Chapter 6
Nash's "One-Page Proof"

KAKUTANI'S SET-VALUED FIXED-POINT THEOREM

Overview. In this chapter we'll study Shizuo Kakutani's extension of the Brouwer Fixed-Point Theorem to maps whose values are *sets*, and we'll show how John Nash used Kakutani's result to provide a very quick proof of his famous Theorem 5.11 on the existence of Nash Equilibrium.

Prerequisites. Undergraduate-level real analysis in $\mathbb{R}^N$, basic notions of set theory. The notions of normed linear spaces and metric spaces pop up, but nothing will be lost by thinking of everything in the context of $\mathbb{R}^N$.

6.1 Multifunctions

A *multifunction*[1] Φ from a set X to a set Y (notation: $\Phi\colon X \rightrightarrows Y$) is a function Φ defined on X whose values are *nonempty subsets* of Y.

We'll quite naturally refer to X as the *domain* of Φ, but perhaps somewhat less naturally will refer to Y as its *range*, and will further abuse terminology by defining $\Phi(E)$, the *image* of $E \subset X$, to be the *union* of the sets $\Phi(x)$ as x ranges through E. Proceeding in this vein, the *graph* of Φ will be the subset of the product set $X \times Y$ defined by

$$\operatorname{graph}(\Phi) = \bigcup_{x\in X} \{x\} \times \Phi(x) = \{(x,y) \in X \times Y \colon y \in \Phi(x)\}.$$

Example 6.1 (Inverse Maps). Perhaps the most commonly encountered set-valued map is the inverse map induced by an ordinary function. For such a function $f\colon X \to Y$, define $f^{-1}\colon Y \rightrightarrows X$ by

$$f^{-1}(y) = \{x \in X \colon f(x) = y\} \qquad (y \in Y).$$

[1] Also called a "correspondence," or simply a "set-valued map."

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For example, if $X = \mathbb{R}$ and $f(x) = x^2$ for each $x \in \mathbb{R}$, then $f^{-1}(y) = \{\sqrt{y}, -\sqrt{y}\}$, a set-valued map whose domain is here taken to be the non-negative real numbers, and whose range is $\mathbb{R}$.

Exercise 6.1. For the general inverse mapping $f^{-1}\colon Y \rightrightarrows X$, what is graph (f^{-1}), and how does it relate to the graph of f? How does this play out for the particular map f^{-1} where $f(x) = x^2$ (x real)?

Exercise 6.2 (A step map). Consider the set-valued map $\Phi\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{+1\} & \text{if } x > 0 \end{cases}$$

What is $\Phi(\mathbb{R})$? Sketch the graph of Φ; how does it differ from that of the "usual" step function?

If $\Phi\colon X \rightrightarrows Y$ where X and Y are metric spaces, we'll say Φ is *closed-valued* if $\Phi(x)$ is closed in Y for every $x \in X$, with similar definitions applying to all other topological or geometric properties (openness, compactness, convexity, ...) that sets may have. We'll say Φ has *closed graph* if graph(Φ) is closed in the product space $X \times Y$. For example, the step map defined in Exercise 6.2 above has closed graph.

Exercise 6.3. Show that if X and Y are metric spaces and $f\colon X \to Y$ is continuous, then the graph of $f^{-1}\colon Y \rightrightarrows X$ is closed in $Y \times X$.

Exercise 6.4. Suppose X and Y are metric spaces. Show that if $\Phi\colon X \rightrightarrows Y$ has closed graph then $\Phi(x)$ is closed for each $x \in X$. Is the converse true?

Example 6.2 (The metric projection). Suppose (X, d) is a metric space and K is a compact subset of X. Then for each $x \in X$ there is an element $\xi \in K$ that is nearest to x, i.e.,

$$\inf_{k \in K} d(x, k) = d(x, \xi).$$

The *metric projection* of X onto K is the set-valued map P_K that associates to a point of X the collection of all these elements of K that are nearest to x.

We've already encountered the metric projection in Sect. 4.4, where $X = \mathbb{R}^N$ and K is both compact and convex. In that setting we called P_K the "closest point map," and showed that it's an "ordinary" function in that each of its values is a singleton.

Exercise 6.5. Prove that the metric projection of a metric space onto a compact subset has closed graph.

6.2 Best Response

The notion of "best response" arose during our work in Chap. 5 on Nash Equilibrium. In this section we'll strip the concept down to its essentials and see how it really concerns a multifunction. We begin with an $M \times N$ real matrix A, and its associated quadratic function

$$u(x,y) = xAy^t \qquad (x \in \mathbb{R}^M, y \in \mathbb{R}^N).$$

Exercise 6.6. Show that u is continuous on $\mathbb{R}^M \times \mathbb{R}^N$.

We'll be concerned only with the values u takes on the cartesian product $\Pi_M \times \Pi_N$ of the standard simplices in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, where—as in the last chapter—we think of each simplex as a collection of probability vectors representing mixed strategies for a two-player game, and $u(x,y)$ as Player I's "payoff" for playing strategy $x \in \Pi_M$ against Player II's strategy $y \in \Pi_N$. Recall that for each $y \in \Pi_N$ the function $u(\cdot,y)$ is continuous on the compact subset Π_M of $\mathbb{R}^M$, hence there exists $x^* \in \Pi_M$ such that $u(x^*,y) = \max_{x \in \Pi_m} u(x,y)$.

We've previously called the probability vector x^* a "best response" for the first player to the second player's strategy y. In this section we'll consider the set $\mathrm{BR}(y)$ of *all* best responses to y, thereby obtaining the "best-response multifunction" $\mathrm{BR}\colon \Pi_N \rightrightarrows \Pi_M$ for the payoff-matrix A.

Proposition 6.3. *Each value of BR is a nonvoid compact, convex subset of* Π_M. *The graph of BR is closed in* $\Pi_N \times \Pi_M$.

Proof. We've already noted that $\mathrm{BR}(y)$ is nonempty for each $y \in \Pi_N$. Let's fix y and write $u_y(\cdot) = u(\cdot,y)$, so u_y is an affine continuous function on Π_M. Let μ be the first player's payoff for some best response (hence for *all* best responses) to the second player's strategy y, i.e., $\mu = u(x^*,y) = u_y(x^*)$ for every $x^* \in \mathrm{BR}(y)$. Thus $\mathrm{BR}(y) = u_y^{-1}(\{\mu\})$. Now for each continuous function the inverse image of a closed set is closed, and for each affine map the inverse image of a convex set is convex (Exercise 5.2, page 59). Thus $\mathrm{BR}(y)$, being the image under u_y^{-1} of the closed, convex singleton $\{\mu\}$, is itself a convex subset of Π_M that is closed, and hence compact, in Π_M.

As for the closed graph, suppose $((y_j,x_j))_{j=1}^{\infty}$ is a sequence of points in the graph of BR that converges in $\Pi_N \times \Pi_M$ to a point (y,x), necessarily also in $\Pi_N \times \Pi_M$. We need to prove that (y,x) belongs to the graph of B, i.e., that x is a best response to y. For each j we know that x_j is a best response to y_j, i.e., if $\xi \in \Pi_M$ is fixed, then $u(x_j,y_j) \geq u(\xi,y_j)$. By the continuity of u (Exercise 6.6) this inequality is preserved as $j \to \infty$, so $u(x,y) \geq u(\xi,y)$. Since this last inequality is true for each $\xi \in \Pi_M$, the strategy x is a best response to y, i.e., (y,x) belongs to the graph of BR. □

6.3 The Kakutani Fixed-Point Theorem

To say $x \in X$ is a *fixed point* of $\Phi\colon X \rightrightarrows X$ means that $x \in \Phi(x)$.

Exercise 6.7 (Fixed points of some multifunctions).

(a) For an "ordinary" function $f\colon X \to X$, what does it mean to say that $x \in X$ is a fixed point of $f^{-1}\colon X \rightrightarrows X$?

(b) What are the fixed points of the step function of Exercise 6.2?

(c) For the metric projection as in Example 6.2, what are the fixed points?

Here is the main theorem of this chapter.

Theorem 6.4 (Kakutani's set-valued fixed-point theorem)**.** *If C is a nonvoid compact, convex subset of $\mathbb{R}^N$ and $\Phi\colon C \rightrightarrows C$ is a multifunction with compact, convex values and closed graph, then Φ has a fixed point.*

Were Φ to have a "continuous selection," i.e., a continuous map $\varphi\colon C \to C$ with $\varphi(x) \in \Phi(x)$ for each $x \in C$, then Kakutani's result would follow immediately from the Brouwer Fixed-Point Theorem. However under the hypotheses of Kakutani's theorem such selections need not always exist. For example, the "step-map" Φ of Exercise 6.2, when restricted to (say) the closed interval $C = [-1,1]$, satisfies those hypotheses, but has no continuous selection. Nevertheless there is an "approximate" version of this argument that will carry the day.

To simplify the statement of this "approximate-selection theorem:" for (X,d) a metric space, $E \subset X$, and $\varepsilon > 0$, define $[E]_\varepsilon = \bigcup_{x\in E} B_\varepsilon(x)$, where $B_\varepsilon(x)$ is the open d-ball in X of radius ε, centered at x. Thus $[E]_\varepsilon$ is the set of all points in X that lie a distance $< \varepsilon$ from E. We'll call $[E]_\varepsilon$ the "ε-expansion of E"; clearly, it's an open subset of X.

Exercise 6.8. Suppose C is a convex subset of $\mathbb{R}^N$ (or for that matter, of a normed linear space). Show that the ε-expansion $[C]_\varepsilon$ is convex for every $\varepsilon > 0$.

When the space in which such expansions take place is the product of two metric spaces, we'll take the "product distance" to be the maximum of the distances in the individual factors. More precisely: if (X,ρ) and (Y,σ) are metric spaces, then the metric in $X \times Y$ will be

$$d((x,y),(x',y')) := \max\,[\rho(x,x'),\sigma(y,y')] \qquad ((x,y),(x',y') \in X \times Y).$$

Exercise 6.9. For the product space $\mathbb{R} \times \mathbb{R}$ (with the usual metric on each factor) describe and sketch the open ε-ball with center at the origin. What's the relationship between this "product metric" and the Euclidean metric? Do they have the same convergent sequences?

Theorem 6.5 (The Approximate-Selection Theorem)**.** *Suppose X is a compact metric space, Y a nonvoid compact, convex subset of a normed linear space, and $\Phi\colon X \rightrightarrows Y$ a multifunction all of whose values are compact and convex. If* graph(Φ) *is closed, then for every $\varepsilon > 0$ there exists a continuous function $f\colon X \to Y$ with* graph$(f) \subset [\text{graph}(\Phi)]_\varepsilon$.

Proof of Kakutani's Theorem. Granting the truth of The Approximate-Selection Theorem, use that theorem to provide, for each positive integer n, a continuous function $f_n\colon C \to C$ whose graph lies in $[\text{graph}\,(\Phi)]_{1/n}$. Brouwer's theorem provides, for each n, a fixed point p_n for f_n, hence the pair (p_n, p_n) lies in graph(f_n), and so lies within $1/n$ of graph(Φ). Since C is compact the sequence of "approximate fixed points" (p_n) has a convergent subsequence—say convergent to $p \in C$. But then the corresponding subsequence of pairs (p_n, p_n) converges in $C \times C$ to (p, p), which lies a distance zero from graph(Φ). Since graph(Φ) is closed, (p, p) belongs to it, i.e., $p \in \Phi(p)$. □

For the proof of the Approximate-Selection Theorem we need to explore the consequences for a multifunction of possessing a closed graph.

Lemma 6.6 (Semicontinuity Lemma)**.** *Suppose X and Y are metric spaces with Y compact. If $\Phi\colon X \rightrightarrows Y$ has closed graph, then for every $\varepsilon > 0$ and $x \in X$ there exists a ball B_x centered at x such that $\Phi(B_x) \subset [\Phi(x)]_\varepsilon$.*

Multifunctions satisfying the conclusion of this Proposition are said to be *upper semicontinuous,* reflecting the fact that for ξ close to x the set $\Phi(\xi)$ "cannot be too large" compared to $\Phi(x)$.

Proof of Lemma. Suppose Φ is *not* upper semicontinuous. Then there exists $\varepsilon > 0$, $x \in X$, and a sequence (x_n) in X convergent to x such that $\Phi(x_n)$ is not contained in $[\Phi(x)]_\varepsilon$, i.e., for each index n there exists $y_n \in \Phi(x_n)$ that lies *at least* ε-distant from $\Phi(x)$. Since Y is compact, we may, upon passing to an appropriate subsequence, assume that (y_n) converges to some point $y \in Y$, which also lies at least ε-distant from $\Phi(x)$. Thus the sequence of pairs (x_n, y_n), which belongs to graph(Φ), converges in $X \times Y$ to the pair (x, y) which is *not* in graph(Φ), so graph(Φ) is not closed. □

Exercise 6.10 (Converse to the Semicontinuity Lemma). Suppose X and Y are metric spaces and that $\Phi\colon X \rightrightarrows Y$ is closed-valued and upper semicontinuous. Show that graph(Φ) is closed.

Proof of Theorem 6.5. Fix $\varepsilon > 0$. For each $x \in X$, Lemma 6.6 provides us with a ball B_x centered at x for which $\Phi(B_x) \subset [\Phi(x)]_\varepsilon$. We may take this ball to have radius $< \varepsilon$. The collection of balls $\frac{1}{2}B_x$ (center still at x, but radius half that of B_x) is an open cover of the compact space X, and so has a finite subcover which, to save on subscripts, we'll write as $\mathscr{B} = \{\frac{1}{2}B_1, \frac{1}{2}B_2, \ldots \frac{1}{2}B_n\}$, where B_j is the original ball centered at $x_j \in X$. By Proposition B.6 of Appendix B there is a partition of unity $\{p_1, p_2, \ldots p_n\}$ subordinate to the covering $\mathscr{B}$, i.e., for each index j the function $p_j : X \to [0,1]$ is continuous and vanishes off $\frac{1}{2}B_j$, while the totality of these functions sums to 1 on X. For each index j choose $y_j \in \Phi(x_j)$; then define $f\colon X \to Y$ by

$$f(x) = \sum_{j=1}^{n} p_j(x) y_j \qquad (x \in X). \tag{6.1}$$

Thus f is continuous on X, with values in the convex hull of $\Phi(X)$. We'll be done if we can show that graph$(f) \subset [\text{graph}\,(\Phi)]_\varepsilon$, i.e., that for $x \in X$ the point $(x, f(x))$ of graph(f) lies within ε of graph(Φ).

Fix $x \in X$ and note that since the "partition function" p_j vanishes identically off $\frac{1}{2}B_j$, the sum on the right-hand side of Eq. (6.1) involves only those indices j for which $x \in \frac{1}{2}B_j$. Let $J = J(x)$ denote this collection of indices, so that $f(x) = \sum_{j \in J} p_j(x) y_j$.

Let m be an index in $J(x)$ corresponding to a ball B_j of largest radius for $j \in J(x)$. Note that for each $j \in J(x)$ our point x lies within radius$(\frac{1}{2}B_j) \leq$ radius$(\frac{1}{2}B_m)$ of the center x_j, so all these points x_j for $j \in J(x)$ lie within radius(B_m) of each other. Thus for each $j \in J(x)$ we have (from our semicontinuity-driven initial choice of balls): $\Phi(x_j) \subset [\Phi(x_m)]_\varepsilon$. In particular the point y_j, chosen to lie in $\Phi(x_j)$, also lies in $[\Phi(x_m)]_\varepsilon$. *Conclusion:* $f(x)$ lies in the convex hull of $[\Phi(x_m)]_\varepsilon$. But $\Phi(x_m)$ is convex, hence so is its ε-expansion $[\Phi(x_m)]_\varepsilon$ (Exercise 6.8). Thus $f(x) \in [\Phi(x_m)]_\varepsilon$, i.e., there exists $y \in \Phi(x_m)$ with $d(f(x), y) < \varepsilon$.

Finally, note that x, being a point of $\frac{1}{2}B_m$, lies within radius$(B_m) < \varepsilon$ of x_m, so by our definition of the metric in $X \times Y$ the point $(x, f(x))$ lies within ε of $(x_m, y) \in$ graph(Φ), i.e., $(x, f(x))$ lies within ε of graph(Φ), as promised. □

Remark. In this proof we used the compactness of Y only to deduce the upper semicontinuity of Φ from the closed-graph hypothesis. Thus we could eliminate this extra assumption on Y by requiring at the outset that Φ be upper semicontinuous.

6.4 Application to Nash Equilibrium

We're now in a position to give Nash's "one-page" proof [85] of Theorem 5.11 on the existence of Nash Equilibria for mixed-strategy extensions of non-cooperative finite games.

As usual, we'll keep notation to a minimum by concentrating on the two-person situation. Here the original "pure strategy" game provides M strategies for Player I and N strategies for Player II. The payoff matrices are A and B for the respective players, and the mixed-strategy payoff functions are: $u_{\mathrm{I}}(x,y) = xAy^t$ for Player I, and $u_{\mathrm{II}}(x,y) = xBy^t$, where $x \in \Pi_M$ and $y \in \Pi_N$. Denote the best response multifunction for Player I by BR_{I}, and for Player II by $\mathrm{BR}_{\mathrm{II}}$. Recall that to say the strategy pair $(x^*, y^*) \in \Pi_M \times \Pi_N$ is a *Nash Equilibrium* means that each is a best response to the other, i.e., that $x^* \in \mathrm{BR}_{\mathrm{I}}(y^*)$ and $y^* \in \mathrm{BR}_{\mathrm{II}}(x^*)$. So if we define $\mathrm{BR}\colon \Pi_M \times \Pi_N \rightrightarrows \Pi_M \times \Pi_N$ by

$$\mathrm{BR}(x,y) = \mathrm{BR}_{\mathrm{I}}(y) \times \mathrm{BR}_{\mathrm{II}}(x) \qquad (x,y) \in \Pi_M \times \Pi_N,$$

then we're saying that our strategy pair (x^*, y^*) is a Nash Equilibrium if and only if it's a fixed point of the multifunction *BR*.

Since the set $\Pi_M \times \Pi_N$ on which *BR* is acting is a compact, convex subset of $\mathbb{R}^{MN}$, Kakutani's Theorem 6.4 will produce the desired fixed point if we can verify that *BR* satisfies its hypotheses.

To this end, note that we've already shown (Proposition 6.3) the individual cartesian factors BR_{I} and $\mathrm{BR}_{\mathrm{II}}$ of BR to have nonvoid compact convex values. Since these properties are preserved by cartesian products the values of BR are also nonvoid, compact, and convex. We've also shown that the factors of BR each have graphs that are closed in their ambient compact product spaces, hence they are compact. Now the graph of BR is homeomorphic via a permutation of coordinates to the cartesian product of the graphs of BR_{I} and $\mathrm{BR}_{\mathrm{II}}$, so it too is compact. Thus Kakutani's theorem applies, and shows that BR has the desired fixed point. □

Notes

Kakutani's Theorem. Theorem 6.5, the approximate selection result that did the heavy lifting in the proof of Kakutani's Theorem (Theorem 6.4) is attributed variously to von Neumann [89] and more recently to Cellina [23]. The proof given here is taken from [16, pp. 59–62].

Kakutani's original proof [58]. This takes place on an N-simplex where Kakutani constructs, for each triangulation, a piecewise-affine approximate selection.

Another famous Kakutani. Michiko Kakutani, the influential New York Times literary critic and 1998 Pulitzer Prize winner, is Shizuo Kakutani's daughter.

von Neumann and Kakutani. As mentioned in the *Notes* to the previous chapter, von Neumann proved a special case of Theorem 6.4 on which he based his proof of the Minimax Theorem. The reference for this is [89].