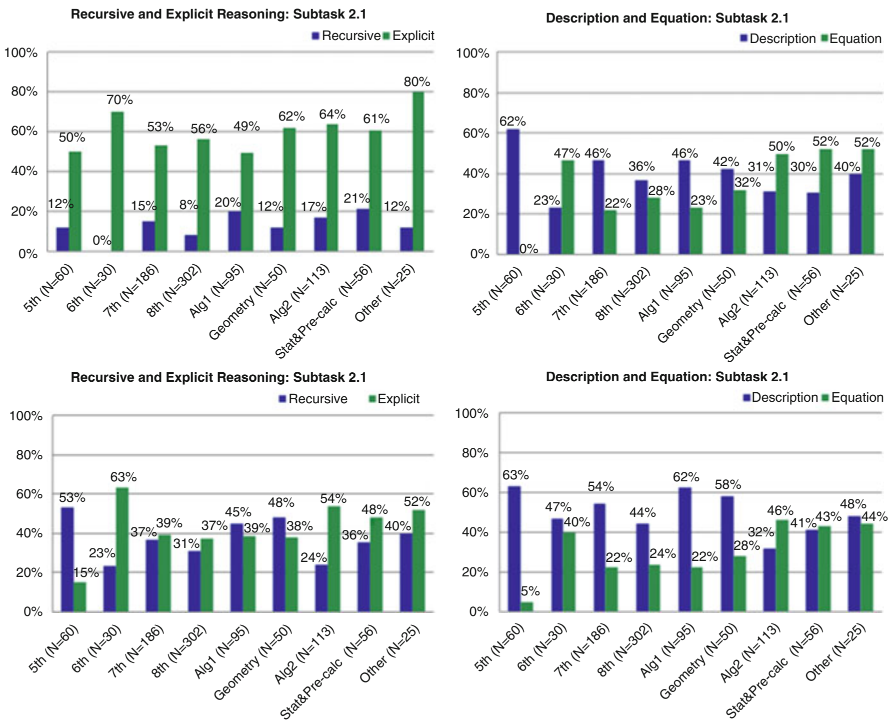


Fig. 3 Graphs depicting student response rates by type and grade

recursively and that used verbal descriptions or symbolic expressions (see Fig. 3). The graphs vividly displayed the ways in which the student work aligned with or deviated from the teachers' predictions. For example, looking across the grades, the graphs revealed not only a trend toward expressing generalizations explicitly and with symbolic expressions but also an unexpected persistence of both recursive reasoning and verbal descriptions.

The findings of the UPDATE analysis were discussed briefly in whole group and then the participants met in grade-alike groups to discuss the findings and graphs as they pertained to their grade level. Teachers were encouraged to identify instructional issues raised by these findings—issues that pertained within their grade level and issues that might pertain across grade levels. Participants actively discussed and debated the findings and possible implications, moving fluidly between the graphs of general findings and the specific student responses that were available to them in the packets examined earlier in the day. Following discussion in grade-alike groups, the participants moved into cross-grade groups that mixed middle school and high school teachers. In these groups, participants discussed what the findings of this analysis suggested about what students were and were not learning from their mathematics instruction at each grade level and also across grade levels in order to increase curricular coherence. In their discussion they considered a number of critical issues related to using pattern generalization problems to assist students in learning to express generalizations algebraically; the complexity of such learning has been examined by several researchers interested in the development of algebraic proficiency (e.g., Lannin, 2005). Thus, the DELTA teachers' cumulative experience with the Apples task served as a springboard to their delving deeply into mathematical curricular issues that directly affected their instructional practice.

Reflecting on the Apples Task: Problem Solving and Professional Learning

Though our presentation of the Apples task experience was necessarily brief and general, we think it embodies several points regarding the use of PISA tasks as stimuli for STEM teacher professional learning. One point is that the experience illustrates the diversity of ways that a PISA task might be used to stimulate teacher engagement and learning. The set of activity settings used in DELTA was extensive, and yet it represents only a sample of possibilities. Readers will undoubtedly be able both to generate other uses of the items for preservice and in-service teacher education settings and to think of variations on the specific activities and formats employed in DELTA. Moreover, it is important to think about the cumulative effects of a sequence of activities. In DELTA, the final professional learning activity appeared to have been critically important, but the experience of project participants in solving the tasks and predicting student solutions on prior occasions almost certainly played an important role in creating the learning opportunities that were manifested on that occasion.

A second point is that PISA tasks can be used as found in PISA or modified to fit the needs of a particular teacher education context. The original PISA Apples task was a challenging mathematical task that treated important mathematics concepts and skills and allowed for many legitimate uses as a stimulus for teacher professional learning.

The modification that was made when the task moved from PISA to DELTA did not reduce or alter the complexity of the original PISA task, yet it turned out to be important for two reasons. First, though it retained the mathematical character of the original PISA task, it made the task more accessible to middle school students who had not yet been taught to write and solve algebraic equations. Second, the modification opened the door to students' use of recursive reasoning to express the generalization. Our hunch is that recursion would have been far less likely to appear in the student work if the original PISA version of question 3.2 had been used, and the salience of recursion in the student work turned out to be a source of surprise for the teachers and thus an opportunity for their learning.

A closely related point is that the mixing of middle school and high school teachers in the participant group was useful for the teachers' work with the Apples task. The hybridity of the participant group made available a range of perspectives on how students might solve the task, generated a rich sample of student work, and supported participants' consideration of cross-grade curricular coherence issues. As Silver and Suh (2014) note, these factors played a role in the learning opportunities available to the DELTA teachers.

A final point is the importance of designing activities in ways that allow teachers, especially secondary school teachers, to move beyond a simple right/wrong evaluation of student work. In DELTA, participants made significant progress when they were presented with specific examples of student responses chosen in advance to represent particular strategies and representations and then directed to examine student responses using criteria that drew their attention toward matters of strategy and away from considerations of correctness.

Coda

I recently presented the original and modified Apples task to another group of mathematics teachers, asking them to predict how they might expect students to solve each version. As was the case with the DELTA teachers, this group also did not predict the use of recursion by students solving the modified version of the task. When I later showed them the graphs displayed in Fig. 3, they were surprised to see the frequent use of recursive approaches by students across the grades.

After some discussion, one of the teachers commented that the appearance of recursion was interesting not only for the reasons discussed earlier in this paper but also because recursive thinking could lead to a quite different solution of question 3.3, even though it was stated identically in both versions of the task. He went on to explain that students would be almost certain to focus on the relationship between n^2 and $8n$ to answer question 3.3 in the original PISA version of the task. In fact, this is the expectation evident in the PISA scoring guide (OECD, 2006, p. 13): "Full Credit. Correct response (apple trees) accompanied by a valid explanation. … algebraic explanation based on the formulas n^2 and 8n." In contrast, students who used a recursive approach to solve question 3.2 on the modified version of the task would be likely to focus on the change in difference between successive entries in the table for the apple trees, noting that the difference increases each time and that after $n=5$ the difference for the apple trees will always exceed the constant difference for the pine trees. Though such an explanation would be unlikely to gain full credit using the PISA scoring guide, it is mathematically correct and quite elegant in its own right.

In this teacher's observation, we see once again the interplay between mathematical problem solving and opportunities for teacher professional learning. PISA tasks should be useful resources in supporting the creation of such opportunities now and in the future.

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Building Thinking Classrooms: Conditions for Problem-Solving

Peter Liljedahl

In this chapter, I first introduce the notion of a thinking classroom and then present the results of over 10 years of research done on the development and maintenance of thinking classrooms. Using a narrative style, I tell the story of how a series of failed experiences in promoting problem-solving in the classroom led first to the notion of a thinking classroom and then to a research project designed to find ways to help teachers build such a classroom. Results indicate that there are a number of relatively easy-to-implement teaching practices that can bypass the normative behaviours of almost any classroom and begin the process of developing a thinking classroom.

Motivation

My work on this paper began over 10 years ago with my research on the AHA! experience and the profound effects that these experiences have on students' beliefs and self-efficacy about mathematics (Liljedahl, 2005). That research showed that even one AHA! experience, on the heels of extended efforts at solving a problem or trying to learn some mathematics, was able to transform the way a student felt about mathematics as well as his or her ability to do mathematics. These were descriptive results. My inclination, however, was to try to find a way to make them prescriptive. The most obvious way to do this was to find a collection of problems that provided enough of a challenge that students would get stuck, and then have a solution, or solution path, appear in a flash of illumination. In hindsight, this approach was overly simplistic. Nonetheless, I implemented a number of these problems in a grade 7 (12–13 year olds) class.

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The teacher I was working with, Ms. Ahn, did the teaching and delivery of problems and I observed. Despite her best intentions the results were abysmal. The students did get stuck, but not, as I had hoped, after a prolonged effort. Instead, they gave up almost as soon as the problem was presented to them and they resisted any effort and encouragement to persist. After three days of constant struggle, Ms. Ahn and I both agreed that it was time to abandon these efforts. Wanting to better understand why our well-intentioned efforts had failed, I decided to observe Ms. Ahn teach her class using her regular style of instruction.

That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not thinking. More alarming was that Ms. Ahn's teaching was predicated on an assumption that the students either could not or would not think. The classroom norms (Yackel & Rasmussen, 2002) that had been established had resulted in, what I now refer to as, a non-thinking classroom. Once I realized this, I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class, I saw the same basic behaviour—an assumption, implicit in the teaching, that the students either could not or would not think. Under such conditions, it was unreasonable to expect that students were going to spontaneously engage in problem-solving enough to get stuck and then persist through being stuck enough to have an AHA! experience.

What was missing for these students, and their teachers, was a central focus in mathematics on thinking. The realization that this was absent in so many classrooms that I visited motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion.

Early Efforts

A thinking classroom must have something to think about. In mathematics, the obvious choice for this is a problem-solving task. Thus, my early efforts to build thinking classrooms were oriented around problem-solving. This is a subtle departure from my earlier efforts in Ms. Ahn's classroom. Illumination-inducing tasks were, as I had learned, too ambitious a step. I needed to begin with students simply engaging in problem-solving. So, I designed and delivered a three session workshop for middle school teachers (ages 10–14) interested in bringing problem-solving into their classrooms. This was not a difficult thing to attract teachers to. At that time, there was increasing focus on problem-solving in both the curriculum and the textbooks. The research on the role of problem-solving as both an end unto itself and as a tool for learning was beginning to creep into the professional discourse of teachers in the region.

The three workshops, each 2 h long, walked teachers through three different aspects of problem-solving. The first session was focused around initiating problem-solving work in the classroom. In this session, teachers experienced a number of easy-to-start problem-solving activities that they could implement in their classrooms—problems that I knew from my own experiences were engaging to students. There were a number of mathematical card tricks to explain, some problems with dice, and a few engaging word problems. This session was called *Just do It*, and the expectation was that teachers did just that—that they brought these tasks into their classrooms and had students just do them. There was to be no assessment and no submission of student work.

The second session was called *Teaching Problem-Solving* and was designed to help teachers emerge from their students' experience a set of heuristics for problem-solving. This was a significant departure from the way teachers were used to teaching heuristics at this grade level. The district had purchased a set of resources built on the principles of Pólya's *How to Solve It* (1957). These resources were pedantic in nature, relying on the direct instruction of these heuristics, one each day, followed by some exercises for students to go through practicing the heuristic of the day. This second workshop was designed to do the opposite. The goal was to help teachers pull from the students the problem-solving strategies that they had used quite naturally in solving the set of problems they had been given since the first workshop, to give names to these strategies and to build a poster of these named strategies as a tool for future problem-solving work. This poster also formed an effective vocabulary for students to use in their group or whole class discussions as well as any mathematical writing assignments.

The third workshop was focused on leveraging the recently acquired skills towards the learning of mathematics and to begin to use problem-solving as a tool for the daily engagement in, and learning of, mathematics. This workshop involved the demonstration of how these new skills could intersect with the curriculum in general and the textbook in particular.

The series of three workshops was offered multiple times and was always well attended. Teachers who came to the first tended, for the most part, to follow through with all three sessions. From all accounts, the teachers followed through with their 'homework' and engaged their students in the activities they had experienced within the workshops. However, initial data collected from interviews and field notes were mixed. Teachers reported things like:

"Some were able to do it."
"They needed a lot of help."
"They loved it."
"They don't know how to work together."
"They got it quickly and didn't want to do anymore."
"They gave up early."

Further probing revealed that teachers who reported that their students loved what I was offering tended to have practices that already involved some level of problem-solving. If there was already a culture of thinking and problem-solving in the classroom, then this was aided by the vocabulary of the problem-solving posters, and the teachers got ideas about how to teach with problem-solving. It also revealed that those teachers who reported that their student gave up or didn't know how to work together mostly had practices devoid of problem-solving and group work. In these classrooms, although some students were able to rise to the task, the majority of the class was unable to do much with the problems—recreating, in essence, what I had seen in Ms. Ahn's class. In short, the experiences that the teachers were having implementing problem-solving in the classroom were being filtered through their already existing classroom norms (Yackel & Rasmussen, 2002).

Classroom norms are a difficult thing to bypass (Yackel & Rasmussen, 2002), even when a teacher is motivated to do so. The teachers that attended these workshops wanted to change their practice, but their initial efforts to do so were not rewarded by comparable changes in their students' problem-solving behaviour. Quite the opposite, many of the teachers I was working with were met with resistance and complaints when they tried to make changes to their practice.

From these experiences, I realized that if I wanted to build thinking classrooms—to help teachers to change their classrooms into thinking classrooms—I needed a set of tools that would allow me, and participating teachers, to bypass any existing classroom norms. These tools needed to be easy to adopt and have the ability to provide the space for students to engage in problem-solving unencumbered by their rehearsed tendencies and approaches when in their mathematics classroom.

This realization moved me to begin a program of research that would explore both the elements of thinking classrooms and the traditional elements of classroom practice that block the development and sustainability of thinking classrooms. I wanted to find a collection of teacher practices that had the ability to break students out of their classroom normative behaviour—practices that could be used not only by myself as a visiting teacher but also by the classroom teacher that had previously entrenched the classroom norms that now needed to be broken.

Thinking Classroom

As mentioned, a *thinking classroom* is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly. As such, a thinking classroom, as I conceive it, will intersect with research on mathematical thinking (Mason, Burton, & Stacey, 1982) and classroom norms (Yackel & Rasmussen, 2002). It will also intersect with notions of a didactic contract (Brousseau, 1984), the emerging understandings of studenting (Fenstermacher, 1986, 1994; Liljedahl & Allan, 2013a, 2013b), knowledge for teaching (Hill, Ball, & Schilling, 2008; Shulman, 1986) and activity theory (Engeström, Miettinen, & Punamäki, 1999).

In fact, the notion of a thinking classroom intersects with all aspects of research on teaching and learning, both within mathematics education and in general. All of these theories can be used to explain aspects of an already thinking classroom, and some of them can even be used to inform us how to begin the process of build a thinking classroom. Many of these theories have been around a long time, and yet non-thinking classrooms abound. As such, I made the decision early on to approach my work not from the perspective of a priori theory but from existing teaching practices.

General Methodology

The research to find the elements and teaching practices that foster, sustain and impede thinking classrooms has been going on for over 10 years. Using a framework of noticing (Mason, 2002),[1] I initially explored my own teaching, as well as the practices of more than 40 classroom mathematics teachers. From this emerged a set of nine elements that permeate mathematics classroom practice—elements that account for most of whether or not a classroom is a thinking or a non-thinking classroom. These nine elements of mathematics teaching became the focus of my research. They are:

1. the type of tasks used and when and how they are used
2. the way in which tasks are given to students
3. how groups are formed, both in general and when students work on tasks
4. student workspace while they work on tasks
5. room organization, both in general and when students work on tasks
6. how questions are answered when students are working on tasks
7. the ways in which hints and extensions are used, while students work on tasks
8. when and how a teacher levels[2] their classroom during or after tasks
9. assessment, both in general and when students work on tasks

Ms. Ahn's class, for example, was one in which:

1. practice tasks were given after she had done a number of worked examples
2. students either copied these from the textbook or from a question written on the board
3. students had the option to self-group to work on the homework assignment when the lesson portion of the class was done

[1] At the time, I was only informed by Mason (2002). Since then, I have been informed by an increasing body of literature on noticing (Fernandez, Llinares, & Valls, 2012; Jacobs, Lamb, & Philipp, 2010; Mason, 2011; Sherin, Jacobs, & Philipp, 2011; van Es, 2011).

[2] Levelling (Schoenfeld, 1985) is a term given to the act of closing of, or interrupting, students' work on tasks for the purposes of bringing the whole of the class (usually) up to certain level of understanding. It is most commonly seen when a teacher ends students work on a task by showing how to solve the task.

4. students worked at their desks, writing in their notebooks
5. students sat in rows with the students' desk facing the board at the front of the classroom
6. students who struggled were helped individually through the solution process, either part way or all the way
7. there were no hints, only answers, and an extension was merely the next practice question on the list
8. when 'enough time' time had passed, Ms. Ahn would demonstrate the solution on the board, sometimes calling on 'the class' to tell her how to proceed
9. assessment was always through individual quizzes and tests

This was not, as determined earlier, a thinking classroom. Each of these elements was something that needed exploring and experimenting with. Many were steeped in tradition and classroom norms (Yackel & Rasmussen, 2002).

Research into each of these was done using design-based methods (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003)[3] within both my own teaching practice as well as the practices of a number of teachers participating in a variety of professional development opportunities. This approach allowed me to vary the teaching around each of the elements, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

This method, although fruitful in the end, presented two challenges. The first had to do with the measurement of effectiveness. To do this, I used what I came to call *proxies for engagement*—observable and measurable (either qualitatively or quantitatively) student behaviours. At first, this included only behaviours that fit the *a priori* definition of a thinking classroom. As the research progressed, however, the list of these proxies grew and changed depending on the element being studied and teaching method being used.

The second challenge had to do with the shift in practice needed when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice did little to shift the behaviours of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a teaching method around a specific element needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. When sitting showed to be ineffective, we tried making the students stand. When levelling to the top failed, we tried levelling to the bottom. When answering questions proved to be ineffective, we stopped answering questions. Each of these

[3] This research is now informed also by Norton and McCloskey (2008) and Anderson and Shattuck (2012).

approaches needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

In what follows, I will first present the results of the research done on two of these elements—student workspace and how groups are formed—both independently and jointly. I then present, in brief, the results of the research done on the remaining seven elements and discuss how all nine elements hold together as a framework to build and maintain thinking classrooms. All of this research is informed dually by data and analysis that looks both on the effect on students and the uptake by teachers.

Student Workspace

The research on student workspace began by looking at the default—students sitting in their desks. It became obvious early in this work that this was not conducive to the building of a thinking classroom. As such, almost immediately, a new space was explored. Following the contrarian approach established early on, the next space to test was to have students standing and working somewhere other than at their desks. The shift to having students work on whiteboards and blackboards was then an obvious extension.

In many classrooms where the research was being done, however, there were not enough whiteboards and blackboards available for all groups to work at. Some students would have to still be seated in their desks. This led to a phase of experimentation with alternative work surfaces, including poster board or flipchart paper attached to the walls and smaller whiteboards laying on desks—with some classrooms using all three at the same time. Whenever this occurred, there was a general sense shared between whatever teachers were in the room, as well as myself, that the vertical whiteboards were superior to any of the other options available to students. These observations led to the following pseudo-quantitative study focusing on this phenomenon.

Participants

The participants for this study were the students in five high school classrooms; two grade 12 ($n=31$, 30), two grade 11 ($n=32$, 31) and one grade 10 ($n=31$).[4] In each of these classes, students were put into groups of two to four and assigned to one of five work surfaces to work on while solving a given problem-solving task.

[4] In Canada, grade 12 students are typically 16–18 years of age, grade 11 students 15–18 and grade 10 students 14–17. The age variance is due to a combination of some students fast-tracking to be a year ahead of their peers and some students repeating or delaying their grade 11 mathematics course.

Participating in this phase of the research were also the five teachers whose classes the research took place in. Most high school mathematics teachers teach anywhere from three to seven different classes. As such, it would have been possible to have gathered all of the data from the classes of a single teacher. In order to diversify the data, however, it was decided that data would be gathered from classes belonging to five different teachers.

These teachers were all participating in one of several learning teams which ran in the fall of 2006 and the spring of 2007. Teachers participated in these teams voluntarily with the hope of improving their practice and their students' level of engagement. Each of these learning teams consisted of between 4 and 6, a 2-h meeting spread over half a school year. Sessions took teachers through a series of activities modelled on my most current knowledge about building and maintaining thinking classrooms. Teachers were asked to implement the activities and teaching methods in their own classrooms between meetings and report back to the team how it went.

The teachers, whose classrooms this data was collected in, were all new to the ideas being presented and, other than having individual students occasionally demonstrate work on the whiteboard at the front of the room, had never used them for whole class activity.

Data

As mentioned, the students, in groups of 2–4, worked on one of five assigned work surfaces: a wall-mounted whiteboard, a whiteboard laying on top of their desks or table, a sheet of flipchart paper taped to the wall, a sheet of flipchart paper laying on top of their desk or table, and their own notebooks at their desks or table. To increase the likelihood that they would work as a group, each group was provided with only one felt or, in the case of working in a notebook, one pen. To measure the effectiveness of each of these surfaces, a series of *proxies for engagement* were established.

It is not possible to measure how much a student is thinking during any activity, or how that thinking is individual or predicated on and with the other members of his or her group. However, there are a variety of proxies for this level of engagement that can be established—*proxies for engagement*. For the research presented here, a variety of objective and subjective proxies were established.

1. *Time to task*
   This was an objective measure of how much time passed between the task being given and the first discernable discussion as a group about the task.
2. *Time to first mathematical notation*
   This was an objective measure of how much time passed between the task being given and the first mathematical notation was made on the work surface.
3. *Eagerness to start*
   This is a subjective measure of how eager a group was to start working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned for no enthusiasm

to begin and a 3 being assigned if every member of the group were wanting to start.

4. *Discussion*
   This is a subjective measure of how much group discussion there was while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned for no discussion and a 3 being assigned for lots of discussion involving all members of the group.
5. *Participation*
   This is a subjective measure of how much participation there was from the group members while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if no members of the group were active in working on the task and a 3 being assigned if all members of the group were participating in the work.
6. *Persistence*
   This is a subjective measure of how persistent a group was while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if the group gave up immediately when a challenge was encountered and a 3 being assigned if the group persisted through multiple challenges.
7. *Non-linearity of work*
   This is a subjective measure of how non-linear groups work was. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if the work was orderly and linear and a 3 being assigned if the work was scattered.
8. *Knowledge mobility*
   This is a subjective measure of how much interaction there was between groups. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if there was no interaction with another group and a 3 being assigned if there were lots of interaction with another group or with many other groups.

These measures, like all measures, are value laden. Some proxies (1, 2, 3, 6) were selected partially from what was observed informally when being in a setting where multiple work surfaces were being utilized. Others proxies (4, 5, 7, 8) were selected specifically because they embody some of what defines a thinking classroom—discussion, participation, non-linear work, and knowledge mobility.

As mentioned, these data were collected in the five aforementioned classes during a group problem-solving activity. Each class was working on a different task. Across the five classes, there were ten groups that worked on a wall-mounted whiteboard, ten that worked on a whiteboard laying on top of their desks or table, nine that worked on flipchart paper taped to the wall, nine that worked on flipchart paper laying on top of their desk or table, and eight that worked in their own notebooks at their desks or table. For each group, the aforementioned measures were collected by a team of 3–5 people: the teacher whose class it was, the researcher (me), as well a number of observing teachers. The data were recorded on a visual representation of the classroom and where the groups were located with no group being measured by more than one person.

Results and Discussion

For the purposes of this chapter, it is sufficient to show only the average scores of this analysis (see Table 1).

The data confirmed the informal observations. Groups are more eager to start and there is more discussion, participation, persistence and non-linearity when they work on the whiteboards. However, there are nuances that deserve further attention. First, although there is no significant difference in the time it takes for the groups to start discussing the problem, there is a big difference between whiteboards and flipchart paper in the time it takes before groups make their first mathematical notation. This is equally true whether groups are standing or sitting. This can be attributed to the non-permanent nature of the whiteboards. With the ease of erasing available to them, students risk more and risk sooner. The contrast to this is the very permanent nature of a felt pen on flipchart paper. For students working on these surfaces, it took a very long time and much discussion before they were willing to risk writing anything down. The notebooks are a familiar surface to students, so this can be discounted with respect to willingness to risk starting.

Although the measures for the whiteboards are far superior to that of the flipchart paper and notebook for the measures of eagerness to start, discussion, and participation, it is worth noting that in each of these cases, the vertical surface scores higher than the horizontal one. Given that the maximum score for any of these measures is 3, it is also worth noting that eagerness scored a perfect 3 for those that were standing. That is, for all ten cases of groups working at a vertical whiteboard, ten independent evaluators gave each of these groups the maximum score. For discussion and participation, eight out of the ten groups received the maximum score. On the same measures, the horizontal whiteboard groups received 3, 3, and 2 maximum scores, respectively. This can be attributed to the fact that sitting, even while working at a whiteboard, still gives students the opportunity to become anonymous, to hide and to not participate. Standing doesn't afford this.

Table 1 Average times and scores on the eight measures

	Vertical whiteboard	Horizontal whiteboard	Vertical paper	Horizontal paper	Notebook
N (groups)	10	10	9	9	8
1. Time to task	12.8 s	13.2 s	12.1 s	14.1 s	13.0 s
2. Time to first notation	20.3 s	23.5 s	2.4 min	2.1 min	18.2 s
3. Eagerness	3.0	2.3	1.2	1.0	0.9
4. Discussion	2.8	2.2	1.5	1.1	0.6
5. Participation	2.8	2.1	1.8	1.6	0.9
6. Persistence	2.6	2.6	1.8	1.9	1.9
7. Non-linearity	2.7	2.9	1.0	1.1	0.8
8. Mobility	2.5	1.2	2.0	1.3	1.2

With respect to non-linearity, it is clear that the whiteboards, either vertical or horizontal, allow a greater freedom to explore the problem across the entirety of the surface. Although the whiteboards provide an ease of erasing that is not afforded on the flipchart, work is rarely erased by the students working on whiteboard surfaces. It seems that rather than erasing to make room for more work, the workspace migrates around the whiteboard surface, representing the chronological nature of problem-solving. In contrast, the groups working on flipchart paper tended to not write any work down until they were clear it would contribute to the logical development of a solution.

Finally, it is worth noting that groups that were standing also were more likely to engage with other groups that were standing close by. Although not measured, it was clear that this was more true for the vertical whiteboard groups. There are a number of reasons for this. Most obvious, vertical surfaces are more visible. However, there were very few observed instances of groups that were sitting down looking up to see what the groups that were standing were doing. Likewise, there were no instances of the students standing, looking at the work of the groups that were sitting. Amongst those that were standing, there was a lot of interaction between those working on whiteboards, and almost none between those working on flipchart paper. Finally, there was very little interaction between those working on flipchart paper and those working on whiteboards. Part of this can be explained by proximity—the whiteboard groups were clustered on one or two whiteboards, while the flipchart people were clustered elsewhere. But it also is the case that the whiteboard groups had little reason to look to the flipchart groups. They worked slower and had little written on their work surfaces. This was also true between the flipchart groups—there was little to look at.

In short, groups that worked on vertical whiteboards demonstrated more thinking classroom behaviour—persistence, discussion, participation and knowledge mobility—than any of the other types of work surfaces. The next most conducive was a horizontal whiteboard. The remaining three were not only not conducive to promoting thinking classroom behaviour but they may actually have inhibited it. From this it is clear that the non-permanence of surfaces is critical for decreasing time to task, as well as improving enthusiasm, discussion, participation, and persistence. It also increases the non-linearity of work which mirrors the actual work of thinking groups. Making these non-permanent surfaces vertical further enhances all of these qualities, as well as fostering inter-group collaboration, something that is needed to move the class from a collection of thinking groups to being a thinking classroom.

Vertical Non-permanent Surfaces: Teacher Uptake

Having this evidence that vertical non-permanent surfaces (VNPS) are so instrumental in the fostering of thinking classroom behaviour, a follow-up study was done with teachers vis-à-vis the use of this work surface. The goal of this follow-up

Table 2 Distribution of participants in VNPS study

	Elementary	Middle	Secondary	Total
Learning team	21	43	41	105
Multi-session workshops	12	28	42	82
Single workshops	35	24	54	113
Total	68	95	137	**300**

study was to see the degree to which teachers, when presented with the idea of non-permanent vertical surfaces, were keen to implement it within their teaching, actually tried it, and continued to use it in their teaching.

Participants

Participants for this portion of the study were 300 in-service teachers of mathematics—elementary, middle and secondary school. They were drawn from three sources over a four-year period (2007–2011): participants in variety of single workshops, participants in a number of multi-session workshops, and participants in learning teams. The breakdown of participants, according to grade levels, and form of contact is represented in Table 2.

There were a number of teachers who attended a combination of learning teams, multi-session workshops and single workshops. In these cases, their data was registered as belonging to the group with the most contact. That is, if they attended a single workshop, as well as being a member of a learning team, their participation was registered as being a member of a learning team.

These participants are only a subset of all the teachers that participated in these learning teams, multi-session workshops, and single workshops. They were selected at random from each group I worked with by approaching them at the end of the first (and sometimes only) session and asking them if they would be willing to have me contact them and, potentially, visit their classrooms.

Data

Data consists primarily of interview data. Each participant was interviewed immediately after a session where they were first introduced to the idea of vertical non-permanent surfaces, 1 week later, and 6 weeks later. These interviews were brief and, depending on when the interview was conducted, was originally designed to gauge the degree to which they were committed to trying, or continuing to use, vertical non-permanent surfaces in their teaching and how they were using them. However, participants wanted to talk about much more than just this. They wanted to discuss innovations they had made, the ways in which this was changing their teaching practice as a whole, the reactions of the students and their colleagues, as well as a variety of other details pertaining to vertical non-permanent surfaces. With time, these impromptu conversations changed the initial interview questions to begin to also probe for these more nuanced details. For the purposes of this chapter, however, only the data pertaining to the original intent will be presented.

In addition to the interview data, there were also field notes from 20 classroom visits. These visits were implemented for the purposes of checking the fidelity of the interview data—to see if what teachers were saying is actually what they were doing. In each case, this proved to be the case. It was clear from these data that teachers were true to their words with respect to their use of vertical non-permanent surfaces. However, these visits, like the interviews, offered much more than what was expected. I saw innovations in implementation, observed the enthusiasm of the students, and witnessed the transformational effect that this was having on the teaching practices of the participants.

Results and Discussion

In general, almost all of the teachers who were introduced to the notion of vertical non-permanent surfaces were determined to try it within their teaching and were committed to keep doing it, even after 6 weeks (see Fig. 1). This is a significant uptake rarely seen in the literature. This is likely due, in part, to the ease with which it is modelled in the various professional development settings. During these sessions, not only are the methods involved easily demonstrated but the teachers immediately feel the impact on themselves as learners when they are put into a group to work on a vertical non-permanent surface.

An interesting result from this aggregated view is that there were more teachers using non-permanent vertical surfaces after 6 weeks than there was after 1 week. This has to do with access to these vertical non-permanent surfaces. Many teachers struggled to find such surfaces. There were some amazing improvisations in this regard, from using windows to bringing in a number of novel surfaces, from shower curtains to glossy wall boards. One teacher even stood her classroom tables on end to achieve the effect. As time went on, teachers were able to convince their administrators to provide them with enough whiteboards that these improvisations no longer became necessary. For some teachers, this took more time than others and speaks to the delayed uptake seen in Fig. 1. However, it also speaks to the persistence with which many teachers pursued this idea with.

A disaggregated look at the data shows that neither the grade levels being taught (see Fig. 2) or the type of professional development setting in which the idea was presented (see Fig. 3) had any significant impact on the uptake.

Literature on teacher change typically implies that sustained change can only be achieved through professional development opportunities with multiple sessions and extended contact. That is, single workshops are not effective mediums for promoting change (Jasper & Taube, 2004; Little & Horn, 2007; Lord, 1994; McClain

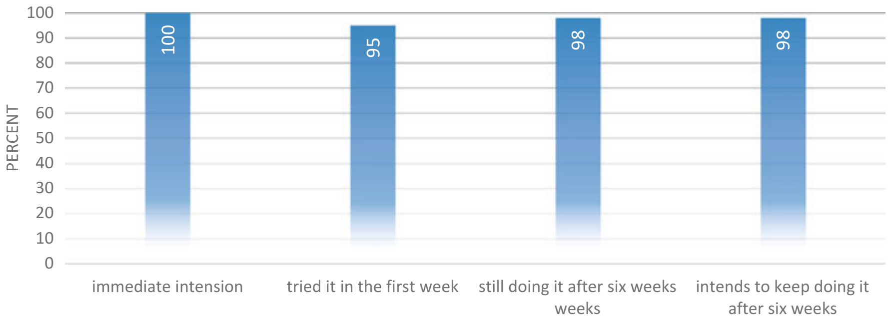

Fig. 1 Uptake of VNPS ($n = 300$)

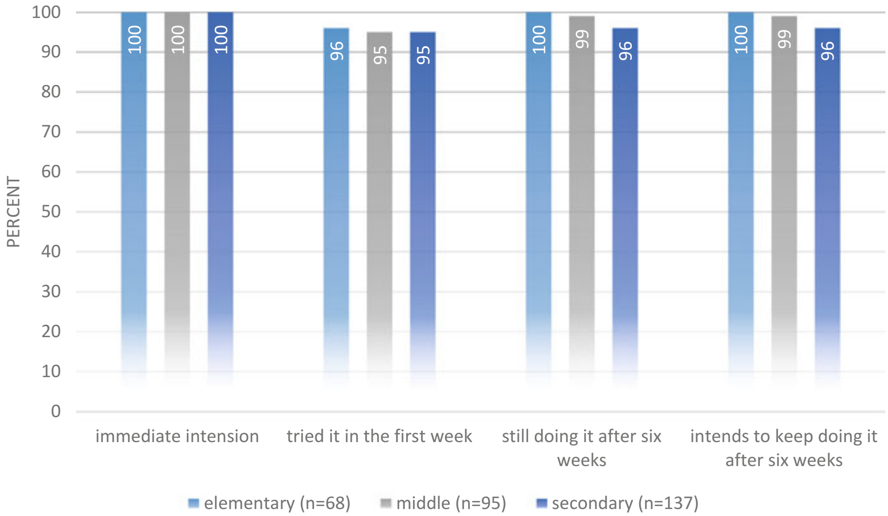

Fig. 2 Uptake of VNPS by grade levels ($n = 300$)

& Cobb, 2004; Middleton, Sawada, Judson, Bloom, & Turley, 2002; Stigler & Hiebert, 1999; Wenger, 1998). The introduction of vertical non-permanent surfaces as a workspace doesn't adhere to these claims. There are many possible reasons for this. The first is that the introduction of non-permanent vertical surfaces was achieved in a single workshop could be, as mentioned, due to the simple fact that it is a relatively easy idea for a workshop leader to model and for workshop participants to experience. Forty five minutes of solving problems in groups standing at a whiteboard coupled with a whole group discussion on the affordances of recreating this within their own classrooms is enough to convince teachers to try it. And trying it leads to a successful implementation. Unlike many other changes that can be made in a teacher's practice, vertical non-permanent surfaces (as demonstrated in

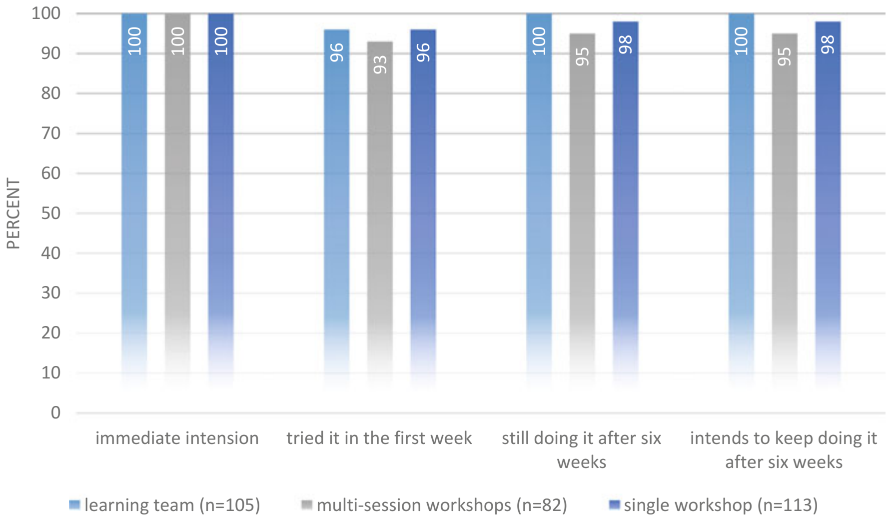

Fig. 3 Uptake of VNPS by professional development setting ($n = 300$)

the first study) was well received by students, was easy to manage at a whole class level, and had an immediate positive effects on classroom thinking behaviour. Together, the ease of modelling coupled with a successful implementation meant that vertical non-permanent surfaces did not need more than a single workshop to change teaching practice.

These possible reasons are supported by the comments of teachers from the interviews after week 1 and week 6. The following comments were chosen from the many collected for their conciseness.

> "*I will never go back to just having students work in their desks*."
> "*How do I get more whiteboards*?"
> "*The principal came into my class … now I'm doing a session for the whole staff on Monday*."
> "*My grade-partner is even starting to do it*."
> "*The kids love it. Especially the windows*."
> "*I had one girl come up and ask when it will be her turn on the windows*."

Not only is the implementation of vertical non-permanent surfaces immediately effective for these teachers, it is also infectious with other teachers quickly latching on to it and administrators quickly seeing the affordances it offers.

But if vertical non-permanent surfaces are the solution, what was the problem? When I began the research on students' workspace, the default was students sitting in desks—sometimes individually in rows, other times clustered in groups. The move from the desks to the vertical workspaces was made, not because I saw something specifically wrong with students being in desks, but rather through adherence to the contrarian approach that was adopted early on in the more general research project. Looking back now at students working in desks, from the perspective of the affordances that having them stand at a non-permanent vertical surface offers, I see more clearly the problems that desks introduced into my efforts to build and maintain thinking classrooms. Primarily, this has to do with anonymity and how desks allow for and even promote this. When students stand at a whiteboard or a window, they are all visible. There is nowhere to hide. When students are in their desks, it is easy for them to become anonymous, hidden and safe—from participating and from contributing. It is not that all students want to be hidden, to not participate, but when the problems gets difficult, when the discussions require more thinking, it is easy for a student to pull back in their participation when they are sitting. Standing in a group makes this more difficult. Not only is it immediately visible to the teacher but it is also clear to the student who is pulling back. To pull back means to step towards the centre of the room, towards the teacher, towards nothing. There is no anonymity in this.

Forming Groups

The research into how best to form groups began, like it did with student work surfaces, by looking at how groups are typically formed in a classroom. In most cases, this is either a strategically planned arrangement decided by the teacher or self-selected groupings of friends as decided by the students. Teachers tend to make groupings in order to meet their educational goals. These may include goals around pedagogy, student productivity, or simply the construction of a peaceful work environment. Meanwhile, students, when given the opportunity, tend to group themselves according to their social goals. This mismatch between educational and social goals in classrooms creates conditions where, no matter how strategic a teacher is in her groupings, some students are unhappy in the failure of that grouping to meet their social goals (Kotsopoulos, 2007; Slavin, 1996).

This disparity results in a decrease in the effectiveness of group work. This led to the exploration of alternative grouping methods. The fact that strategic grouping strategies were often not working, coupled with the contrarian approach of action in such instances, meant that random grouping methods needed to be explored. Working with the same type of population of teachers described above, a variety of random grouping methods were implemented and studied. This preliminary research showed, very quickly, that there was little difference in the effectiveness of strategic groupings and randomized groupings when the randomization was done out of sight of the students. The students assumed that all groupings had a hidden agenda, and merely saying that they were randomly generated was not enough to change classroom behaviour.

However, when the randomization was done in full view of the students, changes were immediately noticed. When randomization was done frequently—twice a day in elementary classrooms and every class in middle and secondary classrooms—the changes in classroom behaviour was profound. Within 2–3 weeks:

- Students became agreeable to work in any group they were placed in.
- There was an elimination of social barriers within the classroom.

- Mobility of knowledge between students increased.
- Reliance on the teacher for answers decreased.
- Reliance on co-constructed intra- and inter-group answers increased.
- Engagement in classroom tasks increased.
- Students became more enthusiastic about mathematics class.

To confirm these observations, one grade 10 (age 15–16) was studied. The details and results of this research have already been published in a chapter entitled *The Affordances of Using Visibly Random Groups in a Mathematics Classroom* (Liljedahl, 2014). What follows is a summary of this research.

The class in which the study was done belonged to Ms. Carley, a teacher with eight years experience who was a participant in one of the learning teams I was leading. Ms. Carley had joined the team because she was dissatisfied with the results of group work in her teaching. She knew that group work was important to learning, but, until now, had felt that her efforts in this regard had been unsuccessful. She was looking for a better way. So, when I suggested to the group that they try using visibly random groups she made an immediate commitment to start using this method in one of her classrooms.

Data consisted of interview transcripts and field notes collected over a 3-month period immediately prior to and during an implementation of visibly random groups in Ms. Carley's class. These data were analysed using analytic induction (Patton, 2002) anchored in the *a priori* and grounded observations from my initial experimentation with random groupings.

These results both confirmed and nuanced the initial observations. Students very quickly shed their anxieties about what groups they were in. They began to collaborate in earnest. After three weeks, a *porosity* developed between group boundaries as both intra- and inter-group collaboration flourished. With this heightened mobilization of knowledge came a decrease in the reliance on the teacher as the *knower* in the room. In the end, there was a marked heightening of enthusiasm and engagement for problem-solving in particular, and in mathematics class in general. In short, Ms. Carley's class became a thinking classroom.

Visibly Random Groupings: Teacher Uptake

Similar to the research on the vertical non-permanent surfaces a pseudo-quantitative study was done on the uptake by teachers on the idea of visibly random groupings (VRG). Tapping into the similar populations of teachers engaged in learning teams, multi-session workshops, and single workshops between 2009 and 2011, a population of 200 teachers were selected to participate (see Table 3).

These teachers were introduced to the idea of visibly random groupings in a similar fashion as above—through modelling and immersion. They were likewise interviewed immediately after their professional development experience, 1 week after their experience, and 6 weeks after. The results of this analysis can be seen in Fig. 4.

Table 3 Distribution of participants in VRG study

	Elementary	Middle	Secondary	Total
Learning team	15	22	31	68
Multi-session workshops	25	19	14	58
Single workshops	10	25	39	74
Total	50	66	84	**200**

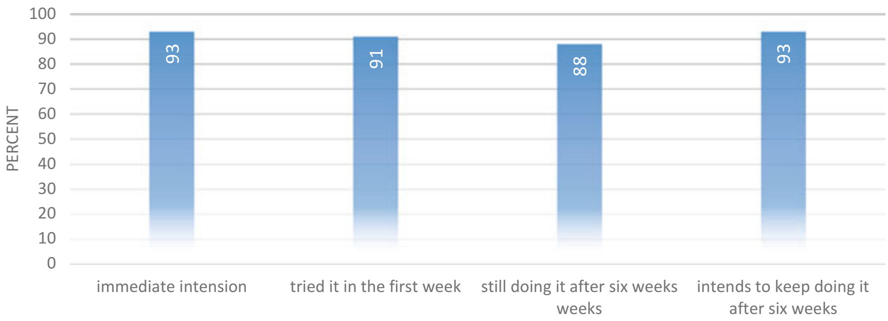

Fig. 4 Uptake of VRG ($n = 200$)

The dip in the uptake between week 1 and week 6 was minor. What was interesting was the uptick in intension after week 6. From the interviews, it became clear that the teachers who had come away from using visibly random groups did so because, after 3–4 weeks, things were working so well that they thought they could now allow the students to work with who they wanted. Once they saw that this was not as effective, they recommitted to going back to random groupings.

Like with vertical non-permanent surfaces, there was no discernible difference in uptake between elementary, middle or secondary teachers. However, unlike the previous study, there was a slight difference depending on the nature of the professional development environment they were participating in (see Fig. 5).

From the interviews, it seemed that although the immediate delivery of the idea was accomplished within a single session, the support of the learning team helped teachers to get on board late if they hesitated in implementing in the 1st week. This explains the uptick in the number of learning team members who started using randomized groups in between the first and the sixth week. This also explains why there was no such uptick amongst the single workshop participants who had no follow-up session, or amongst the multi-session participants who did not have a second session until 8 weeks after the initial idea was presented.

Regardless, there was still a significant uptake by those teachers who only experienced one 90 min session on the use of visibly random groupings. This can be explained in the same way as it was for the vertical non-permanent surfaces—it was easily modelled and the affordances became immediately apparent. As well, the students took to it quickly with little resistance once the participants implemented it within their own classrooms.

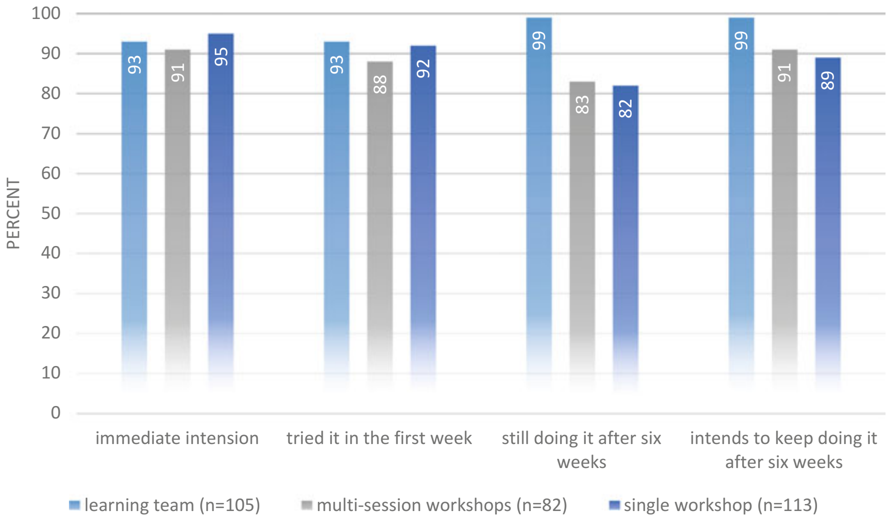

Fig. 5 Uptake of VRG by professional development setting ($n = 200$)

As with the research on the vertical non-permanent surfaces, the research on visibly random groupings included 14 classroom visits. Unlike the research on VNPS, however, the purpose of these visits was not to check the fidelity of the interview data. Rather, it was to see if teachers were continuing to use VRG's even 6–9 months after their last work with me. In each of the 14 visits, I saw a continued use of VRG strategies. And like with my visits in the VNPS research, these visits offered much more than what was expected. I saw innovations in implementation, observed the enthusiasm of the students, and witnessed the transformational effect that this was having on teaching practices.

VNPS and VRG Taken Together: Teacher Uptake

Once it was established that both vertical non-permanent surfaces and visibly random groupings were effective practices for building aspects of a thinking classroom and that these methods had good uptake by teachers, it was easy to bring them together. From a professional development perspective, this is no more difficult than presenting each one separately. VNPS and VRG are easily modelled together, with the participants being put into visibly random groupings to work on vertical non-permanent surfaces. So, this is what was done with a population of teachers similar to the ones described above. From this, 124 participants were followed to gauge the uptake of being exposed to both of these methods simultaneously. The results can be seen in Fig. 6.

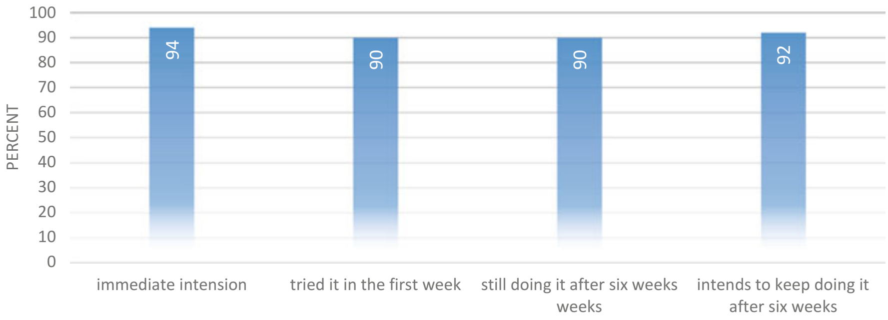

Fig. 6 Uptake of both VNPS and VRG ($n = 124$)

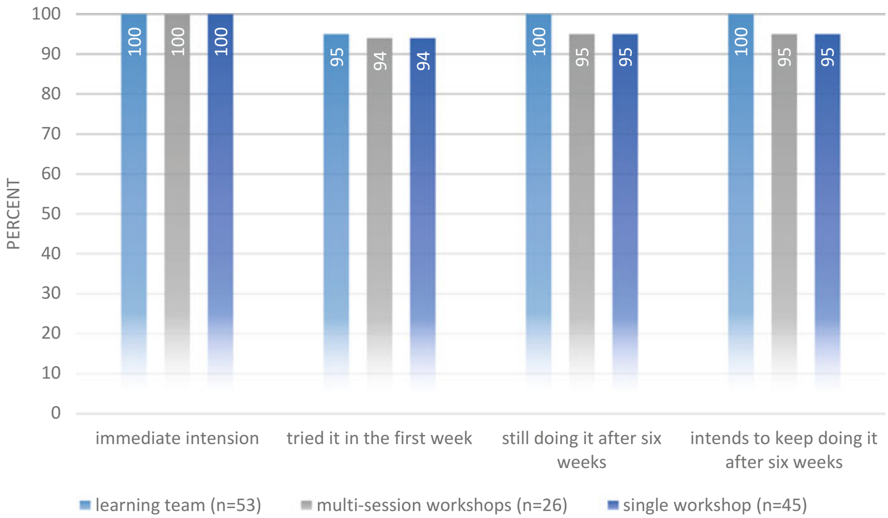

Fig. 7 Uptake of VNPS and VRG by professional development setting ($n = 124$)

Like with visibly random groupings, there was no significant difference in uptake by grade level and a slight difference in uptake as disaggregated by the professional development setting in which the combined methods were presented. Like with visibly random groupings, the teachers in the learning team setting were more consistently implementing the methods presented, whereas those teachers in the single workshop sessions were less likely to get on board late and more likely to drop off early (see Fig. 7). Despite these differences, however, the uptake across for each group was impressive with much enthusiasm for it.

With respect to the effect on students, my observations during ten classroom visits showed the combined benefits of the two interventions. The fact that the students were so comfortable working with each other, coupled with the high visibility of the work afforded by the vertical surfaces, allowed for enhanced intra-group knowledge mobilization. The teachers often commented that they saw huge improvements in the classroom community.

> *"I used to think I had a community in my classroom. Now I see what a community can look like."*

My observation of the student actions during these ten classroom visits confirmed this.

General Findings: All Nine Elements

The results from research on students' workspace and grouping methods are indicative of the findings of research into each of the nine aforementioned elements. From the design-based research on each of these—independently or in conjunction with others—emerged a set of teaching practices that are conducive to either the building, or maintenance, of a thinking classroom. In what follows briefly, these are:

1. *The type of tasks used and when and how they are used*
   Lessons need to begin with good problem-solving tasks. At the early stages of building a thinking classroom, these tasks need to be highly engaging, collaborative tasks that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). Once a thinking classroom is established, the problems need to permeate the entirety of the lesson and emerge rich mathematics (Schoenfeld, 1985) that can be linked to the curriculum content to be 'taught' that day.
2. *The way in which tasks are given to students*
   Tasks need to be given orally. If there are data or diagrams needed, these can be provided on paper, but the instructions pertaining to the activity of the task need to be given orally. This very quickly drives the groups to discuss what is being asked rather than trying to decode instructions on a page.
3. *How groups are formed, both in general and when students work on tasks*
   As presented above, groupings need to be frequent and visibly random. Ideally, at the beginning of every class, a visibly random method is used to assign students to a group of 2–4 for the duration of that class. These groups will work together on any assigned problem-solving tasks, sit together or stand together during any group or whole class discussions.
4. *Student workspace while they work on tasks*
   As discussed, groups of students need to work on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This will make visible all work being done, not just to the teacher but to the groups doing the work. To facilitate discussion, there should be only one felt pen or piece of chalk per group.
5. *Room organization, both in general and when students work on tasks*
   The classroom needs to be de-fronted. The teacher must let go of one wall of the classroom as being the designated teaching space that all desks are oriented

towards. The teacher needs to address the class from a variety of locations within the room and, as much as possible, use all four walls of the classroom. It is best if desks are placed in a random configuration around the room.

6. *How questions are answered when students are working on tasks*
Students only ask three types of questions: (1) proximity questions—asked when the teacher is close; (2) stop-thinking questions—most often of the form 'is this right'; and (3) keep-thinking questions—questions that students ask so they can get back to work. Only the third of these types should be answered. The first two types need to be acknowledged but not answered.
7. *The ways in which hints and extensions are used while students work on tasks*
Once a thinking classroom is established, it needs to be nurtured. This is done primarily through how hints and extensions are given to groups as they work on tasks. Flow (Csíkszentmihályi 1990, 1996) is a good framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it. If their ability is too high, the risk is they get bored. If the challenge is too great, the risk is they become frustrated.
8. *When and how a teacher levels their classroom during or after tasks*
Levelling needs be done at the bottom. When every group has passed a minimum threshold, the teacher needs to engage in discussion about the experience and understanding the whole class now shares. This should involve a reification and formalization of the work done by the groups and often constitutes the 'lesson' for that particular class.
9. *Assessment, both in general and when students work on tasks*
Assessment in a thinking classroom needs to be mostly about the involvement of students in the learning process through efforts to communicate with them where they are and where they are going in their learning. It needs to honour the activities of a thinking classroom through a focus on the processes of learning more so than the products and it needs to include both group work and individual work.

Discussion

However, this research also showed that these are not all equally impactful or purposeful in the building and maintenance of a thinking classroom. Some of these are blunt instruments capable of leveraging significant changes while others are more refined, used for the fine-tuning and maintenance of a thinking classroom. Some are necessary precursors to others. Some are easier to implement by teachers than others, while others are more nuanced, requiring great attention and more practice as a teacher. And some are better received by students than others. From the whole of these results emerged a three-tier hierarchy that represent not only the bluntness and ease of implementation but also an ideal chronology of implementation (see Table 4).

Table 4 Nine elements as chronologically implemented

Stage one	Stage two	Stage three
• Begin lessons with problem-solving tasks	• Oral instructions	• Levelling
• Vertical non-permanent surfaces	• De-fronting the room	• Assessment
• Visibly random groups	• Answering questions	• Managing flow

In the aforementioned research, I presented the results of research into teachers implementing teaching practices from stage one, either separately or together. However, the effect on these teachers is more profound than the numbers and graphs indicated above. This experience with elements in stage one propels them to thirst for more, both in particular and in general. They want more tasks, more examples of how to make random groupings, how to find vertical surfaces. But they also want to know more about assessment, how to ask and answer questions, how to organize their rooms, how to give instructions and how to sustain the engagement they have experienced while at the same time feeling like they are getting through the curriculum. In short, their experience with the teaching methods associated with stage one elements is quite naturally propelling them into wanting to engage in the elements in stages two and three.

These results are not definitive, exhaustive or unique. The teaching methods that emerged as effective for each of these elements emerged as a result of an *a priori* commitment to make change in a contrarian fashion. This continued until positive effects began to emerge, at which point refinements were recursively explored. It is possible that a different approach to the research would have yielded different methods. Different methods could, likewise, emerge a different set of stages optimal for the development of thinking classrooms.

Conclusions

The main goal of this research is about finding ways to build thinking classrooms. One of the sub-goals of this work on building thinking classrooms was to develop methods that not only fostered thinking and collaboration but also bypassed any classroom norms that would potentially inhibit this from happening. Using the methods in stage one while solving problems, either together or separately, was almost universally successful. They worked for any grade, in any class and for any teacher. As such, it can be said that these methods succeeded in bypassing whatever norms existed in the over 600 classrooms in which these methods were tried. Further, they not only bypassed the norms for the students but also the norms of the teachers implementing them. So different were these methods from the existing practices of the teachers participating in the research that they were left with what I have come to call *first-person vicarious experiences*. They are first person because they are living the lesson and observing the results created by their own hands. But the methods are not their own. There has been no time to assimilate them into their own repertoire of practice or into the schema of how they construct meaningful practice. They simply experienced the methods as learners and then were asked to immediately implement them as teachers. As such, they experienced a different way in which their classroom could look and how their students could behave. They experienced, through these other*ly* methods, an other*ly* classroom—a thinking classroom.

The results of this research sound extraordinary. In many ways, they are. It would be tempting to try to attribute these to some special quality of the professional development setting or skill of the facilitator. But these are not the source of these remarkable results. The results, I believe, lie not in what is new but what is not old. The classroom norms that permeate classrooms in North America, and around the world, are so robust, so entrenched, that they transcend the particular classrooms and have become institutional norms (Liu & Liljedahl, 2012). What the methods presented here offer is a violent break from these institutional norms, and in so doing, offer students a chance to be learners much more so than students (Liljedahl & Allan, 2013a, 2013b).

By constructing a thinking classroom, problem-solving becomes not only a means but also an end. A thinking classroom is shot through with rich problems. Implementation of each of the aforementioned methods associated with the nine elements and three stages relies on the ubiquitous use of problem-solving. But at the same time, it also creates a classroom conducive to the collaborative solving of problems.

Afterword

Since this research was completed, I have gone back to visit several of the classrooms of teachers who first took part in the research. These teachers are still using VNPS and VRG as well as having refined their practice around many of the other nine aforementioned elements. Unlike many other professional development initiatives and interventions I have seen implemented over the years, these really seemed to have had a lasting impact on teacher practice. The reason for this seems to come from two sources. First, teachers talk about how much their students like the ‘new’ way of doing mathematics. So much so, in fact, that when they go back to using direct instruction, even for brief periods of time, the students object. The second and more intrinsic reason is that they feel more effective as teachers. Their students are exhibiting the traits that they had been striving for but were unable to achieve through nuanced changes to their initial teaching practice.

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Reaction: Teachers, Problem Posing and Problem-Solving

Kaye Stacey

The six chapters of Part 3 report a variety of initiatives to study how teachers can support their students' solving of mathematical problems. The chapters are from Asia, North and South America and Europe, underlining that improving students' ability to solve mathematical problems is an international endeavour which challenges mathematics educators around the globe. There are local variations, but substantial commonalities arise from the contrast between the widespread perception that mathematics is mainly a set of rules to remember and the inspiring goal of making learning mathematics a creative, problem-solving activity.

Teachers and Problem-Solving

The chapters divide broadly into two groups, with three looking at teachers as problem-solvers and three examining how teaching of, about or through mathematical problem-solving can be better achieved in classrooms. Let us consider the chapters on mathematics teachers first.

Felmer and Perdomo-Diaz present part of a study that examines the support that pre-service university courses provide for teaching mathematical problem-solving. The chapter reports data gathered from 30 new teachers who had graduated from leading universities, and it examines how well they were able to solve two problems, their accompanying affect and self-evaluations. The instruments were very carefully targeted to relevant attributes. Felmer and Perdomo-Diaz selected two problems involving school mathematics, both of which have more than one numerical solution. The large proportion of the sampled teachers not seeking these multiple solutions, and some of their comments, led the authors to conclude that despite

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their years of studying mathematics at university, many teachers have not ever experienced 'working like a mathematician'. This observation about teacher preparation is not unique to Chile, and there have been some influential calls to change this situation. See, for example, Conference Board of the Mathematical Sciences [CBMS] (2001). Stacey (2008) discusses four components of mathematical knowledge needed for teaching, one of which is that teachers should have experienced mathematics in action by solving problems, conducting investigations and modelling the real world. Stacey also shows that the requirements for teacher certification around the world very rarely specify anything about the nature of what is studied (such as giving students the experiences Felmer and Perdomo-Diaz recommend), commonly only specifying the number of mathematics courses studied. Through their analysis of the self-evaluations, Felmer and Perdomo-Diaz also draw attention to another phenomenon—that teachers who perform less well in problem-solving are often not aware of what they have missed. These findings highlight the need for professional learning for practising teachers and the need for it to include experiences of problem-solving.

John Mason's contribution is also concerned with the development of teachers' own mathematical powers. The underlying message is that developing sensitivity to students' problem-solving experience comes from developing sensitivity to one's own. The chapter contains many of Mason's trademark mathematical tasks for 'fostering and sustaining mathematical thinking in others' and many of his memorable and thought-provoking epigrams and allusions to thinkers through the ages. Perspectives are offered on fundamental issues of teaching for, about and through problem-solving, including how to generate interest in a problem (Leung and Leong et al. also confront this), when and how a teacher might intervene in a student's problem-solving and how to develop an atmosphere conducive to conjecturing.

Silver's chapter demonstrates the benefits that teachers gain when they analyse students' mathematical solutions. Teachers were able to delve deeply into issues which were directly related to their classroom practice. Silver recounts how teachers' comments about mathematical activity are very often only evaluative (the students were successful or not, liked a task or not, etc.). Analysing students' responses helped to broaden the range of solution features that teachers noticed. This in turn had instructional power. Teachers saw that students can often creatively solve problems without the standard efficient mathematical techniques, and they also saw the difficulty experienced by some students in using grade-level mathematical knowledge for problem-solving. The chapter documents a successful way for teachers to learn from both their own students' responses and summaries of responses from large samples. The mathematical task discussed in this chapter is a modification of a publically released unit from the OECD's PISA surveys. A further large set of items was released after the 2012 PISA survey (Organisation for Economic Co-operation and Development [OECD], 2013). This provides a rich source of problems set in engaging real-world contexts and involving a variety of mathematical competencies such as using multiple representations.

Implementation Experiments

Three of the chapters in the section are concerned with the classroom implementation of problem-solving and with providing teachers with assistance (training, materials, etc.) to make it sustainable.

Leung Shuk-Kwan reports on a project to assist in integrating problem posing into mathematics instruction in elementary schools in Taiwan. This is now explicitly mentioned in the national curriculum standards. She has established that teachers have little experience with problem posing, and that they improve with practice. The importance of problem posing within problem-solving is underlined by the findings reported in the chapter by Felmer and Perdomo-Diaz that many of their subjects appeared not to pose even the seemingly obvious question of whether all solutions had been found. Working with large groups of teachers to solve practical issues of integration, Leung produced an inventory of tasks and instructional guidelines. The chapter reports in detail on the teaching methods developed by committed teachers working in partnership with the researcher. This has resulted in practical advice to teachers such as to spark problem posing by using concrete things like photographs or manipulatives (this is reminiscent of the advice in Liljedahl's chapter for teachers to present problems orally) and not to give sample problems. The benefits for children were found to be a greater attention to problem mathematical structure and to making contexts realistic and it also provided their teachers with a new window into their knowledge and misunderstandings. The chapter also contains a substantial literature review on problem posing.

Work on the implementation of problem-solving in many countries over several decades has very frequently reported that even when teachers want to change, to make problem-solving and mathematical thinking a more prominent part of their practice, students often resist. The persistence of classroom norms (e.g. where students expect to receive instructions on how to solve predictable problems) operates against teachers who are trying to change their practice. Teachers new to problem-solving (or other educational innovations) find that it 'does not work' and give the innovation up. This has been a widely reported phenomenon for many years (see, e.g. Burkhardt, Groves, Schoenfeld, & Stacey, 1988). Liljedahl's chapter gives a refreshing view on this. Instead of focussing on the deepest—and possibly most difficult—ways in which mathematical problem-solving can become a reality, Liljedahl demonstrates that there are 'low-hanging fruit' by which teachers can disrupt those counterproductive classroom norms relatively easily. The chapter documents the research behind the development of two such innovations: having students stand to work in groups on vertical non-permanent writing surfaces and deciding on the group membership through a random process explicitly known by the students and frequently redone. Liljedahl's chapter also demonstrates that honing the way that these 'low fruit' interventions are best implemented requires solid research, monitoring the frequency of desirable behaviours and long-term uptake.

The chapter by Leong, Tay, Toh, Quek, Toh, and Dindyal is especially interesting from two points of view. Firstly, it aims to infuse a problem-solving approach throughout teaching, and the chapter reports on the research path that has brought the authors to the use of ‘replacement units’. Additionally, the chapter is interesting because of its setting in Singapore, where problem-solving has consistently been promoted as the central plank of the mathematics curriculum for over two decades. The authors observe that problem-solving in the sense of preparing students to tackle very challenging problems has been implemented in Singapore, but that infusion of teaching about and through problem-solving is not yet widespread. Through a 6-year design experiment, Leong and colleagues have identified the obstacles that teachers face in infusing problem-solving throughout their teaching and have converged on practical measures to overcome some of them. They recommend an introductory unit based around problems that are well suited to teaching about mathematical problem-solving so that students develop a language to discuss phases of problem-solving and heuristic strategies. Their approach draws the learning of strategies through the experience of solving problems, with class reflection to draw the strategies from that experience. This successful formula gives depth to learning about strategies (see, e.g. Stacey & Groves, 2006). To promote infusion, the next design phase focussed on ‘infusion problems’, special problems designed to accompany the teaching of difficult topics and guidelines on how to use them. The research highlighted difficulties in using these problems, arising from structural issues (e.g. time pressure), unhelpful classroom norms and teachers puzzling about how the suggested problems fitted logically and developmentally into a teaching sequence. Thus, the current round of experimentation uses complete ‘replacement units’ that aim to achieve all expected content and problem-solving goals within an unchanged time allocation. By research and development of replacement units with cooperating teachers, the authors intend to show by example how problem-solving infusion can be achieved.

Reflections

After reading these chapters, I went back to the Proceedings of the Problem-Solving Theme Group at ICME 5 held in 1984 (Burkhardt et al., 1988). Many of the observations in the chapters of this section echo what is there. We already knew, for example, that teachers would find teaching for, about or through problem-solving to be difficult mathematically (as chapters here attest), difficult pedagogically in meeting the multiple needs of students as they worked on problems and difficult personally because it requires risk-taking and some loss of control (Stacey & Groves, 1988). So, what progress has been made? A simple observation is that many more countries are now involved in problem-solving as attested by these chapters, and they have official curriculum backing. Only 3 of the 36 authors in the ICME 5 proceedings were not from Western countries (Japan, India, South Africa). Furthermore, these chapters illustrate how there are now long-term design research projects working with sizeable groups of ‘ordinary’ teachers to establish practical guidelines to make problem-solving (or as Liljedahl prefers, a thinking classroom) a sustainable part of teaching. This approach is a new contribution. The findings in these chapters will certainly provide advice for international work, but the efforts of the present authors also acknowledge the need for local work with local teachers for local conditions supported by creative research targeting the major obstacles. Problem-solving remains an elusive goal with no silver bullets, but this new phase of research will widen its reach.

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