Chapter 7
The Schauder Fixed-Point Theorem

AN INFINITE DIMENSIONAL BROUWER THEOREM

Overview. Recall that to say a metric space has the *fixed-point property* means that every continuous mapping taking the space into itself must have a fixed point. In Chap. 4 we proved two versions of the Brouwer Fixed-Point Theorem:

THE "BALL" VERSION (Theorem 4.1). *The closed unit ball of* $\mathbb{R}^N$ *has the fixed-point property,*

and the seemingly more general, but in fact equivalent

"CONVEX" VERSION (Theorem 4.5). *Every compact convex subset of* $\mathbb{R}^N$ *has the fixed-point property.*

It turns out that the "ball" version of Brouwer's theorem does not survive the transition to infinitely many dimensions. However all is not lost: the "convex" version *does* survive: compact, convex subsets of normed linear space *do* have the fixed-point property. This is the famous Schauder Fixed-Point Theorem (circa 1930) which will occupy us throughout this chapter. After proving the theorem we'll use it to prove an important generalization of the Picard–Lindelöf Theorem of Chap. 3 (Theorem 3.10). The Schauder Theorem will also be important in the next chapter where it will provide a key step in the proof of Lomonosov's famous theorem on invariant subspaces for linear operators on Banach spaces.

Prerequisites. Basics of normed linear spaces and compactness in metric spaces.

7.1 The Theorem

Theorem 7.1 (The Schauder Fixed-Point Theorem)**.** *In every normed linear space, each compact, convex subset has the fixed-point property.*

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Our proof of Brouwer's theorem depended strongly on the compactness of the closed unit ball of $\mathbb{R}^N$. We'll see in the next chapter (Proposition 8.7) that *no infinite dimensional normed space has this property.* This is easiest to understand for infinite dimensional Hilbert space, where for each orthonormal sequence the distance between two distinct elements is $\sqrt{2}$, hence such a sequence, which belongs to the closed unit ball, has no convergent subsequence. Conclusion: The closed unit ball of an infinite dimensional Hilbert space is non-compact.

It gets worse! Recall that every separable Hilbert space is isometrically isomorphic to ℓ^2, the Hilbert space consisting of square-summable scalar sequences endowed with the norm

$$\|f\| = \left(\sum_{n=1}^{\infty} |f(n)|^2\right)^{1/2} \qquad (f = (f(n))_1^\infty \in \ell^2).$$

Proposition 7.2. *The closed unit ball of the Hilbert space ℓ^2 (hence of every separable Hilbert space) does not have the fixed-point property.*

Proof. Let $(e_n)_1^\infty$ be the standard basis in ℓ^2 (i.e., e_n is the sequence with 1 in the n-th position and zero elsewhere), and define the map T on ℓ^2 by

$$Tf = (1 - \|f\|)e_1 + \sum_{n=1}^{\infty} f(n)e_{n+1} = (1 - \|f\|, f(1), f(2), \ldots) \qquad (f \in \ell^2).$$

Then for $f \in \ell^2$ with $\|f\| \le 1$ we have from the triangle inequality:

$$\|Tf\| \le (1 - \|f\|) + \underbrace{\left\|\sum_{n=1}^{\infty} f(n)e_{n+1}\right\|}_{=\|f\|} = (1 - \|f\|) + \|f\| = 1,$$

so T takes the closed unit ball B of ℓ^2 into itself. Furthermore, if f and g are two vectors in ℓ^2, then a straightforward calculation shows

$$\|Tf - Tg\|^2 = (\|f\| - \|g\|)^2 + \|f - g\|^2 \tag{7.1}$$

from which one deduces that T is continuous on ℓ^2.

Claim. *T has no fixed point in B.*

Proof of Claim. Suppose $f \in B$ *were* a fixed point of T. Upon equating components in the equation $Tf = f$ we would obtain

$$f(n) = 1 - \|f\| \qquad (n \in \mathbb{N}), \tag{7.2}$$

thus exhibiting f as a constant function. But $f \in \ell^2$, so $f(n) \to 0$, hence $f(n) = 0$ for all n. This, along with Eq. (7.2) above, yields the contradiction $0 = 1$. □

Exercise 7.1. Verify the "straightforward calculation" (7.1) and show that it really does establish the continuity claimed above for T.

The key to proving the Schauder Fixed-Point Theorem will be to show that each compact subset of a normed linear space can be "almost" embedded in the convex hull of a finite subset of its points. Such convex hulls are compact (Proposition C.5 of Appendix C), so Brouwer's theorem can be applied to produce approximate fixed points, and hence by Lemma 2.2 (the "Approximate Fixed-Point Lemma," p. 24), an actual fixed point.

7.2 The Proof

Crucial to the proof of Schauder's theorem is the easily proved observation that each compact subset of a metric space can be approximated arbitrarily closely by a finite set. More precisely, for every $\varepsilon > 0$ our compact set K contains an ε-*net*: a finite subset F of K such that for each point $x \in K$ there is a point of F lying within ε of x (Proposition B.3 of Appendix B). This in turn gives rise to an important map called the *Schauder Projection*.

Proposition 7.3 (The Schauder Projection)**.** *Suppose C is a compact convex subset of a normed linear space X. Then given $\varepsilon > 0$ and an ε-net F_ε contained in C, there exists a continuous map P_ε (the* Schauder Projection*) that takes C into the convex hull of F_ε such that $\|P_\varepsilon(x) - x\| < \varepsilon$ for every $x \in C$.*

Proof. We are assuming that $F_\varepsilon = \{x_1, x_2, \ldots x_N\} \subset C \subset \bigcup_{j=1}^N B(x_j, \varepsilon)$, where $B(x_j, \varepsilon)$ denotes the open ball of radius ε centered at the point x_j. By Proposition B.6 (p. 190) there is a partition of unity $(p_1, p_2, \ldots p_N)$ on C subordinate to the covering $\{B(x_j, \varepsilon) \colon 1 \le j \le N\}$. Specifically: $\sum_j p_j \equiv 1$ on C and for each index j the function p_j is non-negative and continuous on C, and identically zero outside $B(x_j, \varepsilon)$.

Now we proceed as in the proof of the "Approximate Selection Theorem" of the last chapter (Theorem 6.5): define the map P_ε on C by

$$P_\varepsilon(x) = \sum_{j=1}^N p_j(x) x_j \qquad (x \in C).$$

For each $x \in C$ the vector $P_\varepsilon(x)$ is a convex combination of the vectors x_j, hence P_ε maps C into the convex hull of F_ε (Proposition C.4 of Appendix C), and since each coefficient function p_j is continuous on C so is P_ε. Moreover

$$\|P_\varepsilon(x) - x\| = \left\| \sum_{j=1}^N p_j(x)(x_j - x) \right\| \le \sum_{j=1}^N p_j(x) \|x_j - x\| \qquad (x \in C),$$

where in the last sum on the right the coefficient $p_j(x)$ is zero whenever $\|x - x_j\|$ is $\ge \varepsilon$. Thus $\|P_\varepsilon(x) - x\| < \varepsilon \sum_j p_j(x) = \varepsilon$ for every $x \in C$, as desired. □

The final estimate above could also be viewed like this: The coefficient $p_j(x)$ vanishes for those vectors $x_j - x$ that lie outside the ball $B(0,\varepsilon)$, so $P_\varepsilon(x) - x$ is a "subconvex" combination of points in that ball, so also lies in that ball.

Proof of the Schauder Fixed-Point Theorem. We're given a compact, convex subset C of a normed space X and a continuous map $f\colon C \to C$. We wish to show that f has a fixed point. By the Approximate Fixed-Point Lemma (Lemma 2.2, p. 24) it's enough to show that: given $\varepsilon > 0$ there exists $x_\varepsilon \in C$ such that $\|f(x_\varepsilon) - x_\varepsilon\| < \varepsilon$.

To this end let $\varepsilon > 0$ be given, choose an ε-net $F_\varepsilon \subset C$, and let P_ε be the Schauder projection of C onto $\operatorname{conv}(F_\varepsilon)$. Then $g_\varepsilon = P_\varepsilon \circ f$ maps C continuously into $\operatorname{conv}(F_\varepsilon)$, and so maps $\operatorname{conv}(F_\varepsilon)$ continuously into itself. Since $\operatorname{conv}(F_\varepsilon)$ is a compact (Proposition C.5 of Appendix C), convex subset of a finite dimensional subspace of X, it is, by Proposition C.9 (p. 197) homeomorphic (even linearly) to a compact, convex subset of a finite dimensional Euclidean space, so by the "Convex" Brouwer Fixed-Point Theorem (Theorem 4.5), g_ε has a fixed point x_ε that lies in $\operatorname{conv}(F_\varepsilon)$, and hence in C. Thus:

$$\|f(x_\varepsilon) - x_\varepsilon\| = \|f(x_\varepsilon) - g_\varepsilon(x_\varepsilon)\| = \|f(x_\varepsilon) - P_\varepsilon(f(x_\varepsilon)\| < \varepsilon,$$

as desired. □

7.3 Generalization to Non-compact Situations

Here's a sobering thought about the Schauder Fixed-Point Theorem: In infinite dimensional normed linear spaces there are not many compact sets. For example, we noted just after our statement of the Schauder Theorem (Theorem 7.1, p. 75) that in such spaces closed balls are never compact, and we gave an argument to prove this for Hilbert space. The exercise below asks you to prove this for the situation we'll encounter in the next section.

Exercise 7.2. Suppose I is a compact interval of the real line. Show that no closed ball in $C(I)$ is compact.

All is not lost, however: thanks to the following result, Schauder's theorem can be applied to non-compact situations—provided that the maps in question have some "built-in compactness." Recall that to say a set in a metric space is "relatively compact" means that its closure is compact.

Corollary 7.4. *Suppose C is a closed convex subset of Banach space and $f\colon C \to C$ is a continuous map. If $f(C)$ is relatively compact in C then f has a fixed point.*

Proof. Since $f(C)$ is a relatively compact subset of the convex set C, Proposition C.6 of Appendix C guarantees that its convex hull is relatively compact. Thus the closure, K, of $\operatorname{conv}(f(C))$ is compact in our Banach space, and since C is closed, $K \subset C$. Thus $f(K) \subset f(C) \subset K$, so Schauder's Theorem applies to the restriction of f to K, and produces the desired fixed point. □

7.4 Application: Initial Value Problems

In Sect. 3.4 we used the Banach Contraction-Mapping Principle to prove that each initial-value problem of the form

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{IVP}$$

with f satisfying appropriate smoothness conditions, has a unique solution on some nontrivial interval centered at x_0. The conditions required by this " Picard–Lindelöf theorem" (Theorem 3.10, p. 35) were that f be defined and continuous on some open subset U of $\mathbb{R}^2$ containing the point (x_0, y_0), and that $\frac{\partial f}{\partial y}$, the partial derivative of f with respect to the second variable, exist on U and be continuous there (or just that f satisfy a uniform Lipschitz condition on U in the second variable). Now, thanks to the Schauder Fixed-Point Theorem, we'll be able to prove the existence of solutions to (IVP) *without having to assume extra second-variable smoothness for* f. However there will be a cost: the solutions need no longer be unique!

Theorem 7.5 (Peano's Theorem)**.** *Suppose f is a real-valued function that is continuous on some open subset of $\mathbb{R}^2$ containing the point (x_0, y_0). Then the initial-value problem* (IVP) *has a solution on some nontrivial interval centered at x_0.*

Proof. By the work of Sect. 3.4 through Lemma 3.9 (p. 34) we know that there is a compact real interval I centered at x_0 such that the Banach space $C(I)$ contains a closed ball $\overline{B}$ centered at the constant function y_0 with $T(\overline{B}) \subset \overline{B}$, where T is the continuous integral operator on $C(I)$ defined by

$$Tu(x) = y_0 + \int_{t=x_0}^{x} f(t, u(t))\, dt \qquad (u \in C(I)).$$

We saw in Sect. 1.3 (p. 4) that a function $y = u(x)$ is a solution on I of (IVP) if and only if it is a fixed point of T. The Schauder Fixed-Point Theorem would immediately provide such a fixed point if only $\overline{B}$ were compact in $C(I)$. Unfortunately Exercise 7.2 above shows that it's not! However, we'll be able to show that $T(\overline{B})$ is relatively compact in $C(I)$, so the existence of a fixed point for T, and therefore of a solution for (IVP), will follow from Corollary 7.4.

To prove this relative compactness it's enough to show, by the Arzela–Ascoli Theorem (Theorem B.8 of Appendix B), that $T(\overline{B})$ is bounded in $C(I)$ and *equicontinuous* on I, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in I \;\; \& \;\; |x - y| < \delta \implies |Tu(x) - Tu(y)| < \varepsilon \qquad \forall u \in \overline{B}.$$

The boundedness of $T(\overline{B})$ has already been established, since $T(\overline{B}) \subset \overline{B}$. As for equicontinuity: fix $u \in \overline{B}$ and note that for each pair of points $x, y \in I$ with $x \le y$ and $|x - y| < \delta$:

$$
\begin{aligned}
|Tu(x) - Tu(y)| &= \left| \int_{t=x_0}^{x} f(t,u(t))\,dt - \int_{t=x_0}^{y} f(t,u(t))\,dt \right| \\
&= \left| \int_{t=x}^{y} f(t,u(t))\,dt \right| \\
&\leq \int_{t=x}^{y} |f(t,u(t))|\,dt \\
&\leq M|x-y| \, .
\end{aligned}
$$

Thus if $\varepsilon > 0$ is given and $|x-y| < \varepsilon/M$, then $|Tu(x) - Tu(y)| < \varepsilon$ for every $u \in \overline{B}$, thus establishing the equicontinuity of $T(\overline{B})$.

Now the equicontinuity of $T(\overline{B})$ carries immediately over to its convex hull $\operatorname{conv}(T(\overline{B}))$, which, being contained in B, is also bounded. Thus $\operatorname{conv}(T(\overline{B}))$ is relatively compact, so K, its closure in $C(I)$, is compact. Since $T(\overline{B}) \subset K$, Schauder's theorem applies to the restriction of T to K, and furnishes the desired fixed point. □

Non-Uniqueness in Peano's Theorem. In contrast to the Banach Contraction-Mapping Principle, Schauder's Theorem makes no claims about uniqueness for the fixed point it produces. The example below shows that non-uniqueness can even occur "naturally." Consider the initial-value problem:

$$
y' = -2y^{1/2}, \; y(0) = 1, \; t \geq 0. \tag{T}
$$

The solution that comes immediately to mind is: $y(t) = (1-t)^2$. Here's another one:

$$
y(t) = \begin{cases} (1-t)^2 & (0 \leq t \leq 1) \\ 0 & (t \geq 1). \end{cases}
$$

The initial-value problem (T) expresses a physical phenomenon discovered by Evangalista Toricelli (1608–1647). *Toricelli's Law* states that water issues from a small hole in the bottom of a container at a rate that is proportional to the square root of the water's depth (see, e.g., [33] for more details). In (T) the function $y(t)$ expresses the depth of the water in the container at time $t \geq 0$. The second solution to (T) is the realistic one for this interpretation; it asserts that the water starts out at $t = 0$ with height 1 and flows out until the container runs dry at $t = 1$, and thereafter stays dry. By contrast, the "obvious" solution $y = (1-t)^2$ for all $t \geq 0$ unrealistically predicts that after the tank runs dry at $t = 1$ it miraculously starts filling up again.

7.5 Application: Multifunctions Again

In the proof of the Kakutani Fixed-Point Theorem (Theorem 6.4, p. 68): if one replaces $\mathbb{R}^N$ with a normed linear space, and Brouwer's Fixed-Point Theorem with Schauder's, then the argument goes through without further change, yielding the following generalization:

Theorem 7.6 (A "Kakutani–Schauder" fixed-point theorem)**.** *If C is a nonvoid compact, convex subset of a normed linear space and $\Phi : C \rightrightarrows C$ is a multifunction with compact, convex values, and closed graph, then Φ has a fixed point.*

Notes

The Schauder Fixed-Point Theorem. Schauder published this result in [107, 1930].

Generalizations of Schauder's theorem. In 1935 Andrey Tychonoff [120] generalized Schauder's Theorem to arbitrary linear topological spaces (see Sect. 9.3, p. 106 for the definition) that are *locally convex,* i.e., for which the neighborhoods of each point have a basis of convex sets. That same year Schauder posed the problem of extending his theorem to complete, metrizable, linear topological spaces. This "Schauder Conjecture" remained open until 2001 when it was settled in the affirmative by Robert Cauty [22]. In [31] Tadeusz Dobrowolski offers an expanded exposition of Cauty's work, along with further references and historical background.

Failure of the fixed-point property for non-compact convex sets. Proposition 7.2, showing that the closed unit ball of infinite dimensional Hilbert space fails to have the fixed-point property, is due to Kakutani[59, 1943]. The generalization to all infinite dimensional normed linear spaces was proved in 1951 by Dugundji [34, Theorem 6.3, p. 362], who showed that in this setting the closed unit ball can always be retracted onto its boundary. Later Victor Klee generalized Dugundji's result even further [62, Sect. 2.3] by showing that *for every metrizable locally convex linear topological space: if a convex set has the fixed-point property, then it must be compact.* Along with Tychonoff's extension of the Schauder theorem, this characterizes for metrizable locally convex spaces the convex sets with the fixed-point property; they are precisely the compact ones.